Morning

Properties of Regular Languages

1. Pumping lemma

Regular language satisfies the pumping lemma. If somebody presents you with fake regular language, use the pumping lemma to show a contradiction.

2. Closure properties

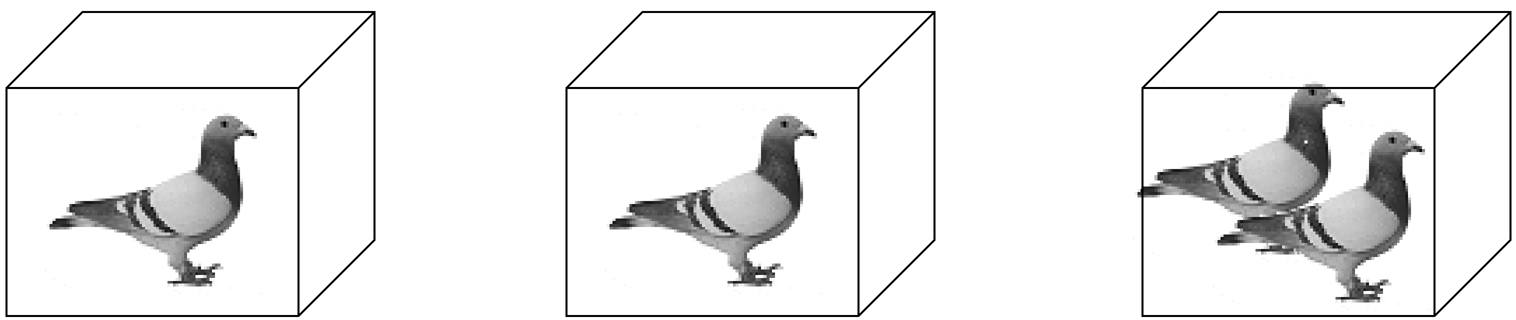
Building automata from components through operations.

