TDA251
Solutions for Assignment 1

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Excrcise 1

Given a graph $G = (V, E)$, let X be an arbitray set of nodes returned by the greedy solution, Y be an independent set with maximum size of nodes. We prove that

$$|X| \geq |Y| / \Delta$$

Denote $X := \{x_1, x_2, \ldots, x_m\}$, where x_i are the nodes selected by the greedy algorithm in sequence. Correspondingly, denote $A_i \subseteq V$ as the set of nodes consists of x_i and the nodes removed together as its neighbours in iteration.
Consider $B_i := A_i \cap Y$, we claim that B_i can not be empty. This is because at least B_i could contain x_i as its element, for by rule of our greedy algorithm, there is no edge between x_i and any other nodes in $V \setminus A_i$, which means if B_i does not contain any other elements different from x_i in A_i, it must contain x_i.
Observe that, in the worst case, B_i has only one element x_i, because x_i has an edge with all of its neighbours. In the best case, B_i has potentially all the neighbours of x_i, supposing there's no edge between any two of these nodes. Thus we have the following inequation

$$1 \leq |B_i| \leq \Delta$$

Thus we have

$$\frac{1}{\Delta} \leq \frac{\left|\{x_i\}\right|}{|B_i|} \leq 1$$

Notice that

$$Y = B_1 \cup B_2 \cup \ldots \cup B_m$$

and

$$B_i \cap B_j = \emptyset, i \neq j$$

Finally we have

$$\left(|X| = \sum_{i=1}^{m} \big|\{x_i\}\big|\right) \geq \left(\sum_{i=1}^{m} \frac{|B_i|}{\Delta} = \frac{|Y|}{\Delta}\right)$$

□

Excrcise 2

A greedy algorithm can be given as follows:

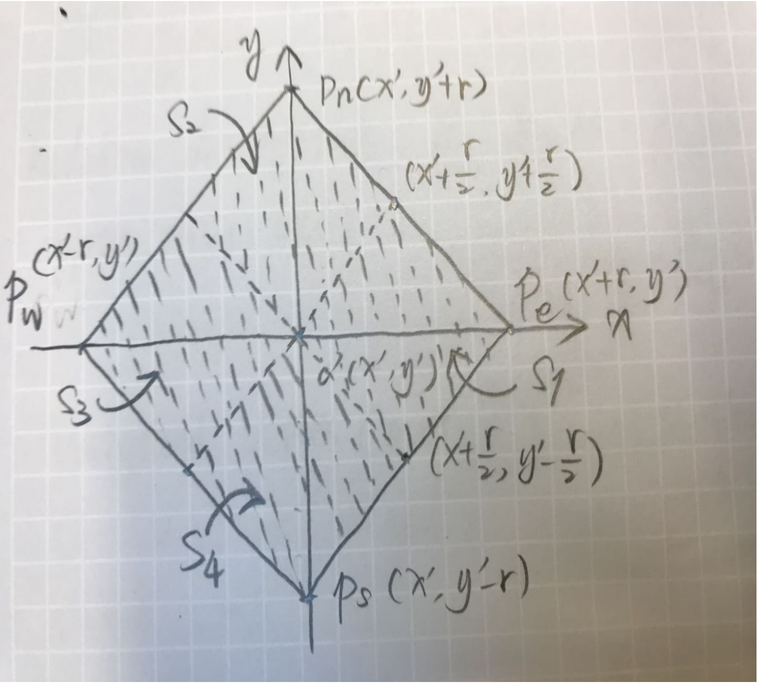
Let S be the site of n sites, S' be the set of remaining sites need to be covered and C be the set of centers.
Initialize $S' = S$, $C = \emptyset$
while $S' \neq \emptyset$ **do**
  Select any site $s \in S'$ and add s in C
  Delete s and all the sites within Manhattan distance r from s from S'
end while
return C

We claim that:

1. The algorithm can run in $O(n^2)$ time
2. Every site has a distance at most r from some center
3. The center set returned has at most $4k$ elements

(1) is true for in the worst case when any two sites have distance larger than r, we have to compare the distance between the selected site s and all the remaining sites in S' with r.
(2) is guaranteed by the way of the deletion of sites that within distance r of the newly selected center site.
(3) is also true for the geometric nature of Manhattan distance. Suppose C^* is the optimal solution that has k centers selected. During the execution of our algorithm, for any newly selected center $c = (x, y)$ there must be a center $c' \in C^*$ that covers it within r distance. Suppose c' has coordinate (x', y'). Observe that the area covered by c' is the square identified by these four points $p_w = (x' - r, y')$, $p_n = (x', y' + r)$, $p_e = (x' + r, y')$, $p_s = (x', y' - r)$. Denote this area as $A_{c'}$, which is depicted in the image below.

- If $c = c'$. Then for the current while-loop, the greedy algorithm does not produce more centers than C^*.



- If c falls into the inner square S_1 denoted by (x', y'),$(x' + \frac{r}{2}, y' + \frac{r}{2})$,$(x' + r, y')$,$(x' + \frac{r}{2}, y' - \frac{r}{2})$. Observe that any two points in this area have Manhattan distance no more than r, which means all sites in this area covered by c' are also covered by c.
- Similarly, if c falls into S_2(denoted by (x', y'), $(x' - \frac{r}{2}, y' + \frac{r}{2})$, $(x', y' + r)$, $(x' + \frac{r}{2}, y' + \frac{r}{2})$), or S_3(denoted by (x', y'), $(x' - \frac{r}{2}, y' - \frac{r}{2})$, $(x' - r, y')$, $(x' - \frac{r}{2}, y' + \frac{r}{2})$), or S_4(denoted by (x', y'), $(x' + \frac{r}{2}, y' - \frac{r}{2})$, $(x', y' - r)$, $(x' - \frac{r}{2}, y' - \frac{r}{2})$), all the sites in that particular area covered by c' would also be covered by c. Thus, during the course of our greedy algorithm, at most 4 sites need to be choosen to cover all the sites originally covered by some center $c' \in C^*$.

It follows that the output of our greedy algorithm has at most $4k$ elements. $\square$