The Hong Kong Polytechnic University

COMP2012 Discrete Mathematics

Assignment 1 - Suggested solutions

Guideline:

- This is the suggested solution only. Some marking is just for your reference.

Questions:

Question 1 **[10 marks]**

You are given n integers $a_1, a_2, \ldots, a_n$.
Let $m = (a_1 + a_2 + \ldots + a_n)/n$.
Prove that there exists some number in $a_1, a_2, \ldots, a_n$ such that it is smaller than or equal to m.

Solution:
Hint: m is the average of the set of numbers $\{a_1, a_2, \ldots, a_n\}$

Proof:
Let M be the minimum of the set $\{a_1, a_2, \ldots, a_n\}$

$$\mathrm{m} = \frac{\sum_i^n a_i}{n} \text{ and each } a_i \geq \min(a_i)$$

Hence,

$$\mathrm{n} \cdot \mathrm{m} = \sum_i^n a_i \geq \sum_i^n \min(a_i) = n \cdot M$$

$$\mathrm{n} \cdot \mathrm{m} \geq n \cdot M$$

$$\mathrm{m} \geq M$$

i.e. the mean (m) is greater or equal to the minimum.
In other words, there are at least 1 number s.t. it is smaller or equal to m.
Q.E.D.

Question 2 **[10 marks]**

Prove the following logic using a truth table.
(a) $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$ (5 marks)
(b) $\neg(p \land q \land r) \equiv \neg p \lor \neg q \lor \neg r$ (5 marks)
(Hint: $p \leftrightarrow q$ means p and q are logically equivalent)

Solution:
You are advised to use T and F to represent the truth values. It is not encouraged to use 0 and 1.

(a) The truth table would look like this:

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \vee \neg q)$
T	T	T	F	F	T	F	T
F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F
F	F	T	T	T	F	T	T

As the 3rd and 8th columns are equal, $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent. Done.

(b) The truth table would look like this:

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \wedge q \wedge r$	$\neg (p \wedge q \wedge r)$	$\neg p \vee \neg q \vee \neg r$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	T	T	T	T
T	F	T	F	T	F	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	T	F	T	T

As the 8th and 9th columns are equal, $\neg (p \vee q \vee r)$ and $\neg p \wedge \neg q \wedge \neg r$ are logically equivalent. Q.E.D.

Question 3 **[10 marks]**

(a) Let the domain D contain all prime numbers ≥ 3 and ≤ 25.

Suggest two propositional functions $P(x)$ and $Q(y)$ so that all the following statements are true at the same time:

- $\exists\, x \in D\ P(x)$
- $\exists\, y \in D\ Q(y)$
- $\neg (\exists\, z \in D\ P(z) \wedge Q(z))$

Prove that your proposed functions P and Q satisfy the above conditions. (5 marks)

(b) **Solve** the following two straight lines L_1 and L_2, where:

$$\begin{cases} L_1: x + y = 0 \\ L_2: x - y = 0 \end{cases}$$ (5 marks)

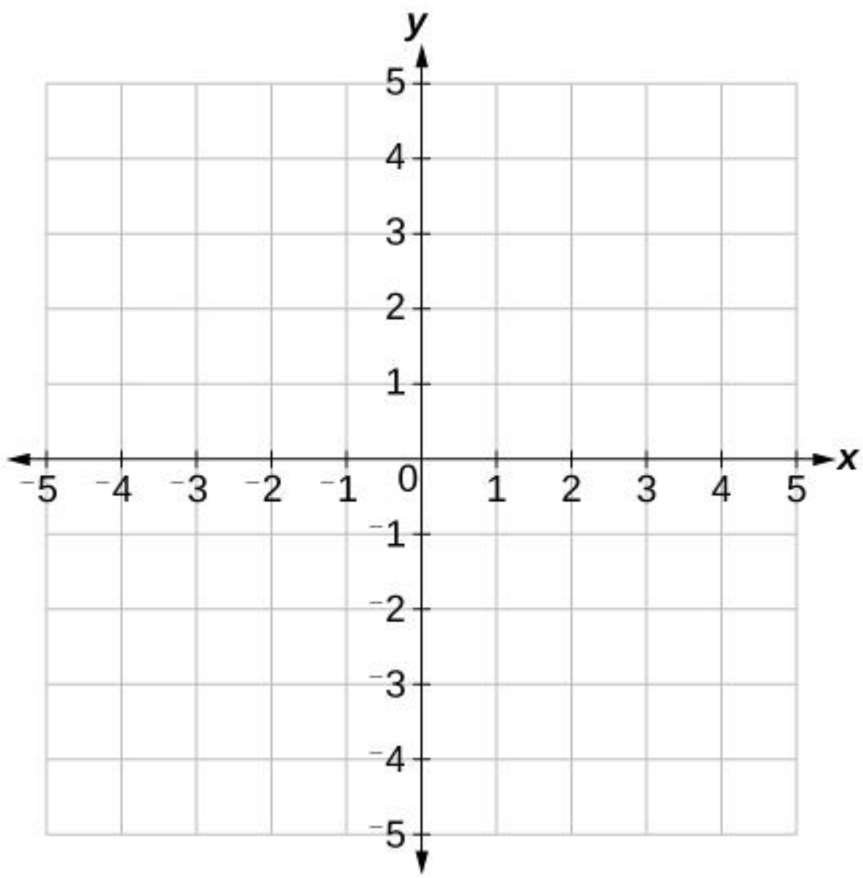
Assume that:

- $x, y \in$ D, where D is a domain {-3, -2, -1, 0, 1, 2, 3}
- Sets A and B contain all the points of L_1 and L_2, respectively.

Solution guide:

- List out all the points (x, y) of each line in the given domain D;
- Solve the equations diagrammatically with the aid of the *coordinate grid* given below. Explain with the concept of set.

- Express your findings with the aid of set builder representation, and write down the final answer in set form.

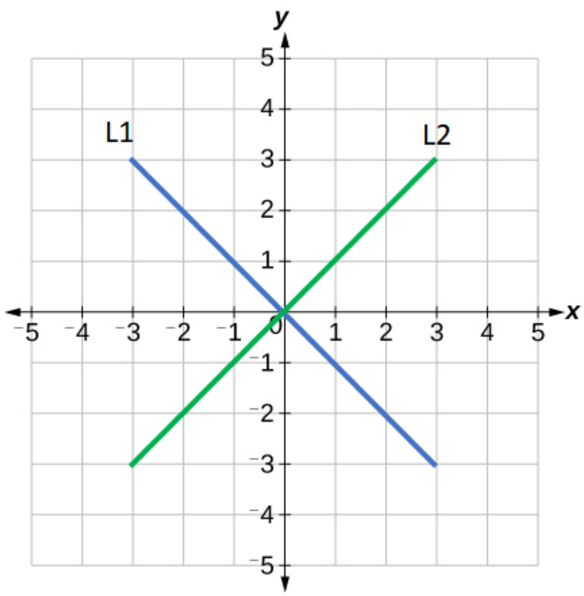


Solution:
(a)
Proof:
Hint: This can be proved by existence.

There are many possible cases, suppose we choose P(x) = {3,7,11,13}, Q(y) = {17,19,23}

Since $P(z) \wedge Q(z)$ = empty set
There are no z exists in $P(z) \wedge Q(z)$
Hence, $\neg (\exists\, z \in D\ P(z) \wedge Q(z))$ is also true

(b)
We have to pick out all those (x, y) satisfying both equations $x + y = 0$ and $x - y = 0$.
Diagrammatically, we found that the solution is just the point of intersection (1 mark) = the interaction of two sets of points (1 mark)



(Correct diagram: containing two lines with intersection 1 mark)
By definition, intersection of any sets $S_1 \cap S_2 = \{x: x \in S_1 \text{ and } x \in S_2\}$
Hence, in this case,
$A \cap B$
$= \{(x, y): x + y = 0 \text{ and } x - y = 0\}$ (set builder representation: 1 mark)
$= \{(x, y): x + y = 0\} \cap \{(x, y): x - y = 0\}$
$= \{(0, 0)\}$ ← Final answer 1 mark

