Answer key to: Introduction to Probability

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Homework 1

1.1 Describe, by listing element(s), the set of positive odd integers less than 8.

ANSWER: $\{1, 3, 5, 7\}$

1.2 Describe, by listing element(s), the set which contains the set of the previous problem.

ANSWER: $\{\ \{1, 3, 5, 7\}\ \}$

1.3 What does the notation "$x \notin A$" assert?

ANSWER: the notation "$x \notin A$" asserts the negation of A is a set and x is an element of A; it evaluates to either true (if A is not a set containing x) or else false (if x is an element of the set A).

1.4 If $A = \{0, 1, 2, 3, 4, 5, 6, 8, 9\}$ has membership predicate α, what is the value of $\alpha(8)$?

ANSWER: $\alpha(8)$ is true, because $8 \in A$.

1.5 What does the notation $A = \{x \mid 7 < x \leq 12\}$ assert?

ANSWER: A is the name of a set having membership predicate "$7 < x \leq 12$"; the set consists of elements x in the universal set for which $7 < x \leq 12$.

1.6 If $\{x \mid \alpha(x)\}$ is the set of the previous problem, then what is $\alpha(x)$?

ANSWER: $\alpha(x) =$ "$7 < x \leq 12$".

1.7 Let $\mathcal{U} = \mathbb{R}$, and let $\alpha(x) =$ "$x = 3 + 1/x$". Describe $\{x \mid \alpha(x)\}$ by listing element(s).

ANSWER: $\{\frac{3+\sqrt{13}}{2}, \frac{3-\sqrt{13}}{2}\}$

1.8 If $\alpha(x) =$ false, then what is $\{x \mid \alpha(x)\}$?

ANSWER: the empty set.

1.9 If C and D are sets, how would one show $C = D$?

ANSWER: one would show that their membership predicates are equal.

1.10 If $\mathcal{U} = \mathbb{R}$, then what is $\{x \mid \exists i \in \mathbb{Z}^+ .\, x < i\}$?

ANSWER: the set of real numbers.

1.11 Show that $(A \subset B) \implies (A \cup B = B)$.

ANSWER: let A, B have respective membership predicates α, β. Appealing to the definition of containment, to say $A \subset B$ is to assert $\alpha \Rightarrow \beta$. The membership predicate of $A \cup B$ is $\alpha \vee \beta$, hence to say $A \cup B = B$ is to assert $\alpha \vee \beta = \beta$. In terms of membership predicates, we are being asked to show

$$(\alpha \Rightarrow \beta) \Longrightarrow (\alpha \vee \beta = \beta)$$

There are many ways to show this (a truth table, for instance), but using tautologies shows that $\alpha \vee \beta = \beta$ can be replaced by $\alpha \Rightarrow \beta$. Hence the expression above can be expressed as

$$(\alpha \Rightarrow \beta) \Longrightarrow (\alpha \Rightarrow \beta)$$

Therefore, if we can show that

$$u \Longrightarrow u \tag{1}$$

is a tautology, we are done (choose u to be $\alpha \Rightarrow \beta$). Expanding the definition of $\Rightarrow$ and using tautologies, equation (1) is $\neg u \vee u \;=\;$ true.

1.12 Show that

$$\Big(\bigcap_{i \in I} A_i\Big)^c \;=\; \bigcup_{i \in I} A_i^c$$

ANSWER: let A_i have membership predicate α_i. The membership predicate of the left hand side is

$$\neg\,(\forall i \in I \,.\, \alpha_i(x))$$

The membership predicate of the right hand side is

$$\exists i \in I \,.\, \neg\alpha_i(x)$$

The equality of these membership predicates is a tautology.

1.13 Show that

$$f^{-1}(A \cup B) \;=\; f^{-1}(A) \cup f^{-1}(B) \tag{2}$$

ANSWER: the membership predicate of $A \cup B$ is $p(x) = \alpha(x) \vee \beta(x)$, thus $f^{-1}(A \cup B)$ has membership predicate

$$p(f(x)) \;= \alpha(f(x)) \vee \beta(f(x))$$

The sets $f^{-1}A$ and $f^{-1}B$ have respective membership predicates $\alpha(f(x))$ and $\beta(f(x))$, hence the membership predicate q of $(f^{-1}A) \cup (f^{-1}B)$ is

$$q(x) \;=\; \alpha(f(x)) \vee \beta(f(x))$$

Thus both sides of (2) have the same membership predicate.

1.14 Let $A = \{x \mid \alpha(x)\}$. Show that $A^c = \alpha^{-1}\{\text{false}\}$.

ANSWER: the membership predicate of A^c — the left hand side — is $\neg\alpha$. Since the membership predicate of $\{\text{false}\}$ is

$$p(x) = \text{“}x = \text{false”}$$

the membership predicate q of $\alpha^{-1}\{\text{false}\}$ — the right hand side — is

$$q(x) \;=\; p(\alpha(x)) \;=\; \text{“}\alpha(x) = \text{false”}$$

We need to show $\neg\alpha = q$. Because α is a predicate, there are only two cases to consider.

Case one: $\alpha(x) = \text{true}$; thus $\neg\alpha(x) = \text{false}$.

$$q(x) \;=\; \text{“}\alpha(x) = \text{false”} \;=\; \text{“true} = \text{false”} \;=\; \text{false}$$

Case two: $\alpha(x) = \text{false}$; thus $\neg\alpha(x) = \text{true}$.

$$q(x) \;=\; \text{“}\alpha(x) = \text{false”} \;=\; \text{“false} = \text{false”} \;=\; \text{true}$$

Therefore, $A^c = \alpha^{-1}\{\text{false}\}$, because both sides have the same membership predicate (i.e., $\neg\alpha = q$).

1.15 Use the tautology $\exists i \,.\, \forall j \,.\, \eta(i,j) \Longrightarrow \forall j \,.\, \exists i \,.\, \eta(i,j)$ to show that

$$f\Big(\bigcap_{i\in I} A_i\Big) \;\subset\; \bigcap_{i\in I} f A_i$$

Let A_i have membership predicate α_i. The membership predicate of the left hand side is

$$p(y) \;=\; \exists x \in \mathcal{U} \,.\, \forall i \in I \,.\, \alpha_i(x) \wedge (y = f(x))$$

The membership predicate of the right hand side is

$$q(y) \;=\; \forall i \in I \,.\, \exists x \in \mathcal{U} \,.\, \alpha_i(x) \wedge (y = f(x))$$

According to the definition of containment, we need to show $p \Rightarrow q$. Considering the tautology

$$\exists x \,.\, \forall i \,.\, \eta(x,i) \;\Longrightarrow\; \forall i \,.\, \exists x \,.\, \eta(x,i)$$

we observe its left hand side becomes $p(y)$ and its right hand side becomes $q(y)$ — hence $p \Rightarrow q$ as required — after choosing

$$\eta(x,i) \;=\; \alpha_i(x) \wedge (y = f(x))$$

1.16 Suppose A and B are countable. Show that $A \cup B$ is countable.

ANSWER: let $I = \{i \mid 0 \le i < 2\}$. By definition, I is finite (hence countable). Let $A_0 = A$ and $A_1 = B$. Since I is countable, and A_i is countable for each $i \in I$, it follows from example 61 that

$$\bigcup_{i\in I} A_i \;=\; A_0 \cup A_1 \;=\; A \cup B$$

is countable.

Homework 2

2.1 If Q is a set, then could Q be a sample space? Explain.

ANSWER: if $Q \neq \emptyset$, then yes; a sample space is a nonempty set.

2.2 If $q \in Q$, then could q be an outcome? Explain.

ANSWER: Q can be a sample space (it is nonempty); in that case q is an outcome.

2.3 Assume the context of the previous question. Could $\{q\}$ be an elementary event? Must $\{q\}$ be an elementary event? Explain.

ANSWER: if Q is a sample space, then (by definition) $\{q\}$ is an elementary event corresponding to Q. However, the sample space is chosen as part of a probability model, and it need not be Q; if the sample space does not contain q, then $\{q\}$ would not be an elementary event.

2.4 Let $\mathcal{A} = \{x \,|\, x \subset \Omega\}$. Is $\mathcal{A}$ a σ-algebra (with respect to Ω)? Explain.

ANSWER: yes. A σ-algebra is a nonempty set of subsets (of Ω), which is closed under complementation and countable unions. Since the empty set is a subset of Ω, $\mathcal{A}$ is nonempty (it contains $\emptyset$). It is closed under complementation because if $A \subset \Omega$, then $A^c \subset \Omega$, and $\mathcal{A}$ contains every subset of Ω. It is closed under (countable) unions because the union of any collection of subsets of Ω is some subset of Ω, and $\mathcal{A}$ contains every subset of Ω. Note that the universal set for $\mathcal{A}$ — i.e., the domain of the membership predicate of $\mathcal{A}$ — is not Ω; sets need not have a common universal set.

2.5 Give a precise, technically correct definition in English prose (your written English should not contain mathematical symbols, and it must be grammatically correct) for what "mutually exclusive" means.

ANSWER: a collection of events is *mutually exclusive* if every pair of different events in the collection have no outcome in common (i.e., the intersection of different events is empty).

