

LLM-Driven Automation for Lyapunov Optimization Design (One Prompt)

User:

Task

- (1) Decompose the given optimization problem into a per-slot decision problem via the Lyapunov optimization framework.
- (2) Analyze the $[O(1/V), O(V)]$ performance-delay tradeoff.

User Input. a base station serves I terminals over discrete time slots $t \in \{0, \dots, T-1\}$. The downlink rate for terminal i during slot t is expressed as

$$r_i(t) = B \log \left(1 + \frac{p_i(t)h_i(t)}{\sum_{j \neq i} p_j(t)h_i(t) + BN_0} \right)$$

where B is the pre-allocated bandwidth of downlink, $p_i(t)$ presents the transmission power to terminal i during slot t , $h_i(t)$ denotes the channel gain between base station and terminal i , N_0 represents the noise power density. It is worth noting that the channel gain is modeled as a stationary process. Additionally, each terminal i has an associated queue $Q_i(t)$ in the base station that stores data awaiting transmission. The queue starts from an initial state 0, and its subsequent states are determined by stochastic data arrival $c_i(t)$ and transmission rate $r_i(t)$:

$$Q_i(t) \triangleq \begin{cases} 0, & t = 0 \\ \max\{Q_i(t-1) - r_i(t-1), 0\} + c_i(t-1), & t \geq 1 \end{cases}$$

The objective of the following model is to minimize the long-term averaged transmission power, while ensuring quality of service and transmission queue stability:

$$\min_{\mathbf{P}} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{i=1}^I \mathbb{E}[p_i(t)] \quad (1)$$

$$\text{s.t. } 0 \leq p_i(t) \leq p_{\max}, \forall i \in \{1, \dots, I\}, t \in \{0, \dots, T-1\} \quad (2)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[r_i(t)] \geq \bar{r}_i, \forall i \in \{1, \dots, I\} \quad (3)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[e_i(t)] \leq \bar{e}_i, \forall i \in \{1, \dots, I\} \quad (4)$$

$$\lim_{t \in \infty} \frac{\mathbb{E}[Q_i(t)]}{t} = 0, \forall i \in \{1, \dots, I\} \quad (5)$$

Constraint (2) sets the transmission power boundary. Constraints (3) and (4) ensure quality of service in terms of transmission speed and bit error rate, respectively. Specifically, the bit error rate $e_i(t)$ depends on $p_i(t)h_i(t)$, with its expression varying by modulation mode. However, to focus on the LLM generation performance and without loss of generality, this paper omits the detailed expression of $e_i(t)$.

Output Format. Return the precise derivation for Lyapunov decomposition and performance analysis. Every step in the mathematical derivation needs to be explained.

DeepSeek:

Lyapunov Optimization Framework Decomposition

Step 1: Define Virtual Queues for Long-Term Constraints

To handle the long-term constraints (2) and (3), introduce **virtual queues**:

- **Rate deficit queue** $Z_i(t)$ for minimum rate constraint \bar{r}_i :

$$Z_i(t+1) = \max \{Z_i(t) + \bar{r}_i - r_i(t), 0\}, \quad Z_i(0) = 0$$

This queue accumulates unmet rate requirements.

- **Error surplus queue** $Y_i(t)$ for maximum error constraint \bar{e}_i :

$$Y_i(t+1) = \max \{Y_i(t) + e_i(t) - \bar{e}_i, 0\}, \quad Y_i(0) = 0$$

This queue accumulates excess bit errors.

Step 2: Define Combined Queue Vector and Lyapunov Function

- **Combined queue vector**:

$$\Theta(t) = [\mathbf{Q}(t), \mathbf{Z}(t), \mathbf{Y}(t)], \quad \text{where } \mathbf{Q}(t) = [Q_i(t)], \mathbf{Z}(t) = [Z_i(t)], \mathbf{Y}(t) = [Y_i(t)]$$

- **Lyapunov function** (measures queue congestion):

$$L(\Theta(t)) = \frac{1}{2} \sum_{i=1}^I [Q_i(t)^2 + Z_i(t)^2 + Y_i(t)^2]$$

This scalar quantifies the overall queue backlog.

Step 3: Define Conditional Lyapunov Drift

- **One-slot drift** $\Delta(\Theta(t))$:

$$\Delta(\Theta(t)) = \mathbb{E} [L(\Theta(t+1)) - L(\Theta(t)) \mid \Theta(t)]$$

This represents the expected change in congestion.

Step 4: Drift-Plus-Penalty Minimization

Augment the drift with the objective (average power minimization) using a tradeoff parameter $V > 0$:

$$\Delta(\Theta(t)) + V \cdot \mathbb{E} \left[\sum_{i=1}^I p_i(t) \mid \Theta(t) \right]$$

Goal: Minimize an upper bound of this expression.

Step 5: Derive Drift Upper Bound

Using the queue update laws and the inequality $(\max\{x, 0\})^2 \leq x^2$:

- **Actual queue** (Q_i):

$$Q_i(t+1)^2 \leq [Q_i(t) - r_i(t) + c_i(t)]^2$$

- **Virtual queues** (Z_i, Y_i):

$$Z_i(t+1)^2 \leq [Z_i(t) + \bar{r}_i - r_i(t)]^2$$

$$Y_i(t+1)^2 \leq [Y_i(t) + e_i(t) - \bar{e}_i]^2$$

Summing these, the Lyapunov drift satisfies:

$$L(\Theta(t+1)) - L(\Theta(t)) \leq C + \sum_{i=1}^I [Q_i(t)(c_i(t) - r_i(t)) + Z_i(t)(\bar{r}_i - r_i(t)) + Y_i(t)(e_i(t) - \bar{e}_i)]$$

where $C = \frac{1}{2} \sum_{i=1}^I [c_{\max}^2 + r_{\max}^2 + \bar{r}_i^2 + r_{\max}^2 + e_{\max}^2 + \bar{e}_i^2]$ is a finite constant (since $c_i(t)$, $r_i(t)$, and $e_i(t)$ are bounded).

Step 6: Minimize Per-Slot Drift-Plus-Penalty

Ignoring constants and terms independent of $\mathbf{p}(t)$, minimize:

$$\mathbb{E} \left[\sum_{i=1}^I \left(V p_i(t) - \underbrace{[Q_i(t) + Z_i(t)] r_i(t)}_{\text{rate reward weight}} + \underbrace{Y_i(t)}_{\text{error penalty weight}} e_i(t) \right) \mid \Theta(t) \right]$$

Given $\Theta(t)$ and channel states $\mathbf{h}(t)$, the **per-slot problem** is:

$$\min_{\mathbf{p}(t)} \sum_{i=1}^I [V p_i(t) - [Q_i(t) + Z_i(t)] r_i(t) + Y_i(t) e_i(t)]$$

subject to $0 \leq p_i(t) \leq p_{\max}$, $\forall i$.

Key properties:

- $r_i(t)$ is concave in $\mathbf{p}(t)$ (log-SINR function).
 - $e_i(t)$ is typically convex in $p_i(t)$ (decreases with SINR).
 - The problem is non-convex due to interference coupling. Approximate solutions (e.g., fractional programming) are used in practice.
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Performance-Delay Tradeoff Analysis

Assumption: Feasibility and Slackness

Assume there exists a stationary policy π^* that:

- Satisfies all constraints strictly with slackness $\epsilon > 0$:

$$\mathbb{E}[r_i^{\pi^*}(t)] \geq \bar{r}_i + \epsilon, \quad \mathbb{E}[e_i^{\pi^*}(t)] \leq \bar{e}_i - \epsilon, \quad \mathbb{E}\left[\sum_i p_i^{\pi^*}(t)\right] = p_{\text{opt}}$$

- p_{opt} is the optimal average power.

Performance Bound

The drift-plus-penalty bound under the per-slot policy is:

$$\Delta(\Theta(t)) + V\mathbb{E}\left[\sum_i p_i(t) \mid \Theta(t)\right] \leq C + Vp_{\text{opt}} - \epsilon \sum_{i=1}^I \mathbb{E}[Q_i(t) + Z_i(t) + Y_i(t) \mid \Theta(t)]$$

Taking expectation, summing over $t \in \{0, \dots, T-1\}$, and dividing by T :

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[\sum_i p_i(t)\right] \leq p_{\text{opt}} + \frac{C}{V} + \frac{\mathbb{E}[L(\Theta(0))]}{VT}$$

As $T \rightarrow \infty$:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[\sum_i p_i(t)\right] \leq p_{\text{opt}} + \frac{C}{V} \quad (O(1/V) \text{ optimality gap})$$

Queue Bound

From the drift inequality:

$$\frac{1}{T} \sum_{t=0}^{T-1} \sum_{i=1}^I \mathbb{E}[Q_i(t) + Z_i(t) + Y_i(t)] \leq \frac{C + Vp_{\text{opt}}}{\epsilon} + \frac{\mathbb{E}[L(\Theta(0))]}{\epsilon T}$$

As $T \rightarrow \infty$, average queue backlog satisfies:

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{i=1}^I \mathbb{E}[Q_i(t) + Z_i(t) + Y_i(t)] \leq \frac{C + Vp_{\text{opt}}}{\epsilon} \quad (O(V) \text{ backlog})$$

Tradeoff Interpretation

- **Power optimality:** Average power is within $O(1/V)$ of optimal.
- **Delay tradeoff:** Average queue backlog (and hence delay by Little's Law) is $O(V)$.
- **Tuning:** Increasing V reduces power consumption but increases delay.