Question 4 **[10 marks]**

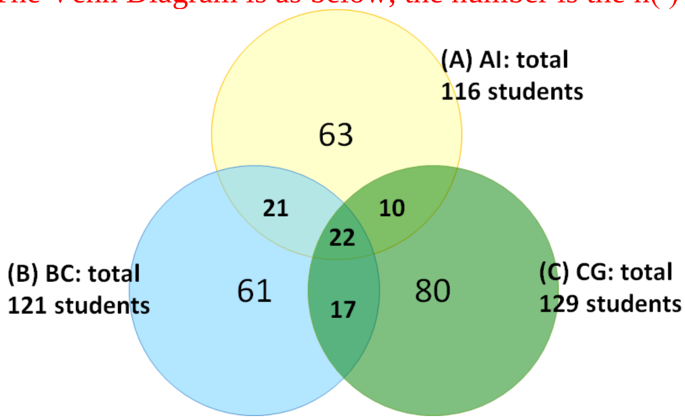
COMP department provides 3 elective courses (AI, Blockchain, and Computer graphics), and all the Year 4 COMP students need to take at least ONE elective course. According to the elective enrollment report of this Academic Year, 116 students have chosen an AI course, 121 students have chosen a Blockchain course, and 129 students have chosen a Computer graphics course. If 43 students take both AI and Blockchain courses; 32 students take both AI and Computer graphics courses; 39 students take both Blockchain and Computer graphics courses, and 22 take all three elective courses.

With the aid of Venn Diagram, show the steps to calculate:

(a) The total number of Year 4 COMP students of this Academic Year. (5 marks)
(b) Assuming a student completes any two elective courses, including Blockchain, he or she will receive a degree certificate with a FinTech major. How many persons, assuming that every student passes the courses, will be eligible to receive this? (5 marks)

<u>Solution:</u>
(a)
Let $A \equiv$ students have taken AI elective;
$B \equiv$ students have taken Blockchain elective;
$C \equiv$ students have taken Computer graphics elective.
Let $n(\cdot)$ = number of a set

From the information, we have:
n(A) = 116, n(B) = 121, n(C) = 129
n(A ∩ B) = 43, n(A ∩ C) = 32, n(B ∩ C) = 39
n(A ∩ B ∩ C) = 22
The Venn Diagram is as below, the number is the n(·) of each colored segment:



Hence, the total number of Year 4 COMP students will be
= 63 + 61 + 80 + 21+ 10 + 17 + 22
= 274

OR
By the principle of Inclusion-Exclusion (Lecture 6)
n(A) + n(B) + n(C) - n(A ∩ B) - n(A ∩ C) - n(B ∩ C) + n(A ∩ B ∩ C)
= 116 + 121 + 129 – 43 – 32 – 39 + 22
= 274

(b) n(Blockchain major) = 21+22+17 = 60

Question 5 **[15 marks]**
Given the table of rules of inferences in Appendix I (at the back page).
(a) Prove the rule "Modus ponens" using a truth table (2 marks)
(b) Given: $p \wedge q$
$(p \vee s) \rightarrow \neg r$
$r \vee t$
Prove: t (5 marks)

(c) Consider the following two statements
S1: If a student is known to be cheating, then he/she will not be passed.
S2: If a student is good, he/she will be passed.

Determine which one of the statements (i) to (iv) follows from S1 and S2 as per sound inference rules of logic. (8 marks)
(i) If a student is known to be cheating, he is good
(ii) If a student is not known to be cheating, he is not good
(iii) If a student is good, he is not known to be cheating
(iv) If a student is not good, he is not known to be cheating

Solution guide:
Starting with writing down premises, i.e. the propositional logic of statements S1, S2, and the given statements (i) to (iv).

Solution:
(a)

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

(b)

1.	$p \wedge q$	*Given*
2.	$(p \vee s) \rightarrow \neg r$	*Given*
3.	$r \vee t$	*Given*
4.	p	*Simplification*
5.	$p \vee s$	*Addition*
6.	$\neg r$	*M.P.*
7.	t	*D.S.*

(c)
Let C = student is cheating
E = student will be passed
K = student is good
Premises
S1: C→¬E (1 mark)
S2: K→E (1 mark)
S(i): C→K (1 mark)
S(ii): ¬C→¬K (1 mark)
S(iii): K→¬C (1 mark)
S(iv): ¬K→¬C (1 mark)

Statement (iii) is correct, while others are wrong (1 mark)
Reason: (1 mark)

K→E	(Given)
E→¬C	(Modus tollens or M.T.)
K→¬C	(Hypothetical syllogism or H.S.)
Q.E.D.	