2.6 What is a probability measure?

ANSWER: A *probability measure* is a function $P : \mathcal{A} \to [0, 1]$ such that

- $P(\Omega) = 1$
- If $\mathcal{C}$ is a countable mutually exclusive collection of events, then

$$P\Big(\bigcup_{A \in \mathcal{C}} A \Big) \;=\; \sum_{A \in \mathcal{C}} P(A)$$

where $\mathcal{A}$ is a $\sigma-$algebra with respect to the sample space Ω.

2.7 Explain the meaning of the following:

If $A_1, A_2, \ldots$ is a countable sequence of events, then

$$\begin{aligned} A_i \uparrow A \quad &\Longrightarrow \quad P(A_i) \uparrow P(A) \\ A_i \downarrow A \quad &\Longrightarrow \quad P(A_i) \downarrow P(A) \end{aligned}$$

ANSWER: the first: if $A_i \subset A_{i+1}$ for all i, then $P(A_i) \leq P(A_{i+1})$ for all i, and

$$\lim_{i \to \infty} P(A_i) \;=\; P\big(\bigcup_i A_i\big)$$

The second: if $A_i \supset A_{i+1}$ for all i, then $P(A_i) \geq P(A_{i+1})$ for all i, and

$$\lim_{i \to \infty} P(A_i) \;=\; P\big(\bigcap_i A_i\big)$$

2.8 What is a Kolmogorov probability space?

ANSWER: A Kolmogorov probability space $(\Omega, \mathcal{A}, P)$ is comprised of a sample space Ω, a σ-algebra $\mathcal{A}$, and a probability measure P.

2.9 Give a precise, technically correct definition in English prose (your written English should not contain mathematical symbols, and it must be grammatically correct) for what "equally likely" means.

ANSWER: Elements of a finite, mutually exclusive collection of events are said to be equally likely if and only if each has the same probability as any other.

2.10 Quality is measured with a score of 0 to 29, and the past three evaluations show a decreasing trend. Rather than evidence of declining quality, consider instead the possibility that evaluations produce random results; model as rolling a fair 30-sided die three times. Give a suitable probability space (faces are numbered $1, 2, \ldots$) and determine the probability each roll will produce a number less than the previous.

ANSWER: Let the sample space be $\Omega \;=\; \{1, \ldots, 30\} \times \{1, \ldots, 30\} \times \{1, \ldots, 30\}$ and let the σ-algebra be $\mathcal{A} = 2^\Omega$ (the set of all subsets of Ω). Let the probability measure P be determined by the condition that elementary events are equally likely. The event in question is

$$A \;=\; \{\omega \,|\, (\omega \in \Omega) \wedge (\omega \text{ has decreasing components})\}$$

and has probability $P(A) \;=\; |A|/|\Omega| \;=\; 4060/27000 \;\approx\; 0.2$.

2.11 A company manufactures 13 types of cars. Prior to full-scale production, 4 of each type are made, and a sample of 5 cars undergo crash testing. The sales department is happy if one of each type remains for display, and quality control is happy if at least 3 types are tested. What has that to do with playing cards? What is the probability sales is happy but quality control is not?

ANSWER: the types correspond to ranks — A, 2, 3, $\ldots$10, J, Q, K — and devices correspond to cards; the testing sample is a five-card hand. If less than 3 types are tested (quality control is not happy), 2 ranks were dealt. If testing leaves one device of each type for display (sales is happy), each rank dealt is represented by either 2 or 3 cards. The probability is that of a full house, ≈ 0.00144.

2.12 A computer chip has 52 pins, and testing consists of setting five pins high and the rest low. If all such patterns need to be checked, what has that to do with playing cards? Suppose the chip has 15 pins, and patterns to be checked are: 4 high, 5 low, and the rest floating. How may tests need to be performed?

ANSWER: make a one-to-one correspondence between pins and cards in a 52-card deck. Set pins high if their corresponding card is in a given five-card hand, and set them low otherwise. The number of test patterns is the number of five-card hands.

Similarly, let there be 15 cards from which 4 are dealt to player "high" and 5 are dealt to player "low". Set pins high if their card was dealt to "high", set them low if their card was dealt to "lo", and let them float otherwise. A hand is determined by where each card goes; it is described by $f : \{1, \ldots, 15\} \to \{\text{high}, \text{low}, \text{undealt}\}$. The number of such functions is the multinomial coefficient

$$\binom{15}{4, 5, 6} \approx 6.3063e + 05$$

2.13 Two matched components are need for two channels in a high precision device, and components have 118 equally likely possible measurements. How many components are needed to have at least 62% probability of finding a matched pair?

ANSWER: Assuming n is the answer, let the sample space be

$$\Omega = \{1, \ldots, 118\}^n$$

Outcome ω is a function mapping the i th component to its measured value ω_i. If M^c is the event that a sample contains no matched components, then

$$|M^c| = \{\omega \,|\, \omega \text{ is injective}\}$$

Appealing to basic counting principles,

$$|M^c| = \frac{118!}{(118 - n)!}, \qquad |\Omega| = 118^n$$

Therefore,

$$P(M) = 1 - P(M^c) = 1 - \frac{118!}{118^n(118 - n)!}$$

Solving for $P(M) \geq 0.62$ yeilds $n = 16$.

2.14 Use basic counting principles to show that if $f : A \to B$ is bijective, then $|A| = |B|$.

ANSWER: A bijective function is both injective and surjective; thus $|A| \leq |B| \leq |A|$ (by basic counting principles).

2.15 Let $t = 2$, $I = \{4, 9\}$, and $x_i = i^2$. Expand and simplify the following macro (show the expansion and simplification process in detail).

$$t! \prod_{j\in I} \ulcorner \sum_{v_j=0}^{\infty} \frac{x_j^{v_j}}{v_j!} \lrcorner \, [t = \sum_{i\in I} v_i]$$

ANSWER:

$$\begin{aligned}
& 2! \prod_{j\in\{4,9\}} \ulcorner \sum_{v_j=0}^{\infty} \frac{(j^2)^{v_j}}{v_j!} \lrcorner \, [2 = \sum_{i\in\{4,9\}} v_i] \\
= \;& 2 \sum_{v_4=0}^{\infty} \frac{(16)^{v_4}}{v_4!} \sum_{v_9=0}^{\infty} \frac{(81)^{v_9}}{v_9!} \, [2 = v_4 + v_9] \\
= \;& 2 \sum_{v_4=0}^{2} \frac{(16)^{v_4}}{v_4!} \frac{(81)^{2-v_4}}{(2-v_4)!} \\
= \;& 2\Big(\frac{(16)^0}{0!} \frac{(81)^2}{2!} + \frac{(16)^1}{1!} \frac{(81)^1}{1!} + \frac{(16)^2}{2!} \frac{(81)^0}{0!} \Big) \\
\approx \;& 9.4090e + 03
\end{aligned}$$

2.16 Example 94 may be extended from three to m types. To streamline notation, denote types by integers; the set of types is $I = \{1, \ldots, m\}$. Given a collection ω of size n, let $n_j(\omega)$ denote how many are type j, and let E be an event with membership predicate $p(n_1(\omega), \ldots, n_m(\omega))$. If type j has probability x_j, then

$$P(E) \;=\; n! \prod_{j\in I} \ulcorner \sum_{v_j=0}^{\infty} \frac{x_j^{v_j}}{v_j!} \lrcorner \, [n = \sum_{i\in I} v_i] \, [p(v_1, \ldots, v_m)]$$

A company tests electronic components, classifying them as type 1 through 4. Past testing suggests the probabilities x_j for the respective types are

$$x_1 = 0.251, \; x_2 = 0.271, \; x_3 = 0.207, \; x_4 = 0.271$$

Suppose 4 components are tested. Estimate the probability that there are not equal numbers of each type.

ANSWER: using the formula above with $m = 4$ and

$$p(v_1, \ldots, v_m) \;=\; [v_1 = \cdots = v_m]$$

gives $\approx 1 - 0.091579 = 0.908421$

2.17 Generalize example 88 as follows. Denote a parallel system comprised of subsystems $s_1, \ldots, s_n$ by $(p, s_1, \ldots, s_n)$. Denote a series system comprised of subsystems $s_1, \ldots, s_n$ by $(s, s_1, \ldots, s_n)$. A subsystem which is a simple component is denoted by the failure probability of that component.

For example, $(s, .2, .3, .1)$ denotes a series system comprised of three components whose failure probabilities are .2, .3, .1. A parallel system comprised of two components with failure probabilities .6, .7 is represented by $(p, .6, .7)$, and a composite system comprised of those two subsystems in series is $(s, (s, .2, .3, .1), (p, .6, .7))$.

What is the failure probability of following system?

$$(s, (p, (s, .23, .24), .17, (s, .29, .1)), (p, (s, .13, .14), (s, .1, .31)), (p, (s, .23, .09), .26, .11))$$

ANSWER:

$$(s, (p, (s, .23, .24), .17, (s, .29, .1)), (p, (s, .13, .14), (s, .1, .31)), (p, (s, .23, .09), .26, .11))$$

$$(s, (p, .4148, .17, .361), (p, .2518, .379), (p, .2993, .26, .11))$$

$$(s, .025456, .095432, .00856)$$

The failure probability is ≈ 0.126005.