Answer key to: Introduction to Probability

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Homework 2

2.1 If Q is a set, then could Q be a sample space? Explain.

ANSWER: if $Q \neq \emptyset$, then yes; a sample space is a nonempty set.

2.2 If $q \in Q$, then could q be an outcome? Explain.

ANSWER: Q can be a sample space (it is nonempty); in that case q is an outcome.

2.3 Assume the context of the previous question. Could $\{q\}$ be an elementary event? Must $\{q\}$ be an elementary event? Explain.

ANSWER: if Q is a sample space, then (by definition) $\{q\}$ is an elementary event corresponding to Q. However, the sample space is chosen as part of a probability model, and it need not be Q; if the sample space does not contain q, then $\{q\}$ would not be an elementary event.

2.4 Let $\mathcal{A} = \{x \,|\, x \subset \Omega\}$. Is $\mathcal{A}$ a σ-algebra (with respect to Ω)? Explain.

ANSWER: yes. A σ-algebra is a nonempty set of subsets (of Ω), which is closed under complementation and countable unions. Since the empty set is a subset of Ω, $\mathcal{A}$ is nonempty (it contains $\emptyset$). It is closed under complementation because if $A \subset \Omega$, then $A^c \subset \Omega$, and $\mathcal{A}$ contains every subset of Ω. It is closed under (countable) unions because the union of any collection of subsets of Ω is some subset of Ω, and $\mathcal{A}$ contains every subset of Ω. Note that the universal set for $\mathcal{A}$ — i.e., the domain of the membership predicate of $\mathcal{A}$ — is not Ω; sets need not have a common universal set.

2.5 Give a precise, technically correct definition in English prose (your written English should not contain mathematical symbols, and it must be grammatically correct) for what "mutually exclusive" means.

ANSWER: a collection of events is *mutually exclusive* if every pair of different events in the collection have no outcome in common (i.e., the intersection of different events is empty).

2.6 What is a probability measure?

ANSWER: A *probability measure* is a function $P : \mathcal{A} \to [0, 1]$ such that

- $P(\Omega) = 1$
- If $\mathcal{C}$ is a countable mutually exclusive collection of events, then

$$P\Big(\bigcup_{A \in \mathcal{C}} A \Big) \;=\; \sum_{A \in \mathcal{C}} P(A)$$

where $\mathcal{A}$ is a $\sigma-$algebra with respect to the sample space Ω.

2.7 Explain the meaning of the following:

If $A_1, A_2, \ldots$ is a countable sequence of events, then

$$\begin{aligned} A_i \uparrow A &\implies P(A_i) \uparrow P(A) \\ A_i \downarrow A &\implies P(A_i) \downarrow P(A) \end{aligned}$$

ANSWER: the first: if $A_i \subset A_{i+1}$ for all i, then $P(A_i) \le P(A_{i+1})$ for all i, and

$$\lim_{i \to \infty} P(A_i) \;=\; P\big(\bigcup_i A_i\big)$$

The second: if $A_i \supset A_{i+1}$ for all i, then $P(A_i) \ge P(A_{i+1})$ for all i, and

$$\lim_{i \to \infty} P(A_i) \;=\; P\big(\bigcap_i A_i\big)$$

2.8 What is a Kolmogorov probability space?

ANSWER: A Kolmogorov probability space $(\Omega, \mathcal{A}, P)$ is comprised of a sample space Ω, a σ-algebra $\mathcal{A}$, and a probability measure P.

2.9 Give a precise, technically correct definition in English prose (your written English should not contain mathematical symbols, and it must be grammatically correct) for what "equally likely" means.

ANSWER: Elements of a finite, mutually exclusive collection of events are said to be equally likely if and only if each has the same probability as any other.

2.10 Quality is measured with a score of 0 to 29, and the past three evaluations show a decreasing trend. Rather than evidence of declining quality, consider instead the possibility that evaluations produce random results; model as rolling a fair 30-sided die three times. Give a suitable probability space (faces are numbered $1, 2, \ldots$) and determine the probability each roll will produce a number less than the previous.