Question 6 **[5 marks]**

Take an original photo by yourself of an everyday object or a scene that you feel is beautiful, which can be fit with a golden spiral. Attach the photo and draw a golden spiral on it.

Solution:

You should provide something like this:

Question 7 **[10 marks]**

Matrix operations

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, E = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

I_d = Identity matrix, O_d = zero matrix (all entries are zero)
(where subscript d is the dimension of the matrix)

Evaluate:

(a) $D^T E$ (2 marks)

(b) Prove whether $A^2 – B^2 = (A – B)(A + B)$ (3 marks)

(c) If $C^3 - 6C^2 + 7C + kI_3 = O_3$, find the value of k (5 marks)

Solution:

(a)

$$D^T E = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} \text{ (1 mark)}$$

$$= \begin{bmatrix} 89 & 98 \\ 116 & 128 \end{bmatrix} \text{ (1 mark)}$$

(b)

Proof: (by disprove)
Let us calculate (A−B)(A+B) as follows using the fact that the matrix product is distributive.
(A−B)(A+B) = A(A+B)−B(A+B) = A^2+AB−BA−B^2 = A^2−B^2+(AB−BA).
Thus if (A−B)(A+B)=A^2−B^2 then AB−BA=O, the zero matrix. Equivalently, AB=BA.

Note that matrix multiplication is not commutative, namely, AB≠BA in general.
Thus we can disprove the statement if we find matrices A and B such that AB≠BA.

OR, By evaluation

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^2 = \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix} \text{ (1 mark)}$$

$$A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -5 & -7 \end{bmatrix} \text{ (1 mark)}$$

$$(A - B)(A + B) = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -9 & -9 \end{bmatrix} \neq A^2 - B^2 \text{ (1 mark)}$$

(c)

$$C^2 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}, C^3 = C^2C = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Putting values, we have:

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2+k & 0 & 0 \\ 0 & -2+k & 0 \\ 0 & 0 & -2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2+k & 0 & 0 \\ 0 & -2+k & 0 \\ 0 & 0 & -2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,
-2 + k = 0
k = 2

Question 8 **[10 marks]**

(a) Determine the time complexity of the following two algorithms which check whether a input number is prime. (8 marks)

IsPrime-A (n)	IsPrime-B (n)

for integer $i \leftarrow 2$ to n-1
    if n mod i ==0 then
        return False
return True
	for integer $i \leftarrow 2$ to floor(sqrt(n))+1
	    if n mod i ==0 then
	        return False
	return True

Hint: You need to write down the run time frequency of each line, and then estimate the final complexity. In addition, you may try to test the pseudocode with some sample *n*.

(b) Which one is more efficient? (1 mark) Why? (1 mark)

Solution:

(a)

IsPrime-A is O(n)

IsPrime-B is O($\sqrt{n}$)

(b)

IsPrime-B is more time efficient as its time complexity is lower then IsPrime-A

O($\sqrt{n}$)< O(n)

Reference:

https://www.youtube.com/watch?v=nO1Y8lR9swI

https://stackoverflow.com/questions/54543956/finding-prime-number-using-the-square-root-method

Question 9 **[10 marks]**

Show that $\forall n \in \mathbb{Z}^+$ (i.e. non-zero positive integer),

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$$

Solution:

Before proving the statement, we should observe that, when n is increased by one, the number of terms of L.H.S. is increased by two, whereas the number of terms on the R.H.S. is only increased by one.

Proof:

Let P(n) denote the above statement.

Basis step:

For n = 1, L.H.S = $1 - \frac{1}{2} = \frac{1}{2}$ = R.H.S. Hence P(1) is true.

Induction step:

Assume P(k) is true for $k \geq 1$, i.e.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2k}=\frac{1}{k+1}+\frac{1}{k+2}-\frac{1}{k+3}+\cdots+\frac{1}{2k}$$

Now for P(k+1),
L.H.S.

$$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2}$$

$$=(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2k})+\frac{1}{2k+1}-\frac{1}{2(k+1)}$$

$$=(\frac{1}{k+1}+\frac{1}{k+2}-\frac{1}{k+3}+\cdots+\frac{1}{2k})+(\frac{1}{2k+1}-\frac{1}{2(k+1)})$$

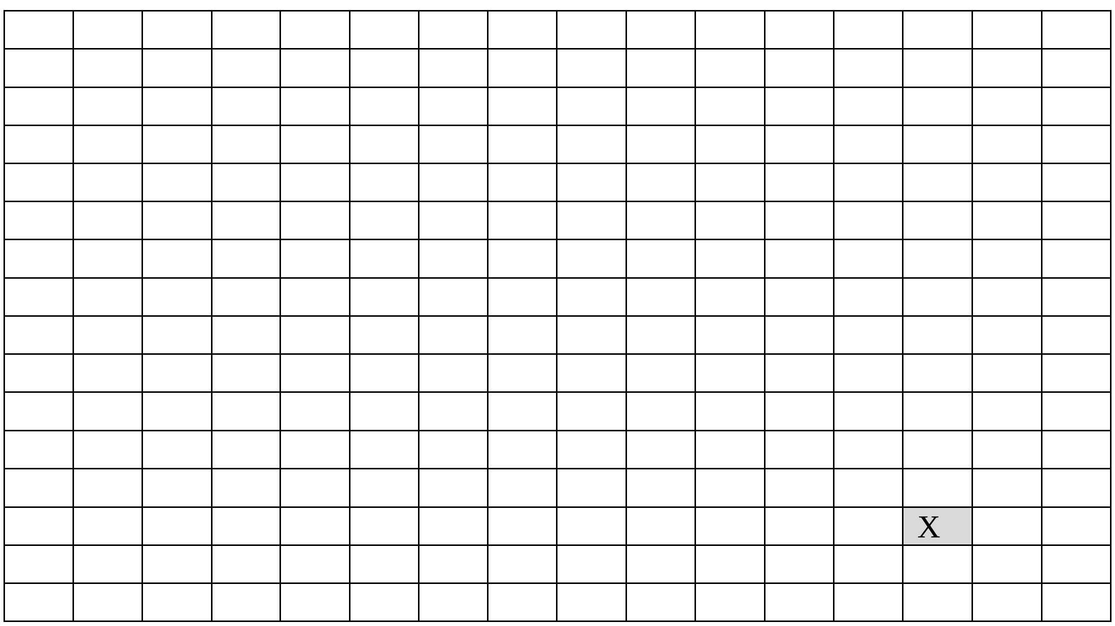
$$=\left(\frac{1}{k+2}-\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2(k+1)}\right)+\frac{1}{k+1}-2(\frac{1}{2(k+1)})$$

$$=\left(\frac{1}{(k+1)+1}-\frac{1}{(k+1)+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2(k+1)}\right)$$

Hence by the principle of mathematical induction, P(n) is true for every positive integer n.

Question 10 **[10 marks]**

We are referring to the tiling problem in “MI: example 3” in the slides of lecture #5. Now, we are given the following 16×16 checkerboard with a missing square (X).



A triomino looks like one of the followings.

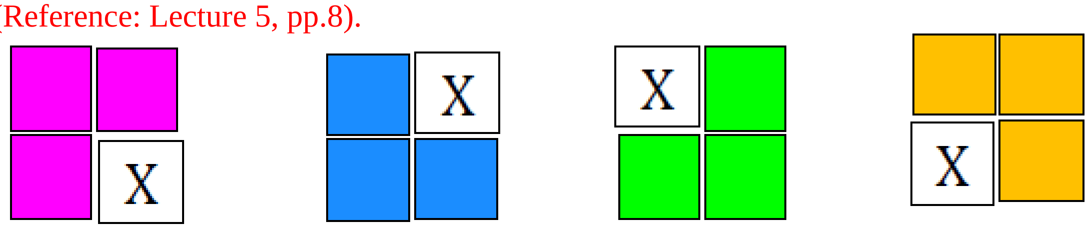
Draw the above checkerboard after tiling it with triominoes. Adjacent triominoes should be filled in different colours (like in the slides).

<u>Solution:</u>
The way of tiling please refer to the solution of lab 5 question 1.

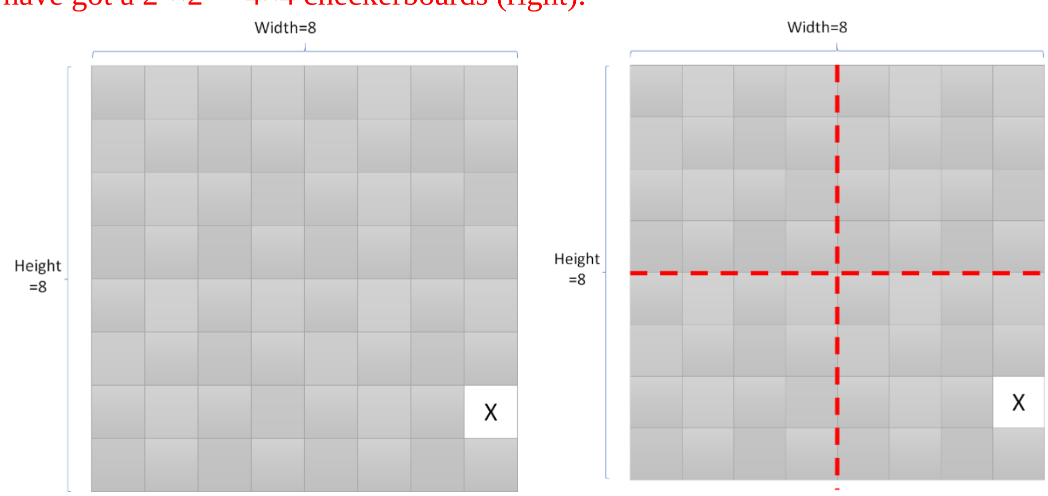
Number of colors should be 4 (note: no more than 4 for all cases!)

Reference to simpler case 8x8 chessboard:

Similarly, like Q1(i). We consider the basis step first.
Basis step, when n=1, we consider 2×2 checkerboard. There are four possible checkboards as shown below. Each checkboard can be tiled using a trimino. (Reference: Lecture 5, pp.8).

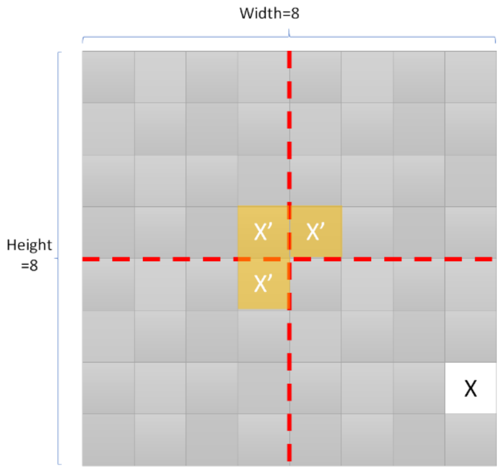


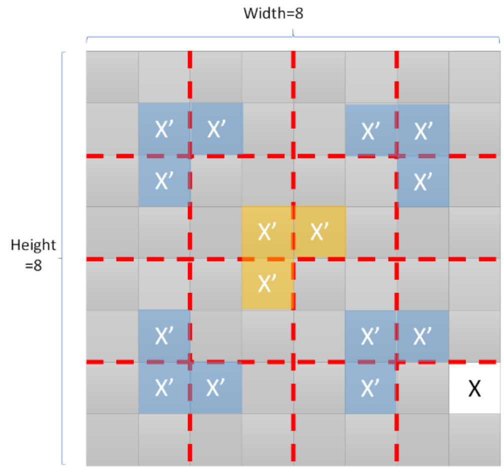
When n=3, we have got a $2^3 \times 2^3$ = 8×8 checkerboard (left).
And then, we try to break the checkerboard into smaller checkerboards: n-1=2, hence we have got a $2^2 \times 2^2$ = 4×4 checkerboards (right).



For each smaller (4×4) checkerboard without missing square, we mark a square as a missing square with a label X' in at the center, and form a new triominoe (left).

We continue to consider n-2=1, hence smaller checkerboards of 2×2 dimensions are formed. Apply the result from Q1(i), we fill the triominoe as right hand side:





Finally, we can fill up the remaining squares with more triominoes.

End of Assignment 1 solution

Appendix I

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution