Advanced Algorithms. Assignment 1

Exercise 1.
In the following, Δ denotes the largest degree of the nodes in a given graph. That is, every node is assumed to have at most Δ neighbors. We assume Δ to be fixed.
An independent set in a graph is a subset X of nodes such that no edges exist between any two nodes of X. The Maximum Independent Set problem asks to find an independent set of maximum size in a given graph. This fundamental problem is known to be NP-complete.
Consider the following rather intuitive greedy algorithm: We take any node, put it in the solution, and remove this node and all its neighbors from the graph. We iterate this step until the graph is empty.

Prove that this algorithm always returns an independent set of size at least $1/\Delta$ of the maximum possible size. (By the way, this is also the best possible guarantee for this algorithm.)

Some remarks and hints – but you may ignore this paragraph:
Specifically you must prove: If X is the greedy solution, and Y is an arbitrary independent set, then $|X| \geq |Y|/\Delta$. It should be rather easy to show $|X| \geq |Y|/(\Delta+1)$, which is acceptable as a "weak" submission. The stronger result $|X| \geq |Y|/\Delta$ is a bit harder to show. Working with upper and lower bounds is probably not helpful here. Instead you might cleverly "assign" certain nodes of X and Y to each other, and then study how the sizes are related.

Find Exercise 2 on the reverse page.

Exercise 2.
In the Center Selection problem we wanted to cover a given set of sites by k centers, such that every site has a distance at most r from some center. We have seen an algorithm that places k centers, but the guaranteed radius has increased to $2r$. It worked in an arbitrary space with a distance function (satisfying the triangle inequality).
Now we consider the opposite problem where the radius is fixed, but the number of centers is flexible. But here the approximation ratio that can be achieved depends on the distance function, and we will treat only one special case.

The actual exercise:

We are given n points (sites) in the 2-dimensional plane with Manhattan distance. That is, the distance between any two points (x, y) and (x', y') is defined by $|x - x'| + |y - y'|$. (In other words, we can walk on horizontal and vertical lines only.)

Propose a polynomial-time algorithm with the following property: If k centers can be placed such that every site has a distance at most r from some center, then your algorithm outputs a set of at most $4k$ centers such that, still, every site has a distance at most r from some center.

(Give coherent and self-contained explanations, not only a quick statement of the main idea. Of course, you need to prove the claimed approximation ratio and some polynomial time bound.)