The Hong Kong Polytechnic University
COMP2012 Discrete Mathematics
Mid-term Test (25%)
Suggested Solutions

[Total 100 marks]

Part A: Multiple Choices (30%, 3 marks each)

Answer ALL questions in this part. Select ONE option only in {A,B,C,D} for an answer.

Question 1

In how many ways, the letters of the word 'STRESS' can be arranged?

(a) 360

(b) 720

(c) 240

(d) 120

Explanation: Required number of arrangements = 6 ! / 3 ! [∵ S has come thrice]

= [6 x 5 x 4 x 3!] /3 ! = 120

Question 2

Find the minimum number of teachers in a college to be sure that four of them are born in the same month.

(a) 37

(b) 3

(c) 4

(d) 12

Explanation: suppose n=12 (pigeon holes)

k+1= 4 i.e k=3

Now the number of pigeons be $k(n+1)$, i.e $k(n+1)$ = 3*12+1 =37

Question 3

Consider the recurrence relation a_1=4, a_n=5_n+a_{n-1}. The value of a_{64} is:

a) 10399

b) 23760

c) 75100

d) 53700

This is trivial: can be solved by simple calculations

Question 4

There are 70 patients admitted to a hospital, of which 29 are diagnosed with typhoid, 32 with malaria, and 14 with both typhoid and malaria. Find the number of patients diagnosed with typhoid or malaria or both.

a) 39

b) 17

c) 47

d) 53

Explanation: By using the inclusion-exclusion principle:

$|T \cup M| = |T| + |M| – |T \cap M| = (29 + 32) – (14) = 47.$

Thus, 47 patients are diagnosed with either typhoid or malaria.

Question 5

Evaluate the matrix C, then deduce the sum of c_{11} and c_{12}.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

a) 58+64

b) 58+139

c) 58+154

d) 64+139

This is trivial: can be solved by simple calculations. See lecture notes for entry positions.

Question 6

Let A and B be the matrices having the same dimensions n-by-m. Given that $A^TA=A$, $BI=I$. Find the solution if $(A^2x-B)=0$.

a) x=-1

b) x=0

c) x=1

d) x=2

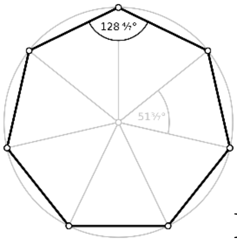
Explanation:

If x is a scalar, then x cannot be 0 and B=/=0

c could be a solution when A is an identity matrix I (one of the possible cases of A). Since $A^TA=A \Longrightarrow A^2=A$, so it is possible,

Question 7

There are four students in the class, namely Amy, Ben, Cony, and Dave. Amy says that a triangle is a bipartite graph. Ben says a pentagon is a bipartite graph. Cony says a square is a bipartite graph. Dave says a heptagon is a bipartite graph. Who among the following is correct?



Hint: a regular heptagon

A) Amy is correct

B) Ben is correct

C) Cony is correct

D) Dave and Cony are both correct

Explanation: We can prove it in the following way. Let ‘1’ be a vertex in the bipartite set X, and let ‘2’ be a vertex in the bipartite set Y. Therefore, the bipartite set X contains all odd numbers and the bipartite set Y contains all even numbers. Now, let us consider a graph of the odd cycle (a triangle). An edge exists from ‘1’ to ‘2’, ‘2’ to ‘3’ and ‘3’ to ‘1’. The latter case (‘3’ to ‘1’) makes an edge to exist in a bipartite set X itself. Therefore, it tells us that graphs with odd cycles are not bipartite, so A (3 sides), B (5 sides), and D (7 sides) are wrong. The answer should be C.

Question 8

For the sentence “I go to school by bus.”. If there are 3, 5, and 11 synonyms for the words “go”, “school”, and “bus”, then how many possible meanings could be given by this sentence?

a) 1

b) 48

c) 88

d) 165

Explanation: This is not a well-designed question, so there are two possible interpretations. Choice A if true if we interpret various combination of synonyms would give one meaning only. But if we interpret synonyms with (a word with multiple meaning, which usually confused by non-English speakers), the possible meaning would be 3*5*11=165

Question 9

A bag contains 25 balls such as 10 balls are red, 7 are white and 8 are blue. What is the minimum number of balls that must be picked up from the bag blindfolded (without replacing any of it) to be assured of picking at least one ball of each colour?

