Math Prerequisites: Practice Exercises

Set

Exercice 1

Writing in extension

Write in extension (i.e. by giving all their elements) the following sets:

$$A = \left\{ integer\ numbers\ between\ \sqrt{2}\ and\ 2\pi \right\}$$

$$B = \left\{ x \in \mathbb{N};\ \exists\ (n, p) \in \mathbb{N} \times \mathbb{N},\ x = \frac{p}{n}\ and\ 1 \leq p \leq 2n \leq 7 \right\}$$

Exercice 2

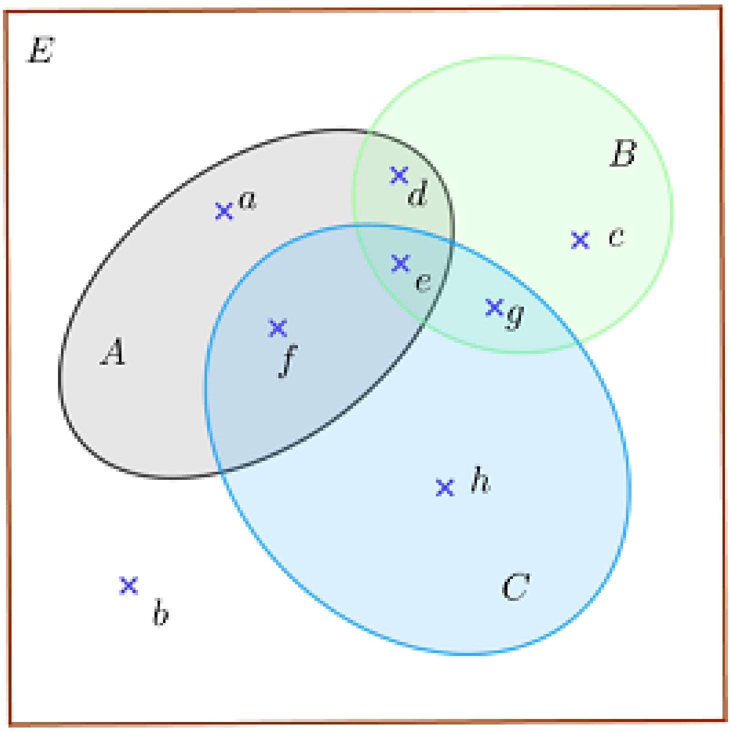
Two descriptions of the same set

Let $A = \left\{ (x, y) \in \mathbb{R}^2;\ 4x - y = 1 \right\}$ $and\ C = \{(t + 1,\ 4t + 3);\ t \in \mathbb{R}\}$
Prove that A = C.

Exercice 3

Venn diagram

We consider the following Venn diagram, with A, B, C three subsets of a set E, and a, b, c, d, e, f, g, h elements of E.



State whether the following statements are true or false:

1. $g \in A \cap \overline{B}$;
2. $g \in \overline{A} \cap \overline{B}$;
3. $g \in \overline{A} \cup \overline{B}$;
4. $f \in C \backslash A$;
5. $e \in \overline{A} \cap \overline{B} \cap \overline{C}$;
6. $\{h, b\} \subset \overline{A} \cap \overline{B}$;
7. $\{a, f\} \subset A \cup C$;

Exercice 4

Subsets of a union

Does the fact that $C \subset A \cup B$ *implies that* $C \subset A$ *or* $C \subset B$?

Exercice 5 three sets

Let A, B, C be three sets such that $A \cup B = B \cap C$. Show that $A \subset B \subset C$.

Exercice 6

Morgan's Laws

Let A, B and C be three subsets of a set E. For $X \subset E$, we denote by X^C the complement of X in E. Prove the following Morgan's laws:

1. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
2. $(A^C)^C = A$
3. $(A \cap B)^C = A^C \cup B^C$
4. $(A \cup B)^C = A^C \cap B^C$.

Exercice 7

Equal union and intersection

Let E be a set and A, B, C three elements of P(E).

1. Prove that, if $A \cap B = A \cup B$, then $A = B$
2. Prove that, if $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$. Is only one of the two conditions sufficient?

Exercice 8

Symmetric difference

Let E a set, A and B two subset of E.

1. Show that $A \Delta B = (A \cap \overline{B}) \cup (B \cap \overline{A})$
2. Calculate $A \Delta A$, $A \Delta \phi$, $A \Delta E$, $A \Delta \overline{A}$
3. Prove that for all A, B, C subset of E, we have: $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$.