The Pigeonhole Principle

4 pigeons

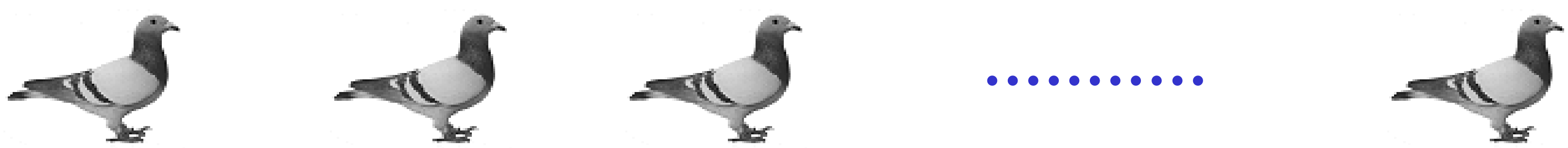
3 pigeonholes

A pigeonhole must contains at least two pigeons



The Pigeonhole Principle

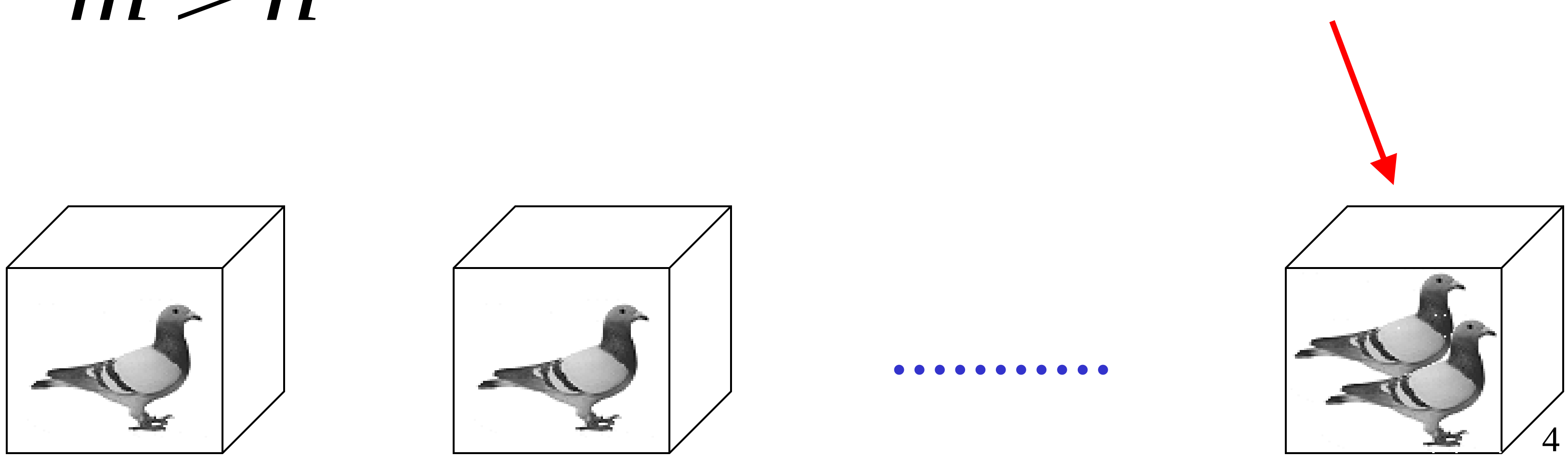
m pigeons



n pigeonholes

There is a pigeonhole
with at least 2 pigeons

$$m > n$$



The DFA Principle

m symbols

$$w = a_1 a_2 \cdots\cdots a_m$$

n states

$$a_n \cdots\cdots a_m \; ?$$

$$m \geq n$$

$$\longrightarrow q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \;\cdots\; \xrightarrow{a_{n-2}} q_{n-1} \xrightarrow{a_{n-1}} q_n \xrightarrow{a_n} ?$$

Property of regular languages

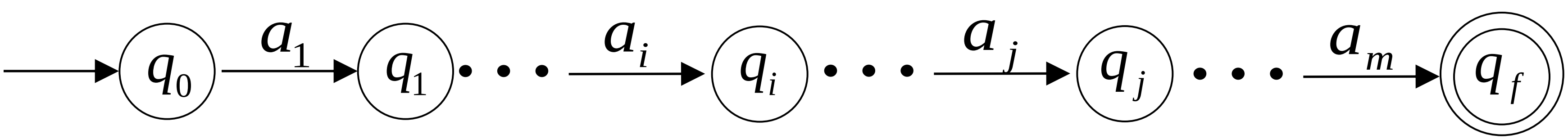
L is a regular language $\Rightarrow \exists$DFA $A : L(A) = L$

