Fair Orientations: Beyond Envy-Freeness

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Abstract

Fair division is an important problem in algorithmic game theory, which has received considerable attention in recent years. In this paper, we focus on an orientation setting where each agent is restricted to receiving items from a specific subset, referred to as their relevant items. The relevance restriction represents a natural constraint in real-world applications and generalizes the classical unconstrained setting. While the orientation setting has been widely studied, existing work has concentrated on envy-freeness. In this paper, we initiate the study of other important fairness concepts such as proportionality, equitability, and their relaxations. We present a comprehensive set of computational results for the allocation of goods, chores, and mixed manna. In particular, for each fairness concept, we consider the computational complexity of deciding the existence of a fair allocation, and the extent to which it can be satisfied and efficiently computed.

1 Introduction

Resource allocation is ubiquitous in everyday life and, due to various concerns, is a challenging problem in game theory, machine learning, multi-agent systems, and more [2]. In particular, fairness is one of the important concerns, especially for the real-world applications that involve human beings, who deserve to be treated without bias. Among the fairness concepts that have been studied, proportionality (PROP) is one of the most prominent, which requires that every agent gets utility at least a given threshold (i.e., their proportional shares) [30]. PROP has been widely adopted as a golden fairness criterion in, for example, cake cutting [14, 28], clustering [13, 24], committee selection [22], participatory budgeting [27], matching [3], sortition [18, 17], and bargaining [21]. Besides PROP, envy-freeness [32] and equitability [9] are also important fairness concepts.

In this paper, we focus on a fundamental model of resource allocation, where each agent is restricted to receiving items from a specific subset, called *relevant* items, and the resulting allocation is also known as an *orientation*. Relevance describes common situations in the real world when certain items are only available to certain agents due to geographic restrictions or demand requirements. A special case under this umbrella is the *graph* orientation problem, where each agent is a vertex in a graph, and the relevant items are the edges incident to it. This problem was first introduced in [15] and has been extensively studied since then. However, most existing research focuses on the envy-based fairness concepts, while proportionality has been overlooked. Informally, an allocation is called envy-free (EF) if everyone gets at least as much value as they would if all the resources were evenly distributed. Since an EF orientation may not exist, two of its relaxations, envy-free up to *one* item (EF1) [10] and envy-free up to *any* item (EFX) [11] are studied instead. EF1 and EFX, respectively, require that the envy from one agent towards another agent can be eliminated if some or any item is removed.

For the graph model with goods, Christodoulou et al. [15] proved that an EFX orientation may not exist, and it is NP-hard to decide whether a graph instance admits an EFX orientation. In contrast to EFX, Deligkas et al. [16] proved that an EF1 orientation always exists even in the general orientation model. When the items are chores, Zhou et al. [36] and Hsu and King [20] showed that although an EFX orientation may not exist, deciding the existence can be done in polynomial time. But EF1 has not been studied for chores. More importantly, other fairness concepts, including PROP, are overlooked in this setting, which motivates our work.

1.1 Our Contribution

In this paper, we study the fair allocation of a set E of m indivisible items among n agents, where items can be goods (which provide positive value), chores (which provide negative value), and a mixture of both. Each agent i has a set of relevant items $E_i \subseteq E$ and can only receive items in E_i. We consider both the general orientation model and two structured cases when the relevance forms a simple graph (i.e., every item is relevant to exactly two agents and $|E_i \cap E_j| \leq 1$) and multigraph (i.e., $E_i \cap E_j$ may contain more than one item). Our primary fairness concept is PROP, and we aim to provide a complete picture of the existence and computational results. We also complement the literature's missing results for envy-freeness and equitability.

Proportional Fairness Assume the valuation function of agent i is $v_i(\cdot)$. The original definition of the proportional share is agent i's average value for all items, i.e., $\frac{1}{n}v_i(E)$, which is widely adopted in the literature. However, in the orientation model, not every item should be considered when defining an agent's proportional share, as the agent is not entitled to receive goods irrelevant to her or irresponsible for completing irrelevant chores. For the case of goods, the literature (on EF-related concepts, e.g., [15, 16]) often sets $v_i(e) = 0$ for all $e \in E \setminus E_i$ so that $v_i(E) = v_i(E_i)$. But this approach can be unfair. For example, if an item is only relevant to agent i, the definition still divides its value among all agents, but it is natural that agent i deserves more than a $\frac{1}{n}$ fraction of the item. This approach is more serious for chores: an irrelevant item (with zero value) appears more favorable than a relevant one (with negative value). If a chore is only relevant to i, i still wants to share it with all other agents.

To address this, we propose a refined definition of proportional share based on the relevance of each item. Specifically, for each item $e \in E$, let n_e be the number of agents to whom e is relevant. Then the proportional value of i for $e \in E_i$ is $\frac{1}{n_e}v_i(e)$, and the proportional share of i is

$$\mathsf{PROP}_i = \sum_{e \in E_i} \frac{1}{n_e} v_i(e),$$

and an allocation is called proportional (PROP) if every agent i's value is no smaller than PROP_i. We summarize our technical results regarding the refined definition of PROP as follows.

Result 1.1 It is NP-complete to decide whether an instance admits a PROP orientation, even for simple graphs containing only goods (or only chores) and agents with bi-valued valuations.

However, if agents have binary valuations, the above problem can be solved in polynomial time, even for general orientation models. We then consider the up to one or any relaxation of PROP. Informally, an orientation is PROP up to one (or any) item, abbreviated as PROP1 (or PROPX), if one (or any) item is added or removed from the considered agent. Although PROPX still cannot be guaranteed, PROP1 allocations always exist and can be satisfied together with fractional Pareto optimality (fPO).

Result 1.2 A PROP1 and fPO orientation exists and can be computed in polynomial time for the general orientation model.

Surprisingly, Result 1.2 is the strongest positive result in the orientation model and PROP1 turns out to be the only fairness concept (among the most widely studied ones) that can be guaranteed. Further, this result also strengthens the unweighted result in [5], where items can be arbitrarily allocated without any relevance constraints and PROP_i adopts the weaker version $\frac{1}{n}v_i(E)$.

Result 1.3 It is NP-complete to decide whether an instance admits a PROPX orientation, even for simple graphs containing only goods (or only chores) and agents with binary valuations.

This result shows a sharp contrast to the unconstrained setting, where PROPX allocations always exist for chores [25]. We additionally identify structures under which PROPX orientations are guaranteed to exist. For example, PROPX orientations always exist if the graph does not contain cycles.

Due to space limitations, the missing proofs of the above results are provided in the appendix.

Equitability and Envy-freeness We then shift our attention to other fairness concepts related to equitability (EQ) and envy-freeness (EF). EQ requires that the subjective values of all agents are the same, i.e., all agents value the bundles they receive equally. Similarly, we are also interested in the up to one/any relaxations, denoted by EQ1 and EQX. Unfortunately, we show that EQ, EQ1 or EQX orientations may not exist, and deciding the existence of such an orientation is intractable.

Result 2.1. It is NP-complete to decide whether an instance admits an EQ, EQ1, or EQX orientation, even for simple graphs.

Finally, we complement the missing results of EF1 orientation for chores. In a sharp contrast to goods, for which an EF1 orientation always exists even in the general orientation model [16], we prove that EF1 orientations may not exist for chores, even when the model is a simple graph. We characterize the necessary and sufficient conditions in simple graphs that ensure EF1 orientations and prove that the decision problem is NP-complete for multigraphs.

Result 2.2. For chores, it is NP-complete to decide whether an instance admits an EF1 orientation in multiple graphs, and is polynomial-time solvable in simple graphs.

Due to space limitations, we present the results on EQ and EF related concepts in the appendix.

1.2 Other Related Work

We refer to the survey [2] for a comprehensive coverage of recent results on fair allocation of indivisible items, and [31] for various constraints that have been studied. In the following, we recall the most relevant work to our paper. Without constraints, EF1 and PROP1 allocations always exist and can be computed in polynomial time for goods, chores and the mixture of goods and chores [8]. Furthermore, PROP1 and Pareto optimal are known to be compatible in the mixed setting [6, 5]. PROPX allocations exist for chores but not for goods [5, 25]. The existence of EFX allocation remains unknown, except for several special cases [12].

Initiated by Christodoulou et al. [15], EFX orientations of relevant items has been extensively studied. Zeng and Mehta [35] characterized the graph structures for which EFX orientations always exist. Kaviani et al. [23] and Amanatidis et al. [4] respectively proved the existence of approximate EFX orientations. Afshinmehr et al. [1] and Hsu [19] extended the study to multi-graphs, and Zhou et al. [36] and Hsu and King [20] generalized the results to chores, the mixture of goods and chores, and other variants of the EFX concept. In contrast to EFX, when the items are goods, Deligkas et al. [16] proved that an EF1 orientation always exists even in the general orientation model. Finally, Li et al. [26] recently bounded the minimum subsidy to achieve envy-freeness.

2 Preliminaries

2.1 The Orientation Model

For any $k \in \mathbb{N}^+$, let $[k] = \{1, \ldots, k\}$. We study the model of allocating a set $E = \{e_1, \ldots, e_m\}$ of m indivisible items to a set $N = \{1, \ldots, n\}$ of n agents. Each agent i has an additive valuation function $v_i : 2^E \to \mathbb{R}$, that is, for any $S \subseteq E$, $v_i(S) = \sum_{e \in S} v_i(\{e\})$. For ease of notation, we write $v_i(e)$ instead of $v_i(\{e\})$. We call a valuation *binary* if $v_i(e) \in \{0, a\}$ for all items $e \in E$, where $a = 1$ or -1. For any $S \subseteq E$, let $|S|$ be the number of items in S. An item is a good (resp., a chore) for an agent if it yields a non-negative (resp., non-positive) value. If the value of an item is zero for an agent, then the item can be a good or a chore. In this paper, we say an instance is a *goods-instance* (or *chores-instance*) if all items are goods (or chores) for all agents. We call it a *mixed-instance* if an item can yield a positive value for one agent but a negative value for another.

