LLM-Driven Automation for Lyapunov Optimization Design

User:

Task. Convert the user-provided optimization problem into the standard form ($\mathbf{P1}$), which requires:

- Objective: long-term average of an expected function

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[f(t)\right]$$

- Constraints: five types
- - long-term average inequalities

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[g_n(t)\right]\leq 0, \forall n\in\{1,\cdots,N\}$$

- - long-term average equalities

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[h_m(t)\right]=0, \forall m\in\{1,\cdots,M\}$$

- - long-term stability

$$\lim_{t\to\infty}\frac{\mathbb{E}[Q_i(t)]}{t}=0, \forall i\in\{1,\cdots,I\}$$

- - instantaneous inequalities

$$a_j(t)\leq 0, \forall j\in\{1,\cdots,J\}, t\in\{0,\cdots,T-1\}$$

- - instantaneous equalities

$$b_k(t)=0, \forall k\in\{1,\cdots,K\}, t\in\{0,\cdots,T-1\}$$

User Input. a base station serves I terminals over discrete time slots $t\in\{0,\cdots,T-1\}$. The downlink rate for terminal i during slot t is expressed as

$$r_i(t)=B\log\left(1+\frac{p_i(t)h_i(t)}{\sum_{j\neq i}p_j(t)h_i(t)+BN_0}\right)$$

where B is the pre-allocated bandwidth of downlink, $p_i(t)$ presents the transmission power to terminal i during slot t, $h_i(t)$ denotes the channel gain between base station and terminal i, N_0 represents the noise power density. It is worth noting that the channel gain is modeled as a stationary process.
Additionally, each terminal i has an associated queue $Q_i(t)$ in the base station that stores data awaiting transmission. The queue starts from an initial state 0, and its subsequent states are determined by stochastic data arrival $c_i(t)$ and transmission rate $r_i(t)$:

$$Q_i(t)\triangleq\begin{cases}0, & t=0\\ \max\{Q_i(t-1)-r_i(t-1),0\}+c_i(t-1), & t\geq 1\end{cases}$$

The objective of the following model is to minimize the long-term averaged transmission power, while ensuring quality of service and transmission queue stability:

$$\min_{\mathbf{P}} \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{i=1}^{I} \mathbb{E}\left[p_i(t)\right] \tag{1}$$

$$\text{s.t.} \quad 0 \leq p_i(t) \leq p_{max}, \forall i \in \{1, \cdots, I\}, t \in \{0, \cdots, T-1\} \tag{2}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[r_i(t)] \geq \bar{r}_i, \forall i \in \{1, \cdots, I\} \tag{3}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[e_i(t)] \leq \bar{e}_i, \forall i \in \{1, \cdots, I\} \tag{4}$$

$$\lim_{t\in\infty} \frac{\mathbb{E}[Q_i(t)]}{t} = 0, \forall i \in \{1, \cdots, I\} \tag{5}$$

Constraint (2) sets the transmission power boundary. Constraints (3) and (4) ensure quality of service in terms of transmission speed and bit error rate, respectively. Specifically, the bit error rate $e_i(t)$ depends on $p_i(t)h_i(t)$, with its expression varying by modulation mode. However, to focus on the LLM generation performance and without loss of generality, this paper omits the detailed expression of $e_i(t)$.

Output Format. Return the reformulated problem ($\mathbf{P1}$) in standard form using LaTeX.

DeepSeek:

The optimization problem is reformulated into the standard form ($\mathbf{P1}$) as follows:

$$\mathbf{P1}: \quad \min_{\mathbf{P}} \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right]$$

Subject to the constraints:

1. **Long-term average inequalities** ($N = 2I$ constraints):

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[g_n(t)\right] \leq 0, \quad \forall n \in \{1, \ldots, 2I\}$$

   where

$$g_n(t) = \begin{cases} \bar{r}_i - r_i(t) & \text{if } n = i \in \{1, \ldots, I\} \\ e_i(t) - \bar{e}_i & \text{if } n = I + i, i \in \{1, \ldots, I\} \end{cases}$$

2. **Long-term average equalities** ($M = 0$):
   No constraints of this type.