Let $A = (\mathrm{Q}, \Sigma, \delta, \mathrm{q}_0, F)$, and $n = |Q|$

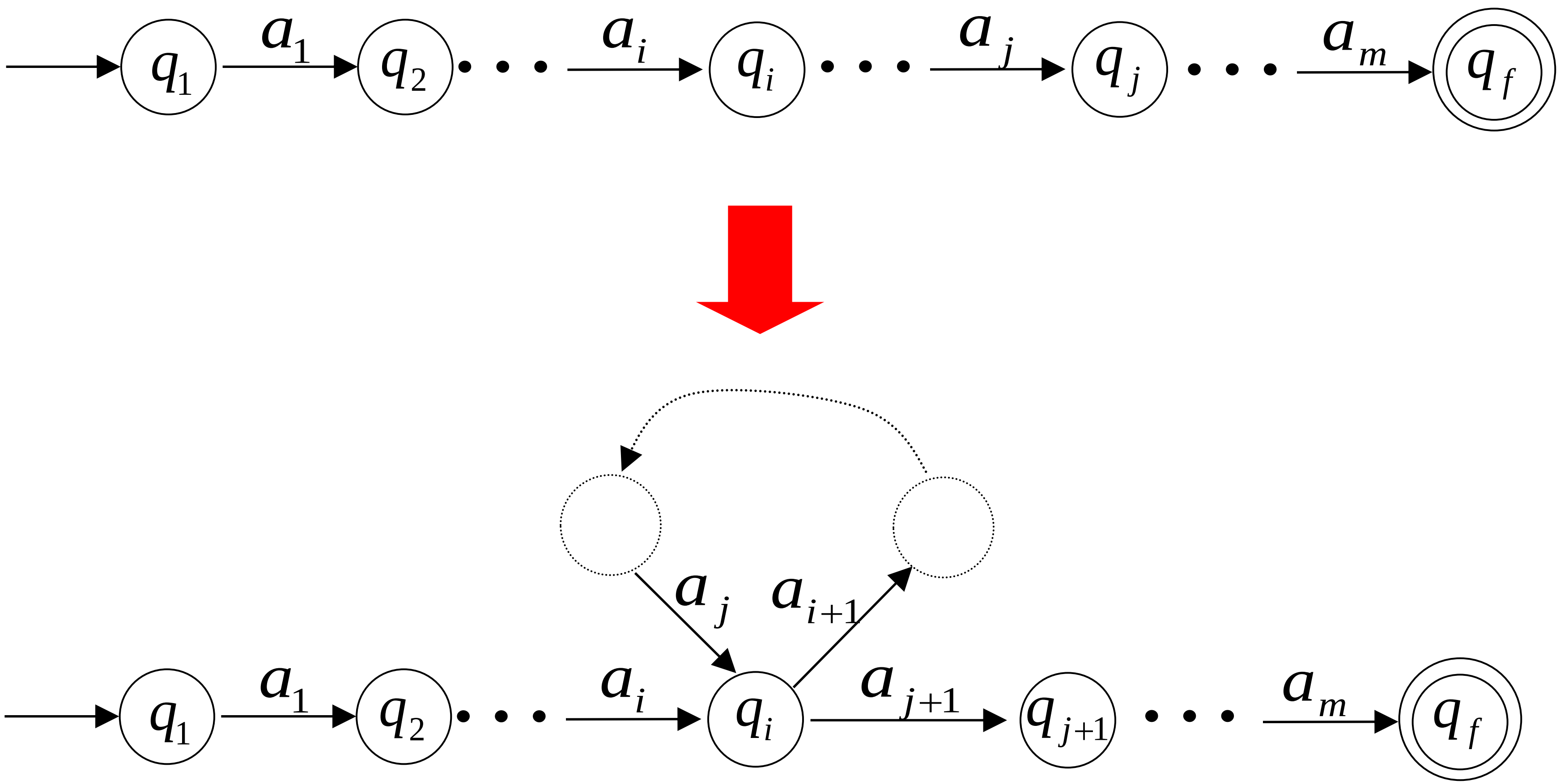
Get $w \in L$, and suppose $w = a_1 a_2 \cdots\cdots a_m, m \geq n$

Let $q_{\mathrm{i}} = \overline{\delta}(q_0, a_1 a_2 \cdots a_i)$

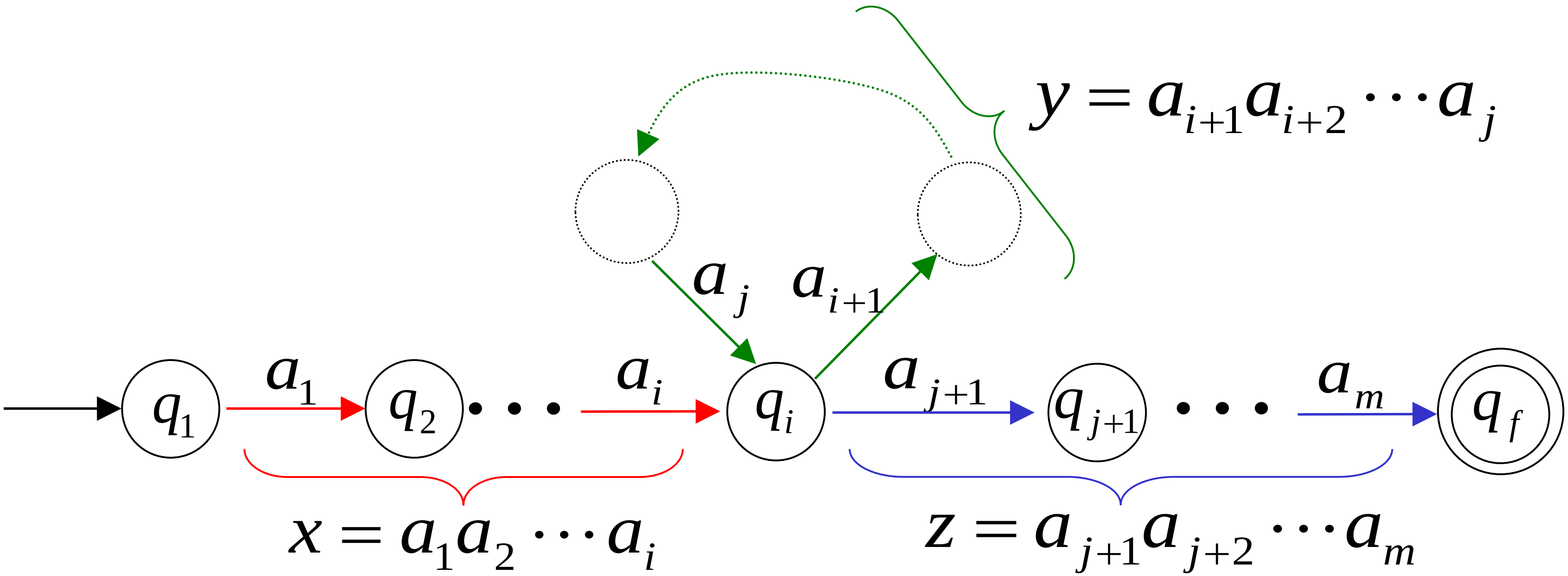
$$\Rightarrow \exists 0 < i < j \leq n : q_i = q_j$$



Property of regular languages



Property of regular languages



$$\Rightarrow w = xyz \quad \begin{cases} |xy| \leq n \\ |y| \geq 1 \text{ or } y \neq \varepsilon \\ xy^k z \in L, \text{ for any } k \geq 0 \end{cases}$$

Pumping Lemma

Pumping lemma for regular languages.

Let L be regular. Then

$\exists n, \forall w \in L : |w| \geq n \Rightarrow w = xyz$ such that

- $y \neq \varepsilon$
- $|xy| \leq n$
- $\forall k \geq 0,\ xy^k z \in L$

Example 6.1 Let $L = \{ 0^n1^n \mid n \geq 0 \}$, is it regular ?

Suppose L is regular. Get $w=0^n1^n \in L$.

By pumping lemma $w=xyz$, $|xy| \leq n$, $y \neq \varepsilon$, and $xy^kz \in L$.

Let $k=0$, then $xz \in L$.

But xz has fewer 0's than 1's, that $xz \notin L$.

It derived a contradiction.

So L is not regular.

Example 6.2 Prove $L = \{vv^R \mid v \in (a,b)^*\}$ is not regular.

Suppose L is regular.

Get $w = a^n b^n b^n a^n \in L$.

for $k=0$, $xz = a^{n-|y|} b^n b^n a^n \in L$.

Example 6.3 Prove $L = \{a^n b^l c^{n+l} \mid n,l \geq 0\}$ is not regular.

Suppose L is regular.

Get $w = a^n b^n c^{2n} \in L$.

for $k=0$, $xz = a^{n-|y|} b^n c^{2n} \in L$.

Closure properties of regular languages

- union : $L \cup M$
- intersection : $L \cap M$
- complement
- difference : $L - M$
- reversal
- closure(star)
- concatenation
- homomorphism
- inverse homomorphism

Closure properties of regular languages

Let L and M be regular.