Exercice 9

Back to the symmetric difference

Let E a set, A and B two subset of E. We recall that

$A \Delta B = (A \cap \overline{B}) \cup (B \cap \overline{A})$

Show that $A \Delta B = B \text{ if and only if } A = \phi$

Exercice 10

Equations and sets

Let E a set, A and B two subset of E. Solve the following equations, of unknown X

1. $A \cup X = B$
2. $A \cap X = B$

Exercice 11

Characteristic function

Let A be a subset of a set E. We call the characteristic function of A the function f of E to the set with two elements {0, 1} such that:

$f(x) = 1 \text{ if } x \in A$

$f(x) = 0 \text{ if } x \notin A$

Let A and B be two subsets of E, f and g their characteristic functions. Show that the following functions are the characteristic functions of sets that will be determined:

1. $1 - f$
2. fg
3. $f + g - fg$

determine the following sets: $f^{-1}(1)$, $f^{-1}(0)$, $f(A)$, $f(\overline{A})$

Exercice 12

Set of subsets

Write the set of all subsets of E = {a, b, c, d}.

Exercice 13

Set of subsets, intersection and union

Let E and F be two sets.

1. Let A be a subset of E∩F. Is A subset of E? of F? Deduce a comparison of P(E∩F) with P(E)∩P(F).
2. Let B be a set which is both contained in E and also in F. Is B contained in E∩F? Deduce a second comparison of P(E∩F) with P(E) ∩ P(F).
3. Prove that P(E)∪P(F) is included in P(E∪F).Give a simple example proving that the converse inclusion is not always true.

Exercice 14

Cartesian product

Let $D = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 \leq 1\}$. Prove that D cannot be written as the Cartesian product of two subsets of $\mathbb{R}$.

Exercice 15

Cartesian product and intersection

Let E and F be two sets, let A, C two subsets of E and B, D two subsets of F. Prove that (A × B) ∩ (C × D) = (A∩C) × (B ∩ D).

Functions

Exercice 1

Verbal definitions

Preliminary1: *In mathematics, a relation is a* ***mathematical statement*** *that describes a connection between various objects.*
A relation can be defined from three elements: a starting set, an ending set and a rule that associates certain elements of the starting set with certain elements of the ending set.
A binary relation between two sets E and F is characterized by a subset of the Cartesian product E × F, i.e. a collection of pairs whose first component is in E and the second in F. The subset G of E × F which characterizes the relation is called ***the graph*** *of the relation.*

Let the set E = {2, 3, 4, 5} , the set F = {1, 2, 4, 8, 16} and the relation $\mathcal{R}$ *: « … is a divisor of … » which associates the elements of E to the elements of F.*

1. Define what a relation is
2. What is the graph of the relation $\mathcal{R}$?
3. Define what a function is
4. Is the relation $\mathcal{R}$ a function?
5. What is the difference between a relation and a function?
6. What is the difference between the input and the output of a function?

Preliminary 2: *Equivalence relation and order relation*
Let R a relation from E to E.We say that the relation R is

- **reflexive** if, for all $x \in E,\ x\, R\, x$
- **Symmetric** if, for all $x,\ y \in E,\ if\ xRy\ then\ yRx$
- **anti-symmetric** if, for all $x,\ y\ \in\ E,\ if\ xRy\ and\ yRx\ then\ x\ = y$

- **transitive** if, for all $x,\ y,\ z \in E\ if\ xRy\ and\ yRz\ then\ xRz$

An equivalence relation is a reflexive, symmetric and transitive relation.
If R is an equivalence relation and x is an element of E, we call equivalence class of x the set of elements y of E such that xRy.

An order relation is a reflexive, anti-symmetric and transitive relation.

1. Let *a* be a strictly positive integer. We define a relation on $\mathbb{R}$ denoted ≡ by
   $x \ \equiv \ y\,[a] \ \Leftrightarrow \ \exists k \in \mathbb{Z},\ x \ = \ y \ + \ ka$
   Show that ≡ is an equivalence relation.
   What is the equivalence class of 1 when a = π

2. Let p be a strictly positive integer. Let define the relation R_p on $\mathbb{Z}$ by
   $n\ R_p m\,[p] \ \Leftrightarrow \ \exists k \in \mathbb{Z},\ n \ = \ m \ + \ kp$
   Show that R_p is an equivalence relation.
   What is the equivalence class of 7 when p = 11.
1. Let A be a nonempty set and P(A) the set of subsets of A. We define on P(A) the relation R defined by:
   $X\ R\ Y \ \Leftrightarrow \ X \subseteq \ Y$
   Show that R is an order relation

Exercice 2

Algebraic

Determine whether the relations below represent a function. (x, y) stand for x is in relation with y

1. $\{(a, b),\ (c, d), (a, c)\}$ $\ the\ elements\ a,\ b,\ c\ are\ two\ by\ two\ distinct$
2. $\{(a, b),\ (b, c),\ (c, c)\}$ $\ the\ elements\ a,\ b,\ c\ are\ two\ by\ two\ distinct$
3. {(*Mbog Bassong, Cosmogenese 1*), (*Cheikh Anta Diop, Nation Naigre et Culture*), (*Cheikh Anta Diop, Civilisation ou Barbarisme*))}

Exercice 3

Algebraic

Determine whether the relations below between x and y represent functions.

