Gauging the strength of inductive theorem provers

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Automated inductive theorem proving

- History in computer science dating back to the 1970ies

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- **Given method M, which theorems can M prove?**

What can mathematical logic contribute?

- Rich landscape of inductive theories and knowledge about them
  - various induction schemes
  - various induction rules
  - with/without parameters
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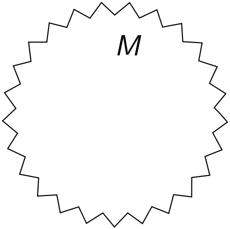
- Theory usually specified by set of axioms, e.g., Peano arithmetic (PA) axiomatised by Q and first-order induction scheme
- $T \vdash \sigma$ iff there is a T proof of σ.
- $T \nvdash \sigma$ iff there is a model $\mathcal{M} \models T$ s.t. $\mathcal{M} \not\models \sigma$.

The typical situation

- Given method M for inductive theorem proving . . .



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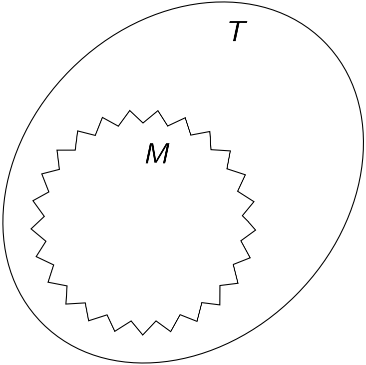

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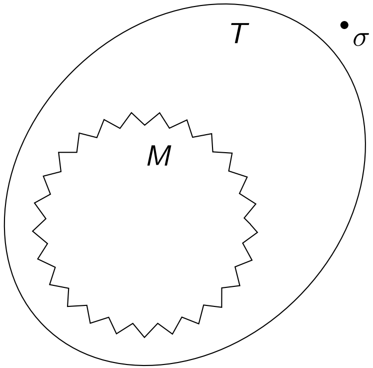

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- Straightforward results,
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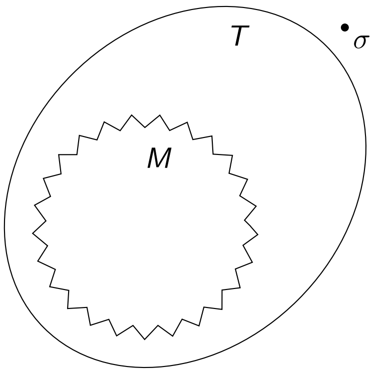

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- Straightforward results,
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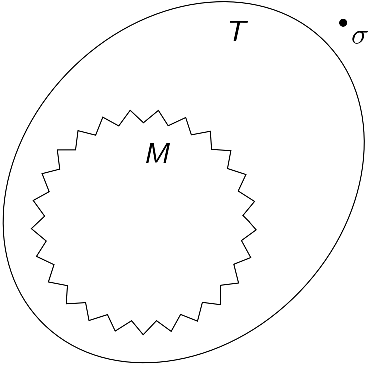
- **Practically meaningful** unprovability results

▶ Challenge problems



Outline

Induction in saturation theorem proving

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- **Definition.** $\mathcal{S}$ *sound* if $C \in \mathcal{C}^\omega$ implies $\mathcal{C} \models C$
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- **Definition.** $\mathcal{S}$ *sound* if $C \in \mathcal{C}^\omega$ implies $\mathcal{C} \models C$
- **Definition.** $\mathcal{S}$ *refutationally complete* if $\mathcal{C} \models \bot$ implies $\bot \in \mathcal{C}^\omega$
- Adding induction to $\mathcal{S}$:

$$\frac{}{\mathrm{CNF}(\mathrm{sk}^\exists(I_x\varphi(x)))}$$

where $\mathrm{sk}^\exists$ is Skolemisation, $I_x\varphi(x)$ is the induction axiom for $\varphi(x)$:

$$\varphi(0) \land \forall x\,(\varphi(x) \to \varphi(s(x))) \to \forall x\,\varphi(x)$$