In the general *orientation model*, each agent i has a non-empty set of *relevant* items $E_i \subseteq E$. For each item $e \in E$, let $N_e = \{i \in N \mid e \in E_i\}$ be the set of agents to whom e is relevant, and let $n_e = |N_e|$. We assume that N_e is non-empty for every e, as otherwise, this item can be removed. The orientation model is general and incorporates the classic unconstrained setting, where $E_i = E$ and $N_e = N$ for all $i \in N$ and $e \in E$. We are also interested in two structured cases. In the *simple graph* orientation model, $n_e = 2$ for all e and $|E_i \cap E_j| \leq 1$ for all $i \neq j$. That is, the model can be described as a simple graph $G = (N, E)$, where each vertex is an agent and each edge is an item that is relevant to the two agents incident to this edge. Similarly, in the *multigraph* orientation model,

$n_e = 2$ for every e but the number of items in $E_i \cap E_j$ is not limited. That is, there may be multiple edges between any two agents (vertices). In this paper, when the instance is a graph or multigraph, we use the terminologies vertex i and agent i, and edge e and item e, interchangeably.

2.2 Solution Concepts

In the orientation model, each item can only be allocated to an agent to whom the item is relevant. Formally, an *orientation* is denoted by $\pi = (\pi_1, \dots, \pi_n)$, where $\pi_i \subseteq E_i$, and for any $i \neq j$, $\pi_i \cap \pi_j = \emptyset$ and $\bigcup_{i \in N} \pi_i = E$. Below, we introduce proportional fairness for mixed instances. Let $\mathsf{PROP}_i = \sum_{e \in E_i} \frac{1}{n_e} \cdot v_i(e)$ be the refined proportional share for agent i.

Definition 1 *An orientation $\pi = (\pi_1, \dots, \pi_n)$ is proportional (PROP) if for every agent $i \in N$, $v_i(\pi_i) \geq \mathsf{PROP}_i$.*

Definition 2 *An orientation $\pi = (\pi_1, \dots, \pi_n)$ is proportional up to any item (PROPX) if for every agent $i \in N$, either (1) $v_i(\pi_i) \geq \mathsf{PROP}_i$, or (2) $v_i(\pi_i \cup \{e\}) \geq \mathsf{PROP}_i$ for any $e \in E_i \setminus \pi_i$ such that $v_i(e) \geq 0$ and $v_i(\pi_i \setminus \{e\}) \geq \mathsf{PROP}_i$ for any $e \in \pi_i$ such that $v_i(e) \leq 0$.*

Definition 3 *An orientation $\pi = (\pi_1, \dots, \pi_n)$ is proportional up to one item (PROP1) if for every agent $i \in N$, one of the following three holds: (1) $v_i(\pi_i) \geq \mathsf{PROP}_i$, (2) $v_i(\pi_i \cup \{e\}) \geq \mathsf{PROP}_i$ for some item $e \in E_i \setminus \pi_i$, or (3) $v_i(\pi_i \setminus \{e\}) \geq \mathsf{PROP}_i$ for some item $e \in \pi_i$.*

It is easy to see that a PROP orientation is PROPX, and a PROPX orientation is PROP1. The above definition can be directly applied to goods- and chores- instances. For example, for goods-instances, an orientation is PROPX if for every agent i, $v_i(\pi_i \cup \{e\}) \geq \mathsf{PROP}_i$ for any $e \in E_i \setminus \pi_i$; For chores-instances, an orientation is PROPX if for every agent i, $v_i(\pi_i \setminus \{e\}) \geq \mathsf{PROP}_i$ for any $e \in \pi_i$. Regarding PROP1, the quantifier of e in the prior two definitions is changed to existence.

In this paper, we also consider equitability (EQ) and envy-freeness (EF), which do not solely depend on a predefined share but also depend on the allocations to the other agents. Briefly, an orientation $\pi = (\pi_1, \dots, \pi_n)$ is equitable (EQ) if $v_i(\pi_i) = v_j(\pi_j)$ for any two agents $i, j \in N$, and envy-free (EF) if $v_i(\pi_i) \geq v_i(\pi_j)$ for any two agents $i, j \in N$. That is, in EQ orientations, the agents have the same value, while in EF orientations, every agent has the largest value in their own allocations. The up to one or any relaxations of EQ and EF are similar to PROP, which are deferred to the appendix, due to the page limit. We also defer the results on these concepts to the appendix.

Finally, we present the definition of Pareto optimality. An orientation $\pi = (\pi_1, \dots, \pi_n)$ *Pareto dominates* another orientation $\pi' = (\pi'_1, \dots, \pi'_n)$ if for any $i \in [n]$, $v_i(\pi_i) \geq v_i(\pi'_i)$ and at least one inequality is strict. An orientation π is Pareto optimal (PO) if no integral orientation Pareto dominates it, and is fPO if no fractional orientation Pareto dominates it, where in an integral orientation, each item is fully allocated to one agent, and in a fractional orientation, a fraction of item can be assigned to some agent. Formally, in a fractional orientation $\pi = (\pi_1, \dots, \pi_n)$, $\pi_i = (\pi_{i,e})_{e \in E}$ where $0 \leq \pi_{i,e} \leq 1$ represents the portion of e allocated to agent i. Notice that π is an orientation requires that for any i, e, $\pi_{i,e} > 0$ if and only if $e \in E_i$ and $\sum_{i \in N_e} \pi_{i,e} = 1$.

3 PROP Fairness

Similar to the classical fair division of indivisible items, PROP is not always satisfiable even for simple graph instances. Consider an item and two agents, and the item is relevant and yields positive value to both agents. A natural question is whether we can efficiently compute a PROP orientation when it exists. Unfortunately, the answer is no even for restricted instances. In the following, if agents' total value of their relevant items are identical, we say the valuations are *normalized*.

We derive the reduction from 2P2N-3SAT problem, known to be NP-complete [7, 34]. A 2P2N-3SAT instance contains a Boolean formula in conjunctive normal form consisting of the set of variables $X = \{x_1, \dots, x_s\}$ and the set of clauses $C = \{C_1, \dots, C_t\}$. Each variable appears exactly twice as a positive literal and exactly twice as a negative literal in the formula, and each clause contains three distinct literals, i.e., $3t = 4s$. Denote by $L = \bigcup_{j=1}^{s} \{x_j^1, x_j^2, \bar{x}_j^1, \bar{x}_j^2\}$ the set of literals and by $C(\ell)$ the clause that contains literal ℓ.

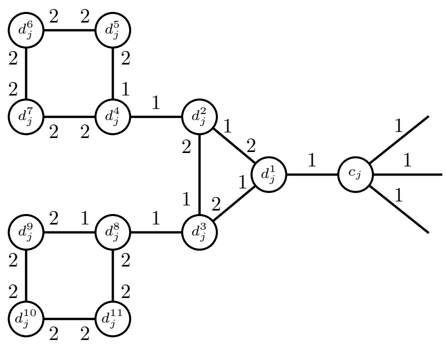

Figure 1: The illustration for goods-instance of the clause gadget for C_j. For each edge, if there is only one label, it represents the value of the edge to both endpoints. If there are two labels, the label closer to a vertex represents that vertex's value for the edge.

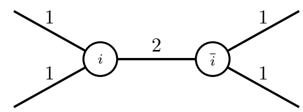

Figure 2: The illustration for goods-instance of the variable gadget for x_i. The label on each edge represents the value of the edge to both endpoints

Theorem 1 *Deciding the existence of PROP orientations is NP-complete, even for simple graphs where (1) all items are goods, valuations are normalized, and $v_i(e) \in \{1, 2\}$ for all i and $e \in E_i$; or all items are chores, valuations are normalized and $v_i(e) \in \{-1, -2\}$ for all i and $e \in E_i$.*

Proof. The problem is in NP as given an orientation, one can verify whether it is PROP or not in time polynomial in n. Next, we derive the NP-completeness by a reduction from 2P2N-3SAT. We first present the reduction for goods and then will adapt the reduction to chores.

Given a 2P2N-3SAT instance, we create a goods-instance as follows:

- for each variable x_i, create two vertices i and $\bar{i}$ and one edge $(i, \bar{i})$ with value 2 to both of them, as illustrated in Figure 2;
- for each clause C_j, create a vertex c_j and 11 dummy vertices $d_j^1, \ldots, d_j^{11}$. Create edge (d_j^1, c_j) with value 1 for d_j^1 and c_j. Moreover, the construction of edges with two endpoints being dummy vertices and their values are illustrated in Figure 1.
- for each clause C_j, create 3 edges based on the following rule: if C_j includes literal x_i^k (resp., $\bar{x}_i^{k'}$), create edge (c_j, i) (resp., $(c_j, \bar{i})$) with value 1 for both incident agents, as illustrated in Figures 1 and 2 (edges without an endpoint);

The created instance has $2s + 12t$ vertices and $s + 17t$ edges. For any i and $e \in E_i$, $v_i(e) \in \{1, 2\}$ holds. Moreover, each agent has a total value of 4 for her incident edges (normalized valuations), and thus, the proportional share of each agent is 2.

Suppose that there exists a truth assignment that satisfies all clauses in C. For $c_j, d_j^1, \ldots, d_j^{11}$, observe that we can satisfy each d_j^r with her proportional share even when (d_j^1, c_j) is allocated to c_j; allocate edges in the cycle with 3 vertices d_j^1, d_j^2, d_j^3 in a clockwise manner, and allocate edges in the cycles with 4 vertices in an anticlockwise manner. Edges (d_j^2, d_j^4) and (d_j^3, d_j^8) are arbitrarily allocated to their incident vertices. Thus, we create a partial assignment where each d_j^r receives value at least 2 and each c_j receives value 1.

For each variable x_i, if x_i is `True`, allocate $(i, \bar{i})$ to vertex i. Then allocate the other two edges incident to i to the vertices corresponding to $C(x_i^1)$ and $C(x_i^2)$; recall that $C(x_i^1)$ refers to the clause that contains literal x_i^1. For vertex $\bar{i}$, allocate her the two edges with value 1. Similarly, if $\bar{x}_i$ is `True`, allocate $(i, \bar{i})$ to vertex $\bar{i}$ and allocate the other two edges incident to $\bar{i}$ to the vertices corresponding to $C(\bar{x}_i^1)$ and $C(\bar{x}_i^2)$, and for vertex i, allocate her the two edges with value 1. At this point, for any $i \in [s]$, both i and $\bar{i}$ receive value 2.

