LECTURE 4: Bayesian Decision Theory

- The Likelihood Ratio Test
- The Probability of Error
- The Bayes Risk
- Bayes, MAP and ML Criteria
- Multi-class problems
- Discriminant Functions

Likelihood Ratio Test (LRT)

- **Assume we are to classify an object based on the evidence provided by a measurement (or feature vector) x**
- **Would you agree that a reasonable decision rule would be the following?**
  - **"Choose the class that is most 'probable' given the observed feature vector x"**
    - **More formally**: Evaluate the posterior probability of each class $P(\omega_i|x)$ and choose the class with largest $P(\omega_i|x)$
- **Let us examine the implications of this decision rule for a 2-class problem**
  - In this case the decision rule becomes

$$\begin{array}{ll} \text{if } P(\omega_1 | x) > P(\omega_2 | x) & \text{choose } \omega_1 \\ \text{else} & \text{choose } \omega_2 \end{array}$$

    - Or, in a more compact form

$$P(\omega_1 | x) \underset{\omega_2}{\overset{\omega_1}{\gtrless}} P(\omega_2 | x)$$

  - Applying Bayes Rule

$$\frac{P(x | \omega_1)P(\omega_1)}{P(x)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{P(x | \omega_2)P(\omega_2)}{P(x)}$$

  - P(x) does not affect the decision rule so it can be eliminated*. Rearranging the previous expression

$$\boxed{\Lambda(x) = \frac{P(x | \omega_1)}{P(x | \omega_2)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{P(\omega_2)}{P(\omega_1)}}$$

  - The term $\Lambda(x)$ is called the **likelihood ratio**, and the decision rule is known as the **likelihood ratio test**

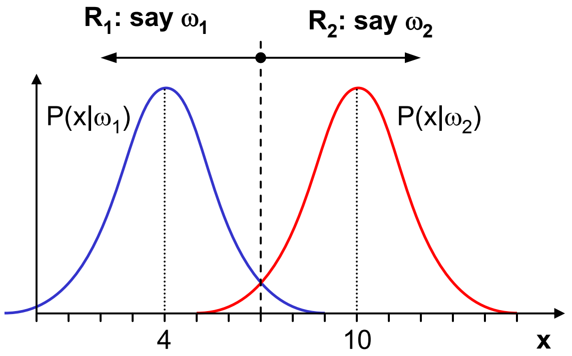
****P(x) can be disregarded in the decision rule since it is constant regardless of class ω_i. However, P(x) will be needed if we want to estimate the posterior $P(\omega_i|x)$ which, unlike $P(x|\omega_i)P(x)$, is a true probability value and, therefore, gives us an estimate of the "goodness" of our decision.***

Likelihood Ratio Test: an example

- **Given a classification problem with the following class conditional densities, derive a decision rule based on the Likelihood Ratio Test (assume <u>equal priors</u>)**

$$P(x|\omega_1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-4)^2} \qquad P(x|\omega_2)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-10)^2}$$

- **Solution**
  - Substituting the given likelihoods and priors into the LRT expression: $\Lambda(x)=\dfrac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-4)^2}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-10)^2}}\ \mathop{\gtrless}\limits_{\omega_2}^{\omega_1}\ \dfrac{1}{1}$
  - Simplifying the LRT expression: $\Lambda(x)=\dfrac{e^{-\frac{1}{2}(x-4)^2}}{e^{-\frac{1}{2}(x-10)^2}}\ \mathop{\gtrless}\limits_{\omega_2}^{\omega_1}\ 1$
  - Changing signs and taking logs: $(x-4)^2-(x-10)^2\ \mathop{\lessgtr}\limits_{\omega_2}^{\omega_1}\ 0$
  - Which yields: $x\ \mathop{\lessgtr}\limits_{\omega_2}^{\omega_1}\ 7$
  - This LRT result makes sense from an intuitive point of view since the likelihoods are identical and differ only in their mean value



- **How would the LRT decision rule change if, say, the priors were such that $P(\omega_1)=2P(\omega_2)$?**

The probability of error (1)

- **The performance of any decision rule can be measured by its <u>probability of error</u> P[error] which, making use of the Theorem of total probability (Lecture 2), can be broken up into**

$$P[\text{error}] = \sum_{i=1}^{C} P[\text{error} \mid \omega_i] P[\omega_i]$$

- **The class conditional probability of error P[error|ω_i] can be expressed as**

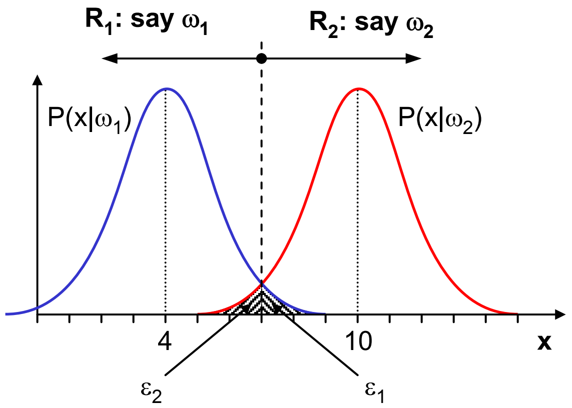
$$P[\text{error} \mid \omega_i] = P[\text{choose } \omega_j \mid \omega_i] = \int_{R_j} P(x \mid \omega_i)\,dx$$

- **So, for our 2-class problem, the probability of error becomes**

$$P[\text{error}] = P[\omega_1] \underbrace{\int_{R_2} P(x \mid \omega_1)\,dx}_{\varepsilon_1} + P[\omega_2] \underbrace{\int_{R_1} P(x \mid \omega_2)\,dx}_{\varepsilon_2}$$

  - where ε_i is the integral of the likelihood $P(x|\omega_i)$ over the region R_j where we choose ω_j

- **For the decision rule of the previous example, the integrals ε_1 and ε_2 are depicted below**
  - Since we assumed equal priors, then P[error] = $(\varepsilon_1 + \varepsilon_2)/2$



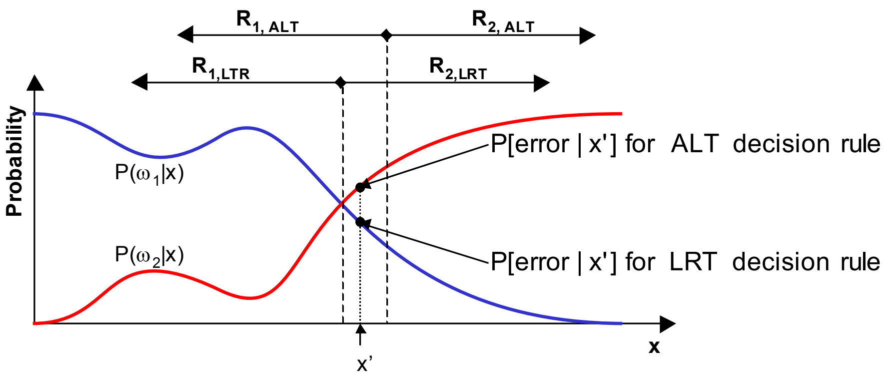
- **Compute the probability for the example above**

The probability of error (2)

- **Now that we can measure the performance of a decision rule we can ask the following question: How good is the Likelihood Ratio Test decision rule?**
  - For this purpose it is convenient to express P[error] in terms of the posterior P[error|x]

$$P[\text{error}] = \int_{-\infty}^{+\infty} P[\text{error} \mid x] P(x) dx$$

  - The optimal decision rule will minimize P[error|x] for every value of x, so that the integral above is minimized
  - At each point x', P[error|x'] is equal to $P[\omega_i|x']$ when we choose the other class ω_j
    - This is depicted in the following figure:



  - From the figure it becomes clear that, for any value of x', the Likelihood Ratio Test decision rule will always have a lower P[error|x']
    - Therefore, when we integrate over the real line, the LRT decision rule will yield a lower P[error]

> For any given problem, the minimum probability of error is achieved by the Likelihood Ratio Test decision rule. This probability of error is called the **Bayes Error Rate** and is the **BEST** any classifier can do.

The Bayes Risk (1)

- **So far we have assumed that the penalty of misclassifying a class ω_1 example as class ω_2 is the same as the reciprocal. In general, this is not the case:**
  - For example, misclassifying a cancer sufferer as a healthy patient is a much more serious problem than the other way around
- **This concept can be formalized in terms of a cost function C_{ij}**
  - C_{ij} represents the cost of choosing class ω_i when class ω_j is the true class
- **We define the Bayes Risk as the expected value of the cost**

$$\Re = E[C] = \sum_{i=1}^{2}\sum_{j=1}^{2} C_{ij} \cdot P[\text{choose } \omega_i \text{ and } x \in \omega_j] = \sum_{i=1}^{2}\sum_{j=1}^{2} C_{ij} \cdot P[x \in R_i \mid \omega_j] \cdot P[\omega_j]$$

- **What is the decision rule that minimizes the Bayes Risk?**
  - First notice that

$$P[x \in R_i \mid \omega_j] = \int_{R_i} P(x \mid \omega_j)dx$$

  - We can express the Bayes Risk as

$$\Re = \int_{R_1} \left[C_{11} \cdot P[\omega_1] \cdot P(x \mid \omega_1) + C_{12} \cdot P[\omega_2] \cdot P(x \mid \omega_2)\right]dx +$$
$$\int_{R_2} \left[C_{21} \cdot P[\omega_1] \cdot P(x \mid \omega_1) + C_{22} \cdot P[\omega_2] \cdot P(x \mid \omega_2)\right]dx$$

  - Then we note that, for either likelihood, one can write:

$$\int_{R_1} P(x \mid \omega_i)dx + \int_{R_2} P(x \mid \omega_i)dx = \int_{R_1 \cup R_2} P(x \mid \omega_i)dx = 1$$

The Bayes Risk (2)

- Merging the last equation into the Bayes Risk expression yields

$$
\begin{aligned}
\Re = {} & C_{11}P[\omega_1]\int_{R_1} P(x|\omega_1)dx + C_{12}P[\omega_2]\int_{R_1} P(x|\omega_2)dx + \\
& + C_{21}P[\omega_1]\int_{R_2} P(x|\omega_1)dx + C_{22}P[\omega_2]\int_{R_2} P(x|\omega_2)dx + \\
& + C_{21}P[\omega_1]\int_{R_1} P(x|\omega_1)dx + C_{22}P[\omega_2]\int_{R_1} P(x|\omega_2)dx + \\
& - C_{21}P[\omega_1]\int_{R_1} P(x|\omega_1)dx - C_{22}P[\omega_2]\int_{R_1} P(x|\omega_2)dx
\end{aligned}
$$

- Now we cancel out all the integrals over R_2

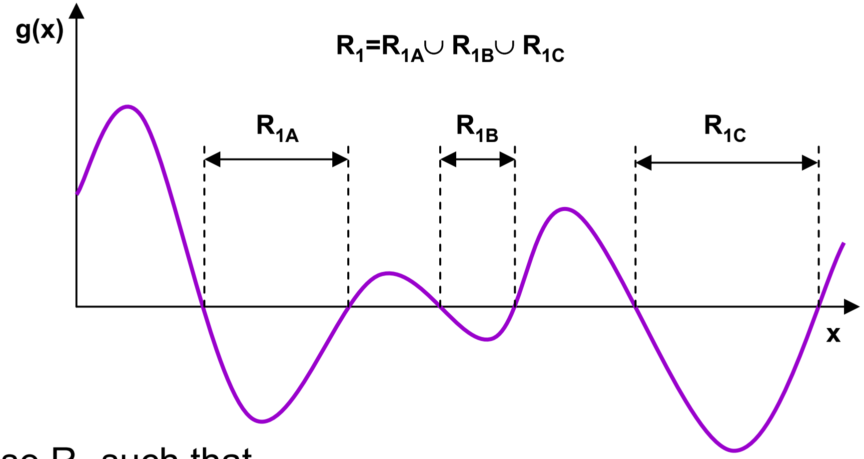
$$
\begin{aligned}
\Re = {} & C_{21}P[\omega_1] + C_{22}P[\omega_2] + \\
& + \underset{\overset{\vee}{0}}{(C_{12} - C_{22})}P[\omega_2]\int_{R_1} P(x|\omega_2)dx - \underset{\overset{\vee}{0}}{(C_{21} - C_{11})}P[\omega_1]\int_{R_1} P(x|\omega_1)dx
\end{aligned}
$$

- The first two terms are constant as far as our minimization is concerned since they do not depend on R_1, so we will be seeking a decision region R_1 that minimizes:

$$
\begin{aligned}
R_1 & = \arg\min\left\{\int_{R_1}\left[(C_{12} - C_{22})P[\omega_2]P(x|\omega_2) - (C_{21} - C_{11})P[\omega_1]P(x|\omega_1)\right]dx\right\} \\
& = \arg\min\left\{\int_{R_1} g(x)dx\right\}
\end{aligned}
$$

The Bayes Risk (3)

- **Let's forget about the actual expression of g(x) to develop some intuition for what kind of decision region R_1 we are looking for**
  - Intuitively, we will select for R_1 those regions that minimize the integral $\int_{R_1} g(x)dx$
    - In other words, those regions where g(x)<0



  - So we will choose R_1 such that

$$(C_{21}-C_{11})P[\omega_1]P(x\,|\,\omega_1)\overset{\omega_1}{>}(C_{12}-C_{22})P[\omega_2]P(x\,|\,\omega_2)$$

  - And rearranging

$$\boxed{\frac{P(x\,|\,\omega_1)}{P(x\,|\,\omega_2)}\underset{\omega_2}{\overset{\omega_1}{\gtrless}}\frac{(C_{12}-C_{22})}{(C_{21}-C_{11})}\frac{P[\omega_2]}{P[\omega_1]}}$$

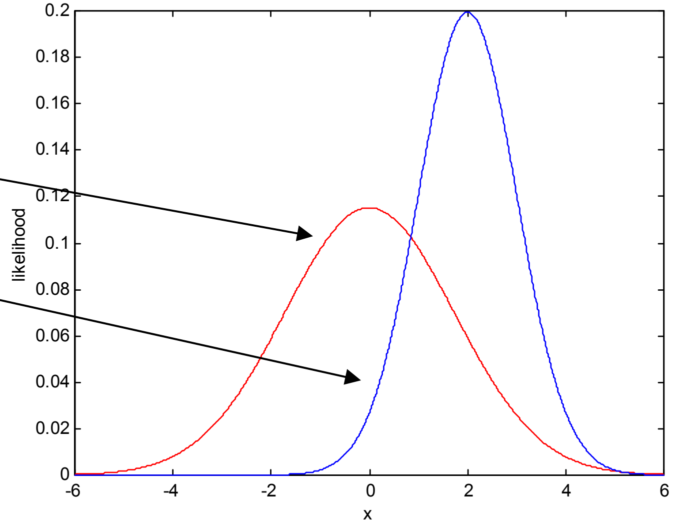
  - Therefore, minimization of the Bayes Risk also leads to a **Likelihood Ratio Test**

The Bayes Risk: an example

- **Consider a classification problem with two classes defined by the following likelihood functions**

$$P(x|\omega_1)=\frac{1}{\sqrt{2\pi}\sqrt{3}}e^{-\frac{1}{2}\frac{x^2}{3}}$$

$$P(x|\omega_2)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-2)^2}$$



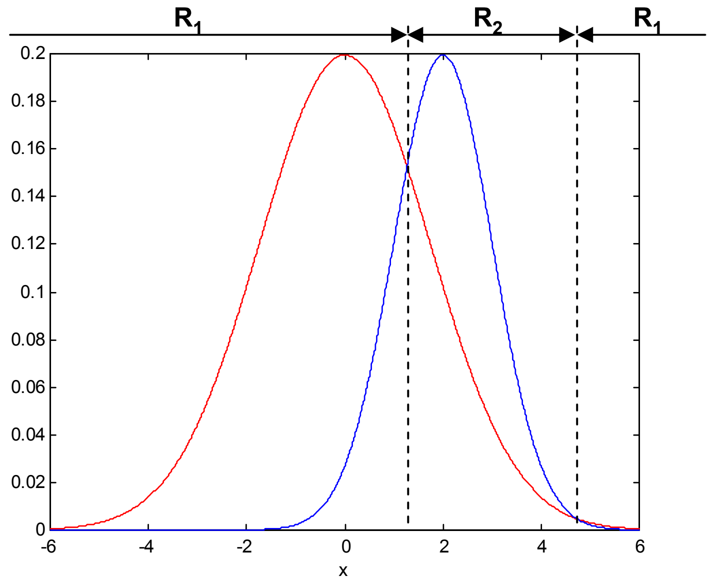
  - Sketch the two densities
  - What is the likelihood ratio?
  - Assume $P[\omega_1]=P[\omega_2]=0.5$, $C_{11}=C_{22}=0$, $C_{12}=1$ and $C_{21}=3^{1/2}$. Determine a decision rule that minimizes the probability of error

$$\Lambda(x)=\frac{\frac{1}{\sqrt{2\pi}\sqrt{3}}e^{-\frac{1}{2}\frac{x^2}{3}}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-2)^2}}\underset{\omega_2}{\overset{\omega_1}{\gtrless}}\frac{1}{\sqrt{3}}$$

$$\frac{e^{-\frac{1}{2}\frac{x^2}{3}}}{e^{-\frac{1}{2}(x-2)^2}}\underset{\omega_2}{\overset{\omega_1}{\gtrless}}1$$

$$-\frac{1}{2}\frac{x^2}{3}+\frac{1}{2}(x-2)^2\underset{\omega_2}{\overset{\omega_1}{\gtrless}}0$$

$$2x^2-12x+12\underset{\omega_2}{\overset{\omega_1}{\gtrless}}0\Rightarrow x=4.73,1.27$$



Variations of the Likelihood Ratio Test (1)

- **The LRT decision rule that minimizes the Bayes Risk is commonly called the <u>Bayes Criterion</u>**

$$\Lambda(x)=\frac{P(x|\omega_1)}{P(x|\omega_2)}\mathop{\gtrless}_{\omega_2}^{\omega_1}\frac{(C_{12}-C_{22})}{(C_{21}-C_{11})}\frac{P[\omega_2]}{P[\omega_1]}$$ **Bayes criterion**

- **Many times we will simply be interested in minimizing the probability of error, which is a special case of the Bayes Criterion that uses the so-called symmetrical or zero-one cost function. This version of the LRT decision rule is referred to as the <u>Maximum A Posteriori Criterion</u>, since it seeks to maximize the posterior $P(\omega_i|x)$**

$$C_{ij}=\begin{cases}0 & i=j\\ 1 & i\neq j\end{cases}\Rightarrow \Lambda(x)=\frac{P(x|\omega_1)}{P(x|\omega_2)}\mathop{\gtrless}_{\omega_2}^{\omega_1}\frac{P(\omega_2)}{P(\omega_1)}\Leftrightarrow\frac{P(\omega_1|x)}{P(\omega_2|x)}\mathop{\gtrless}_{\omega_2}^{\omega_1}1$$ **Maximum A Posteriori (MAP) Criterion**

- **Finally, for the case of equal priors $P[\omega_i]=1/2$, and the zero-one cost function the LTR decision rule is called the <u>Maximum Likelihood Criterion</u>, since it will minimize the likelihood $P(x|\omega_i)$**

$$\left.\begin{array}{l} C_{ij}=\begin{cases}0 & i=j\\ 1 & i\neq j\end{cases}\\ P(\omega_i)=\frac{1}{C}\quad\forall i\end{array}\right\}\Rightarrow \Lambda(x)=\frac{P(x|\omega_1)}{P(x|\omega_2)}\mathop{\gtrless}_{\omega_2}^{\omega_1}1$$ **Maximum Likelihood (ML) Criterion**

Variations of the Likelihood Ratio Test (2)

- **Two more decision rules are commonly cited in the related literature**
  - The **Neyman-Pearson Criterion**, used in Detection and Estimation Theory, which also leads to an LRT decision rule, fixes one class error probabilities, say $\varepsilon_1 < \alpha$, and seeks to minimize the other
    - For instance, for the sea-bass/salmon classification problem of Lecture 1, there may be some kind of government regulation that we must not misclassify more than 1% of salmon as sea bass
    - The Neyman-Pearson Criterion is very attractive since it does not require knowledge of priors and cost function
  - The **Minimax Criterion**, used in Game Theory, is derived from the Bayes criterion, and seeks to minimize the maximum Bayes Risk
    - The Minimax Criterion does nor require knowledge of the priors, but it needs a cost function
  - For more information on these methods, the reader is referred to "Detection, Estimation and Modulation Theory", by H.L. van Trees, the classical reference in this field

Minimum P[error] rule for multi-class problems

- **The decision rule that minimizes P[error] generalizes very easily to multi-class problems**
  - For clarity in the derivation, the probability of error is better expressed in terms of the probability of making a correct assignment

$$P[\text{error}] = 1 - P[\text{correct}]$$

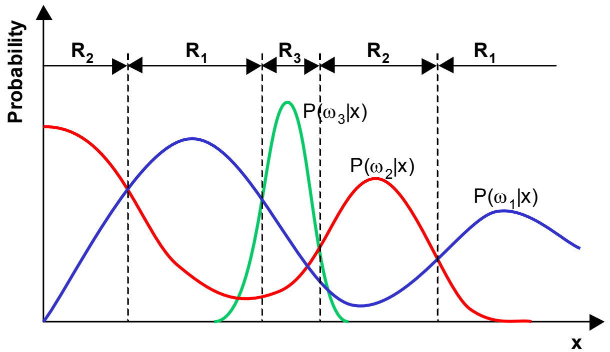
  - The probability of making a correct assignment is

$$P[\text{correct}] = \sum_{i=1}^{C} P(\omega_i) \int_{R_i} P(x|\omega_i)dx$$

  - The problem of minimizing P[error] is equivalent to that of maximizing P[correct]. Expressing P[correct] in terms of the posteriors:

$$P[\text{correct}] = \sum_{i=1}^{C} P(\omega_i) \int_{R_i} P(x|\omega_i)dx = \sum_{i=1}^{C} \int_{R_i} P(x|\omega_i)P(\omega_i)dx = \sum_{i=1}^{C} \underbrace{\int_{R_i} P(\omega_i|x)P(x)dx}_{\Im_i}$$

  - In order to maximize P[correct], we will have to maximize each of the integrals $\Im_i$. In turn, each integral $\Im_i$ will be maximized by choosing the class ω_i that yields the maximum $P[\omega_i|x]$
    $\Rightarrow$ we will define R_i to be the region where $P[\omega_i|x]$ is maximum



- **Therefore, the decision rule that minimizes P[error] is the MAP Criterion**

Minimum Bayes Risk for multi-class problems

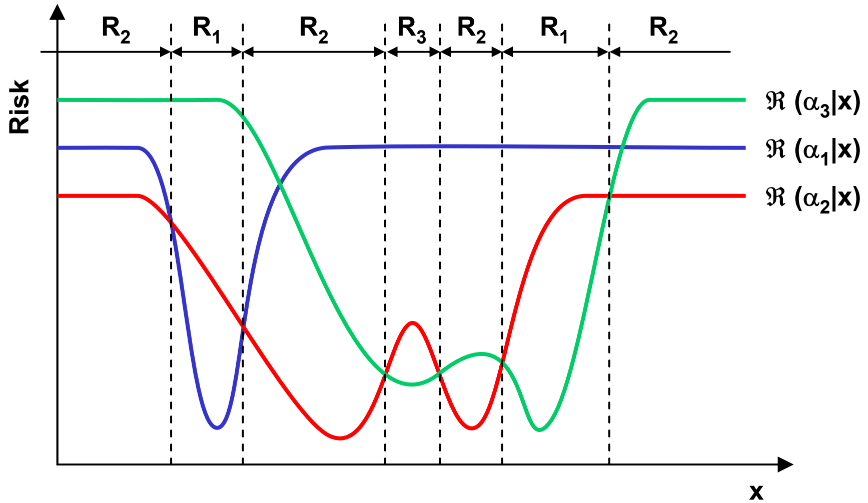
- **To determine which decision rule yields the minimum Bayes Risk for the multi-class problem we will use a slightly different formulation**
  - We will denote by α_i the decision to choose class ω_i,
  - We will denote by $\alpha(x)$ the overall decision rule that maps features x into classes ω_i: $\alpha(x)\rightarrow\{\alpha_1, \alpha_2, \ldots, \alpha_C\}$
- **The (conditional) risk $\Re(\alpha_i|x)$ of assigning a feature x to class ω_i is**

$$\Re(\alpha(x)\rightarrow\alpha_i) = \Re(\alpha_i \mid x) = \sum_{j=1}^{C} C_{ij} P(\omega_j \mid x)$$

- **And the Bayes Risk associated with the decision rule $\alpha(x)$ is**

$$\Re(\alpha(x)) = \int \Re(\alpha(x) \mid x) P(x) dx$$

- **In order to minimize this expression, we will have to minimize the conditional risk $\Re(\alpha(x)|x)$ at each point x in the feature space, which in turn is equivalent to choosing ω_i such that $\Re(\alpha_i|x)$ is minimum**

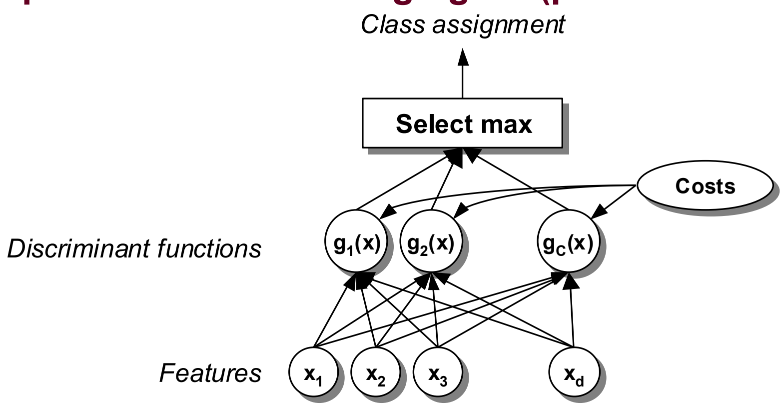


Discriminant functions

- **All the decision rules we have presented in this lecture have the same structure**
  - At each point x in feature space choose class ω_i which maximizes (or minimizes) some measure $\mathbf{g_i(x)}$
- **This structure can be formalized with a set of discriminant functions $g_i(x)$, i=1..C, and the following decision rule**

$$\boxed{\text{"assign x to class } \omega_i \text{ if } g_i(x) > g_j(x) \;\; \forall j \neq i\text{"}}$$

- **Therefore, we can visualize the decision rule as a network or machine that computes C discriminant functions and selects the category corresponding to the largest discriminant. Such network is depicted in the following figure (presented already in Lecture 1)**



- **Finally, we express the three basic decision rules: Bayes, MAP and ML in terms of Discriminant Functions to show the generality of this formulation**

Criterion	Discriminant Function
Bayes	$g_i(x)=-\Re(\alpha_i\|x)$
MAP	$g_i(x)=P(\omega_i\|x)$
ML	$g_i(x)=P(x\|\omega_i)$