ANSWER: Let the sample space be $\Omega \;=\; \{1, \ldots, 30\} \times \{1, \ldots, 30\} \times \{1, \ldots, 30\}$ and let the σ-algebra be $\mathcal{A} = 2^{\Omega}$ (the set of all subsets of Ω). Let the probability measure P be determined by the condition that elementary events are equally likely. The event in question is

$$A \;=\; \{\omega \,|\, (\omega \in \Omega) \wedge (\omega \text{ has decreasing components})\}$$

and has probability $P(A) \;=\; |A|/|\Omega| \;=\; 4060/27000 \;\approx\; 0.2$.

2.11 A company manufactures 13 types of cars. Prior to full-scale production, 4 of each type are made, and a sample of 5 cars undergo crash testing. The sales department is happy if one of each type remains for display, and quality control is happy if at least 3 types are tested. What has that to do with playing cards? What is the probability sales is happy but quality control is not?

ANSWER: the types correspond to ranks — A, 2, 3, $\ldots$10, J, Q, K — and devices correspond to cards; the testing sample is a five-card hand. If less than 3 types are tested (quality control is not happy), 2 ranks were dealt. If testing leaves one device of each type for display (sales is happy), each rank dealt is represented by either 2 or 3 cards. The probability is that of a full house, ≈ 0.00144.

2.12 A computer chip has 52 pins, and testing consists of setting five pins high and the rest low. If all such patterns need to be checked, what has that to do with playing cards? Suppose the chip has 15 pins, and patterns to be checked are: 4 high, 5 low, and the rest floating. How may tests need to be performed?

ANSWER: make a one-to-one correspondence between pins and cards in a 52-card deck. Set pins high if their corresponding card is in a given five-card hand, and set them low otherwise. The number of test patterns is the number of five-card hands.

Similarly, let there be 15 cards from which 4 are dealt to player "high" and 5 are dealt to player "low". Set pins high if their card was dealt to "high", set them low if their card was dealt to "lo", and let them float otherwise. A hand is determined by where each card goes; it is described by $f : \{1, \ldots, 15\} \to \{\text{high}, \text{low}, \text{undealt}\}$. The number of such functions is the multinomial coefficient

$$\binom{15}{4,5,6} \approx 6.3063e+05$$

2.13 Two matched components are need for two channels in a high precision device, and components have 118 equally likely possible measurements. How many components are needed to have at least 62% probability of finding a matched pair?

ANSWER: Assuming n is the answer, let the sample space be

$$\Omega = \{1, \ldots, 118\}^n$$

Outcome ω is a function mapping the i th component to its measured value ω_i. If M^c is the event that a sample contains no matched components, then

$$M^c = \{\omega \,|\, \omega \text{ is injective}\}$$

Appealing to basic counting principles,

$$|M^c| = \frac{118!}{(118-n)!}, \qquad |\Omega| = 118^n$$

Therefore,

$$P(M) = 1 - P(M^c) = 1 - \frac{118!}{118^n(118-n)!}$$

Solving for $P(M) \geq 0.62$ yeilds $n = 16$.

2.14 Use basic counting principles to show that if $f : A \to B$ is bijective, then $|A| = |B|$.

ANSWER: A bijective function is both injective and surjective; thus $|A| \leq |B| \leq |A|$ (by basic counting principles).

2.15 Let $t = 2$, $I = \{4, 9\}$, and $x_i = i^2$. Expand and simplify the following macro (show the expansion and simplification process in detail).

$$t! \prod_{j\in I} \ulcorner \sum_{v_j=0}^{\infty} \frac{x_j^{v_j}}{v_j!} \lrcorner \, [t = \sum_{i\in I} v_i]$$

ANSWER:

$$\begin{aligned}
& 2! \prod_{j\in\{4,9\}} \ulcorner \sum_{v_j=0}^{\infty} \frac{(j^2)^{v_j}}{v_j!} \lrcorner \, [2 = \sum_{i\in\{4,9\}} v_i] \\
= \; & 2 \sum_{v_4=0}^{\infty} \frac{(16)^{v_4}}{v_4!} \sum_{v_9=0}^{\infty} \frac{(81)^{v_9}}{v_9!} \, [2 = v_4 + v_9] \\
= \; & 2 \sum_{v_4=0}^{2} \frac{(16)^{v_4}}{v_4!} \frac{(81)^{2-v_4}}{(2-v_4)!} \\
= \; & 2 \Big(\frac{(16)^0}{0!} \frac{(81)^2}{2!} + \frac{(16)^1}{1!} \frac{(81)^1}{1!} + \frac{(16)^2}{2!} \frac{(81)^0}{0!} \Big) \\
\approx \; & 9.4090e + 03
\end{aligned}$$