Single clause induction

- **Example.** Vampire prover [Voronkov et al. '20]: single clause induction

$$\frac{\overline{L(a)} \lor C}{\mathrm{CNF}(\mathrm{sk}^{\exists}(I_x L(x)))}\ \mathrm{SCIND}$$

  a constant symbol, $L(x)$ literal, $L(a)$ variable-free

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- **Example.** $\mathcal{S} + \mathrm{SCIND}$ refutes

$$\{x + 0 = 0, x + s(y) = s(x + y), c + (c + c) \neq (c + c) + c\}$$

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Theorem

$\mathcal{S}$ sound saturation system, $\mathcal{C}$ clause set. If $\mathcal{S} + \mathrm{SCIND}$ refutes $\mathcal{C}$ then the theory $\mathcal{C} + \mathrm{Literal}(L)\text{-}\mathrm{IND}$ is inconsistent.

Proof Sketch.

Essentially by a proof translation (incl. Skolemisation). □

Challenge Problems (1/2)

Challenge Problem (Even/Odd)

$$
\begin{aligned}
\text{Language: } & 0/0, s/1, E/1, O/1 \\
\text{Axioms: } & 0 \neq s(x), s(x) = s(y) \rightarrow x = y, \\
& E(0), E(x) \rightarrow O(s(x)), O(x) \rightarrow E(s(x)) \\
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Even/Odd is not provable by induction on literals.

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Proof Sketch.

Model $\mathcal{M}$ with domain $(\{0\} \times \mathbb{N}) \cup (\{1\} \times \mathbb{Z})$ and

$$\begin{aligned}
0^{\mathcal{M}} &= (0,0) & E^{\mathcal{M}} &= \{(0,n) \mid n \text{ is even}\} \\
s^{\mathcal{M}}(b,n) &= (b, n+1) & O^{\mathcal{M}} &= \{(0,n) \mid n \text{ is odd}\}
\end{aligned}$$

□

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Corollary

$\mathcal{S}$ sound saturation system. Then $\mathcal{S} + \mathrm{SCIND}$ does not prove Even/Odd.

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Remark

Vampire with multi-clause induction proves Even/Odd.

Challenge Problems (2/2)

Challenge Problem (C_2)

Language: $0/0, s/1, p/1, +/2$

Axioms: $s(x) \neq 0, p(0) = 0, p(s(x)) = x, x + 0 = x, x + s(y) = s(x + y)$

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Challenge Problem ($D_{2,1}$)

Language and axioms as in C_2

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Theorem (Shoenfield 1958)

Open induction does not prove C_2 nor $D_{2,1}$.

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Outline

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- Abstraction of *n*-clause calculus [Kersani, Peltier '13; Kersani '14]

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- Variants

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- Variants
- **Example.** CSC proves Even/Odd.

Logical Characterisation

- Induction with and without parameters

$$\forall \overline{z}\, \big(\varphi(0,\overline{z}) \land \forall x\, (\varphi(x,\overline{z}) \to \varphi(s(x),\overline{z})) \to \forall x\, \varphi(x,\overline{z})\big)$$

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- **Definition.** Γ set of formulas, define

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where $\varphi(x) \in \Gamma$.

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$$[T,R] = T + \{\varphi \mid T \vdash \Gamma, \Gamma/\varphi \in R\}.$$

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Theorem (H, Vierling 2022)

$\mathcal{D}$ is refuted by a CSC iff the theory $\mathcal{D} + [\emptyset, \exists_1\text{-IND}_{\eta}^{\mathrm{R}-}]$ is inconsistent.