Then the following languages are all regular :

- Union : $L \cup M$
- Concatenation : LM
- Closure : L^*
- Difference : $L - M$

- Union : $L \cup M$

Suppose $L(A)=L$, $L(B)=M$

Let $A= (Q_1,\Sigma_1,\delta_1,q_1,F_1)$, $B= (Q_2,\Sigma_2,\delta_2,q_2,F_2)$

$$C = (Q_1 \cup Q_2 \cup \{q_0\}, \Sigma_1 \cup \Sigma_2, \delta, q_0, F_1 \cup F_2)$$

$$\delta : \delta(q_0, \varepsilon) = \{q_1, q_2\}$$

$$\delta(q, a) = \delta_1(q,a), \forall (q, a) \in Q_1 \times \Sigma_1$$

$$\delta(q, a) = \delta_2(q,a), \forall (q, a) \in Q_2 \times \Sigma_2$$

Then $L(C)= L \cup M$

Closure properties of regular languages

- Reversal $L^R = \{w^R \mid w \in L\}$

Convert $A(L)$ into $A(L^R)$ by :

- Reverse all the arcs of $A(L)$
- Convert start state of $A(L)$ to accepting state of $A(L^R)$
- Create a new state as start state of $A(L^R)$ with ε-transitions to all the accepting states of $A(L)$

Closure properties of regular languages

- Complement

$$\overline{L} = \{w \mid w \in \Sigma^* \text{ and } w \notin L\}$$

Let DFA $A=(Q, \Sigma, \delta, q_0, F)$, and $L(A)=L$

Let DFA $B=(Q, \Sigma, \delta, q_0, S)$, and $S=Q-F$

Then $L(B)= \overline{L}$

Closure properties of regular languages

- Intersection : $L \cap M$

Suppose $L(A)=L$, $L(B)=M$

Let $A= (Q_1,\Sigma,\delta_1,q_1,F_1)$, $B= (Q_2,\Sigma,\delta_2,q_2,F_2)$

$$C = (Q_1 \times Q_2 , \Sigma, \delta , (q_1, q_2), F_1 \times F_2)$$

$$\delta : (Q_1 \times Q_2) \times \Sigma \rightarrow Q_1 \times Q_2$$

$$\delta ((p, q), a) = (\delta_1(p,a), \delta_2(q,a))$$

Then $L(C)= L \cap M$

Homomorphism

$h: \quad \Sigma^* \to \Gamma^*$

Let $w = a_1 a_2 \ldots\ldots a_n \in \Sigma^*$, then

$h(w) = h(a_1)h(a_2)\ldots\ldots h(a_n)$

Let $\Sigma = \{0, 1\}$, $\Gamma = \{a, b\}$, $h(0) = ab$, $h(1) = \varepsilon$

$h(0110) = h(0)h(1)h(1)h(0) = ab\varepsilon\varepsilon ab = abab$

$h(L) = \{ h(w) \mid w \text{ is in } L \}$

Homomorphism

Regular language is closed under homomorphism.

Assume r is a regular expression.

For any symbol a of r ,h(a) is a regular expression

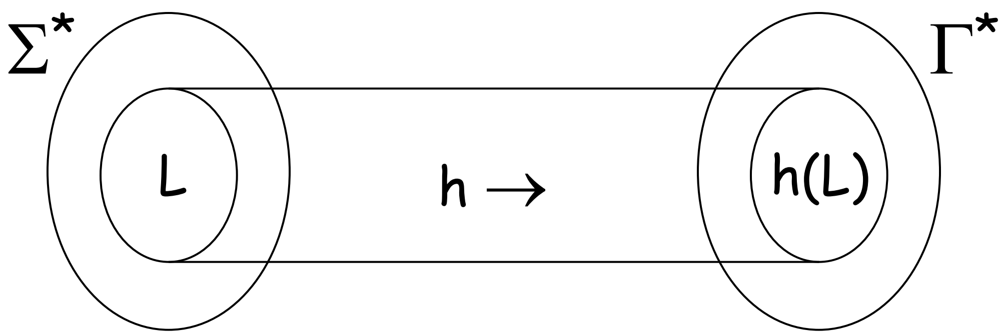
So is h(r).

It says that L(h(r)) is regular.

Inverse Homomorphism

$h: \quad \Sigma^* \rightarrow \Gamma^*$

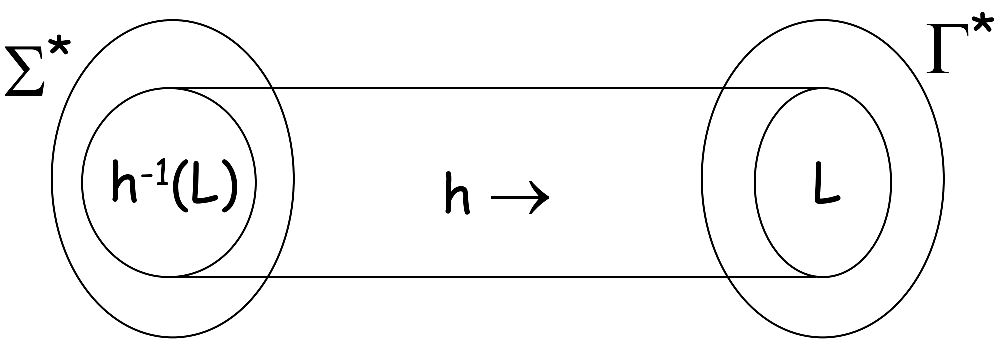
$h^{-1}(L) = \{ w \mid h(w) \text{ is in } L \}$



$\forall w \in L \Rightarrow h(w) \in h(L)$

$\forall v \in h(L)$

$\Rightarrow \exists w \in L : h(w) = v$



$\forall w \in h^{-1}(L) \Rightarrow h(w) \in L$

$\forall v \in L \Rightarrow$

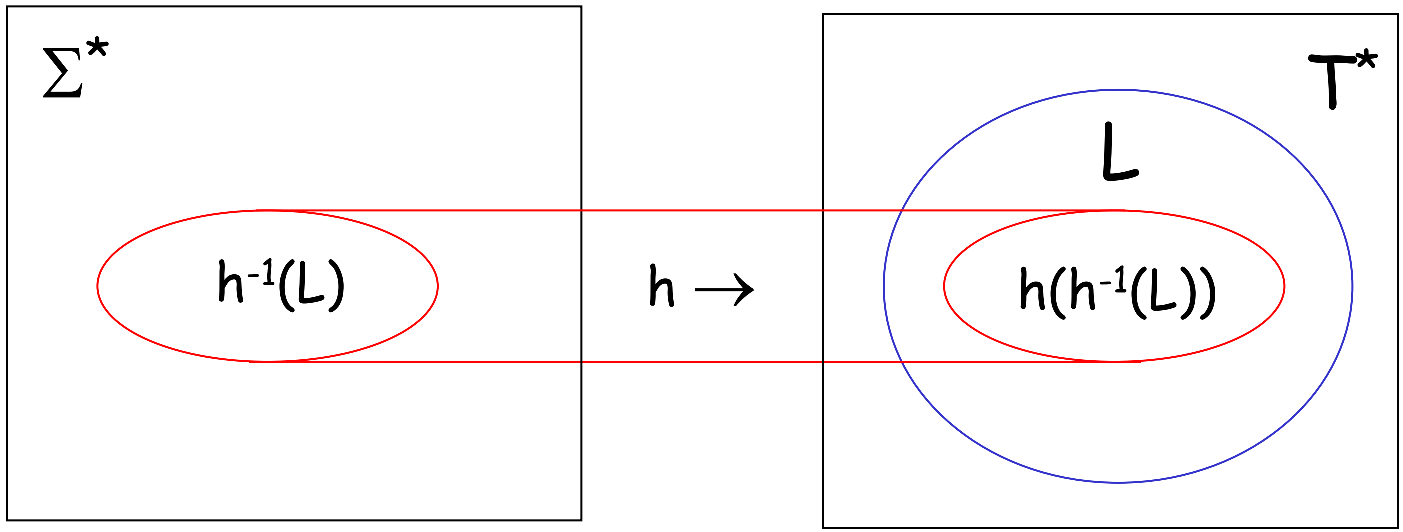
$\exists w \in h^{-1}(L) : h(w) = v$?