3. **Long-term stability** (I constraints):

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[Q_i(t)\right]}{t} = 0, \quad \forall i \in \{1, \ldots, I\}$$

4. **Instantaneous inequalities** ($J = 2I$ constraints):

$$a_j(t) \leq 0, \quad \forall j \in \{1, \ldots, 2I\}, t \in \{0, \ldots, T-1\}$$

   where

$$a_j(t) = \begin{cases} -p_i(t) & \text{if } j = 2i-1, i \in \{1, \dots, I\} \\ p_i(t) - p_{\max} & \text{if } j = 2i, i \in \{1, \dots, I\} \end{cases}$$

5. **Instantaneous equalities** ($K = 0$):
   No constraints of this type.

Explanation:

- **Objective**: Minimizes the long-term average expected total transmission power, with $f(t) = \sum_{i=1}^{I} p_i(t))$.
- **Long-term average inequalities**:
  - For $n = 1$ to I: Ensures minimum average rate $\bar{r}_i$ (via $g_n(t) = \bar{r}_i - r_i(t)$).
  - For $n = I + 1$ to $2I$: Ensures maximum average bit error rate $\bar{e}_i$ (via $g_n(t) = e_i(t) - \bar{e}_i$).

- **Long-term stability**: Guarantees queue stability for all terminals (I constraints).
- **Instantaneous inequalities**:
  - For each i, $j = 2i - 1$ enforces $p_i(t) \geq 0$ (via $a_j(t) = -p_i(t)$).
  - For each i, $j = 2i$ enforces $p_i(t) \leq p_{\max}$ (via $a_j(t) = p_i(t) - p_{\max}$).

- **No equality constraints** ($M = 0$, $K = 0$) are present.

This reformulation adheres strictly to the required standard form ($\mathbf{P1}$).

User:

Task. Decompose the long-term stochastic optimization problem ($\mathbf{P1}$) into a per-slot decision problem ($\mathbf{P2}$) via the Lyapunov optimization framework. Notice that the second moments of $c_i(t)$ (arrival rate of actual queue), $d_i(t)$ (departure rate of actual queue), $g_n(t)$ (long-term inequality variable) and $h_m(t)$ (long-term equality variable)are bounded.

User Input. The converted strand form (P1).

Output Format. Note that the reader is new to Lyapunov optimization and may not be familiar with the concept. Please return the precise derivation of decomposition, especially the drift bound derivation, and the per-slot problem ($\mathbf{P2}$) in LaTeX. Each step in the mathematical derivation needs to be explained.

DeepSeek:

Decomposition via Lyapunov Optimization Framework

We decompose the long-term stochastic problem ($\mathbf{P1}$) into a per-slot problem ($\mathbf{P2}$) using Lyapunov optimization. The steps are:

Step 1: Define Virtual Queues for Long-Term Constraints

For each long-term average inequality constraint $g_n(t) \leq 0$, define a virtual queue $Z_n(t)$:

$$Z_n(t+1) = \max\left\{Z_n(t) + g_n(t), 0\right\}, \quad \forall n \in \{1, \dots, 2I\}$$

- **Interpretation**:
  $Z_n(t)$ tracks the "backlog" of unsatisfied constraint n. Stability of $Z_n(t)$ (i.e., $\lim_{t\to\infty} \frac{\mathbb{E}[Z_n(t)]}{t} = 0$) ensures the long-term average constraint holds.

Step 2: Combined Queue Vector

Define the combined queue vector $\mathbf{\Theta}(t) = [\mathbf{Q}(t), \mathbf{Z}(t)]$, where:

- $\mathbf{Q}(t) = [Q_1(t), \ldots, Q_I(t)]$: Physical data queues.
- $\mathbf{Z}(t) = [Z_1(t), \ldots, Z_{2I}(t)]$: Virtual queues for constraints.

Step 3: Lyapunov Function

Define the Lyapunov function to measure queue congestion:

$$L(\mathbf{\Theta}(t)) = \frac{1}{2}\left[\sum_{i=1}^{I} Q_i^2(t) + \sum_{n=1}^{2I} Z_n^2(t)\right]$$

- **Properties**:
  $L(\mathbf{\Theta}(t)) \geq 0$ and $L(\mathbf{\Theta}(t)) = 0$ iff all queues are empty.