Challenge problem

Challenge Problem ($E_{0,1,2}$)

Language: $0/0, s/1, p/1, +/2$

Axioms: $0 \neq s(x), p(0) = 0, p(s(x)) = x, x + 0 = x, x + s(y) = s(x + y),$
$x + y = y + x, x + (y + z) = (x + y) + z$

Goal: $\forall x\,(x + 0 = x + x \rightarrow x = 0)$

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$\exists_1$-IND$^-$ *does not prove* $E_{0,1,2}$.

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Proof Sketch.

Countermodel $\mathcal{M}$, domain $\{(i, n) \in \mathbb{N} \times \mathbb{Z} \mid i = 0 \text{ implies } n \in \mathbb{N}\}$

$$0^{\mathcal{M}} = (0, 0) \qquad p^{\mathcal{M}}((0, n)) = (0, n \dot{-} 1)$$

$$s^{\mathcal{M}}(i, n) = (i, n + 1) \qquad p^{\mathcal{M}}((i, n)) = (i, n - 1) \text{ if } i > 0$$

$$(i, n) +^{\mathcal{M}} (j, m) = (\max(i, j), n + m) \qquad \square$$

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Corollary

CSC does not prove $E_{0,1,2}$.

Outline

Cancellation

- The natural numbers satisfy:
  - Left cancellation C_{l}: $x + y = x + z \rightarrow y = z$
  - Right cancellation C_{r}: $y + x = z + x \rightarrow y = z$
  - **Observation.** Open induction proves C_{l} and C_{r}.

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- Sequences with nil, cons and concatenation $\frown$ satsify:
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- Provable by list induction?

$$\forall \bar{z} \left(\varphi(\mathrm{nil}, \bar{z}) \land \forall x \, \forall Y \left(\varphi(Y, \bar{z}) \to \varphi(\mathrm{cons}(x, Y), \bar{z})\right) \to \forall X \, \varphi(X, \bar{z})\right)$$

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- **Observation.** Open (list) induction proves left cancellation for lists.

Challenge Problem (1/2)

Challenge Problem (Right List Cancellation)

Language: nil : list, cons : $\iota \times$ list $\rightarrow$ list, $\frown$: list $\times$ list $\rightarrow$ list

Axioms: $\mathrm{nil} \neq \mathrm{cons}(x, X), \mathrm{cons}(x, X) = \mathrm{cons}(y, Y) \rightarrow x = y \wedge X = Y,$

$\mathrm{nil} \frown Y = Y, \mathrm{cons}(x, X) \frown Y = \mathrm{cons}(x, X \frown Y)$

Goal: $\forall X \, \forall Y \, \forall Z \, (Y \frown X = Z \frown X \rightarrow Y = Z)$

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Theorem (H, Vierling 2024)

Open induction does not prove Right List Cancellation.

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& \mathrm{nil} \frown Y = Y, \mathrm{cons}(x, X) \frown Y = \mathrm{cons}(x, X \frown Y) \\
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Open induction does not prove Right List Cancellation.

Proof Sketch.

Idea: $(a) \frown (a, a, a, \ldots) = (a, a, a, \ldots) = \mathrm{nil} \frown (a, a, a, \ldots)$ but $(a) \neq \mathit{nil}$

$\square$

Challenge Problem (1/2)

Challenge Problem (Right List Cancellation)

Language: $\text{nil} : \text{list}, \text{cons} : \iota \times \text{list} \to \text{list}, \frown : \text{list} \times \text{list} \to \text{list}$

Axioms: $\text{nil} \neq \text{cons}(x, X), \text{cons}(x, X) = \text{cons}(y, Y) \to x = y \land X = Y,$

$\text{nil} \frown Y = Y, \text{cons}(x, X) \frown Y = \text{cons}(x, X \frown Y)$

Goal: $\forall X \, \forall Y \, \forall Z \, (Y \frown X = Z \frown X \to Y = Z)$

Theorem (H, Vierling 2024)

Open induction does not prove Right List Cancellation.

Proof Sketch.

Idea: $(a) \frown (a, a, a, \ldots) = (a, a, a, \ldots) = \text{nil} \frown (a, a, a, \ldots)$ but $(a) \neq nil$

$\mathcal{M}$: (a certain set of) sequences of length $< \omega^3$ $\square$

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Theorem (H, Vierling 2024)

Open induction does not prove Right List Cancellation.

Remark

$\forall_1$*-induction proves Right List Cancellation.*

Challenge Problem (2/2)

Challenge Problem (Right List Decomposition)

Language: $\mathrm{nil} : \mathrm{list}, \mathrm{cons} : \iota \times \mathrm{list} \to \mathrm{list}, \frown : \mathrm{list} \times \mathrm{list} \to \mathrm{list}$

Axioms: $\mathrm{nil} \neq \mathrm{cons}(x, X), \mathrm{cons}(x, X) = \mathrm{cons}(y, Y) \to x = y \land X = Y,$
$\mathrm{nil} \frown Y = Y, \mathrm{cons}(x, X) \frown Y = \mathrm{cons}(x, X \frown Y)$

Goal: $\forall X \, (X = \mathrm{nil} \lor \exists Y \, \exists z \, X = Y \frown \mathrm{cons}(z, \mathrm{nil}))$

Challenge Problem (2/2)

Challenge Problem (Right List Decomposition)

Language: nil : list, cons : $\iota \times$ list $\to$ list, $\frown$: list $\times$ list $\to$ list

Axioms: $\mathrm{nil} \neq \mathrm{cons}(x, X), \mathrm{cons}(x, X) = \mathrm{cons}(y, Y) \to x = y \wedge X = Y,$
$\mathrm{nil} \frown Y = Y, \mathrm{cons}(x, X) \frown Y = \mathrm{cons}(x, X \frown Y)$

Goal: $\forall X\, (X = \mathrm{nil} \vee \exists Y\, \exists z\, X = Y \frown \mathrm{cons}(z, \mathrm{nil}))$

Theorem (H, Vierling 2024)

Open induction does not prove Right List Decomposition.

Challenge Problem (2/2)

Challenge Problem (Right List Decomposition)

Language: $\mathrm{nil} : \mathrm{list}, \mathrm{cons} : \iota \times \mathrm{list} \to \mathrm{list}, \frown : \mathrm{list} \times \mathrm{list} \to \mathrm{list}$

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Theorem (H, Vierling 2024)

Open induction does not prove Right List Decomposition.

Observation

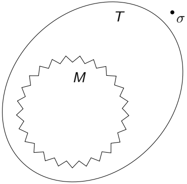
Open induction proves **Left** *List Decomposition.*

$$\forall X\, (X = \mathrm{nil} \vee \exists Y\, \exists z\, X = \mathrm{cons}(z, \mathrm{nil}) \frown Y)$$

Outline

Conclusion

- Strategy for analysing method M:
  - Find upper bound T for strength of M
  - (Practically relevant) statement σ unprovable in T
- T overapproximates M



Conclusion

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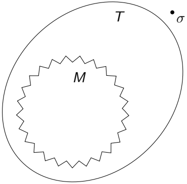

- Logical foundations of automated inductive theorem proving
- Challenge problems

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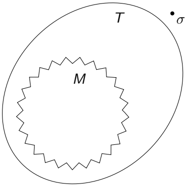

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  Where is my prover in this landscape of theories?

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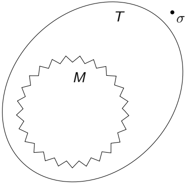

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Future Work:

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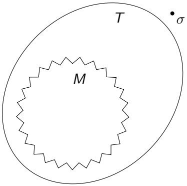

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- More on other inductive datatypes: lists, trees, etc.

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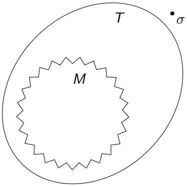

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Future Work:

- More on other inductive datatypes: lists, trees, etc.
- Analyticity

Thank you!

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