Morning

Properties of Context-free Languages

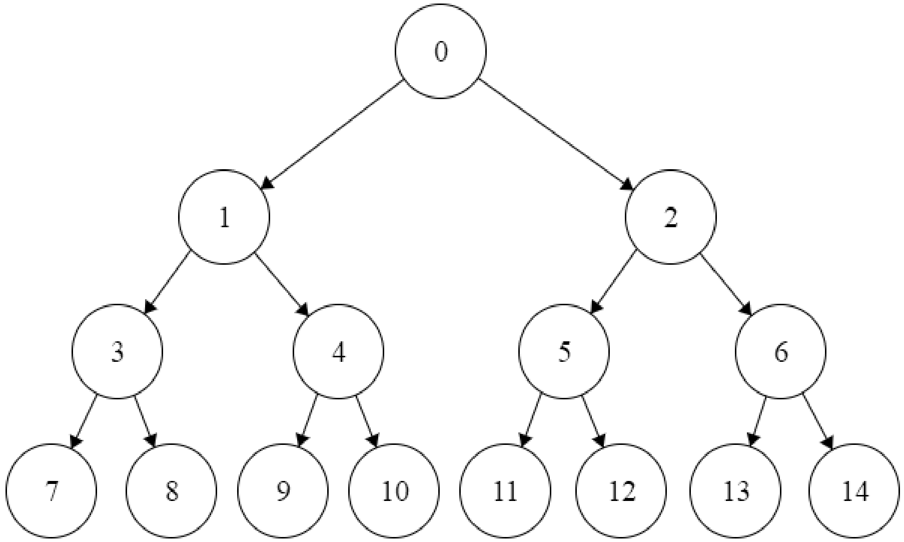
- Pumping lemma
- Closure properties

Full binary tree

A binary tree T is full if each node is either a leaf or possesses exactly two child-nodes.

$$\text{leafnum} = 2^h$$

$$h = \log_2(\text{leafnum})$$



$h = 3$, $\text{leafnum} = 2^3 = 8$

Chomsky Normal Form

All productions are one of following two forms :

