Mathematics prerequisites

Elements of Set Theory

Definition

A set is a collection of pairwise distinct objects called elements.
A set can be defined in two ways:

- in extension: we give the list of elements;
- in comprehension: we give a common property verified by the elements of the set.

Example
Let E be the set of even integers between 0 and 10. Then the elements of E are 0, 2, 4, 6, 8 et 10. We write E = {0, 2, 4, 6, 8, 10}.

Notes

To say that a mathematical object x is an element of a set A, we write: $x \in A$. When x is not an element of A, we write: $x \notin A$.
With E = {0, 2, 4, 6, 8, 10}, we have : $4 \in E$ et $5 \notin E$.

Definition Empty Set

There exists a set which does not contain any element, it is the empty set denoted $\varnothing$.

Set Cardinality

The cardinality of a set is the number of elements in that set. The cardinal of a set A is denoted card(A)

Inclusion

Set A is a subset of Set B if all the elements of A are elements of B, in other words
$\forall x, x \in A \Rightarrow x \in B$

We note it $A \subseteq B$ (A included in B).
Example: $\{0, 1, 2\} \subseteq \{0, 1, 2, 3\} \subseteq N \subseteq Z \subseteq Q \subseteq R \subseteq C$

Definition

Two sets A and B are equal when they have the same elements. We write : $A = B$.
Thereby, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
Example: If $E = \{0, 2, 4, 6, 8, 10\}$ and $F = \{6, 8, 10, 0, 2, 4, 6, 8\}$, we have : $E = F$.

Set of subset

Let A be a set, the set of parts of A denoted P(A) is the set of subsets of A.

Note

We always have:

- $\varnothing \in P(A)$ because $\varnothing \subseteq A$,
- $A \in P(A)$ because $A \subseteq A$.

Set Operations

Intersection

If A and B are two sets, we denote $A \cap B$ the set of mathematical objects that belong to A and B. $A \cap B$ reads « A inter B » or « the intersection of A and B ». We note that $A \cap B$ is a subset of A and a subset of B.

Example if $E = \{0, 2, 4, 6, 8, 10\}$ and $F = \{3, 10, 2, 8, 8, 5\}$, then $E \cap F = \{2, 8, 10\}$.

Union

If A and B are two sets, we denote $A \cup B$ the set of mathematical objects that belong to A or B. $A \cup B$ reads «A union B» or «the union of A and B». Note that A and B are subsets of $A \cup B$.

Example if $E = \{0, 2, 4, 6, 8, 10\}$ and $F = \{3, 10, 2, 8, 8, 5\}$, then $E \cup F = \{0, 2, 3, 4, 5, 6, 8, 10\}$.

Properties

Idempotence : $A \cup A = A$
Commutativity : $A \cup B = B \cup A$
Associativity : $A \cup (B \cup C) = (A \cup B) \cup C$

Idempotence : $A \cap A = A$
Commutativity : $A \cap B = B \cap A$
Associativity : $A \cap (B \cap C) = (A \cap B) \cap C$
Neutral element : $A \cup \varnothing = A$
Neutral element : $A \cap \Omega = A$

Distributivity
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ et $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Difference

A\B = {elements in A but not in B}

Symmetric difference

$A \Delta B$ = {elements in $A \cup B$ but not in $A \cap B$} = $(A \cup B) \setminus (A \cap B)$

Complementary

$\overline{A} = \Omega \setminus A$

Properties

Involution : $\overline{\overline{A}} = A$
Morgan's Law : $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Cartesian product

The Cartesian product of two sets A and B noted $A \times B$ is the set defined by:
$A \times B = \{(a, b)$ where $a \in A$ et $b \in B\}$.
$A_1 \times \cdots \times A_k = \{(a_1, \ldots, a_k)$ where $a_i \in A_i$ for all $i \in \{1, \ldots, k\}\}$.

Example :

For the RGB computer color coding system (“Red, Green, Blue”), a color is an element of $[0, 255] \times [0, 255] \times [0, 255] = [0, 255]^3$
two colors that have the same triplet are equal; we can define color sets:

$\{\text{predominantly green color}\} = \{(r , g , b) : g \geq (r + b)\}$

Application to Haskell

In programming languages, like Haskell, some objects are declared/defined with a certain data type:

- Bool is interpreted as the set {True, False},
- Int is interpreted as the set of integers between -2^{63} et 2^{63} -1.
- Integer is interpreted as the set of relative integers.
- Float is interpreted as the set of single precision floating point numbers.
- String is interpreted as the set of character strings

The symmetric difference ($A \triangle B$) corresponds to the symbol | in Haskell

Overview of functions

Definitions

Let A and B two non empty sets. Let $\mathcal{R}$ be a relation from A to B. La relation $\mathcal{R}$ is a function if any element x of A is related to at most one element of B. i.e: For any element x of A, for any element y of B and for any element z of B if x is related to y and x is related to z then y = z.

$\forall\, x \in\ A, \forall\, y \in\ B, \forall\, z \in B,\ \ x\, R\, y\ \ \wedge\, x\, R\, z\ \Rightarrow y\ \ = z$

In other words, the same causes are said to produce the same effects.

Note

Let f be a function f from A to B. if x is related to y by f, then:

- We denote f (x) = y.
- y is called image of x by f and x is called antecedent of y by f.
- A is called the starting set and B is called the ending set.
- We write $f\!:\ A\ \rightarrow\ B$

  $x\ \rightarrow\ f(x)$

Example

Let E be the relation from the IR set of real numbers to the Z set of relative integers which to any real number associates its integer part (the largest of the integers smaller than this number). The relation E is a function. 3.2 is related to 3.

$E(3, 2)\ \ =\ \ 3,\ E(-\ 6, 2)\ \ =\ \ 7$. At any time, $E(3, 2)$ Always gives 3.

$E\!:\ IR\ \rightarrow\ Z$

$x\ \rightarrow\ E(x)$

Counterexample

Consider the relationship $\Re$ of Z (set of relative integers) toward Z which to any relative integer x associates the relative integer y such that x is the square of y. This relation is not a function. Because :

4 is the square of 2. So 4 $\Re$ 2

4 is the square of -2. So 4 $\Re$ -2

4 is related to more than one element (-2 and 2). The relation $\Re$ is therefore not a function.

The same causes do not produce the same effects. Indeed, if $\Re$ is a function, what would be $\Re(4)$? 2 or -2? At one time we will have 2, at another time we will have -2!

Definitions

Let f be a function from set A to set B. The image of the function f, denoted Im(f) is the subset of B made up of all the images of the elements of the starting set A.

The inverse image of an element y of the target set B, denoted $f^{-1}(y)$, is the set of elements of A whose image by the function f is y.

The inverse image of a part P of the target set B, denoted f^{-1} (P), is the set of elements of A whose image by the function f is contained in P.

Let f be the function defined by $f(x) = x^2 + 2$, complete relationships
f(0) = …
$f^{-1}(6)$ = ……
Im(f) = ……
$f^{-1}([11; 27])$ = ……

Definition

The definition set of a function f is the set of elements x for which f(x) exists. We denote D(f) this set (or simply Df).

Function operations

Let f and g be two functions defined from A to B. we consider that on B are defined the operators $+, \ -, \ \times, \div$. We define their sum, difference, product and quotient by stating that:

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(f \times g)(x) = f(x) \times g(x)$

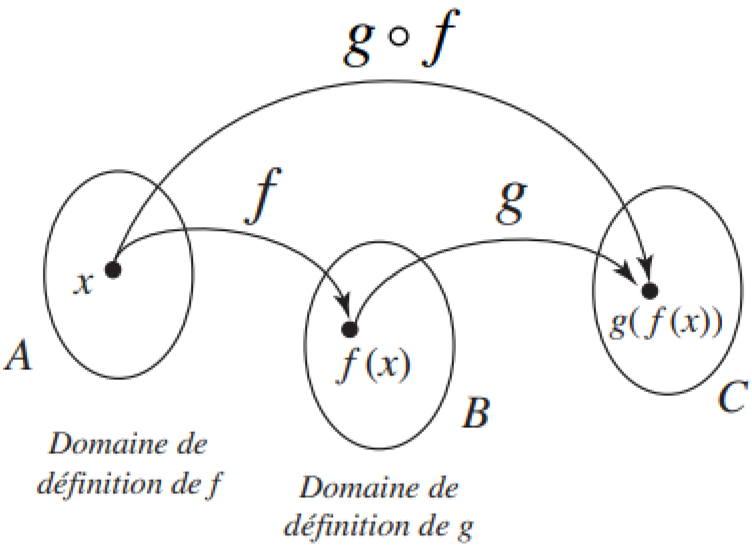
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Function composition

Definition

The composite function $(g \circ f)$ of two functions f and g is defined by $(g \circ f)(x) = g(f(x))$

The definition set of $g \circ f$ is the set of all x in the definition set of f such that f(x) is in the definition set of g.



Example (to be completed)

1. f (x) = x^2 and g(x) = 2x + 1 then
   a. (f ∘ g)(x) = …
   b. (g ∘ f)(x) = ...
2. f (x) = x^2 + 1 et g(x) = $\frac{1}{x}$ then
   a. g (f (x)) = …
   b. f (g(x)) = …
3. f (x) = $\sqrt{x}$ et g(x) = 2x – 6 then

a. $(f \circ g)(x) = \ldots$
b. $(g \circ f)(x) =$

Extension and restriction of a function

Definition

Let f be a function defined from a set A to a set B. Let C be a subset of A ($C \subseteq A$). The restriction of f on C denoted $f|_C$ is the function defined from C to B such that for all

$x \in C, f|_C(x) = f(x)$

Let f be a function defined from a set A to a set B. Let C be a set such that C contains A ($A \subseteq C$). Let g be a function from C to B. the function h from C to B defined by:

- $\forall x \in A, h(x) = f(x)$
- $\forall x \in C \backslash A, h(x) = g(x)$

is called extension of the function f on C.

Multivariate function

Definition

Let f be a function from a set A to a set B. f is a multivariate function if A is a Cartesian product of at least two sets. That is to say there exists k greater than or equal to 2 such that
$A = A_1 \times \cdots \times A_k = \{(a_1, \ldots, a_k)$ where $a_i \in A_i$ pour tout $i \in \{1, \ldots, k\}\}$.

$$f: A \rightarrow B$$
$$\left(x_1, x_{2,}..., x_k\right) \rightarrow f\left(x_1, x_{2,}..., x_k\right)$$

Example

$$f: IR^2 \rightarrow IR$$
$$(x, y) \rightarrow \sqrt{x^2 + y^2}$$

Parametric functions

Let m be an element of a set denoted by M. Let f be a function from A to B. The function f is parametrized by m if $\forall x \in A, f(x)\ is\ expressed\ in\ terms\ of\ m$. We denote f_m.

Example

Let m be a real number.

$f : IR \rightarrow IR$

$\quad x \rightarrow f(x) = 2x + m$

f is a function parameterized by m.

Let k be a real number.

$f : IR \rightarrow IR$

$\quad x \rightarrow \sqrt{x^2 + k^2}$ is a function parameterized by k

Note

if f_m is a function parameterized in m from A to B (with m belonging to M), then f_m comes from the function g defined as:

$g : A \times M \rightarrow B$

$\quad (x, m) \rightarrow f_m(x)$

f_m (x) is obtained by fixing the variable m in the function g.

Application to Haskell

This is applied in Haskell through the following concepts:

- Currying,
- Higher Order Function,
- Parameterized Functions