A) 10

B) 18

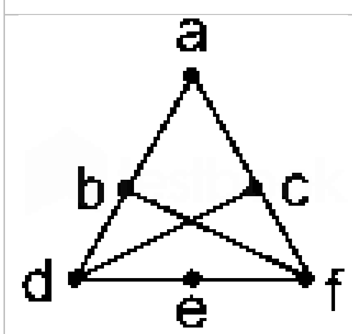
C) 63

D) 35

Explanation: Consider three buckets red, white and blue and we want the total number of balls such that each bucket contain at least one ball. Now consider the state of picking up a ball without replacement : (normally you consider the worst case scenario in these cases) Starting 10 balls all are red and thus goes to bucket name Red. Now again picking up the ball gives 7 balls which are of same colour and put all of them in a bucket named White. The next pick will definitely be of different colour thus: we picked 10 + 7 + 1 = 18.

Question 10

Which of the following Graphs is(are) planar?

<table>
<tr><th>G1:</th><th>G2:</th><th>G3:</th></tr>
<tr>
<td>
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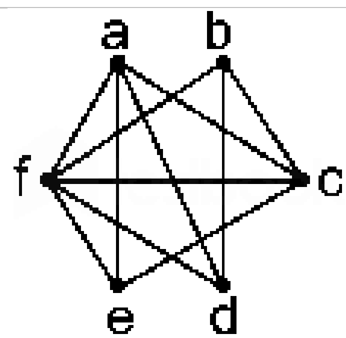
<td>
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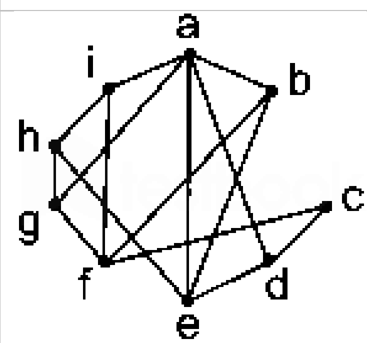
<td>
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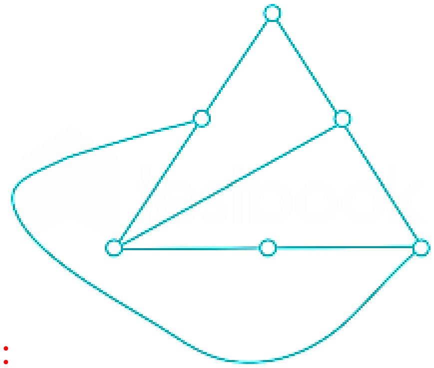
A) G1 and G2 only

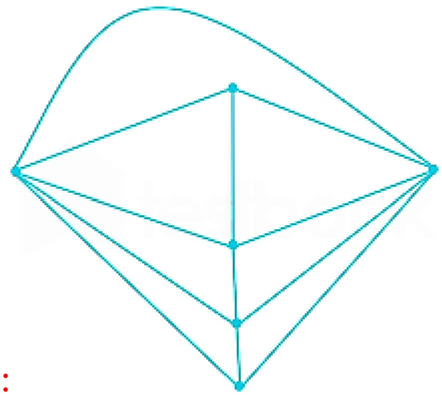
B) G2 and G3 only

C) G1 only

D) G2 only

Explanation: A graph is said to be planar if it can be drawn so that no edges intersect each other, only intersect at the endpoints. G1 and G2 can be redrawn as the following left and right figures, respectively:

G1: 

G2: 

But Figure G3 cannot be redrawn in such a way that no edges intersect each other.

Part B: Short Questions (50 marks)

You need to answer ALL five questions in this part.

Question 11 **[10 marks]**

From the following premises show that *t* is a conclusion.

1. ¬p ∧ q
2. r→p
3. ¬r→s
4. s→t

Solution:

Step	**Reason**
1. ¬p ∧ q	From premise #1 (0.5 for step + 0.5 for reason = 1 mark)
2. ¬p	Using the Simplification Rule on (step #1) (0.5 for step +1 for rule = 1.5 marks)
3. r→p	From premise #2 (1 mark)
4. ¬r	Using the Modus Tollens Rule on (step #2) and (step 3) (1.5 mark)
5. ¬r→s	From premise #3 (1 mark)
6. s	Using the Modus Ponens Rule on (step #4) and (step #5) (1.5 mark)
7. s→t	Using premise #4 (1 mark)
8. t	Using the Modus Ponens Rule on (step #6) and (step #7) (1.5 mark)

Conclusion is optional, but #8 step (final step) has to be correct.

Question 12 **[10 marks]**

Given the letter t to represent the statement "Bill is tall," y for "Bill is young," and s for "Bill is strong."

Translate these statements into symbols. (2 marks each)

(a) Bill is (not old) and tall.

(b) Bill is tall, but neither young nor strong.

(c) Bill is short, strong and old.