1. $A \to BC\ ,\quad A, B, C \in V$

2. $A \to a\ ,\quad a \in T$

Chomsky Normal Form

Convert following CFG into CNF

$S \rightarrow ABa$, $A \rightarrow aab$, $B \rightarrow Ac$

$S \rightarrow AC$, $A \rightarrow DE$, $B \rightarrow AF$

$C \rightarrow BD$, $D \rightarrow a$, $F \rightarrow c$

$E \rightarrow DG$, $G \rightarrow b$

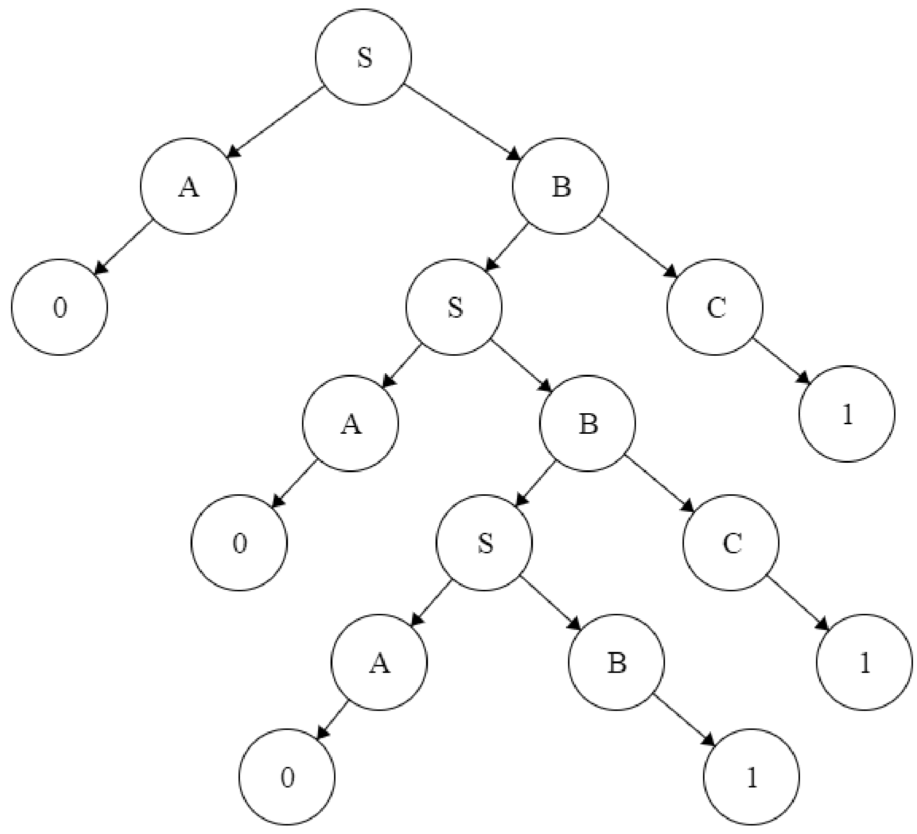
Chomsky Normal Form

Convert following CFG into CNF

$S \rightarrow 0S1 \mid \varepsilon$

$A \rightarrow 0, \quad C \rightarrow 1$

$S \rightarrow AB, \; B \rightarrow 1 \mid SC$



Pumping Lemma

Let L be a CFL . Then there exists some positive integer n such that any $w \in L$ with $|w| \geq n$ can be de-composed as

$$w = uvxyz$$

with

$$|vxy| \leq n$$

and

$$|vy| \geq 1$$

such that

$$uv^i xy^i z \in L$$

for all i=0,1,2,……

Pumping Lemma

L is a CFL $\Rightarrow$ There is a CFG $G=(V,T,R,S)$: $L(G) = L$.

V is finite $\Rightarrow$ $m=|V|$

$|\alpha|$ is finite $\forall\ A\to\alpha \Rightarrow k = \max\{\ |\alpha| \text{ for all } A\to\alpha\ \}$

Let $n=k^m$

For any $w\in L$ with $|w|\geq n$, there must be some variable A that appears at least two times in the path of parse tree.

That is : $S \stackrel{*}{\Rightarrow} uAz \stackrel{*}{\Rightarrow} uvAyz \stackrel{*}{\Rightarrow} w$

Pumping Lemma - CNF

$m = |V|$

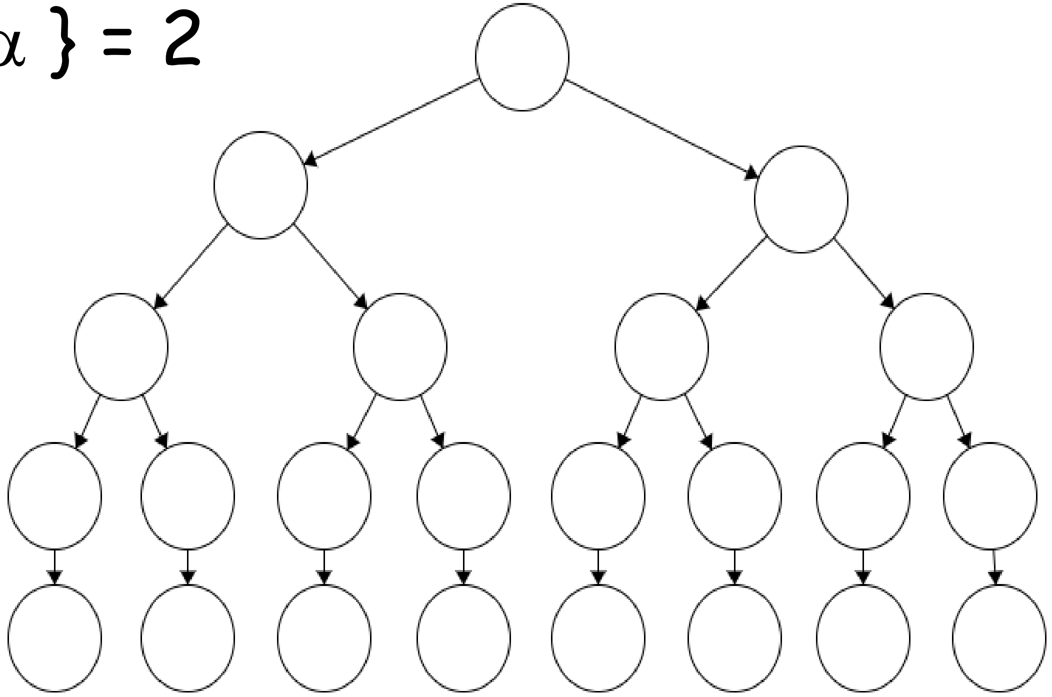
$k = \max\{ |\alpha| : A \to \alpha \} = 2$