Step 4: Conditional Lyapunov Drift

Define the one-slot conditional drift:

$$\Delta(\mathbf{\Theta}(t)) = \mathbb{E}\left[L(\mathbf{\Theta}(t+1)) - L(\mathbf{\Theta}(t)) | \mathbf{\Theta}(t)\right]$$

- **Goal**: Minimize $\Delta(\mathbf{\Theta}(t))$ to maintain queue stability.

Step 5: Drift Bound Derivation

Expand $\Delta(\mathbf{\Theta}(t))$ using queue dynamics:

$$L(\mathbf{\Theta}(t+1)) - L(\mathbf{\Theta}(t)) = \frac{1}{2}\sum_{i=1}^{I}\left[Q_i^2(t+1) - Q_i^2(t)\right] + \frac{1}{2}\sum_{n=1}^{2I}\left[Z_n^2(t+1) - Z_n^2(t)\right]$$

Bound for Physical Queues ($Q_i(t)$):
From $Q_i(t+1) = \max\{Q_i(t) - r_i(t), 0\} + c_i(t)$:

$$Q_i^2(t+1) \leq [Q_i(t) - r_i(t)]^2 + c_i^2(t) + 2c_i(t)\max\{Q_i(t) - r_i(t), 0\}$$

Using $\max\{Q_i(t) - r_i(t), 0\} \leq Q_i(t)$:

$$Q_i^2(t+1) - Q_i^2(t) \leq c_i^2(t) + r_i^2(t) + 2Q_i(t)(c_i(t) - r_i(t))$$

Bound for Virtual Queues ($Z_n(t)$):
From $Z_n(t+1) = \max\{Z_n(t) + g_n(t), 0\}$:

$$Z_n^2(t+1) \leq [Z_n(t) + g_n(t)]^2 = Z_n^2(t) + g_n^2(t) + 2Z_n(t)g_n(t)$$

$$Z_n^2(t+1) - Z_n^2(t) \leq g_n^2(t) + 2Z_n(t)g_n(t)$$

Summing the Bounds:

$$L(\mathbf{\Theta}(t+1)) - L(\mathbf{\Theta}(t)) \leq \frac{1}{2}\sum_{i=1}^{I}\left[c_i^2(t) + r_i^2(t)\right] + \frac{1}{2}\sum_{n=1}^{2I} g_n^2(t) + \sum_{i=1}^{I} Q_i(t)(c_i(t) - r_i(t)) + \sum_{n=1}^{2I} Z_n(t)g_n(t)$$

Since second moments of $c_i(t)$, $r_i(t)$, and $g_n(t)$ are bounded, there exists a finite constant $C > 0$ such that:

$$\mathbb{E}\left[\frac{1}{2}\sum_{i=1}^{I}\left[c_i^2(t) + r_i^2(t)\right] + \frac{1}{2}\sum_{n=1}^{2I} g_n^2(t)|\mathbf{\Theta}(t)\right] \leq C$$

Thus:

$$\Delta(\mathbf{\Theta}(t)) \leq C + \mathbb{E}\left[\sum_{i=1}^{I} Q_i(t)(c_i(t) - r_i(t)) + \sum_{n=1}^{2I} Z_n(t)g_n(t)|\mathbf{\Theta}(t)\right]$$

Step 6: Drift-Plus-Penalty Minimization

Introduce the objective function with a weight $V > 0$:

$$\Delta(\mathbf{\Theta}(t)) + V\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)|\mathbf{\Theta}(t)\right] \leq C + \mathbb{E}\left[\sum_{i=1}^{I} Q_i(t)(c_i(t) - r_i(t)) + \sum_{n=1}^{2I} Z_n(t)g_n(t) + V\sum_{i=1}^{I} p_i(t)|\mathbf{\Theta}(t)\right]$$

Minimizing the right-hand side (RHS) every slot t stabilizes queues and optimizes the objective.

