COMP2012 Discrete Mathematics Notes - Counting

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October 30, 2023

1 The Product Rule

If a task can be decomposed into a sequence of two sub-tasks, there are n_1 ways to do the first sub-task, and n_2 ways to do the second sub-task, then there are $n_1 \cdot n_2$ ways to do this task.

$$|S_1 \times S_2| = |S_1| \times |S_2|$$

2 The Sum Rule

If a task can be done in either n_1 ways or in n_2 ways, and none of these ways are the same, then there are $n_1 + n_2$ ways to do this task.

$$|S_1 \cup S_2| = |S_1| + |S_2|$$

3 The Subtraction Rule

If a task can be done in either n_1 ways or in n_2 ways, and the number of common ways is n_{common}, then there are $n_1 + n_2 - n_{common}$ ways to do this task.

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$$

4 The Principle of Inclusion-Exclusion

Given n sets

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum_{1 \le i \le n} |S_i| - \sum_{1 \le i \le j \le n} |S_i \cup S_j| + \ldots + (-1)^{n-1} |S_1 \cap S_2 \cap \ldots S_n|$$

5 The Division Rule

Suppose that a set S is the union of n disjoint subsets, where each subset contains exactly d elements, we get:

$$n = |S| / d$$

6 The Pigeonhole Principle

Let k be a positive integer. If $k + 1$ (or more) objects are placed into k boxes, then there exists at least one box that contains at least two objects. Remainders are usually used to construct the "boxes".

7 The Generalized Pigeonhole Principle

Let N and k be positive integers. If n objects are placed into k boxes, then there exists at least one box that contains at least $\lceil n/k \rceil$ objects The generalized pigeonhole principle can not be adapted when the amount of objects in a specific box is required.

8 Permutation

$$P(n,r) = n(n-1)\dots(n-r+1) = {}^{n!}\!/\!_{(n-r)!}$$

9 Circular Permutation

Arrange n objects out of m around a fix circle, the number of permutations is:

$${}^{P(m,n)}\!/\!_{n}$$

10 Combination

$$C(n,r) = {}^{n(n-1)\dots(n-r+1)}\!/\!_{r!} = {}^{n!}\!/\!_{r!(n-r)!}$$

11 The Binomial Theorem

Let x, y be variables, and n be a non-negative integer, we have:

$$(x+y)^n = \sum_{r=0}^{n} C(n,r) \cdot x^{n-r} y^r$$

12 Pascal's Identity

$$C(n+1,r) = C(n,r-1) + C(n,r)$$

13 Permutation with Repetition

For a set with n elements, the number of permutations to r positions with repetition is:

$$n^r$$

14 Combination with Repetition

For a set with n elements, the number of combinations to r positions with repetition is:

$$C(n+r-1,r)$$

Note that the number of combinations with repetition can not be computed by dividing the number of permutations with repetition by $n!$.

To understand the formula, regard the "position" r as balls and the "choice" n as baskets. Pour the balls into the baskets. The amount of balls in a basket means that the times of the "choice" be chosen, and a "choice" can never be chosen. This is the first-stage abstraction. The second-stage abstraction is that, put $n-1$ sticks and r balls to form n brackets with balls (two sticks on the edge). Hence, there are $n-1+r$ locations for sticks and balls to occupy.