Translate these statements into English. (2 marks each)

(d) $\neg y \vee t$

(e) $s \wedge \neg (t \wedge \neg s)$

Solution:

(2 marks each correct answer, partially correct: at most 1 mark):

(a) $y \wedge t$

(b) $t \wedge \neg (y \vee s)$

(c) $\neg t \wedge s \wedge \neg y$

(d) Bill is not young or Bull is tall / Either Bill is old or he is tall (either one correct)

(e) Bill is strong and not both tall and weak

Question 13 **[10 marks]**

Prove that (A – B) ∩ (B – A) = Φ by contradiction

Proof:

(A – B) ∩ (B – A) = Φ can be rewritten as $(A \cap \bar{B}) \cap (B \cap \bar{A}) = \Phi$ (2 marks)

Assume $(A \cap \bar{B}) \cap (B \cap \bar{A}) \neq \Phi$ (2 marks)

$\exists x \in U \; s.t. \; x \in ((A \cap \bar{B}) \cap (B \cap \bar{A}))$ (1 mark)

$x \in A \cap \bar{B} \;\; and \;\; x \in (B \cap \bar{A})$

$x \in A$ and $x \in \bar{B}$ *and* $x \in B$ and $x \in \bar{A}$) (1 mark)

$x \in A$ and $x \notin B$ *and* $x \in B$ and $x \notin A$) Contradiction! (2 marks)

Conclusion: $(A \cap \bar{B}) \cap (B \cap \bar{A}) = \Phi$ and hence (A – B) ∩ (B – A) = Φ (2 marks)

Question 14 **[10 marks]**

Suppose g_n is a recursively defined sequence given by $g_1=1$, $g_2=2$, $g_3=6$, and $g_n=(n^3-3n^2+2n)g_{n-3}$ for all $n>4$.

Prove for all $n \in \mathbb{Z}^+$, $g_n=n!$

(Remember to include the method's name that was employed in the proof.)

Solution:

Theorem. For all $n \in \mathbb{Z}^+$, $g_n=n!$ (1 mark)

Proof.

We verify that $g_1=1=1!$, so the claim is true when $n=1$. (1 mark for basis step)

$g_2=2=2\text{x}1=2!$ → True (1 mark)

$g_3=6=3\text{x}2\text{x}1=3!$ → True (1 mark)

$g_4=(4^3-3\text{x}4^2+2\text{x}4)1!=(64-48+8)1!=24=4\text{x}3\text{x}2\text{x}1=4!$ → True (1 mark)

(it is ok to test g_5 instead of g_4)

Now we let $k \geq 1$, and assume now that $g_i = i!$ for all intergers $1 \leq i \leq k$. (1 mark, correct induction step)

(Note: this is a recurrence function, so likely we can consider this is a strong induction problem)

Now since $1 \leq k - 2 \leq k$,

$$
\begin{aligned}
g_{k+1} &= [(k+1)^3 - 3(k+1)^2 + 2(k+1)]g_{k-2} \\
&= [(k+1)^3 - 3(k+1)^2 + 2(k+1)](k-2)! \\
&= [k^3 - k](k-2)! \\
&= [(k+1)k(k-1)](k-2)! \\
&= (k+1)!
\end{aligned}
$$
(3 marks for calculations)

Thus, by strong induction, (1 mark for method)

$g_n = n!$ for all $n \in \mathbb{Z}^+$ is true. (Q.E.D.) (1 mark for conclusion)

Question 15 **[10 marks]**

A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if:

(a) one particular woman must be excluded from the committee; (3 marks)

(b) there is to be a majority of women (7 marks)

<u>Solution:</u>

(a)

$C_5^9 = \frac{9!}{9!(9-4)!} = \frac{9!}{9!5!}$ (1 mark for correct formulation)

$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 5!} = 126$ (1 mark for calculation steps)

= 126 (1 mark for correct final answer)

(b)

In this 5-person committee, #women is more than #men (1 mark)

Case 1: 4 women: C_4^4 & 1 man: C_1^6 (1 mark for correct formulation)

Hence: $C_4^4 \cdot C_1^6 = \frac{4!}{0! \cdot 4!} \cdot \frac{6 \cdot 5!}{5! \cdot 1!} = 1 \cdot 6 = 6$ (1 mark)

Case 2: 3 women: C_3^4 & 2 man: C_2^6 (1 mark for correct formulation)

Hence: $C_3^4 \cdot C_2^6 = \frac{4 \cdot 3!}{1! \cdot 3!} \cdot \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2 \cdot 1} = 4 \cdot 15 = 60$ (1 mark)

Adding both cases together we have $C_4^4 \cdot C_1^6 + C_3^4 \cdot C_2^6 = 6 + 60$ (1 mark)

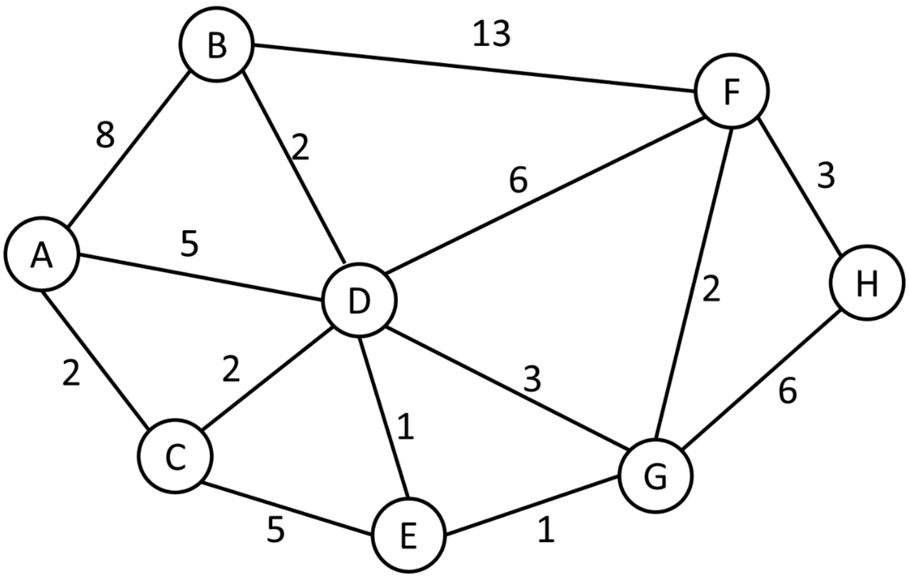
$= 66$ (1 mark)

Part C: Long Questions (20 marks)

You need to answer ALL one question in this part.

Question 16

Given a graph *G*, as below:



(a) Write down the adjacency List of Graph *G*. (4 marks)

(b) Start from A, determine the traversal order using DFS (3 marks)

(c) Start from A, determine the traversal order using BFS (3 marks)

(d) Apply Dijkstra's Algorithm to determine the shortest path and its length from A to H. (10 marks)

Solution:

(a) The adjacency list is as follows:

A: B:8 | C:2 | D:5 (0.5 mark)

B: A:8 | D:2 | F:13 (0.5 mark)

C: A:2 | D:2 | E:5 (0.5 mark)

D: A:5 | B:2 | C:2 | E:1 | F:6 | G:3 (0.5 mark)

E: C:5 | D:1 | G:1 (0.5 mark)

F: B:13 | D:6 | G:2 | H:3 (0.5 mark)

G: D:3 | E:1 | F:2 | H:6 (0.5 mark)

H: F:3 | G:6 (0.5 mark)

(For each row, if more than half is correct, then count it as correct)

(b)

DFS: Traversal order: A B D C E G F H (3 marks)

If A is not start point, deduct 1 mark.

No need steps.

(c)

BFS: Traversal order: A B C D F E G H (3 marks)

If A is not start point, deduct 1 mark.

No need steps.

(d)

Q = (A:0), (B:∞), (C:∞), (D:∞), (E:∞), (F:∞), (G:∞), (H:∞) (init, 1 mark)

[Iteration 1] (1 mark)

Extract vertex A

Update vertices B, C, D

Q = (C:2), (B:8), (D:5), (E:∞), (F:∞), (G:∞), (H:∞)

[Iteration 2] (1 mark)

Extract vertex C

Update vertices D, E

Q = (D:4), (E:5), (B:8), (F:∞), (G:∞), (H:∞)

[Iteration 3] (1 mark)

Extract vertex D

Update vertices D, E

Q = (E:5), (B:6), (G:7), (F:10), (H:∞)

[Iteration 4] (1 mark)

Extract vertex E

Update vertices G

Q = (B:6), (G:6), (F:10), (H:∞)

[Iteration 5] (1 mark)

Extract vertex B

Update vertices: None

Q = (G:6), (F:10), (H:∞)

[Iteration 6] (1 mark)

Extract vertex G

Update vertices: F, H

Q = (F:8), (H:12)

[Iteration 7] (1 mark)

Extract vertex F

Update vertices: H

Q = (H:11)

[Iteration 8] (1 mark)

Extract vertex H

Update vertices: None

Q = Φ

Hence: Shortest path length is 11: A⇒C⇒D⇒E⇒G⇒F⇒H (1 mark)

(B is not included. But it doesn't matter)

End of Mid-term Test

Appendix I

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution