COMP2012 Discrete Mathematics Notes - Graphs

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1 Definition

$G = (V, E)$, V is a set of vertices and E is a set of edges. $|V|$ denotes the size of V and so does $|E|$. Example: $(u,\ v)$ is the edge between two connected vertices (with direction).

$$V = \{1, 2, 3, 4\}$$

$$E = (1, 2), (2, 3), (3, 4), (2, 4)$$

The degree of a vertex v, denoted by $deg(v)$, is the number of edges connected to it.

2 Adjacency List

The adjacency list of vertex u is:

$$adj[u] = v : (u, v) \in E$$

$adj[u]$ contains each vertex v adjacent to u. The storage space of the graph structure is $O(|V| + |E|)$. Hence, it is used to represent a sparse graph where $|E| \ll |V|^2$.

3 Adjacency Matrix

It is a matrix with size $|V|^2$, where

$$a_{i,j} = \begin{cases} 0 & if\ (i, j) \in E \\ 1 & otherwise \end{cases} \tag{1}$$

The storage space of the graph structure is $O(|V|^2)$. Hence, it is used to represent a dense graph, where $|E|$ is close to $|V|^2$.

4 Terminology

Let $G = (V, E)$ be an undirected graph, we have:

$$\sum_{v \in V} deg(v) = 2|E|,\ |V_{deg(v)\ is\ odd,\ v \in V}|\ is\ even$$

If G is a directed graph. The in-degree of v, denoted by $deg^-(v)$, is the number of edges with v as their end vertex. The out-degree of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex. We get:

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v)$$

5 Bipartite graphs

A bipartite graph is a simple graph whose vertex set V can be divided into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2. A simple graph is bipartite if and only if it is 2-colorable (i.e., possible t o assign one of two colors to each vertex so that no two adjacent vertices have the same color).

6 Graph Isomorphism

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists an one-to-one and onto function $F : V_1 \to V_2$ so that

$$a \; and \; b \; are]; adjacent \; in \; G_1 \leftrightarrow F(a) \; and \; F(b) \; are \; adjacent \; in \; G_2$$

7 Connectivity

A vertex v s called a cut vertex if its removal from a connected graph makes the graph disconnected. So dose an edge e when it is a cur edge. A subset V' (of the vertex set V) is called vertex cut if its removal from a connected graph makes the graph disconnected.

8 Depth-First Search

Algorithm 1: Depth-First Search - Initialization

```
Data: Graph G = (V, E), root vertex s
Result: Graph Traversal Initialization
for u ∈ G.V do
    u.depth ← ∞                         // Unseen vertex has depth as ∞
    u.parent ← null
end
s.depth ← 0
DFS(G, s)
```

Algorithm 2: Depth-First Search - Recursion $DFS(G, u)$

```
Data: Graph G = (V, E), vertex u
Result: Graph Traversal
for v ∈ G.adj[u] do
    if v.depth = ∞ then
                    // Base case: adjacent vertex v has depth ∞, thus v has been seen
        v.depth ← u.depth + 1
        v.parent ← u
        DFS(G, v)
    end
end
```

9 Breadth-First Search

Algorithm 3: Breadth-First Search

Data: Graph $G = (V, E)$, root vertex s
Result: Graph Traversal Initialization

```
for u ∈ G.V \ {s} do
    u.depth ← ∞
    u.parent ← null
end
s.depth ← 0
s.parent ← null
Q ← a new queue
Enqueue(Q, s)
while Q is not empty do
    u ← Dequeue(Q)
    for v ∈ G.adj[u] do
        if v.depth = ∞ then
            v.depth ← u.depth + 1
            v.parent ← u
            Enqueue(Q, v)
        end
    end
end
```

10 Euler Paths and Circuits

An Euler path is a simple path that contains every edge of a graph without repetition, and an An Euler circuit can be regarded as an Euler path that begins and ends at the same vertex.

1. A connected graph with at least two vertices has an Euler circuit if and only if each vertex has an even degree.
2. A connected graph with at least two vertices has an Euler path if and only if it has exactly two vertices of odd degree.

11 Hamilton Paths and Circuits

A Hamilton path is a simple path that passes through every vertex of a graph. A Hamilton circuit can be regarded as a Hamilton path that begins and ends at the same vertex.

11.1 Ore's Theorem

Let G be a graph with n vertices, $n \geq 3$ G has a Hamilton circuit if:

$$\forall u, v \in V, v \notin adj[u], deg(u) + deg(v) \geq n$$

The theorem is sufficient but without necessity.

12 Shortest Path - Dijkstra's Algorithm

Algorithm 4: Dijkstra's Algorithm

```
Data: Graph G = (V, E), root vertex s
Result: Shortest Path Search Initialization
for v ∈ G.V do
    v.distance ← ∞                    // the best path distance from s to v found so far
    v.parent ← null                   // parent of v on such a path
end
s.distance ← 0
Q ← a new priority queue
Enqueue(Q, V)
while Q is not empty do
    u ← minExtract(Q)
        // find and dequeue the vertex currently with the minimum distance from the root
    for v ∈ adj[u] do
        if u.distance + edgeWeight(u, v) < v.distance then
            v.distance ← u.distance + edge − weight(u, v)
            v.parent ← u
            keyDecrease(Q, v, v.distance)
                // update v's key (temporary minimum distance from the root) to the current
                // v.distance
        end
    end
end
```

13 Planar Graphs

A graph is called a planar if it can be drawn without any crossing of edges.

13.1 Euler's Formula

Let G be a connected planar graph with v vertices and e edges. The number of regions in planar representation:

$$e - v + 2$$

14 Graph Coloring

he chromatic number of a graph G, $\chi(G)$, is the least number of colors required to color G.

1. If G is a circular graph with n vertices, where $n \geq 3$:

$$\chi(G) = \begin{cases} 2 & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases} \tag{2}$$

2. The four color theorem:

$$If\ G\ is\ a\ planar\ graph, \chi(G) \leq 4$$