1. $x = y^3$
2. $y = \pm\sqrt{(1 - x)}$
3. $y = x^3$
4. $y^2 = x^2$

Exercice 4

Evaluating functions

Evaluate the function $f(x) = 2x - 5$; $f(x) = \sqrt{2 - x} + 5$; $f(x) = \frac{6x-1}{5x+2}$ (apply the fonction to an argument) at the indicated value

1. $f(-3)$
2. $f(2)$
3. $f(-a)$
4. $-f(a)$
5. $f(a + h)$
6. $f(\frac{-2}{5})$

Exercice 5

Images and Antecedents

Given the function $k(t) = 2t - 1$

1. Evaluate $k(2)$
2. Solve $k(t) = 3$
3. What can we deduce from the above results?

Exercice 6

Domain of a function

Find the domain of definition for each of the following functions:

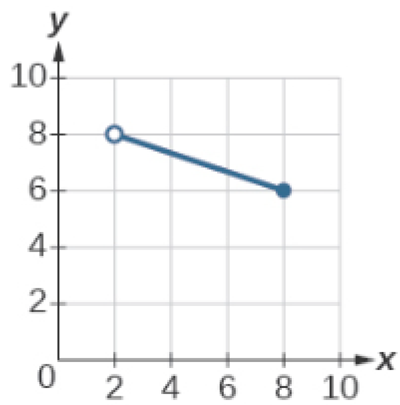
1. $f(x) = 5 - 2x^2$
2. $f(x) = \sqrt{4 - 3x}$
3. $f(x) = \frac{9}{(x-6)}$

Exercice 7

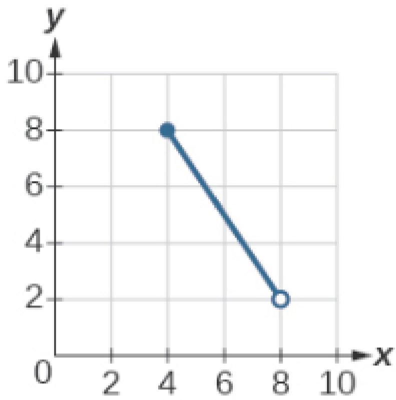
Domain and Range of a function

Find the domain of definition and range for each of the following functions:

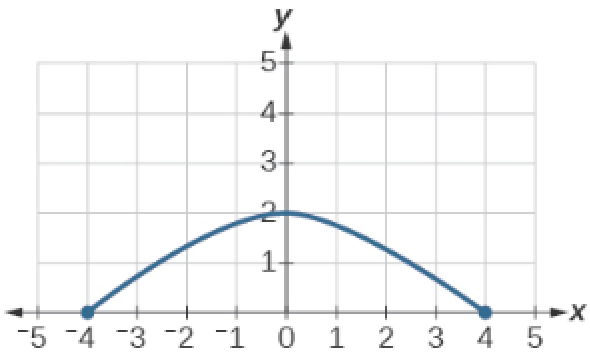
1. Function 1



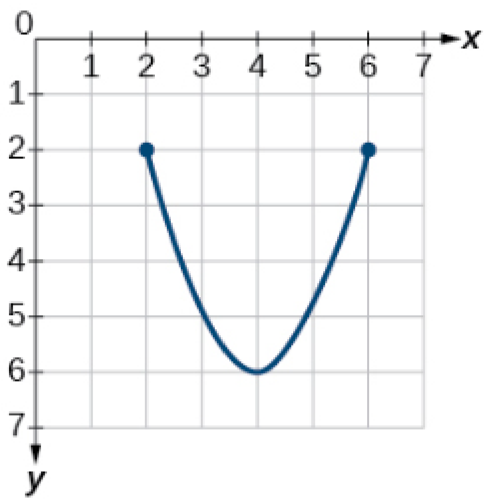
2. Function 2



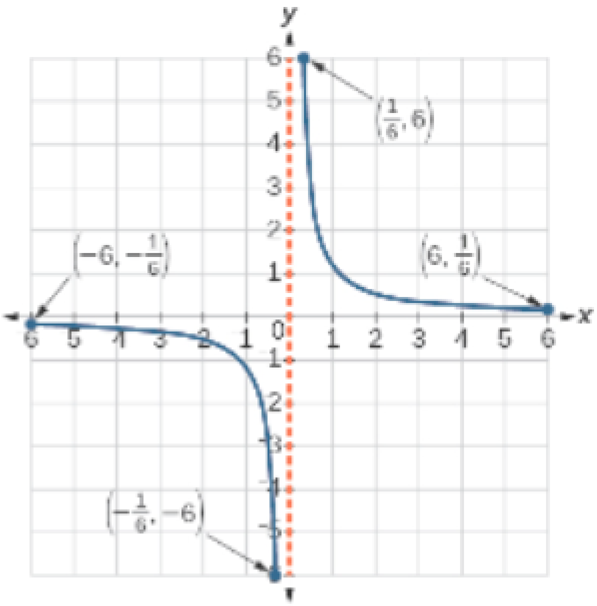
3. Function 3



4. Function 4



5. Function 5



Exercice 8

Real World Application

The height h of a projectile is a function of the time t it is in the air. The height in feet for seconds is given by the function: $h(t) = -16t^2 + 96t$

1. What is the domain of the function?
2. What does the domain mean in the context of the problem?

Exercice 9

Real World Application

The cost in MaatCoin (a local African token) for digitizing (into a NFArt) a stolen artwork is given by the function: *cost (numberOfPieces) = 10 * numberOfPieces + 500*

1. Calculate the fixed cost. The fixed cost is determined when zero items are produces
2. What is the cost for making 12 items?
3. How many items could I make when given 600 MatCoins?

4. Let suppose the maximum production cost allowed is 15000 MaatCoins. What is the domain and range of the cost function?

Exercice 10

Real World Application: Parametric functions

The cost in MaatCoin (a local African token) in digitizing (into a NFArt) a stolen artwork is given by the function: cost m of (numberOfPieces) = m * numberOfPieces + 500 where m represents the unitary price of a given piece of artwork.

1. Redo Exercise 7 with m fixed to:
   a. m is 20
   b. m is 3
   c. m is 7

Exercice 11

Local Context Application

The amount of garbage (plastic wrappers, putrefied food, …etc), G, produced by a city in Cameroon with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in millions of people.

The town of Douala has a population of 5,000,000 and produces 2,500 tons of garbage each week. Express this information in terms of the function f.Douala has a population of 5,000,000 and produces 2,500 tons of garbage each week. Express this information in terms of the function f .
Explain the meaning of the statement $f(5) = 2$.

How can we turn this problem into a parametric function where the parameter represents the amount of garbage per city?

Exercice 12

Local Context Application

The number of cubic meters of dirt, D , needed to cover a garden with an area a in square meters is given by $D = g(a)$.

A garden with area 5000 m^2 requires 50 m^3 of dirt.

1. Express this information in terms of the function g .
2. Explain the meaning of the statement $g(100) = 1$.

Exercice 13

Defining Operation / Composition of Functions

Assuming f and g are well defined functions,

1. How do you define the function $\frac{f}{g}$ given f and g are functions
2. How do you define the function $f - g$ given f and g are functions
3. How do you define the function $f + g$ given f and g are functions
4. How do you define the function $f * g$ given f and g are functions
5. How do you define $f \circ g$?
6. How do you find the domain for the composition of two functions ?

Exercice 14

Composition of Functions

Given $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$, find the following:

1. $f(g(2))$
2. $f(g(x))$
3. $g(f(x))$
4. $(g \circ g)$(x)
5. $(f \circ f)$(−2)

Exercice 15

Composition of Functions: Application

1. The function $D(p)$ gives the number of items that will be demanded when the price is p . The production cost is $C(x)$ the cost of producing x items. To determine the cost of production when the price is 6 MaatCoins, which of the following should we use?
   a. Evaluate $D(C(6))$
   b. Evaluate $C(D(6))$
   c. Solve $D(C(x)) = 6$
   d. Solve $D(C(p)) = 6$

2. The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?
   a. Evaluate $A(m(4))$
   b. Evaluate $m(A(4))$
   c. Solve $A(m(t)) = 4$

3. A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register.
   a. Write a price $P(x)$ function that computes the final price of the item in terms of the original price x.
      (Hint: Use function composition to find your answer.)

4. A raindrop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t + 2}$,
   a. Find the area of the ripple as a function of time.

b. Find the area of the ripple at $t = 2$

5. A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).

6. The radius , in inches, of a spherical balloon is related to the volume V, , by $r(V) = \sqrt[3]{\frac{3v}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.
   a. Find the composite function $r(V(t))$.
   b. Find the exact time when the radius reaches 10 inches

7. The number of bacteria in a refrigerated food product is given by $N(T) = 23T - 56T + 1, \quad 23 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.
   a. Find the composite function $N(T(t))$
   b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

References:

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