Step 7: Per-Slot Problem ($\mathbf{P2}$)

At each slot t, given $\mathbf{\Theta}(t)$, observe $c_i(t)$, $h_i(t)$, and solve:

$$\mathbf{P2}: \quad \min_{\{p_i(t)\}} \quad \sum_{i=1}^{I} Q_i(t)(c_i(t) - r_i(t)) + \sum_{n=1}^{2I} Z_n(t)g_n(t) + V\sum_{i=1}^{I} p_i(t)$$

subject to:

$$0 \leq p_i(t) \leq p_{\max}, \quad \forall i \in \{1, \ldots, I\}$$

Substitute $g_n(t)$:
Define $g_n(t)$ explicitly:

- For $n = i \in \{1, \ldots, I\}$ (rate constraints):

$$g_i(t) = \bar{r}_i - r_i(t)$$

- For $n = I + i \in \{I+1, \ldots, 2I\}$ (error constraints):

$$g_{I+i}(t) = e_i(t) - \bar{e}_i$$

Rewrite Objective:

$$
\begin{aligned}
\text{RHS} &= \sum_{i=1}^{I} Q_i(t)(c_i(t) - r_i(t)) + \sum_{i=1}^{I} Z_i(t)(\bar{r}_i - r_i(t)) + \sum_{i=1}^{I} Z_{I+i}(t)(e_i(t) - \bar{e}_i) + V \sum_{i=1}^{I} p_i(t) \\
&= \underbrace{\sum_{i=1}^{I} Q_i(t)c_i(t) + \sum_{i=1}^{I} Z_i(t)\bar{r}_i - \sum_{i=1}^{I} Z_{I+i}(t)\bar{e}_i}_{\text{Constants (independent of } p_i(t))} \\
&\quad + \sum_{i=1}^{I} \left[-(Q_i(t) + Z_i(t))r_i(t) + Vp_i(t) + Z_{I+i}(t)e_i(t) \right]
\end{aligned}
$$

Dropping constants, the per-slot problem simplifies to:

$$
\mathbf{P2}: \quad \min_{\{p_i(t)\}} \quad \sum_{i=1}^{I} \left[Vp_i(t) - (Q_i(t) + Z_i(t))r_i(t) + Z_{I+i}(t)e_i(t) \right]
$$

Step 8: Queue Updates

After solving $\mathbf{P2}$ at slot t:

1. **Physical Queues**:

$$
Q_i(t+1) = \max\left\{Q_i(t) - r_i(t), 0\right\} + c_i(t), \quad \forall i
$$

2. **Virtual Queues**:

$$
\begin{aligned}
Z_i(t+1) &= \max\left\{Z_i(t) + \bar{r}_i - r_i(t), 0\right\}, \qquad \forall i \in \{1, \dots, I\} \\
Z_{I+i}(t+1) &= \max\left\{Z_{I+i}(t) + e_i(t) - \bar{e}_i, 0\right\}, \quad \forall i \in \{1, \dots, I\}
\end{aligned}
$$

Final Per-Slot Problem ($\mathbf{P2}$)

$$
\boxed{
\begin{aligned}
&\qquad\qquad\qquad \mathbf{P2}: \\
&\min_{\{p_i(t)\}_{i=1}^{I}} \quad \sum_{i=1}^{I} \left[Vp_i(t) - (Q_i(t) + Z_i(t))\, r_i(t) + Z_{I+i}(t)e_i(t) \right] \\
&\text{s.t.} \qquad 0 \le p_i(t) \le p_{\max}, \quad \forall i \in \{1, \dots, I\} \\
&\text{where} \\
&\qquad\quad r_i(t) = B \log\left(1 + \frac{p_i(t)h_i(t)}{\sum_{j \ne i} p_j(t)h_i(t) + BN_0}\right) \\
&\qquad\quad e_i(t) = f_{\text{BER}}(p_i(t), h_i(t)) \quad \text{(omitted function)}
\end{aligned}
}
$$

Key Insights:

- **Tradeoff Control**: Parameter $V \ge 0$ balances queue stability and power minimization.
- **Decoupling**: $\mathbf{P2}$ depends only on current states ($\mathbf{\Theta}(t)$, $c_i(t)$, $h_i(t)$), not distributions.
- **Implementation**: Solve $\mathbf{P2}$ per slot, update queues, and repeat.

User:

Task. Analyze the performance gap between the original problem ($\mathbf{P1}$) and the per-slot solution ($\mathbf{P2}$) by rigorously deriving the $O(1/V)$ objective suboptimality and $O(V)$ queue backlog bounds. Use the drift-plus-penalty framework with these key components:

- Long-term attributes may be affected by the random event $\omega(t)$ and decision $x(t)$, i.e. $f(t) = f(\omega(t), x(t))$, $g_n(t) = g_n(\omega(t), x(t))$, $h_m(t) = h_m(\omega(t), x(t))$, $c_i(t) = c_i(\omega(t), x(t))$, $d_i(t) = d_i(\omega(t), x(t))$
- Random events $\omega(t)$ are i.i.d. over slots t
- Define ω-only policy that make decisions based on $\omega(t)$ only. Assume there exists an ω-only policy $x^\star(t)$ satisfies:
-
    - $\mathbb{E}[f(\omega(t), x^\star(t))] = f^\star$, where $f^\star$ is the optimal solution of ($\mathbf{P1}$)
-
    - $\mathbb{E}[g_n(\omega(t), x^\star(t))] \leq -\delta, \exists \delta > 0$
-
    - $\mathbb{E}[h_m(\omega(t), x^\star(t))] = 0$
-
    - $\mathbb{E}[c_i(\omega(t), x^\star(t)) - d_i(\omega(t), x^\star(t))] \leq -\varepsilon, \exists \varepsilon > 0$
- The expectation of $f(t)$ is bounded within the interval $[f_{min}, f_{max}]$.

User Input. The converted standard form ($\mathbf{P1}$) and per-slot problem ($\mathbf{P2}$) .

Output Format. Notice that the reader is a newcomer who is not familiar with Lyapunov optimization. Please return the precise derivation of performance analysis in LaTeX. Every step in the mathematical derivation needs to be explained.

DeepSeek:

Performance Analysis of the Drift-Plus-Penalty Algorithm

We analyze the performance gap between the original problem ($\mathbf{P1}$) and the per-slot solution ($\mathbf{P2}$) using the drift-plus-penalty framework. The goal is to derive:

- **Objective suboptimality**: $O(1/V)$ gap from the optimal solution.
- **Queue backlog**: $O(V)$ bound.

Step 1: Key Definitions and Assumptions

1. **System Dynamics**:
    - Random event $\omega(t)$ is i.i.d. over slots t.
    - Decision variables $x(t)$ include power allocations $\{p_i(t)\}_{i=1}^{I}$.
    - Attributes:
        - Objective: $f(t) = \sum_{i=1}^{I} p_i(t)$
        - Inequality constraints: $g_n(t) = \begin{cases} \bar{r}_i - r_i(t), & n = i \\ e_i(t) - \bar{e}_i, & n = I + i \end{cases}$
        - Physical queues: $c_i(t)$ (arrivals), $d_i(t) = r_i(t)$ (service rates).

2. ω**-Only Policy**:

   - There exists an ω-only policy $x^\star(t)$ that:

     - Achieves optimal objective: $\mathbb{E}[f(\omega(t), x^\star(t))] = f^\star$

     - Satisfies strict feasibility for constraints:

$$\mathbb{E}[g_n(\omega(t), x^\star(t))] \leq -\delta, \quad \exists \delta > 0, \quad \forall n$$

$$\mathbb{E}[c_i(\omega(t), x^\star(t)) - r_i(\omega(t), x^\star(t))] \leq -\varepsilon, \quad \exists \varepsilon > 0, \quad \forall i$$

- Bounded objective: $f_{\min} \leq f(t) \leq f_{\max}$.

Step 2: Drift-Plus-Penalty Bound

Recall the drift-plus-penalty bound derived earlier:

$$\Delta(\mathbf{\Theta}(t)) + V\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)|\mathbf{\Theta}(t)\right] \leq C + \mathbb{E}\left[\sum_{i=1}^{I} Q_i(t)(c_i(t) - r_i(t)) + \sum_{n=1}^{2I} Z_n(t)g_n(t) + V\sum_{i=1}^{I} p_i(t)|\mathbf{\Theta}(t)\right]$$

where C is a finite constant from bounded second moments.

Step 3: Comparison with ω-Only Policy

Since $\mathbf{P2}$ minimizes the right-hand side (RHS) over all feasible decisions, including $x^\star(t)$:

$$\text{RHS under } x(t) \leq \text{RHS under } x^\star(t)$$

Taking conditional expectations:

$$\mathbb{E}\left[\text{RHS}|\mathbf{\Theta}(t)\right] \leq \mathbb{E}\left[\text{RHS under } x^\star(t)|\mathbf{\Theta}(t)\right]$$

Substitute the ω-only policy properties:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{I} Q_i(t)(c_i(t) - r_i(t)) + \sum_{n=1}^{2I} Z_n(t)g_n(t) + V\sum_{i=1}^{I} p_i(t)|\mathbf{\Theta}(t)\right] \\
&\leq \sum_{i=1}^{I} Q_i(t)\underbrace{\mathbb{E}[c_i(t) - r_i(t)|\mathbf{\Theta}(t)]}_{\leq -\varepsilon} + \sum_{n=1}^{2I} Z_n(t)\underbrace{\mathbb{E}[g_n(t)|\mathbf{\Theta}(t)]}_{\leq -\delta} + \underbrace{V\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)|\mathbf{\Theta}(t)\right]}_{=f^\star} \\
&\leq -\varepsilon\sum_{i=1}^{I} Q_i(t) - \delta\sum_{n=1}^{2I} Z_n(t) + Vf^\star
\end{aligned}$$

Thus:

$$\Delta(\mathbf{\Theta}(t)) + V\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)|\mathbf{\Theta}(t)\right] \leq C - \varepsilon\sum_{i=1}^{I} Q_i(t) - \delta\sum_{n=1}^{2I} Z_n(t) + Vf^\star$$

Step 4: Expected Drift-Plus-Penalty

Take expectation over $\boldsymbol{\Theta}(t)$:

$$\mathbb{E}\left[\Delta(\boldsymbol{\Theta}(t))\right] + V\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right] \le C - \varepsilon\sum_{i=1}^{I}\mathbb{E}[Q_i(t)] - \delta\sum_{n=1}^{2I}\mathbb{E}[Z_n(t)] + Vf^\star$$

Sum over $t = 0$ to $T-1$:

$$\sum_{t=0}^{T-1}\mathbb{E}\left[\Delta(\boldsymbol{\Theta}(t))\right] + V\sum_{t=0}^{T-1}\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right] \le CT + Vf^\star T - \varepsilon\sum_{t=0}^{T-1}\sum_{i=1}^{I}\mathbb{E}[Q_i(t)] - \delta\sum_{t=0}^{T-1}\sum_{n=1}^{2I}\mathbb{E}[Z_n(t)]$$

Step 5: Telescope Lyapunov Drift

The Lyapunov drift telescopes:

$$\sum_{t=0}^{T-1}\mathbb{E}\left[\Delta(\boldsymbol{\Theta}(t))\right] = \mathbb{E}[L(\boldsymbol{\Theta}(T))] - \mathbb{E}[L(\boldsymbol{\Theta}(0))]$$

Since $L(\boldsymbol{\Theta}(T)) \ge 0$ and $\mathbb{E}[L(\boldsymbol{\Theta}(0))] = L_0$ (finite initial value):

$$\mathbb{E}[L(\boldsymbol{\Theta}(T))] - L_0 + V\sum_{t=0}^{T-1}\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right] \le CT + Vf^\star T - \varepsilon\sum_{t=0}^{T-1}\sum_{i=1}^{I}\mathbb{E}[Q_i(t)] - \delta\sum_{t=0}^{T-1}\sum_{n=1}^{2I}\mathbb{E}[Z_n(t)]$$

Step 6: Objective Suboptimality ($O(1/V)$)

Rearrange for the objective:

$$V\sum_{t=0}^{T-1}\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right] \le Vf^\star T + CT + L_0 + \underbrace{\varepsilon\sum_{t=0}^{T-1}\sum_{i=1}^{I}\mathbb{E}[Q_i(t)] + \delta\sum_{t=0}^{T-1}\sum_{n=1}^{2I}\mathbb{E}[Z_n(t)]}_{\ge 0}$$

Drop non-negative terms:

$$V\sum_{t=0}^{T-1}\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right] \le Vf^\star T + CT + L_0$$

Divide by VT:

$$\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right] \le f^\star + \frac{C}{V} + \frac{L_0}{VT}$$

Take $\limsup_{T\to\infty}$:

$$\boxed{\limsup_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[\sum_{i=1}^{I} p_i(t)\right] \le f^\star + \frac{C}{V}}$$

This shows an $O(1/V)$ optimality gap.