For each vertex c_j, as clause C_j is satisfied, c_j receives one additional incident edge besides (d_j^1, c_j), resulting in a value of at least 2. Therefore, we create a PROP orientation.

Next, for the reserve direction, suppose that there exists a PROP orientation π. We now create a truth assignment of $\{x_i\}_{i\in[t]}$ as follows: if $(i,\bar{i})$ is allocated to vertex i, then set x_i to `True`; and otherwise, if $(i,\bar{i})$ is allocated to vertex $\bar{i}$, set $\bar{x}_i$ to `True`. Such a truth assignment ensures that exactly one of $x_i, \bar{x}_i$ is set to `True`, and hence, the truth assignment is valid. Next, we prove that the assignment satisfies all clauses.

For a contradiction, suppose that there exists a clause $C_{j'}$ not satisfied. By the created truth assignment, each of the three vertices corresponding to the three literals in $C_{j'}$ does not receive the edge with value 2. Consequently, the three edges connecting $c_{j'}$ to variable vertices must not be allocated to $c_{j'}$ so that the three vertices corresponding to the three literals in $C_{j'}$ can satisfy PROP. Hence, the value of $c_{j'}$ is at most 1, meaning that the orientation is not PROP, deriving the desired contradiction. Therefore, the created truth assignment satisfies all clauses.

As for the chores-instance, we use the same graph construction, but each agent's value on the edges are mapped to the corresponding negative values. If an edge has value 1 (resp., 2) for a vertex in the reduction for goods, we now change it to -1 (resp., -2). For any i and $e \in E_i$, $v_i(e) \in \{-1,-2\}$ holds. Moreover, each agent has a total value of -4 for her incident edges (normalized valuations), and thus, the proportional share of each agent is -2.

Suppose there is a truth assignment that satisfies all clauses in C. For $c_j, d_j^1, \dots, d_j^{11}$, observe that we can satisfy each d_j^r her proportional share even when (d_j^1, c_j) is allocated to d_j^1; allocate edges in the cycle with 3 vertices d_j^1, d_j^2, d_j^3 in an anticlockwise direction, and allocate edges in cycles with 4 vertices in a clockwise direction. Edges (d_j^2, d_j^4) and (d_j^3, d_j^8) are arbitrarily allocated to their incident vertices. At this point, each d_j^r satisfies her proportional share, and her value is unchanged thereafter.

For each x_i, if x_i is `True`, allocate $(i,\bar{i})$ to $\bar{i}$, and allocate i the other two edges incident to her; if $\bar{x}_i$ is `True`, allocate $(i,\bar{i})$ to i, and allocate $\bar{i}$ the other two edges incident to her. At this point, for any $i \in [s]$, both i and $\bar{i}$ receive value -2. For each vertex c_j, since clause C_j is satisfied, at least one incident edge other than (d_j^1, c_j) is not allocated to her, which implies that her value is at least -2. Therefore, the created orientation is PROP.

Next, for the reverse direction, suppose that there exists a PROP orientation π. We create a truth assignment of $\{x_i\}_{i\in[t]}$ as follows: if $(i,\bar{i})$ is allocated to vertex i, then set $\bar{x}_i$ to `True`, and otherwise, set x_i to `True`. Such a truth assignment ensures that exactly one of $x_i, \bar{x}_i$ is set to `True`, and hence, the truth assignment is valid. Moreover, if some clause $C_{j'}$ is not satisfied, then in π, $c_{j'}$ receives value at most -3, violating PROP. Therefore, the created truth assignment satisfies all clauses. ∎

To complement the hardness result, we show that for binary valuations and goods-instance (or chores-instance), one can compute in polynomial time a PROP orientation when it exists.

Theorem 2 *For the general orientation model, one can in polynomial time determine whether a PROP orientation exists or not, and compute one if it exists.*

Proof. [Proof Sketch] We begin with the goods-instance. If some item has value zero for all relevant agents, arbitrarily allocate item to the relevant agents. If some item has value one for exactly one relevant agent, allocate the item to that agent. At this point, let π' be the current partial assignment, and for each i, let a_i be the current value of agent i. Moreover, each of the unallocated items has value one for at least two agents relevant to that item.

Next, create a bipartite graph with two parts of vertices X and Y. For each unallocated item with respect to π', create a vertex in X. For each agent i, create a number $\lceil \mathsf{PROP}_i \rceil - a_i$ of vertices in Y. For each such a vertex, connect it to the vertex $x \in X$ if agent i is relevant to the item corresponding to x and has value one for it. We can show that a PROP orientation exists if and only if there exists a Y-perfect matching in the created bipartite graph. Then by applying the matching algorithm, one can determine whether a PROP orientation exists or not and compute one when it exists in time polynomial in n and m .

The idea is similar for chores-instances. If some item has value zero for some relevant agent, then allocate it to that agent. At this point, each of the unallocated item has value -1 for every relevant agent. Next, we create a bipartite graph $(X \cup Y)$. For each unallocated item, create a vertex in X. For each agent i, create a number $\lfloor |\mathsf{PROP}_i| \rfloor$ of vertices in Y. For each such an agent vertex, connect it to the vertex $x \in X$ if agent i is relevant to the item corresponding to x. We can prove that a PROP orientation exists if and only if there exists a X-perfect matching in the created bipartite graph. ■

4 PROP1 Fairness

We now consider the notion of PROP1. The main result is that for the general orientation model and the mixed-instance, one can always compute a PROP1 and fPO orientation in polynomial time. We introduce extra notations. For a given orientation $\pi = (\pi_1, \ldots, \pi_n)$, we also write $\pi = (\pi_{i,e})_{i \in [n], e \in E}$, where $\pi_{i,e}$ represents the portion of e allocated to agent i. Note that π is an orientation requires that for any i, e, $\pi_{i,e} > 0$ if and only if $e \in E_i$.

To compute the integral PROP1 and fPO orientation, we start from a fractional proportional orientation created as follows: for any e and any $i \in N_e$, let $\pi_{i,e} = \frac{1}{n_e}$. In this fractional orientation, each agent receives her proportional share. Then we find another fPO fractional orientation that Pareto improves the initial proportional orientation. Moreover, the new fractional orientation has acyclic *undirected consumption* graph: a bipartite graph in which vertices on one side are the agents and vertices on the other side are the items, and there is an edge between agent i and item e if and only if $\pi_{i,e} > 0$. To show the existence of such a fPO fractional orientation, we follow steps by steps the techniques in [29] and adapt them to the orientation model. Last, we round the fractional orientation to a PROP1 and fPO integral orientation.

Below we present the lemma of the intermediate step, whose proofs are deferred to the appendix.

Lemma 1 *For any orientation π, one can compute in polynomial time a fPO orientation π^* such that: (i) π^* either Pareto dominates π or gives every agent the same value as π, and (ii) the undirected consumption graph $\mathcal{CG}_{\pi^*}$ is acyclic.*

Theorem 3 *For the general orientation model and the mixed-instance, we can compute a PROP1 and fPO orientation in polynomial time.*

Proof. Let π be the orientation where $\pi_{i,e} = \frac{1}{n_e}$ for all $i \in [n]$ and all $e \in E_i$. One can verify that in π, each agent i receives exactly her proportional share. Based on Lemma 1, we find another orientation π^* where every agent is weakly better off, and hence, each agent i receives at least her proportional share. Moreover, π^* is fPO and the undirected consumption graph $\mathcal{CG}_{\pi^*}$ is acyclic.

We round π^* to an integral orientation that is fPO and PROP1. Consider an *undirected shared graph* $\mathcal{SG}_{\pi^*}$ defined as follows: create a vertex for each agent, and there is an edge between vertices i and j if and only if there exists e such that $\pi^*_{i,e}, \pi^*_{j,e} > 0$. In other words, the existence of an edge between i and j in $\mathcal{SG}_{\pi^*}$ means agents i and j share an item in π^*. Moreover, as π^* is fPO, each shared item must result in the value with the same sign (positive, negative, or zero) for both its endpoints; otherwise, we can increase the value of an agent without harming others, contradicting fPO.

As $\mathcal{CG}_{\pi^*}$ is acyclic, it is not hard to verify that $\mathcal{SG}_{\pi^*}$ is also acyclic, and hence, $\mathcal{SG}_{\pi^*}$ is a forest. For each component of $\mathcal{SG}_{\pi^*}$, we form it into a rooted tree. Then starting from the leaves to the root, we visit every vertex. For each vertex i, we let agent i receive fully the item with non-negative value shared with the agent having a higher depth than that of agent i, and let agent i receive fully the item with negative value shared with the agent having lower depth than that of agent i.

Let π' be the resulting integral orientation. We claim that π' is PROP1, as compared to π^*, the value of each agent i for non-negative item weakly increases and there exists at most one item with negative value that is partially allocated to i in π^* but is fully allocated to i in π'. After removing such a negative-valued item, agent i has value at least her proportional share, and therefore, π' is PROP1.

Finally, we prove that π' is fPO. Varian [33] proved that an allocation π is fPO if and only if there exist strictly positive weights $\{\lambda_i\}_{i \in [n]}$ of agents such that π maxmizes weighted welfare $\sum_{i \in [n]} \lambda_i v_i(\pi_i)$.We show that this characterization holds for the general orientation model and the mixed-instance in the appendix. Since orientation π^* is fPO, π^* maxmizes weighted welfare $\sum_{i \in [n]} \lambda^*_i v_i(\pi^*_i)$ for some strictly positive weight $\{\lambda^*_i\}_{i \in [n]}$ of agents. Hence in π^*, each item is always allocated to the agent with the highest weighted value with respect to $\{\lambda^*_i\}_{i \in [n]}$. After rounding π^* to π', it is still the case that every item is allocated to the agent with the highest weighted value, and thus, π' maximizes the weighted welfare for the weights $\{\lambda_i^*\}_{i\in[n]}$ and is fPO. ■

5 PROPX Fairness

After establishing the existence of fPO and PROP1 orientation, we are now concerned with the PROPX fairness, a notion stricter than PROP1. In sharp contrast, the results for PROPX are mostly negative. Specifically, a PROPX orientation is not guaranteed to exist even for simple graphs and binary valuations.

Proposition 1 *PROPX orientations may not exist, even for simple graphs and binary valuations.*

Proof. Let us begin with the goods-instance and consider the instance illustrated in Figure 3. Fix an arbitrary orientation π, and due to symmetry, assume that $(1, 2)$ is allocated to agent 2. Now we focus on agents 1,3, and 4. To ensure that agent 1 satisfies PROPX, both edges $(1, 3)$ and $(1, 4)$ must be allocated to agent 1. As a consequence, one of agents 3 and 4 receives $\emptyset$ and violates PROPX.

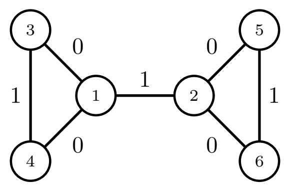

Figure 3: The illustration of the goods-instance where PROPX does not exist. The label on each edge represents the value of the edge to both endpoints

As for chores-instance, we consider the same graph construction and transform agents' valuations into their negative. By arguments similar to that of goods, one can verify that no orientation is PROPX. ■

Given the non-existence, a natural question is whether one can efficiently compute a PROPX orientation when it exists. Unfortunately, the answer is no even for simple graphs and binary valuations.

Theorem 4 *For both goods- and chores-instances, deciding the existence of PROPX orientations is NP-complete, even when the valuations are binary.*

Proof. The problem is in NP as given an orientation, one can verify whether it is PROPX or not in polynomial time. We derive the NP-completeness by a reduction from 2P2N-3SAT. We first present the reduction for goods and then will adapt the reduction to chores.

Given a 2P2N-3SAT instance, we create a goods-instance as follows:

- for each clause C_j, create vertex c_j and 6 dummy vertices $d_j^1, \dots, d_j^6$. Create edges (c_j, d_j^1) and (c_j, d_j^4), and moreover, create edges with both endpoints being dummy vertices, as illustrated in Figure 4;
- for each variable x_i, create two vertices i and $\bar{i}$, and an edge $(i, \bar{i})$ with value 1 for both $i, \bar{i}$, as illustrated in Figure 5;
- for each clause C_j, create three more edges: if C_j includes literal x_i^k (resp., $\bar{x}_i^{k'}$), add edge (c_j, i) (resp. $(c_j, \bar{i})$) with value 1 for both incident agents, as illustrated in Figures 4 and 5.

The created instance has $2s + 7t$ vertices and $s + 11t$ edges, and moreover, each agent's valuation is binary. We first claim that in a PROPX orientation (if exists), for each c_j, edge (c_j, d_j^1) must be allocated to d_j^1. For a contradiction, assume that this is not the case. We now focus on agents d_j^1, d_j^2, d_j^3. Without loss of generality, assume edge (d_j^2, d_j^3) is allocated to d_j^2 in the orientation. Then in order to make d_j^3 satisfy PROPX, we need to allocate (d_j^1, d_j^3) to her, which then makes d_j^1 violates PROPX, as d_j^1 has value zero in the orientation and edge (d_j^1, d_j^3) (with value zero) is not allocated to

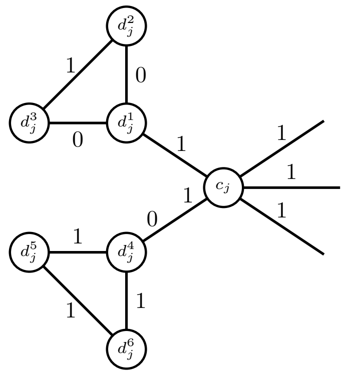

Figure 4: The illustration for goods-instance of the clause gadget for C_j. For each edge, if there is only one label, it represents the value of the edge to both endpoints. If there are two labels, the label closer to a vertex represents that vertex's value for the edge.

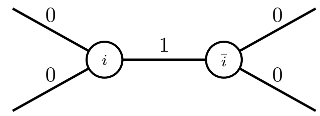

Figure 5: The illustration for goods-instance of the variable gadget for x_i. The label of the edge represents the value of the edge to both of the endpoints.

her but her proportional share is $\frac{1}{2}$. We below show that there exists a truth assignment satisfying all clauses if and only if there exists a PROPX orientation.

Suppose that there exists a truth assignment satisfying all clauses. If x_i is set to be `True`, we allocate $(i, \bar{i})$ to i. Then, we allocate $\bar{i}$ all edges, currently unallocated, incident to her. Moreover, assign the two unallocated edges incident to i to the vertices corresponding to $C(x_i^1)$ and $C(x_i^2)$. Similarly, if $\bar{x}_i$ is set to be `True`, we orient these edges in the reverse direction. Then agents i and $\bar{i}$ satisfy PROPX.

For each vertex c_j and its dummy vertices, allocate (c_j, d_j^1) to d_j^1 and allocate (c_j, d_j^4) to c_j. One can allocate the edge with both endpoints being dummy vertices in a way such that each d_j^r satisfies PROPX. At this point, c_j should receive at least one more edge to achieve PROPX. Note that the truth assignment satisfying clause C_j, and thus, the edge connecting c_j to the vertex corresponding the true literal in C_j is allocated to c_j. Therefore, the created orientation is PROPX.

Next, for the reverse direction, suppose that there exists a PROPX orientation π. We now create a truth assignment as follows: if $(i, \bar{i})$ is allocated to vertex i, we set x_i to `True`; otherwise we set $\bar{x}_i$ to `True`. Such a truth assignment ensures that exactly one of x_i and $\bar{x}_i$ is set to `True`, and hence, the truth assignment is valid. For a contradiction, we suppose that there exists a clause C_j that is not satisfied. Then since we have proved that (c_j, d_j^1) must be allocated to d_j^1, vertex c_j has value at most 1 in π, as C_j is not satisfied. Accordingly, c_j does not satisfy PROPX, deriving the desired contradiction.

For chores-instances, we use the same graph construction, but each agent's valuations are negated. The detailed proofs for chores-instances are deferred to the appendix. ∎

6 Conclusion

In this paper, we study the fair orientation problem and give a complete set of computational results on PROP, EQ, and EF related concepts. We propose a refined definition for PROP and show that PROP1 is the only fairness concept that can always be satisfied among the interested ones. In addition, we design a polynomial-time algorithm that computes a PROP1 and PO orientation even when the items are a mixture of goods and chores. For the other fairness concepts, including PROP, PROPX, EQ, EQ1, EQX, and EF1, we prove that they are not always satisfiable and the decision problems are NP-complete even in restricted settings. We complement these results by identifying the conditions under which they can be satisfied. In the future, it would be interesting to investigate other valuations, constraints, and fairness concepts.

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NeurIPS Technical Appendices

A Additional Definitions

We first provide the missing definitions related to equitability and envy-freeness.

Definition 4 (EQ) *An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is equitable (EQ) if $v_i(\pi_i) = v_j(\pi_j)$ for any two agents $i, j \in N$.*

Definition 5 (EQX) *An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is equitable up to any item (EQX) if for any two agents $i, j \in N$, either (1) $v_i(\pi_i) = v_j(\pi_j)$, or (2) $v_i(\pi_i) \geq v_j(\pi_j \setminus \{e\})$ holds for every item $e \in \pi_j$ with $v_j(e) > 0$, and $v_i(\pi_i \setminus \{e\}) \geq v_j(\pi_j)$ holds for every item $e \in \pi_i$ with $v_i(e) < 0$.*

Definition 6 (EQ1) *An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is equitable up to one item (EQ1) if for any two agents $i, j \in N$, one of the following three holds: (1) $v_i(\pi_i) = v_j(\pi_j)$, (2) there exists $e \in \pi_j$ such that $v_i(\pi_i) \geq v_j(\pi_j \setminus \{e\})$, or (3) there exists $e \in \pi_i$ such that $v_i(\pi_i \setminus \{e\}) \geq v_j(\pi_j)$.*

It is easy to see that an EQ orientation is EQX, and an EQX orientation is EQ1.

Definition 7 (EF) *An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is envy-free (EF) if for any two agents $i, j \in N$, $v_i(\pi_i) \geq v_i(\pi_j)$ holds.*

Definition 8 (EF1) *An orientation $\pi = (\pi_1, \ldots, \pi_n)$ is envy-free up to one item (EF1) if for any two agents $i, j \in N$, one of the following three holds: (1) $v_i(\pi_i) \geq v_i(\pi_j)$, (2) there exists $e \in \pi_j$ such that $v_i(\pi_i) \geq v_i(\pi_j \setminus \{e\})$, or (3) there exists $e \in \pi_i$ such that $v_i(\pi_i \setminus \{e\}) \geq v_i(\pi_j)$.*

An EF orientation is also EF1.

B Equitable Fairness

In this section, we shift our focus to fairness concepts related to the equitability. Throughout this section, agents' valuations are always normalized, i.e., the total value of items is identical for each agent. We first provide an instance for which EQ1 orientations do not exist. Since EQ implies EQX, which implies EQ1, there are no EQ or EQX orientations in this instance.

Proposition 2 *An EQ1 orientation is not guaranteed to exist even for simple graphs.*

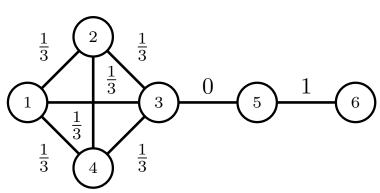

Figure 6: The illustration of the goods-instance where EQ1 does not exist. The label on each edge represents the value of the edge to both endpoints.

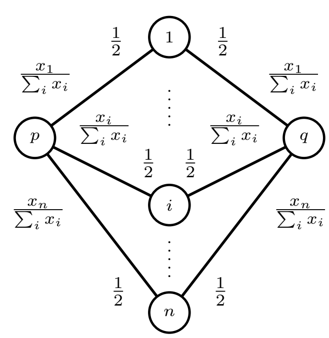

Figure 7: The illustration of the goods-instance for the reduction from EQ. The label closer to a vertex represents that vertex's value for the edge.

Proof. In the example shown in Figure 6, there will be one agent between 5 and 6 receiving no good; suppose that this agent is 5. However, as there are 6 edges among 1, 2, 3, and 4, by the pigeonhole principle, at least one of them will receive two or more edges. Suppose this agent is 1, then we have $v_1(\pi_1 \setminus \{e\}) = \frac{1}{3} > v_5(\pi_5) = 0$ for all $e \in \pi_1$, which means that EQ1 does not hold between 1 and 5. As for the chores-instance, we use the same graph while changing each agent's value on each edge to the corresponding negative value. In that case, we have $v_1(\pi_1 \setminus \{e\}) = -\frac{1}{3} < v_5(\pi_5) = 0$ for all $e \in \pi_1$, which violates EQ1. ■

The non-existence results show the sharp contrast between the orientation model and the classical fair division model, where EQ1 and EQX allocations always exist when agents' valuations are additive. Given the nonexistence, it is natural to ask whether one can compute an EQ, EQ1, or EQX orientation when it does exist. Unfortunately, the answer is negative. Below, we derive reductions from the PARTITION problem: given a set $\{x_1, \ldots, x_n\}$ of n integers whose sum is $2T$, can $[n]$ be partitioned into two sets I_1 and I_2 such that $\sum_{j \in I_1} x_j = \sum_{j \in I_2} x_j = T$?

Theorem 5 *Determining whether an EQ orientation exists on a simple graph in both goods- and chores-instances is NP-complete, even for instances with additive valuations.*

Proof. The decision problem is in NP as given an orientation, one can decide whether it is EQ in polynomial time. We derive the reduction from the PARTITION problem and begin with the goods-instance. We create a simple graph with $n+2$ vertices $\{1, \ldots, n, p, q\}$ and $2n$ edges. The created graph and the valuations of agents are illustrated in Figure 7. One can verify that the instance is normalized as the total value of items for each agent is equal to one.

We claim that in an EQ orientation, each vertex i with $i \in [n]$ must receive exactly one incident edge. If some vertex $i' \in [n]$ receives two incident edges, the value of agent i' is one, but the value of p is less than 1, violating EQ. If some vertex $i' \in [n]$ receives no edge, then the value of agent i' is zero, but the value of p is positive, violating EQ. Thus in an EQ orientation, the value of each agent must be $\frac{1}{2}$. Then the set of items received by p and q in an EQ orientation map a solution of the PARTITION problem. Therefore, an EQ orientation exists if and only if the answer to the PARTITION instance is a yes-instance.

For the reduction of the chores-instance, we create the same graph but negate agents' valuations. By similar arguments, we can prove that in an EQ orientation, each vertex $i \in [n]$ must receive exactly one incident edge. Therefore, an EQ orientation exists if and only if the answer to the PARTITION instance is a yes-instance. ■

Theorem 6 *Determining whether an EQX orientation exists on a simple graph in both goods- and chores-instances is NP-complete, even for instances with additive valuations.*

Proof. The decision problem is in NP as given an orientation, one can decide whether it is EQX in polynomial time. We derive the reduction from the PARTITION problem and begin with the goods-instance.

We create a goods-instance of a simple graph, as illustrated in Figure 8:

- create vertices $1, \ldots, n$ and vertices p, q; for each $j \in [n] \cup \{p, q\}$, create two dummy vertices k_j, k'_j;
- for each $j \in [n] \cup \{p, q\}$, create edges $(j, k_j), (j, k'_j), (k_j, k'_j)$, and moreover, for each $j \in [n]$, create edges (j, p) and (j, q).

We define the agents' valuations as follows:

- for each $j \in [n] \cup \{p, q\}$, let $v_j((j, k_j)) = v_j((j, k'_j)) = \epsilon$, $v_{k_j}((j, k_j)) = v_{k'_j}((j, k'_j)) = \frac{1}{2} - \epsilon$, and $v_{k_j}((k_j, k'_j)) = v_{k'_j}((k_j, k'_j)) = \frac{1}{2} + \epsilon$;
- for each $i \in [n]$, let $v_i((p, i)) = v_i((q, i)) = \frac{1}{2} - \epsilon$;
- for p and q, let $v_p((p, i)) = v_q((q, i)) = \frac{x_i}{\sum_i x_i}(1 - 2\epsilon)$ for all $i \in [n]$,

where $\epsilon > 0$ is arbitrarily small. As the total value of each agent is one, the created instance is normalized.

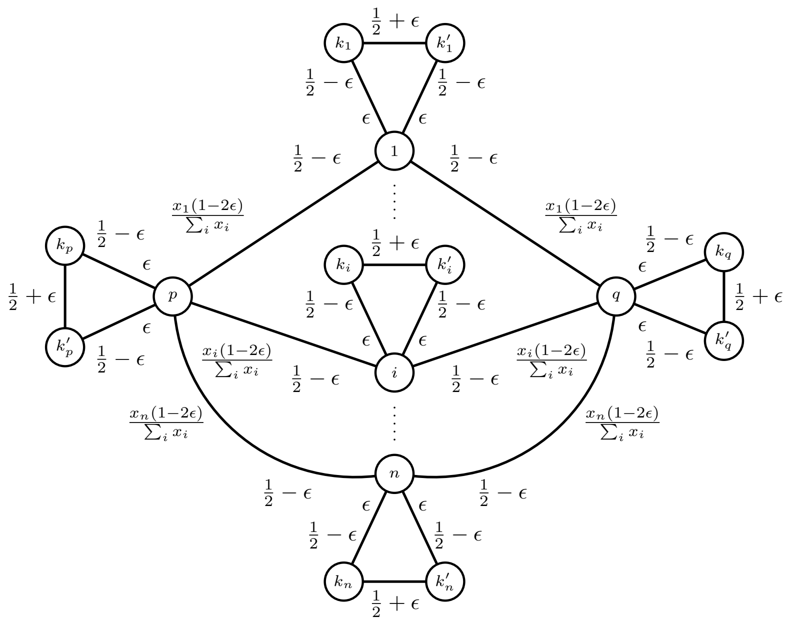

Figure 8: Illustration of the goods-instance for reduction from EQX. The label closer to a vertex represents that vertex's value for the edge.

We claim that in an EQX orientation π, each $j \in [n] \cup \{p, q\}$ must receive exactly one of edges (j, k_j) and (j, k_j'). Suppose not. If j receives neither (j, k_j) nor (j, k_j'), then EQX is violated between k_j and k_j'; assume without loss of generality that (k_j, k_j') is allocated to k_j, then k_j' violates EQX when compared to k_j as $v_{k_j}(\pi_{k_j} \setminus \{(p, k_j)\}) = \frac{1}{2} + \epsilon > v_{k_j'}(\pi_{k_j'}) = v_{k_j'}((j, k_j')) = \frac{1}{2} - \epsilon$. If j receives both of (j, k_j) and (j, k_j'), then one of k_j and k_j' receives no edges and violates EQX when compared to j.

Given the above claim, we further show that in an EQX orientation π, each agent $i \in [n]$ must receive exactly one of the edges (p, i) and (q, i). Suppose not. If i receives both of (p, i) and (q, i), then k_i, k_i' violate EQX when compared to i, as after removing the edge with value ϵ for i (such an edge exists due to the above claim), the value of i is still $1 - 2\epsilon > \frac{1}{2} + \epsilon$. If i receives neither (p, i) nor (q, i), then i violates EQX when compared to p as after removing the edge with value ϵ for p, the value of p is greater than 2ϵ.

Suppose that the PARTITION instance is a yes-instance and I_1 and I_2 are a solution. For each $i \in [n]$, if $i \in I_1$, allocate (i, p) to p and if $i \in I_2$, allocate (i, q) to q. Next for each $i \in [n]$, allocate i the other incident edge with value $\frac{1}{2} - \epsilon$ for her. For each $j \in [n] \cup \{p, q\}$, allocate (k_j, j) to j, (k_j', j) to k_j', and (k_j', k_j) to k_j. At this point, all edges are allocated, and the orientation is EQX, as I_1 and I_2 are a solution to the PARTITION instance.

For the reverse direction, suppose that there exists an EQX orientation π. Let $S_p := \{i \in [n] \mid (i, p) \in \pi_p\}$ and $S_q := \{i \in [n] \mid (i, q) \in \pi_q\}$. We prove that S_p and S_q must be a solution to the PARTITION instance. First, as each i receives exactly one from edges (i, p) and (i, q), it holds that $S_p \cup S_q = [n]$. Assume for the contradiction that S_p and S_q are not a solution, and then, assume $\sum_{i \in S_p} x_i < \sum_{i \in S_q} x_i$. Moreover, as x_i's are integers, we have $\sum_{i \in S_p} x_i \leq T - 1$, and thus, $v_p(\pi_p) \leq \frac{T-1}{2T}(1 - 2\epsilon) + \epsilon = \frac{1}{2} + \frac{\epsilon}{T} - \frac{1}{2T} < \frac{1}{2} - \epsilon$, where the last inequality transition is due to that $\epsilon \ll \frac{1}{T}$. For any $i \in [n]$, as π_i contains an edge with value ϵ and an edge with value $\frac{1}{2} - \epsilon$, agent p violates EQX when compared i, contradicting that π is an EQX orientation.

As for the chores-instance, we consider the same graph creation but negate agents' valuations. We claim that in an EQX orientation π, each $j \in [n] \cup \{p, q\}$ must receive exactly one of edges (j, k_j) and (j, k_j'). If j receives both of (j, k_j) and (j, k_j'), then one of k_j and k_j' receives no edges in π (let k_j be such a vertex), and j violates EQX when compared to k_j. If j receives neither (j, k_j) nor (j, k_j'), then EQX is violated when comparing k_j and k_j'. Based on the claim, we further show that in

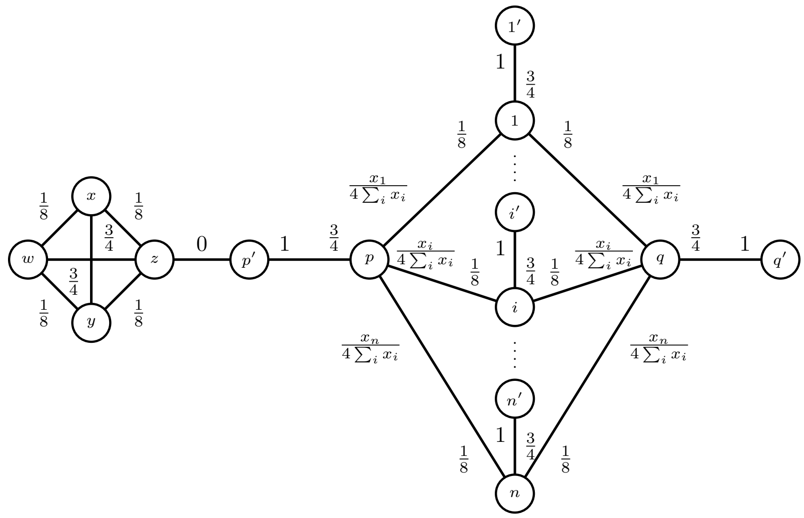

Figure 9: Illustration of the goods-instance for reduction from EQ1. For the edge with one label, the label is the value of that edge for both endpoints. If an edge has two labels, the label closer to a vertex represents that vertex's value for the edge.

an EQX orientation π, each $i \in [n]$ should receive exactly one of edges (p, i) and (q, i); otherwise, i violates EQX when compared to the vertex with larger value between k_i and k'_i. Then by the argument similar to that of goods, one can verify that there exists an EQX orientation in the created instance if and only if the PARTITION instance is a yes-instance. ■

Theorem 7 *Determining whether an EQ1 orientation exists on a simple graph in both goods- and chores-instances is NP-complete, even for instances with additive valuations.*

Proof. The decision problem is in NP as given an orientation, one can decide whether it is EQ1 in polynomial time. We derive the reduction from the PARTITION problem and begin with the goods-instance.

We create a goods-instance of a simple graph, as illustrated in Figure 9:

- create vertices $1, \dots, n$ and vertices $1', \dots, n'$, and for each $i \in [n]$, create edge (i, i');
- create vertices p, p', q, q', and create edges (p, p') and (q, q'). Moreover, for each $i \in [n]$, create edges (p, i) and (q, i);
- create vertices w, x, y, z and edges such that these four vertices form K_4 (i.e., a complete graph with 4 vertices). Moreover, create edge (z, p').

We define the agents' valuations as follows:

- for each agent $i \in [n]$, $v_i((i, i')) = \frac{3}{4}$, $v_{i'}((i, i')) = 1$, $v_i((p, i)) = v_i((q, i)) = \frac{1}{8}$, and $v_p((p, i)) = v_q((q, i)) = \frac{x_i}{4 \sum_i x_i}$;
- $v_p((p, p')) = v_q((q, q')) = \frac{3}{4}$, $v_{p'}((p, p')) = v'_q((q, q')) = 1$, and $v_{p'}((z, p')) = 0$;
- $v_w((w, z)) = v_z((w, z)) = v_x((x, y)) = v_y((x, y)) = \frac{3}{4}$, and $v_a((a, b)) = v_b((a, b)) = \frac{1}{8}$ for every pairs of vertices $(a, b) \in \{(w, x), (x, z), (z, y), (y, w)\}$.

We first claim that in an EQ1 orientation π, each agent has a value at least $\frac{1}{8}$. Let us focus on w, x, y, z, and in any orientation (and hence in π), there must be an agent receive two edges, with one having value $\frac{1}{8}$ and the other one having value $\frac{3}{4}$. Thus, after removing the edge with value $\frac{3}{4}$, that agent still has value $\frac{1}{8}$, which makes that in EQ1 orientation π, each agent should receive value at least $\frac{1}{8}$. Then edges (p', p) and (q', q) must be allocated to p' and q' respectively, and for each $i \in [n]$, edge (i', i) should be allocated to i'. Moreover, for each $i \in [n]$, vertex i should receive at least one (indeed exactly one) of (i, p) and (i, q). Then the total value of edges can be allocated to p and q is $\frac{1}{4}$, which makes in π, the value of p and q should be $\frac{1}{8}$. Thus, there exists a subset of $\{x_1, \ldots, x_n\}$ with total value T. Therefore, there exists an EQ1 orientation if and only if the PARTITION instance is a yes-instance.

As for the chores-instance, we consider the same graph creation but negate agents' valuations. Similarly, in an any orientation, there must an agent among w, x, y, z receiving at least one edge with value $-\frac{1}{8}$ and one edge with value $-\frac{3}{4}$. Thus in an EQ1 orientation π, every agent should receive value at most $-\frac{1}{8}$. Then edges (p', p) and (q', q) must be allocated to p' and q' respectively. Moreover, for each $i \in [n]$, edge (i', i) should be allocated to i'. Then one can verify that the unallocated edges can make both of p, q have value no greater than $-\frac{1}{8}$ if and only if the PARTITION instance is a yes-instance. Therefore, we establish an equivalence between the existence of EQ1 orientation and a yes-instance of the PARTITION problem. ■

C EF1 Orientations for Chores

We finally move to envy-freeness. A complete computational study of EFX fairness for goods, chores and mixed manna were provided in [15, 36]. Thus, in this section, we only focus on the EF1 fairness. For goods, it is known that an EF1 orientation always exists even in the general orientation model [16]. In a sharp contrast, we prove that EF1 orientations may not exist for chores, even when the model is a simple graph.

When verifying EF1, agent i compares the value of her bundle with her value for the bundles of other agents. For item $e \notin E_i$, v_i is defined to be zero in [15, 36]. When considering EF1, we follow this definition. Moreover, we say an edge is *objectively negative* if the edge results in negative values for both their endpoints.

Proposition 3 *In a chores-instance, there is an EF1 orientation on a simple graph if and only if the number of objectively negative edges is no greater than the number of vertices in the graph.*

Proof. We begin with a claim that if there exists an EF1 orientation π, then each agent should receive at most one objectively negative edge. Suppose not, and assume that i receives two objectively negative edges (i, j) and (i, k) in π. Then i violates EF1 when compared to j, as after removing any edge in π, the value of i is negative but i values π_j at zero (as (i, j) is in π_i, then $\pi_j \cap E_i = \emptyset$); recall that i values e at zero for all $e \notin E_i$. Thus based on the claim, if there exists an EF1 orientation, then the number of the objective negative edges is at most the number of vertices.

For the "if" direction, when the number of objectively negative edges is at most the number of vertices, we can compute an EF1 orientation. First, allocate all edges incident to an agent having zero value on it to such an agent. Then, if there are edges remaining unallocated, they must be objectively negative; they will form one or more components. If a component formed by the objectively negative edges contains no cycle, i.e., is a tree, we can allocate every edge downwards to the corresponding child vertex by selecting an arbitrary vertex as the root of the tree. If a component contains a cycle, we can allocate every edge in the cycle to an endpoint in the same direction, and the remaining edges. Repeat the process until all edges are allocated. The procedures terminate in at most m rounds as in each round, at least one edge is allocated. ■

The above constructive proof is indeed a polynomial time algorithm for deciding the existence of EF1 orientation for simple graphs and compute one when it exists. On the contrary, for multigraphs, the decision problem becomes computationally intractable.

Theorem 8 *For the chores-instance, determining whether an EF1 orientation exists in a multigraph is NP-complete.*

Proof. The decision problem is in NP as given an orientation, one can decide whether it is EF1 in polynomial time. We below derive the reduction from the PARTITION problem. Create a chores-instance with 3 vertices and $n + 4$ edges as follows:

- create vertices $1, 2, 3$;

- create n edges $e_1, \dots, e_n$ between 1 and 2. Moreover, create two edges e_{n+1}, e_{n+2} (resp., e_{n+3}, e_{n+4}) between 1 and 3 (resp., between 2 and 3);
- for each $k \in [4]$, the value of e_{n+k} is $-2T-1$ for both of its endpoints, and for each $i \in [n]$, the value of e_i is $-x_i$ for both of its endpoints.

We claim that in an EF1 orientation π, agent 3 cannot receive both e_{n+1}, e_{n+2} or both e_{n+3}, e_{n+4}. Suppose that 3 receives both e_{n+1}, e_{n+2}, then agent 3 would value 1's bundle at zero and violates EF1. The same reasoning applies to e_{n+3}, e_{n+4}. Similarly, agent 1 (resp., agent 2) cannot receive both e_{n+1}, e_{n+2} (resp., e_{n+3}, e_{n+4}). Therefore, in an EF1 orientation π, agent 1 (resp., agent 2) must receive exactly one of e_{n+1}, e_{n+2} (resp., e_{n+3}, e_{n+4}). Then at this point, 3 satisfies EF1 no matter how the remaining edges are allocated.

For agent 1 and 2, they do not violate EF1 when compared to agent 3, as the total value of $e_1, \dots, e_n$ is greater than $-2T-1$. When verifying the EF1 condition between agents 1 and 2, edge e_{n+k} for some k would always be hypothetically removed. Therefore, there exists an EF1 orientation if and only if the PARTITION instance is a yes-instance. ■

D Missing Materials for Section 3

Theorem 2 [Restated]. *For the general orientation model, one can in polynomial time determine whether a PROP orientation exists or not, and compute one if it exists.*

Proof. We begin with the goods-instance. If some item has value zero for all relevant agents, arbitrarily allocate the item to the relevant agents. If some item has value one for exactly one relevant agent, allocate the item to that agent. At this point, let π' be the current partial assignment, and for each i, let a_i be the current value of agent i. Moreover, each of the unallocated items has value one for at least two agents relevant to that item.

Next, create a bipartite graph with two parts of vertices X and Y. For each unallocated item with respect to π', create a vertex in X. For each agent i, create a number $\lceil \mathsf{PROP}_i \rceil - a_i$ of vertices in Y. For each such a vertex, connect it to the vertex $x \in X$ if agent i is relevant to the item corresponding to x and has value one for it. We below prove that a PROP orientation exists if and only if there exists a Y-perfect matching in the created bipartite graph.

For the "if" direction, based on the Y-perfect matching, one can extend π' to another (possibly partial) assignment where each agent i receives her proportional share. Moreover, unallocated items (if any) make agents weakly better off as items are goods. Thus, a PROP orientation exists. For the "only if" direction, denote by $Q \subseteq E$ the set of items with value one for at least two of their relevant agents. In the PROP orientation, each agent i must receive value at least $\lceil \mathsf{PROP}_i \rceil - a_i$ when restricting to Q as items $E \setminus Q$ can increase the value of agent i by at most a_i. Hence, one can convert the assignment of Q in the PROP to a Y-perfect matching.

By applying the matching algorithm, one can determine whether a PROP orientation exists or not and compute one when it exists in time polynomial in n and m .

As for the chores-instance, the idea is similar. First, if some item has value zero for some relevant agent, then allocate it to that agent. At this point, each of the unallocated items has value -1 for every relevant agent. Next, we create a bipartite graph $(X \cup Y)$. For each unallocated item, create a vertex in X. For each agent i, create a number $\lfloor |\mathsf{PROP}_i| \rfloor$ of vertices in Y. For each such agent vertex, connect it to the vertex $x \in X$ if agent i is relevant to the item corresponding to x. By the arguments similar to those for the goods-instance, one can verify that a PROP orientation exists if and only if there exists a X-perfect matching in the created bipartite graph. Moreover, when a X-perfect matching exists, we can convert the matching to a PROP orientation. ■

E Missing Materials for Section 4

Throughout this section, we say that an item e is:

- a chore if $v_i(e) < 0$ for all $i \in N_e$;
- neutral if $v_i(e) = 0$ for at least one $i \in N_e$ and $v_j(e) \leq 0$ for all $j \in N_e$;

- a good if $v_i(e) > 0$ for at least one $i \in N_e$;
- a pure good if $v_i(e) > 0$ for all $i \in N_e$.

An orientation π is *non-malicious* if each good e is consumed by agents $i \in N_e$ with $v_i(e) > 0$ and each neutral item e is consumed by agents $i \in N_e$ with $v_i(e) = 0$. Every fPO orientation is clearly non-malicious.

We will consider the agent-object graphs – bipartite graphs in which the nodes on one side are the agents and the nodes on the other side are the objects. In the *(undirected) consumption-graph* $\mathcal{CG}_\pi$ of an orientation π, there is an edge between agent $i \in [n]$ and an item $e \in E$ if and only if $\pi_{i,e} > 0$; note as π is an orientation, if there is an edge between i and e, then $i \in N_e$ and $e \in E_i$.

The *weighted directed consumption-graph* $\overrightarrow{\mathcal{CG}}_\pi$ of an orientation π is constructed as follows. There is an edge $(i \to e)$ with weight $w_{i \to e} = |v_i(e)|$ if one of the two conditions holds:

- $\pi_{i,e} > 0, v_i(e) \geq 0$ and $e \in E_i$;
- $\pi_{i,e} < 1, v_i(e) < 0$ and $e \in E_i$.

The opposite edge $(e \to i)$ with weight $w_{e \to i} = \frac{1}{|v_i(e)|}$ is included in $\overrightarrow{\mathcal{CG}}_\pi$ in one of the two cases:

- $\pi_{i,e} > 0, v_i(e) < 0$ and $e \in E_i$;
- $\pi_{i,e} < 1, v_i(e) > 0$ and $e \in E_i$.

The *product* of a directed path P in $\overrightarrow{\mathcal{CG}}_\pi$, denoted $\Pi(P)$, is the product of weights of edges in P. In particular, the product of a cycle $C = (i_1 \to e_1 \to \cdots \to e_L \to i_{L+1} = i_1)$ is

$$\Pi(C) = \prod_{k=1}^{L} (w_{i_k \to e_k} \cdot w_{e_k \to i_{k+1}}).$$

Closely following steps by steps the proofs in [29], we present the characterisation of fPO orientation.

Lemma 2 *Given an orientation π, the following three properties are equivalent*

(i) π is fPO;

(ii) π is non-malicious and its directed consumption graph $\overrightarrow{\mathcal{CG}}_\pi$ has no cycle C with $\Pi(C) < 1$.

(iii) there is a vector of weights $\lambda = (\lambda_i)_{i \in [n]}$ with $\lambda_i > 0, i \in [n]$, such that $\pi_{i,e} > 0$ implies $\lambda_i v_i(e) \geq \lambda_j v_j(e)$ for all $i, j \in [n]$ and $e \in E$.

Proof. (i) $\implies$ (ii).

If π is fPO but malicious, then reallocating items in a non-malicious way strictly improves the value of some agents without harming the others. Thus, an fPO orientation π must be non-malicious.

We now show that there are no directed cycles $C = (i_1 \to e_1 \to i_2 \to e_2 \to \ldots \to i_L \to e_L \to i_{L+1} = i_1)$ in $\overrightarrow{\mathcal{CG}}_\pi$ with $\Pi(C) < 1$. For the sake of contradiction, assume that C is such a cycle. We show how to construct an exchange of items among the agents in C such that their value strictly increases without affecting the other agents. This will contradict the Pareto-optimality of π.

Define $R := \Pi(C)^{\frac{1}{L}}$. As $\Pi(C) < 1$, then $R < 1$ holds. For each $k \in [L]$, there are edges $i_k \to e_k$ and $e_k \to i_{k+1}$. Then by the definition of $\overrightarrow{\mathcal{CG}}_\pi$,

- either i_k receives a positive amount of e_k and both i_k and i_{k+1} have positive value for e_k,
- or i_{k+1} has positive amount of e_{k+1} and both i_k and i_{k+1} have negative value for e_k.

Now we describe the reallocation. For each $k \in [L]$, agent i_k give a small positive amount ϵ_k of e_k to i_{k+1} in the case of good or i_{k+1} gives ϵ_k fraction of e_k to i_k in the case of chore where $\epsilon_k \in (0, h_k]$

and $h_k = \pi_{i_k,e_k}$ for a good and $h_k = \pi_{i_{k+1},e_k}$ for a chore. Then each i_k loses a value of $\epsilon_k \cdot |v_{i_k}(e_k)|$ and gains a value of $\epsilon_{k-1} \cdot |v_{i_k}(e_{k-1})|$, resulting in a net change of $\epsilon_{k-1} \cdot |v_{i_k}(e_{k-1})| - \epsilon_k \cdot |v_{i_k}(e_k)|$ in the value of agent i_k. In order to guarantee that every agent in C strictly gains from the reallocation, it is sufficient to choose $\epsilon_1, \ldots, \epsilon_k$ such that the following inequalities hold for all $k \in [L]$:

$$\epsilon_{k-1} \cdot |v_{i_k}(e_{k-1})| - \epsilon_k \cdot |v_{i_k}(e_k)| > 0 \iff \frac{\epsilon_{k-1}}{\epsilon_k} > \frac{|v_{i_k}(e_k)|}{|v_{i_k}(e_{k-1})|}.$$

For any $\epsilon_1 > 0$, define $\epsilon_k = \epsilon_{k-1} \cdot R \cdot \frac{|v_{i_k}(e_{k-1})|}{|v_{i_k}(e_k)|}$ for every $k \in \{2, \ldots, L\}$. Since $R < 1$, the above inequality is satisfied for each $k \in \{2, \ldots, L\}$. It remains to show that it is also satisfied for $k = 1$. Note that

$$\begin{aligned}\epsilon_L &= \epsilon_1 R^{L-1} \prod_{k=2}^{L} \frac{|v_{i_k}(e_{k-1})|}{|v_{i_k}(e_k)|} = \epsilon_1 R^{L-1} \frac{|v_{i_1}(e_1)|}{|v_{i_1}(e_L)|} \prod_{k=1}^{L} \frac{|v_{i_k}(e_{k-1})|}{|v_{i_k}(e_k)|} \\ &= \epsilon_1 \frac{R^{L-1}}{\Pi(C)} \frac{|v_{i_1}(e_1)|}{|v_{i_1}(e_L)|} = \epsilon_1 R^{-1} \frac{|v_{i_1}(e_1)|}{|v_{i_1}(e_L)|},\end{aligned}$$

which implies

$$\frac{\epsilon_1}{\epsilon_L} = R \cdot \frac{|v_{i_1}(e_L)|}{|v_{i_1}(e_1)|} < \frac{|v_{i_1}(e_L)|}{|v_{i_1}(e_1)|},$$

as required. Therefore, we can choose ϵ_1 sufficiently small so that $\epsilon_k \leq h_k$ for all $k \in [L]$ and such a reallocation is feasible.

(ii) $\implies$ (iii).

We assume that $\overrightarrow{\mathcal{CG}}_\pi$ contains no directed cycles C with $\Pi(C) < 1$ and π is non-malicious. We prove below the existence of weights λ_i's satisfying (iii).

Starting from $\overrightarrow{\mathcal{CG}}_\pi$, for each pair of distinct agents $i, j \in [n]$, add directed edges $i \to j$ with weight

$$w_{i \to j} = \left(\max \left\{ 1, |v_k(e)|, \frac{1}{|v_k(e)|} : e \in E, k \in N_e, v_k(e) \neq 0 \right\} \right)^{2(n-1)}.$$

Note that each new edge has the same weight. Let the resulting new graph be $\overrightarrow{G}$, and we claim that $\overrightarrow{G}$ has no cycle C with weight $\pi(C) < 1$. For the sake of contradiction, assume that C be such a cycle in $\overrightarrow{G}$ with $\pi(C) < 1$. Then C must contain at least one new edge. Each agent vertex appears once in C, the cycle C contains at most $2n - 2$ old edges. If none of the old edges in C has weight zero, then by the definition of the weight of the new edge, $\pi(C) \geq 1$ holds, a contradiction. Old edges with weight zero are only possible from an agent to an item (suppose from some agent i to some item e), and moreover, $\pi_{i,e} > 0$ and $v_i(e) = 0$. However, as π is non-malicious, such e has no outgoing edges, and thus, edge $i \to e$ cannot be a part of any cycle.

Fix an arbitrary agent (suppose agent 1). For every other agent $j \in [n]$, let $P_{1,j}$ be a directed path from 1 to j in $\overrightarrow{G}$, for which the product $\Pi(P_{1,j})$ is minimal. Note that the minimum is well-defined and is attained on an acyclic path, as by the construction, there are no cycles with a product smaller than 1.

Define $\lambda_j := \Pi(P_{1,j})$ for all $j \neq 1$ and $\lambda_1 := 1$. We now prove that these weights satisfy (iii), i.e., $\pi_{i,e} > 0$ implies $\lambda_i v_i(e) \geq \lambda_j v_j(e)$ for all $j \in N_e$. Fix i, e with $\pi_{i,e} > 0$ and some j with $j \in N_e$. As π is non-malicious, we can, without loss of generality, assume that both i, j agree whether e is a good or a chore, i.e., $v_i(e) \cdot v_j(e) > 0$; if i, j disagree, then by the non-maliciousness, $\lambda_i v_i(e) \geq \lambda_j v_j(e)$ holds for any $\lambda_i, \lambda_j > 0$.

If e is a good (i.e., $v_i(e) > 0$ and $v_j(e) > 0$), then there is an edge $i \to e$ (as $\pi_{i,e} > 0$, $v_i(e) > 0$, and $i \in N_e$) and an edge $e \to j$ (as $\pi_{j,e} < 1$, $v_j(e) > 0$, and $j \in N_e$). Consider the optimal path $P_{1,i}$ and the concatenated path $Q_{1,j} = P_{1,i} \to e \to j$. The path As the path $P_{1,j}$ has the minimal product among all paths from 1 to j, we have

$$\Pi(Q_{1,j}) \geq \Pi(P_{1,j}) \iff \Pi(P_{1,i}) \cdot \frac{v_i(e)}{v_j(e)} \geq \Pi(P_{1,j}) \iff \lambda_i v_i(e) \geq \lambda_j v_j(e).$$

If e is a chore (i.e., $v_i(e) < 0$ and $v_j(e) < 0$), then there is an edge $j \to e$ (as $v_j(e) < 0$, $\pi_{j,e} < 1$, and $j \in N_e$) and an edge $e \to i$ (as $\pi_{i,e} > 0$, $v_i(e) < 0$, and $i \in N_e$). Define $Q_{1,i}$ as $P_{1,j} \to e \to i$ and get

$$\Pi(Q_{1,i}) \geq \Pi(P_{1,i}) \iff \Pi(P_{1,j}) \cdot \frac{|v_j(e)|}{|v_i(e)|} \geq \Pi(P_{1,i}) \iff \lambda_j |v_j(e)| \geq \lambda_i |v_i(e)| \iff \lambda_i v_i(e) \geq \lambda_j v_j(e).$$

(iii) $\implies$ (i).

As in π, each item e is allocated to the agent i with the highest $\lambda_i v_i(e)$ among all agents in N_e, then π maximizes $\sum_{e \in E} \sum_{i \in N_e} \lambda_i v_i(e)$ over all (fractional) orientations. Since λ_i's are positive, π is fPO because any Pareto-improvement must increase the weighted sum. ■

We now prove Lemma 1.

Lemma 1 [Restated]. *For any orientation π, one can compute in polynomial time a fPO orientation π^* such that: (i) π^* either Pareto dominates π or gives every agent the same value as π, and (ii) the undirected consumption graph $\mathcal{CG}_{\pi^*}$ is acyclic.*

Proof. If π is malicious, implement the following reallocation:

- for each $e \in E$ with $\max_{i \in N_e} v_i(e) > 0$, reallocate the share of agents with $v_j(e) \leq 0$ to an agent $i \in N_e$ with $v_i(e) > 0$;
- for each $e \in E$ with $\max_{i \in N_e} v_i(e) = 0$, reallocate the share of agents with $v_j(e) < 0$ to an agent i with $v_i(e) = 0$.

Let the resulting non-malicious orientation be π'.

We now describe reallocation items that eliminates the cycle (if any). Let us call a cycle $C = (i_1 \to e_1 \to i_2 \to e_2 \to \ldots \to i_L \to e_L \to i_{L+1} = i_1)$ in the directed graph $\overrightarrow{\mathcal{CG}}_{\pi'}$ *simple* if each node is visited at most once and for any $i \in [n]$ and $e \in E_i$, only one of edges $i \to e$ or $e \to i$ is contained in the cycle.

We claim that if there is a simple cycle C in $\overrightarrow{\mathcal{CG}}_{\pi'}$ with $\Pi(C) \leq 1$, then C can be eliminated by reallocation of items. The idea of eliminating the cycle is similar to the reallocation in the proof of Lemma 2. As edges $i_k \to e_k$ and $e_k \to i_{k+1}$ exist in $\overrightarrow{\mathcal{CG}}_{\pi'}$, the values $v_{i_k}(e_k)$ and $v_{i_{k+1}}(e_k)$ are both non-zero and have the same sign due to the construction of $\overrightarrow{\mathcal{CG}}_{\pi'}$. We implement the following reallocation of items:

- if $v_{i_k}(e_k) > 0$ and $v_{i_{k+1}}(e_k) > 0$, then take ϵ_k amount of e_k from i_k and give it to i_{k+1}, where $0 < \epsilon \leq h_k$ with $h_k = \pi'_{i_k, e_k}$;
- if $v_{i_k}(e_k) < 0$ and $v_{i_{k+1}}(e_k) < 0$, then transfer ϵ_k of e_k from i_{k+1} to i_k where $0 < \epsilon \leq h_k$ with $h_k = \pi'_{i_{k+1}, e_k}$.

Then amounts ϵ_k's are selected in a way such that $\epsilon_k |v_{i_k}(e_k)| = \epsilon_{k+1} |v_{i_k}(e_{k+1})|$ for every $k \in [L-1]$. Thus for each $k = 2, \ldots, L$, the value of agent i_k remains indifferent after the reallocation, while agent i_1 is weakly better off because the condition of $\Pi(C) \leq 1$. Note that after reallocating items, the resulting allocation is still an orientation. It is not hard to verify that we can select ϵ_k's as large as possible so that one of edges $i_k \to e_k$ in $\overrightarrow{\mathcal{CG}}_{\pi'}$ can be removed.

Repeat this reallocation until there are no simple cycles with $\Pi(C) \leq 1$. As at least one edge is removed each time in the undirected graph of the underlying orientation and there are at most mn edges, we need to most $(n-1)m$ repetitions for achieving the orientation where no simple cycle has product at most 1. Let the resulting orientation be π^*.

By the construction, π^* weakly improves the value of each agent in π; is non-malicious; has no cycle C in $\overrightarrow{\mathcal{CG}}_{\pi^*}$ with $\Pi(C) < 1$. By Lemma 2, π^* is fPO.

We now prove that $\mathcal{CG}_{\pi^*}$ is acyclic. For a contradiction, assume that there is a cycle C in $\mathcal{CG}_{\pi^*}$. Then in the directed graph $\overrightarrow{\mathcal{CG}}_{\pi^*}$, there are two cycles: C passed in one direction and in the opposite direction. Let them be $\overrightarrow{C}$ and $\overleftarrow{C}$. As $\Pi(\overrightarrow{C}) \cdot \Pi(\overleftarrow{C}) = 1$, one of them has the product of at most 1, and indeed, by fPO it must hold that $\Pi(\overrightarrow{C}) = \Pi(\overleftarrow{C}) = 1$. However, all such cycles were eliminated in the previous stages.

For the running time of the algorithm, constructing the non-malicious orientation, finding the cycles with $\Pi(C) \leq 1$, and resolving/eliminating the cycle takes time polynomial in m and n. As cycle-elimination repeats at most $(n-1)m$ times, the total running time is polynomial in n and m. ■

F Missing Materials for Section 5

Theorem 4 [Restated]. *For both goods- and chores-instances, deciding the existence of PROPX orientations is NP-complete, even when the valuations are binary.*

Proof. [Proof for the chores-instance] We create the graph identical to that for the goods-instance. For agents' valuations, let edge (c_j, d_j^1) result in value -1 for d_j^1 and value zero for c_j for all j. For each of the other edges, if their value for a vertex is one in the goods-instance, change it to -1; if their value for a vertex is zero in the goods-instance, it remains the same.

We first claim that in a PROPX orientation (if it exists), for each c_j, edge (c_j, d_j^1) must be allocated to c_j. Assume for the contradiction that this is not the case. We now focus on agents d_j^1, d_j^2, d_j^3. Without loss of generality, assume edge (d_j^2, d_j^3) is allocated to d_j^2 in the orientation. Then in order to make d_j^2 satisfy PROPX, we need to allocated (d_j^2, d_j^1) to d_j^1, which means that d_j^1 cannot receive (c_j, d_j^1). Then in a PROPX orientation, the value of c_j must be at most -2 as her proportional share is -2 and she always receives an edge with value zero. We prove below that there exists a truth assignment satisfying all clauses if and only if there exists a PROPX orientation.

Suppose that there exists a truth assignment satisfying all clauses. If x_i is set to `True`, we allocate $(i, \bar{i})$ to $\bar{i}$. Then we allocate i the two edges with value zero for her and allocate the two unallocated edges incident to $\bar{i}$ to the vertices corresponding to $C(\bar{x}_i^1)$ and $C(\bar{x}_i^2)$. Similarly, if $\bar{x}_i$ is set `True`, we orient these edges in the reverse direction. Then at this point, both agents i and $\bar{i}$ satisfy PROPX.

For each vertex c_j and its dummy vertices, allocate (c_j, d_j^1) to c_j and allocate (c_j, d_j^4) to d_j^4. One can allocate the edges with both endpoints being dummy vertices in a way such that each d_j^r satisfies PROPX. Then all edges are allocated and c_j has value at least -2 as the edge connecting c_j to the vertex corresponding to the true literal in C_j is not allocated to c_j. Therefore, the created orientation is PROPX.

Next for the reverse direction, suppose that there exists a PROPX orientation π. We now create a truth assignment as follows: if $(i, \bar{i})$ is allocated to i, set $\bar{x}_i$ to `True`; otherwise, set x_i to `True`. Such a truth assignment ensures that exactly one of x_i and $\bar{x}_i$ is set to `True`, and hence, the truth assignment is valid. For a contradiction, suppose that there exists a clause C_j that is not satisfied. Then c_j has value at most -3 as each edge connecting c_j to the variable vertices with literals in C_j must be allocated to c_j. As edge (c_j, d_j^1) is allocated to c_j, vertex c_j violates PROPX, a contradiction. ■