$n = k^m = 2^m$

$\forall w \in L$ with $|w| \geq n$

$|w| = 2^m$

$\Rightarrow h \geq m+1$



$h = 4$, leafnum $= 2^{h-1} = 8$

Pumping Lemma - CNF

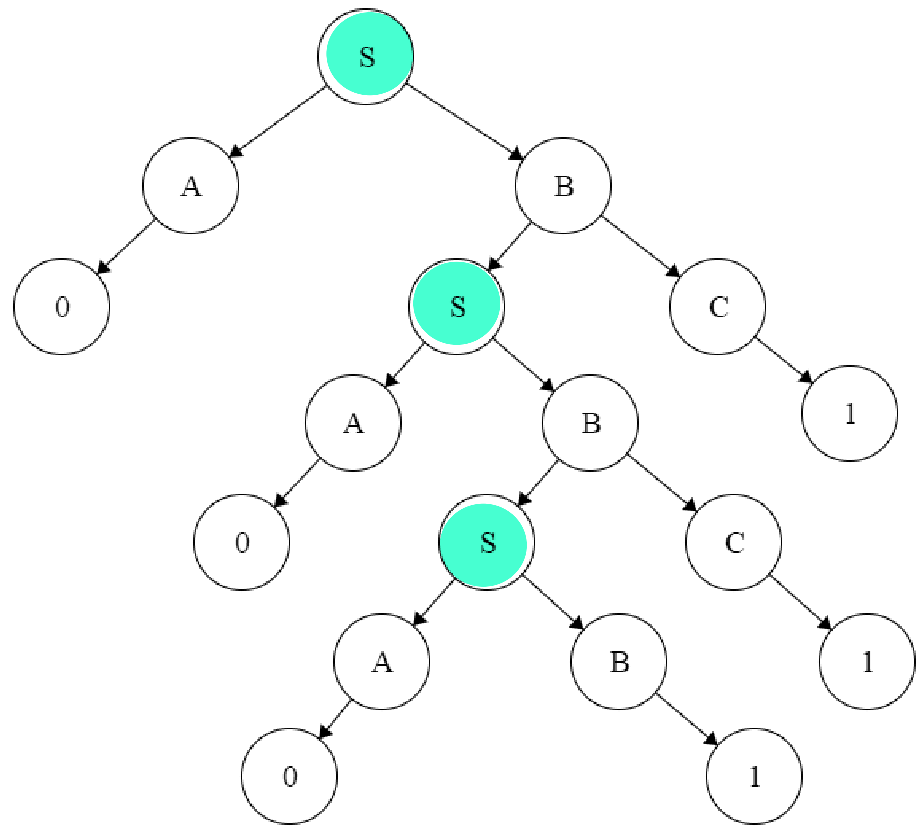
$S \to 0S1 \mid \varepsilon$

$A \to 0, \quad C \to 1$

$S \to AB, \; B \to 1 \mid SC$

$|w| = 2^m$

$\Rightarrow h > m+1 = |V| + 1$



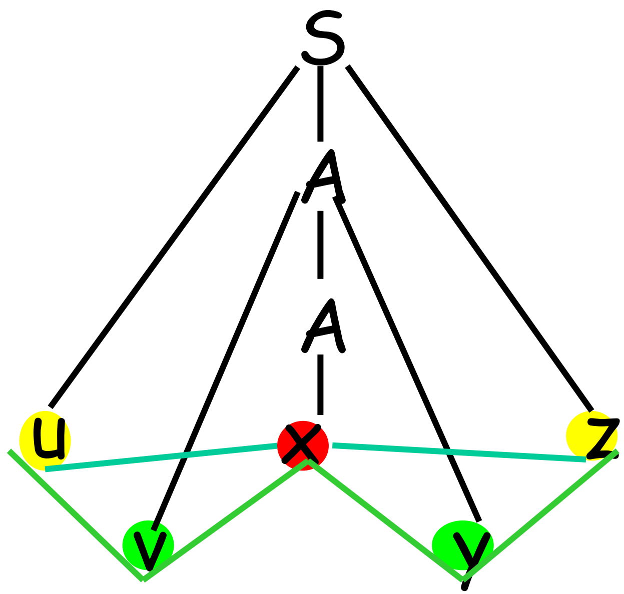
h = leafnum

Pumping Lemma

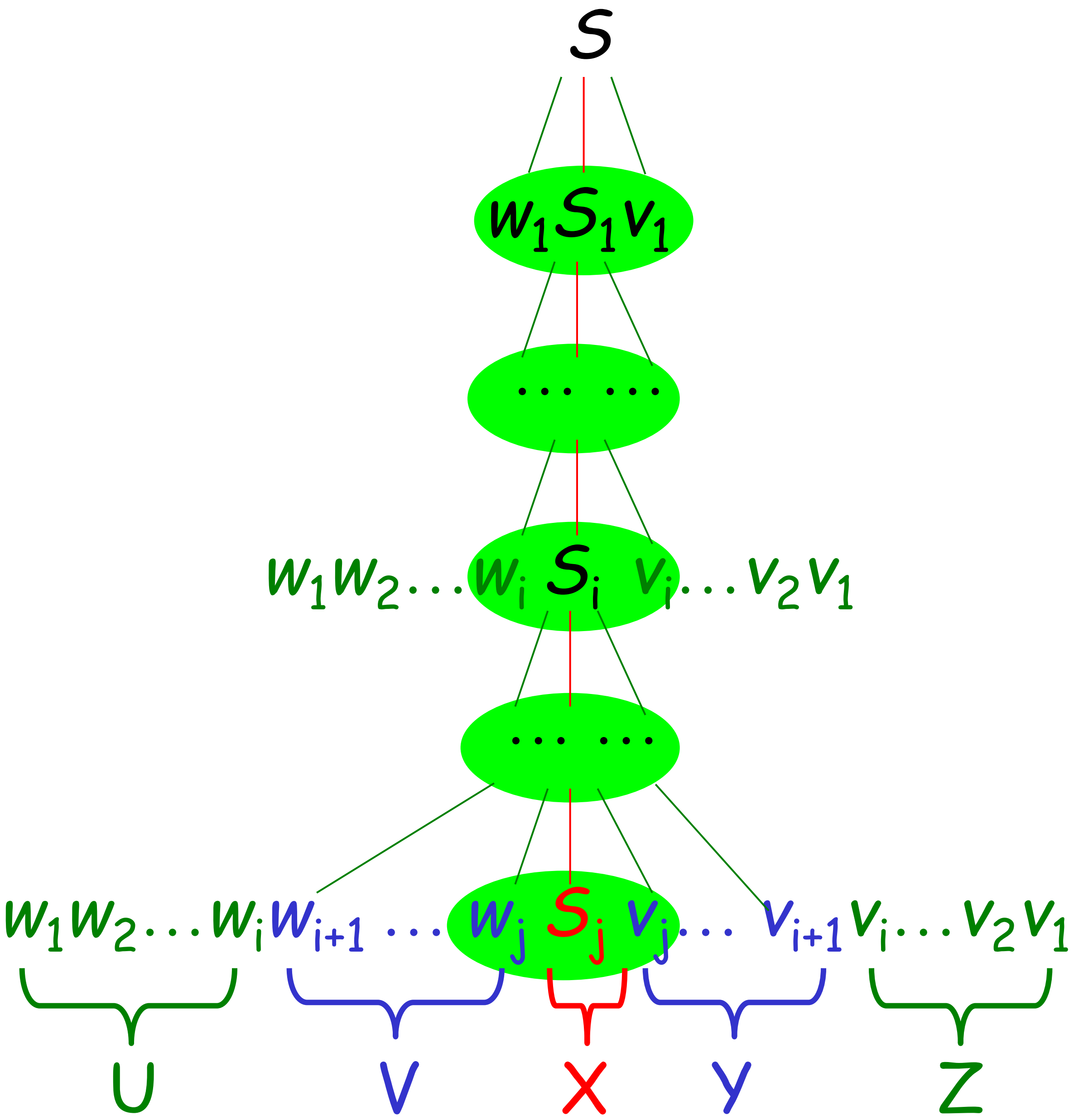
$|w| = 2^m \Rightarrow h \geq m+1$

$\exists$ Path : $A_0\text{-}A_1\text{-}\ldots\ldots\text{-}A_i\text{-}\ldots\ldots\text{-}A_j\text{-}\ldots\ldots\text{-}A_m\text{-}\ldots\ldots\text{-}A_l$

such that $A_i = A_j$ $(i < j)$



$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} w$


S
$w_1 S_1 v_1$
$\cdots\ \cdots$
$w_1 w_2 \ldots w_i\ S_i\ v_i \ldots v_2 v_1$
$\cdots\ \cdots$
$w_1 w_2 \ldots w_i\, w_{i+1} \ldots w_j\ S_j\ v_j \ldots v_{i+1}\, v_i \ldots v_2 v_1$
U
V
X
Y
Z

$$S_i = S_j$$

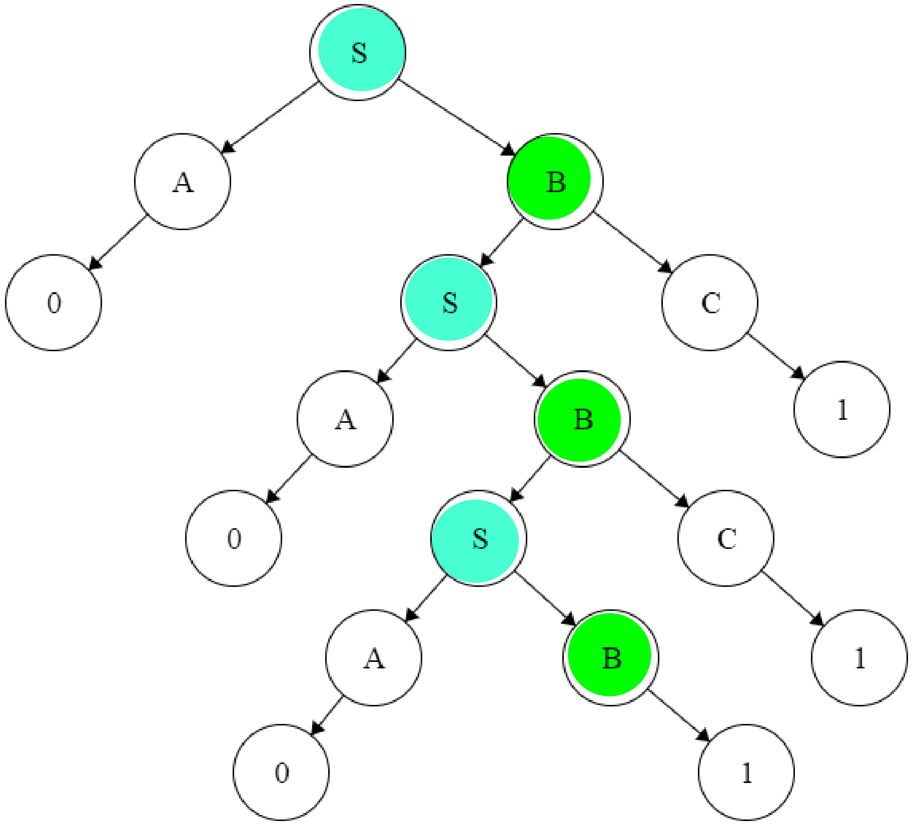
Example 1

$S \to AB,\ B \to 1 \mid SC$

$A \to 0,\ \ C \to 1$

$S \overset{*}{\Rightarrow} 0B1 \overset{*}{\Rightarrow} 000111$

$S \overset{*}{\Rightarrow} 01$



leafnum = h

Example 2 "Not CFL"

$$L = \{ ww \mid w \in \{0,1\}^* \}$$

Suppose L is CFL.

By pumping lemma there exist a constant n, $\forall w \in L$, where $|w| \geq n$, w can be broken into five strings, $w=uvxyz$, such that $|vxy| \leq n$, $vy \neq \varepsilon$, and $uv^k xy^k z \in L$.

Get $w=0^n1^n0^n1^n \in L$. Then $uvxyz=0^n1^n0^n1^n$.

7 cases in 2 groups of the position of vy.

Each derives a contradiction

So L is not CFL.

Example 2 "Not CFL"

Get $w=0^n1^n0^n1^n \in L$. Then $uvxyz=0^n1^n0^n1^n$.

7 cases in 2 groups of the position of vy.

Group 1

$0^n 1^n 0^n 1^n$

Group 2

$0^n 1^n 0^n 1^n$

Example 3 "Not CFL"

$$L = \{ 0^n 1^m \mid n = m^2 \}$$

Closure properties

- union : $L \cup M$
- concatenation
- closure(star)
- reversal
- intersection : $L \cap M$
- complement
- difference : $L - M$
- homomorphism
- inverse homomorphism

Union

If L_1 and L_2 are CFL , then so is $L_1 \cup L_2$.

Let $G(L_1)=(V_1,T_1,R_1,S_1)$, $G(L_2)=(V_2,T_2,R_2,S_2)$

Then $G(L_1 \cup L_2) = (V_1 \cup V_2 \cup \{S\}, T_1 \cup T_2, R, S)$

$R = \{S \to S_1 \mid S_2\} \cup R_1 \cup R_2$

Concatenation

If L_1 and L_2 are CFL , then so is $L_1 L_2$.

Let $G(L_1)=(V_1,T_1,R_1,S_1)$, $G(L_2)=(V_2,T_2,R_2,S_2)$

Then $G(L_1\ L_2) = (V_1 \cup V_2 \cup \{S\}, T_1 \cup T_2, R, S)$

$R = \{S \to S_1\ S_2\} \cup R_1 \cup R_2$

Star

If L is a CFL , then so is L^* .

Let $G(L) = (V,T,R,S)$

Then $G(L^*) = (V,T, \{S \rightarrow SS | \varepsilon\} \cup R,S)$

Reversal

If L is a CFL , then so is L^R.

Let $G(L)=(V,T,R,S)$

Then $G(L^R)=(V,T, \{A\to\alpha^R \mid A\to\alpha\in R\},S)$

Intersection

CFL is not closed under intersection .

$$L_1 = \{ a^n b^n c^m \mid n \geq 0, m \geq 0 \}$$

$$L_2 = \{ a^n b^m c^m \mid n \geq 0, m \geq 0 \}$$

$$L_1 \cap L_2 = \{ a^n b^n c^n \mid n \geq 0 \}$$

Intersection

If L_1 is a CFL and L_2 is a RL , then $L_1 \cap L_2$ is CFL.

$P(L_1) = (Q_1, \Sigma_1, \Gamma, \delta_1, q_1, z_0, F_1)$

$A(L_2) = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$

$P(L_1 \cap L_2) = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, \Gamma, \delta, (q_1, q_2), z_0, F_1 \times F_2)$

$$\delta((q,p),a,X)=((r,s), \alpha)$$

where $\delta_1(q,a,X)=(r,\alpha)$, $\delta_2(p,a)=s$

Complement

Example 4

$L = \{ 0^n 1^n \mid n \geq 0, n \neq 100 \}$. Show that L is CFL.

Example 5

$L = \{ w \mid w \in \{a,b,c\}^*, n_a(w)=n_b(w)=n_c(w) \}$

Show that L is not context-free.

Good good study

day day up !