COMP2021 Discrete Mathematics Assignment 1

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Question 1 *Proof* : Suppose that $\forall i \in \{1, 2, \ldots, n\}, a_i > m$, namely, $\nexists a_i \leq m$, then $\sum_{i=1}^{n} a_i > nm$. A contradiction is implied: $m = \frac{1}{n}\sum_{i=1}^{n} a_i > \frac{nm}{n} = m$. Therefore, $\exists i \in \{1, 2, \ldots, n\}, a_i \leq m$. This concludes the proof.

Question 2

(a) *Proof* : The truth table of Question 2 (a):

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	T	F	F	T	F	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	T

Therefore, the given proporisition $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ is true.

(b) *Proof* : The truth table of Question 2 (b):

p	q	r	$p \wedge q \wedge r$	$\neg(p \wedge q \wedge r)$	$\neg p$	$\neg q$	$\neg r$	$\neg p \vee \neg q \vee \neg r$
T	T	T	T	F	F	F	F	F
T	T	F	F	T	F	F	T	T
T	F	T	F	T	F	T	F	T
T	F	F	F	T	F	T	T	T
F	T	T	F	T	T	F	F	T
F	T	F	F	T	T	F	T	T
F	F	T	F	T	T	T	F	T
F	F	F	F	T	T	T	T	T

Therefore, the given proporisition $\neg(p \wedge q \wedge r) \equiv \neg p \vee \neg q \vee \neg r$ is true.

Question 3

(a) Let $P(x) :=$ "*x equals to* 5", $Q(y) :=$ "*y equals to* 7".

Proof : The proporsition can be divided into three sections:

(1) Let $x = 5$, and it satisfies the proporsition $x \in D, P(x)$ *is true*. The proporsition $\exists x, P(x)$ *is true* is true.

(2) Let $x = 7$, and it satisfies the proporsition $x \in D, Q(x)$ *is true*. The proporsition $\exists x, Q(x)$ *is true* is true.

(3) If $\neg(\exists x \in D, P(x) \wedge Q(x))$ is true, then $\exists x \in D, P(x) \wedge Q(x)$ is false. Therefore, the proporsition $\exists x \in D, P(x)$ *and* $Q(x)$ *shall not be true at the same time* shall be true. Suppose that the proporsition $\exists x \in D, P(x)$ *and* $Q(x)$ *are true at the same time* is true, namely, x equals to 5 and x equals to 7. However, $5 \neq 7$, and the supposed proporsition will never be true. Hence, $\neg(\exists x \in D, P(x) \wedge Q(x))$ is true.

The proof is concluded.

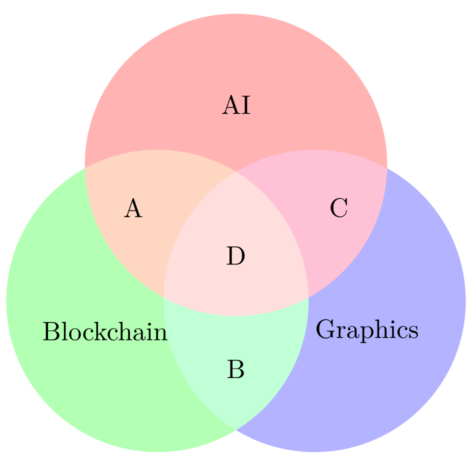
(b) There are three steps to solve the problem:

(1) Considering the line L_1. For $x \in D$, we get the set of points $S_1 = \{(-3, 3), (-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2), (3, -3)\}$, where all points in the set S_1 are on the line L_1.

(2) Considering the line L_2. For $x \in D$, we get the set of points $S_2 = \{(-3, -3), (-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), (3, 3)\}$, where all points in the set S_2 are on the line L_2.

(3) The intersection of the two sets, namely, $S_1 \cap S_2 = (0, 0)$. The solutions of the system of equations should be the elements of the intersection, namely, $(0, 0)$.

Question 4 The Venn Diagram of the situation is as below:



In the Venn Diagram, denote the set of students taking Blockchain and AI by A, the set of students taking Blockchain and Computer Graphics by B, the set of students taking AI and Computer Graphics by C, and the set of students taking all the subjects by D. Hence, $|A| = 43, |B| = 39, |C| = 32$, and $|D| = 22$. Additionally, for the number of studnets in each subject, denote them by $|Blockchain| = 121, |AI| = 116, |Graphics| = 129$.

(a) The sum of the number of students in each subject, minused by the sum of students who attend two subjects at the same time, plusing the number of students who attend three classes at the same time, equals to the total number of students. Therefore, there are $|Blockchain| + |AI| + |Graphics| - (|A| + |B| + |C|) + |D| = 121 + 116 + 129 - (43 + 39 + 32) + 22 = 274$ students in total.

(b) From the Venn Diagram, it is obvious that the the eligible students are those who belongs to A and B. To calculate the number of them, it is supposed to minus those taking all three subjects. Therefore, there are $|A| + |B| - |D| = 43 + 39 - 22 = 60$ eligible students to receice the certificate.

Question 5

(a) The truth table for Modus ponens is as below:

p	q	$p \to q$	$p \land (p \to q)$	$p \land (p \to q) \to q$
T	T	T	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(b) *Proof* : Because of the Simplication interference, $p \land q \Rightarrow p, q$. Then, according to Addition, $p \Rightarrow p \lor s$. Since $(p \lor s) \to \neg r$, $\neg r$ can be implied. Meanwhile, $r \lor t$ holds, thus t can be implied due to Disjunctive syllogism.

(c) Set p as the proporsition "*a student is known to be cheating*$''$, q as "*a student is good*$''$, and r as "*a student will be passed*$''$. Thus, S_1 can be represented as $p \to \neg r$, and S_2 as $q \to r$.

(1) The statement (i) can be represented as $if\ p\ then\ q$. (i) can not hold, as $(p \land (p \to \neg r)) \implies \neg r\ (Modus\ ponens)$, and $(\neg r \land (q \to r)) \implies \neg q\ (Modus\ tollens)$. Hence, we get a completely opposite result "$astudentisnotgood''$.

(2) The statement (ii) can be represented as $if\ \neg p\ then\ \neg q$. However, no relationship among the inference rules of logic can be found between $\neg p$ and $p \to \neg r$ or $q \to r$. It can be concluded that this statement is not valid.

(3) The statement (iii) can be represented as $if\ q\ then\ \neg p$. The statement shall be valid: $(q \land (q \to r)) \implies r\ (Modus\ ponens)$, and $(r \land (p \to \neg r)) \implies \neg p\ (Modus\ tollens)$. That is, "*a student is not known to be cheating*$''$ can be implied from the given statement.

(4) The statement (iv) can be represented as $if\ \neg p\ then\ \neg q$. It is not logically consistent, for no per sound inference rule can be found between $\neg q$ and $q \to r$ or $p \to \neg r$.

Question 6 The picture of my mouse:

Figure 1: My Logitech M575 ERGO Mouse with its golden-spiral-like shape

Question 7

(a) $D^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $D^T \times E = \begin{bmatrix} 1\times 7+3\times 9+5\times 11 & 1\times 8+3\times 10+5\times 12 \\ 2\times 7+4\times 9+6\times 11 & 2\times 8+4\times 10+6\times 12 \end{bmatrix} = \begin{bmatrix} 89 & 98 \\ 116 & 128 \end{bmatrix}$.

(b) $A^2 = \begin{bmatrix} 1\times 1+2\times 0 & 0\times 1+(-1)\times 0 \\ 1\times 2+2\times(-1) & 0\times 2+(-1)\times(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B^2 = \begin{bmatrix} 2\times 2+1\times(-1) & (-1)\times 2+3\times(-1) \\ 2\times 1+1\times 3 & (-1)\times 1+3\times 3 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$, $A^2 - B^2 = \begin{bmatrix} 1-3 & 0-(-5) \\ 0-5 & 1-8 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -5 & -7 \end{bmatrix}$.

$A - B = \begin{bmatrix} 1-2 & 0-(-1) \\ 2-1 & -1-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -4 \end{bmatrix}$, $A + B = \begin{bmatrix} 1+2 & 0+(-1) \\ 2+1 & -1+3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix}$, $(A-B)(A+B) = \begin{bmatrix} -1\times 3+1\times 3 & -1\times(-1)+1\times 2 \\ 1\times 3+-4\times 3 & 1\times(-1)+(-4)\times 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -9 & -9 \end{bmatrix}$.

Therefore, $A^2 - B^2 \neq (A-B)(A+B)$.

(c)

$$C^2 = \begin{bmatrix} 1\times 1+0\times 0+2\times 2 & 1\times 0+0\times 2+2\times 0 & 1\times 2+0\times 1+2\times 3 \\ 0\times 1+2\times 0+1\times 2 & 0\times 0+2\times 2+1\times 0 & 0\times 2+2\times 1+1\times 3 \\ 2\times 1+0\times 0+3\times 2 & 2\times 0+0\times 2+3\times 0 & 2\times 2+0\times 1+3\times 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix},$$

$$C^3 = C^2C = \begin{bmatrix} 5\times 1+0\times 0+8\times 2 & 5\times 0+0\times 2+8\times 0 & 5\times 2+0\times 1+8\times 3 \\ 2\times 1+4\times 0+5\times 2 & 2\times 0+4\times 2+5\times 0 & 2\times 2+4\times 1+5\times 3 \\ 8\times 1+0\times 0+13\times 2 & 8\times 0+0\times 2+13\times 0 & 8\times 2+0\times 1+13\times 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}.$$

Therefore,

$$C^3-6C^2+7C+kI_3 = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 5\times 6 & 0\times 6 & 8\times 6 \\ 2\times 6 & 4\times 6 & 5\times 6 \\ 8\times 6 & 0\times 6 & 13\times 6 \end{bmatrix} + \begin{bmatrix} 0\times 7 & 0\times 7 & 2\times 7 \\ 0\times 7 & 2\times 7 & 1\times 7 \\ 2\times 7 & 0\times 7 & 3\times 7 \end{bmatrix} + \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} k-2 & 0 & 0 \\ 0 & k-2 & 0 \\ 0 & 0 & k-2 \end{bmatrix}.$$

$C^3-6C^2+kI_3 = O_3$ implies $k-2=0, k=2$. Hence, the value of k is 2.

Question 8

(a) The frequency of each line is given below:

Algorithm 1: ISprime-A(n)

```
for integer i ← 2 to n − 1 do
                                                  // C1 frequency: at most n - 2
    if n mod i == 0 then
                                                  // C2 frequency: at most n - 2
        return False
                                                  // C3 frequency: at most 1
    return True
                                                  // C4 frequency: at most 1
```

The completely of the first algorithm is $(C_1+C_2)\times(n-2)+C_3+C_4 \Rightarrow \Theta(n)$.

Algorithm 2: ISprime-B(n)

```
for integer i ← 2 to floor(sqrt(n)) + 1 do
                                                  // C1 frequency: at most sqrt(n)
    if n mod i == 0 then
                                                  // C2 frequency: at most sqrt(n)
        return False
                                                  // C3 frequency: at most 1
    return True
                                                  // C4 frequency: at most 1
```

The completely of the second algorithm is $(C_1+C_2)\times(\sqrt{n})+C_3+C_4 \Rightarrow \Theta(\sqrt{n})$.

(b) The second algorithm is more efficient than the first. When $n \in \mathbb{Z}_+$, n is always greater than or equal to $\sqrt{n}$, then generally, the first algorithm holds an greater complexity than the second.

Question 9 *Proof* : If $k=1, LHS = 1-\frac{1}{2} = \frac{1}{2}, RHS = \frac{1}{2}$, thus $LHS = RHS$. Assume that when $k=n$, the proporsition is true, namely, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2k} = \frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}$. If $n=k+1, LHS = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2} = \frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2} = \frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}+(\frac{1}{k+1}-\frac{1}{2k+2}) = \frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$, while $RHS = \frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$. Hence, $LHS = RHS$. The proporsition is also true if $n=k+1$, based on the proporsition is true when $k=n$. This concludes the proof.

Question 10 Pictures explaining the procedure are given below:

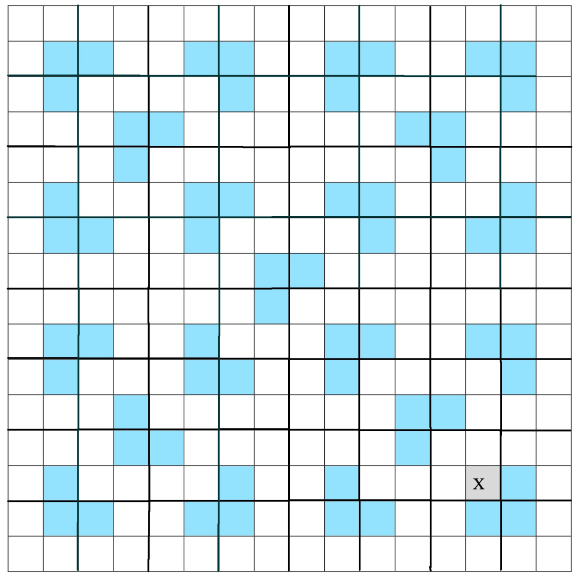


Figure 2: Divide the checkerboard in to $8 \times 8, 4 \times 4$, and 2×2 squares, and place the triominoes to ensure that one block of every square is occupied.

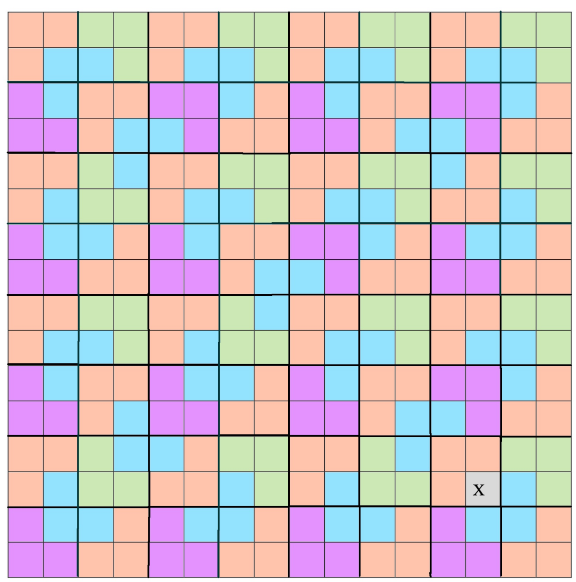


Figure 3: Tile the checkerboard with other triominoes.