2.16 Example 94 may be extended from three to m types. To streamline notation, denote types by integers; the set of types is $I = \{1, \ldots, m\}$. Given a collection ω of size n, let $n_j(\omega)$ denote how many are type j, and let E be an event with membership predicate $p(n_1(\omega), \ldots, n_m(\omega))$. If type j has probability x_j, then

$$P(E) \; = \; n! \prod_{j\in I} \ulcorner \sum_{v_j=0}^{\infty} \frac{x_j^{v_j}}{v_j!} \lrcorner \, [n = \sum_{i\in I} v_i] \, [p(v_1, \ldots, v_m)]$$

A company tests electronic components, classifying them as type 1 through 4. Past testing suggests the probabilities x_j for the respective types are

$$x_1 = 0.251, \; x_2 = 0.271, \; x_3 = 0.207, \; x_4 = 0.271$$

Suppose 4 components are tested. Estimate the probability that there are not equal numbers of each type.

ANSWER: using the formula above with $m = 4$ and

$$p(v_1, \ldots, v_m) \; = \; [v_1 = \cdots = v_m]$$

gives $\approx 1 - 0.091579 = 0.908421$

2.17 Generalize example 88 as follows. Denote a parallel system comprised of subsystems $s_1, \ldots, s_n$ by $(p, s_1, \ldots, s_n)$. Denote a series system comprised of subsystems $s_1, \ldots, s_n$ by $(s, s_1, \ldots, s_n)$. A subsystem which is a simple component is denoted by the failure probability of that component.

For example, $(s, .2, .3, .1)$ denotes a series system comprised of three components whose failure probabilities are .2, .3, .1. A parallel system comprised of two components with failure probabilities .6, .7 is represented by $(p, .6, .7)$, and a composite system comprised of those two subsystems in series is $(s, (s, .2, .3, .1), (p, .6, .7))$.

What is the failure probability of following system?

$$(s, (p, (s, .23, .24), .17, (s, .29, .1)), (p, (s, .13, .14), (s, .1, .31)), (p, (s, .23, .09), .26, .11))$$

ANSWER:

$$(s, (p, (s, .23, .24), .17, (s, .29, .1)), (p, (s, .13, .14), (s, .1, .31)), (p, (s, .23, .09), .26, .11))$$

$$(s, (p, .4148, .17, .361), (p, .2518, .379), (p, .2993, .26, .11))$$

$$(s, .025456, .095432, .00856)$$

The failure probability is ≈ 0.126005.

Homework 3

3.1 An unreliable test is performed twelve times (model as independent experiments). What is the probability of obtaining at least ten incorrect results if a test is only 47.0% reliable?

ANSWER: appeal to example 112 with $n = 12$, where the two $(m = 2)$ elemetary events are correct and incorrect. The probability of obtaining at least 10 incorrect results is

$$\sum_{v_1=0}^{12} \sum_{v_2=10}^{12} \binom{12}{v_1,\, v_2} (0.470)^{v_1} (1 - 0.470)^{v_2} \;\approx\; 3.1217e-02$$

3.2 A disk contains sectors, 0.9% of which are bad; disk I/O will fail if it involves a bad sector. Model an I/O operation involving n sectors as independent experiments, each of which fails with probability 0.009. How many sectors may an I/O operation involve if the probability of success is at least 0.56?

ANSWER: appeal to example 112, where the two $(m = 2)$ elemetary events are good sector and bad sector. The probability of no bad sectors — a successful I/O operation — is the right hand side below

$$0.56 \;\leq\; \binom{n}{n,\, 0} (0.991)^n (0.009)^0 \;=\; (0.991)^n$$

Solving yields $n \leq 64.1$.

3.3 Assume a test will always detect a certain disease provided a person has it, but the test has a false-positive rate of 1.9%. Suppose the prevalence of the disease is 7.8%. If you test positive for the disease, what is the probability that you actually have it?

ANSWER: let events D, T^+, T^-, correspond to having the disease, testing positive, and testing negative (respectively).

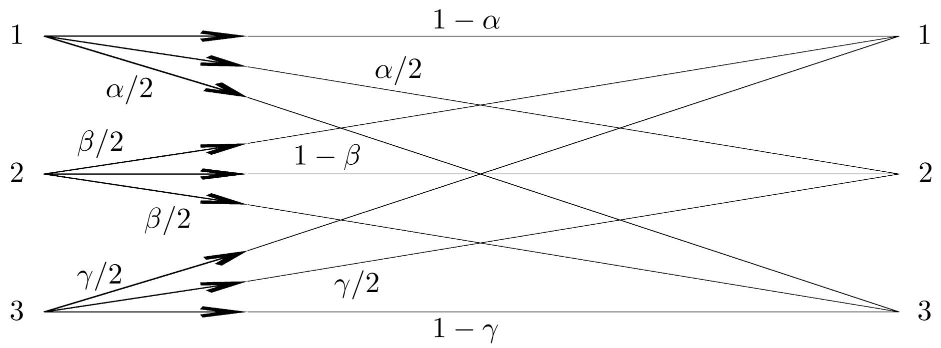
$$\begin{aligned}
P(D|T^+) &= \frac{P(T^+|D)P(D)}{P(T^+)} \;=\; \frac{(1)(0.078)}{P(T^+)} \\
P(T^+) &= P(T^+|D)P(D) + P(T^+|D^c)P(D^c) \;=\; (1)(0.078) + (0.019)(1 - 0.078) \\
P(D|T^+) &= \frac{(1)(0.078)}{(1)(0.078) + (0.019)(1 - 0.078)} \;\approx\; 8.1660e-01
\end{aligned}$$

3.4 An experiment consists of testing four circuits randomly chosen without replacement (model as a sequence of independent experiments) from a production run of size 516. If there are only 349 good circuits, what is the probability that the first circuit passes testing and all remaining tests will fail?

ANSWER:

$$\Big(\prod_{j=0}^{0} \frac{349 - j}{516 - j}\Big)\Big(\prod_{j=1}^{3} \frac{516 - 349 - (j - 1)}{516 - j}\Big)$$

3.5 In the ternary communication channel below, 3 is sent two times as often as 2, and 2 is sent four times more frequently than 1. What is the conditional probability a 1 was sent given that a 2 is observed?



ANSWER: if S_i is the event that i was sent, R_i is the event that i was received, then given information is

$$P(R_1|S_1) = 1-\alpha, \quad P(R_2|S_1) = P(R_3|S_1) = \alpha/2$$
$$P(R_2|S_2) = 1-\beta, \quad P(R_1|S_2) = P(R_3|S_2) = \beta/2$$
$$P(R_3|S_3) = 1-\gamma, \quad P(R_1|S_3) = P(R_2|S_3) = \gamma/2$$
$$P(S_3) = 2P(S_2), \quad P(S_2) = 4P(S_1), \quad 1 = P(S_1) + P(S_2) + P(S_3)$$

Solving the last three equations above yields

$$P(S_3) = 8/13, \quad P(S_2) = 4/13, \quad P(S_1) = 1/13$$

Using Bayes' theorem,

$$P(S_1|R_2) = \frac{P(R_2|S_1)P(S_1)}{P(R_2)} = \frac{(\alpha/2)(1/13)}{P(R_2)}$$

By the law of total probability,

$$\begin{aligned} P(R_2) &= P(R_2|S_3)P(S_3) + P(R_2|S_2)P(S_2) + P(R_2|S_1)P(S_1) \\ &= (\gamma/2)(8/13) + (1-\beta)(4/13) + (\alpha/2)(1/13) \end{aligned}$$

Therefore,

$$P(S_1|R_2) = \frac{(\alpha/2)(1/13)}{(\gamma/2)(8/13) + (1-\beta)(4/13) + (\alpha/2)(1/13)}$$

3.6 Machines A, B, C in a semiconductor facility manufacture 3%, 7%, and 90% of the total chips made (respectively). Suppose their respective defective rates are 1.14%, 2.56%, and 0.24%. A chip, drawn randomly from their combined output, is found to be defective. What is the probability the defective chip was manufactured by machine A? by machine B? by machine C?

ANSWER: to streamline notation, abbreviate the events that a device was produced by machine A, machine B, machine C by A, B, C (respectively), and let D be the event that a device is defective. Appealing to the law of total probability,

$$\begin{aligned} P(D) &= P(D \mid A)\, P(A) + P(D \mid B)\, P(B) + P(D \mid C)\, P(C) \\ &= (0.0114)(0.03) + (0.0256)(0.07) + (0.0024)(0.90) \\ &\approx 4.2940e-03 \end{aligned}$$

Using Bayes' theorem,

$$\begin{aligned} P(A \mid D) &= \frac{P(D \mid A)\, P(A)}{P(D)} \approx (0.0114)(0.03)/0.004294 \approx 7.9646e-02 \\ P(B \mid D) &= \frac{P(D \mid B)\, P(B)}{P(D)} \approx (0.0256)(0.07)/0.004294 \approx 4.1733e-01 \\ P(C \mid D) &= \frac{P(D \mid C)\, P(C)}{P(D)} \approx (0.0024)(0.90)/0.004294 \approx 5.0303e-01 \end{aligned}$$

3.7 A manufacturer of integrated circuits estimates that of the chips produced, 77% meet specifications, 19% need repair, and 4% need to be discarded. If 11 chips are chosen for inspection, estimate the probability that:

- 8 meet specs, 2 need repair, and 1 should be discarded.
- At least 5 need repair.

Hint: make use of the result to example 112.

ANSWER: model as $n = 11$ independent experiments where for each chip there are three possible outcomes; the probability of being good is $p_1 = 0.77$, the probability of needing repair is $p_2 = 0.19$, the probability of being discarded is $p_3 = 0.04$.

The probability that 8 meet specs, 2 need repair, and 1 should be discarded is

$$\binom{11}{8, 2, 1} (0.77)^8 (.0.19)^2 (0.04)^1 \approx 8.8328e-02$$

The probability that at least 5 need repair is

$$\sum_{n_1=0}^{11} \sum_{n_1=0}^{11} \sum_{n_1=0}^{11} \binom{11}{n_1, n_2, n_3} (0.77)^{n_1} (0.19)^{n_2} (0.04)^{n_3} [n_2 \geq 5] \approx 4.1318e-02$$

3.8 A medical breathing apparatus used for anesthesia fails with probability p. Failure means patient death unless a monitor alerts the physician. Suppose the monitor fails with probability q, and system component failures are independent events. A corporation's cost-containment expert argues that if $q > p$, then installation of the monitor is useless. Show the corporation values profits over saving lives, by computing the probability of a patient dying with and without the monitor in place.

ANSWER: without the monitor, the probability of death is p since death is equivalent to breathing apparatus failure. With the monitor, the probability of death is pq since both failure events must occur and the events are independent (the probability that they both fail is the product of their probabilities). Assuming the monitor fails with probability $q < 1$ (a reasonable assumption), the monitor will save lives (according to the probability model) because $pq < p$.

3.9 For the probability space of example 118, verify that each door is equally likely to be the prize door, p_i is the probability you initially choose door i, each of the two unchosen doors are equally likely to be opened, and your initial choice is independent of which door has the prize.

ANSWER: let D_j be the event that the prize is behind door j, let U_k be the event that unchosen door k is opened, and let C_i be the event of initially choosing door i.

ω	$P(\{\omega\})$	D_1	D_2	D_3	U_1	U_2	U_3	C_1	C_2	C_3
$\langle 1,1,2,3\rangle$	$p_1/6$	✓				✓		✓		
$\langle 1,1,3,2\rangle$	$p_1/6$	✓					✓	✓		
$\langle 1,2,1,3\rangle$	$p_2/6$	✓			✓				✓	
$\langle 1,2,3,1\rangle$	$p_2/6$	✓					✓		✓	
$\langle 1,3,1,2\rangle$	$p_3/6$	✓			✓					✓
$\langle 1,3,2,1\rangle$	$p_3/6$	✓				✓				✓
$\langle 2,1,2,3\rangle$	$p_1/6$		✓			✓		✓		
$\langle 2,1,3,2\rangle$	$p_1/6$		✓				✓	✓		
$\langle 2,2,1,3\rangle$	$p_2/6$		✓		✓				✓	
$\langle 2,2,3,1\rangle$	$p_2/6$		✓				✓		✓	
$\langle 2,3,1,2\rangle$	$p_3/6$		✓		✓					✓
$\langle 2,3,2,1\rangle$	$p_3/6$		✓			✓				✓
$\langle 3,1,2,3\rangle$	$p_1/6$			✓		✓		✓		
$\langle 3,1,3,2\rangle$	$p_1/6$			✓			✓	✓		
$\langle 3,2,1,3\rangle$	$p_2/6$			✓	✓				✓	
$\langle 3,2,3,1\rangle$	$p_2/6$			✓			✓		✓	
$\langle 3,3,1,2\rangle$	$p_3/6$			✓	✓					✓
$\langle 3,3,2,1\rangle$	$p_3/6$			✓		✓				✓

Summing probabilities corresponding to D_j gives $(p_1 + p_2 + p_3)/3 = 1/3$ for each j (doors are equally likely to be the prize door). Summing probabilities corresponding to C_i gives $6(p_i/6) = p_i$. Given i, summing probabilities corresponding to $C_i \cap U_k$ gives $3[k \neq i](p_i/6) = [k \neq i]p_i/2$ for each k (unchosen doors are equally likely to be opened). Independence of C_i and D_j requires $P(C_i D_j) = P(C_i)P(D_j) = p_i/3$ for each i and j; summing corresponding probabilities gives $2(p_i/6) = p_i/3$ as required.

3.10 Let $(\Omega, \mathcal{A}, P)$ be the probability space of example 118, and let B denote the event that opening an unchosen door does not reveal the prize. In the conditional probability space $(B, \mathcal{A} \cap B, P|B)$, when is your initial choice independent of which door has the prize?

ANSWER: let $(\Omega, \mathcal{A}, P)$ and D_j, U_k, C_i be as in the previous problem. We need to determine when

$$(P|B)(C_i D_j) \;=\; (P|B)(C_i)\,(P|B)(D_j)$$

The left side is $P(C_i D_j B)/P(B)$ and the right side is $P(C_i B)P(D_j B)/P(B)^2$, thus we are concerned wth determining when the following holds

$$P(B)P(C_i D_j B) \;=\; P(C_i B)P(D_j B)$$

ω	$P(\{\omega\})$	B	D_1	D_2	D_3	C_1	C_2	C_3
$\langle 1,1,2,3\rangle$	$p_1/6$	✓	✓			✓		
$\langle 1,1,3,2\rangle$	$p_1/6$	✓	✓			✓		
$\langle 1,2,1,3\rangle$	$p_2/6$		✓				✓	
$\langle 1,2,3,1\rangle$	$p_2/6$	✓	✓				✓	
$\langle 1,3,1,2\rangle$	$p_3/6$		✓					✓
$\langle 1,3,2,1\rangle$	$p_3/6$	✓	✓					✓
$\langle 2,1,2,3\rangle$	$p_1/6$			✓		✓		
$\langle 2,1,3,2\rangle$	$p_1/6$	✓		✓		✓		
$\langle 2,2,1,3\rangle$	$p_2/6$	✓		✓			✓	
$\langle 2,2,3,1\rangle$	$p_2/6$	✓		✓			✓	
$\langle 2,3,1,2\rangle$	$p_3/6$	✓		✓				✓
$\langle 2,3,2,1\rangle$	$p_3/6$			✓				✓
$\langle 3,1,2,3\rangle$	$p_1/6$	✓			✓	✓		
$\langle 3,1,3,2\rangle$	$p_1/6$				✓	✓		
$\langle 3,2,1,3\rangle$	$p_2/6$	✓			✓		✓	
$\langle 3,2,3,1\rangle$	$p_2/6$				✓		✓	
$\langle 3,3,1,2\rangle$	$p_3/6$	✓			✓			✓
$\langle 3,3,2,1\rangle$	$p_3/6$	✓			✓			✓

Summing up the probabilities for B gives $4(p_1 + p_2 + p_3)/6 = 2/3$. Summing up the probabilities for $C_i B$ gives $4(p_i/6) = 2p_i/3$. Summing up the probabilities for $D_j B$ gives $p_j/6 + (p_1 + p_2 + p_3)/6 = (1 + p_j)/6$. Summing up the probabilities for $C_i D_j B$ gives $[i = j]p_i/6 + p_i/6 = (1 + [i = j])p_i/6$. It follows that we are concerned with

$$\frac{2}{3}\,\frac{1 + [i = j]}{6}\,p_i \;=\; \frac{2p_i}{3}\,\frac{1 + p_j}{6}$$

Let $p_i \neq 0$ (some door is initially chosen); simplifying the above yields

$$[i = j] \;=\; p_j$$

In other words, door i is initially chosen with probability one; all other doors have zero probability of being initially chosen — but in that case doors are *not* equally likely to have the prize, $(P|B)(D_j) = (1 + p_j)/4$.

3.11 Simpson's paradox (example 115) may be expressed in terms of how conditioning can effect correlation. Events A and B are *positively correlated* with respect to the probability measure P if $P(A \cap B) > P(A)P(B)$, and they are *negatively correlated* if $P(A \cap B) < P(A)P(B)$.

The inequality $P(A \,|\, B^c) > P(A \,|\, B)$ is equivalent to saying A and B are negatively correlated, as the following demonstrates:

$$\begin{aligned} P(A \,|\, B^c) \;>\; P(A \,|\, B) \quad &\Longleftrightarrow \quad \frac{P(AB^c)}{P(B^c)} \;>\; \frac{P(AB)}{P(B)} \\ &\Longleftrightarrow \quad \frac{P(A) - P(AB)}{1 - P(B)} \;>\; \frac{P(AB)}{P(B)} \\ &\Longleftrightarrow \quad (P(A) - P(AB))P(B) \;>\; P(AB)(1 - P(B)) \\ &\Longleftrightarrow \quad P(A)P(B) \;>\; P(AB) \end{aligned}$$

Reversing inequalities in the above shows that the positive correlation of A and B is equivalent to $P(A \,|\, B^c) < P(A \,|\, B)$.

Use the above to recast example 115 as saying: given a probability space $(\Omega, \mathcal{A}, P)$ and negatively correlated events A and B (with respect to P), it might happen that there exists an event C for which A and B are positively correlated with respect to both $P|C$ and $P|C^c$.

ANSWER: note that

$$P(U \,|\, VW) \;=\; \frac{P(UVW)/P(W)}{P(VW)/P(W)} \;=\; \frac{P(UV \,|\, W)}{P(V \,|\, W)} \;=\; \frac{(P|W)(UV)}{(P|W)(V)} \;=\; (P|W)(U \,|\, V)$$

Hence to say A and B are positively correlated with respect to $P|C$ is equivalent to

$$P(A \,|\, BC) \;=\; (P|C)(A \,|\, B) \;>\; (P|C)(A \,|\, B^c) \;=\; P(A \,|\, B^c C)$$

To say A and B are positively correlated with respect to $P|C^c$ is equivalent to

$$P(A \,|\, BC^c) \;=\; (P|C^c)(A \,|\, B) \;>\; (P|C^c)(A \,|\, B^c) \;=\; P(A \,|\, B^c C^c)$$

If A and B are negatively correlated (with respect to P) then

$$P(A \,|\, B) \;<\; P(A \,|\, B^c)$$

That the inequalities displayed above can be true simultaneously is the message of example 115.

3.12 If F_t denotes the event that failure happens at or before time t, then $P(F_t\,|F_s^c)$ is the probability failure happens by time t given it has not occurred by time s. The limit as $s\uparrow t$ (as s increases to t) of the rate of change with respect to t of that conditional probability is the *hazard rate* (assuming differentiability),

$$h(t) \;=\; \lim_{s\uparrow t} \frac{\partial}{\partial t} P(F_t\,|F_s^c)$$

Note that if $s < t$, then $F_s \subset F_t$, and therefore

$$P(F_t) \;=\; P(F_t(F_s \cup F_s^c)) \;=\; P(F_s \cup F_t F_s^c) \;=\; P(F_s) + P(F_t F_s^c)$$

Hence $P(F_t F_s^c) = \mathcal{F}(t) - \mathcal{F}(s)$, where $\mathcal{F}(x) = P(F_x)$. Use this observation to obtain a formula for the hazard rate, expressing it in terms of $\mathcal{F}$.

ANSWER: Since

$$P(F_t\,|F_s^c) \;=\; \frac{P(F_t F_s^c)}{P(F_s^c)} \;=\; \frac{\mathcal{F}(t) - \mathcal{F}(s)}{1 - \mathcal{F}(s)}$$

it follows that

$$h(t) \;=\; \lim_{s\uparrow t} \frac{\partial}{\partial t} \frac{\mathcal{F}(t) - \mathcal{F}(s)}{1 - \mathcal{F}(s)} \;=\; \lim_{s\uparrow t} \frac{\mathcal{F}'(t)}{1 - \mathcal{F}(s)} \;=\; \frac{\mathcal{F}'(t)}{1 - \mathcal{F}(t)}$$