Example 6.4

Let $\Sigma=\{ a, b \}$, $\Gamma=\{ 0, 1 \}$, $h(a)=01$, $h(b)=10$

Let $L=L((00+1)^*)$ then

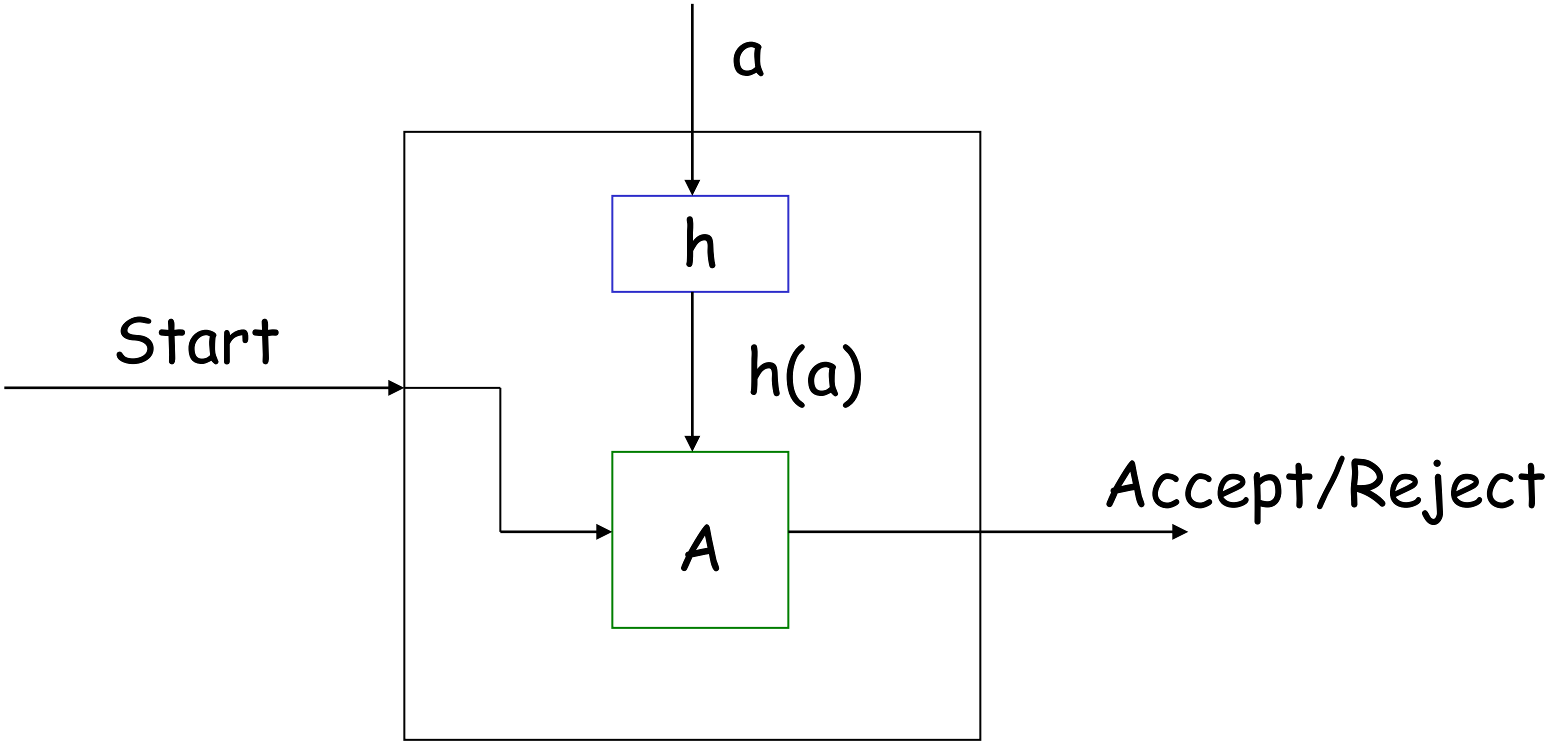
$h^{-1}(L)=L((ba)^*)$



$h^{-1}(L)=\{ba\}^*$, $h(h^{-1}(L))=\{1001\}^* \subset L=\{00,1\}^*$

Inverse Homomorphism

RL is closed under inverse homomorphism.



$$A = (Q, T, \delta, q_0, F), \quad B = (Q, \Sigma, \gamma, q_0, F)$$

$$where\ \ \gamma(q, a) = \hat{\delta}(q, h(a))$$

Good good study

day day up !