Step 7: Queue Backlog Bound ($O(V)$)

From the telescoped inequality, drop $V\sum_t \mathbb{E}[\sum_i p_i(t)] \geq 0$:

$$\varepsilon \sum_{t=0}^{T-1}\sum_{i=1}^{I} \mathbb{E}[Q_i(t)] + \delta \sum_{t=0}^{T-1}\sum_{n=1}^{2I} \mathbb{E}[Z_n(t)] \leq Vf^\star T + CT + L_0$$

Use $\sum_{t=0}^{T-1} \mathbb{E}[Q_i(t)] \geq T \cdot \min_t \mathbb{E}[Q_i(t)]$ and isolate:

$$\frac{1}{T}\sum_{t=0}^{T-1}\left[\varepsilon \sum_{i=1}^{I} \mathbb{E}[Q_i(t)] + \delta \sum_{n=1}^{2I} \mathbb{E}[Z_n(t)]\right] \leq Vf^\star + C + \frac{L_0}{T}$$

As $T \to \infty$:

$$\varepsilon \cdot \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1}\sum_{i=1}^{I} \mathbb{E}[Q_i(t)] + \delta \cdot \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1}\sum_{n=1}^{2I} \mathbb{E}[Z_n(t)] \leq Vf^\star + C$$

Since $\varepsilon, \delta > 0$:

$$\boxed{\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[Q_i(t)] \leq \frac{Vf^\star + C}{\varepsilon}, \quad \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[Z_n(t)] \leq \frac{Vf^\star + C}{\delta}}$$

This shows queue backlogs are $O(V)$.

Step 8: Constraint Satisfaction

For virtual queues $Z_n(t)$, the update is:

$$Z_n(t+1) \geq Z_n(t) + g_n(t)$$

Sum from $t = 0$ to $T-1$:

$$Z_n(T) - Z_n(0) \geq \sum_{t=0}^{T-1} g_n(t)$$

Divide by T:

$$\frac{1}{T}\sum_{t=0}^{T-1} g_n(t) \leq \frac{Z_n(T) - Z_n(0)}{T}$$

Take expectation:

$$\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[g_n(t)] \leq \frac{\mathbb{E}[Z_n(T)]}{T} - \frac{\mathbb{E}[Z_n(0)]}{T}$$

From Lyapunov function boundedness:

$$\mathbb{E}[Z_n^2(T)] \leq 2\mathbb{E}[L(\mathbf{\Theta}(T))] \leq 2\left(L_0 + (Vf^\star + C)T\right)$$

By Jensen's inequality:

$$\mathbb{E}[Z_n(T)] \leq \sqrt{2(L_0 + (Vf^\star + C)T)}$$

Thus:

$$\frac{\mathbb{E}[Z_n(T)]}{T} \leq \sqrt{\frac{2(L_0 + (Vf^\star + C)T)}{T^2}} = \sqrt{\frac{2(L_0 + Vf^\star + C)}{T}} \to 0 \quad \text{as} \quad T \to \infty$$

Finally:

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[g_n(t)] \leq 0$$

This ensures long-term constraints hold:

- For $n = i$: $\liminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[r_i(t)] \geq \bar{r}_i$
- For $n = I + i$: $\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[e_i(t)] \leq \bar{e}_i$

Physical queue stability follows similarly from $\mathbb{E}[Q_i(T)]/T \to 0$.

Summary of Performance Bounds

1. **Objective Suboptimality**:

$$\text{Time-average power} \leq f^\star + \frac{C}{V} + O\left(\frac{1}{T}\right)$$

2. **Queue Backlog**:

$$\text{Time-average } \mathbb{E}[Q_i(t)] = O(V), \quad \text{Time-average } \mathbb{E}[Z_n(t)] = O(V)$$

3. **Constraint Satisfaction**:

All long-term constraints hold asymptotically.