Chapter 1

1. In case of an accident, there is a high chance of getting lost. The transportation cost is very high each time. However, if the infrastructure is set once, it will be very easy to use it repeatedly. Time for wireless transmission is negligible as signals travel at the speed of light.

2. Advantages of bursty data communication

   (a) Pulses are made very narrow, so multipaths are resolvable

   (b) The transmission device needs to be switched on for less time.

   Disadvantages

   (a) Bandwidth required is very high

   (b) Peak transmit power can be very high.

3. $P_b = 10^{-12}$
   $\frac{1}{2\overline{\gamma}} = 10^{-12}$
   $\overline{\gamma} = \frac{10^{12}}{2} = 5 \times 10^{11}$ (very high)

4. Geo: 35,786 Km above earth $\Rightarrow RTT = \frac{2\times35786\times10^3}{c} = 0.2386s$
   Meo: 8,000- 20,000 Km above earth $\Rightarrow RTT = \frac{2\times8000\times10^3}{c} = 0.0533s$
   Leo: 500- 2,000 Km above earth $\Rightarrow RTT = \frac{2\times500\times10^3}{c} = 0.0033s$
   Only Leo satellites as delay $= 3.3ms < 30ms$

5.

6. optimum no. of data user = d
   optimum no. of voice user = v
   Three different cases:
   Case 1: d=0, v=6
   $\Rightarrow revenue = 60.80.2 = 0.96$

   Case 2: d=1, v=3
   revenue = [prob. of having one data user]×(revenue of having one data user)
   + [prob. of having two data user]×(revenue of having two data user)
   + [prob. of having one voice user]×(revenue of having one voice user)
   + [prob. of having two voice user]×(revenue of having two voice user)
   + [prob. of having three or more voice user]×(revenue in this case)

$$\Rightarrow \quad 0.5^2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \times \$1 + 0.5^2 \times \$1 + \begin{pmatrix} 6 \\ 1 \end{pmatrix} 0.8 \times 0.2^5 \times \$0.2 + \begin{pmatrix} 6 \\ 2 \end{pmatrix} 0.8^2 \times 0.2^4 \times \$0.4 +$$

$$\left[1 - \begin{pmatrix} 6 \\ 1 \end{pmatrix} 0.8 \times 0.2^5 \times \$0.2 - \begin{pmatrix} 6 \\ 2 \end{pmatrix} 0.8^2 \times 0.2^4 \times \$0.4\right] \times \$0.6$$

$$\Rightarrow \$1.35$$

   Case 3: d=2, v=0
   revenue $= 2 \times 0.5 = \$1$
   So the best case is case 2, which is to allocate 60kHz to data and 60kHz to voice.

7.

8. 1. Hand-off becomes a big problem.
   2. Inter-cell interference is very high and should be mitigated to get reasonable SINR.
   3. Infrastructure cost is another problem.

9. Smaller the reuse distance, larger the number of users who can use the same system resource and so capacity (data rate per unit bandwidth) increases.

10. (a) 100 cells, 100 users/cell $\Rightarrow$ 10,000 users

    (b) 100 users/cell $\Rightarrow$ 2500 cells required
    $\frac{100km^2}{Area/cell} = 2500 cells \Rightarrow \frac{Area}{cell} = .04km^2$

    (c) From Rappaport or iteration of formula, we get that 100 $\frac{channels}{cell} \Rightarrow 89 \frac{channels}{cell}$ $@P_b = .02$
    Each subscriber generates $\frac{1}{30}$ of an Erlang of traffic.
    Thus, each cell can support $30 \times 89 = 2670$ subscribers
    Macrocell: $2670 \times 100 \Rightarrow 267,000$ subscribers
    Microcell: 6,675,000 subscribers

    (d) Macrocell: \$50 M
    Microcell: \$1.25 B

    (e) Macrocell: \$13.35 M/month $\Rightarrow$ 3.75 months *approx* 4 months to recoop
    Microcell: \$333.75 M/month $\Rightarrow$ 3.75 months *approx* 4 months to recoop

11. One CDPD line : 19.2Kbps
    average $W_{imax} \sim 40Mbps$
    $\therefore$ number of CDPD lines $\sim 2 \times 10^3$

Chapter 2

1.

$$
\begin{aligned}
P_r &= P_t\left[\frac{\sqrt{G_l}\lambda}{4\pi d}\right]^2 \quad \lambda = c/f_c = 0.06 \\
10^{-3} &= P_t\left[\frac{\lambda}{4\pi 10}\right]^2 \Rightarrow P_t = 4.39KW \\
10^{-3} &= P_t\left[\frac{\lambda}{4\pi 100}\right]^2 \Rightarrow P_t = 438.65KW
\end{aligned}
$$

Attenuation is very high for high frequencies

2. d= 100m
h_t = 10m
h_r = 2m
delay spread $= \tau = \frac{x+x'-l}{c} = 1.33\times$

3. $\Delta\phi = \frac{2\pi(x'+x-l)}{\lambda}$

$$
\begin{aligned}
x'+x-l &= \sqrt{(h_t+h_r)^2+d^2} - \sqrt{(h_t-h_r)^2+d^2} \\
&= d\left[\sqrt{\left(\frac{h_t+h_r}{d}\right)^2+1} - \sqrt{\left(\frac{h_t-h_r}{d}\right)^2+1}\right]
\end{aligned}
$$

$d \gg h_t, h_r,$ we need to keep only first order terms

$$
\begin{aligned}
&\sim d\left\{\left[\frac{1}{2}\sqrt{\left(\frac{h_t+h_r}{d}\right)^2+1}\right] - \left[\frac{1}{2}\sqrt{\left(\frac{h_t-h_r}{d}\right)^2+1}\right]\right\} \\
&= \frac{2(h_t+h_r)}{d} \\
\Delta\phi &\sim \frac{2\pi}{\lambda}\frac{2(h_t+h_r)}{d}
\end{aligned}
$$

4. Signal nulls occur when $\Delta\phi = (2n+1)\pi$

$$\begin{aligned}
\frac{2\pi(x'+x-l)}{\lambda} &= (2n+1)\pi \\
\frac{2\pi}{\lambda}\left[\sqrt{(h_t+h_r)^2+d^2} - \sqrt{(h_t-h_r)^2+d^2}\right] &= \pi(2n+1) \\
\sqrt{(h_t+h_r)^2+d^2} - \sqrt{(h_t-h_r)^2+d^2} &= \frac{\lambda}{2}(2n+1) \\
\text{Let m } = (2n+1) & \\
\sqrt{(h_t+h_r)^2+d^2} &= m\frac{\lambda}{2} + \sqrt{(h_t-h_r)^2+d^2} \\
\text{square both sides} & \\
(h_t+h_r)^2+d^2 &= m^2\frac{\lambda^2}{4} + (h_t-h_r)^2 + d^2 + m\lambda\sqrt{(h_t-h_r)^2+d^2} \\
x = (h_t+h_r)^2,\ y = (h_t-h_r)^2,\ x-y = 4h_th_r & \\
x &= m^2\frac{\lambda^2}{4} + y + m\lambda\sqrt{y+d^2} \\
\Rightarrow d &= \sqrt{\left[\frac{1}{m\lambda}\left(x - m^2\frac{\lambda^2}{4} - y\right)\right]^2 - y} \\
d &= \sqrt{\left(\frac{4h_th_r}{(2n+1)\lambda} - \frac{(2n+1)\lambda}{4}\right)^2 - (h_t-h_r)^2} \quad ,n \in \mathbb{Z}
\end{aligned}$$

5. $h_t = 20m$
   $h_r = 3m$
   $f_c = 2GHz \quad \lambda = \frac{c}{f_c} = 0.15$
   $d_c = \frac{4h_th_r}{\lambda} = 1600m = 1.6Km$
   This is a good radius for suburban cell radius as user density is low so cells can be kept fairly large. Also, shadowing is less due to fewer obstacles.

6. Think of the building as a plane in $\mathbb{R}^3$
   The length of the normal to the building from the top of Tx antenna $= h_t$
   The length of the normal to the building from the top of Rx antenna $= h_r$
   In this situation the 2 ray model is same as that analyzed in the book.

7. $h(t) = \alpha_1\delta(t-\tau) + \alpha_2\delta(t-(\tau+0.22\mu s))$
   $G_r = G_l = 1$
   $h_t = h_r = 8m$
   $f_c = 900MHz, \lambda = c/f_c = 1/3$
   $R = -1$

$$\begin{aligned}
\text{delay spread} = \frac{x+x'-l}{c} &= 0.022 \times 10^{-6}s \\
\Rightarrow \frac{2\sqrt{8^2+\left(\frac{d}{2}\right)^2} - d}{c} &= 0.022 \times 10^{-6}s \\
\Rightarrow d = 16.1m & \\
\therefore \tau = \frac{d}{c} = 53.67ns &
\end{aligned}$$

$$\alpha_1 = \left(\frac{\lambda}{4\pi}\frac{\sqrt{G_l}}{l}\right)^2 = 2.71 \times 10^{-6}$$

$$\alpha_2 = \left(\frac{\lambda}{4\pi}\frac{\sqrt{RG_r}}{x+x'}\right)^2 = 1.37 \times 10^{-6}$$

8. A program to plot the figures is shown below. The power versus distance curves and a plot of the phase difference between the two paths is shown on the following page. From the plots it can be seen that as G_r (gain of reflected path) is decreased, the asymptotic behavior of P_r tends toward d^{-2} from d^{-4}, which makes sense since the effect of reflected path is reduced and it is more like having only a LOS path. Also the variation of power before and around dc is reduced because the strength of the reflected path decreases as G_r decreases. Also note that the the received power actually increases with distance up to some point. This is because for very small distances (i.e. d = 1), the reflected path is approximately two times the LOS path, making the phase difference very small. Since R = -1, this causes the two paths to nearly cancel each other out. When the phase difference becomes 180 degrees, the first local maxima is achieved. Additionally, the lengths of both paths are initially dominated by the difference between the antenna heights (which is 35 meters). Thus, the powers of both paths are roughly constant for small values of d, and the dominant factor is the phase difference between the paths.

```
clear all;
close all;
ht=50;
hr=15;
f=900e6;
c=3e8;
lambda=c/f;
GR=[1,.316,.1,.01];
Gl=1;
R=-1;
counter=1;
figure(1);
d=[1:1:100000];
l=(d.^2+(ht-hr)^2).^.5;
r=(d.^2+(ht+hr)^2).^.5;
phd=2*pi/lambda*(r-l);
dc=4*ht*hr/lambda;
dnew=[dc:1:100000];

for counter = 1:1:4,
    Gr=GR(counter);
    Vec=Gl./l+R*Gr./r.*exp(phd*sqrt(-1));
    Pr=(lambda/4/pi)^2*(abs(Vec)).^2;
    subplot(2,2,counter);
    plot(10*log10(d),10*log10(Pr)-10*log10(Pr(1)));
    hold on;
    plot(10*log10(dnew),-20*log10(dnew));
    plot(10*log10(dnew),-40*log10(dnew));
end
hold off
```

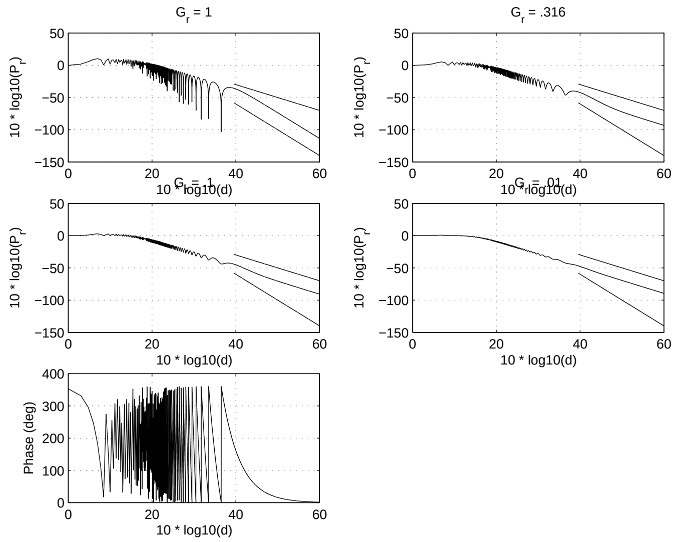

Figure 1: Problem 8

9. As indicated in the text, the power fall off with distance for the 10-ray model is d^{-2} for relatively large distances

10. The delay spread is dictated by the ray reaching last $d = \sqrt{(500/6)^2 + 10^2} = 83.93m$
Total distance = 6d = 503.59m
$\tau_0 = 503.59/c = 1.68\mu s$
L.O.S ray d = 500m
$\tau_0 = 500/c = 1.67\mu s$
$\therefore$ delay spread = $0.01\mu s$

11. $f_c = 900MHz$
$\lambda = 1/3m$
$G = 1$ radar cross section $20dBm^2 = 10\log_1 0\sigma \Rightarrow \sigma = 100$
d=1 , $s = s' = \sqrt{(0.5d)^2 + (0.5d)^2} = d\sqrt{0.5} = \sqrt{0.5}$
Path loss due to scattering

$$\frac{P_r}{P_t} = \left[\frac{\lambda\sqrt{G\sigma}}{(4\pi)^{3/2}ss'}\right]^2 = 0.0224 = -16.498dB$$

Path loss due to reflection (using 2 ray model)

$$\frac{P_r}{P_t} = \left(\frac{R\sqrt{G}}{s+s'}\right)^2\left(\frac{\lambda}{4\pi}\right)^2 = 3.52\times 10^{-4} = -34.54dB$$

$d = 10$ $P_{scattering} = -56.5dB$ $P_{reflection} = -54.54dB$
$d = 100$ $P_{scattering} = -96.5dB$ $P_{reflection} = -74.54dB$
$d = 1000$ $P_{scattering} = -136.5dB$ $P_{reflection} = -94.54dB$
Notice that scattered rays over long distances result in tremendous path loss

12.

$$P_r = P_t K\left(\frac{d_0}{d}\right)^\gamma \quad \rightarrow \text{simplified}$$

$$P_r = P_t\left(\frac{\sqrt{G_l}}{4\pi}\right)^2\left(\frac{\lambda}{d}\right)^2 \quad \rightarrow \text{free space}$$

$\therefore$ when $K = \left(\frac{\sqrt{G_l}}{4\pi}\right)^2$ and $d_0 = \lambda$
The two models are equal.

13. $P_{noise} = -160dBm$
$f_c = 1GHz, d_0 = 1m,\ K = (\lambda/4\pi d_0)^2 = 5.7 \times 10^{-4}, \lambda = 0.3, \gamma = 4$
We want $SNR_{recd} = 20dB = 100$
$\because$ Noise power is 10^{-19}

$$P = P_t K \left(\frac{d_0}{d}\right)^\gamma$$

$$10^{-17} = 10K \left(\frac{0.3}{d}\right)^4$$

$$d \leq 260.7m$$

14. d = distance between cells with reused freq
p = transmit power of all the mobiles

$$\left(\frac{S}{I}\right)_{\text{uplink}} \geq 20dB$$

(a) Min. S/I will result when main user is at A and Interferers are at B
d_A = distance between A and base station #1 = $\sqrt{2}km$ d_B = distance between B and base station #1 = $\sqrt{2}km$

$$\left(\frac{S}{I}\right)_{\text{min}} = \frac{P\left[\frac{G\lambda}{4\pi d_A}\right]^2}{2P\left[\frac{G\lambda}{4\pi d_B}\right]^2} = \frac{d_B^2}{2d_A^2} = \frac{(d_{min}-1)^2}{4} = 100$$

$\Rightarrow d_{min} - 1 = 20km \Rightarrow d_{min} = 21km$ since integer number of cells should be accommodated in distance d $\Rightarrow d_{min} = 22km$

(b)

$$\frac{P_\gamma}{P_u} = k\left[\frac{d_0}{d}\right]^\gamma \Rightarrow \left(\frac{S}{I}\right)_{\text{min}} = \frac{Pk\left[\frac{d_0}{d_A}\right]^\gamma}{2Pk\left[\frac{d_0}{d_B}\right]^\gamma} =$$

$$\frac{1}{2}\left[\frac{d_B}{d_A}\right]^\gamma = \frac{1}{2}\left[\frac{d_{min}-1}{\sqrt{2}}\right]^\gamma = \frac{1}{2}\left[\frac{d_{min}-1}{\sqrt{2}}\right]^3 = 100$$

$\Rightarrow d_{min} = 9.27 \Rightarrow$ with the same argument $\Rightarrow d_{min} = 10km$

(c)

$$\left(\frac{S}{I}\right)_{\text{min}} = \frac{k\left[\frac{d_0}{d_A}\right]_A^\gamma}{2k\left[\frac{d_0}{d_B}\right]_B^\gamma} = \frac{(d_{min}-1)^4}{0.04} = 100$$

$\Rightarrow d_{min} = 2.41km \Rightarrow$ with the same argument $d_{min} = 4km$

15. $f_c = 900MHz, h_t = 20m, h_r = 5m, d = 100m$

Large urban city	$PL_{largecity} = 353.52dB$
small urban city	$PL_{smallcity} = 325.99dB$
suburb	$PL_{suburb} = 207.8769dB$
rural area	$PL_{ruralarea/countryside} = 70.9278dB$

As seen , path loss is higher in the presence of multiple reflectors, diffractors and scatterers

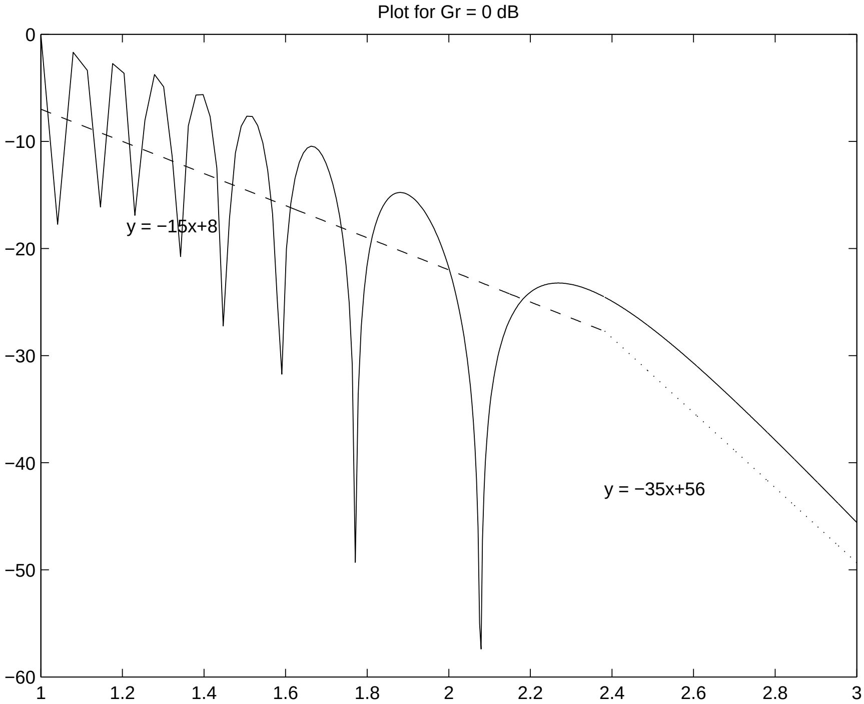

Figure 2: Problem 16

16. Piecewise linear model for 2-path model. See Fig 2

17. $P_r = P_t - P_L(d) - \sum_i^3 FAF_i - \sum_j^2 PAF_j$
FAF =(5,10,6),   PAF =(3.4,3.4)

$$P_L(d) K \left(\frac{d_0}{d}\right)_0^\gamma = 10^{-8} = -8dB$$

$$-110 = P_t - 80 - 5 - 10 - 6 - 3.4 - 3.4$$

$$\Rightarrow P_t = -2.2dBm$$

18. (a) $\frac{P_r}{P_t} dB = 10 \log_{10} K - 10r \log_1 0 \frac{d}{d_0}$
using least squares we get
$10 \log_{10} K = -29.42dB$
$\gamma = 4$

(b) PL(2Km) $= 10 \log_{10} K - 10r \log_{10} d = -161.76dB$

(c) Receiver power can be assumed to be Gaussian with variance $\sigma^2_{\psi dB}$

$$X \sim N(0, \sigma^2_{\psi dB})$$

$$Prob(X < -10) = Prob\left(\frac{X}{\sigma_{\psi dB}} < \frac{-10}{\sigma_{\psi dB}}\right) = 6.512 \times 10^{-4}$$

19. Assume free space path loss parameters
$f_c = 900MHz \rightarrow \lambda = 1/3m$
$\sigma_{\psi dB} = 6$
$SNR_{recd} = 15dB$
$P_t = 1W$

$g = 3dB$
$P_{noise} = -40dBm \Rightarrow P_{recvd} = -55dB$
Suppose we choose a cell of radius d

$$\begin{aligned} \mu(d) &= P_{recvd}(\text{due to path loss alone}) \\ &= P_t\left(\frac{\sqrt{G_l}\lambda}{4\pi d}\right)^2 = \frac{1.4\times 1-^{-3}}{d^2} \end{aligned}$$

$$\mu_{dB} = 10\log_1 0(\mu(d))$$
$$P(P_{recd}(d) > -55) = 0.9$$

$$\begin{aligned} P\left(\frac{P_{recd}(d)-\mu_{dB}}{\sigma_{\psi dB}} > \frac{-55-\mu_{dB}}{6}\right) &= 0.9 \\ \Rightarrow \frac{-55-\mu_{dB}}{6} &= -1.282 \\ \Rightarrow \mu_{dB} &= -47.308 \\ \Rightarrow \mu(d) &= 1.86\times 10^{-5} \\ \Rightarrow d &= 8.68m \end{aligned}$$

20. MATLAB CODE

```
Xc = 20;
ss = .01;
y = wgn(1,200*(1/ss));
for i = 1:length(y)
    x(i) = y(i);
    for j = 1:i
        x(i) = x(i)+exp(-(i-j)/Xc)*y(j);
    end
end
```

21. Outage Prob. = Prob. [received $power_{dB} \leqslant Tp_{dB}$]
Tp = 10dB

(a)

$$outageprob. = 1 - Q\left(\frac{Tp-\mu_\psi}{\sigma_\psi}\right) = 1 - Q\left(\frac{-5}{8}\right) = Q\left(\frac{5}{8}\right) = 26\%$$

(b) $\sigma_\psi = 4$dB, outage prob $< 1\% \Rightarrow$

$$Q\left(\frac{Tp-\mu_\psi}{\sigma_\psi}\right) > 99\% \Rightarrow \frac{Tp-\mu_\psi}{\sigma_\psi} < -2.33 \Rightarrow$$

$$\mu_\psi \geq 19.32dB$$

(c)

$$\sigma_\psi = 12dB, \frac{Tp-\mu_\psi}{\sigma_\psi} < -6.99 \Rightarrow \mu_\psi \geq 37.8dB$$

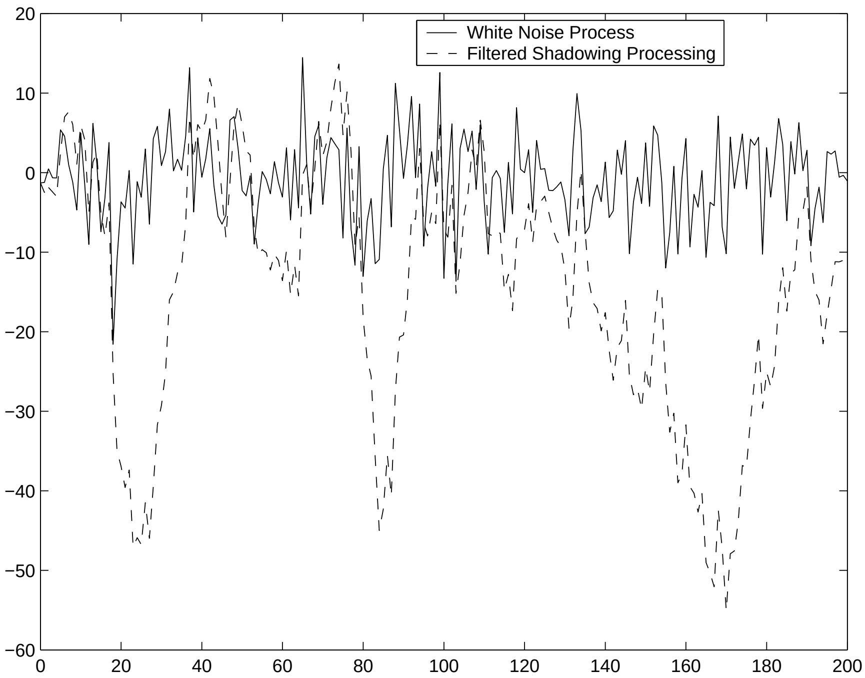

Figure 3: Problem 20

(d) For mitigating the effect of shadowing, we can use macroscopic diversity. The idea in macroscopic diversity is to send the message from different base stations to achieve uncorrelated shadowing. In this way the probability of power outage will be less because both base stations are unlikely to experience an outage at the same time, if they are uncorrelated.

22.

$$C = \frac{2}{R^2} \int_{r=0}^{R} rQ\left(a + b\ln\frac{r}{R}\right) dr$$

To perform integration by parts, we let $du = rdr$ and $v = Q\left(a + b\ln\frac{r}{R}\right)$. Then $u = \frac{1}{2}r^2$ and

$$dv = \frac{\partial}{\partial r}Q\left(a + b\ln\frac{r}{R}\right) = \frac{\partial}{\partial x}Q(x)|_{x=a+b\ln(r/R)}\frac{\partial}{\partial r}\left(a + b\ln\frac{r}{R}\right) = \frac{-1}{\sqrt{2\pi}}\exp(-k^2/2)\frac{b}{r}dr. \tag{1}$$

where $k = a + b\ln\frac{r}{R}$. Then we get

$$\begin{aligned}
C &= \frac{2}{R^2}\left[\frac{1}{2}r^2 Q\left(a+b\ln\frac{r}{R}\right)\right]_{r=0}^{R} + \frac{2}{R^2}\int\limits_{r=0}^{R}\frac{1}{2}r^2\frac{1}{\sqrt{2\pi}}\exp(-k^2/2)\frac{b}{r}dr && (2)\\
&= Q(a) + \frac{1}{R^2}\int\limits_{r=0}^{R} r^2\frac{1}{\sqrt{2\pi}}\exp(-k^2/2)\frac{b}{r}dr && (3)\\
&= Q(a) + \frac{1}{R^2}\int\limits_{r=0}^{R}\frac{1}{\sqrt{2\pi}}R^2\exp\left(\frac{2(k-a)}{b}\right)\exp(-k^2/2)\frac{b}{r}dr && (4)\\
&= Q(a) + \int\limits_{k=-\infty}^{a}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-k^2}{2}+\frac{2k}{b}-\frac{2a}{b}\right)dk && (5)\\
&= Q(a) + \exp\left(\frac{-2a}{b}+\frac{2}{b^2}\right)\int\limits_{k=-\infty}^{a}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(k-\frac{2}{b})^2\right)dk && (6)\\
&= Q(a) + \exp\left(\frac{2-2ab}{b^2}\right)\left[1-Q\left(\frac{a-\frac{2}{b}}{1}\right)\right] && (7)\\
&= Q(a) + \exp\left(\frac{2-2ab}{b^2}\right)Q\left(\frac{2-ab}{b}\right) && (8)\\
& && (9)
\end{aligned}$$

Since $Q(-x) = 1 - Q(x)$.

23. $\gamma = 3$
$d_0 = 1$
$k = 0dB$
$\sigma = 4dB$
$R = 100m$
$P_t = 80mW P_{min} = -100dBm = -130dB$

$$\overline{P_\gamma}(R) = P_t K\left(\frac{d_0}{d}\right)^\gamma = 80\times 10^{-9} = -70.97dB$$

$$a = \frac{P_{min} - \overline{P_\gamma}(R)}{\sigma} = 14.7575$$

$$b = \frac{10\gamma\log_{10}e}{\sigma_{\psi dB}} = 3.2572$$

$$c = Q(a) + e^{\frac{2-2ab}{b^2}}Q\left(\frac{2-2ab}{b}\right) \simeq 1$$

24. $\gamma = 6$
$\sigma = 8$

$$\overline{P_\gamma}(R) = 20 + P_{min}$$

$$a = -20/8 = -2.5$$

$$b = \frac{10\times 6\times\log_{10}e}{8} = 20.3871$$

$$c = 0.9938$$

25.

$\gamma/\sigma_{\psi_{\text{dB}}}$	2	4	6
4	0.7728	0.8587	0.8977
8	0.6786	0.7728	0.8255
12	0.6302	0.7170	0.7728

Since $\overline{P_r}(r) \geq P_{min}$ for all $r \leq R$, the probability of non-outage is proportional to $Q\left(\frac{-1}{\sigma}\right)$, and thus decreases as a function of σ. Therefore, C decreases as a function of σ. Since the average power at the boundary of the cell is fixed, C increases with γ, because it forces higher transmit power, hence more received power at $r < R$. Due to these forces, we have minimum coverage when $\gamma = 2$ and $\sigma = 12$. By a similar argument, we have maximum coverage when $\gamma = 6$ and $\sigma = 4$. The same can also be seen from this figure:

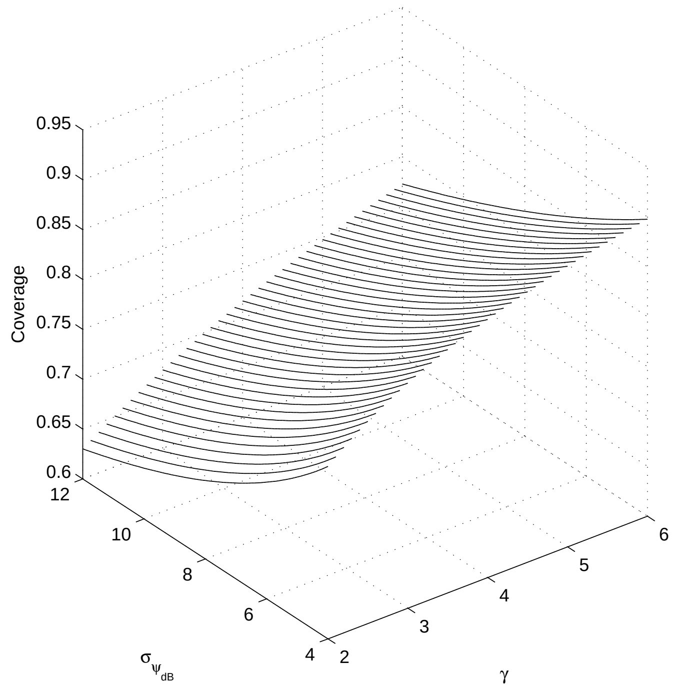

Figure 4: Problem 25

The value of coverage for middle point of typical values i.e. $\gamma = 4$ and $\sigma = 8$ can be seen from the table or the figure to be 0.7728.

Chapter 3

1. $d = vt$
   $r + r' = d + \frac{2h^2}{d}$
   Equivalent low-pass channel impulse response is given by
   $$c(\tau, t) = \alpha_0(t)e^{-j\phi_0(t)}\delta(\tau - \tau_0(t)) + \alpha_1(t)e^{-j\phi_1(t)}\delta(\tau - \tau_1(t))$$
   $\alpha_0(t) = \frac{\lambda\sqrt{G_l}}{4\pi d}$ with $d = vt$
   $\phi_0(t) = 2\pi f_c \tau_0(t) - \phi_{D_0}$
   $\tau_0(t) = d/c$
   $\phi_{D_0} = \int_t 2\pi f_{D_0}(t)dt$
   $f_{D_0}(t) = \frac{v}{\lambda}\cos\theta_0(t)$
   $\theta_0(t) = 0 \ \forall t$
   $\alpha_1(t) = \frac{\lambda R\sqrt{G_l}}{4\pi(r+r')} = \frac{\lambda R\sqrt{G_l}}{4\pi(d+\frac{2h^2}{d})}$ with $d = vt$
   $\phi_1(t) = 2\pi f_c \tau_1(t) - \phi_{D_1}$
   $\tau_1(t) = (r + r')/c = (d + \frac{2h^2}{d})/c$
   $\phi_{D_1} = \int_t 2\pi f_{D_1}(t)dt$
   $f_{D_1}(t) = \frac{v}{\lambda}\cos\theta_1(t)$
   $\theta_1(t) = \pi - \arctan\frac{h}{d/2} \ \forall t$

2. For the 2 ray model:
   $$\tau_0 = \frac{l}{c}$$
   $$\tau_1 = \frac{x + x'}{c}$$
   $$\therefore \text{delay spread}(T_m) = \frac{x + x' - l}{c} = \frac{\sqrt{(h_t + h_r)^2 + d^2} - \sqrt{(h_t - h_r)^2 + d^2}}{c}$$
   when $d \gg (h_t + h_r)$
   $$T_m = \frac{1}{c}\frac{2h_t h_r}{d}$$
   $$h_t = 10m,\ h_r = 4m,\ d = 100m$$
   $$\therefore T_m = 2.67 \times 10^{-9}s$$

3. Delay for LOS component $= \tau_0 = 23$ ns
   Delay for First Multipath component $= \tau_1 = 48$ ns
   Delay for Second Multipath component $= \tau_2 = 67$ ns

   $\tau_c =$ Delay for the multipath component to which the demodulator synchronizes.
   $$T_m = \max_m \tau_m - \tau_c$$
   So, when $\tau_c = \tau_0$, $T_m = 44$ ns. When $\tau_c = \tau_1$, $T_m = 19$ ns.

4. $f_c = 10^9 Hz$
   $\tau_{n,min} = \frac{10}{3\times 10^8} s$
   $\therefore \min f_c \tau_n = \frac{10^{10}}{3\times 10^8} = 33 \gg 1$

5. Use CDF strategy.

$$F_z(z) = P[x^2+y^2 \leq z^2] = \int\int_{x^2+y^2\leq z^2} \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y)^2}{2\sigma^2}} dxdy = \int_0^{2\pi}\int_0^z \frac{1}{2\pi\sigma^2} e^{\frac{-r^2}{2\sigma^2}} rdrd\theta = 1 - e^{\frac{-z^2}{2\sigma^2}} (z \geq 0)$$

$$\frac{df_z(z)}{dz} = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} \rightarrow Rayleigh$$

   For Power:

$$F_{z^2}(z) = P[Z \leq \sqrt{z}] = 1 - e^{\frac{-z^2}{2\sigma^2}}$$

$$f_z(z) = \frac{1}{2\sigma^2} e^{\frac{-z}{2\sigma^2}} \rightarrow Exponential$$

6. For Rayleigh fading channel

$$Pr(P_r < P_0) = 1 - e^{-P_0/2\sigma^2}$$

$$2\sigma^2 = -80dBm,\ P_0 = -95dBm,\ Pr(P_r < P_0) = 0.0311$$

$$2\sigma^2 = -80dBm,\ P_0 = -90dBm,\ Pr(P_r < P_0) = 0.0952$$

7. For Rayleigh fading channel

$$P_{outage} = 1 - e^{-P_0/2\sigma^2}$$

$$0.01 = 1 - e^{-P_0/P_r}$$

$$\therefore P_r = -60dBm$$

8. $2\sigma^2$ = -80dBm = 10^{-11}
   Target Power P_0 = -80 dBm = 10^{-11}
   Avg. power in LOS component = s^2 = -80dBm = 10^{-11}

$$Pr[z^2 \leq 10^{-11}] = Pr[z \leq \frac{10^{-5}}{\sqrt{10}}]$$

   Let $z_0 = \frac{10^{-5}}{\sqrt{10}}$

$$= \int_0^{z_0} \frac{z}{\sigma^2} e^{-\frac{-(z^2+s^2)}{2\sigma^2}} I_0\left(\frac{zs}{\sigma^2}\right) dz, \quad z \geq 0$$

$$= 0.3457$$

   To evaluate this, we use Matlab and $I_0(x)$ = besseli(0,x). Sample Code is given:

```
clear P0 = 1e-11; s2 = 1e-11; sigma2 = (1e-11)/2; z0 =
sqrt(1e-11); ss = z0/1e7; z = [0:ss:z0]; pdf =
(z/sigma2).*exp(-(z.^2+s2)/(2*sigma2)).*besseli(0,z.*(sqrt(s2)/sigma2));
int_pr = sum(pdf)*ss;
```

9. CDF of Ricean distribution is

$$F_Z^{\text{Ricean}}(z) = \int_0^z p_Z^{\text{Ricean}}(z)$$

where

$$p_Z^{\text{Ricean}}(z) = \frac{2z(K+1)}{Pr}\exp\left[-K - \frac{(K+1)z^2}{Pr}\right] I_0\left(2z\sqrt{\frac{K(K+1)}{Pr}}\right), \quad z \geq 0$$

For the Nakagami-m approximation to Ricean distribution, we set the Nakagami m parameter to be $(K+1)^2/(2K+1)$. CDF of Nakagami-m distribution is

$$F_Z^{\text{Nakagami-m}}(z) = \int_0^z p_Z^{\text{Nakagami-m}}(z)$$

where

$$p_Z^{\text{Nakagami-m}}(z) = \frac{2m^m z^{2m-1}}{\Gamma(m)Pr^m}\exp\left[\frac{-mz^2}{Pr}\right], \quad z \geq 0, \quad m \geq 0.5$$

We need to plot the two CDF curves for K = 1,5,10 and Pr =1 (we can choose any value for Pr as it is the same for both the distributions and our aim is to compare them). Sample code is given:

```
z = [0:0.01:3]; K = 10; m = (K+1)^2/(2*K+1); Pr = 1; pdfR =
((2*z*(K+1))/Pr).*exp(-K-((K+1).*(z.^2))/Pr).*besseli(0,(2*sqrt((K*(K+1))/Pr))*z);
pdfN = ((2*m^m*z.^(2*m-1))/(gamma(m)*Pr^m)).*exp(-(m/Pr)*z.^2);
for i = 1:length(z)
    cdfR(i) = 0.01*sum(pdfR(1:i));
    cdfN(i) = 0.01*sum(pdfN(1:i));
end plot(z,cdfR); hold on plot(z,cdfN,'b--'); figure;
plot(z,pdfR); hold on plot(z,pdfN,'b--');
```

As seen from the curves, the Nakagami-m approximation becomes better as K increases. Also, for a fixed value of K and x, prob(γ<x) for x large is always greater for the Ricean distribution. This is seen from the tail behavior of the two distributions in their pdf, where the tail of Nakagami-distribution is always above the Ricean distribution.

10. (a) W = average received power
    Z_i = Shadowing over link i
    P_{ri} = Received power in dBW, which is Gaussian with mean W, variance σ^2

    (b)

$$P_{outage} = P\left[P_{r,1} < T \cap P_{r,2} < T\right] = P\left[P_{r,1} < T\right] P[P_{r,2} < T] \; \textit{since } Z_1, Z_2 \textit{ independent}$$

$$= \left[Q\left(\frac{W-T}{\sigma}\right)\right]^2 = \left[Q\left(\frac{\triangle}{\sigma}\right)\right]^2$$

    (c)

$$P_{out} = \int_{-\infty}^{\infty} P\left[P_{r,1} \leq T \;,\; P_{r,2} < T|Y=y\right] f_y\left(y\right) dy$$

$$P_{r,1}|Y=y \sim N\left(W+by, a^2\sigma^2\right), \; \textit{and} \; [P_{r,1}|Y=y] \perp [P_{r,2}|Y=y]$$

$$P_{outage} = \int_{-\infty}^{\infty} \left[Q\left(\frac{W+by-T}{a\sigma}\right)\right]^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

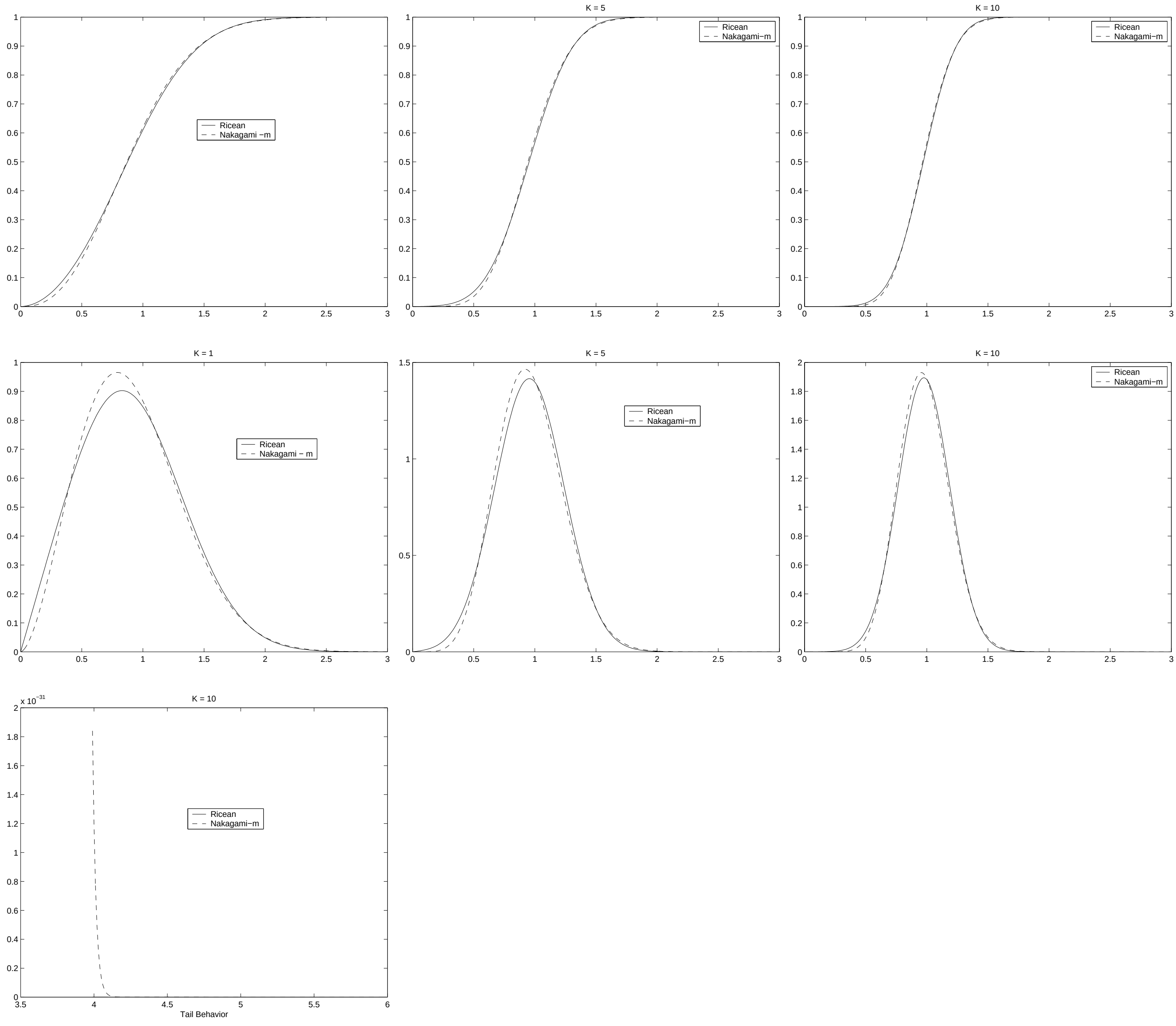

Figure 1: The CDF and PDF for $K = 1, 5, 10$ and the Tail Behavior

let $\frac{y}{\sigma}= u$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[Q\left(\frac{W-T+bu\sigma}{a\sigma}\right)\right]^2 e^{\frac{-u^2}{2}} du \;=\; \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[Q\left(\frac{\triangle+by\sigma}{a\sigma}\right)\right]^2 e\frac{-y^2}{2} dy$$

(d) Let a = b = $\frac{1}{\sqrt{2}}$, $\sigma = 8$, $\triangle = 5$. With independent fading we get

$$P_{out}= \left[Q\left(\frac{5}{8}\right)\right]^2 = \; 0.0708.$$

With correlated fading we get $P_{out}=$ 0.1316.
Conclusion : Independent shadowing is prefarable for diversity.

11. There are many acceptable techniques for this problem. Sample code for both the stochastic technique(preferred) and the Jake's technique are included.

Jakes: Summing of appropriate sine waves

```
%Jake's Method
close all; clear all;
%choose N=30
N=30; M=0.5*(N/2-1); Wn(M)=0; beta(M)=0;
%We choose 1000 samples/sec
ritemp(M,2001)=0; rqtemp(M,2001)=0; rialpha(1,2001)=0; fm=[1 10
100]; Wm=2*pi*fm; for i=1:3
    for n=1:1:M
        for t=0:0.001:2
            %Wn(i)=Wm*cos(2*pi*i/N)
            Wn(n)=Wm(i)*cos(2*pi*n/N);
            %beta(i)=pi*i/M
            beta(n)=pi*n/M;
            %ritemp(i,20001)=2*cos(beta(i))*cos(Wn(i)*t)
            %rqtemp(i,20001)=2*sin(beta(i))*cos(Wn(i)*t)
            ritemp(n,1000*t+1)=2*cos(beta(n))*cos(Wn(n)*t);
            rqtemp(n,1000*t+1)=2*sin(beta(n))*cos(Wn(n)*t);
            %Because we choose alpha=0,we get sin(alpha)=0 and cos(alpha)=1
            %rialpha=(cos(Wm*t)/sqrt(2))*2*cos(alpha)=2*cos(Wm*t)/sqrt(2)
            %rqalpha=(cos(Wm*t)/sqrt(2))*2*sin(alpha)=0
            rialpha(1,1000*t+1)=2*cos(Wm(i)*t)/sqrt(2);
        end
    end
    %summarize ritemp(i) and rialpha
    ri=sum(ritemp)+rialpha;
    %summarize rqtemp(i)
    rq=sum(rqtemp);
    %r=sqrt(ri^2+rq^2)
    r=sqrt(ri.^2+rq.^2);
    %find the envelope average
    mean=sum(r)/2001;
    subplot(3,1,i);
```

```
    time=0:0.001:2;
    %plot the figure and shift the envelope average to 0dB
    plot(time,(10*log10(r)-10*log10(mean)));
    titlename=['fd = ' int2str(fm(i)) ' Hz'];
    title(titlename);
    xlabel('time(second)');
    ylabel('Envelope(dB)');
end
```

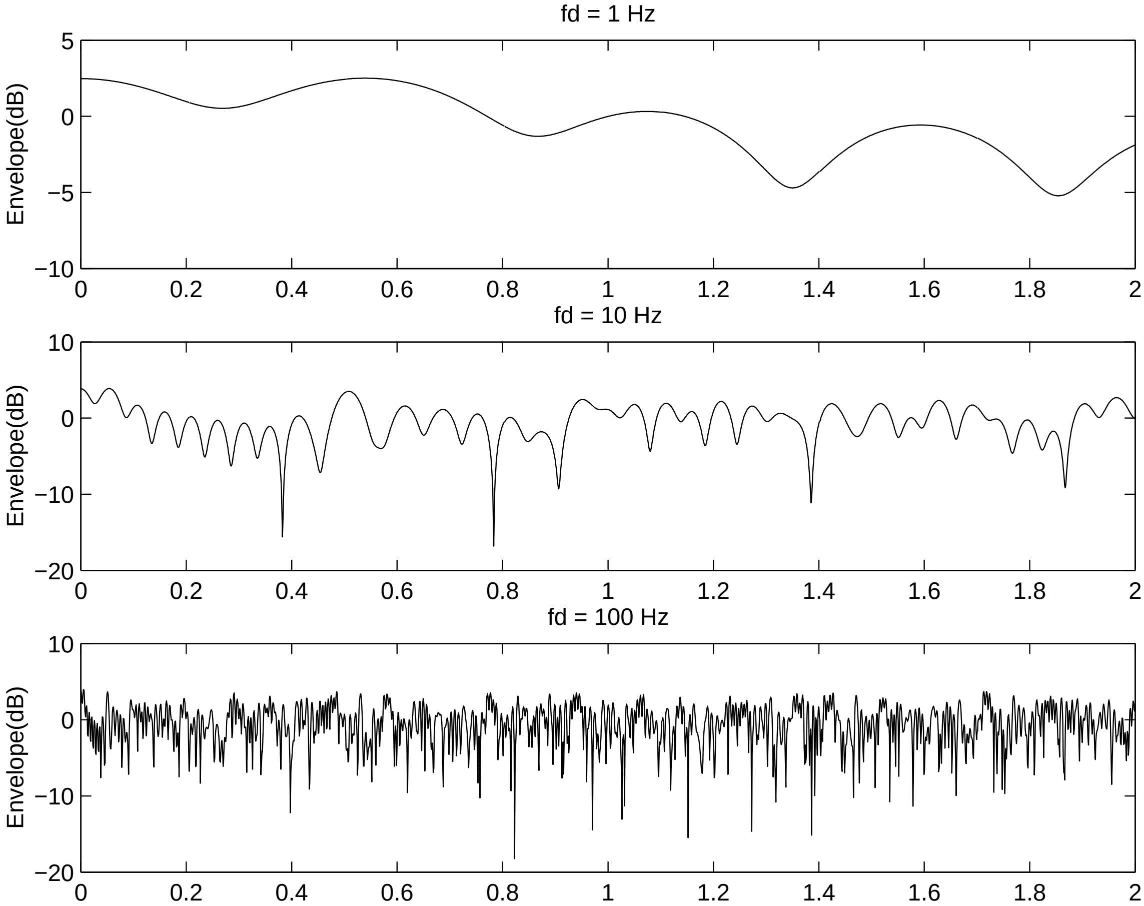

Figure 2: Problem 11

Stochastic: Usually two guassian R.V.'s are filtered by the Doppler Spectrum and summed. Can also just do a Rayleigh distribution with an adequate LPF, although the above technique is prefered.

```
function [Ts, z_dB] = rayleigh_fading(f_D, t, f_s)
%
% function [Ts, z_dB] = rayleigh_fading(f_D, t, f_s)
% generates a Rayleigh fading signal for given Doppler frequency f_D,
% during the time perios [0, t], with sampling frequency f_s >= 1000Hz.
%
% Input(s)
%   -- f_D  : [Hz] [1x1 double] Doppler frequency
%   -- t    : simulation time interval length, time interval [0,t]
%   -- f_s  : [Hz] sampling frequency, set to 1000 if smaller.
% Output(s)
%   -- Ts   : [Sec][1xN double] time instances for the Rayleigh signal
%   -- z_dB : [dB] [1xN double] Rayleigh fading signal
%

% Required parameters
if f_s < 1000;
  f_s = 1000;                 % [Hz]  Minumum required sampling rate
end;
```

```
N = ceil( t*f_s );           % Number of samples

% Ts contains the time instances at which z_dB is specified
Ts = linspace(0,t,N);

if mod( N, 2) == 1
  N = N+1;                   % Use even number of samples in calculation
end;
f = linspace(-f_s,f_s,N);    % [Hz]  Frequency samples used in calculation

% Generate complex Gaussian samples with line spectra in frequency domain
% Inphase :
Gfi_p = randn(2,N/2); CGfi_p = Gfi_p(1,:)+i*Gfi_p(2,:); CGfi = [
conj(fliplr(CGfi_p)) CGfi_p ];

% Quadrature :
Gfq_p = randn(2,N/2); CGfq_p = Gfq_p(1,:)+i*Gfq_p(2,:); CGfq = [
conj(fliplr(CGfq_p)) CGfq_p ];

% Generate fading spectrum, this is used to shape the Gaussian line spectra
omega_p = 1;   % this makes sure that the average received envelop can be 0dB
S_r = omega_p/4/pi./(f_D*sqrt(1-(f/f_D).^2));

% Take care of samples outside the Doppler frequency range, let them be 0
idx1 = find(f>=f_D); idx2 = find(f<=-f_D); S_r(idx1) = 0;
S_r(idx2) = 0; S_r(idx1(1)) = S_r(idx1(1)-1);
S_r(idx2(length(idx2))) = S_r(idx2(length(idx2))+1);

% Generate r_I(t) and r_Q(t) using inverse FFT:
r_I = N*ifft(CGfi.*sqrt(S_r)); r_Q = -i*N*ifft(CGfq.*sqrt(S_r));

% Finally, generate the Rayleigh distributed signal envelope
z = sqrt(abs(r_I).^2+abs(r_Q).^2); z_dB = 20*log10(z);

% Return correct number of points
z_dB = z_dB(1:length(Ts));
```

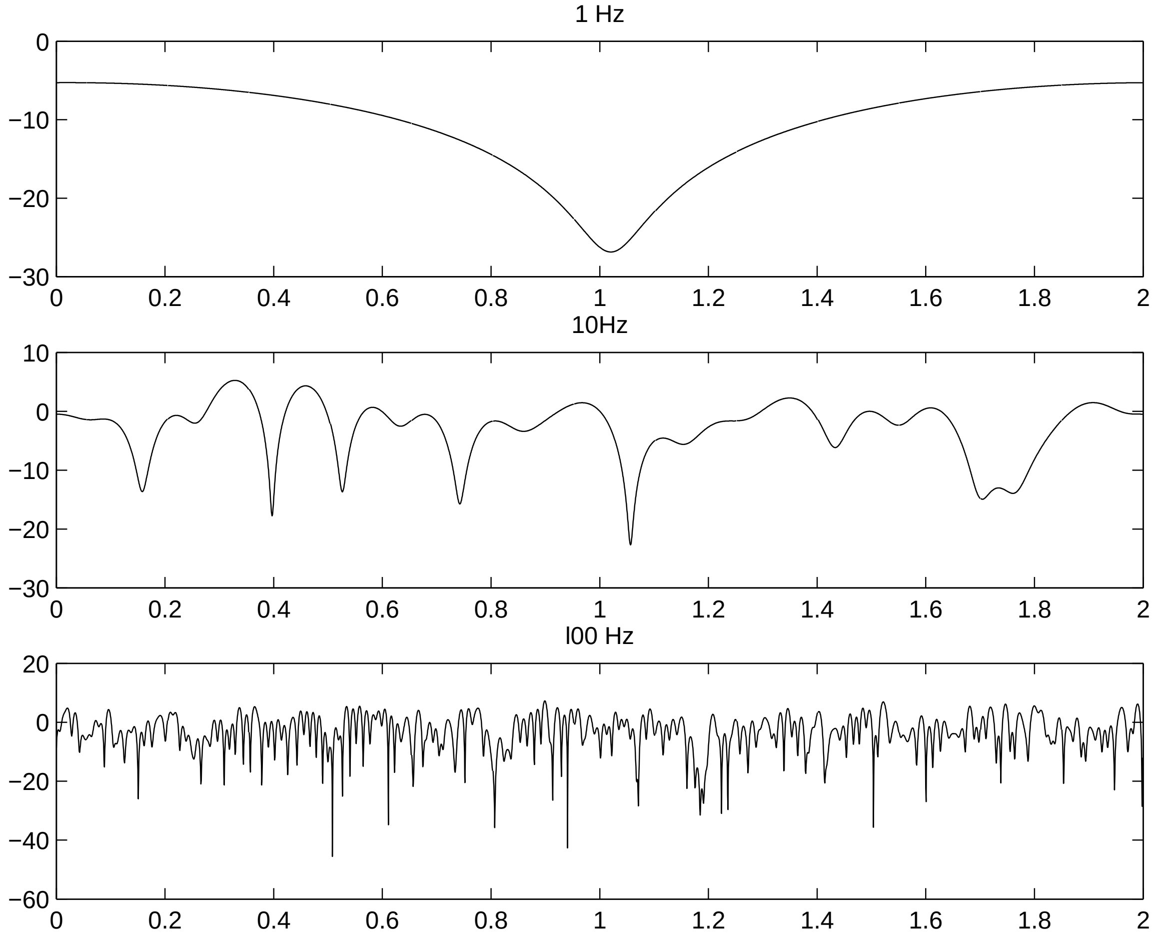

Figure 3: Problem 11

12. $P_r = 30dBm$
    $f_D = 10Hz$

$$P_0 = 0dBm,\ \bar{t}_z = \frac{e^{\rho^2} - 1}{\rho f_D \sqrt{2\pi}} = 0.0013s$$
$$P_0 = 15dBm,\ \bar{t}_z = 0.0072s$$
$$P_0 = 25dBm,\ \bar{t}_z = 0.0264s$$

13. In the reader, we found the level crossing rate below a level by taking an average of the number of times the level was crossed over a large period of time. It is easy to convince that the level crossing rate above a level will have the same expression as eq. 3.44 in reader because to go below a level again, we first need to go above it. There will be some discrepancy at the end points, but for a large enough T it will not effect the result. So we have

$$L_Z(above) = L_Z(below) = \sqrt{2\pi} f_D \rho e^{-\rho^2}$$

And,

$$\overline{t_Z}(above) = \frac{p(z > Z)}{L_Z(above)}$$
$$p(z > Z) = 1 - p(z \leq Z) = 1 - (1 - e^{-\rho^2}) = e^{-\rho^2}$$
$$\overline{t_Z}(above) = \frac{1}{\sqrt{2\pi} f_D \rho}$$

The values of $\overline{t_Z}(above)$ for $f_D = $ 10,50,80 Hz are 0.0224s, 0.0045s, 0.0028s respectively. Notice that as f_D increases, $\overline{t_Z}(above)$ reduces.

14. Note: The problem has been solved for $T_s = 10\mu s$
$P_r = 10dB$
$f_D = 80Hz$

$$
\begin{array}{lrl}
R_1: & -\infty \leq \gamma \leq -10dB, & \pi_1 = 9.95 \times 10^{-3} \\
R_2: & -10dB \leq \gamma \leq 0dB, & \pi_2 = 0.085 \\
R_3: & 0dB \leq \gamma \leq 5dB, & \pi_3 = 0.176 \\
R_4: & 5dB \leq \gamma \leq 10dB, & \pi_4 = 0.361 \\
R_5: & 10dB \leq \gamma \leq 15dB, & \pi_5 = 0.325 \\
R_6: & 15dB \leq \gamma \leq 20dB, & \pi_6 = 0.042 \\
R_7: & 20dB \leq \gamma \leq 30dB, & \pi_7 = 4.54 \times 10^{-5} \\
R_8: & 30dB \leq \gamma \leq \infty, & \pi_8 = 3.72 \times 10^{-44}
\end{array}
$$

$N_j \rightarrow$ level crossing rate at level A_j

$$
\begin{array}{ll}
N_1 = 0, & \rho = \sqrt{\frac{0}{10}} \\
N_2 = 19.85, & \rho = \sqrt{\frac{0.1}{10}} \\
N_3 = 57.38, & \rho = \sqrt{\frac{1}{10}} \\
N_4 = 82.19, & \rho = \sqrt{\frac{10^{0.5}}{10}} \\
N_5 = 73.77, & \rho = \sqrt{\frac{10}{10}} \\
N_6 = 15.09, & \rho = \sqrt{\frac{10^{1.5}}{10}} \\
N_7 = 0.03, & \rho = \sqrt{\frac{10^2}{10}} \\
N_8 = 0, & \rho = \sqrt{\frac{10^3}{10}}
\end{array}
$$

```
MATLAB CODE:
N = [0 19.85 57.38 82.19 73.77 15.09 .03 0];

Pi = [9.95e-3 .085 .176 .361 .325 .042 4.54e-5 3.72e-44];

T = 10e-3;
for i = 1:8
    if i == 1
        p(i,1) = 0;
        p(i,2) = (N(i+1)*T)/Pi(i);
        p(i,3) = 1-(p(i,1)+p(i,2));
    elseif i == 8
        p(i,1) = (N(i)*T)/Pi(i);
        p(i,2) = 0;
        p(i,3) = 1-(p(i,1)+p(i,2));
    else
        p(i,1) = (N(i)*T)/Pi(i);
        p(i,2) = (N(i+1)*T)/Pi(i);
        p(i,3) = 1-(p(i,1)+p(i,2));
```

```
    end
end

% p =
%
%          0    0.0199    0.9801
%     0.0023    0.0068    0.9909
%     0.0033    0.0047    0.9921
%     0.0023    0.0020    0.9957
%     0.0023    0.0005    0.9973
%     0.0036    0.0000    0.9964
%     0.0066         0    0.9934
%          0         0    1.0000
```

15. (a)

$$S(\tau,\rho) = \begin{cases} \alpha_1\delta(\tau) & \rho = 70Hz \\ \alpha_2\delta(\tau - 0.022\mu sec) & \rho = 49.5Hz \\ 0 & else \end{cases}$$

The antenna setup is shown in Fig. 15

From the figure, the distance travelled by the LOS ray is d and the distance travelled by the first multipath component is

$$2\sqrt{\left(\frac{d}{2}\right)^2 + 64}$$

Given this setup, we can plot the arrival of the LOS ray and the multipath ray that bounces off the ground on a time axis as shown in Fig. 15

So we have

$$2\sqrt{\left(\frac{d}{2}\right)^2 + 8^2} - d = 0.022 \times 10^{-6} 3 \times 10^8$$

$$\Rightarrow 4\left(\frac{d^2}{4} + 8^2\right) = 6.6^2 + d^2 + 2d(6.6)$$

$$\Rightarrow d = 16.1m$$

$f_D = v\cos(\theta)/\lambda$. $v = f_D\lambda/\cos(\theta)$. For the LOS ray, $\theta = 0$ and for the multipath component, $\theta = 45^o$. We can use either of these rays and the corresponding f_D value to get $v = 23.33$m/s.

(b)

$$d_c = \frac{4h_t h_r}{\lambda}$$

$d_c = 768$ m. Since $d \ll d_c$, power fall-off is proportional to d^{-2}.

(c) $T_m = 0.022\mu$s, $B^{-1} = 0.33\mu$s. Since $T_m \ll B^{-1}$, we have flat fading.

16. (a) Outdoor, since delay spread ≈ 10 μsec.

Consider that 10 μsec $\Rightarrow$ d = ct = 3km difference between length of first and last path

(b) Scattering function

$$S(\tau,\rho) = F_{\Delta t}[A_c(\tau,\Delta t)]$$

$= \frac{1}{W} rect\left(\frac{1}{W}\rho\right)$ for $0 \le \tau \le 10\mu sec$

The Scattering function is plotted in Fig. 16

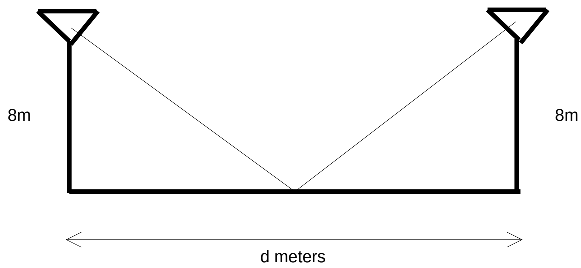

Figure 4: Antenna Setting

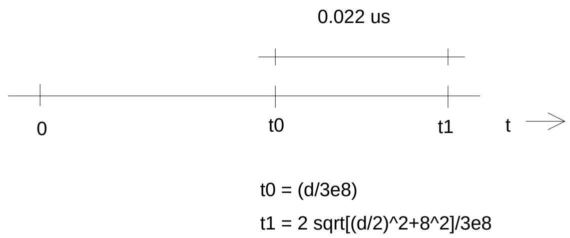

Figure 5: Time Axis for Ray Arrival

(c) Avg Delay Spread $= \frac{\int_0^\infty \tau A_c(\tau)d\tau}{\int_0^\infty A_c(\tau)d\tau} = 5\mu\text{sec}$

RMS Delay Spread $= \sqrt{\frac{\int_0^\infty (\tau-\mu_{Tm})^2 A_c(\tau)d\tau}{\int_0^\infty A_c(\tau)d\tau}} = 2.89\mu\text{sec}$

Doppler Spread $= \frac{W}{2} = 50$ Hz

(d) $\beta_u >$ Coherence BW $\Rightarrow$ Freq. Selective Fading $\approx \frac{1}{T_m} = 10^5 \Rightarrow \beta_u >= 10^5$ kHz
Can also use μ_{Tm} or σ_{Tm} instead of T_m

(e) Rayleigh fading, since receiver power is evenly distributed relative to delay; no dominant LOS path

(f) $t_R = \frac{e^{\rho^2}-1}{\rho F_D\sqrt{2\pi}}$ with $\rho = 1$, $f_D = \frac{W}{2} \to\ t_r = .0137$ sec

(g) Notice that the fade duration never becomes more than twice the average. So, if we choose our data rate such that a single symbol spans the average fade duration, in the worst case two symbols will span the fade duration. So our code can correct for the lost symbols and we will have error-free transmission. So $\frac{1}{t_R} = 72.94$ symbols/sec

17. (a) $T_m \approx .1\text{msec} = 100\mu\text{sec}$
$B_d \approx .1\text{Hz}$
Answers based on μ_{T_m} or σ_{T_m} are fine too. Notice, that based on the choice of either T_m, μ_{T_m} or σ_{T_m}, the remaining answers will be different too.

(b) $B_c \approx \frac{1}{T_m} = 10\text{kHz}$
$\Delta f > 10\text{kHz}$ for $u_1 \perp u_2$

(c) $(\Delta t)_c = 10\text{s}$

(d) $3\text{kHz} < B_c \Rightarrow$ Flat
$30\text{kHz} > B_c \Rightarrow$ Freq. Selective

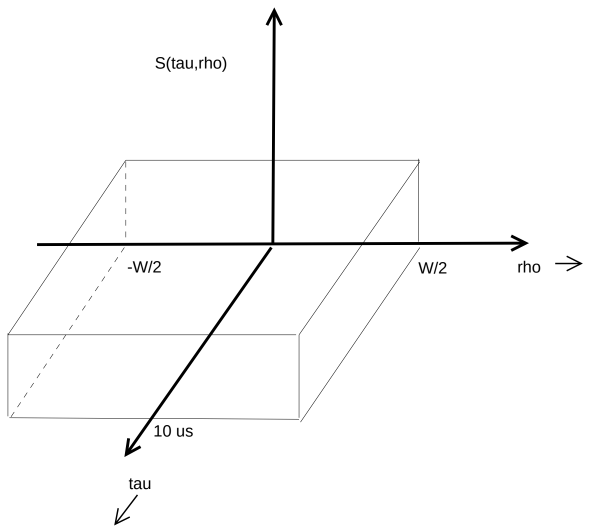

Figure 6: Scattering Function

Chapter 4

1. $C = B\log_2\left(1+\frac{S}{N_0 B}\right)$
   $C = \frac{\log_2 \;\; 1+\frac{S}{N_0B}}{\frac{1}{B}}$
   As $B \to \infty$ by L'Hospital's rule
   $$C = \frac{S}{N_0}\frac{1}{\ln 2}$$

2. $B = 50$ MHz
   $P = 10$ mW
   $N_0 = 2\ \times 10^{-9}$ W/Hz
   $N = N_0 B$
   $C = 6.87$ Mbps.

   $P_{\text{new}} = 20$ mW, $C = 13.15$ Mbps (for $x \ll 1$, $\log(1+x) \approx x$)
   $B = 100$ MHz, Notice that both the bandwidth and noise power will increase. So $C = 7$ Mbps.

3. $P_{noise} = 0.1mW$
   $B = 20MHz$

   (a) $C_{user1\to\text{base station}} = 0.933B = 18.66Mbps$
   (b) $C_{user2\to\text{base station}} = 3.46B = 69.2Mbps$

4. (a) Ergodic Capacity (with Rcvr CSI only)= $B[\sum_{i=1}^{6}\log_2(1+\gamma_i)p(\gamma_i)] = 2.8831 \times B = 57.66$ Mbps.

   (b) $p_{\text{out}} = Pr(\gamma < \gamma_{\text{min}})$
   $C_o =$ (1-p_{out})$B\log_2(1+\gamma_{min})$
   For
   $\gamma_{\text{min}} > 20$dB, $p_{\text{out}} = 1$, $C_o = 0$
   15dB $< \gamma_{\text{min}} <$ 20dB, $p_{\text{out}} = .9$, $C_o = 0.1B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx$ 20dB.
   10dB $< \gamma_{\text{min}} <$ 15dB, $p_{\text{out}} = .75$, $C_o = 0.25B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx$ 15dB.
   5dB $< \gamma_{\text{min}} <$ 10dB, $p_{\text{out}} = .5$, $C_o = 0.5B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx$ 10dB.
   0dB $< \gamma_{\text{min}} <$ 5dB, $p_{\text{out}} = .35$, $C_o = 0.65B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx$ 5dB.
   -5dB $< \gamma_{\text{min}} <$ 0dB, $p_{\text{out}} = .1$, $C_o = 0.9B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx$ 0dB.
   $\gamma_{\text{min}} < -5$dB, $p_{\text{out}} = 0$, $C_o = B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx$ -5dB.

   Plot is shown in Fig. 1. Maximum at $\gamma_{\text{min}} = 10$dB, p_{out}=0.5 and $C_o = 34.59$ Mbps.

5. (a) We suppose that all channel states are used
   $$\frac{1}{\gamma_0} = 1 + \sum_{i=1}^{4}\frac{1}{\gamma_i}p_i \Rightarrow \;\; \gamma_0 = 0.8109$$
   $$\frac{1}{\gamma_0} - \frac{1}{\gamma_4} > 0 \therefore true$$
   $$\frac{S(\gamma_i)}{\overline{S}} = \frac{1}{\gamma_0} - \frac{1}{\gamma_i}$$

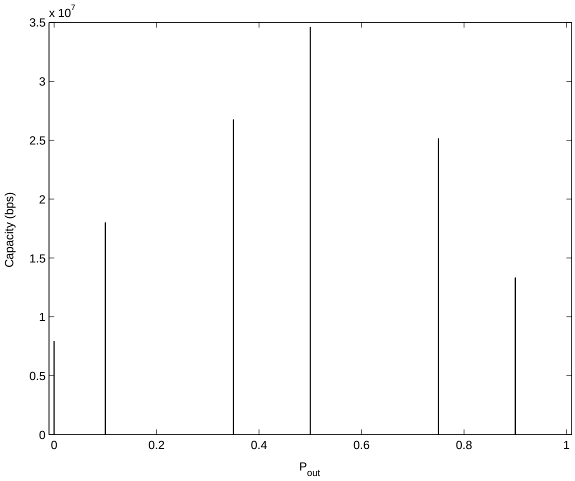

Figure 1: Capacity vs P_{out}

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 1.2322 & \gamma = \gamma_1 \\ 1.2232 & \gamma = \gamma_2 \\ 1.1332 & \gamma = \gamma_3 \\ 0.2332 & \gamma = \gamma_4 \end{cases}$$

$$\frac{C}{B} = \sum_{i=1}^{4} \log_2 \left(\frac{\gamma_i}{\gamma_0} \right) p(\gamma_i) = 5.2853 bps/Hz$$

(b) $\sigma = \frac{1}{E[1/\gamma]} = 4.2882$
$\frac{S(\gamma_i)}{\overline{S}} = \frac{\sigma}{\gamma_i}$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 0.0043 & \gamma = \gamma_1 \\ 0.0029 & \gamma = \gamma_2 \\ 0.4288 & \gamma = \gamma_3 \\ 4.2882 & \gamma = \gamma_4 \end{cases}$$

$$\frac{C}{B} = \log_2(1+\sigma) = 2.4028 bps/Hz$$

(c) To have $p_{out} = 0.1$ or 0.01 we will have to use all the sub-channels as leaving any of these will result in a p_{out} of at least 0.2 $\therefore$ truncated channel power control policy and associated spectral efficiency are the same as the zero-outage case in part b .

To have p_{out} that maximizes C with truncated channel inversion, we get

$$\max \frac{C}{B} = 4.1462 bps/Hz \quad p_{out} = 0.5$$

6. (a)

$$SNR_{recvd} = \frac{P_\gamma(d)}{P_{noise}} = \begin{cases} 10dB & w.p.\ 0.4 \\ 5dB & w.p.\ 0.3 \\ 0dB & w.p.\ 0.2 \\ -10dB & w.p.\ 0.1 \end{cases}$$

Assume all channel states are used

$$\frac{1}{\gamma_0} = 1 + \sum_{i=1}^{4} \frac{1}{\gamma_i} p_i \Rightarrow \ \gamma_0 = 0.4283 > 0.1 \quad \therefore \text{not possible}$$

Now assume only the best 3 channel states are used

$$\frac{0.9}{\gamma_0} = 1 + \sum_{i=1}^{3} \frac{1}{\gamma_i} p_i \Rightarrow \gamma_0 = 0.6742 < 1 \qquad \therefore \text{ok!}$$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 1.3832 & \gamma = \gamma_1 = 10 \\ 1.1670 & \gamma = \gamma_2 = 3.1623 \\ 0.4832 & \gamma = \gamma_3 = 1 \\ 0 & \gamma = \gamma_4 = 0.1 \end{cases}$$

$$C/B = 2.3389 bps/Hz$$

(b) $\sigma = 0.7491$
$C/B = \log_2(1+\sigma) = 0.8066 bps/Hz$

(c) $\left(\frac{C}{B}\right)_{max} = 2.1510 bps/Hz$ obtained by using the best 2 channel states.
With $p_{out} = 0.1 + 0.2 = 0.3$

7. (a) Maximize capacity given by

$$C = \max_{S(\gamma): \int S(\gamma) p(\gamma) d\gamma = \overline{S}} \int_\gamma B \log\left(1 + \frac{S(\gamma)\gamma}{\overline{S}}\right) p(\gamma) d\gamma.$$

Construct the Lagrangian function

$$\mathcal{L} = \int_\gamma B \log\left(1 + \frac{S(\gamma)\gamma}{\overline{S}}\right) p(\gamma) d\gamma - \lambda \int \frac{S(\gamma)}{\overline{S}} p(\gamma) d\gamma$$

Taking derivative with respect to $S(\gamma)$, (refer to discussion section notes) and setting it to zero, we obtain,

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma} & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

Now, the threshold value must satisfy

$$\int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) p(\gamma) d\gamma = 1$$

Evaluating this with $p(\gamma) = \frac{1}{10} e^{-\gamma/10}$, we have

$$\begin{aligned} 1 &= \frac{1}{10\gamma_0} \int_{\gamma_0}^{\infty} e^{-\gamma/10} d\gamma - \frac{1}{10} \int_{\gamma_0}^{\infty} \frac{e^{-\gamma/10}}{\gamma} d\gamma & (1) \\ &= \frac{1}{\gamma_0} e^{-\gamma_0/10} - \frac{1}{10} \int_{\frac{\gamma_0}{10}}^{\infty} \frac{e^{-\gamma}}{\gamma} d\gamma & (2) \\ &= \frac{1}{\gamma_0} e^{-\gamma_0/10} - \frac{1}{10} \text{EXPINT}(\gamma_0/10) & (3) \end{aligned}$$

where EXPINT is as defined in matlab. This gives $\gamma_0 = 0.7676$. The power adaptation becomes

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{0.7676} - \frac{1}{\gamma} & \gamma \geq 0.7676 \\ 0 & \gamma < 0.7676 \end{cases}$$

(b) Capacity can be computed as

$$C/B = \frac{1}{10}\int_{0.7676}^{\infty} \log\left(\gamma/0.7676\right) e^{-\gamma/10} d\gamma = 2.0649 \quad \text{nats/sec/Hz}.$$

Note that I computed all capacites in nats/sec/Hz. This is because I took the natural log. In order to get the capacity values in bits/sec/Hz, the capacity numbers simply need to be divided by natural log of 2.

(c) AWGN capacity $C/B = \log(1 + 10) = 2.3979$ nats/sec/Hz.

(d) Capacity when only receiver knows γ

$$C/B = \frac{1}{10}\int_{0}^{\infty} \log\left(1+\gamma\right) e^{-\gamma/10} d\gamma = 2.0150 \quad \text{nats/sec/Hz}.$$

(e) Capacity using channel inversion is ZERO because the channel can not be inverted with finite average power. Threshold for outage probability 0.05 is computed as

$$\frac{1}{10}\int_{\gamma 0}^{\infty} e^{-\gamma/10} d\gamma = 0.95$$

which gives $\gamma_0 = 0.5129$. This gives us the capacity with truncated channel inversion as

$$\begin{aligned} C/B &= \log\left[1 + \frac{1}{\frac{1}{10}\int_{\gamma_0}^{\infty} \frac{1}{\gamma} e^{-\gamma/10} d\gamma}\right] * 0.95 && (4)\\ &= \log\left[1 + \frac{1}{\frac{1}{10}\text{EXPINT}(\gamma_0/10)}\right] * 0.95 && (5)\\ &= 1.5463 \quad \text{nats/s/Hz}. && (6) \end{aligned}$$

(f) Channel Mean=-5 dB = 0.3162. So for perfect channel knowledge at transmitter and receiver we compute $\gamma_0 = 0.22765$ which gives capacity $C/B = 0.36$ nats/sec/Hz.
With AWGN, $C/B = \log(1 + 0.3162) = 0.2748$ nats/sec/Hz.
With channel known only to the receiver $C/B = 0.2510$ nats/sec/Hz.

Capacity with AWGN is always greater than or equal to the capacity when only the reciever knows the channel. This can be shown using Jensen's inequality. However the capacity when the transmitter knows the channel as well and can adapt its power, can be higher than AWGN capacity specially at low SNR. At low SNR, the knowledge of fading helps to use the low SNR more efficiently.

8. (a) If neither transmitter nor receiver knows when the interferer is on, they must transmit assuming worst case, i.e. as if the interferer was on all the time,

$$C = B\log\left(1 + \frac{\overline{S}}{N_0 B + \overline{I}}\right) = 10.7Kbps.$$

(b) Suppose we transmit at power S_1 when jammer is off and S_2 when jammer is off,

$$C = B\max\left[\log\left(1 + \frac{S_1}{N_o B}\right)0.75 + \log\left(1 + \frac{S_2}{N_o B + \overline{I}}\right)0.25\right]$$

subject to

$$0.75 S_1 + 0.25 S_2 = \overline{S}.$$

This gives $S_1 = 12.25mW$, $S_2 = 3.25mW$ and $C = 53.21Kbps$.

(c) The jammer should transmit $-x(t)$ to completely cancel off the signal.

$\overline{S} = 10$mW
$N_0 = .001\ \mu$W/Hz
$B = 10$ MHz

Now we compute the SNR's as:

$$\gamma_j = \frac{|H_j|^2 \overline{S}}{N_0 B}$$

This gives: $\gamma_1 = \frac{|1|^2 10^{-3}}{0.001 \times 10^{-6} 10 \times 10^6} = 1$, $\gamma_2 = .25$, $\gamma_3 = 4$, $\gamma_4 = 0.0625$

To compute γ_0 we use the power constraint as:

$$\sum_j \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_j} \right)_+ = 1$$

First assume that $\gamma_0 < 0.0625$, then we have

$$\frac{4}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{.25} + \frac{1}{4} + \frac{1}{.0625} \right)$$
$$\Rightarrow \gamma_0 = .1798 > 0.0625$$

So, our assumption was wrong. Now we assume that $0.0625 < \gamma_0 < .25$, then

$$\frac{3}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{.25} + \frac{1}{4} \right)$$
$$\Rightarrow \gamma_0 = .48 > 0.25$$

So, our assumption was wrong again. Next we assume that $0.25 < \gamma_0 < 1$, then

$$\frac{2}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{4} \right)$$
$$\Rightarrow \gamma_0 = .8889 < 1$$

This time our assumption was right. So we get that only two sub-bands each of bandwidth 10 MHz are used for transmission and the remaining two with lesser SNR's are left unused.

Now, we can find capacity as:

$$C = \sum_{j:\gamma_j \geq \gamma_0} B \log_2 \left(\frac{\gamma_j}{\gamma_0} \right)$$

This gives us, C = 23.4 Mbps.

9. Suppose transmit power is P_t, interference power is P_{int} and noise power is P_{noise}.
In the first strategy $C/B = \log_2 \left(1 + \frac{P_t}{P_{int} + P_{noise}}\right)$
In the second strategy $C/B = \log_2 \left(1 + \frac{P_t - P_{int}}{P_{noise}}\right)$
Assuming that the transmitter transmits -x[k] added to its message always, power remaining for actual messages is $P_t - P_{int}$
The first or second strategy may be better depending on

$$\frac{P_t}{P_{int} + P_{noise}} \gtrless \frac{P_t - P_{int}}{P_{noise}} \Rightarrow P_t - P_{int} - P_{noise} \gtrless 0$$

P_t is generally greater than $P_{int} + P_{noise}$, and so strategy 2 is usually better.

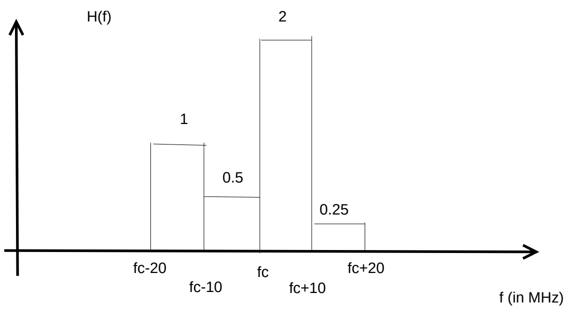

Figure 2: Problem 11

10. We show this for the case of a discrete fading distribution

$$C = \Sigma \log\left(1 + \frac{(1+j)^2 P_j}{N_0 B}\right)$$

$$\mathcal{L} = \sum_i \log\left(1 + \frac{(1+j)^2 P_j}{N_0 B}\right) - d_j \left(\sum_j P_j - P\right)$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial P_j} &= 0 \\
\Rightarrow \frac{(1+j)^2 P_j}{N_0 B} &= \frac{1}{\lambda} \frac{(1+j)^2}{N_0 B} - 1 \\
\text{let} \gamma_j = \frac{(1+j)^2 P}{N_0 B} & \\
\Rightarrow \frac{P_j}{P} &= \frac{1}{\lambda P} - \frac{1}{\gamma_j} \\
\text{denote} \frac{1}{\gamma_0} = \frac{1}{\lambda P} & \\
\therefore \frac{P_j}{P} &= \frac{1}{\gamma_0} - \frac{1}{\gamma_j} \\
\text{subject to the constraint} & \\
\frac{\Sigma P_j}{P} &= 1
\end{aligned}$$

11. $\overline{S} = 10\text{mW}$
$N_0 = .001\ \mu\text{W/Hz}$
$B = 10$ MHz

Now we compute the SNR's as:

$$\gamma_j = \frac{|H_j|^2 \overline{S}}{N_0 B}$$

This gives: $\gamma_1 = \frac{|1|^2 10^{-3}}{0.001 \times 10^{-6} 10 \times 10^6} = 1$, $\gamma_2 = .25$, $\gamma_3 = 4$, $\gamma_4 = 0.0625$

To compute γ_0 we use the power constraint as:

$$\sum_j \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_j}\right)_+ = 1$$

First assume that $\gamma_0 < 0.0625$, then we have

$$\frac{4}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{.25} + \frac{1}{4} + \frac{1}{.0625}\right)$$

$$\Rightarrow \gamma_0 = .1798 > 0.0625$$

So, our assumption was wrong. Now we assume that $0.0625 < \gamma_0 < .25$, then

$$\frac{3}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{.25} + \frac{1}{4}\right)$$

$$\Rightarrow \gamma_0 = .48 > 0.25$$

So, our assumption was wrong again. Next we assume that $0.25 < \gamma_0 < 1$, then

$$\frac{2}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{4}\right)$$

$$\Rightarrow \gamma_0 = .8889 < 1$$

This time our assumption was right. So we get that only two sub-bands each of bandwidth 10 MHz are used for transmission and the remaining two with lesser SNR's are left unused.

Now, we can find capacity as:

$$C = \sum_{j:\gamma_j \geq \gamma_0} B \log_2\left(\frac{\gamma_j}{\gamma_0}\right)$$

This gives us, C = 23.4 Mbps.

12. For the case of a discrete number of frequency bands each with a flat frequency response, the problem can be stated as

$$\max_{s.t. \sum_i P(f_i) \leq P} \sum_i \log_2\left(1 + \frac{|H(f_i)|^2 P(f_i)}{N_0}\right)$$

denote $\gamma(f_i) = \frac{|H(f_i)|^2 P(f_i)}{N_0}$

$$L = \sum_i \log_2\left(1 + \gamma(f_i)\frac{P(f_i)}{P}\right) + \lambda(\sum P(f_i))$$

denote $x_i = \frac{P(f_i)}{P}$, the problem is similar to problem 10

$$\Rightarrow x_i^\star = \frac{1}{\gamma_0} - \frac{1}{\gamma(f_i)}$$

where γ_0 is found from the constraints

$$\sum_i \left(\frac{1}{\gamma_0} - \frac{1}{\gamma(f_i)}\right) = 1 \text{ and } \frac{1}{\gamma_0} - \frac{1}{\gamma(f_i)} \geq 0 \forall i$$

13. (a) C=13.98Mbps

MATLAB

```
Gammabar = [1 .5 .125];
ss = .001;
P = 30e-3;
N0 = .001e-6;

Bc = 4e6;
Pnoise = N0*Bc;
hsquare = [ss:ss:10*max(Gammabar)];
gamma = hsquare*(P/Pnoise);

for i = 1:length(Gammabar)
    pgamma(i,:) = (1/Gammabar(i))*exp(-hsquare/Gammabar(i));
end

gamma0v = [1:.01:2];
for j = 1:length(gamma0v)
    gamma0 = gamma0v(j);
    sumP(j) = 0;
    for i = 1:length(Gammabar)
        a = gamma.*(gamma>gamma0);
        [b,c] = max(a>0);
        gammac = a(find(a));
        pgammac = pgamma(i,c:length(gamma));
        Pj_by_P = (1/gamma0)-(1./gammac);
        sumP(j) = sumP(j) + sum(Pj_by_P.*pgammac)*ss;
    end
end
[b,c] = min(abs((sumP-1)));
gamma0ch = gamma0v(c);

C = 0;
for i = 1:length(Gammabar)
    a = gamma.*(gamma>gamma0ch);
    [b,c] = max(a>0);
    gammac = a(find(a));
    pgammac = pgamma(i,c:length(gamma));
    C = C + Bc*ss*sum(log2(gammac/gamma0ch).*pgammac);
end
```

(b) C=13.27Mbps

MATLAB

```
Gammabarv = [1 .5 .125];
ss = .001;
Pt = 30e-3;
N0 = .001e-6;

Bc = 4e6;
Pnoise = N0*Bc;
```

```
P = Pt/3;
for k = 1:length(Gammabarv)
    Gammabar = Gammabarv(k);
    hsquare = [ss:ss:10*Gammabar];
    gamma = hsquare*(P/Pnoise);
    pgamma = (1/Gammabar)*exp(-hsquare/Gammabar);
    gamma0v = [.01:.01:1];
    for j = 1:length(gamma0v)
        gamma0 = gamma0v(j);
        a = gamma.*(gamma>gamma0);
        [b,c] = max(a>0);
        gammac = a(find(a));
        pgammac = pgamma(c:length(gamma));
        Pj_by_P = (1/gamma0)-(1./gammac);
        sumP(j) = sum(Pj_by_P.*pgammac)*ss;
    end
    [b,c] = min(abs((sumP-1)));
    gamma0ch = gamma0v(c);
    a = gamma.*(gamma>gamma0ch);
    [b,c] = max(a>0);
    gammac = a(find(a));
    pgammac = pgamma(c:length(gamma));
    C(k) = Bc*ss*sum(log2(gammac/gamma0ch).*pgammac);
end Ctot = sum(C);
```

Chapter 5

1. $s_i(t) = \sum_k s_{ik}\phi_k(t)$
   $s_j(t) = \sum_k s_{jk}\phi_k(t)$
   where $\{\phi_k(t)\}$ forms an orthonormal basis on the interval [0,T]

$$\begin{aligned}\int_0^T [s_i(t) - s_j(t)]^2 dt &= \int_0^T \left(\sum_m s_{im}\phi_m(t) - \sum_m s_{jm}\phi_m(t)\right)^2 dt \\ &= \int_0^T \left(\sum_m (s_{im} - s_{jm})\, \phi_m(t)\right)^2 dt\end{aligned}$$

   Notice all the cross terms will integrate to 0 due to orthonormal property. So we get

$$\begin{aligned}&= \int_0^T \sum_m (s_{im} - s_{jm})^2\, \phi_m(t)\phi_m(t) dt \\ &= \sum_m (s_{im} - s_{jm})^2 = ||s_i - s_j||^2\end{aligned}$$

2. Consider

$$\phi_1 = \frac{1}{\sqrt{T}}\left[\sin\left(\frac{2\pi t}{T}\right) + \cos\left(\frac{2\pi t}{T}\right)\right]$$

$$\phi_2 = \frac{1}{\sqrt{T}}\left[\sin\left(\frac{2\pi t}{T}\right) - \cos\left(\frac{2\pi t}{T}\right)\right]$$

$$\int_0^T \phi_1(t)\phi_2(t) = 0, \quad \int_0^T \phi_1^2(t) = 1, \quad \int_0^T \phi_2^2(t) = 1$$

3.

$$s'_m(t) = s_m(t) - \frac{1}{M}\sum_{i=1}^M s_i(t)$$

$$\varepsilon' = \int_0^T s'^2_m(t) dt = \int_0^T \left(s_m(t) - \frac{1}{M}\sum_{i=1}^M s_i(t)\right)^2 dt$$

   s_i's are orthonormal so the cross terms integrate to 0 and we get

$$\varepsilon' = \int_0^T s_m^2(t)dt - \int_0^T \frac{1}{M}s_m^2(t)dt = \varepsilon - \frac{1}{M}\varepsilon = \varepsilon\frac{(M-1)}{M}$$

$$< s'_m(t), s'_n(t) >= \int_0^T \left(s_m(t) - \frac{1}{M}\sum_{i=1}^M s_i(t)\right)\left(s_n(t) - \frac{1}{M}\sum_{i=1}^M s_i(t)\right) dt = -\frac{\varepsilon}{M}$$

4. (a)

$$\begin{aligned}< f_1(t), f_2(t) >= \int_0^T f_1(t)f_2(t)dt &= 0 \\ < f_1(t), f_3(t) >= \int_0^T f_1(t)f_3(t)dt &= 0 \\ < f_2(t), f_3(t) >= \int_0^T f_2(t)f_3(t)dt &= 0\end{aligned}$$

$\Rightarrow f_1(t), f_2(t), f_3(t)$ are orthogonal

(b)

$$
\begin{aligned}
x(t) &= af_1(t) + bf_2(t) + cf_3(t) \\
0 \le t \le 1 : x(t) &= \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c = -1 \\
1 \le t \le 2 : x(t) &= \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c = 1 \\
2 \le t \le 3 : x(t) &= -\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c = 1 \\
3 \le t \le 4 : x(t) &= -\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c = 3 \\
\Rightarrow a = -2, \quad b = 2, \quad c = -2 & \\
\Rightarrow x(t) &= -2f_1(t) + 2f_2(t) - 2f_3(t)
\end{aligned}
$$

5. (a) A set of orthonormal basis functions is

In this set the given waveforms can be written as
s_1 = [1 2 -1 -1]
s_2 = [1 -1 1 -1]
s_3 = [-2 1 1 1]
s_4 = [1 -2 -2 2]
now we can see using Matlab or otherwise that the dimensionality is 4

(b) done in part a

(c)

$$
\begin{aligned}
||s_1 - s_2||^2 &= 14 \\
||s_1 - s_3||^2 &= 22 \\
||s_1 - s_4||^2 &= 27 \\
||s_2 - s_3||^2 &= 14 \\
||s_2 - s_4||^2 &= 19 \\
||s_3 - s_4||^2 &= 31 \\
||s_1||^2 &= 10 \\
||s_2||^2 &= 4 \\
||s_3||^2 &= 6 \\
||s_4||^2 &= 13
\end{aligned}
$$

The minimum distance between any pair is $\sqrt{14}$

6. From 5.28 we have
$\widehat{m} = m_i$ corresponding to $\widehat{s_i} = \arg\max_{s_i} p(\gamma|s_i)p(s_i)$

$$
\begin{aligned}
\max_{s_i} L(s_i) &= p(\gamma|s_i)p(s_i) \\
\max_{s_i} l(s_i) = logL(s_i) &= \log p(\gamma|s_i) + \log p(s_i) \\
&= \max_{s_i} - \underbrace{\frac{N}{2}\log(\pi N_0)}_{constant} - \frac{1}{N_0}\sum_{j=1}^{N}(\gamma_j - s_{ij})^2 + \log p(s_i)
\end{aligned}
$$

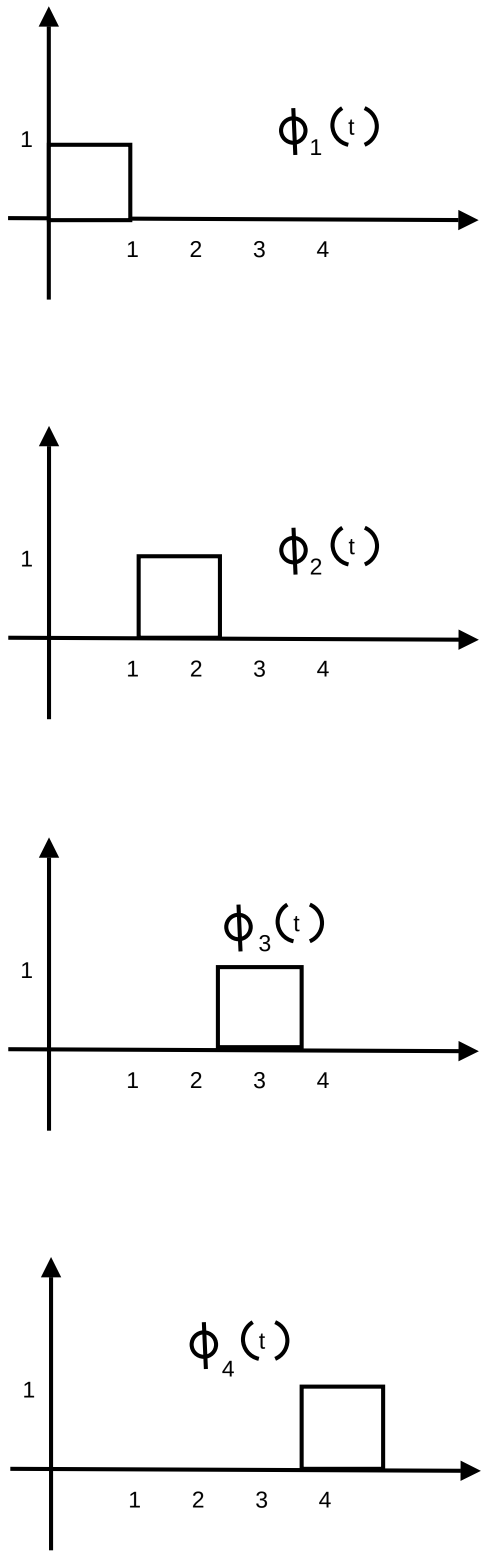

Figure 1: Problem 5

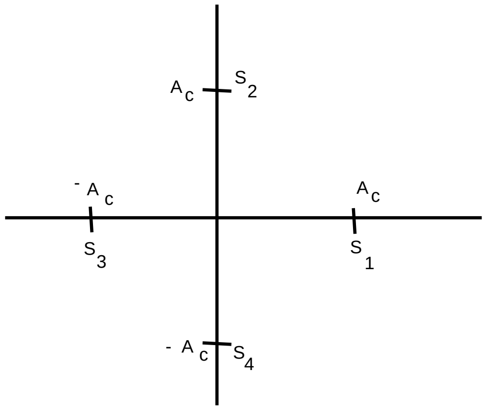

Figure 2: Problem 6

$$
\begin{aligned}
&= \max_{s_i} -\frac{1}{N_0}||\gamma_j - s_{ij}||^2 + \log p(s_i) \\
&= \min_{s_i} \frac{1}{N_0}||\gamma_j - s_{ij}||^2 + \log 1/p(s_i)
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= (A_c, 0) \\
s_2 &= (0, A_c) \\
s_3 &= (-A_c, 0) \\
s_4 &= (0, -A_c) \\
p(s_1) &= p(s_3) = 0.2 \\
p(s_2) &= p(s_4) = 0.3
\end{aligned}
$$

$$
\therefore z_i = \left\{ x \in \Re^2 | \frac{1}{N_0}||x - s_i||^2 + \log 1/p(s_i) < \frac{1}{N_0}||x - s_j||^2 + \log 1/p(s_j) \forall i \neq j \right\}
$$

which can be further solved using Matlab or otherwise far a given value of A_c and N_0

7. $n_r(t) = n(t) - \sum_{j=1}^{N} n_j \phi_j(t)$
$\gamma_j = s_{ij} + n_j$
We know $\phi_{\rightarrow}\phi_N$ span the signal space. Suppose we add (M-N) additional basis vectors so that $\phi_1 \dots \phi_M$ span the noise space . This can always be done for some M(may be infinite). Also M¿N

$$
n(t) = \sum_{k=1}^{M} n_k \phi_k(t) \quad \text{where} \phi_i \text{form an orthonormal set}
$$

then

$$
E[n_r(t_k) r_j] = E\left[\left(\sum_{k=1}^{M} n_k \phi_k(t_k) - \sum_{p=1}^{N} n_p \phi_p(t_p)\right)(s_{ij} + n_j)\right]
$$

Since the signal is always independent of noise and white noise components in the orthogonal directions are independent too, we have

$$
E[n_r(t_k) r_j] = E[n_j^2 \phi_j(t_k)] - E[n_j^2 \phi_j(t_k)] = 0 \quad , \text{for } j = 1 \dots N
$$

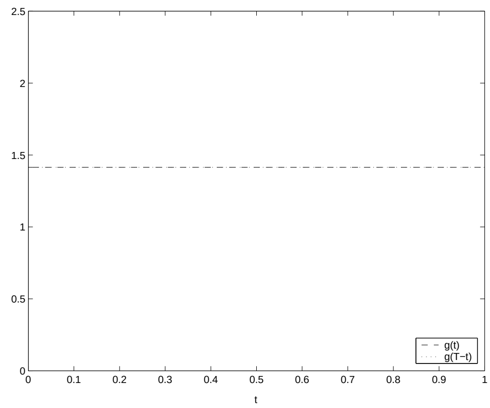

Figure 3: Problem 9a

8. Suppose $s_m(t)$ is transmitted and N_t is noise added. If the k^{th} filter is $h_k(t)$, the output of the k^{th} filter is

$$y_k(t) = \int_{-\infty}^{\infty} (s_m(\tau) + N_\tau) h_k(t-\tau) d\tau$$

sampling at time T gives

$$y_k(T) = \int_{-\infty}^{\infty} s_m(\tau) h_k(T-\tau) d\tau + \int_{-\infty}^{\infty} N_\tau h_k(T-\tau) d\tau$$

Denote noise contribution as $\nu_k = \int_{-\infty}^{\infty} N_\tau h_k(T-\tau) d\tau$

$$E[\nu_k] = 0$$

$$\sigma_{\nu_k}^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |h_k(T-\tau)|^2 d\tau$$

Signal energy $= \left(\int_{-\infty}^{\infty} s_m(\tau) h_k(T-\tau) d\tau \right)^2$

$$SNR = \frac{\left(\int_{-\infty}^{\infty} s_m(\tau) h_k(T-\tau) d\tau \right)^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |h_k(T-\tau)|^2 d\tau}$$

use Cauchy-Schwartz inequality to get upper bound on SNR as

$$SNR \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |s_m(\tau)|^2 d\tau$$

with equality iff

$$h_k^{opt}(T-\tau) = \alpha \overline{s_m(\tau)} \Rightarrow h_k^{opt}(t) = \alpha \overline{s_m(T-t)}$$

which is the required result for matched filter

9. (a) $g(t) = \sqrt{\frac{2}{T}} \quad 0 \leq t \leq T$
   $g(T-t) = \sqrt{\frac{2}{T}} \quad 0 \leq t \leq T$
   plotted for T=1 , integral value = 2/T = 2

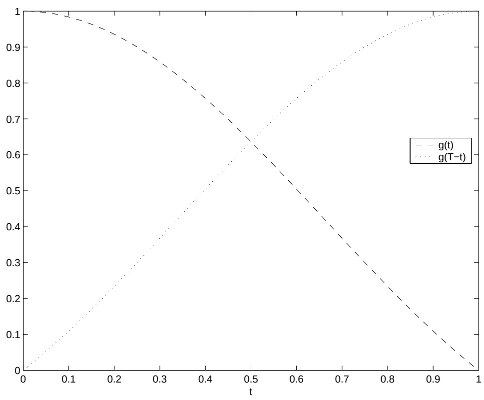

Figure 4: Problem 9b

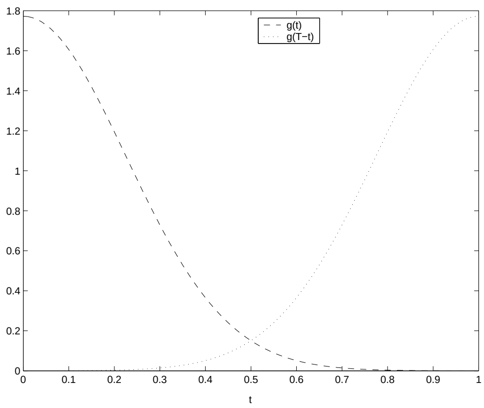

Figure 5: Problem 9c

(b) $g(t) = sinc(t) \quad 0 \le t \le T$
$g(T-t) = sinc(T-t) \quad 0 \le t \le T$
plotted for T=1 , integral value = 0.2470

(c) $g(t) = \frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 t^2/\alpha^2} \quad 0 \le t \le T$
$g(T-t) == \frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 (T-t)^2/\alpha^2} \quad 0 \le t \le T$
plotted for T=1 , integral value = 0.009

```
MATTLAB CODE
T = 1; alpha = 1;

t = [0:.01:T];
%% Part a)
g = repmat(sqrt(2/T),1,length(t));
gm=repmat(sqrt(2/T),1,length(t));
int_a = sum(g.*gm)*.01;
plot(t,g,'b--'); hold on; plot(t,gm,'b:');

%% Part b)
figure;
g = sinc(t);
gm = sinc(T-t);
```

```
int_b = sum(g.*gm)*.01;
plot(t,g,'b--');
hold on;
plot(t,gm,'b:');

%% Part c)
figure; g = (sqrt(pi)/alpha)*exp(-((pi)^2*t.^2)/alpha^2);
gm=(sqrt(pi)/alpha)*exp(-((pi)^2*(T-t).^2)/alpha^2);;
int_c=sum(g.*gm)*.01; plot(t,g,'b--');
hold on;
plot(t,gm,'b:');
```

10. For Fig 5.4 $\gamma_k = \int_0^T \gamma(\tau)\phi_k(\tau)d\tau$
For Fig 5. $\gamma_k = \int_0^T \gamma(\tau)\phi_k(T-(T-\tau))d\tau = \int_0^T \gamma(\tau)\phi_k(\tau)d\tau$ which is the same as above.

11. (5.40) gives $\frac{1}{4}\sum_{i=1}^4 \sum_{j=1,j\neq i}^4 Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right) = 4.1 \times 10^{-9}$
(5.43) gives $(4-1)Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 2.3 \times 10^{-8}$
(5.44) gives $\frac{(4-1)}{\sqrt{\pi}} \exp\left(-\frac{d_{min}^2}{4N_0}\right) = 1.9 \times 10^{-7}$
(5.45) gives $M_{d_{min}} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 2Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = 1.5 \times 10^{-8}$

```
MATLAB CODE
 Ac = 4;
 s(1,:) = [Ac 0];
 s(2,:) = [0 2*Ac];
 s(3,:) = [0 -2*Ac];
 s(4,:) = [-Ac 0];

sume = 0; for i = 1:4
    for j = 1:4
        if j ~= i
            d(i) = norm(s(i,:)-s(j,:));
            sume = sume+Q(d(i)/sqrt(2));
        end
    end
end E1 = .25*sume;
dmin = min(d);

E2 = 3*Q(dmin/sqrt(2));
E3 = (3/sqrt(pi))*exp(-dmin^2/4);
E4=2*Q(dmin/sqrt(2));
```

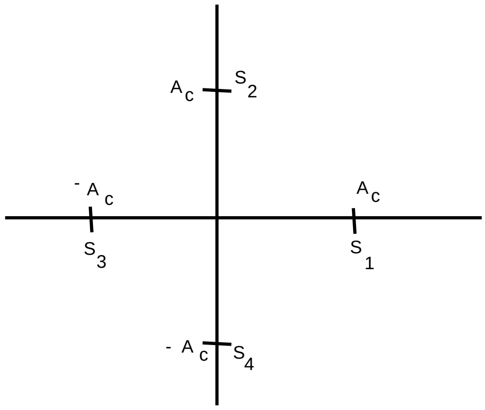

Figure 6: Problem 11

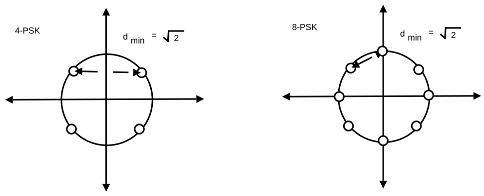

Figure 7: Problem 13

12.

$$
\begin{aligned}
\gamma_1(t) &= \int_\tau \gamma(\tau)\cos(2\pi f_c\tau+\phi)g(t-(T-\tau))d\tau \\
&= \int_\tau [s_i(\tau)+n(\tau)]\cos(2\pi f_c\tau+\phi)g(t-(T-\tau))d\tau \\
s_i(\tau) &= s_{i1}g(\tau)\cos(2\pi f_c\tau+\phi_0)+s_{i2}g(\tau)\sin(2\pi f_c\tau+\phi_0) \\
\therefore \gamma_1(t) &= \int_\tau \frac{s_{i1}}{2}[\cos(4\pi f_c\tau+\phi+\phi_0)+\cos(\phi-\phi_0)]g(\tau)g(t-(T-\tau))d\tau + \\
&\int_\tau \frac{s_{i2}}{2}[\cos(4\pi f_c\tau+\phi+\phi_0)-\cos(\phi-\phi_0)]g(\tau)g(t-(T-\tau))d\tau + \\
&\int_\tau n(\tau)\cos(2\pi f_c\tau+\phi)g(t-(T-\tau))d\tau
\end{aligned}
$$

where $\phi-\phi_0=\Delta\phi$
Similrly we can find $\gamma_2(t)$. Notice that terms involving f_c will integrate to 0 approximately as $f_cT \gg 1$

13. For 4PSK $d_{min}=\sqrt{2\varepsilon} \Rightarrow \varepsilon_{4PSK}=1$
For 8PSK $d_{min}=\sqrt{\varepsilon+\varepsilon-2\varepsilon\cos(\pi/4)} \Rightarrow \varepsilon_{8PSK}=\frac{1}{1-\cos(\pi/4)}=3.4142$
extra energy factor = 3.4142 = 5.33dB

14. For square QAM constellations, it is easy to derive that

$$s_l=\frac{d_{min}^2}{6}(2^{2l}-1)=\left(\frac{d_{min}}{\sqrt(2)}\right)\frac{1}{3}(4^l-1)$$

$$\therefore s_l \propto \frac{4^l}{3}$$

$$\therefore s_{l+1} \propto \frac{44^l}{3}=4s_l$$

(16 QAM) For MQAM, $s_l(l=2) = 2.5d_{min}^2$
(4 PAM) For MPAM, $s_l(l=2) = 1.25d_{min}^2$
(16 PSK) For MPSK, $s_l(l=2) = \frac{d_{min}^2}{2(1-\cos(\pi/8))} = 6.5685d_{min}^2$

15. M points are separated by an angle $\frac{2\pi}{M}$
If $|\Delta|\phi > \frac{1}{2}\frac{2\pi}{M}$, we will go into the decision region for another adjacent symbol and so will make a detection error

16. Gray encoding of bit sequence to phase transitions:
We first draw the figure and write down bit sequences for each phase in a way that exactly 1 bit changes between nearest neighbors. We get the following table

Bit Sequence	Phase transition
000	0
001	$\pi/4$
010	$3\pi/4$
011	$\pi/2$
100	$-\pi/4$
101	$-\pi/2$
110	π
111	$-3\pi/4$

The resulting encoding of the given sequence is as follows:

Bit Sequence	Mapped Symbol
	s(k-1) $= Ae^{j\pi/4}$
101	$s(k) = Ae^{-j\pi/4}$
110	$s(k+1) = Ae^{j3\pi/4}$
100	$s(k+2) = Ae^{j\pi/2}$
101	$s(k+3) = A$
110	$s(k+4) = Ae^{j\pi}$

17. (a) $a = 0.7071A$
$b = 1.366A$
(b) $A^2 = r^2\left(2 - 2\cos\frac{\pi}{4}\right)$
$r = 1.3066A$
(c) Avg power of 8PSK $= r^2 = 1.7071A^2$
Avg power of 8 QAM $= 1.1830A^2$
The 8QAM constellation has a lower average power by a factor of 1.443 (1.593 dB)
(d) See Fig 10
(e) We have 3 bits per symbol $\therefore$ symbol rate = 30 Msymbols/sec

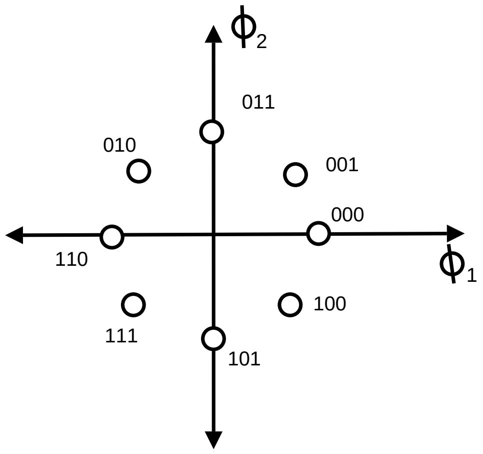

Figure 8: Problem 16

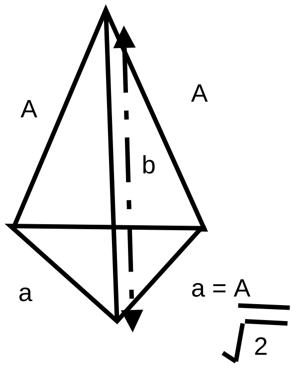

Figure 9: Problem 17a

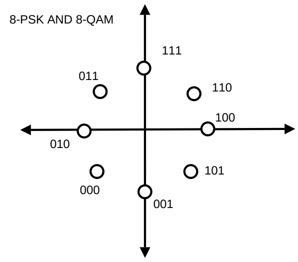

Figure 10: Problem 17d

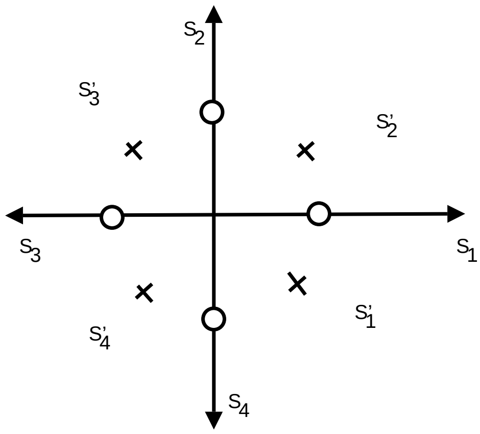

Figure 11: Problem 18a

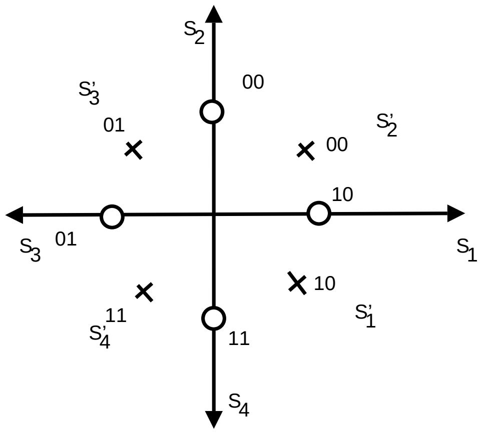

Figure 12: Problem 18b

18. (a) one set is on the axis and the other is $\pi/4-$ offset. If the current symbol uses a point of $\otimes$, the next symbol must come from $\odot$ and vice versa

    (b) See Fig 12

    (c) 0 1 00 10 01 11 10 01 01 Assume we start from the $\odot$ points
    s_3 s'_2 s_1 s'_3 s_4 s'_1 s_3 s'_3

    (d) See Fig 13

$b_1 b_0$	Phase change from previous symbol
00	$\pi/4$
01	$3\pi/4$
11	$-3\pi/4$
10	$-\pi/4$

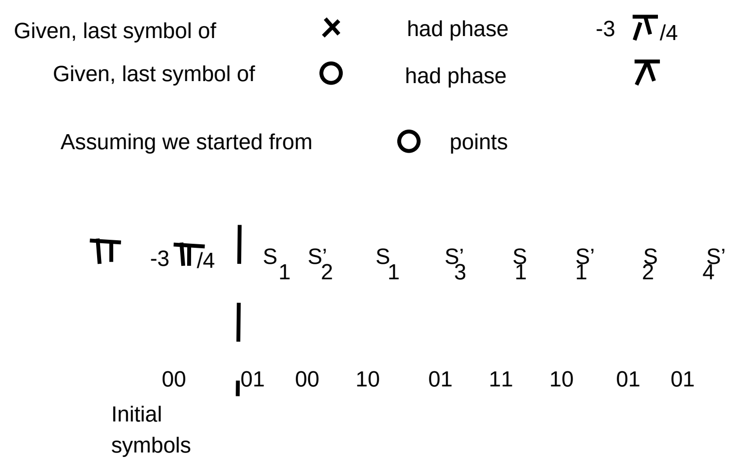

Figure 13: Problem 18d

19.

$$\int_0^T 2\cos(2\pi f_i t)\cos(2\pi f_j t)dt = 0 \Rightarrow \underbrace{\int_0^T 2\cos\left(2\pi(f_i+f_j)t\right)}_{A} + \underbrace{\int_0^T 2\cos\left(2\pi(f_i-f_j)t\right)}_{B}$$

$$A \to 0 \quad , \text{as } f_i + f_j \gg 1$$

$$B = \sin\left(2\pi(f_i - f_j)T\right) \quad \text{is 0 first time for } 2\pi(f_i - f_j)T = \pm\pi \Rightarrow |f_i - f_j| = 0.5T$$

```
MATLAB CODE
gamma_dB = [0:.01:60];
gamma = 10.^(gamma_dB/10);

vBdT = .01;
x = 2*pi*vBdT;
rho_C = besselj(0,x);
Pb_bar=.5*((1+gamma*(1-rho_C))./(1+gamma));
semilogy(gamma_dB,Pb_bar);

hold on;
vBdT = .001;
x = 2*pi*vBdT;
rho_C = besselj(0,x);
Pb_bar= .5*((1+gamma*(1-rho_C))./(1+gamma));
semilogy(gamma_dB,Pb_bar,'b:');

vBdT = .0001;
x = 2*pi*vBdT;
rho_C = besselj(0,x);
Pb_bar=.5*((1+gamma*(1-rho_C))./(1+gamma));
semilogy(gamma_dB,Pb_bar,'b--');
```

20.

$$p(kT) = \left\{ \begin{array}{ll} p_0 = p(0) & k = 0 \\ 0 & k \neq 0 \end{array} \right\} \ldots (a)$$

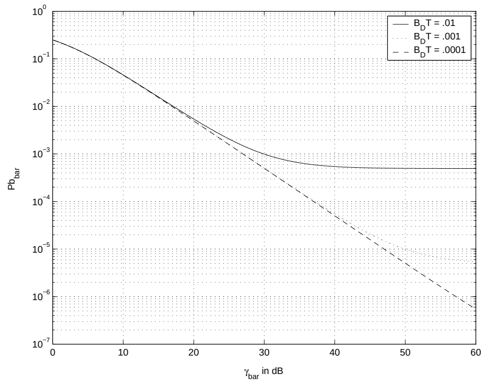

Figure 14: Problem 19

$$\begin{aligned}
p(t) &= \int_{-\infty}^{\infty} P(f)e^{j2\pi ft}df \\
p(kT) &= \int_{-\infty}^{\infty} P(f)e^{j2\pi fkT}df \\
p(kT) &= \sum_{m=-\infty}^{\infty}\int_{(2m-1)/2T}^{(2m+1)/2T} P(f)e^{j2\pi fkT}df \\
&= \sum_{m=-\infty}^{\infty}\int_{-1/2T}^{1/2T} P\left(f+\frac{m}{T}\right)e^{j2\pi fkT}df \\
&= \int_{-1/2T}^{1/2T}\sum_{m=-\infty}^{\infty} P\left(f+\frac{m}{T}\right)e^{j2\pi fkT}df \\
p(kT) &= \int_{-1/2T}^{1/2T} Q(f)e^{j2\pi fkT}df \quad \ldots 1
\end{aligned}$$

Q(f) is periodic with period 1/T and therefore it can be expanded in terms of Fourier coefficients

$$\begin{aligned}
Q(f) &= \sum_{n=-\infty}^{\infty} q_n e^{j2\pi fnT} \\
\text{where } q_n &= T\int_{-1/2T}^{1/2T} Q(f)e^{-j2\pi fnT}df \quad \ldots 2
\end{aligned}$$

Compare 1 and 2 to get

$$q_n = Tp(-nT)$$

$\therefore$ 'a' translates to

$$q_n = \begin{cases} p_0T & n=0 \\ 0 & n\neq 0 \end{cases}$$

But this means that $Q(f) = p_0T$ or $\sum_{l=-\infty}^{\infty} P\left(f + l/T\right) = p_0T$

21. Gaussian pulse is given as

$$g(t) = \frac{\sqrt{\pi}}{\alpha}e^{-\pi^2t^2/\alpha^2}$$

Notice that g(t) never goes to 0 except at $\pm\infty$
$\therefore$ Nyquist criterion

$$g(kT) = \begin{cases} g_0 & k=0 \\ 0 & k\neq 0 \end{cases}$$

cannot be satisfied for any finite T.

Chapter 6

1. (a) For sinc pulse, $B = \frac{1}{2T_s} \Rightarrow T_s = \frac{1}{2B} = 5 \times 10^{-5}s$

   (b) $SNR = \frac{P_b}{N_0 B} = 10$
   Since 4-QAM is multilevel signalling
   $SNR = \frac{P_b}{N_0 B} = \frac{E_s}{N_0 B T_s} = \frac{2E_s}{N_0 B} \quad (\because BT_s = \frac{1}{2})$
   $\therefore$ SNR per symbol $= \frac{E_s}{N_0} = 5$
   SNR per bit $= \frac{E_b}{N_0} = 2.5$ (a symbol has 2 bits in 4QAM)

   (c) SNR per symbol remains the same as before $= \frac{E_s}{N_0} = 5$
   SNR per bit is halved as now there are 4 bits in a symbol $\frac{E_b}{N_0} = 1.25$

2. $p_0 = 0.3, p_1 = 0.7$

   (a)

$$
\begin{aligned}
P_e &= Pr(0 \text{ detected, 1 sent — 1 sent})p(1 \text{ sent}) + Pr(1 \text{ detected, 0 sent — 0 sent})p(0 \text{ sent}) \\
&= 0.7Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) + 0.3Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) = Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)
\end{aligned}
$$

   $d_{min} = 2A$

$$
= Q\left(\sqrt{\frac{2A^2}{N_0}}\right)
$$

   (b)

$$
p(\hat{m} = 0|m = 1)p(m = 1) = p(\hat{m} = 1|m = 0)p(m = 0)
$$

$$
0.7Q\left(\frac{A+a}{\sqrt{\frac{N_0}{2}}}\right) = 0.3Q\left(\frac{A-a}{\sqrt{\frac{N_0}{2}}}\right) \quad , a > 0
$$

   Solving gives us 'a' for a given A and N_0

   (c)

$$
p(\hat{m} = 0|m = 1)p(m = 1) = p(\hat{m} = 1|m = 0)p(m = 0)
$$

$$
0.7Q\left(\frac{A}{\sqrt{\frac{N_0}{2}}}\right) = 0.3Q\left(\frac{B}{\sqrt{\frac{N_0}{2}}}\right) \quad , a > 0
$$

   Clearly $A > B$, for a given A we can find B

   (d) Take $\frac{E_b}{N_0} = \frac{A^2}{N_0} = 10$
   In part a) $P_e = 3.87 \times 10^{-6}$
   In part b) a=0.0203 $P_e = 3.53 \times 10^{-6}$
   In part c) B=0.9587 $P_e = 5.42 \times 10^{-6}$
   Clearly part (b) is the best way to decode.

```
MATLAB CODE:
A = 1;
N0 = .1;
a = [0:.00001:1];
t1 = .7*Q(A/sqrt(N0/2));
```

```
t2=.3*Q(a/sqrt(N0/2));
diff = abs(t1-t2);
[c,d] = min(diff);
a(d)
c
```

3. $s(t) = \pm g(t)\cos 2\pi f_c t$
   $r = \hat{r}\cos\Delta\phi$
   where $\hat{r}$ is the signal after the sampler if there was no phase offset. Once again, the threshold that minimizes P_e is 0 as $(\cos\Delta\phi)$ acts as a scaling factor for both +1 and -1 levels. P_e however increases as numerator is reduced due to multiplication by $\cos\Delta\phi$

$$P_e = Q\left(\frac{d_{min}\cos\Delta\phi}{\sqrt{2N_0}}\right)$$

4.

$$\begin{aligned}
A_c^2\int_0^{T_b}\cos^2 2\pi f_c t dt &= A_c^2\int_0^{T_b}\frac{1+\cos 4\pi f_c t}{2} \\
&= A_c^2\left[\frac{T_b}{2}+\underbrace{\frac{\sin(4\pi f_c T_b)}{8\pi f_c}}_{\to 0 \text{ as } f_c \gg 1}\right] \\
&= \frac{A_c^2 T_b}{2} = 1 \\
x(t) &= 1+n(t)
\end{aligned}$$

Let prob 1 sent $=p_1$ and prob 0 sent $=p_0$

$$\begin{aligned}
P_e &= \frac{1}{6}[1.p_1+0.p_0]+\frac{2}{6}[0.p_1+0.p_0]+\frac{2}{6}[0.p_1+0.p_0]+ \\
&\quad \frac{1}{6}[0.p_1+1.p_0] \\
&= \frac{1}{6}[p_1+p_0] = \frac{1}{6} \quad (\because p_1+p_0=1 \text{ always })
\end{aligned}$$

5. We will use the approximation $P_e \sim$ (average number of nearest neighbors).$Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$
   where number of nearest neighbors = total number of points taht share decision boundary

   (a) 12 inner points have 5 neighbors
       4 outer points have 3 neighbors
       avg number of neighbors = 4.5
       $P_e = 4.5Q\left(\frac{2a}{\sqrt{2N_0}}\right)$
   (b) 16QAM, $P_e = 4\left(1-\frac{1}{4}\right)Q\left(\frac{2a}{\sqrt{2N_0}}\right) = 3Q\left(\frac{2a}{\sqrt{2N_0}}\right)$
   (c) $P_e \sim \frac{2\times3+3\times2}{5}Q\left(\frac{2a}{\sqrt{2N_0}}\right) = 2.4Q\left(\frac{2a}{\sqrt{2N_0}}\right)$
   (d) $P_e \sim \frac{1\times4+4\times3+4\times2}{9}Q\left(\frac{3a}{\sqrt{2N_0}}\right) = 2.67Q\left(\frac{3a}{\sqrt{2N_0}}\right)$

6.

$$P_{s,\text{exact}} = 1-\left(1-\frac{2(\sqrt{M}-1)}{\sqrt{M}}Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)\right)^2$$

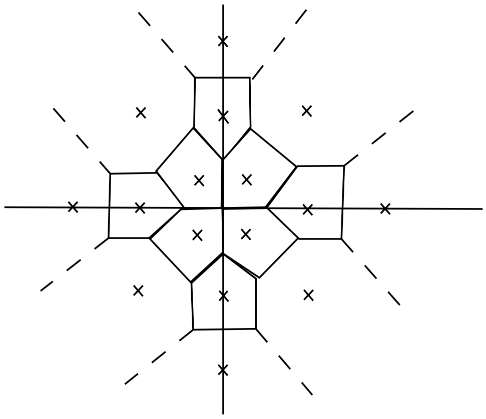

Figure 1: Problem 5

$$P_{s,\text{approx}} = \frac{4(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$$

approximation is better for high SNRs as then the multiplication factor is not important and P_e is dictated by the coefficient of the Q function which are same.

```
MATLAB CODE:
gamma_db = [0:.01:25];
gamma = 10.^(gamma_db/10);
M = 16;

Ps_exact=1-exp(2*log((1-((2*(sqrt(M)-1))/(sqrt(M)))*Q(sqrt((3*gamma)/(M-1))))));
Ps_approx = ((4*(sqrt(M)-1))/sqrt(M))*Q(sqrt((3*gamma)/(M-1)));
semilogy(gamma_db, Ps_exact);
hold on
semilogy(gamma_db,Ps_approx,'b:');
```

7. See figure. The approximation error decreases with SNR because the approximate formula is based on nearest neighbor approximation which becomes more realistic at higher SNR. The nearest neighbor bound over-estimates the error rate because it over-counts the probability that the transmitted signal is mistaken for something other than its nearest neighbors. At high SNR, this is very unlikely and this over-counting becomes negligible.

8. (a)

$$I_x(a) = \int_0^\infty \frac{e^{-at^2}}{x^2+t^2} dt$$

since the integral converges we can interchange integral and derivative for a¿0

$$\begin{aligned} \frac{\partial I_x(a)}{\partial a} &= \int_0^\infty \frac{-te^{-at^2}}{x^2+t^2} dt \\ x^2 I_x(a) - \frac{\partial I_x(a)}{\partial a} &= \int_0^\infty \frac{(x^2+t^2)e^{-at^2}}{x^2+t^2} dt = \int_0^\infty e^{-at^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{a}} \end{aligned}$$

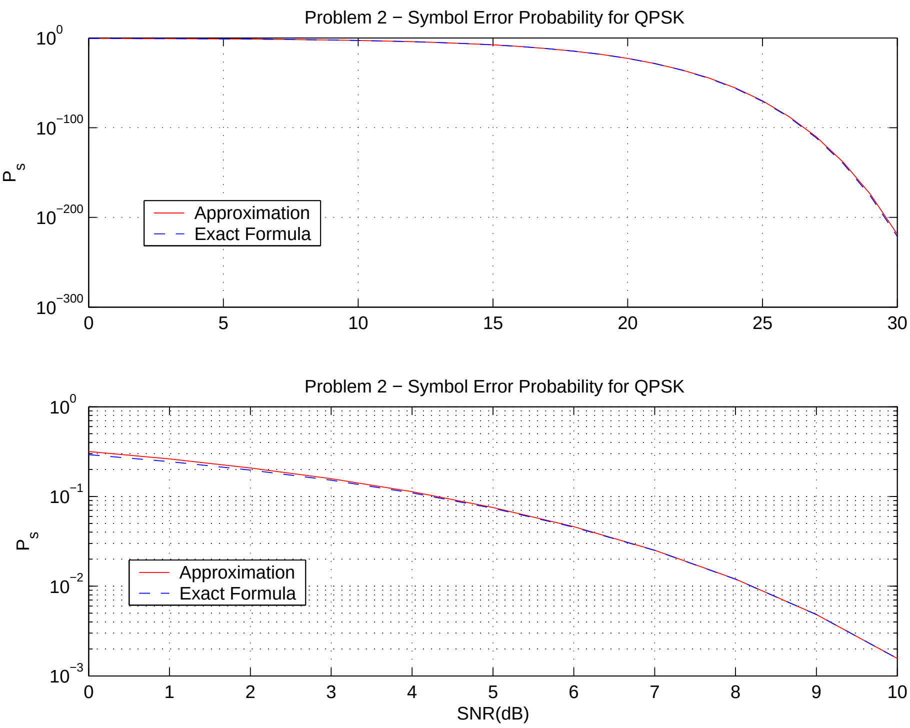

Figure 2: Problem 7

(b) Let $I_x(a) = y$, we get

$$\begin{aligned}
y' - x^2 y &= -\frac{1}{2}\sqrt{\frac{\pi}{a}} \\
\text{comparing with} & \\
y' + P(a)y &= Q(a) \\
P(a) = -x^2 \quad &, \quad Q(a) = -\frac{1}{2}\sqrt{\frac{\pi}{a}} \\
I.F. = e^{\int P(u)u} = e^{-x^2 a} & \\
\therefore e^{-x^2 a} y &= \int -\frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-x^2 u} du \\
\text{solving we get} & \\
y &= \frac{\pi}{2x} e^{ax^2} erfc(x\sqrt{a})
\end{aligned}$$

(c)

$$\begin{aligned}
erfc(x\sqrt{a}) &= I_x(a)\frac{2x}{\pi}e^{-ax^2} = \frac{2x}{\pi}e^{-ax^2}\int_0^\infty \frac{e^{-at^2}}{x^2+t^2}dt \\
a = 1 & \\
erfc(x) &= \frac{2x}{\pi}e^{-ax^2}\int_0^\infty \frac{e^{-at^2}}{x^2+t^2}dt \\
&= \frac{2}{\pi}\int_0^{\pi/2} e^{-x^2/sin^2\theta} d\theta \\
Q(x) = \frac{1}{2}erfc(x/\sqrt{2}) &= \frac{1}{\pi}\int_0^{\pi/2} e^{-x^2/2sin^2\theta} d\theta
\end{aligned}$$

9. $\overline{P} = 100W$
$N_0 = 4W, \quad SNR = 25$
$P_e = Q(\sqrt{2\gamma}) = Q(\sqrt{50}) = 7.687 \times 10^{-13}$
data requires $P_e \sim 10^{-6}$
voice requires $P_e \sim 10^{-3}$
so it can be used for both.
with fading $P_e = \frac{1}{4\overline{\gamma}_b} = 0.01$
So the system can't be used for data at all. It can be used for very low quality voice.

10. $T_s = 15\mu sec$
at 1mph $T_c = \frac{1}{B_d} = \frac{1}{v/\lambda} = 0.74s \gg T_s$
$\therefore$ outage probability is a good measure.
at 10 mph $T_c = 0.074s \gg T_s$ $\therefore$ outage probability is a good measure.
at 100 mph $T_c = 0.0074s = 7400\mu s > 15\mu s$ outage or outage combined with average prob of error can be a good measure.

11.

$$\begin{aligned} M_\gamma(s) &= \int_0^\infty e^{s\gamma} p(\gamma) d\gamma \\ &= \int_0^\infty e^{s\gamma} \frac{1}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}} d\gamma \\ &= \frac{1}{1 - \overline{\gamma} s} \end{aligned}$$

12. (a) When there is path loss alone, $d = \sqrt{100^2 + 500^2} = 100\sqrt{6} \times 10^3$
$P_e = \frac{1}{2} e^{-\gamma_b} \Rightarrow \gamma_b = 13.1224$
$\frac{P_\gamma}{N_0 B} = 13.1224 \Rightarrow P_\gamma = 1.3122 \times 10^{-14}$
$\frac{P_\gamma}{P_t} = \left[\frac{\sqrt{G}\lambda}{4\pi d}\right]^2 \Rightarrow 4.8488W$

(b)

$$x = 1.3122 \times 10^{-14} = -138.82dB$$

$$P_{\gamma,dB} \sim N(\mu P_\gamma, 8), \sigma_{dB} = 8$$

$$\begin{aligned} P(P_{\gamma,dB} \geq x) &= 0.9 \\ P\left(\frac{P_{\gamma,dB} - \mu P_\gamma}{8} \geq \frac{x - \mu P_\gamma}{8}\right) &= 0.9 \\ \Rightarrow Q\left(\frac{x - \mu P_\gamma}{8}\right) &= 0.9 \\ \Rightarrow \frac{x - \mu P_\gamma}{8} &= -1.2816 \end{aligned}$$

$$\Rightarrow \mu P_\gamma = -128.5672dB = 1.39 \times 10^{-13}$$

13. (a) Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos C$ with a,b = $\sqrt{E_s}$, c = d_{min}, C = $\Theta = 22.5$
c = $d_{min} = \sqrt{2E_s(1 - \cos 22.5)} = .39\sqrt{E_s}$
Can also use formula from reader

(b) $P_s = \alpha_m Q\left(\sqrt{\beta_m \gamma_s}\right) = 2Q\left(\sqrt{\frac{{d_{min}}^2}{2N_o}}\right) = 2Q(\sqrt{.076\gamma_s})$
$\alpha_m = 2,\ \beta_m = .076$

(c) $\overline{P_e} = \int\limits_0^\infty P_s(\gamma_s) f(\gamma_s) d\gamma_s$

$= \int\limits_0^\infty \alpha_m Q(\sqrt{\beta_m \gamma_s}) f(\gamma_s) d\gamma_s$

Using alternative Q form

$= \frac{\alpha_m}{\pi} \int\limits_0^{\frac{\pi}{2}} \left(1 + \frac{g\overline{\gamma_s}}{(\sin\phi)^2}\right)^{-1} d\phi$ with g $= \frac{\beta_m}{2}$

$= \frac{\alpha_m}{2}\left[1 - \sqrt{\frac{g\overline{\gamma_s}}{1+g\overline{\gamma_s}}}\right] = 1 - \sqrt{\frac{.038\overline{\gamma_s}}{1+.038\overline{\gamma_s}}} = \frac{1}{.076\overline{\gamma_s}}$, where we have used an integral table to evaluate the integral

(d) $P_d = \frac{P_s}{4}$

(e) BPSK: $\overline{P}_b = \frac{1}{4\overline{\gamma}_b} = 10^{-3}$, $\Rightarrow \overline{\gamma_b} = 250$, 16PSK: From above get $\overline{\gamma_s} = 3289.5$

Penalty $= \frac{3289.5}{250} = 11.2$dB

Also will accept $\gamma_b(16PSK) = 822 \Rightarrow = 5.2dB$

14.

$$\begin{aligned}
\overline{P}_b &= \int_0^\infty P_b(\gamma) p(\gamma) d\gamma \\
P_b(\gamma) = \frac{1}{2} e^{-\gamma} & \\
\overline{P}_b &= \frac{1}{2}\int_0^\infty e^{-\gamma b} p_\gamma(\gamma) d\gamma = \frac{1}{2}\mathcal{M} \\
\text{But from 6.65} & \\
\mathcal{M}_\gamma(s) &= \left(1 - \frac{s\gamma}{m}\right)^{-m} \\
\therefore \overline{P}_b &= \frac{1}{2}\left(1 + \frac{\gamma}{m}\right)^{-m} \\
\text{For} \quad M = 4, \quad \gamma = 10 & \\
\overline{P}_b &= 3.33 \times 10^{-3}
\end{aligned}$$

15.
```
%Script used to plot the average probability of bit error for BPSK modulation in
%Nakagami fading m = 1, 2, 4.
%Initializations
avg_SNR = [0:0.1:20]; gamma_b_bar = 10.^(avg_SNR/10); m = [1 2 4];
line = ['-k', '-r', '-b']

for i = 1:size(m,2)
   for j = 1:size(gamma_b_bar, 2)
        Pb_bar(i,j) = (1/pi)*quad8('nakag_MGF',0,pi/2,[],[],gamma_b_bar(j),m(i),1);
   end
   figure(1);
   semilogy(avg_SNR, Pb_bar(i,:), line(i));
   hold on;
end

xlabel('Average SNR ( gamma_b ) in dB'); ylabel('Average bit error
probability ( P_b ) '); title('Plots of P_b for BPSK modulation in
Nagakami fading for m = 1, 2, 4'); legend('m = 1', 'm = 2', 'm =
4');
```

```
function out = nakag_MGF(phi, gamma_b_bar, m, g);
%This function calculates the m-Nakagami MGF function for the specified values of phi.
%phi can be a vector. Gamma_b_bar is the average SNR per bit, m is the Nakagami parameter
%and g is given by Pb(gamma_b) = aQ(sqrt(2*g*gamma_b)).

out = (1 + gamma_b_bar./(m*(sin(phi).^2)) ).^(-m);
```

SNR = 10dB

M	BER
1	2.33×10^{-2}
2	5.53×10^{-3}
4	1.03×10^{-3}

16. For DPSK in Rayleigh fading, $\overline{P_b} = \frac{1}{2\overline{\gamma_b}} \Rightarrow \overline{\gamma_b} = 500$
$N_oB = 3 \times 10^{-12} mW \Rightarrow P_{target} = \overline{\gamma}_b N_0 B = 1.5 \times 10^{-9} mW$ = -88.24 dBm

Now, consider shadowing:
$P_{out} = P[P_r < P_{target}] = P[\Psi < P_{target} - \overline{P_r}] = \Phi\left(\frac{P_{target}-\overline{P_r}}{\sigma}\right)$
$\Rightarrow \Phi^{-1}(.01) = 2.327 = \frac{P_{target}-\overline{P_r}}{\sigma}$
$\overline{P_r} = -74.28$ dBm $= 3.73 \times 10^{-8}$ mW $= P_t\left(\frac{\lambda}{4\pi d}\right)^2$
$\Rightarrow$ d = 1372.4 m

17. (a)

$$\gamma_1 = \begin{cases} 0 & \text{w.p. } 1/3 \\ 30 & \text{w.p. } 2/3 \end{cases}$$

$$\gamma_2 = \begin{cases} 5 & \text{w.p. } 1/2 \\ 10 & \text{w.p. } 1/2 \end{cases}$$

In MRC, $\gamma_\Sigma = \gamma_1 + \gamma_2$. So,

$$\gamma_\Sigma = \begin{cases} 5 & \text{w.p. } 1/6 \\ 10 & \text{w.p. } 1/6 \\ 35 & \text{w.p. } 1/3 \\ 40 & \text{w.p. } 1/3 \end{cases}$$

(b) Optimal Strategy is water-filling with power adaptation:

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}, & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

Notice that we will denote γ_Σ by γ only hereon to lighten notation. We first assume $\gamma_0 < 5$,

$$\sum_{i=1}^{4} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i}\right) p_i = 1$$

$$\Rightarrow \frac{1}{\gamma_0} = 1 + \sum_{i=1}^{4} \frac{p_i}{\gamma_i}$$

$$\Rightarrow \gamma_0 = 0.9365 < 5$$

So we found the correct value of γ_0.

$$C = B\sum_{i=1}^{4} \log_2\left(\frac{\gamma_i}{\gamma_0}\right) p_i$$

C = 451.91 Kbps

(c) Without, receiver knowledge, the capacity of the channel is given by:

$$C = B\sum_{i=1}^{4} \log_2(1+\gamma_i)p_i$$

C = 451.66 Kbps

Notice that we have denote γ_Σ by γ to lighten notation.

18. (a)

$$\begin{aligned} s(k) &= s(k-1) \\ z(k-1) &= g_{k-1}s(k-1) + n(k-1) \\ z(k) &= g_k s(k) + n(k) \end{aligned}$$

From equation 5.63 , the input to the phase comparator is

$$\begin{aligned} z(k)z^\star(k-1) &= g_k g_(k-1)^\star s(k)s^\star(k-1) + g_k s(k-1)n_{k-1}^\star + \\ & \quad g_(k-1)^\star s^\star(k-1)n_k + n_k n_{k-1}^\star \end{aligned}$$

but $s(k) = s(k-1)$

$$s(k)s^\star(k-1) = |s_k|^2 = 1 \quad \text{(normalized)}$$

(b)

$$\begin{aligned} \widetilde{n}_k &= s_{k-1}^\star n_k \\ \widetilde{n}_{k-1} &= s_{k-1}^\star n_{k-1} \\ \widetilde{z} &= g_k g_{k-1}^\star + g_k \widetilde{n}_{k-1}^\star + g_{k-1}^\star \widetilde{n}_k \\ \phi_x(s) = \frac{p_1 p_2}{(s-p_1)(s-p_2)} &= \frac{A}{s-p_1} + \frac{B}{s-p_2} \\ A = (s-p_1)\phi_x(s)|_{s=p_1} &= \frac{p_1 p_2}{p_1 - p_2} \\ B = (s-p_2)\phi_x(s)|_{s=p_2} &= \frac{p_1 p_2}{p_2 - p_1} \end{aligned}$$

(c) Relevant part of the pdf

$$\phi_x(s) = \frac{p_1 p_2}{(p_2-p_1)(s-p_2)}$$

$$\therefore p_x(x) = \frac{p_1 p_2}{(p_2-p_1)}\mathcal{L}^{-1}\left(\frac{1}{(s-p_2)}\right) = \frac{p_1 p_2}{(p_2-p_1)}e^{p_2 x} \quad , x < 0$$

(d)

$$P_b = prob(x<0) = \frac{p_1 p_2}{(p_2-p_1)}\int_{-\infty}^{0} e^{p_2 x}dx = -\frac{p_1}{p_2-p_1}$$

(e)

$$p_2 - p_1 = \frac{1}{2N_0[\overline{\gamma}_b(1-\rho_c)+1]} + \frac{1}{2N_0[\overline{\gamma}_b(1+\rho_c)+1]} = \frac{\overline{\gamma}_b + 1}{N_0[\overline{\gamma}_b(1-\rho_c)+1][\overline{\gamma}_b(1+\rho_c)+1]}$$

$$\therefore \overline{P}_b = \frac{\overline{\gamma}_b(1-\rho_c)+1}{2(\overline{\gamma}_b+1)}$$

(f) $\rho_c = 1$

$$\therefore \overline{P}_b = \frac{1}{2(\overline{\gamma}_b+1)}$$

19. $\overline{\gamma}_b$ 0 to 60dB
$\rho_c = J_0(2\pi B_D T)$ with $B_D T = 0.01, 0.001, 0.0001$
where J_0 is 0 order Bessel function of 1^{st} kind.

$$\overline{P}_b = \frac{1}{2}\left[\frac{1+\overline{\gamma}_b(1-\rho_c)}{1+\overline{\gamma}_b}\right]$$

when $B_D T = 0.01$, floor can be seen about $\overline{\gamma}_b = 40dB$
when $B_D T = 0.001$, floor can be seen about $\overline{\gamma}_b = 60dB$
when $B_D T = 0.0001$, floor can be seen between $\overline{\gamma}_b = 0$ to $60dB$

20. Data rate = 40 Kbps
Since DQPSK has 2 bits per symbol. $\therefore T_s = \frac{2}{40\times 10^3} = 5\times 10^{-5} sec$

DQPSK
Gaussian Doppler power spectrum, $\rho_c = e^{-(\pi B_D T)^2}$
$B_D = 80Hz$
Rician fading $K = 2$
$\rho_c = 0.9998$

$$\overline{P}_{floor} = \frac{1}{2}\left[1 - \sqrt{\frac{(\rho_c/\sqrt{2})^2}{1-(\rho_c/\sqrt{2})^2}}\right]\exp\left[-\frac{(2-\sqrt{2})K/2}{1-\rho_c/\sqrt{2}}\right] = 2.138\times 10^{-5}$$

21. <u>ISI:</u>

<u>Formula based approach:</u>

$$P_{floor} = \left(\frac{\sigma T_m}{T_s}\right)^2$$

Since its Rayleigh fading, we can assume that $\sigma_{T_m} \approx \mu_{T_m} = 100$ns
$P_{floor} \leq 10^{-4}$
which gives us

$$\left(\frac{\sigma T_m}{T_s}\right)^2 \leq 10^{-4}$$

$$T_s \geq \frac{\sigma_{T_m}}{\sqrt{P_{floor}}} = 10\mu sec$$

So, $T_s \geq 10\mu s$. $T_b \geq 5\mu s$. $R_b \leq 200$ Kbps.

Thumb - Rule approach:

μ_t = 100 nsec will determine ISI. As long as $T_s \gg \mu_T$, ISI will be negligible. Let $T_s \geq 10\ \mu_T$. Then $R \leq \frac{2bits}{symbol}\frac{1}{T_s}\frac{symbols}{sec} = 2$Mbps

Doppler:

$B_D = 80$ Hz

$$P_{floor} = 10^{-4} \geq \frac{1}{2}\left[1 - \sqrt{\frac{\left(\rho_c/\sqrt{2}\right)^2}{1-\left(\rho_c/\sqrt{2}\right)^2}}\right]$$

$$\Rightarrow \rho_c \geq 0.9999$$

But ρ_c for uniform scattering is $J_0(2\pi B_D T_s)$, so

$$\rho_c = J_0(2\pi B_D T_s) = 1 - (\pi f_D T_s)^2 \geq 0.9999$$

$$\Rightarrow T_s \leq 39.79\mu s$$

$T_b \leq 19.89\mu s$. $R_b \geq 50.26$ Kbps.

Combining the two, we have 50.26 Kbps $\leq R_b \leq$ 200 Kbps (or 2 Mbps).

22. From figure 6.5
with $P_b = 10^{-3}$, $\quad d = \theta_{T_m}/T_s$, $\quad \theta_{T_m} = 3\mu s$
BPSK

$$d = 5 \times 10^{-2}$$

$$T_s = 60\mu sec$$

$$R = 1/T_s = 16.7Kbps$$

QPSK

$$d = 4 \times 10^{-2}$$

$$T_s = 75\mu sec$$

$$R = 2/T_s = 26.7Kbps$$

MSK

$$d = 3 \times 10^{-2}$$

$$T_s = 100\mu sec$$

$$R = 2/T_s = 20Kbps$$

Chapter 7

1. $P_s = 10^{-3}$
   QPSK, $P_s = 2Q(\sqrt{\gamma_s}) \leq 10^{-3}$, $\gamma_s \geq \gamma_0 = 10.8276$.

$$P_{out}(\gamma_0) = \prod_{i=1}^{M} \left(1 - e^{-\frac{\gamma_0}{\overline{\gamma}_i}}\right)$$

   $\overline{\gamma}_1 = 10$, $\overline{\gamma}_2 = 31.6228$, $\overline{\gamma}_3 = 100$.

   $M = 1$
   $P_{out} = \left(1 - e^{-\frac{\gamma_0}{\overline{\gamma}_1}}\right) = 0.6613$

   $M = 2$
   $P_{out} = \left(1 - e^{-\frac{\gamma_0}{\overline{\gamma}_1}}\right)\left(1 - e^{-\frac{\gamma_0}{\overline{\gamma}_2}}\right) = 0.1917$

   $M = 3$
   $P_{out} = \left(1 - e^{-\frac{\gamma_0}{\overline{\gamma}_1}}\right)\left(1 - e^{-\frac{\gamma_0}{\overline{\gamma}_2}}\right)\left(1 - e^{-\frac{\gamma_0}{\overline{\gamma}_3}}\right) = 0.0197$

2. $p_{\gamma_\Sigma}(\gamma) = \frac{M}{\overline{\gamma}}\left[1 - e^{-\gamma/\overline{\gamma}}\right]^{M-1} e^{-\gamma/\overline{\gamma}}$
   $\overline{\gamma} = 10dB = 10$
   as we increase M, the mass in the pdf keeps on shifting to higher values of γ and so we have higher values of γ and hence lower probability of error.

   MATLAB CODE

```
gamma = [0:.1:60];
gamma_bar = 10;
M = [1 2 4 8 10];
fori=1:length(M)
    pgamma(i,:) = (M(i)/gamma_bar)*(1-exp(-gamma/gamma_bar)).^...
                  (M(i)-1).*(exp(-gamma/gamma_bar));
end
```

3.

$$\begin{aligned}
\overline{P}_b &= \int_0^\infty \frac{1}{2} e^{-\gamma} p_{\gamma_\Sigma}(\gamma) d\gamma \\
&= \int_0^\infty \frac{1}{2} e^{-\gamma} \frac{M}{\overline{\gamma}} \left[1 - e^{-\gamma/\overline{\gamma}}\right]^{M-1} e^{-\gamma/\overline{\gamma}} d\gamma \\
&= \frac{M}{2\overline{\gamma}} \int_0^\infty e^{-(1+1/\overline{\gamma})\gamma} \left[1 - e^{-\gamma/\overline{\gamma}}\right]^{M-1} d\gamma \\
&= \frac{M}{2\overline{\gamma}} \sum_{n=0}^{M-1} \binom{M-1}{n} (-1)^n e^{-(1+1/\overline{\gamma})\gamma} d\gamma \\
&= \frac{M}{2} \sum_{n=0}^{M-1} \binom{M-1}{n} (-1)^n \frac{1}{1+n+\overline{\gamma}} = \text{desired expression}
\end{aligned}$$

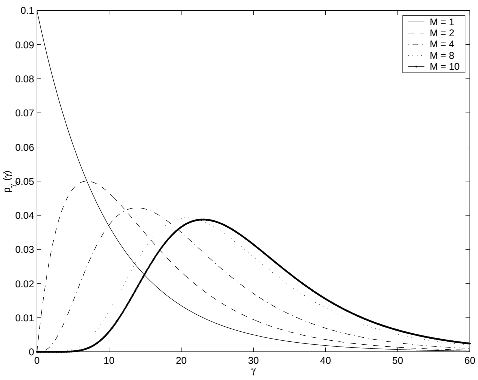

Figure 1: Problem 2

4.

$$p_{\gamma_\Sigma}(\gamma) = \begin{cases} Pr\{\gamma_2 < \gamma_\tau, \gamma_1 < \gamma\} & \gamma < \gamma_\tau \\ Pr\{\gamma_\tau \le \gamma_1 \le \gamma\} + Pr\{\gamma_2 < \gamma_\tau, \gamma_1 < \gamma\} & \gamma > \gamma_\tau \end{cases}$$

If the distribution is iid this reduces to

$$p_{\gamma_\Sigma}(\gamma) = \begin{cases} P_{\gamma_1}(\gamma) P_{\gamma_2}(\gamma_\tau) & \gamma < \gamma_\tau \\ Pr\{\gamma_\tau \le \gamma_1 \le \gamma\} + P_{\gamma_1}(\gamma) P_{\gamma_2}(\gamma_\tau) & \gamma > \gamma_\tau \end{cases}$$

5.

$$\overline{P}_b = \int_0^\infty \frac{1}{2} e^{-\gamma} p_{\gamma_\Sigma}(\gamma) d\gamma$$

$$p_{\gamma_\Sigma}(\gamma) = \begin{cases} \left(1 - e^{-\gamma_T/\overline{\gamma}}\right) \frac{1}{\overline{\gamma}} e^{-\gamma_r/\overline{\gamma}} & \gamma < \gamma_T \\ \left(2 - e^{-\gamma_T/\overline{\gamma}}\right) \frac{1}{\overline{\gamma}} e^{-\gamma_r/\overline{\gamma}} & \gamma > \gamma_T \end{cases}$$

$$\begin{aligned} \overline{P}_b &= \frac{1}{2\overline{\gamma}} \left(1 - e^{-\gamma_T/\overline{\gamma}}\right) \int_0^{\gamma_T} e^{-\gamma/\overline{\gamma}} e^{-\gamma} d\gamma + \frac{1}{2\overline{\gamma}} \left(2 - e^{-\gamma_r/\overline{\gamma}}\right) \int_{\gamma_T}^\infty e^{-\gamma/\overline{\gamma}} e^{-\gamma} d\gamma \\ &= \frac{1}{2(\overline{\gamma}+1)} \left(1 - e^{-\gamma_T/\overline{\gamma}} + e^{-\gamma_T} e^{-\gamma_T/\overline{\gamma}}\right) \end{aligned}$$

6.

	$\overline{P}_b$	$\overline{P}_b(10dB)$	$\overline{P}_b(20dB)$
no diversity	$\frac{1}{2(\overline{\gamma}+1)}$	0.0455	0.0050
SC(M=2)	$\frac{M}{2} \sum_{m=0}^{M-1} (-1)^m \frac{\binom{M-1}{m}}{1+m+\overline{\gamma}}$	0.0076	9.7×10^{-5}
SSC	$\frac{1}{2(\overline{\gamma}+1)} \left(1 - e^{-\gamma_T/\overline{\gamma}} + e^{-\gamma_T} e^{-\gamma_T/\overline{\gamma}}\right)$	0.0129	2.7×10^{-4}

As SNR increases SSC approaches SC

7. See

```
MATLAB CODE:
gammab_dB = [0:.1:20];
gammab = 10.^(gammab_dB/10);
M= 2;
```

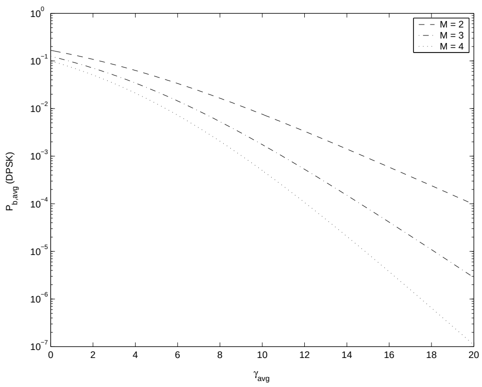

Figure 2: Problem 7

```
for j = 1:length(gammab)
    Pbs(j) = 0
    for m = 0:M-1
        f = factorial(M-1)/(factorial(m)*factorial(M-1-m));
        Pbs(j) = Pbs(j) + (M/2)*((-1)^m)*f*(1/(1+m+gammab(j)));
    end
end
semilogy(gammab_dB,Pbs,'b--')
hold on

M = 3;
for j = 1:length(gammab)
    Pbs(j) = 0
    for m = 0:M-1
        f = factorial(M-1)/(factorial(m)*factorial(M-1-m));
        Pbs(j) = Pbs(j) + (M/2)*((-1)^m)*f*(1/(1+m+gammab(j)));
    end
end
semilogy(gammab_dB,Pbs,'b-.');
hold on

M = 4;
for j = 1:length(gammab)
    Pbs(j) = 0
    for m = 0:M-1
        f = factorial(M-1)/(factorial(m)*factorial(M-1-m));
        Pbs(j) = Pbs(j) + (M/2)*((-1)^m)*f*(1/(1+m+gammab(j)));
    end
end
semilogy(gammab_dB,Pbs,'b:');
hold on
```

8.

$$\gamma_\Sigma = \frac{1}{N_0}\frac{\left(\sum_{i=1}^M a_i\gamma_i\right)^2}{\sum_{i=1}^M a_i^2} \le \frac{1}{N_0}\frac{\sum a_i^2 \sum \gamma_i^2}{\sum a_i^2} = \frac{\sum \gamma_i^2}{N_0}$$

Where the inequality above follows from Cauchy-Schwartz condition. Equality holds if $a_i = c\gamma_i$ where c is a constant

9. (a) $\gamma_i = 10$ dB $= 10,\ 1 \le i \le N$
   $N = 1,\ \gamma = 10,\ M = 4$
   $P_b = .2e^{-1.5\frac{\gamma}{(M-1)}} = .2e^{-15/3} = 0.0013.$
   (b) In MRC, $\gamma_\Sigma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$.
   So $\gamma_\Sigma = 10N$

$$P_b = .2e^{-1.5\frac{\gamma_\Sigma}{(M-1)}} = .2e^{-5N} \le 10^{-6}$$
$$\Rightarrow N \ge 2.4412$$

   So, take $N = 3$, $P_b = 6.12\ \times 10^{-8} \le 10^{-6}$.

10. Denote $N(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\quad, Q'(x) = -N(x)$

$$\begin{aligned}
\overline{P}_b &= \int_0^\infty Q(\sqrt{2\gamma})dP(\gamma) \\
Q(\infty) = 0, \quad P(0) = 0 & \\
\frac{d}{d\gamma}Q(\sqrt{2\gamma}) &= -N(\sqrt{2\gamma})\frac{\sqrt{2}}{2\sqrt{\gamma}} = -\frac{1}{\sqrt{2\pi}}e^{-\gamma}\frac{1}{2\sqrt{\gamma}} \\
\overline{P}_b &= \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\gamma}\frac{1}{2\sqrt{\gamma}}P(\gamma)d\gamma \\
P(\gamma) &= 1 - e^{-\gamma/\overline{\gamma}}\sum_{k=1}^{M}\frac{(\gamma/\overline{\gamma})^{k-1}}{(k-1)!} \\
1 \quad \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\gamma}\frac{1}{2\sqrt{\gamma}}d\gamma &= \frac{1}{2} \\
2 \quad \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\gamma}\frac{1}{2\sqrt{\gamma}}e^{-\gamma/\overline{\gamma}}\sum_{k=1}^{M}\frac{(\gamma/\overline{\gamma})^{k-1}}{(k-1)!}d\gamma &= \sum_{k=1}^{M}\frac{1}{(k-1)!}\left[\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-\gamma\ 1+\frac{1}{\overline{\gamma}}}\ \gamma^{-1/2}\left(\frac{\gamma}{\overline{\gamma}}\right)^{k-1}d\gamma\right] \\
\text{Denote } A = \left(1 + \frac{1}{\overline{\gamma}}\right)^{-1/2} & \\
&= \sum_{m=0}^{M-1}\frac{1}{m!}\frac{1}{2\sqrt{pi}}\int_0^\infty e^{-\gamma/A^2}\gamma^{-1/2}\left(\frac{\gamma}{\overline{\gamma}}\right)^m d\gamma \\
\text{let } \gamma/A^2 = u & \\
&= \sum_{m=0}^{M-1}\frac{1}{m!}\frac{1}{2\sqrt{pi}}\int_0^\infty e^{-u}\frac{u^{-1/2}}{A}\left(\frac{uA^2}{\overline{\gamma}}\right)^m A^2du \\
&= \frac{A}{2} + \sum_{m=1}^{M-1}\binom{2m-1}{m}\frac{A^{2m}}{2^{2m}}\frac{A}{\overline{\gamma}^m} \\
\overline{P}_b &= \frac{1-A}{2} - \sum_{m=1}^{M-1}\binom{2m-1}{m}\frac{A^{2m+1}}{2^{2m}\overline{\gamma}^m}
\end{aligned}$$

11.

$$\text{Denote} N(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad Q'(x) = [1-\phi(x)]' = -N(x)$$

$$\overline{P_b} = \int_0^\infty Q(\sqrt{2\gamma}) dP(\gamma) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\gamma} \frac{1}{\sqrt{2\gamma}} P(\gamma) d\gamma$$

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\gamma} \frac{1}{\sqrt{2\gamma}} d\gamma = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \quad (1)$$

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\gamma} \frac{1}{\sqrt{2\gamma}} e^{-2\gamma/\overline{\gamma}} d\gamma = \frac{1}{2\sqrt{1+\frac{2}{\overline{\gamma}}}} \quad (2)$$

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\gamma} \frac{1}{\sqrt{2\gamma}} \sqrt{\frac{\pi\gamma}{\overline{\gamma}}} e^{-\gamma/\overline{\gamma}} \left(1 - 2Q\left(\sqrt{\frac{2\gamma}{\overline{\gamma}}}\right)\right) d\gamma = \frac{1}{2\sqrt{\overline{\gamma}}} \frac{1}{B\sqrt{A\overline{\gamma}}} \quad (3)$$

$$\text{where } A = 1 + \frac{2}{\overline{\gamma}}, \quad B = 1 + \frac{1}{\overline{\gamma}}$$

$$\text{overall } \overline{P}_b = \frac{1}{2}\left[1 - \sqrt{1 - \frac{1}{(1+\overline{\gamma})^2}}\right]$$

12.

	$\overline{P}_b$	$\overline{P}_b(10dB)$	$\overline{P}_b(20dB)$
no diversity	$\frac{1}{2}\left[1-\sqrt{\frac{\overline{\gamma}_b}{1+\overline{\gamma}_b}}\right]$	0.0233	0.0025
two branch SC	$\int Q(\sqrt{2\gamma}) p_{\gamma_\Sigma} d\gamma$	0.0030	3.67×10^{-5}
two branch SSC	$\int Q(\sqrt{2\gamma}) p_{\gamma_\Sigma} d\gamma$	0.0057	1.186×10^{-4}
two branch EGC	$\int Q(\sqrt{2\gamma}) p_{\gamma_\Sigma} d\gamma$	0.0021	2.45×10^{-5}
two branch MRC	$\int Q(\sqrt{2\gamma}) p_{\gamma_\Sigma} d\gamma$	0.0016	0.84×10^{-5}

As the branch SNR increases the performance of all diversity combining schemes approaches the same.

```
MATLAB CODE:
gammatv = [.01:.1:10];
gammab = 100;
gamma = [0:.01:50*gammab];
for i = 1:length(gammatv)
    gammat = gammatv(i);
    gamma1 = [0:.01:gammat];
    gamma2 = [gammat+.01:.01:50*gammab];
    tointeg1 = Q(sqrt(2*gamma1)).*((1/gammab)*(1-exp(-gammat/gammab)).*exp(-gamma1/gammab));
    tointeg2 = Q(sqrt(2*gamma2)).*((1/gammab)*(2-exp(-gammat/gammab)).*exp(-gamma2/gammab));
    anssum(i) = sum(tointeg1)*.01+sum(tointeg2)*.01;
end
```

13.
```
gammab_dB = [10];
gammab = 10.^(gammab_dB/10);
Gamma=sqrt(gammab./(gammab+1));
pb_mrc =(((1-Gamma)/2).^2).*(((1+Gamma)/2).^0+2*((1+Gamma)/2).^1);
pb_egc = .5*(1-sqrt(1-(1./(1+gammab)).^2));
```

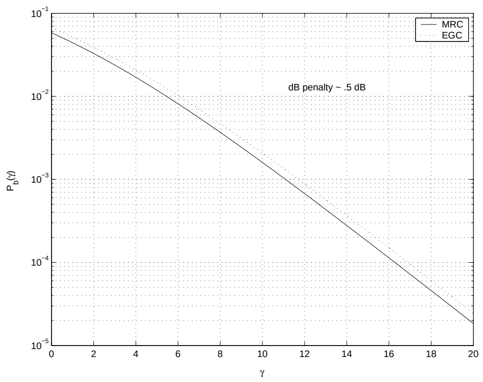

Figure 3: Problem 13

14. $10^{-3} = P_b = Q(\sqrt{2\gamma_b}) \Rightarrow 4.75, \overline{\gamma} = 10$
MRC $P_{out} = 1 - e^{-\gamma_0/\overline{\gamma}} \sum_{k=1}^{M} \frac{(\gamma_0/\overline{\gamma})^{k-1}}{(k-1)!} = 0.0827$
ECG $P_{out} = 1 - e^{-2\gamma_R} - \sqrt{\pi\gamma_R} e^{-\gamma_R}(1 - 2Q(\sqrt{2\gamma_R})) = 0.1041 > P_{out,MRC}$

15. $\overline{P}_{b,MRC} = 0.0016 < 0.0021\overline{P}_{b,EGC}$

16. If each branch has $\overline{\gamma} = 10dB$ Rayleigh
$\gamma_\Sigma =$ overall recvd SNR $= \frac{\gamma_1+\gamma_2}{2} \sim \frac{\gamma e^{-\gamma/(\overline{\gamma}/2)}}{(\overline{\gamma}/2)^2} \gamma \geq 0$
BPSK
$$\overline{P}_b = \int_0^\infty Q(\sqrt{2\gamma}) p_{\gamma_\Sigma} d\gamma = 0.0055$$

17. $p(\gamma)$ where $\int_0^\infty p(\gamma) e^{-x\gamma} d\gamma = \frac{0.01\overline{\gamma}}{\sqrt{x}}$
we will use MGF approach

$$\overline{P}_b = \frac{1}{\pi} \int_0^{\pi/2} \Pi_{i=1}^2 M_{\gamma_i} \left(-\frac{1}{\sin^2\phi}\right) d\phi$$
$$= \frac{1}{\pi} \int_0^{\pi/2} (0.01\overline{\gamma} \sin\phi)^2 d\phi$$
$$= \frac{(0.01\overline{\gamma})^2}{4} = 0.0025$$

18.
$$\overline{P}_b = \left(\frac{1-\pi}{2}\right)^3 \sum_{m=0}^{2} \left(\begin{array}{c} l+m \\ m \end{array}\right) \left(\frac{1+\pi}{2}\right)^m; \quad \pi = \sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}$$

Nakagami-2 fading
$$M_\gamma\left(-\frac{1}{\sin^2\phi}\right) = \left(1 + \frac{\overline{\gamma}}{2\sin^2\phi}\right)^{-2}$$
$$\overline{P}_b = \frac{1}{\pi} \int_0^{\pi/2} \left(M_\gamma\left(-\frac{1}{\sin^2\phi}\right)\right)^3 d\phi, \overline{\gamma} = 10^{1.5} = 5.12 \times 10^{-9}$$

```
MATLAB CODE:

gammab = 10^(1.5);
Gamma = sqrt(gammab./(gammab+1));
```

```
sumf = 0;

for m = 0:2
    f = factorial(2+m)/(factorial(2)*factorial(m));
    sumf = sumf+f*((1+Gamma)/2)^m;
end
pb_rayleigh = ((1-Gamma)/2)^3*sumf;
phi = [0.001:.001:pi/2];
sumvec = (1+(gammab./(2*(sin(phi).^2)))).^(-6);
pb_nakagami = (1/pi)*sum(sumvec)*.001;
```

19.

$$\overline{P}_b = \frac{1}{\pi}\int_0^{\pi/2}\left(1+\frac{\overline{\gamma}}{2\sin^2\phi}\right)^{-2}\left(1+\frac{\overline{\gamma}}{\sin^2\phi}\right)^{-1}d\phi$$

```
gammab_dB = [5:.1:20];
gammabvec = 10.^(gammab_dB/10);

for i = 1:length(gammabvec)
    gammab = gammabvec(i);
    phi = [0.001:.001:pi/2];
    sumvec = ((1+(gammab./(2*(sin(phi).^2)))).^(-2)).*((1+...
             (gammab./(1*(sin(phi).^2)))).^(-1));
    pb_nakagami(i) = (1/pi)*sum(sumvec)*.001;
end
```

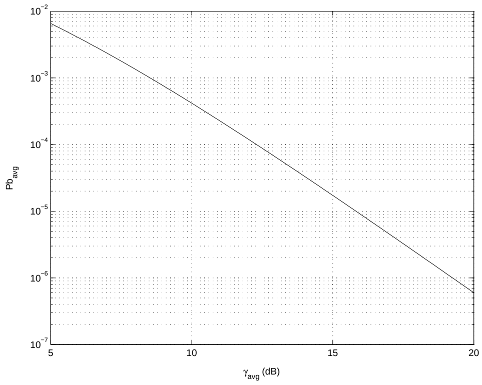

Figure 4: Problem 19

20.

$$P_b = \frac{2}{3}Q\left(\sqrt{2\gamma_b(3)}\sin\left(\frac{\pi}{8}\right)\right)$$

$$\alpha = 2/3, \quad g = 3\sin^2\left(\frac{\pi}{8}\right)$$

$$M_\gamma\left(-\frac{g}{\sin^2\phi}\right) = \left(1+\frac{g\overline{\gamma}}{\sin^2\phi}\right)^{-1}$$

$$\overline{P}_b = \frac{\alpha}{\pi}\int_0^{\pi/2}\left(1+\frac{g\overline{\gamma}}{\sin^2\phi}\right)^{-M}d\phi$$

MATLAB CODE:

```
M = [1 2 4 8];
alpha = 2/3; g = 3*sin(pi/8)^2;

gammab_dB = [5:.1:20];
gammabvec = 10.^(gammab_dB/10);

for k = 1:length(M)
    for i = 1:length(gammabvec)
        gammab = gammabvec(i);
        phi = [0.001:.001:pi/2];
        sumvec = ((1+((g*gammab)./(1*(sin(phi).^2)))).^(-M(k)));
        pb_nakagami(k,i) = (alpha/pi)*sum(sumvec)*.001;
    end
end
```

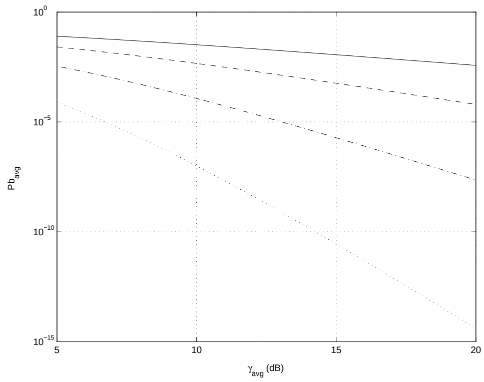

Figure 5: Problem 20

21.

$$\begin{aligned}
Q(z) &= \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{z^2}{\sin^2\phi}\right] d\phi \quad , z > 0 \\
Q^2(z) &= \frac{1}{\pi}\int_0^{\pi/4} \exp\left[-\frac{z^2}{2\sin^2\phi}\right] d\phi \quad , z > 0 \\
P_s(\gamma_s) &= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2} \exp\left[-\frac{g\gamma_s}{\sin^2\phi}\right] d\phi - \\
& \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}^2\right)\int_0^{\pi/4} \exp\left[-\frac{g\gamma_s}{\sin^2\phi}\right] d\phi \\
\overline{P}_s &= \int_0^{\infty} P_s(\gamma_\Sigma) p_{\gamma_\Sigma}(\gamma_\Sigma) d\gamma_\Sigma \\
&= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\int_0^{\infty} exp\left(\frac{g\gamma_\Sigma}{\sin^2\phi}\right) p_{\gamma_\Sigma}(\gamma) d\gamma_\Sigma d\phi - \\
& \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4}\int_0^{\infty} exp\left(\frac{g\gamma_\Sigma}{\sin^2\phi}\right) p_{\gamma_\Sigma}(\gamma) d\gamma_\Sigma d\phi \\
\text{But } \gamma_\Sigma = \gamma_1+\gamma_2+\ldots+\gamma_M = \Sigma\gamma_i & \\
&= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2} \Pi_{i=1}^{M} M_{\gamma_i}\left(-\frac{g}{\sin^2\phi}\right) d\phi - \\
& \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4} \Pi_{i=1}^{M} M_{\gamma_i}\left(-\frac{g}{\sin^2\phi}\right) d\phi
\end{aligned}$$

22. Rayleigh: $M_{\gamma_s}(s) = (1-s\overline{\gamma}_s)^{-1}$
Rician: $M_{\gamma_s}(s) = \frac{1+k}{1+k-s\overline{\gamma}_s}\exp\left(\frac{k_s\overline{\gamma}_s}{1+k-s\overline{\gamma}_s}\right)$
MPSK

$$\overline{P}_s = \int_0^{(M-1)\pi/M} M\gamma_s\left(-\frac{g}{\sin^2\phi}\right) d\phi \rightarrow \text{no diversity}$$

Three branch diversity

$$\overline{P}_s = \frac{1}{\pi}\int_0^{(M-1)\pi/M}\left(1+\frac{g\overline{\gamma}}{\sin^2\phi}\right)^{-1}\left[\frac{(1+k)\sin^2\phi}{(1+k)\sin^2\phi+g\overline{\gamma}_s}\exp\left(-\frac{k\overline{\gamma}_s g}{(1+k)\sin^2\phi+g\overline{\gamma}_s}\right)\right]^2 d\phi$$

$g = \sin^2\left(\frac{\pi}{16}\right)$

$$= 0.1670$$

MQAM:
Formula derived in previous problem with $g = \frac{1.5}{16-1} = \frac{1.5}{15}$
$\overline{P}_s = 0.0553$

```
MATLAB CODE:

gammab_dB = 10;
gammab = 10.^(gammab_dB/10);
K = 2;
```

```
g = sin(pi/16)^2;
phi = [0.001:.001:pi*(15/16)];

sumvec=((1+((g*gammab)./(sin(phi).^2))).^(-1)).*((((...
        (1+K)*sin(phi).^2)./((1+K)*sin(phi).^2+...
         g*gammab)).*exp(-(K*gammab*g)./((1+K)*sin(phi).^2+g*gammab))).^2);
pb_mrc_psk = (1/pi)*sum(sumvec)*.001;

g = 1.5/(16-1);
phi1 = [0.001:.001:pi/2];
phi2 = [0.001:.001:pi/4];

sumvec1=((1+((g*gammab)./(sin(phi1).^2))).^...
        (-1)).*(((((1+K)*sin(phi1).^2)./((1+K)*...
        sin(phi1).^2+g*gammab)).*exp(-(K*gammab*g)./((...
        1+K)*sin(phi1).^2+g*gammab))).^2);
sumvec2=((1+((g*gammab)./(sin(phi2).^2))).^(-1)).*((((...
         (1+K)*sin(phi2).^2)./((1+K)*sin(phi2).^2+...
        g*gammab)).*exp(-(K*gammab*g)./((1+K)*sin(phi2).^2+g*gammab))).^2);
pb_mrc_qam = (4/pi)*(1-(1/sqrt(16)))*sum(sumvec1)*.001 - ...
             (4/pi)*(1-(1/sqrt(16)))^2*sum(sumvec2)*.001;
```

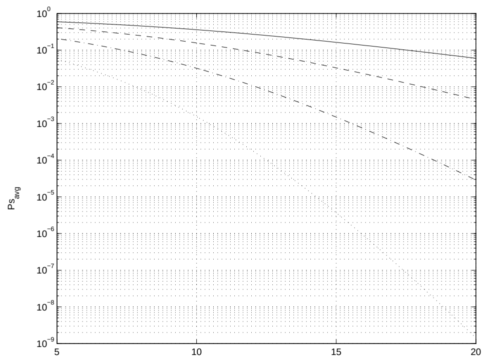

Figure 6: Problem 22

23. MATLAB CODE:

```
M = [1 2 4 8];
alpha = 2/3;
g = 1.5/(16-1);

gammab_dB = [5:.1:20];
gammabvec = 10.^(gammab_dB/10);

for k = 1:length(M)
    for i = 1:length(gammabvec)
        gammab = gammabvec(i);
        phi1 = [0.001:.001:pi/2];
```

```
        phi2 = [0.001:.001:pi/4];
        sumvec1 = ((1+((g*gammab)./(1*(sin(phi1).^2)))).^(-M(k)));
        sumvec2 = ((1+((g*gammab)./(1*(sin(phi2).^2)))).^(-M(k)));
        pb_mrc_qam(k,i) = (4/pi)*(1-(1/sqrt(16)))*sum(sumvec1)*.001 - ...
                          (4/pi)*(1-(1/sqrt(16)))^2*sum(sumvec2)*.001;
    end
end
```

Chapter 9

1.

$$\overline{P_s} = \frac{1}{\pi}\int_0^{\frac{7}{8}\pi}\left(1+\frac{g\overline{\gamma}_s}{\sin^2\phi}\right)^{-1}d\phi \qquad g=\sin^2\left(\frac{\pi}{8}\right)$$
$$\overline{P}_s = \overline{P}_b\log_2 8 = 3\overline{P}_b = 3\times 10^{-3}$$
$$\gamma = 1119$$
$$\frac{R_b}{B} = \frac{3R_s}{B} = 3$$

2. $P_{out} = P(\gamma < \gamma_0) = 1 - e^{-\gamma_0/100} = 0.1$
$$\gamma_0 = 10.536$$
$$\sigma = 1/E_{\gamma_0}[1/\gamma] = \frac{1}{\int_{\gamma_0}^{\infty}\frac{1}{\gamma}P(\gamma)} = 56.3125 = 17.506dB$$
For 4-QAM, SNR =10.35dB
16-QAM, SNR = 17.35dB
$\therefore$ Maximum constellation size is 16-QAM

3. For QPSK
$$P_b = Q\left(\sqrt{\sigma}\right) = 10^{-3}$$
$$\sigma = 10.35dB = 10.85$$
Settin $\sigma = 1/E_{\gamma_0}[1/\gamma]$ and solving for γ_0
$$\gamma_0 = 0.0056$$
$$P_{out} = 1 - e^{-\gamma_0/100} = 5.6\times 10^{-5}$$

4. (a) M=4
$$P_b = Q(\sqrt{\sigma}) = 10^{-3} \qquad \therefore \sigma = 10.35dB$$
M=16
$$\sigma = 17.35dB$$
$\therefore$

$\gamma < 10.35dB$	no transmission
$10.35dB < \gamma < 17.35dB$	4-QAM
$\gamma > 17.35dB$	16-QAM

$$\gamma_0 = 10.35dB$$

(b)

$$\begin{aligned} P(4QAM) &= P(10.35dB < \gamma < 17.35dB) \\ &= \int_{10.35dB}^{17.35dB}\frac{1}{100}e^{-\gamma/100}d\gamma \\ &= 0.3164 \\ P(16QAM) &= P(\gamma > 17.35dB) \\ &= e^{-17.5dB/100} \\ &= 0.5809 \end{aligned}$$

$\therefore R_b = 0.3164\times 2 + 0.5809\times 4 = 2.9564bps/Hz$

(c)

$$\overline{P}_e = \frac{\int_0^{10} P(\gamma)Q(\sqrt{\gamma})d\gamma}{\int_0^{\gamma_0} P(\gamma)d\gamma} = 0.0242$$

5. For BPSK

$$P_b = Q(\sqrt{2\gamma}) = 10^{-3}$$
$$\gamma = 4.77 = 6.78dB$$

For QPSK

$$P_b = Q(\sqrt{\gamma}) = 10^{-3}$$
$$\gamma = 4.77 = 10.35dB$$

For 8PSK

$$P_b \approx 0.67Q\left(\sqrt{2\gamma}\sin(\pi/8)\right) = 10^{-3}$$
$$\gamma = 14.78dB$$

$$\begin{array}{rl} \gamma > 14.78dB & \text{no code 8PSK} \\ 10.35dB < \gamma < 14.78dB & \text{no code QPSK} \\ 6.78dB < \gamma < 10.35dB & \text{no code BPSK} \\ 3.78dB < \gamma < 6.78dB & \text{1st code BPSK} \\ 2.78dB < \gamma < 3.78dB & \text{2nd code BPSK} \\ 1.78dB < \gamma < 2.78dB & \text{3rd code BPSK} \\ \gamma < 1.78dB & \text{no transmission} \end{array}$$

For Rayleigh fading

$$\begin{aligned} R &= P(\gamma > 14.78dB).3 + P(10.35dB < \gamma < 14.78dB).2 \\ &\quad + P(6.78dB < \gamma < 10.35dB).1 + P(3.78dB < \gamma < 6.78dB)\frac{1}{2} \\ &\quad + P(2.78dB < \gamma < 3.78dB)\frac{1}{3} + P(1.78dB < \gamma < 2.78dB)\frac{1}{4} \\ &= 2.6061bps/Hz \end{aligned}$$

6.

$$J = \int \log_2\left(1 + \frac{K\gamma S(\gamma)}{\overline{S}}\right)P(\gamma)d\gamma - \lambda \int S(\gamma)P(\gamma)d\gamma$$
$$\frac{\partial J}{\partial S(\gamma)} = 0$$
$$\Rightarrow \frac{1}{1 + \frac{K\gamma S(\gamma)}{\overline{S}}}\frac{K\gamma}{\overline{S}} - \lambda = 0$$
$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma K} & \gamma \geq \gamma_0/K \\ 0 & \gamma < \gamma_0/K \end{cases}$$
$$M(\gamma) = 1 + \gamma K\frac{S(\gamma)}{\overline{S}} = 1 + \gamma K\left(\frac{1}{\gamma_0} - \frac{1}{\gamma K}\right) = \frac{\gamma K}{\gamma_0}$$
$$\therefore M(\gamma) = \gamma/\gamma_K$$
$$\therefore \log_2 M(\gamma) = \log_2(\gamma/\gamma_K)$$

7. (a) $\overline{P}_e = \int_0^\infty \frac{1}{\overline{\gamma}}e^{-\gamma/\overline{\gamma}}2e^{-1.5\gamma/(M-1)}d\gamma$

(b) Using (a)
Numerically, M=2 $\therefore BPSK \quad \overline{P}_e = 0.025$
Specral efficiency = 1bps/Hz

(c) In Fig 9.3 , Average SNR = 20dB
Spectral efficiency = 4bps/Hz
$\therefore$ Adaptive modulation has higher spectral efficiency.

8. (a)

$$P_b \le .2e^{-\frac{1.5\gamma}{(M-1)}\frac{S(\gamma)}{\overline{S}}}$$

$$M(\gamma) = 1 + \frac{1.5\gamma}{-\ln(5P_b)}\frac{S(\gamma)}{\overline{S}} = 1 + \gamma K \frac{S(\gamma)}{\overline{S}}$$

where

$$K = \frac{1.5}{-\ln(5P_b)}$$

We maximize spectral efficiency by maximizing:

$$\mathbb{E}[\log_2 M(\gamma)] = \int \log_2\left(1 + \gamma K \frac{S(\gamma)}{\overline{s}}\right) p(\gamma) d\gamma$$

subject to:

$$\int S(\gamma) p(\gamma) d\gamma = \overline{S}$$

This gives the water-filling solution for optimal power adaptation as:

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma K} & \gamma \ge \frac{\gamma_0}{K} \\ 0 & \gamma < \frac{\gamma_0}{K} \end{cases}$$

This gives that the optimal rate adaptation is

$$M(\gamma) = \frac{\gamma}{\gamma_K}, \quad \gamma \ge \gamma_K$$

where $\gamma_K = \gamma_0/K$.
To find the cut-off, we use the average power constraint equation as

$$\int_{\gamma_K}^{\infty} \frac{1}{K}\left(\frac{1}{\gamma_K} - \frac{1}{\gamma}\right)\frac{1}{\overline{\gamma}} e^{-\frac{\gamma}{\overline{\gamma}}} d\gamma = 1$$

For $\overline{\gamma} = 20$ dB $= 100$ and $P_b = 10^{-4}$, Using Matlab we get $K = 0.1973$, $\gamma_K = 4.3$, $\gamma_0 = 0.8486$.

(b)

$$\frac{R}{B} = \mathbb{E}[\log_2 M(\gamma)] = \int \log_2\left(1 + \gamma K \frac{S(\gamma)}{\overline{S}}\right) p(\gamma) d\gamma = \int_{\gamma_K}^{\infty} \log_2\left(\frac{\gamma}{\gamma_K}\right)\frac{1}{\overline{\gamma}} e^{-\frac{\gamma}{\overline{\gamma}}} d\gamma$$

Using Matlab we get 3.7681 bits/sec/Hz

(c)

$$M(\gamma) = 1 + \gamma K \frac{S(\gamma)}{\overline{S}}$$

For truncated channel inversion, the power adaptation is given as

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{\sigma}{\gamma} & \gamma \ge \gamma_0 \\ 0 & o.w. \end{cases}$$

where

$$\sigma = \frac{1}{\mathbb{E}_{\gamma_0}[1/\gamma]} = \frac{1}{\int_{\gamma_0}^{\infty} \frac{1}{\gamma}\frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} d\gamma}$$

The value of γ_0 is chosen so as to maximize spectral efficiency i.e.

$$\frac{R}{B} = \max_{\gamma_0} \log_2(1 + K\sigma)p(\gamma > \gamma_0)$$

$$= \max_{\gamma_0} \log_2 \left(1 + K\frac{1}{\mathbb{E}_{\gamma_0}[1/\gamma]}\right) p(\gamma > \gamma_0)$$

Using Matlab we get, $\frac{R}{B} = 3.3628$ at $\gamma_0 = 23.18$ and $p_{\text{out}} = 0.2069$.

9. Assume a target BER of 10^{-3}

(a)

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma K} & \gamma \geq \gamma_0/K \\ 0 & \gamma < \gamma_0/K \end{cases}$$

Try 4 channels

$$\begin{aligned} \gamma_0 &= \frac{1}{\frac{1}{K}\sum \frac{1}{\gamma_i}P(\gamma_i) + 1} \\ &= 0.6491 \\ K &= \frac{-1.5}{\ln(5P_b)} = 0.2831 \\ \therefore \gamma_K = \gamma_0/K = 2.2928 \\ \gamma_1 &= 10^{0.5} > 0.6491 \end{aligned}$$

$\therefore$ we should use 4 channels
$\therefore$ optimal power and rate adaptation

$$\frac{S(\gamma)}{\overline{S}} = \frac{1}{0.6491} - \frac{1}{0.2831\gamma}$$

Instant Rate

$$\log_2 M(\gamma) = \log_2 (\gamma/2.2928)$$

(b) Average spectral density

$$\sum P_i \log_2 (\gamma/2.2928) = 2.457 bps/Hz$$

(c) Truncated channel inversion
First we use 4 channels

$$E\left(1/\gamma\right) = \sum_{i=1}^{4} \frac{1}{\gamma_i} P_i = 0.1530$$

$$\frac{R}{B} = \log_2\left(1 + \frac{K}{E(1/\gamma)}\right) = \log_2\left(1 + \frac{0.2831}{0.1530}\right) = 1.5111$$

Second, we use 3 channels

$$E\left(1/\gamma\right) = \sum_{i=1}^{4} \frac{1}{\gamma_i} P_i = 0.0265$$

$$\frac{R}{B} = \log_2\left(1 + \frac{K}{E(1/\gamma)}\right) 0.6 = 2.1278$$

Now, we use 2 channels

$$E\left(1/\gamma\right) = \sum_{i=1}^{4} \frac{1}{\gamma_i} P_i = 0.0065$$

$$\frac{R}{B} = \log_2\left(1 + \frac{K}{E(1/\gamma)}\right) 0.4 = 2.1910$$

Finally, we use 1 channel

$$\frac{R}{B} = \log_2\left(1 + \frac{K}{E(1/\gamma)}\right) 0.2 = 2.0936$$

$\therefore$ we use 2 channels.
Power control policy

$$\frac{S(\gamma)}{\overline{S}} = \frac{1}{\gamma E(1/\gamma)} = \frac{153.84}{\gamma}, \quad \gamma = 15dB, 20dB$$

$$\frac{R}{B} = 2.191 bps/Hz$$

This is less than waterfilling

10. $\overline{\gamma} = 100$, $P_b = 10^{-3}$

Constellation restricted to $\mathcal{M} = \{0, 2, 4, 16, 64, 256\}$

$$\frac{R}{B} = \max_{\gamma_0} \log_2 \lfloor \left(1 + K \frac{1}{\mathbb{E}_{\gamma_0}[1/\gamma]}\right) \rfloor_{\mathcal{M}} p(\gamma > \gamma_0)$$

Define

$$x = \left(1 + K \frac{1}{\mathbb{E}_{\gamma_0}[1/\gamma]}\right)$$

Using Matlab:

For the unrestricted case, we get, $(R/B)_{\max} = 3.76$ at $\gamma_0^* = 20.25$ and $x(\gamma_0^*) = 24.37$. As for the continuous case, the R/B curve is a strictly concave function of γ_0, we know that the best choices of M are either 16 or 64 as $16 \le x(\gamma_0^*) = 24.37 \le 64$.

For $x = 16$, we get $\gamma_0 = 9.4$ and corresponding (R/B) as 3.64

For $x = 64$, we get $\gamma_0 = 60.5$ and corresponding (R/B) as 3.27.

So, we choose $M = 16$ and corresponding spectral efficiency is 3.36 bits/sec/Hz.

11. (a)

$\gamma < 20$	no transmission
$20 < \gamma < 40$	M=2
$40 < \gamma < 160$	M=4
$160 < \gamma < 640$	M=16
$640 < \gamma < \infty$	M=64

From table 9.2,
$\overline{\gamma} = 20dB, \quad f_D = 100Hz$

$\widehat{f_D} = 80Hz$ is 4/5 of $f_D = 100Hz$
$\therefore \overline{\widehat{\tau}} = \frac{f_D}{\widehat{f_D}}\overline{\tau} = \frac{5}{4}\overline{\tau}_j$

$$\therefore \overline{\tau} = [0.737\ 0.301\ 1.06\ 2.28\ 3.84]5/4 \text{ ms}$$

$$\overline{\tau} = [0.9213\ 0.3762\ 1.3250\ 2.85\ 1.7875] \text{ ms}$$

number of symbols $= \overline{\tau}/T_s$
Spectral efficiency:

$$\begin{aligned}\frac{R}{B} &= P(20 < \gamma < 40)0.1 + P(40 < \gamma < 160)0.2 \\ &\quad + P(160 < \gamma < 640)0.4 + P(640 < \gamma < \infty)0.8 \\ &= 1.8995 bps/Hz\end{aligned}$$

(b)

$$\overline{P}_b = \sum_{j=1}^{N-1} \frac{2(\sqrt{M_j}-1)}{\sqrt{M_j}\log_2 M_j} Q\left(\sqrt{\frac{3(M_j-1)}{K(M_j-1)}}\right) \int_{\gamma_K^\star M_{j-1}}^{\gamma_K^\star M_j} P(\gamma)d\gamma = 2.25 \times 10^{-4}$$

$\therefore$ the exact BER is better than the target BER

12. (a) $\overline{\gamma} = 100$
$B = 30$ KHz
$f_D = 80$ Hz
$P_b = 10^{-3}$

$$\epsilon = \frac{\hat{\gamma}}{\gamma}$$

$$\overline{P}_b = \int_0^\infty \frac{1}{5}(5P_b)^{\frac{1}{\epsilon}} p(\epsilon)d\epsilon = \int_{0.5}^{1.5} \frac{1}{5}(5P_b)^{\frac{1}{\epsilon}} p(\epsilon)d\epsilon = 0.0017$$

where the last integral was evaluated using Matlab.
Another, more analytical way to solve this is a method proposed by William Wu, a student enrolled for the course as:

$$\bar{P}_b \approxeq .2 \int_{0.5}^{1.5} (5P_b)^{1/\epsilon} d\epsilon$$

This integral cannot be evaluated in closed form, but it can be expressed in terms of a special function; namely, the exponential integral.
Consider the integral $S = \int_{K_1}^{K_2} a^{1/x}dx$, where a is a real positive number. Then $a = e^p$ for some $p \in \mathbb{R}$. Rewriting the integral in terms of p, and then integrating by parts, we have

$$\begin{aligned} S = \int_{K_1}^{K_2} a^{1/x}dx = \int_{K_1}^{K_2} e^{p/x}dx &= e^{p/x}\cdot x\Big|_{K_1}^{K_2} - \int_{K_1}^{K_2} xe^{p/x}\frac{-p}{x^2}dx \\ &= e^{p/x}\cdot x\Big|_{K_1}^{K_2} + \underbrace{\int_{K_1}^{K_2} \frac{p}{x}e^{p/x}dx.}_{\beta} \end{aligned}$$

With regards to the second integral "β", let $u = -p/x$. Then $x = -\frac{p}{u}$, and $dx = \frac{p}{u^2}du$, yielding

$$\beta = \int_{K_3}^{K_4} (-u)e^{-u}\frac{p}{u^2}du = -p\int_{K_3}^{K_4} \frac{e^{-u}}{u}du$$

where $K_3 = \frac{-p}{K_1}$ and $K_4 = \frac{-p}{K_2}$. Applying the data provided in this problem, $p \approxeq -5.298, K_3 \approxeq 10.596$, and $K_4 \approxeq 3.532$. Reversing the orientation of integration yields

$$\beta = p\int_{K_4}^{K_3} \frac{e^{-u}}{u}du = p(E_1(K_4) - E_1(K_3))$$

where E_1 is the exponential integral function $E_1(x) \triangleq \int_x^\infty \frac{e^{-t}}{t}dt$. Conclusively, using $K_1 = 0.5, K_2 = 1.5$, and $P_b = 10^{-3}$, we have

$$\bar{P}_b \approxeq (1/5)\left[e^{p/x}\cdot x\Big|_{K_1}^{K_2} + p(E_1(K_4) - E_1(K_3))\right] \approxeq 1.674\times 10^{-3}$$

where $E_1(x)$ was evaluated using the `expint()` function in MATLAB.

(b)

$$\hat{\gamma}(t) = \gamma(t-\tau) + \gamma_\epsilon(t)$$

$$\begin{aligned} P_b(\gamma(t),\hat{\gamma}(t)) &\leq 0.2\text{exp}\left[\frac{-1.5\gamma(t)}{M(\hat{\gamma}(t))-1}\frac{S(\hat{\gamma}(t))}{\overline{S}}\right] && (1)\\ &= 0.2(5P_b)^{\frac{\gamma(t)}{\hat{\gamma}(t)}} && (2)\\ &= = 0.2(5P_b)^{\frac{1}{\delta+\epsilon}} && (3)\end{aligned}$$

where

$$\delta = \frac{\gamma(t-\tau)}{\gamma(t)}$$

$$\epsilon = \frac{\gamma_\epsilon(t)}{\gamma(t)}$$

IF the fading process is stationary t does not matter. In general $\gamma(t-\tau)$ and $\gamma_\epsilon(t)$ are not independent of $\gamma(t)$.

In further calculation we drop t as process stationary

$$\overline{P}_b \leq \int_0^\infty \int \gamma_0{}^\infty 0.2(5P_b)^{\frac{\gamma}{\hat{\gamma}}}p(\gamma,\hat{\gamma})d\hat{\gamma}d\gamma$$

Notice that $\hat{\gamma}$ integrates only from γ_0 as the transmitter which has knowledge of $\hat{\gamma}$ only does not transmit if $\hat{\gamma}$ is less than γ_0, where as γ can vary over its entire range.

$$\overline{P}_b \leq \int_0^\infty \left(\int \gamma_0{}^\infty 0.2(5P_b)^{\frac{\gamma}{\hat{\gamma}}}p(\hat{\gamma}|\gamma)d\hat{\gamma}\right)p(\gamma)d\gamma$$

where we have used Baye's rule to relate the joint distribution to the marginal distribution as

$$p(\gamma,\hat{\gamma}) = p(\hat{\gamma}|\gamma)p(\gamma)$$

Now using

$$\frac{\gamma}{\hat{\gamma}} = \frac{1}{\delta+\epsilon}$$

we get

$$\overline{P}_b \leq \int_0^\infty \left(\int 0^\infty 0.2(5P_b)^{\frac{1}{\delta+\epsilon}}p\left((\delta+\epsilon)|\gamma\right)d(\delta+\epsilon)\right)p(\gamma)d\gamma$$

where if $\hat{\gamma} \in \{\gamma_0, \infty\}$ and $\gamma \in \{0, \infty\}$, then $(\delta + \epsilon) = \frac{\hat{\gamma}}{\gamma} \in \{0, \infty\}$.
Now IF, $\gamma(t-\tau)|\gamma$ and $\gamma_\epsilon|\gamma$ are independent then we can write

$$p((\delta+\epsilon)|\gamma) = p(\delta|\gamma) \otimes p(\epsilon|\gamma)$$

Further, IF $\gamma(t-\tau)|\gamma$ only depends on $p(\tau)$, we only need the distribution of $p(\tau)$ and $p(\gamma_\epsilon|\gamma)$.

13. Suppose target $P_e = 10^{-3}$
$K_c = KG_c = 3dB \times 0.2831 = 0.5662$

$$\therefore \frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma K_c} & \gamma \geq \gamma_0/K_c \\ 0 & \gamma < \gamma_0/K_c \end{cases}$$

$$\int \frac{S(\gamma)}{\overline{S}} = 1 \Rightarrow \gamma_0 = 0.93$$

$$\therefore \frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{0.93} - \frac{1}{0.5562\gamma} & \gamma \geq 1.67 \\ 0 & \gamma < 1.67 \end{cases}$$

$$\frac{R}{B} = \int_{\gamma K_c}^{\infty} \log_2\left(\frac{\gamma}{\gamma K_c}\right) p(\gamma) d\gamma = 5.1bps/Hz$$

14.

$$\begin{aligned} P_b &\approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\gamma}{M-1}}\right) \\ &\approx \frac{4}{\log_2 M \sqrt{\frac{6\pi\gamma}{M-1}}} \exp\left(-\frac{3\gamma}{2(M-1)}\right) \end{aligned}$$

$$\therefore C_1 = \frac{4}{\log_2 M \sqrt{\frac{6\pi\gamma}{M-1}}}$$

$$C_2 = -1.5$$

$$C_3 = 1$$

$$C_4 = 1$$

15.

$$\begin{aligned} J &= \int_0^\infty K(\gamma) p(\gamma) d\gamma + \gamma \left[\int_0^\infty \frac{S(\gamma)}{\overline{S}} P(\gamma) d\gamma\right] \\ &= \int_0^\infty \frac{1}{C_3} \log_2\left(C_4 - \frac{C_2 \gamma}{\ln(\gamma_b/C_1)} \frac{S(\gamma)}{\overline{S}}\right) p(\gamma) d\gamma + \lambda\left(\frac{S(\gamma)}{\overline{S}} p(\gamma) d\gamma\right) \end{aligned}$$

$$\frac{\partial J}{\partial S(\gamma)} 0 \Rightarrow \frac{1/\ln 2}{C_4 - \frac{C_2\gamma}{\ln(p_b/C_1)} \frac{S(\gamma)}{\overline{S}}} \frac{C_2 \gamma}{\ln(P_b/C_1)\overline{S}} + \frac{\lambda}{\overline{S}}$$

$$\Rightarrow \frac{S(\gamma)}{\overline{S}} = \begin{cases} -\frac{1}{3(\ln 2)\lambda \overline{S}} - \frac{1}{\gamma K} & S(\gamma) \geq 0, K(\gamma) \geq 0 \\ 0 & else \end{cases}$$

16. (a) $\overline{\gamma} = 30$ dB $= 1000$
$P_b = 10^{-7}$

	c_1	c_2	c_3	c_4
1st Bound	0.05	6	1.9	1
2nd Bound	0.2	7	1.9	-1
3rd Bound	0.25	8	1.94	0

We use eq. 9.46 in the reader

$$\frac{1}{K} = -\frac{c_4}{c_2} \ln\left(\frac{P_b}{c_1}\right)$$
$$K = -\frac{c_2}{c_4 \ln\left(\frac{P_b}{c_1}\right)}$$

For the 1st bound:
$K = 0.4572$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0 K} - \frac{1}{\gamma K} & \gamma > \gamma_0 \\ 0 & o.w. \end{cases} \tag{4}$$

K is known to be 0.4572, γ_0 is found using the constraint

$$\int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0 K} - \frac{1}{\gamma K}\right) p(\gamma) d\gamma = 1$$

Using Matlab, we get $\gamma_0 = 2.16$

For the 2nd bound:
K = -0.4825

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} -\frac{1}{\gamma_0 K} - \frac{1}{\gamma K} & \gamma > \gamma_0 \\ 0 & o.w. \end{cases} \tag{5}$$

K is known to be -0.4825, γ_0 is found using the constraint

$$\int_{\gamma_0}^{\infty} \left(-\frac{1}{\gamma_0 K} - \frac{1}{\gamma K}\right) p(\gamma) d\gamma = 1$$

Using Matlab, we get $\gamma_0 = 2.10$

For the 3rd bound:
$K = \infty$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{K_0}{\gamma_0} & \gamma > \gamma_0 \\ 0 & o.w. \end{cases} \tag{6}$$

K_0 is $\frac{-\ln(P_b/c_1)}{c_2} = 1.8415$, γ_0 is found using the constraint

$$\int_{\gamma_0}^{\infty} \left(\frac{K_0}{\gamma_0}\right) p(\gamma) d\gamma = 1$$

Using Matlab, we get $\gamma_0 = 1.838$

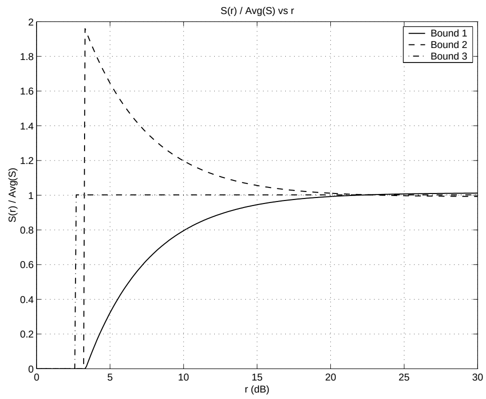

Figure 1: Problem 16, $S(\gamma)/\overline{S}$

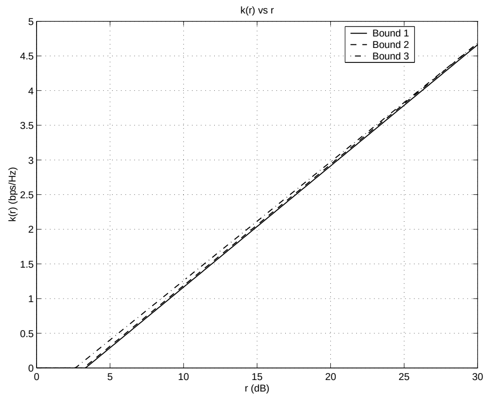

Figure 2: Problem 16, $k(\gamma)$

(b)

$$k(\gamma) = \begin{cases} \frac{1}{c_3}\log_2\left(\frac{\gamma}{\gamma_0}\right) & \gamma \geq \gamma_0 \\ 0 & o.w. \end{cases} \tag{7}$$

The plots for $S(\gamma)/\overline{S}$ and $k(\gamma)$ are given below:

For bound 1, the cutoff is dictated by the power constraint, where as for bounds 2 and 3, where K is either negative or infinite, the cutoff is dictated by the rate positivity constraint.

(c) By looking at the curves we notice that for Low SNR

1st Bound: Water-filling
2nd Bound: Channel-Inversion
3rd Bound: Constant Power

For high SNR's, all the three bounds have the same power adaptation.

The rates are almost the same for all the three bounds.

17. $\mathcal{M} = \{M_0 = 0, \ldots M_{N-1}\}$ where M_0 is no data transmission
$k_j = \log_2 M_j \quad, j > 0$
$k_0 = 0$
k_j assigned to $R_j = [\gamma_j - 1, \gamma_j) \quad, j = 0, \ldots N-1$

$\gamma_{-1} = 0$
$\gamma_{N-1} = \infty$

$$P_b(\gamma) \approx c_1 \exp\left[\frac{-c_1\gamma\frac{S(\gamma)}{\overline{S}}}{2^{c_3k_j} - c_4}\right]$$

Setting target value $P_b = P(\gamma)$

$$\ln\left(\frac{P_b}{c_1}\right) = \frac{-c_1\gamma\frac{S(\gamma)}{\overline{S}}}{2^{c_3k_j} - c_4}$$

$$\Rightarrow \frac{S(\gamma)}{\overline{S}} = \frac{\frac{-\ln(P_b/c_1)}{c_2}(2^{c_3k_j} - c_4)}{\gamma} = \frac{h(k_j)}{\gamma}$$

$$\mathcal{L} = \sum_{j=1}^{N-1} k_j \int_{\gamma_{j-1}}^{\gamma_j} p(\gamma)d\gamma + \lambda\left[\sum_{j=1}^{N-1}\int_{\gamma_{j-1}}^{\gamma_j} \frac{h(k_j)}{\gamma}p(\gamma)d\gamma - 1\right]$$

$\frac{\partial\mathcal{L}}{\partial\gamma_j} = 0 \quad 0 \le j \le N-2$

$$\frac{\partial\mathcal{L}}{\partial\gamma_0} = -k_1p(\gamma_0) - \lambda\frac{h(k_1)}{\gamma_0}p(\gamma_0) = 0$$

$$\gamma_0 = \rho\frac{h(k_1)}{k_1} \quad \text{for some constant}\rho = -\lambda$$

for j > 0

$$\frac{\partial\mathcal{L}}{\partial\gamma_0} = k_jp(\gamma_j) - k_{j+1}p(\gamma_j) + \lambda\left[\frac{h(k_j)}{\gamma_j}p(\gamma_j) - \frac{h(k_{j+1})}{\gamma_j}p(\gamma_j)\right] = 0$$

$$\gamma_j = \frac{h(k_{j+1}) - h(k_j)}{k_{j+1} - k_j}\rho \quad 1 \le j \le N-2$$

For some $\rho = -\lambda$ where ρ is found from the average power constraint as:

$$\sum_{j=1}^{N-1}\int_{\gamma_{j-1}}^{\gamma_j}\frac{h(k_j)}{\gamma}p(\gamma)d\gamma = 1$$

18.

$$J = \sum_{j=1}^{N-1} k_j\int_{\gamma_{j-1}}^{\gamma_j} p(\gamma)d\gamma + \lambda_1\left[\sum_{j=1}^{N-1} k_j\int_{\gamma_{j-1}}^{\gamma_j}(P_b(\gamma) - \overline{P}_b)p(\gamma)d\gamma\right] + \lambda_2\left[\int_{\gamma_0}^{\infty} S(\gamma)p(\gamma)d\gamma - \overline{S}\right]$$

$$\frac{\partial J}{\partial S(\gamma)} = 0$$

Notice that $P_b(\gamma)$ is a function of $S(\gamma)$

$$\lambda_1\frac{\partial P_b(\gamma)}{S(\gamma)}p(\gamma) + \lambda_2 p(\gamma) = 0 \quad , \gamma_{j-1} \le \gamma \le \gamma_j$$

$$\Rightarrow \frac{\partial P_b(\gamma)}{S(\gamma)} = \frac{-\lambda_2}{k_j\lambda_1}$$

$$P_b(\gamma) = c_1 \exp\left[\frac{-c_1\gamma\frac{S(\gamma)}{\overline{S}}}{2^{c_3k_j} - c_4}\right]$$

$$\frac{\partial P_b(\gamma)}{S(\gamma)} = P_b(\gamma)\frac{-c_2\gamma\frac{1}{\overline{S}}}{2^{c_3k_j} - c_4} = \frac{-\lambda_2}{k_j\lambda_1} \Rightarrow P_b(\gamma) = \lambda\frac{f(k_j)}{\gamma k_j} \quad \gamma_{j-1} \leq \gamma \leq \gamma_j$$

where $\lambda = \frac{\overline{S}\lambda_2}{c_2\lambda_1}$

$$\lambda\frac{f(k_j)}{\gamma k_j} = c_1 \exp\left[\frac{-c_2\gamma S(\gamma)/\overline{S}}{f(k_j)}\right]$$

$$\frac{-c_2\gamma S(\gamma)/\overline{S}}{f(k_j)} = \ln\left[\lambda\frac{f(k_j)}{c_1\gamma k_j}\right]$$

$$\therefore S(\gamma) = S_j(\gamma) \quad , \gamma_{j-1} \leq \gamma \leq \gamma_j$$

where $\frac{S_j(\gamma)}{\overline{S}} = \frac{f(k_j)}{-\gamma c_2}\ln\left[\lambda\frac{f(k_j)}{c_1\gamma k_j}\right]$

$$\frac{\partial J}{\partial \gamma_j} = 0 \quad 0 \leq j \leq N-2$$

$$\Rightarrow (k_j - k_{j+1}) + \lambda_1[k_j(P_b(\gamma_j) - \overline{P}_b) - k_{j+1}(P_b(\gamma_j) - \overline{P}_b)] + \lambda_2[S_j(\gamma_j) - S_{j+1}(\gamma_j)] = 0$$

$$P_b(\gamma_j) = \overline{P}_b - \frac{1}{\lambda_1} - \frac{\lambda_2}{\lambda_1}\frac{S_{j+1}(\gamma_j) - S_j(\gamma_j)}{(k_{j+1} - k_j)}$$

Substituting for $P_b(\gamma_j)$ and $S_j(\gamma_j)$ we et an equation between γ_j, λ_1 and λ_2

$$\frac{\lambda f(k_j)}{\gamma_j k_j} = \overline{P}_b - \frac{1}{\lambda_1} - \frac{\lambda_2}{\lambda_1}\frac{\frac{f(k_{j+1})}{-\gamma_j c_2}\ln\left[\frac{\lambda f(k_{j+1})}{c_1\gamma_j k_{j+1}}\right] - \frac{f(k_j)}{-\gamma_j c_2}\ln\left[\frac{\lambda f(k_j)}{c_1\gamma_j k_j}\right]}{(k_{j+1} - k_j)}$$

where $\lambda = \frac{\overline{S}\lambda_2}{c_2\lambda_1}$
Notice that λ_1 and λ_2 are found using the constraints

$$\sum_{j=1}^{N-1}\int_{\gamma_{j-1}}^{\gamma_j}\frac{f_i(\gamma)}{\overline{S}}p(\gamma)d\gamma = 1$$

and

$$\sum_{j=1}^{N-1} k_j\int_{\gamma_{j-1}}^{\gamma_j}(P_b(\gamma_j) - \overline{P}_b)p(\gamma)d\gamma = 0$$

19. $\overline{\gamma} = 20\text{dB} = 100$
$P_b = 10^{-4}$
$c_1 = 0.05,\ c_2 = 6,\ c_3 = 1.9,\ c_4 = 1$

$\mathcal{M} = \{0, 2, 4, 8, 16\}$
$\gamma_{-1} = 0$
$\gamma_{(N-1)} = \gamma_4 = \infty$
$N = 5.$

$$\frac{S(\gamma)}{\overline{S}} = \frac{h(k_j)}{\gamma}, \quad \gamma_{j-1} \leq \gamma\gamma_j, , j = 0, 1, 2, 3, 4$$

where

$$h(k_j) = -\frac{\ln(P_b/c_1)}{c_2}\left(2^{c_3 k_j} - c_4\right), , j = 0, 1, 2, 3, 4$$

$k_j = \log_2 M_j,\ j = 0, 1, 2, 3, 4$

Region boundaries are γ_0, γ_1, γ_2 and γ_4 given by

$$\gamma_0 = \rho\frac{h(k_1)}{k_1}$$

$$\gamma_j = \rho\frac{h(k_{j+1}) - h(k_j)}{k_{j+1} - k_j}, j = 1, 2, 3$$

Find ρ using:

$$\sum_{j=1}^{N-1}\int_{\gamma_{j-1}}^{\gamma_j}\frac{h(k_j)}{\gamma}p(\gamma)d\gamma = 1$$

Using Matlab, we find that $\rho = 0.7$

Once we know the region boundaries (as we know ρ), we find the spectral efficiency as:

$$\frac{R}{B} = \sum_{j=1}^{N-1}\int_{\gamma_{j-1}}^{\gamma_j} k_j p(\gamma)d\gamma \tag{8}$$

$$= \sum_{j=1}^{N-1} k_j \int_{\gamma_{j-1}}^{\gamma_j} p(\gamma)d\gamma \tag{9}$$

$$= \sum_{j=1}^{N-1} k_j p(j) \tag{10}$$

where

$$p(j) = \int_{\gamma_{j-1}}^{\gamma_j} p(\gamma)d\gamma$$

Using Matlab we find that $R/B = 3.002$ bits/sec/Hz.

20. From 9.84 and 9.85

$$S(\gamma) = S_j(\gamma) \quad , \gamma_{j-1} \leq \gamma \leq \gamma_j$$

where

$$\frac{S_j(\gamma)}{\overline{S}} \ln\left[\frac{\lambda f(k_j)}{c_1\gamma k_j}\right]\frac{f(k_j)}{-\gamma c_2} \quad ,j=1,\dots N-1$$

$$P_{b_j}(\gamma)=\frac{\lambda f(k_j)}{\gamma k_j} \quad \gamma_{j-1}\leq\gamma\leq\gamma_j \quad ,j=1\dots N-1$$

Solving approximately

$$\gamma_{j-1}=\frac{f(k_j)}{k_j}\rho \quad ,j=1\dots N-1$$

Solve for ρ and λ using

$$\sum_{j=1}^{N-1}\int_{\gamma_{j-1}}^{\gamma_j}\frac{S_j(\gamma)}{\overline{S}}p(\gamma)d\gamma=1$$

and

$$\sum_{j=1}^{N-1}k_j\int_{\gamma_{j-1}}^{\gamma_j}(P_{b_j}(\gamma)-\overline{P}_b)p(\gamma)d\gamma=0$$

using Matlab we get one solution as

$$\rho=3 \quad \lambda=4\times10^{-6}$$

optimal region boundaries 0, 8.1964, 19.3932, 50.9842, 144.7588, ∞
$P_{out}=0.0787$ ASE = 2.5810 bits/sec/Hz

```
MATLAB

M = [0 2 4 8 16];
N = 5; k = log2(M);
k(1) = 0;
c1 = .05;
c2 = 6;
c3 = 1.9;
c4 = 1;
Pb_bar = 1e-4;
gamma_bar = 100;

f = (2.^(c3*k))-c4;

lambda1 = 1e4; ss = .1; ss1 = .0001; ss2 = .1; count1 = 1; for
lambda = ss1:ss1:ss1
    count2 = 1;
    for rho = 1:ss2:2
        gamma_bnd = [f./k]*rho;
        gamma_bnd(1) = 0;
        gamma_bnd(6) = 100*gamma_bar;
        for i = 1:N
            a = [gamma_bnd(i):ss/10:gamma_bnd(i+1)];
            loi(i) = length(a);
        end
        mloi = max(loi);
        gamma = zeros(5,mloi);
```

```
        for i = 1:N
            a = [gamma_bnd(i):ss/10:gamma_bnd(i+1)];
            gamma(i,:) = [a zeros(1,mloi-length(a))];
        end

        sum_power(count1,count2) = -1;
        for i = 1:N-1
            S_by_S_bar = -log((lambda*f(i+1))./(c1*gamma(i+1,1:loi(i))*k(...
                         i+1))).*(f(i+1)./(gamma(i+1,1:loi(i))*c2));
            if min(S_by_S_bar)<0
                stop;
            end
            p_gamma = (1/gamma_bar)*exp(-gamma(i+1,1:loi(i))/gamma_bar);
            sum_int(i) = (ss/10)*sum(S_by_S_bar.*p_gamma);
            sum_power(count1,count2) = sum_power(count1,count2)+sum_int(i);
        end
        sum_Pb(count1,count2) = 0;
        for i = 1:N-1
            p_gamma = (1/gamma_bar)*exp(-gamma(i+1,1:loi(i))/gamma_bar);
            Pb_gamma = repmat(Pb_bar-(1/lambda1), 1, length(p_gamma));
            Pb_int(i) = (ss/10)*k(i+1)*sum((Pb_gamma-Pb_bar).*p_gamma);
            sum_Pb(count1,count2) = sum_Pb(count1,count2)+Pb_int(i);
        end
        count2 = count2+1;
    end
    count1 = count1+1;
end

PART b
M = [0 2 4 8 16];
N = 5;
k = log2(M);
k(1) = 0;
c1 = .05;
c2 = 6;
c3 = 1.9;
c4 = 1;
Pb_bar = 1e-4;
gamma_bar = 100;

f = (2.^(c3*k))-c4;

lambda1 = 1e4; ss = .1;
    count2 = 1;
    for rho = 15:1:20
        gamma_bnd = [f./k]*rho;
        gamma_bnd(1) = 0;
        gamma_bnd(6) = 100*gamma_bar;
        for i = 1:N
            a = [gamma_bnd(i):ss/10:gamma_bnd(i+1)];
```

```
        loi(i) = length(a);
    end
    mloi = max(loi);
    gamma = zeros(5,mloi);
    for i = 1:N
        a = [gamma_bnd(i):ss/10:gamma_bnd(i+1)];
        gamma(i,:) = [a zeros(1,mloi-length(a))];
    end

    sum_Pb(count2) = 0;
    for i = 1:N-1
        p_gamma = (1/gamma_bar)*exp(-gamma(i+1,1:loi(i))/gamma_bar);
        Pb_gamma = repmat(Pb_bar-(1/lambda1), 1, length(p_gamma));
        Pb_int(i) = (ss/10)*k(i+1)*sum((Pb_gamma-Pb_bar).*p_gamma);
        sum_Pb(count2) = sum_Pb(count2)+Pb_int(i);
    end
    count2 = count2+1;
end
```

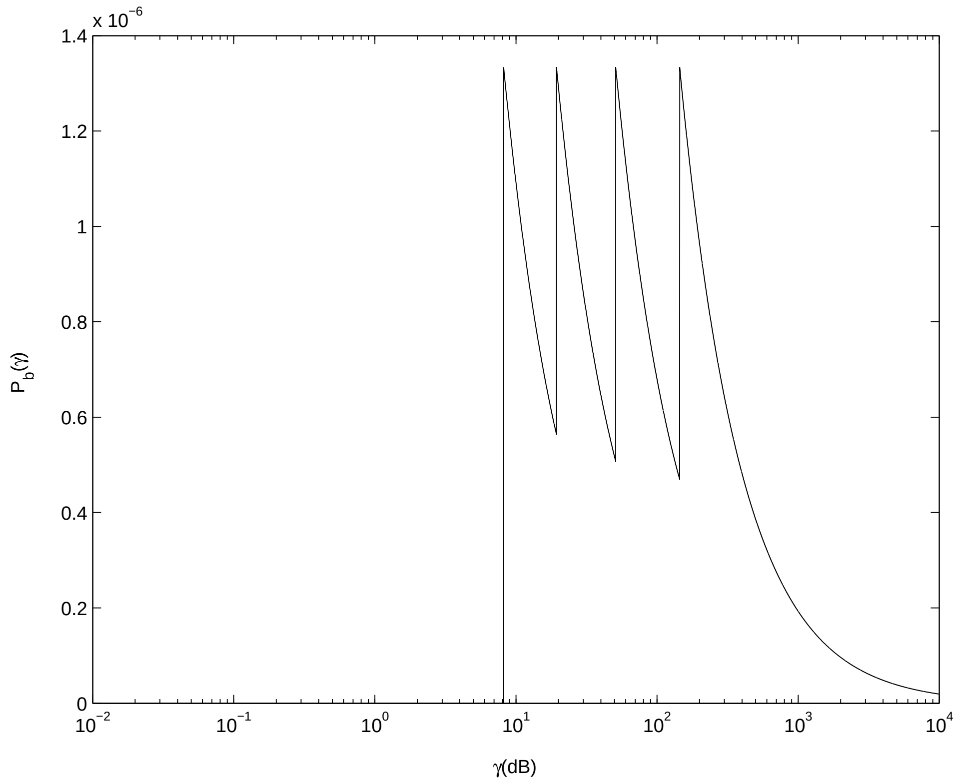

Figure 3: Problem 20

21. $C_1 = 0.05 \quad C_2 = 6 \quad C_3 = 1.9 \quad C_4 = 1$

$$K = \frac{C_2}{C_1/\overline{P}_b - 1}$$

$$\overline{P_b}(\overline{\gamma}) = \frac{c_1}{\frac{c_2\overline{\gamma}S(\overline{\gamma})\overline{S}}{2^{c_3 k(\overline{\gamma})} - c_4} + 1}$$

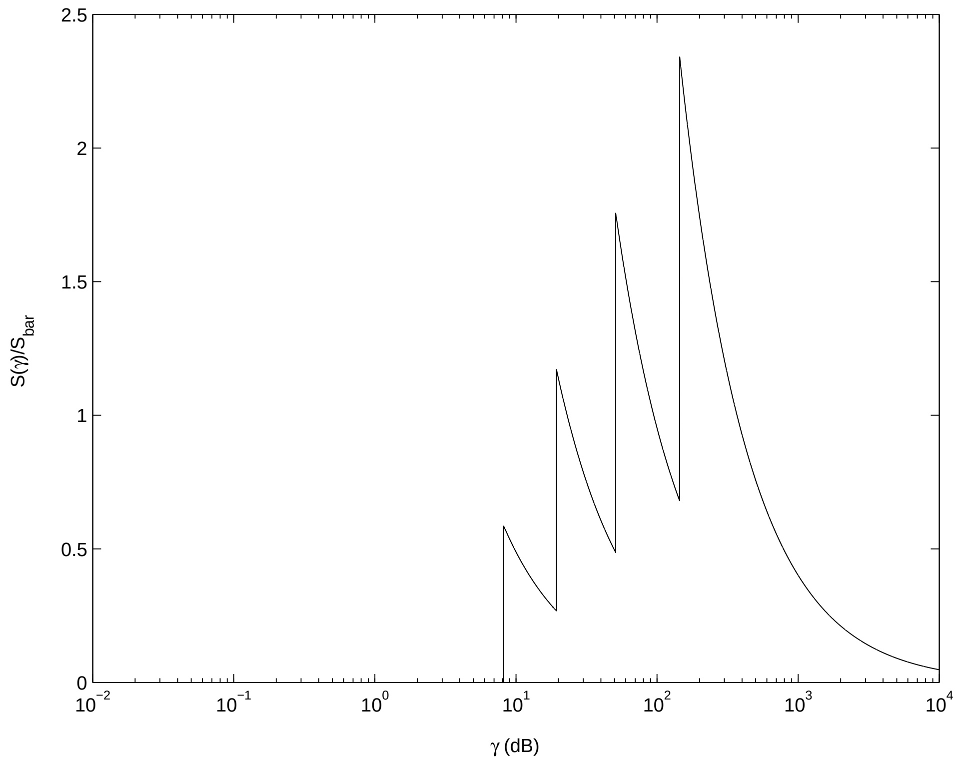

Figure 4: Problem 20

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\overline{\gamma}_0} - \frac{C_4}{\gamma K} & \overline{\gamma} \geq C_4 \overline{\gamma}_0 / K \\ 0 & else \end{cases}$$

$$k(\overline{\gamma}) = \frac{1}{c_3} \log_2(K\overline{\gamma}/\overline{\gamma_0})$$

$$\frac{R}{B} = \int_{C_4 \gamma_0 / K}^{\infty} \log_2 \left(\frac{K\overline{\gamma}}{\overline{\gamma}_0} \right) p(\overline{\gamma}) d\overline{\gamma}$$

Matlab gives

$$\overline{\gamma}_b = 0.92, \quad P_{out} = 10^{-16}, \quad R/B = 1.964 bits/sec/Hz$$

```
MATLAB CODE:

c1 = .05;
c2 = 6;
c3 = 1.9;
c4 = 1;
Pb_bar = 1e-3;

K = c2/((c1/Pb_bar)-1);

meu_psi_db = 20;
sigma_psi_db = 8;
sigma = 10^(sigma_psi_db/10);
meu = 10^(meu_psi_db/10);
```

```
S = sqrt(log(exp(log(sigma^2)-2*log(meu))+1));

M = (2*log(meu)-S^2)/2;

gamma0_bar = [.9:.01:1];
ss1 = .1;
for k1 = 1:length(gamma0_bar)
    a = [(c4/K)*gamma0_bar(k1):ss1:1000*meu];
    b = (1./(S*sqrt(2*pi)*a)).*exp(-(log(a)-M).^2/(2*S^2));
    S_by_S_bar = (1/gamma0_bar(k1)) - ((c4/K)./a);
    sum_int(k1) = ss1*sum(S_by_S_bar.*b);
end

[m,n] = min(abs(sum_int-1));
gamma0_bar_chosen = gamma0_bar(n);

a = [(c4/K)*gamma0_bar_chosen:ss1:100*meu];

b = (1./(S*sqrt(2*pi)*a)).*exp(-(log(a)-M).^2/(2*S^2));

S_by_S_bar = (1/gamma0_bar_chosen) - ((c4/K)./a);

kgamma = (1/c3)*log2(K*(a/gamma0_bar_chosen));

ASE = sum(b.*kgamma)*ss1;
Pout = 1-(sum(b)*ss1);
```

22. Notice that from 9.91 to 9.97 remains unchanged
(9.97) $\rightarrow$
We maximize spectral efficiency by maximizing

$$E[k(\overline{\gamma})] = \int_0^\infty \frac{1}{c_3}\log_2\left[c_4 + \frac{K\overline{\gamma}S(\overline{\gamma})}{\overline{S}}\right]p(\overline{\gamma})d\overline{\gamma}$$

subject to

$$\int_0^\infty \frac{S(\overline{\gamma})}{\overline{S}}p(\overline{\gamma})d\overline{\gamma} = 1$$

for $c_4 < 0$, $\quad K = \frac{c_2}{c_1/\overline{P}_b - 1}$
we get the solution similar to 9.54 as

$$\frac{S(\overline{\gamma})}{\overline{S}} = \begin{cases} \frac{1}{\overline{\gamma}_0} - \frac{c_4}{K\overline{\gamma}} & \gamma \geq \gamma_0\frac{-c_4}{K} \\ 0 & o.w. \end{cases}$$

where $\overline{\gamma}_0$ is determined by power constraint
Notice that it is like channel inversion as c_4 is negative
Also, optimal rate adaptation is

$$k(\overline{\gamma}) = \frac{1}{c_3}\log_2\left(\frac{-c_4\overline{\gamma}}{\overline{\gamma}_0}\right)$$

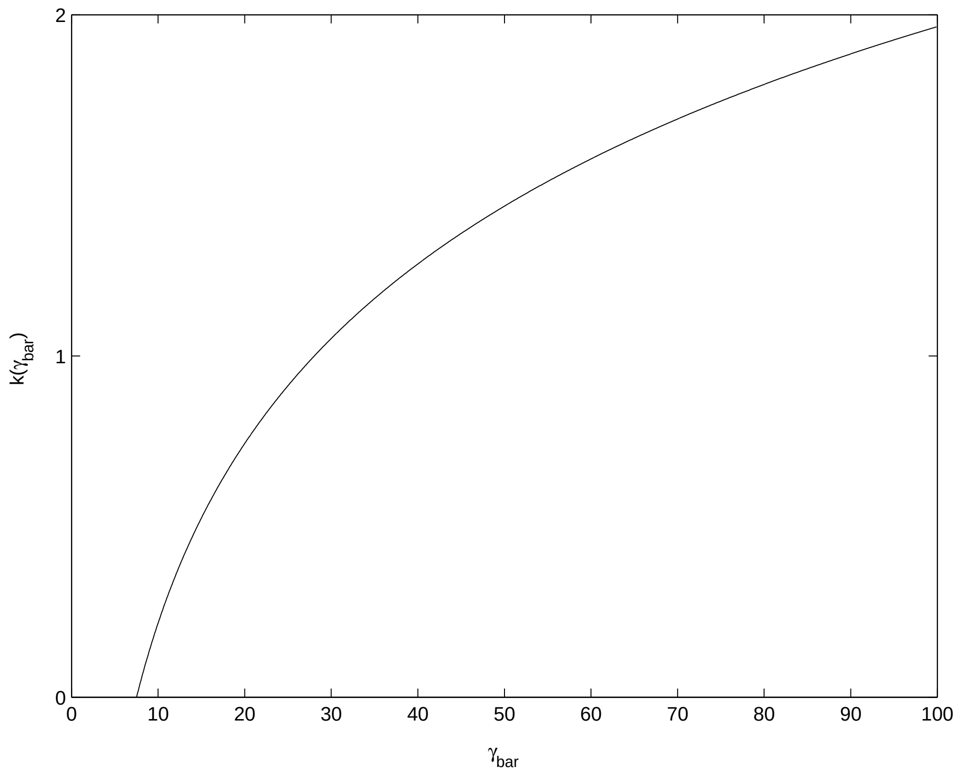

Figure 5: Problem 21

23. Similar to the previous problem, we get the optimal adaptation as

$$\frac{S(\overline{\gamma})}{\overline{S}} = \begin{cases} \frac{K_0}{\overline{\gamma}_0} & \overline{\gamma} \geq \overline{\gamma}_0 \\ 0 & o.w. \end{cases}$$

where $K_0 = \frac{c_1/\overline{P}_b - 1}{c_2}$ and γ_0 is found from power constraint.
Notice that it is an on-off power transmission scheme.
Optimal rate adaptation is given as

$$k(\overline{\gamma}) = \frac{1}{c_3} \log_2 \left(\frac{\overline{\gamma}}{\overline{\gamma}_0} \right)$$

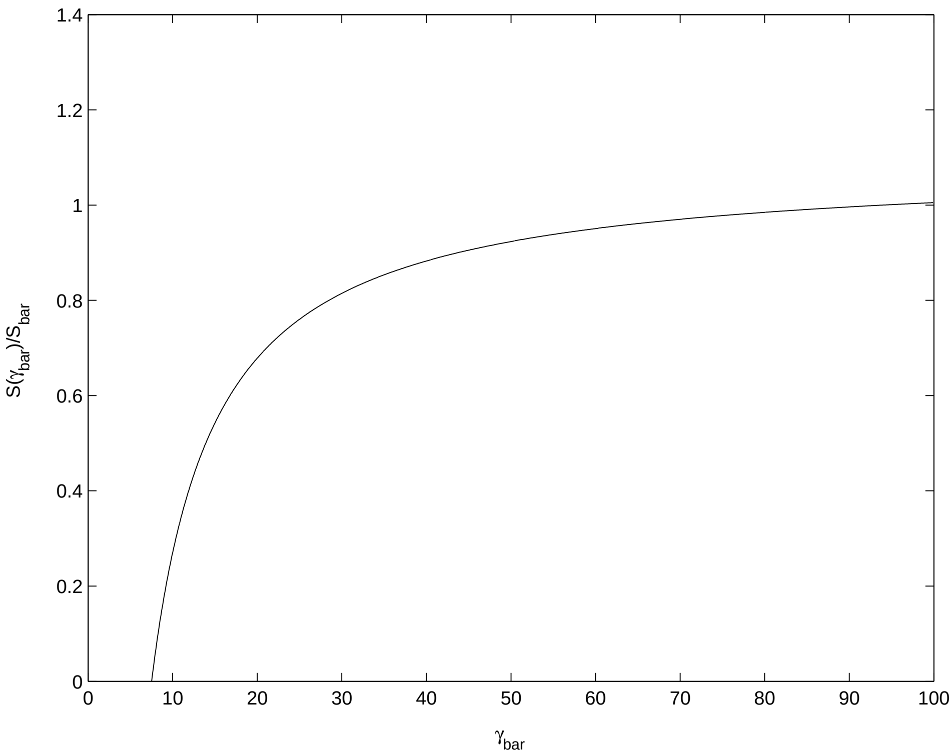

Figure 6: Problem 21

Chapter 10

1. (a)

$$\begin{aligned}\overline{(AA^H)^T} &= \overline{(A^H)^T.A^T}\\ &= \overline{\overline{(A^T)}^T}\,\overline{A^T}\\ &= AA^H\\ \therefore (AA^H)^H &= AA^H\end{aligned}$$

For AA^H, $\quad \lambda = \overline{\lambda}$, i.e. eigen-values are real

$$AA^H = Q\Lambda Q^H$$

(b) $X^H AA^H X = (X^H A)(X^H A)^H = \|X^H A\| \geq 0$
$\therefore AA^H$ is positive semidefinite.

(c) $I_M + AA^H = I_M + Q\Lambda Q^H = Q(I + \Lambda)Q^H$
A^H positive semidefinite $\Rightarrow \lambda_i \geq 0 \forall i$
$\therefore 1 + \lambda_i > 0 \forall i$
$\therefore I_M + AA^H$ positive definite

(d)

$$\begin{aligned}\det[I_M + AA^H] &= \det[I_M + Q\Lambda Q^H]\\ &= \det[Q(I_M + \Lambda_M)Q^H]\\ &= \det[I_M + \Lambda_M]\\ &= \Pi_{i=1}^{Rank(A)}(1 + \lambda_i)\end{aligned}$$

$$\begin{aligned}\det[I_N + A^H A] &= \det[I_N + \widetilde{Q}\Lambda\widetilde{Q}^H]\\ &= \det[\widetilde{Q}(I_N + \Lambda_N)\widetilde{Q}^H]\\ &= \det[I_N + \Lambda_N]\\ &= \Pi_{i=1}^{Rank(A)}(1 + \lambda_i)\end{aligned}$$

$\because AA^H \quad$ and $\quad A^H A$ have the same eigen-value
$\therefore \det[I_M + AA^H] = \det[I_N + A^H A]$

2. $H = U\Sigma V^T$

$$U = \begin{bmatrix} -0.4793 & 0.8685 & -0.1298\\ -0.5896 & -0.4272 & -0.6855\\ -0.6508 & -0.2513 & 0.7164 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1.7034 & 0 & 0\\ 0 & 0.7152 & 0\\ 0 & 0 & 0.1302 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.3458 & 0.6849 & 0.4263 \\ -0.5708 & 0.2191 & 0.0708 \\ -0.7116 & -0.6109 & 0.0145 \\ -0.2198 & 0.3311 & -0.9017 \end{bmatrix}$$

3. $H = U\Sigma V^T$
   Let

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Check the rank of each matrix
   rank(H_I) = 3
   $\therefore$ multiplexing gain = 3
   rank(H_2) = 4
   $\therefore$ multiplexing gain = 4

5.

$$C = \sum_{i=1}^{R_H} \log_2 \left(1 + \frac{\lambda_i \rho}{M_t}\right)$$

   Constraint $\sum V_i = \rho \sum \lambda_i =$ constant

$$\therefore \frac{\partial C}{\partial \lambda_i} = \frac{\rho}{M_t \ln 2} \frac{1}{(1 + \frac{\lambda_i \rho}{M_t})} - \frac{\rho}{M_t \ln 2} \frac{1}{(1 + \frac{\lambda_i \rho}{M_t})} = 0$$

$$\Rightarrow \lambda_i = \lambda_j$$

   $\therefore$ when all R_H singular values are equal, this capacity is maximized.

6. (a) Any method to show H $\approx U\Lambda$V is acceptable. For example:
   D = $\begin{bmatrix} .13 & .08 & .11 \\ .05 & .09 & .14 \\ .23 & .13 & .10 \end{bmatrix}$ where : $d_{ij} = \left|\frac{H_{ij} - H}{H_{ij}}\right| \times 100$

   (b) precoding filter M = V^{-1}
   shaping filter F = U^{-1}
   F = $\begin{bmatrix} -.5195 & -.3460 & -.7813 \\ -.0251 & -.9078 & .4188 \\ -.8540 & .2373 & .4629 \end{bmatrix}$
   M = $\begin{bmatrix} -.2407 & -.8894 & .3887 \\ -.4727 & -.2423 & -.8473 \\ -.8478 & .3876 & .3622 \end{bmatrix}$
   Thus $\overline{Y}$ = F(H)M$\overline{X}$ + F$\overline{N}$ = $U^* U \Lambda$V$V^* \overline{X} + U^* \overline{N}$
   $= \Lambda \overline{X} + U^* \overline{N}$

   (c) $\frac{P_i}{P} = \frac{1}{\gamma_o} - \frac{1}{\gamma_i}$ for $\frac{1}{\gamma_i} > \frac{1}{\gamma_o}$, 0 else
   $\gamma_i = \frac{\lambda_i{}^2 P}{N_o B}$ = 94.5 for i = 1, 6.86 for i = 2, .68 for i = 3
   Assume $\gamma_2 > \gamma_0 > \gamma_3$ since γ_3 = .68 is clearly too small for data transmission

$\sum \frac{P_i}{P} = 1 \Rightarrow \frac{2}{\gamma_0} - \frac{1}{\gamma_1} - \frac{1}{\gamma_2} = 1 \Rightarrow \gamma_0 = 1.73$
$\frac{P_1}{P} = .5676$ $\frac{P_2}{P} = .4324$
C $= B\left[\log_2\left(1+\gamma_1\frac{P_1}{P}\right) + \log_2\left(1+\gamma_2\frac{P_2}{P}\right)\right]$
$= 775.9$ kbps

(d) With equal weight beamforming, the beamforming vector is given by $c = \frac{1}{\sqrt{(3)}}[1\ 1\ 1]$. The SNR is then given by:

$$SNR = \frac{c^H H^H H c}{N_0 B} = (.78)(100) = 78. \qquad (1)$$

This gives a capacity of 630.35 kbps. The SNR achieved with beamforming is smaller than the best channel in part (c). If we had chosen c to equal the eigenvector corresponding to the best eigenvalue, then the SNR with beamforming would be equal to the largest SNR in part(c). The beamforming SNR for the given c is greater than the two smallest eigenvalues in part(c) because the channel matrix has one large eigenvalue and two very small eigenvalues.

7. $C = \max B \log_2 \det[I_{M\gamma} + HR_X H^H]$
$R_X : T_\gamma(R_X) = \rho$ If the channel is known to the transmitter, it will perform an SVD decomposition of H as

$$H = U\Sigma V$$

$$HR_X H^H = (U\Sigma V)R_X(U\Sigma V)^H$$

By Hadamard's inequality we have that for $A \in \Re^{n\times n}$

$$\det(A) \leq \Pi_{i=1}^n A_{ii}$$

with equality iff A is diagonal.
We choose R_X to be diagonal, say $= \Omega$ then

$$\det(I_{MR} + HR_X H^H) = \det(I + \Omega\Sigma^2)$$

$$\therefore C = \max_{\sum_i \rho_i \leq \rho} B\sigma_i \log_2(1 + \lambda_i\rho_i)$$

where $\sqrt{\lambda_i}$ are the singular values.

8. The capacity of the channel is found by the decomposition of the channel into R_H parallel channels, where R_H is the rank of the channel matric H.

$$C = \max_{\rho_i : \sum_i \rho_i \leq \rho} \sum_i B \log_2(1 + \lambda_i\rho_i)$$

where $\sqrt{\lambda_i}$ are the R_H non-zero singular values of the channel matrix H and ρ is the SNR constraint.

$$\gamma_i = \lambda_i \rho$$

Then the optimal power allocation is given as

$$\frac{P_i}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_i} & \gamma_i \geq \gamma_0 \\ 0 & \gamma_i < \gamma_0 \end{cases} \qquad (2)$$

for some cut-off value γ_0. The resulting capacity is given as

$$C = \sum_{i:\gamma_i \geq \gamma_0} B \log_2(\gamma_i/\gamma_0)$$

For

$$H = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$R_H = 3$, $\gamma_1 = 80$, $\gamma_2 = 40$, $\gamma_3 = 40$. We first assume that γ_0 is less than the minimum γ_i which is 40.

$$\gamma_0 = \frac{3}{1 + \sum_{i=1}^{3} \frac{1}{\gamma_i}}$$

which gives $\gamma_0 = 2.8236 < \min_i \gamma_i$, hence the assumption was correct.

$$\frac{C}{B} = 12.4732\,\text{bits/sec/Hz}$$

For

$$H = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$R_H = 4$, $\gamma_1 = 40$, $\gamma_2 = 40$, $\gamma_3 = 40$, $\gamma_4 = 40$. We first assume that γ_0 is less than the minimum γ_i which is 40.

$$\gamma_0 = \frac{4}{1 + \sum_{i=1}^{4} \frac{1}{\gamma_i}}$$

which gives $\gamma_0 = 3.6780 < \min_i \gamma_i$, hence the assumption was correct.

$$\frac{C}{B} = 13.7720\,\text{bits/sec/Hz}$$

9. $H = \begin{bmatrix} h_{11} & \dots & h_{1M_t} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ h_{M_r 1} & \dots & h_{M_r M_t} \end{bmatrix}_{M_r \times M_t}$

Denote $G = HH^T$

$$
\begin{aligned}
\lim_{M_t\to\infty} \frac{1}{M_t} G_{ii} &= \lim_{M_t\to\infty} \frac{1}{M_t} [h_{i1} \ldots h_{iM_t}] \begin{bmatrix} \overline{h_{i1}} \\ . \\ . \\ . \\ \overline{h_{iM_t}} \end{bmatrix} \\
&= \lim_{M_t\to\infty} \frac{1}{M_t} \sum_{j=1}^{M_t} \|h_{ij}\|^2 \\
&= E_j \|h_{ij}\|^2 \\
&= \sigma^2 \\
&= 1 \quad \forall i \\
\lim_{M_t\to\infty, i\neq j} \frac{1}{M_t} G_{ij} &= \lim_{M_t\to\infty} \frac{1}{M_t} [h_{i1} \ldots h_{iM_t}] \begin{bmatrix} \overline{h_{j1}} \\ . \\ . \\ . \\ \overline{h_{jM_t}} \end{bmatrix} \\
&= \lim_{M_t\to\infty} \frac{1}{M_t} \sum_{k=1}^{M_t} h_{ik}\overline{h_{jk}} \\
&= E_k h_{ik}\overline{h_{jk}} \\
&= E_k(h_{ik}) E_k(\overline{h_{jk}}) \\
&= 0 \quad \forall i, j, i \neq j \\
\therefore \lim_{M\to\infty} \frac{1}{M} HH^T &= I_M \\
\therefore \lim_{M\to\infty} B \log_2 \det \left[I_M + \frac{\rho}{M} HH^T \right] &= B \log_2 \det [I_M + \rho I_M] \\
&= B \log_2 [1+\rho] \det I_M \\
&= MB \log_2 [1+\rho]
\end{aligned}
$$

10. We find the capacity by randomly generating 10^3 channel instantiations and then averaging over it. We assume that distribution is uniform over the instantiations.

    MATLAB CODE

```
clear;
clc;
Mt = 1;
Mr = 1;
rho_dB = [0:25];

rho = 10.^(rho_dB/10);
for k = 1:length(rho)
    for i = 1:100
        H = wgn(Mr,Mt,0,'dBW','complex');
        [F, L, M] = svd(H);
        for j = 1:min(Mt,Mr)
```

```
            sigma(j) = L(j,j);
        end
        sigma_used = sigma(1:rank(H));
        gamma = rho(k)*sigma_used;
        %% Now we do water filling\
        gammatemp = gamma;
        gammatemp1 = gammatemp;
        gamma0 = 1e3;
        while gamma0 > gammatemp1(length(gammatemp1));
            gammatemp1 = gammatemp;
            gamma0 = length(gammatemp1)/(1+sum(1./gammatemp1));
            gammatemp = gammatemp(1:length(gammatemp)-1);
        end
        C(i) = sum(log2(gammatemp1./gamma0));
    end
    Cergodic(k) = mean(C);
end
```

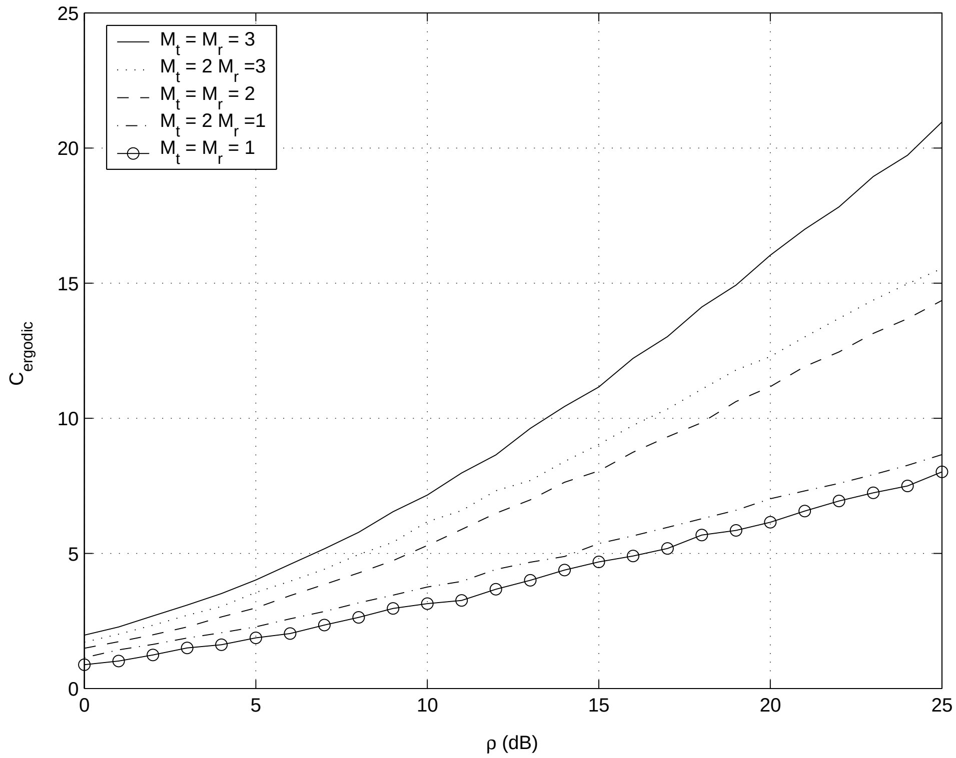

Figure 1: Problem 10

11. We find the capacity by randomly generating 10^4 channel instantiations and then averaging over it. We assume that distribution is uniform over the instantiations.

```
MATLAB CODE

clear;
clc;
Mt = 1;
```

```
Mr = 1;
rho_dB = [0:30];

rho = 10.^(rho_dB/10);
for k = 1:length(rho)
    for i = 1:1000
        H = wgn(Mr,Mt,0,'dBW','complex');
        [F, L, M] = svd(H);
        for j = 1:min(Mt,Mr)
            sigma(j) = L(j,j);
        end
        sigma_used = sigma(1:rank(H));
        gamma = rho(k)*sigma_used;
        C(i) = sum(log2(1+gamma/Mt));
    end
    Cout(k) = mean(C);
    pout = sum(C<Cout(k))/length(C);
    while pout > .01
        Cout(k) = Cout(k)-.1;
        pout = sum(C<Cout(k))/length(C);
    end
    if Cout(k)<0;
        Cout(k) = 0;
    end
end
```

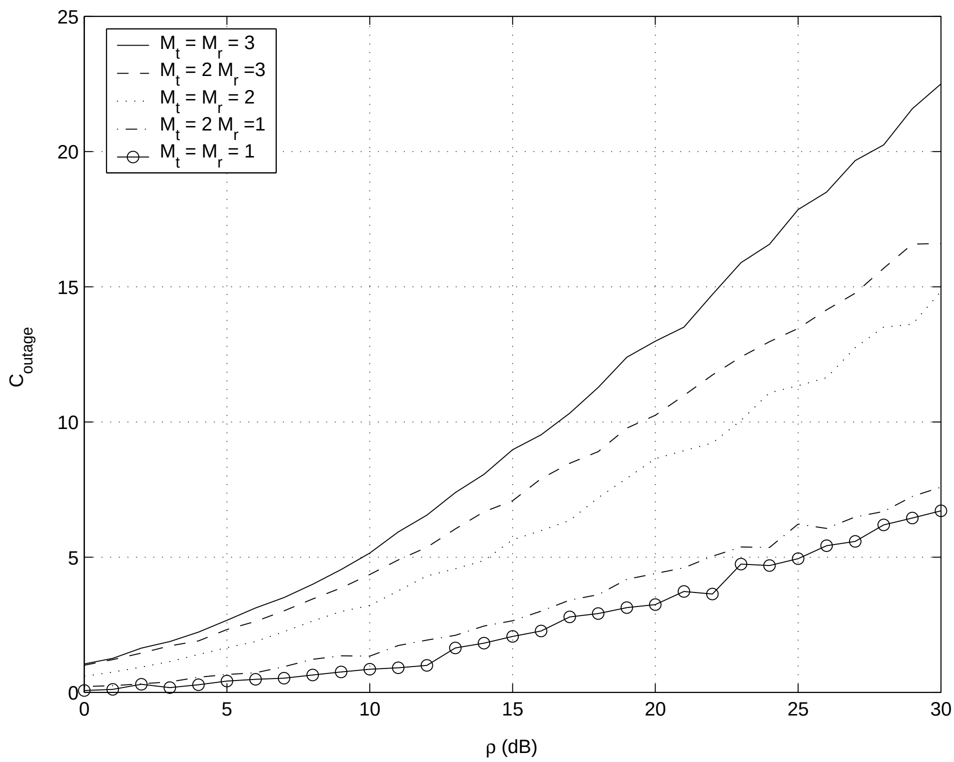

Figure 2: Problem 11

12.

$$\begin{aligned} P\left(u^{\star}n < X\right) &= P\left(\sum_{i=1}^{M_r} u_i n_i < X\right) \\ &= \sum_{i=1}^{M_r} u_i P\left(n_i < X\right) \\ &= P\left(n_i < X\right) \end{aligned}$$

$\therefore$ the statistics of $u^{\star}n$ are the same as the statistics of each of these elements

13.

$$\begin{aligned} \Sigma_x &= \|u^{\star}Hvx\| \\ &= \|u^{\star}Hv\|^2\|x\|^2 \\ &= v^H H^H u^{\star H} u^{\star}Hv\|x\|^2 \\ &= v^H H^H Hv\|x\|^2 \\ &= v^H Q^H Qv\|x\|^2 \\ &\leq \lambda_{max}\|x\|^2 \end{aligned}$$

with equality when u, v are the principal left and right singular vectors of the channel matrix H

$$\therefore SNR_{max} = \lambda_{max}\frac{\|x\|^2}{N} = \lambda_{max}\rho$$

14.

$$H = \begin{bmatrix} 0.1 & 0.5 & 0.9 \\ 0.3 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}$$

When both the transmitter and the receiver know the channel, for beamforming, u and v correspond to the principal singular vectors (or the singular vectors corresponding to the maximum singular value of H). Notice that the singular values of H are the square root of the eigen values of HH^H (Wishart Matrix).

Using Matlab, we get that the maximum singular value of H is 1.4480 and the singular vectors corresponding to this value are

$$u_{\text{opt}} = \begin{bmatrix} -0.7101 \\ -0.4641 \\ -0.5294 \end{bmatrix}$$

and

$$v_{\text{opt}} = \begin{bmatrix} -0.1818 \\ -0.4190 \\ -0.8896 \end{bmatrix}$$

It is easy to check that $u_{\text{opt}}^T u_{\text{opt}} = 1$ and $v_{\text{opt}}^T v_{\text{opt}} = 1$ and that

$$u_{\text{opt}}^T * H * v_{\text{opt}} = 1.4480$$

Since, during beamforming from eq. 10.17 in reader,

$$y = (u^T Hv)x + u^T n$$

and for a given transmit SNR of ρ, the received SNR is given as

$$\mathrm{SNR}_{\mathrm{rcvd}} = \rho(u_{\mathrm{opt}}^T H v_{\mathrm{opt}})^2$$

since, u_{opt} has norm 1, noise power is not increased. For, $\rho = 1$, SNR is simply $(1.4480)^2 = 2.0968$.

When the channel is not known to the transmitter, it allocates equal power to all the antennas and so the precoding vector (or the optimal weights) at the transmitter is given as

$$v_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Define

$$h = Hv_2$$

So, eq. 10.17 in the reader can be written as

$$y = (u^T h)x + u^T n$$

To maximize SNR we need to find a u_2 of norm 1 such that $(u'h)$ is maximized.

Using Matlab, we get that the maximum singular value of h is 1.2477 and the singular vector corresponding to this value is

$$u_2 = \begin{bmatrix} 0.6941 \\ 0.5090 \\ 0.5090 \end{bmatrix}$$

It is easy to check that $u_2^T u_2 = 1$ and that

$$u_2^T * h = 1.2477$$

Alternatively, from MRC concept we know that:

$$u_2 = \frac{h^H}{||h||} = \begin{bmatrix} 0.6941 \\ 0.5090 \\ 0.5090 \end{bmatrix}$$

where $||h||$ is the L_2 norm of h.

For a given transmit SNR of ρ, the received SNR is given as

$$\mathrm{SNR}_{\mathrm{rcvd}} = \rho(u_2^T H v_2)^2$$

since, u_2 has norm 1, noise power is not increased. For, $\rho = 1$, SNR is simply $(1.2477)^2 = 1.5567$.

15. (a) $\rho = 10$ dB $= 10$

$$P_e = \rho^{-d}$$

So to have $P_e \leq 10^{-3}$, we should have $d \geq 3$, or at least $d = 3$. Solving the equation that relates diversity gain d to multiplexing gain r at high SNR's we get

$$d = (M_r - r)(M_t - r)$$

$$\Rightarrow 3 = (8-r)(4-r)$$

Solving for r we get

$$r = 3.35 \,\text{or}\, 8.64$$

We have that $r \leq \min\{M_r, M_t\}$, so $r \leq 4$ and so $r = 3.35$. But we know that r has to be an integer. So, we take the nearest integer which is smaller than the calculated value of r, which gives us $\boxed{r{=}3}$.

No credits for this part:

If we are allowed to assume that equations 10.23 and 10.24 hold at finite SNR's too and we are given that we can use base 2 for logarithms, we can find the data rate as

$$R = r\log_2(\rho) = 9.96\,\text{bits/s/Hz}$$

(b) With, $r = 3$, we can find d as

$$d = (M_r - r)(M_t - r) = (8-3)(4-3) = 5$$

For this value of d,

$$P_e = \rho^{-d} = 10^{-5}$$

16. According to SVD of h
$\sqrt{\lambda} = 1.242$
$\therefore C/B = \log_2(1+\lambda\rho) = \log_2(1+1.242^2.10) = 4.038bps/Hz$

17.

$$\mathbf{H} = \begin{bmatrix} .3 & .5 \\ .7 & .2 \end{bmatrix} = \begin{bmatrix} -.5946 & .8041 \\ -.8041 & .5946 \end{bmatrix} \begin{bmatrix} .8713 & 0 \\ 0 & .3328 \end{bmatrix} \begin{bmatrix} -.8507 & .5757 \\ -.5757 & -.8507 \end{bmatrix}$$

P= 10mW
$N_0 = 10^{-9}$ W/Hz
$B = 100$ KHz

(a) When $\mathbf{H}$ is known both at the transmitter and at the receiver, the transmitter will use the optimal precoding filter and the receiver will use the optimal shaping filter to decompose the MIMO channel into 2 parallel channels. We can then do water-filling over the two parallel channels available to get capacity.

Finding the γ_i's

$$\gamma_1 = \frac{\lambda_1^2 P}{N_0 B} = 75.92$$

$$\gamma_2 = \frac{\lambda_2^2 P}{N_0 B} = 11.08$$

Finding γ_0

Now, we have to find the cutoff value γ_0. First assume that γ_0 is less than both γ_1 and γ_2. Then

$$\left(\frac{1}{\gamma_0} - \frac{1}{\gamma_1}\right) + \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_2}\right) = 1$$

$$\Rightarrow \frac{2}{\gamma_0} = 1 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$$
$$\Rightarrow \gamma_0 = \frac{1}{1 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2}} = 1.81$$

which is less than both γ_1 and γ_2 values so our assumption was correct.

Finding capacity

Now we can use the capacity expression as

$$C = \sum_{i=1}^{2} B \log_2 \left(\frac{\gamma_i}{\gamma_0} \right) = 800 \text{ Kbps}$$

(b) **Total is**

Essentially we have two parallel channels after the precoding filter and the shaping filter are used at the transmitter and receiver respectively.

$$M(\gamma) = 1 + \gamma K \frac{S(\gamma)}{\bar{S}}$$

Finding K

$$K = \frac{-1.5}{\ln(5P_b)} = .283$$

$\gamma_K = \gamma_0/K$.

Finding γ_0 or γ_K

We now find the cut-off γ_0. First assume that $\gamma_0 < \{\gamma_1, \gamma_2\}$. Notice that γ_1 and γ_2 have already been calculated in part (a) as $\gamma_1 = 75.92$ and $\gamma_2 = 11.08$.

$$\left(\frac{1}{\gamma_0} - \frac{1}{\gamma_1 K} \right) + \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_2 K} \right) = 1$$
$$\Rightarrow \frac{2}{\gamma_0} = 1 + \frac{1}{\gamma_1 K} + \frac{1}{\gamma_2 K}$$
$$\Rightarrow \gamma_0 = \frac{1}{1 + \frac{1}{K} \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right)} = 1.4649$$

which is less than both γ_1 and γ_2 values, so our assumption was correct.

$$\gamma_K = \gamma_0/K = 5.1742$$

Finding Rate R

Therefore the total rate, R is given as

$$R = B \left[\log_2 \left(\frac{\gamma_1}{\gamma_K} \right) + \log_2 \left(\frac{\gamma_2}{\gamma_K} \right) \right]$$

$$\Rightarrow R = B4.97$$

This gives that $\boxed{R\text{=}497.36 \text{ Kbps}}$ (Obviously less than ergodic capacity).

(c) Since now we use beamforming to get diversity only, the transmitter and the receiver use the principal left and right eigen vectors of the **Wishart Matrix** $\mathbf{HH}^H$.

Once this is done the SNR at the combiner output is simply $\lambda_{\max}\rho$, where $\lambda_{\max}$ is the maximum eigen value of the **Wishart Matrix** $\mathbf{HH}^H$ and ρ is $\frac{P}{N_0 B}$

Finding γ_s

As given in the question, $\lambda_{\max}$ is 0.7592 and ρ was calculated to be 100. So we get that $\boxed{\gamma_s = 75.92}$.

Finding P_b

When using BPSK, $\gamma_s = \gamma_b$. Now we can use the expression for P_b for BPSK

$$P_b = Q\left(\sqrt{2\gamma_b}\right) = Q\left(\sqrt{2 \times 75.92}\right) = \begin{cases} 0 & \text{Using the approx. given in the Ques.} \\ 3.4 \times 10^{-35} & \text{Using Matlab} \end{cases}$$

Credit is given for either value.

Finding Rate R

Since we are using BPSK and are given that $B = 1/T_b$, we get the rate using BPSK to be $\boxed{R\text{=}100 \text{ Kbps}}$.

Comparing with previous part

Comparing with part (b), we can see that the rate R decreases by 397.36 Kbps and the P_b improves as P_b is now $3.4 \times 10^{-35} \sim 0$ whereas earlier it was 10^{-3}.

(d) Therefore we see that we can tradeoff rate for robustness of the system. If we are willing to decrease the rate at which we transmit, we can get more diversity advantage i.e. one strong channel which gives a much less value of P_b.

18. (a)

```
clear;
clc;
Mt = 4;
Mr = Mt;
rho_dB = [0:20];

rho = 10.^(rho_dB/10);
for k = 1:length(rho)
    for i = 1:1000
        H = wgn(Mr,Mt,0,'dBW','complex');
        [F, L, M] = svd(H);
        for j = 1:min(Mt,Mr)
```

```
                sigma(j) = L(j,j);
            end
            sigma_used = sigma(1:rank(H));
            gamma = rho(k)*sigma_used;
            %% Now we do water filling\
            gammatemp = gamma;
            gammatemp1 = gammatemp;
            gamma0 = 1e3;
            while gamma0 > gammatemp1(length(gammatemp1));
                gammatemp1 = gammatemp;
                gamma0 = length(gammatemp1)/(1+sum(1./gammatemp1));
                gammatemp = gammatemp(1:length(gammatemp)-1);
            end
            C(i) = sum(log2(gammatemp1./gamma0));
        end
        Cergodic(k) = mean(C);
    end
```

(b)
```
    clear;
    clc;
    Mt = 4;
    Mr = Mt;
    rho_dB = [0:20];

    rho = 10.^(rho_dB/10);
    for k = 1:length(rho)
        for i = 1:1000
            H = wgn(Mr,Mt,0,'dBW','complex');
            [F, L, M] = svd(H);
            for j = 1:min(Mt,Mr)
                sigma(j) = L(j,j);
            end
            sigma_used = sigma(1:rank(H));
            gamma = rho(k)*sigma_used;
            C(i) = sum(log2(1+gamma/Mt));
        end
        Cout(k) = mean(C);
    end
```

19. using Matlab we get $C_{out} = 7.8320$

```
MATLAB CODE

clear;
clc;
Mt = 4;
Mr = Mt;
rho_dB = 10;
rho = 10.^(rho_dB/10);

for k = 1:length(rho)
    for i = 1:1000
```

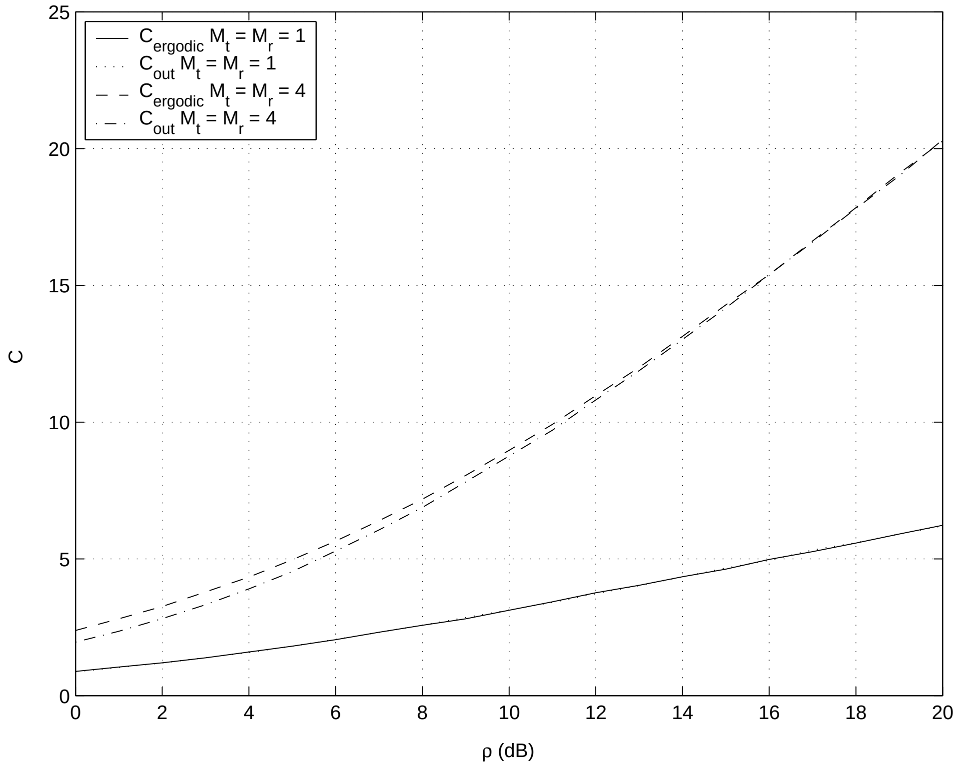

Figure 3: Problem 18

```
        H = wgn(Mr,Mt,0,'dBW','complex');
        [F, L, M] = svd(H);
        for j = 1:min(Mt,Mr)
            sigma(j) = L(j,j);
        end
        sigma_used = sigma(1:rank(H));
        gamma = rho(k)*sigma_used;
        C(i) = sum(log2(1+gamma/Mt));
    end
    Cout(k) = mean(C);
    pout = sum(C<Cout(k))/length(C);
    while pout > .1
        Cout(k) = Cout(k)-.01;
        pout = sum(C<Cout(k))/length(C);
    end
    if Cout(k)<0;
        Cout(k) = 0;
    end
end
```

20. As μ increases, the span of cdf becomes narrower and so capacity starts converging to a single number.

```
MATLAB CODE

clear;
clc;
```

```
Mt = 8;
Mr = Mt;
rho_dB = 10;
rho = 10.^(rho_dB/10);

for i = 1:1000
    H = wgn(Mr,Mt,0,'dBW','complex');
    [F, L, M] = svd(H);
    for j = 1:min(Mt,Mr)
        sigma(j) = L(j,j);
    end
    sigma_used = sigma(1:rank(H));
    gamma = rho*sigma_used;
    C(i) = sum(log2(1+gamma/Mt));
end
[f,x] = ecdf(C);
```

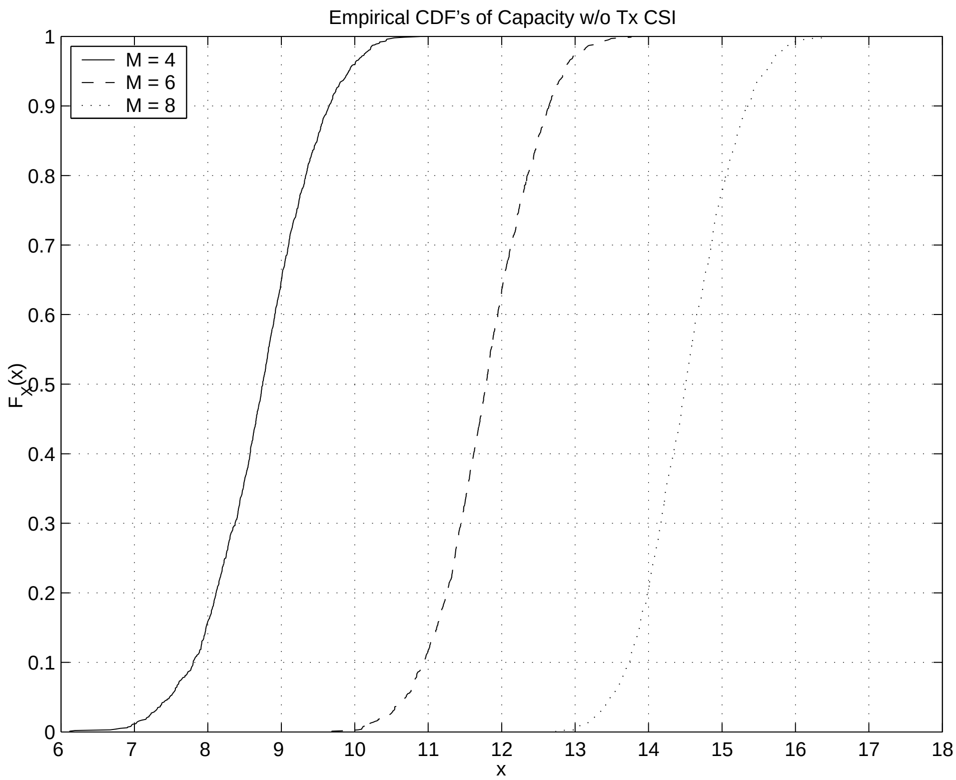

Figure 4: Problem 20

Chapter 11

1. See Fig 1

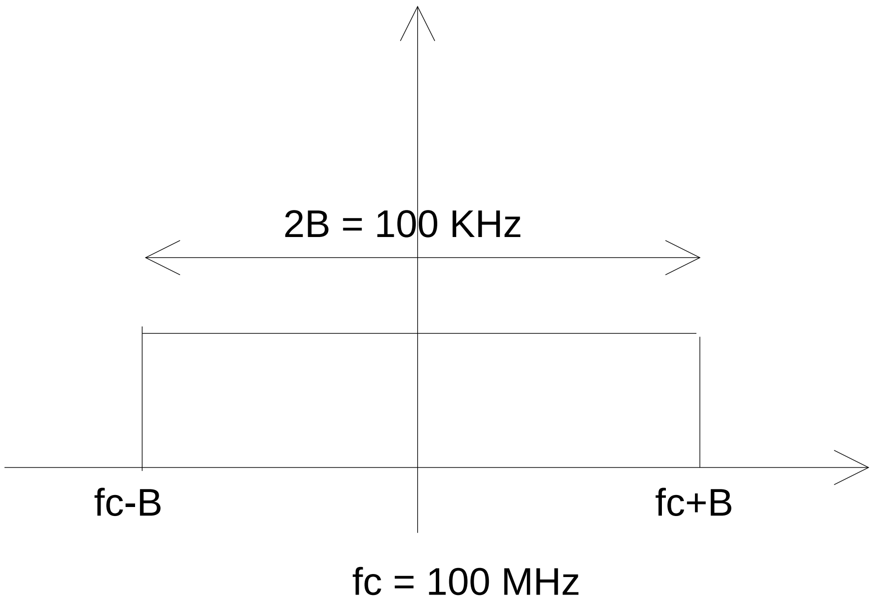

Figure 1: Band of interest.

$B = 50$ KHz, $f_c = 100$ MHz

$$H_{\text{eq}}(f) = \frac{1}{H(f)} = f$$

Noise PSD $= N_0$ W/Hz. Using this we get

$$\begin{aligned}
\text{Noise Power} &= \int_{f_c-B}^{f_c+B} N_0 |H_{\text{eq}}(f)|^2 df && (1)\\
&= N_0 \int_{f_c-B}^{f_c+B} f^2 df && (2)\\
&= N_0 \frac{f^3}{3}\Big]_{(f_c-B)}^{(f_c+B)} && (3)\\
&= \frac{N_0}{3}(f_c+B)^3 - (f_c-B)^3 && (4)\\
&= 10^{21} N_0 \,\text{W} && (5)
\end{aligned}$$

Without the equalizer, the noise power will be $2BN_0 = 10^5 N_0$ W. As seen from the noise power values, there is tremendous noise enhancement and so the equalizer will **not** improve system performance.

2. (a) For the first channel:
ISI power over a bit time $= A^2 T_b / T_b = A^2$ For the 2nd channel:
ISI power over a bit time $= \frac{A^2}{T_b} \sum_{n=1}^{\infty} \int_{nT_b}^{(n+1)T_b} e^{-t/T_m} dt = 2e^{-1/2} A^2$

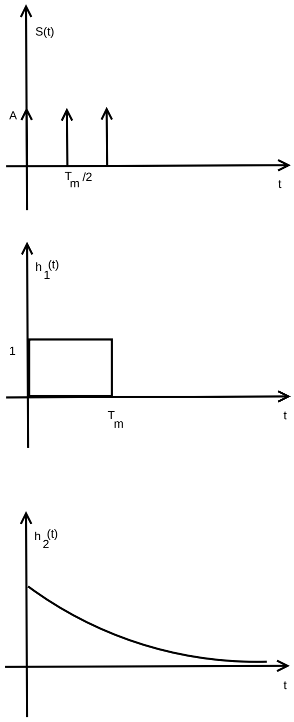

Figure 2: Problem 2a

(b) No ISI: pulse interval $= 11/2\mu s = 5.5\mu s$
$\therefore$ Data rate $= 1/5.5\mu s = 181.8Kbps$
If baseband signal =100KHz: pulse width $= 10\mu s$
Data rate $= 2/10\mu s + 10\mu s = 100Kbps$

3. (a)

$$h(t) = \begin{cases} e^{-\frac{t}{\tau}} & t \geq 0 \\ 0 & o.w. \end{cases} \tag{6}$$

$\tau = 6\ \mu$ sec

$$H_{\mathrm{eq}}(f) = \frac{1}{H(f)}$$

$$\begin{aligned} H(f) &= \int_0^\infty e^{-\frac{t}{\tau}} e^{-j2\pi f t} dt & (7) \\ &= \frac{1}{\frac{1}{\tau} + j2\pi f} & (8) \end{aligned}$$

Hence,

$$H_{\mathrm{eq}}(f) = \frac{1}{\tau} + j2\pi f$$

(b)

$$\frac{\mathrm{SNR_{eq}}}{\mathrm{SNR_{ISI}}} = \frac{\frac{\int_{-B}^{B} S_x(f)|H(f)|^2|H_{eq}(f)|^2 df}{\int_{-B}^{B} N_0|H_{eq}(f)|^2 df}}{\frac{\int_{-B}^{B} S_x(f)|H(f)|^2 df}{2BN_0}}$$

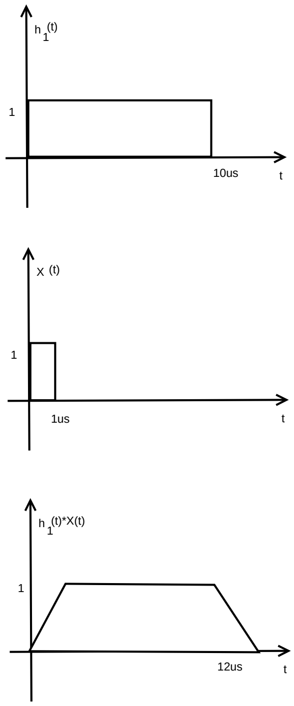

Figure 3: Problem 2b

Assume $S_x(f) = S,\ -B \le f \le B \Rightarrow$

$$\frac{\frac{2BS}{N_0\ \frac{2B}{\tau^2}+\frac{8\pi^2}{3}B^3}}{\frac{S\int_{-B}^{B}|H(f)|^2 df}{2BN_0}} = \frac{\frac{2B}{\frac{1}{\tau^2}+\frac{4\pi^2}{3}B^2}}{1.617\times 10^{-6}} = 0.9364 = -0.28 \text{ dB}$$

(c)

$$\begin{aligned}
h[n] &= 1 \ + \ e^{\frac{-T_s}{\tau}}\delta[n-1] + e^{\frac{-2T_s}{\tau}}\delta[n-2] + ... \\
H(z) &= 1 + e^{\frac{-T_s}{\tau}}z^{-1} + e^{\frac{-2T_s}{\tau}}z^{-2} + e^{\frac{-3T_s}{\tau}}z^{-3} + ... \\
&= \sum_{n=0}^{\infty}\left(e^{-\frac{T_s}{\tau}}z^{-1}\right)^n = \frac{z}{z-e^{-\frac{T_s}{\tau}}} = \frac{1}{1-e^{-\frac{T_s}{\tau}}z^{-1}}
\end{aligned}$$

$\Rightarrow H_{eq}(z) = \frac{1}{H(z)+N_0}$. Now, we need to use some approximation to come up with the filter tap coefficient values. If we assume $N_0 \approxeq 0$ (the zero-forcing assumption), we get $H_{eq}(z) = 1-e^{-\frac{T_s}{\tau}}z^{-1}$. Thus, a two tap filter is sufficient. For $T_s = \frac{1}{30}$ ms, we have a_0=1, $a_1 = -0.0039$ as the tap coefficient values. Any other reasonable way is also accepted.

4. $\omega_i = c_i$ where $\{c_i\}$ is the inverse Z- transform of 1/F(z)
Show that this choice of tap weights minimizes

$$\left|\frac{1}{F(z)} - (\omega_{-N}z^N + \ldots + \omega_N z^{-N})\right|^2 \quad \ldots (1)$$

at z $= e^{j\omega}$
If F(z) is of length 2 and monic, say $F(z) = 1 - a_1 z$ then

$$\frac{1}{F(z)} = 1 + a_1 z^{-1} + a_1^2 z^{-2} + \ldots \quad \text{where } c_1 = a_1,\ c_2 = a_1^2,\ a_1 < 1$$

It is easy to see that the coefficients become smaller and smaller. So if we had the opportunity to cancel any (2N+1) coefficients we will cancel the ones that are closest to z^0. Hence we get that $\omega_i = c_i$ minimizes (1). The result can be similarly proved for length of F(z) greate than 2 or non-monic.

5. (a) $H_{eq}(f) = \frac{1}{H(f)}$ for ZF equalizer

$$H_{eq}(f) = \begin{cases} 1 & f_c - 20MHz \le f < f_c - 10MHz \\ 2 & f_c - 10MHz \le f < f_c \\ 0.5 & f_c \le f < f_c + 10MHz \\ 4 & f_c + 10MHz \le f < f_c + 20MHz \\ 0 & o.w. \end{cases} \tag{9}$$

   (b) S=10mW Signal power
   $N = N_0[1^2 \times 10MHz + 2^2 \times 10MHz + 0.5^2 \times 10MHz + 4^2 \times 10MHz] = 0.2125mW$
   $\therefore SNR = 47.0588 = 16.73dB$

   (c) $T_s = 0.0125\mu sec$
   $P_b \le 0.2e^{-1.5\gamma/M-1}$ or $M \le 1 + \frac{1.5SNR}{-ln(5P_b)}$
   for $P_b = 10^{-3}$ $M \le 14.3228$ ($M \ge 4$ thus using the formula is reasonable)
   $R = \frac{\log_2 M}{T_s} = 307.2193Mbps$

   (d) We use M=4 non overlapping subchannels, each with B=10MHz bandwidth
   1: $f_c - 20MHz \le f < f_c - 10MHz$ $\quad \alpha_1 = 1$
   2: $f_c - 10MHz \le f < f_c$ $\quad \alpha_2 = 0.5$
   3: $f_c \le f < f_c + 10MHz$ $\quad \alpha_3 = 2$
   4: $f_c + 10MHz \le f < f_c + 20MHz$ $\quad \alpha_4 = 0.25$
   Power optimization: $\gamma_i = \frac{P\alpha_i^2}{N_0 B}$ for $i = 1, 2, 3, 4$
   $\gamma_1 = 1000 \quad \gamma_2 = 250 \quad \gamma_3 = 4000 \quad \gamma_4 = 62.5$
   for $P_b = 10^{-3}$ $K = 0.2831$

$$\frac{P_i}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{K\gamma_i} & \gamma_i \ge \gamma_0/K \\ 0 & \gamma_i \le \gamma_0/K \end{cases} \tag{10}$$

   We can see that all subchannels will be used and

$$P_1 = 2.6523 \quad P_2 = 2.5464 \quad P_3 = 2.6788 \quad P_4 = 2.1225$$

   and

$$\gamma_0 = 3.7207$$

   thus $R = 2B\sum_{i=1}^{4} \log(K\gamma_i/\gamma_0) = 419.9711Mbps$

6. (a)

$$\mathfrak{F}\{f(t)\} = \begin{cases} T & |f| < 1/T \\ 0 & o.w. \end{cases}$$

$$\begin{aligned} F_Z(f) &= \frac{1}{T_S}\sum_{n=-\infty}^{\infty} F\left(f + \frac{n}{T_s}\right) \\ &= 1 \end{aligned}$$

   $\therefore$ folded spectrum of f(t) is flat.

(b)

$$
\begin{aligned}
y_k &= y(kT+t_0) \\
&= \sum_{i=-\infty}^{\infty} X_i f(kT+t_0-iT) \\
&= \sum_{i=-N}^{N} X_i f\left((k-i)T+t_0\right) \\
&= X_k sinc(t_0) + \underbrace{\sum_{i=-N+k, i\neq k}^{N+k} X_i f\left((k-i)T+t_0\right)}_{ISI}
\end{aligned}
$$

(c)

$$
\begin{aligned}
ISI &= \sum_{i=-N+k, i\neq k}^{N+k} X_i \frac{\sin\left(\pi(k-i)+t_0/T\pi\right)}{\pi(k-i)+t_0/T\pi} \\
&= sin(\pi t_0/T) \sum_{i=-N, i\neq 0}^{N} \frac{1}{\pi t_0/T - \pi i} \\
&= sin(\pi t_0/T) \sum_{i=1}^{N} \left[\frac{-1}{\pi t_0/T - \pi i} + \frac{1}{\pi t_0/T - \pi i}\right] \\
&= \frac{2}{\pi} sin(\pi t_0/T) \sum_{n=1}^{N} \frac{n}{n^2 - t_0^2/T^2}
\end{aligned}
$$

Thus, ISI $\rightarrow \infty$ as N $\rightarrow \infty$

7. $g_m(t) = g^\star(-t) = g(t) = sinc(t/T_S), |t| < T_s$
   Noise whitening filter : $\frac{1}{G_m^\star(1/z^\star)}$

8. $J_{min} = 1 - \sum_{j=-\infty}^{\infty} c_j f_{-j}$

$$\begin{aligned}
B(z) &= C(z)F(z) \\
&= \frac{F(z)F^\star(z^{-1})}{F(z)F^\star(z^{-1}) + N_0} \\
&= \frac{X(z)}{X(z) + N_0} \\
\therefore b_0 &= \frac{1}{2\pi j} \oint \frac{B(z)}{z} dz \\
&= \frac{1}{2\pi j} \oint \frac{X(z)}{z[X(z) + N_0]} dz \\
&= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0} d\omega
\end{aligned}$$

$$\therefore J_{min} = 1 - \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0} d\omega$$

$$\begin{aligned}
&= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{X(e^{j\omega T}) + N_0} d\omega \\
&= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{T^{-1} \sum_{n=-\infty}^{\infty} |H(\omega + 2\pi n/T)|^2 + N_0} d\omega \\
&= T_s \int_{-0.5T_s}^{-0.5T_s} \frac{N_0}{F_\Sigma(f) + N_0} df
\end{aligned}$$

9.

$$\begin{aligned}
V_W J &= \left(\frac{\partial J}{\partial w_0}, \ldots, \frac{\partial J}{\partial w_N} \right) \\
J &= w^T M_v w^\star - 2\Re\{V_d w^\star\} + 1
\end{aligned}$$

$$\therefore \frac{\partial J}{\partial \overline{w}} = 2M_v w^T - 2V_d$$

$$\begin{aligned}
\frac{\partial J}{\partial \overline{w}} = 0 &\Rightarrow 2M_v w^T = 2V_d \\
&\Rightarrow w_{opt} = \left(M_v^T\right)^{-1} V_d^H
\end{aligned}$$

10.

$$\begin{aligned}
J_{min} &= T_s \int_{-0.5T_s}^{0.5T_s} \frac{N_0}{F_\Sigma(f) + N_0} df \\
\because \frac{N_0}{F_\Sigma(f) + N_0} &\geq 0 \quad \therefore J_{min} \geq 0 \\
\frac{N_0}{F_\Sigma(f) + N_0} &\leq \frac{N_0}{N_0} = 1 \\
\therefore J_{min} &\leq T_s \int_{-0.5T_s}^{0.5T_s} 1 df = 1 \\
\therefore 0 \leq &\ J_{min} \leq 1
\end{aligned}$$

11.

$$\begin{aligned} F_\Sigma(f) &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} F\left(f+\frac{n}{T_s}\right) \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} 1+0.5e^{-j2\pi\ f+\frac{n}{T_s}} + 0.3e^{-j4\pi\ f+\frac{n}{T_s}} \end{aligned}$$

MMSE equalizer :

$$J_{min} = T_s \int_{-0.5/T_s}^{0.5/T_s} \frac{N_0}{F_\Sigma(f)+N_0} df$$

DF equalizer :

$$J_{min} = exp\left\{ T_s \int_{-0.5/T_s}^{0.5/T_s} \ln\left[\frac{N_0}{F_\Sigma(f)+N_0}\right]\right\} df$$

12. (a) G(f) is a sinc(), so theoretically infinite. But 2/T is also acceptable (Null to Null bandwidth)
(b) $\tau \gg T$ is more likely since $T = 10^{-9} sec$
As long as $\tau > T_b$, get ISI and so, frequency selective fading
(c) Require $T_b = T_m + T \Rightarrow R = \frac{1}{T_m+T} = 49997.5bps$
(d) $H_{eq}(z) = \frac{1}{F(z)}$ for ZF equalizer
$\Rightarrow H_{eq}(z) = \frac{1}{d_0+d_1z^{-1}+d_2z^{-2}}$
Long division yields the first 2 taps as

$$w_0 = 1/\alpha_0$$
$$w_1 = -\alpha_1/\alpha_0^2$$

13. (a)

$$H_{zf}(f) = \frac{1}{H(f)} = \begin{cases} 1 & 0 \le f \le 10\text{KHz} \\ 2 & 10\text{KHz} \le f \le 20\text{KHz} \\ 3 & 20\text{KHz} \le f \le 30\text{KHz} \\ 4 & 30\text{KHz} \le f \le 40\text{KHz} \\ 5 & 40\text{KHz} \le f \le 50\text{KHz} \end{cases}$$

(b) The noise spectrum at the output of the filter is given by $N(f) = N_0|H_{eq}(f)|^2$, and the noise power is given by the integral of $N(f)$ from -50 kHz to 50 kHz:

$$\begin{aligned} N = \int_{f=-50kHz}^{50kHz} N(f)df &= 2N_0 \int_{f=0kHz}^{50kHz} |H_{eq}(f)|^2 df \\ &= 2N_0(1+4+9+16+25)(10kHz) \\ &= 1.1\text{mW} \end{aligned}$$

(c) The noise spectrum at the output of the filter is given by $N(f) = \frac{N_0}{(H(f)+\alpha)^2}$, and the noise power is given by the integral of $N(f)$ from -50 kHz to 50 kHz. For $\alpha = .5$ we get

$$N = 2N_0(.44+1+1.44+1.78+2.04))(10kHz) = 0.134 \text{ mW}$$

For $\alpha = 1$ we get

$$N = 2N_0(.25+.44+.56+.64+.69))(10kHz) = 0.0516 \text{ mW}$$

(d) As α increases, the frequency response $H_{eq}(f)$ decreases for all f. Thus, the noise power decreases, but the signal power decreases as well. The factor α should be chosen to balance maximizing the SNR and minimizing distortion, which also depends on the spectrum of the input signal (which is not given here).

(e) As $\alpha \to \infty$, the noise power goes to 0 because $H_{eq}(f) \to 0$ for all f. However, the signal power also goes to zero.

14. The equalizer must be retrained because the channel de-correlates. In fact it has to be retrained at least every channel correlation time.
Benefits of training

(a) Use detected data to adjust the equalizer coefficients. Can work without training information

(b) eliminate ISI.

15. $N = 4$

LMS-DFE: $2N+1$ operations/iteration $\Rightarrow$ 9 operations/iteration

RLS: $2.5(N)^2 + 4.5N$ operations/iteration $\Rightarrow$ 58 operations/iteration

Each iteration, one bit sent. The bit time is different for LMS-DFE/RLS, T_b (LMS-DFE) $<T_b$ (RLS). But time to convergence is faster for RLS.

Case 1: $f_D = 100$ Hz $\Rightarrow (\Delta t_c) \equiv 10$ msec, must retrain every 5 msec.

LMS-DFE: R $= \frac{10^7}{9}$ - $\frac{1000 \text{ bits}}{5 \text{ msecs}} = 911$ Kbps

RLS: R $= \frac{10^7}{58}$- $\frac{50 \text{ bits}}{5 \text{ msec}} = 162$ Kbps

Case2: $f_D = 1000$ Hz $\Rightarrow$ retrain every 0.5 msec

$R_{\text{LMS-DFE}} = 0$ bps

$R_{\text{RLS}} = 72.4$ Kbps

16. In the adaptive method, we start with some initial value of tap coefficients $\mathbf{W_0}$ and then use the steepest descent method

$$\mathbf{W_{k+1}} = \mathbf{W_K} - \mathbf{\Delta G_K} \quad \ldots (\mathbf{1})$$

where Δ is some small positive number and $\mathbf{G_K}$ is the gradient of MSE $=$ E$|\hat{d}_k - \hat{\hat{d}}_k|^2$ is $\mathbf{RW_k} - \mathbf{p}$ (Notice that 11.37 was a solution of gradient =0 , $\therefore \mathbf{RW} = \mathbf{p}$)

$$\therefore \mathbf{G_K} = \mathbf{RW_k} - \mathbf{p} = -\mathbf{E}[\varepsilon_\mathbf{k} \mathbf{Y_k}^\star]$$

where $\mathbf{Y_k} = [\mathbf{y_{k+L}} \ldots \mathbf{y_{K-L}}]^\mathbf{T}$ and $\varepsilon_k = \hat{\hat{d}}_k - \hat{d}_k$
Approximately (1) can be rewritten as

$$\mathbf{W_{k+1}} = \mathbf{W_k} + \mathbf{\Delta} \varepsilon_\mathbf{k} \mathbf{Y_k}^\star$$

Chapter 12

1. (a) $\psi_i = \cos(2\pi j/T_N t + \phi_j)$
   To form a set of orthonormal basis on $[0, T_N]$
   We need, $\int_0^{T_N} \psi_j \psi_k dt = 0$

$$\begin{aligned}
\int_0^{T_N} \psi_j \psi_k dt &= \int_0^{T_N} \cos(2\pi j/T_N t + \phi_j)\cos(2\pi k/T_N t + \phi_k)\, dt \\
&= \int_0^{T_N} \frac{1}{2}\cos(2\pi(j+k)/T_N t + \phi_j + \phi_k) + \frac{1}{2}\cos(2\pi(j-k)/T_N t + \phi_j - \phi_k)\, dt \\
&= \frac{1}{2}\frac{T_N}{2\pi(j+k)}\left[\sin(2\pi(j+k) + \phi_j + \phi_k) - \sin(\phi_j + \phi_k)\right] \\
&+ \frac{1}{2}\frac{T_N}{2\pi(j-k)}\left[\sin(2\pi(j-k) + \phi_j - \phi_k) - \sin(\phi_j - \phi_k)\right] \\
&= 0
\end{aligned}$$

   $\Rightarrow$ j and k are integers
   $\Rightarrow$ The minimum separation for sub-carriers $\cos(2\pi j/T_N t + \phi_j)$is $1/T_N$ for any ϕ_j

   (b) If $\phi_j = 0 \forall$j

$$\int_0^{T_N} \psi_j \psi_k dt = \frac{1}{2}\frac{T_N}{2\pi(j+k)}\sin 2\pi(j+k) + \frac{1}{2}\frac{T_N}{2\pi(j-k)}\sin 2\pi(j-k) = 0$$

   $\Rightarrow 2\pi(j+k) = l_1\pi \quad 2\pi(j-k) = l_2\pi \quad l_1, l_2 \in \mathbb{Z}$
   $\Rightarrow$ j and k are multiples of 1/2
   $\Rightarrow$ The minimum separation:$1/2T_N$

2. (a) $T_N = 1/B_N = 10T_m = 10/B_c = 10/10KHz = 1ms$

   (b) $B = \frac{N(1+\beta+\epsilon)}{T_N} = \frac{128(1+1.5+0.1)}{1ms} = 333KHz$

   (c) $B = \frac{N+\beta+\epsilon}{T_N} = \frac{128+1.5+0.1}{1ms} = 129.6KHz$
   $\therefore$ The total bandwidth using overlapping carriers is less than half of the non overlapping bandwidth.

3.

$$\begin{aligned}
x_3[n] &= x_2[n] \otimes x_1[n] \\
&= \sum_{m=0}^{N-1} x_2[m]x_1[(n-m)_N] \\
X_3[k] &= \sum_{n=0}^{N-1} x_3[n]\omega_N^{kn} \\
&= \sum_{n=0}^{N-1}\left[\sum_{m=0}^{N-1} x_2[m]x_1[(n-m)_N]\right]\omega_N^{kn}
\end{aligned}$$

interchange order of summation

$$
\begin{aligned}
X_3[k] &= \sum_{m=0}^{N-1} x_2[m] \left[\sum_{n=0}^{N-1} x_1[(n-m)_N] \omega_N^{kn} \right] \\
&= \sum_{m=0}^{N-1} x_2[m] X_1[k] \omega_N^{km} \\
&= X_1[k] \sum_{m=0}^{N-1} x_2[m] \omega_N^{km} \\
&= X_1[k] X_2[k]
\end{aligned}
$$

$\therefore$ circular convolution of discrete-time sequences leads to multiplication of their DFTs.

4. (a) For FDM, the number of subchannels $= \frac{B}{(\Delta f)_c} = 5$

$T_s = 10\ \mu sec$, R $= \frac{1}{10\ \mu sec} = 0.1$ Mbps

(b) $\bar{P}_{b_n} \ = \ \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_{b_n}}{\bar{\gamma}_{b_n}+1}}\ \right]$ where $\gamma_{b_n} = 1000/\text{n}$

n	$\bar{\gamma}_{b_n}$	$\bar{P}_{b_n}$
1	1000	2.5×10^{-4}
2	500	5×10^{-4}
3	333	7.5×10^{-4}
4	250	10^{-3}
5	200	1.25×10^{-3}

BER after decoding $= \sum_{i=3}^{5} P_r\ [i\ channels\ in\ error] = 3.5 \times 10^{-9}$

The total date rate of the system is the same as the data rate of any of the subcarriers (since they all have the same bits transmitted over them) $\Rightarrow$ R $= 0.1$ Mbps

(c) **Since it is not specified which equation to use for calculation of SNR, all answers based on any correct equation in the reader are being given full credit. What is given below is just one way to do the problem. Your answer can be totally different but we still give credit for it.**

For BPSK in Rayleigh fading:

$$\overline{P}_b = \frac{1}{2}\left[1 - \sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\right]$$

For $\overline{P}_b \leq 10^{-3}$, we get $\text{SNR}_{\min} = 248.75$.

For higher order QAM's we use the equation given in the reader:

$$\overline{P}_s = \frac{\alpha_M}{2}\left(1 - \sqrt{\frac{0.5\beta_M\overline{\gamma}_s}{1+0.5\beta_M\overline{\gamma}_s}}\right)$$

which gives

$$\overline{P}_b \approx \frac{\alpha_M}{2\log_2 M}\left(1 - \sqrt{\frac{0.5\beta_M\overline{\gamma}_s}{1+0.5\beta_M\overline{\gamma}_s}}\right)$$

For 4-QAM (since its a rectangular constellation), from Table 6.1 we have $\alpha_M = 1$ and $\beta_M = 1$, so we get $\text{SNR}_{\min} = 249.25$. However using the exact equation 6.81, we get that for 4-QAM, $\text{SNR}_{\min}$ $= 453$, which makes more sense as it is much greater than that required for BPSK. Hence we will use this value.

For 8-QAM (since its a non-rectangular constellation), from Table 6.1 we have $\alpha_M = 4$ and $\beta_M = 3/7$, so we get $\text{SNR}_{\min} = 1552$, which is higher than the SNR on any branch and so should not be used.

Hence:

n	$\bar{\gamma}_{s_n}$	Max(M)	$\bar{P}_b$
1	1000	4	0.454×10^{-3}
2	500	4	0.906×10^{-3}
3	333	2	0.748×10^{-3}
4	250	2	0.997×10^{-3}
5	200	0	1.25×10^{-3} with BPSK

$R = 0.1 \sum_n \log_2(M) = 0.6$ Mbps

5. (a) If the baseband bandwidth is 100 Khz, then at the carrier frequency they have bandwidth of $B = 200$ KHz.

   For flat fading we need the coherence bandwidth to be much greater than the bandwidth of the signal. Therefore $B_c \geq 10B = 2$ MHz.

   For independent fading, we want the channel between two carriers to be uncorrelated. That is, we want $B_c < \Delta B = 200$ KHz.

   If the fading between the different channels is correlated, they will all tend to have fades at the same instants. Therefore, coding over sub-channels will not work because all channels will tend to fade at same time and the code will not be able to correct all these errors.

   (b) We have
$$\text{BER} \leq .2e^{-1.5\gamma(M-1)}$$
   which means that
$$M \leq 1 + \frac{1.5\gamma}{-\ln 5\text{BER}} = 1 + 0.283\gamma$$
   For the first sub-channel, this gives $M \leq 4.5672$, which means we use 4-QAM.

   For the second sub-channel, $M \leq 8.108$, which means we use 8-QAM.

   For the third sub-channel, $M \leq 18.856$, which means we use 16-QAM.

   Therefore, at each symbol time we will transmit 9bits.

   We have that $T_s = 1/B = 10\ \mu$s, means we transmit at 900 Kbps total.

   (c) To achieve the same data rate, we will need 3bits/symbol per sub-channel, that is 8-QAM constellation.

   To achieve this, we need a minimum SNR of 13.93 dB per sub-channel.

   In the first sub-channel we must increase the power by 2.93dB, giving a transmit power of $P_{t_1} =$ 196.33 mW.

   For the second sub-channel we need 0.07 dB less power, that is $P_{t_1} = 98.40$ mW.

   For the third sub-channel we need 4.07 dB less, that is $P_{t_2} = 39.17$ mW.

   The total transmit power is now, $P_t = 333.90$ mW. Therefore we need to increase the transmit power by 33.90 mW with respect to case b).

6. $T_c = 20\mu s$
   $\therefore B_c = 1/T_c = 50KHz$
   $\therefore B_N = B_c/2 = 25KHz$
   $B = NB_N = 8 \times 25KHz = 200KHz$
   $SNR = 20dB, \quad$ target BER $= 10^{-3}$
   For MQAM $P_e \approx 0.2e^{-1.5\gamma/M-1}$

$\Rightarrow M = 1 + -1.5\gamma/\ln 5P_e = 29.31$
$\therefore M = 16$
$\therefore R = NR_N = N\log_2 M/T_N = N\log_2 M\frac{B_N}{1+\beta} = 400Kbps$

7.

$$\begin{bmatrix} y_{N-1} \\ y_{N-2} \\ . \\ . \\ . \\ y_0 \end{bmatrix} = \begin{bmatrix} h_0 & h_1 & \dots & h_\mu & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & h_0 & \dots & h_{\mu-1} & h_\mu & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & h_0 & \dots & \dots & \dots & h_\mu \end{bmatrix} \begin{bmatrix} x_{N-1} \\ . \\ x_0 \\ x_{-1} \\ . \\ x_{-\mu} \end{bmatrix} + \begin{bmatrix} \nu_{N-1} \\ . \\ . \\ . \\ . \\ \nu_0 \end{bmatrix}$$

$$= \begin{bmatrix} h_0 & \dots & h_\mu & 0 & \dots & 0 \\ 0 & h_0 & \dots & h_{\mu-1} & h_\mu & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & h_\mu \\ 0 & \dots & \dots & \dots & \dots & h_0 \end{bmatrix} \begin{bmatrix} x_{N-1} \\ . \\ . \\ . \\ . \\ x_0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ h_2 & h_3 & \dots & h_{\mu-2} & \dots & \dots \\ h_1 & h_2 & \dots & h_{\mu-1} & \dots & \dots \end{bmatrix} \begin{bmatrix} x_{N-1} \\ . \\ . \\ . \\ . \\ x_0 \end{bmatrix} + \begin{bmatrix} \nu_{N-1} \\ . \\ . \\ . \\ . \\ \nu_0 \end{bmatrix}$$

$$= \begin{bmatrix} h_0 & \dots & h_\mu & 0 & \dots & 0 \\ 0 & h_0 & \dots & h_{\mu-1} & h_\mu & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ h_2 & h_3 & \dots & h_{\mu-2} & \dots & h_\mu \\ h_1 & h_2 & \dots & h_{\mu-1} & \dots & h_0 \end{bmatrix} \begin{bmatrix} x_{N-1} \\ . \\ . \\ . \\ . \\ x_0 \end{bmatrix} + \begin{bmatrix} \nu_{N-1} \\ . \\ . \\ . \\ . \\ \nu_0 \end{bmatrix}$$

8. DFT:

$$\begin{aligned} X[i] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\omega_N^{in} \\ X[0] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]1 \\ X[1] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\omega_N^{n} \\ \dots\dots & \quad \dots\dots \\ X[N-1] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\omega_N^{(N-1)n} \end{aligned}$$

$$\therefore \overline{X}(n) = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \dots & \omega_N^{N-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \dots & \omega_N^{(N-1)^2} \end{bmatrix} .\overline{x}(n)$$

9. (a) First row of Q = [1 1 . . . 1]
$HQ(1,:)^T = H[1\ 1\ .\ .\ .\ 1]^T = \sum_{i=0}^{\mu} h_i [1\ 1\ .\ .\ .\ 1]^T$
$\therefore$ First row of Q is an eigenvector of H with $\lambda_0 = \sum_{i=0}^{\mu} h_i$

(b) Second row of Q = $\left[\ 1 \quad \omega_N \quad \omega_N^2 \quad \dots \quad \dots \quad \omega_N^{N-1}\ \right]$

$$\begin{aligned} HQ(2,:)^T &= H\left[\ 1 \quad \omega_N \quad \omega_N^2 \quad \dots \quad \dots \quad \omega_N^{N-1}\ \right]^T \\ &= \sum_{i=0}^{\mu} h_i \omega_N^i \left[\ 1 \quad \omega_N \quad \omega_N^2 \quad \dots \quad \dots \quad \omega_N^{N-1}\ \right]^T \\ &= \sum_{i=0}^{\mu} h_i \omega_N^i Q(2,:)^T \end{aligned}$$

$\therefore$ second row of Q is an eigenvector of H with $\lambda_1 = \sum_{i=0}^{\mu} h_i \omega_N^i$

(c) For k^{th} row of Q = $\left[\ 1 \quad \omega_N^{k-1} \quad \omega_N^{2(k-1)} \quad \dots \quad \dots \quad \omega_N^{(N-1)(k-1)}\ \right]$

$$\begin{aligned} HQ(k,:)^T &= H\left[\ 1 \quad \omega_N^{k-1} \quad \omega_N^{2(k-1)} \quad \dots \quad \dots \quad \omega_N^{(N-1)(k-1)}\ \right]^T \\ &= \sum_{i=0}^{\mu} h_i \omega_N^{i(k-1)} \left[\ 1 \quad \omega_N^{k-1} \quad \omega_N^{2(k-1)} \quad \dots \quad \dots \quad \omega_N^{(N-1)(k-1)}\ \right]^T \\ &= \sum_{i=0}^{\mu} h_i \omega_N^{i(k-1)} Q(k,:)^T \end{aligned}$$

$\therefore \forall$k, k^{th} row of Q is an eigenvector of H with eigenvalue $\lambda_k = \sum_{i=0}^{\mu} h_i \omega_N^{i(k-1)}$

10. $\widetilde{x}[n] = \underbrace{0 \ldots\ldots 0}_{\mu} x_0 \ldots\ldots x_{N-1}$

For $\mu \le n \le N$

$$\begin{aligned} y[n] &= \widetilde{x}[n] \star h[n] \\ &= \sum_{k=1}^{\mu} h[k]\widetilde{x}[n-k] \\ &= \sum_{k=1}^{\mu} h[k]x[n-k] \\ &= \sum_{k=1}^{\mu} h[k]x[n-k]_N \\ &= x[n] \otimes h[n] \end{aligned}$$

For $n < \mu$

$$\begin{aligned} y[n] &= \widetilde{x}[n] \star h[n] + \widetilde{x}[n+N] \star h[n+N] \\ &= \sum_{k=1}^{\mu} h[k]\widetilde{x}[n-k] + \sum_{k=n+1}^{\mu} h[k]\widetilde{x}[N+n-k] \\ &= \sum_{k=1}^{\mu} h[k]x[n-k] + \sum_{k=n+1}^{\mu} h[k]x[n-k]_N \\ &= \sum_{k=1}^{\mu} h[k]x[n-k]_N \\ &= x[n] \otimes h[n] \end{aligned}$$

11. (a)

$$H = \begin{bmatrix} .7 & .5 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .7 & .5 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .7 & .5 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 & .5 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .7 & .5 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .7 & .5 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .7 & .5 & .3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .7 & .5 & .3 \end{bmatrix}$$

$Y = Hx + \nu$

(b)

$$\widetilde{H} = \begin{bmatrix} .7 & .5 & .3 & 0 & 0 & 0 & 0 & 0 \\ 0 & .7 & .5 & .3 & 0 & 0 & 0 & 0 \\ 0 & 0 & .7 & .5 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 & .5 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & .7 & .5 & .3 & 0 \\ 0 & 0 & 0 & 0 & 0 & .7 & .5 & .3 \\ .3 & 0 & 0 & 0 & 0 & 0 & .7 & .5 \\ .5 & .3 & 0 & 0 & 0 & 0 & 0 & .7 \end{bmatrix}$$

$\widetilde{H} = M\Lambda M^H$

(c) The flat fading channel gains are the diagonal elements of the matrix Λ

```
MATLAB CODE

clear all;

H=[ .7 .5 .3 0 0 0 0 0
    0 .7 .5 .3 0 0 0 0
    0 0 .7 .5 .3 0 0 0
    0 0 0 .7 .5 .3 0 0
    0 0 0 0 .7 .5 .3 0
    0 0 0 0 0 .7 .5 .3
    .3 0 0 0 0 0 .7 .5
    .5 .3 0 0 0 0 0 .7 ]

 [V,D] = eig(H)

 V*D*V'-H
```

12. (a) $\mu = 4$

The VC system doesn't require a cyclic prefix to make the subchannels orthogonal.

(b)

$$\begin{bmatrix} y_{255} \\ y_{254} \\ . \\ . \\ . \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 0.7 & 0.3 & 0.2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0.6 & 0.7 & 0.3 & 0.2 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 1 & 0.6 & 0.7 & 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} x_{255} \\ . \\ . \\ . \\ . \\ x_{-4} \end{bmatrix} + \begin{bmatrix} \nu_{255} \\ . \\ . \\ . \\ . \\ \nu_0 \end{bmatrix}$$

$$H = U\Sigma V^H$$

Singular values (using 'svd' in Matlab)

(c) use 'svd' in Matlab

13. $R_{min} = 52\text{sub-carriers} \times \frac{1/2\text{bit}}{\text{coded bit}} \times \frac{1\text{ coded bit}}{\text{sub-carrrier symbol}} \times \frac{1\text{ sub-carrrier symbol}}{4\times 10^{-6}\text{sec}} = 6.5Mbps$

$R_{max} = 52\text{sub-carriers} \times \frac{3/4\text{bit}}{\text{coded bit}} \times \frac{6\text{ coded bit}}{\text{sub-carrrier symbol}} \times \frac{1\text{ sub-carrrier symbol}}{4\times 10^{-6}\text{sec}} = 58.5Mbps$

14.

$$\begin{aligned}
R &= \frac{48}{2}\text{sub-carriers} \times \frac{1/2\text{bit}}{\text{coded bit}} \times \frac{1\text{ coded bit}}{\text{sub-carrrier symbol}} \times \frac{1\text{ sub-carrrier symbol}}{4\times 10^{-6}\text{sec}} \\
&+ \frac{48}{2}\text{sub-carriers} \times \frac{3/4\text{bit}}{\text{coded bit}} \times \frac{6\text{ coded bit}}{\text{sub-carrrier symbol}} \times \frac{1\text{ sub-carrrier symbol}}{4\times 10^{-6}\text{sec}} \\
&= 7.5Mbps
\end{aligned}$$

15. $x(t) = sinc\left(\frac{t}{T}\right)\left[\frac{\cos\left(\beta\frac{\pi t}{T}\right)}{1-\frac{2\beta t}{T}}\right]$

$$X(\omega) = \begin{cases} T & |\omega| \le \frac{\pi}{T}(1-\beta) \\ \frac{T}{2}\left[1-\sin\left(\frac{T}{2\beta}\left(|\omega|-\frac{\pi}{T}\right)\right)\right] & \frac{\pi}{T}(1-\beta) \le |\omega| \le \frac{\pi}{T}(1+\beta) \\ 0 & |\omega| \ge \frac{\pi}{T}(1+\beta) \end{cases}$$

$$\max_t |x(t)|^2 = s^2(t)|_{t=0} = 1$$

$$\begin{aligned}
E_t\left[|X(t)|^2\right] &= \frac{1}{2\pi T}E_\omega\left[|X(\omega)|^2\right] \\
&= \frac{1}{2\pi T}2\int_{\frac{\pi}{T}(1-\beta)}^{\frac{\pi}{T}(1+\beta)} |\frac{T}{2}\left[1-\sin\left(\frac{T}{2\beta}\left(|\omega|-\frac{\pi}{T}\right)\right)\right]|^2 d\omega + \frac{1}{2\pi T}T^2\frac{\pi}{T}(1-\beta) \\
&= 1-0.5\beta
\end{aligned}$$

$$\therefore PAR = \frac{\max_t |x(t)|^2}{E_t\left[|X(t)|^2\right]} = \frac{1}{1-0.5\beta}$$

$$\begin{cases} \beta = 0 \\ \beta = 1 \\ \beta = 2 \end{cases}, \quad PAR = \begin{cases} 1 \\ 2 \\ \infty \end{cases}$$ This pulse shape is less sensitive to timing errors.

16.

$$
\begin{aligned}
\lim_{\delta\to 0} &= \lim_{\delta\to 0} \frac{T_N\left(1-e^{-j2\pi(\delta+m)}\right)}{j2\pi(\delta+m)} \\
&\approx \lim_{\delta\to 0} \frac{T_N\left(1-e^{-j2\pi\delta}\right)}{j2\pi m} \\
&\approx \frac{T_N\left(j2\pi\delta\right)}{j2\pi m} \\
&= \frac{T_N\delta}{m} \\
\therefore ICI_i &= \sum_{m\neq i} |I_m|^2 \\
&= \sum_{m\neq i} \left(\frac{T_N\delta}{m}\right)^2 \\
&= \left[\sum_{m\neq i} \left(\frac{1}{m}\right)^2\right] (T_N\delta)^2 \\
\therefore C_0 &= \sum_{m\neq i} \left(\frac{1}{m}\right)^2
\end{aligned}
$$

Chapter 13

1. $x_i = \int_0^T x(t)s(t)dt = \sum_{j=1}^N s_{ij}^2 + I_j s_{ij})$ (see [1] for details)

   (a)
$$E[x_i/s_i(t)] = E_s \quad (\text{ to show })$$

$$s_i(t) = \sum_j s_{ij}\phi_j(t)$$

   By linearity of expectation,
$$E[(\sum_{j=1}^N s_{ij}^2 + I_j s_{ij})/s_i(t)] = E[(\sum_{j=1}^N s_{ij}^2/s_i(t)] + E[I_j s_{ij})/s_i(t)] \quad (1)$$

   s_{ij} are all zero mean and variance E_s/N
   Also, the interfering signal is independent of the transmitted signal. Thus, the equation (1) above, evaluates to
$$\frac{E_s}{N}N + E[I_j]0 = E_s$$
   Notice that the knowledge of s_i is needed to get the correlation to work out. If s_i is not known, correlation will do the right thing only 1/M of the time as we see in the next part.

   (b)
$$\begin{aligned} E[x_i] &= \sum_{j=1}^N E[s_{ij}^2] \\ &= \sum_{j=1}^N E[s_{ij}^2/s_{ij}]p(s_{ij}) \\ &= \frac{E_s}{M} \end{aligned}$$

   (c)
$$\begin{aligned} Var[x_i/s_i(t)] &= E[x_i^2/s_i(t)] - E[x_i/s_i(t)] \\ &= E[x_i^2/s_i(t)] \\ &= E\left[\sum_{k,l} I_k I_l \overline{s_{ik}s_{il}}\right] \\ &= \sum_k i_k^2 E[s_{ik}^2] = \frac{E_s}{N}\sum_k I_k^2 \\ &= \frac{E_s}{N}E_j \end{aligned}$$

(d) As in part b,
$Var[x_i] = \frac{E_s E_j}{NM}$ (As correlator gives non-zero expected output only 1/M of the time)

(e) $SIR = \frac{E^2[x_i]}{Var[x_i]} = \frac{(E_s/M)^2}{\frac{E_s E_j}{NM}} = \frac{E_s}{E_j}\frac{N}{M}$

2. Matlab

```
fc = 100e6;
Ts = 1e-6;
ss = 100;
t = [Ts/ss:Ts/ss:2*Ts];

s = [ones(1,100) -1*ones(1,100)]; sc = [ones(1,10) -1*ones(1,10)];
for i = 1:9
    sc = [sc ones(1,10) -1*ones(1,10)];
end
car = cos(2*pi*fc*t);
x = s.*sc.*car;
```

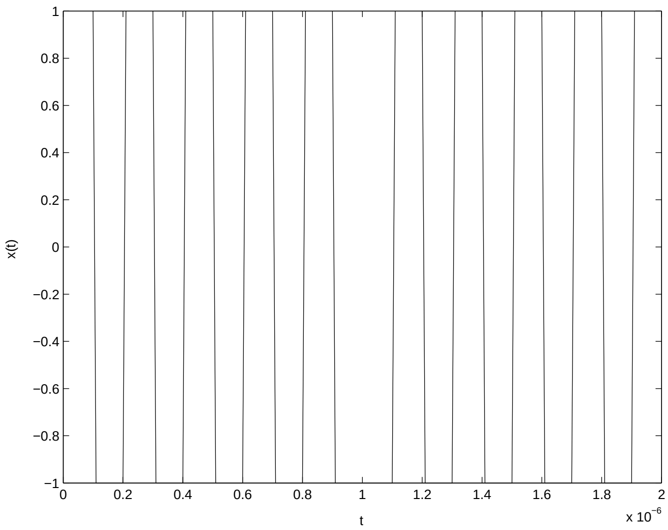

Figure 1: Problem 2

3. $\tau = 10\mu s$

(a) No fading if hop rate greater than $1/e$

$$\begin{aligned} \tau_c &< \tau \\ R_c = \frac{1}{\tau_c} = hoprate &> \frac{1}{e} = 100KHz \end{aligned}$$

(b) $\tau_c = 50\mu s$
$\tau_s = 0.5ms$
Since number of reflected paths get averaged over one symbol duration, we have flat fading.

(c) $\tau_c = 50\mu s$
$\tau_s = 0.5\mu s$
System has severe frequency-selective fading.

4. $n'(t) = n(t)s_c(t)$

   (a)

$$\begin{aligned}
\rho_{n'}(\tau) &= E[n'(t)\bigstar n'(-t)] \quad \text{(Real n'(t))} \\
&= E[(n(t)s_c(t))\bigstar(n(-t)s_c(-t))] \\
&= E\left[\int n(\tau)s_c(\tau)n(t+\tau)s_c(t+\tau)d\tau\right] \\
&= \int \rho_n(\tau)\rho_c(\tau)d\tau \qquad\qquad [\,PSD = \mathfrak{F}(\rho_{n'}(\tau))\,]
\end{aligned}$$

   (b) If $\rho_c(\tau) = \delta(\tau) \Rightarrow \rho_{n'}(\tau) = \rho_n(\tau)$

   (c) If n(t)is AWGN then $\rho_n(\tau) = \frac{N_0}{2}\delta(\tau)$
   $\rho_{n'}(\tau) = \frac{N_0}{2}\rho_c(\tau)$
   As $N \to \infty, \quad \rho_{n'}(\tau) = \frac{N_0}{2}\delta(\tau) \quad$ (same as $\rho_n(\tau)$)

5. $s_c(t)$ is real and periodic. $\rho_c(t)$ is periodic with the same period.

$$\begin{aligned}
\rho_c(t) &= \frac{1}{T}\int_T s_c(\tau)s_c(t+\tau)d\tau \qquad \left(\int_T \rightarrow \text{ integral over any interval of length T}\right) \\
\rho_c(-t) &= \frac{1}{T}\int_T s_c(\tau)s_c(-t+\tau)d\tau \\
\mathit{Let} \quad x = -t+\tau & \\
\rho_c(-t) &= \frac{1}{T}\int_T s_c(x+t)s_c(x)dx \\
&= \rho_c(t)
\end{aligned}$$

   To prove that maximum is at $t = 0$,

$$\rho_c(t) = \int s_c(\tau)s_c(t+\tau)d\tau \leq \int s_c^2(\tau)d\tau = \rho_c(0)$$

   Hence maximum is at 0.

6.

$$\begin{aligned}
\frac{1}{T}\int_0^T s_c(t-\tau_0)s_c(t-\tau_1)dt &= \frac{1}{T}\int_T s_c(t-\tau_0)s_c(t-\tau_1)dt \\
\text{Let} \quad x = t-\tau_0 & \\
&= \frac{1}{T}\int_T s_c(x)s_c(x+\tau_0-\tau_1)dx \\
&= \rho_c(\tau_0-\tau_1) = \rho_c(\tau_1-\tau_0)
\end{aligned}$$

7. $s_c(t)$ is periodic with period T
   $\Rightarrow s_c(t \pm T) = s_c(t)$

$$\begin{aligned}
\rho_c(t) &= \frac{1}{T}\int_0^T s_c(\tau)s_c(t+\tau)d\tau \\
\rho_c(t+T) &= \frac{1}{T}\int_0^T s_c(\tau)s_c(t+T+\tau)d\tau \\
\text{But } s_c(t+T+\tau) = s_c(t+\tau) & \\
\Rightarrow \rho_c(t+T) &= \frac{1}{T}\int_0^T s_c(\tau)s_c(t+\tau)d\tau = \rho_c(t)
\end{aligned}$$

8. We are given in (13.19) that

$$\rho_c(\tau) = \begin{cases} 1 - \frac{|\tau|(1+1/N)}{T_c} & |\tau| \leq T_c \\ -1/N & |\tau| \geq T_c \end{cases}$$

periodic with period

$$NT_c = T_s$$

one period $\rightarrow$

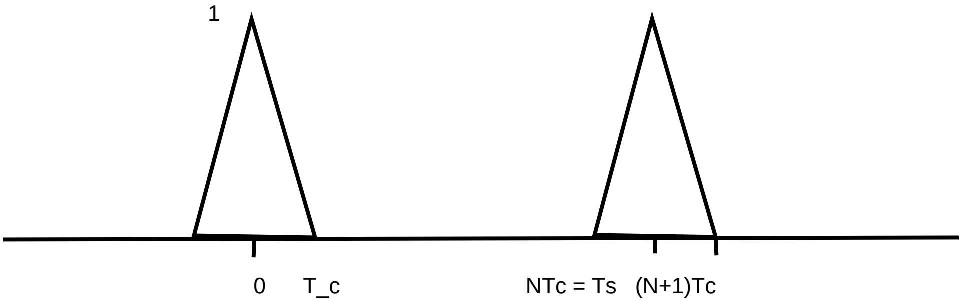

Figure 2: Problem 8

$$\begin{array}{rcl} \left(1+\dfrac{1}{N}\right)\Lambda(t) - \dfrac{1}{N} & \leftrightarrow & \left(\dfrac{N+1}{N}\right) sinc^2(f) - \dfrac{1}{N}\delta(f) \\ \left(1+\dfrac{1}{N}\right)\Lambda(t/T_c) - \dfrac{1}{N} & \leftrightarrow & T_c\left[(1+\dfrac{1}{N})sinc^2(fT_c) - \dfrac{1}{N}\delta(fT_c)\right] \end{array}$$

make periodic with period NT_c

$$\sum_{k=-\infty}^{\infty} \rho(t-kNT_c) \quad \leftrightarrow \quad \sum_{m=-\infty}^{\infty} \frac{1}{NT_c} S(f)|_{f=\frac{m}{NT_c}} \delta(f - \frac{m}{NT_c})$$

$$\therefore \rho_{sc}(f) = \sum_{m=-\infty}^{\infty}\left[\frac{N+1}{N^2} sinc^2(\frac{m}{n})\delta(f-\frac{m}{T_s}) - \underbrace{\frac{1}{N^2}\delta(m)}_{\text{neglect for large N}}\right]$$

MATLAB

```
clear;
Ts = 1e-6;
spread = 1000;
ss = 10;

f = [-1*spread*(1/Ts): ss/Ts: spread*(1/Ts)];
N = 100;

m = -1*spread:ss:spread;
for i = 1:length(m)
    Psc(i) = ((N+1)/N^2)*(sinc(m(i)/N))^2 - (1/N^2)*(m(i)==0);
```

```
end
plot(f*Ts,Psc,'bo')
```

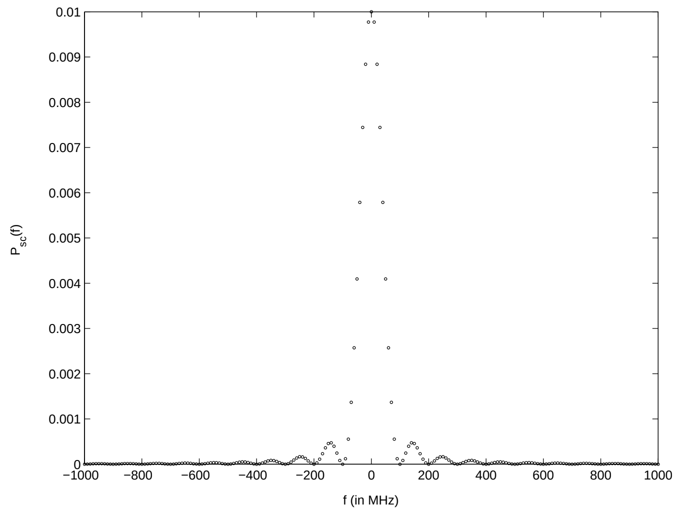

Figure 3: Problem 8

9. Both m-sequences and random binary spreading sequences have balanced run-length and shift properties.
First notice that all these properties are trivially true for a random binary sequence (rbs)
1. balanced: since a rbs is generated by coin tosses using a fair coin $\therefore E[heads] = E[0's] = N/2 = E[1's] = E[tails]$ after N coin tosses.
2. run-length:

$$\text{prob of having (single 1 or single 0 in a row)} = 1/2$$

$$\text{prob of having (2 1's or 2 0's in a row)} = 1/4$$

$$\text{prob of having (k 1's or k 0's in a row)} = 1/2\text{k}$$

3. Shift: Comparing a rbs with a shifted version of itself is same as comparing two independent coin tosses, so prob(2 independent coin tosses result in same outcome) = 1/2 = prob(2 independent coin tosses result in different outcome)
$\therefore$ half of the positions will be same and the remaining half will be different

Now we show the same for m-sequences
1. balanced: Consider a shift register generator of length 'r'. The shift register passes through all possible nonzero states. Of these states $2^{r-1} = \frac{1}{2}(N+1)$ have a one in the right-most position, and $2^{r-1} - 1$ have a zero in the right-most position. Thus there is one more one than zeros in the output sequence.
2. run-length: Consider a shift register of length 'r'. There can be no run of ones having length l$\geq$ r since this would require that the all-one shift register state be followed by another all-ones state. This cannot occur since each shift register state occurs once and only once during N cycles. Thus there is a single run of r consecutive ones and this run is preceded by a zero and followed by a zero.

A run of r-1 ones must be preceded and followed by a zero. This requires that the shift register state which is r-1 ones followed by a 0 be followed immediately by the state which is a 0 followed by r-1 ones. These two states are also passed through in the the generation of the run of r ones, where they are separated by the all-ones state. Since each state occurs only once, there can be no run of r-1 ones. A run of r-1 zeros must be preceded by and followed by 1's. Thus the shift register must pass through the state which is a 1 followed by r-1 zeros. This state occurs only once so there is a single run of r-1 zeros.
Now consider a run of k ones where $1 \leq k \leq r-1$. Each run of k ones must be preceded by and followed by a 0. Thus the shift register must pass through the state which is a 0 followed by k ones followed by a 0, with r-k-2 remaining positions taking arbitrary values. There are 2^{r-k-2} possible ways to complete these remaining positions in the shift register, so there are 2^{r-k-2} runs of k ones. Similarly, there are 2^{r-k-2} runs of k zeros.
3. Shift: We first prove a property known as "shift and add". This says that the modulo-2 sum of an m-sequence and any phase shift of the same sequence is another phase of the same m-sequence.
Proof: Consider a shift register. Since different initial conditions result in a different phase of the same sequence, two phases b(D) and b'(D) of the same sequence can be written as b(D)=a(D)/g(D) and b'(D)=a'(D)/g(D), where a(D) and a'(D) are distinct initial conditions. The modulo-2 sum b(D) + b'(D) =[a(D)+a'(D)]/g(D) = a"(D)/g(D). Since the modulo-2 sum o any two distinct initial conditions is a third distinct initial condition, a"(D)/g(D) =b"(D) is a third distinct phase of the original sequence b(D).
Now we prove the claim by contradiction. Suppose they do not match and mismatch at exactly half the locations. Then their sum will have more 1's and 0's. sum is also an m-sequence which is supposed to have equal 1's and 0's; thus we have a contradiction.

10. $h(t) = \alpha_0\delta(t-\tau_0) + \alpha_1\delta(t-\tau_1)$
neglecting noise, the signal input to the synchronizer is given as
$z(t) = \alpha_0 x(t-\tau_0)s_c(t-\tau_0)\cos^2(2\Pi f_c(t-\tau_0)) + \alpha_1 x(t-\tau_1)s_c(t-\tau_1)\cos(2\Pi f_c(t-\tau_1))\cos(2\Pi f_c(t-\tau_0))$
We have assumed that the demodulator got synchronized to the first multipath
Assuming x(t) remains the same x_k over an interval [0,T] we have

$$\begin{aligned} w(\tau) &= \frac{1}{T}\int_0^T \alpha_0 x_k s_c(t-\tau_0)s_c(t-\tau)\cos^2(2\Pi f_c(t-\tau_0))dt \\ &\quad +\frac{1}{T}\int_0^T \alpha_1 x_k s_c(t-\tau_1)s_c(t-\tau)\cos(2\Pi f_c(t-\tau_1))\cos(2\Pi f_c(t-\tau_0))dt \\ &\sim \frac{0.5x_k}{T}\alpha_0\rho_c(\tau-\tau_0) + \frac{0.5x_k}{T}\alpha_1\rho_c(\tau-\tau_1)\cos(2\Pi f_c(\tau_0-\tau_1)) \end{aligned}$$

For binary transmission $x_k = \pm 1$
So Costa loop will try to maximize $|w(\tau)|$ and will synchronize to the first component if α_0 is higher or the second component of $\alpha_1\cos(2\Pi f_c(\tau_0-\tau_1))$ is higher.

A similar analysis can be done if demodulator synchronizes to the other multipath component. However, notice that the demodulator will also synchronize to the multipath component for which α_1 is greater. Essentially, both the demodulator and the Costa loop will synchronize to the first multipath component if α_0 is higher and to the second multipath component otherwise.

11. $P_b = 10^{-6}$
For DPSK, we know that the BER is given by $\frac{1}{2}e^{-\gamma_b}$, which for a maximum BER of 10^{-6}, gives a value of minimum receive SNR of $\gamma_{\min} = 13.12$.

We assume that the noise statistics are the same after de-spreading and that the possible interference is modeled as white noise.

We have that

$$\text{SNR}_{\text{(after-despreading)}} = K\rho_c^2(\tau)\text{SNR}_{\text{(before-despreading)}}$$

For the first branch, $\rho_c(0) = 1$, for the second branch $\rho_c(T_c/4) = 0.733$ and for the third branch $\rho_c(T_c/3) = 0.644$. Therefore, the SNR's after de-spreading are: $\text{SNR}_1 = 150$, $\text{SNR}_2 = 80.667$, $\text{SNR}_3 = 62.296$.

With selection combining, the outage probability is given by

$$P_{\text{out}} = (1 - e^{-\frac{\gamma_{\min}}{\overline{\gamma}_1}})(1 - e^{-\frac{\gamma_{\min}}{\overline{\gamma}_2}})(1 - e^{-\frac{\gamma_{\min}}{\overline{\gamma}_3}}) = 0.24\%$$

12. (a) $E(r_{\sum}) = E(r_1) + E(r_2) = 9 + 9\rho^2 = 11.9438$

    (b) $p(r_{\sum}) = p(r_1)\bigstar p(r_2)$
    a=1/6. See Fig 4

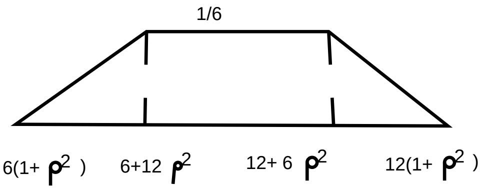

Figure 4: Problem 12b

    (c) $P_b = \frac{1}{2}\exp^{-r_b} \Rightarrow r_b = 8.5172$
    height of $\Delta = \frac{1}{6}\left(\frac{6\rho^2}{8.5172-7.9625}\right)^{-1} = 0.0471$
    $\therefore P_{out} = \frac{1}{2}(0.0471)(8.5172 - 7.9625) = 0.0131$

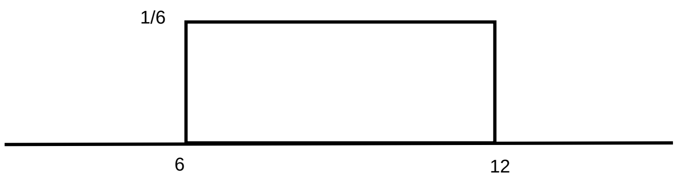

Figure 5: Problem 12

13. (a) For the multipaths to be attenuated by $-1/N$, we need $\tau_1 \geq T_c$ and $\tau_1 \leq T_b - T_c$. Similarly, we need $\tau_2 \geq T_c$ and $\tau_2 \leq T_b - T_c$.

    (b) Instantaneous BER in DPSK is given by $P_b = \frac{1}{2}e^{-\gamma_b}$. For $P_b = 10^{-3}$, this corresponds to $\gamma_b = 6.21$. While the channel has average power $\overline{\gamma}$, the outage probability in Rayleigh fading is given by $P_{out} = 1 - e^{-\gamma_0/\overline{\gamma}}$. Note that α_0 is Rayleigh distributed with average power 5 with probability .5 and the channel is Rayleigh distributed with average power 10 with probabilty .5. This is not equivalent to Rayleigh fading with average power 7.5. While the channel has average power 5 the outage probability is $1 - e^{-6.21/5}$, and while the channel has average power 10 the outage probability is $1 - e^{-6.21/10}$. The overall outage probability is given by

$$P_{out} = .5(1 - e^{-6.21/5} + 1 - e^{-6.21/10}) = .587$$

    (c) The outage probability when using selection combining is the product of each of outage probability of each branch. For branch 2, the outage probability is $.5(1 + 1 - e^{-6.21/20}) = .633$. For branch

3, the outage probability is $.75(1-e^{-6.21/5})+0.25(1-e^{-6.21/10})=0.649$. Therefore, the overall outage probability is given by:

$$P_{out}=P_{out,1}P_{out,2}P_{out,3}=(0.587)(0.633)(0.649)=0.241$$

(d) Again, the outage probability of a 2-branch RAKE with selection combining is given by the product of the outage probability of each branch. Therefore, we should select the two branches with the smallest outage probabilities, i.e. branches 0 and 1. The corresponding outage probability is

$$P_{out}=P_{out,1}P_{out,2}=(0.587)(0.633)=0.37$$

14. Following along the same lines as the previous question, we know that outage on the ith branch is given as:

$$P_{out}=\begin{cases}1 & w.p.\ \ 0.25\\ 1-e^{-\frac{\gamma_{\min}}{\overline{\gamma}_i}} & w.p.\ \ 0.75\end{cases}$$

For DPSK, we know that the BER is given by $\frac{1}{2}e^{-\gamma_b}$, which for a maximum BER of 10−3, gives a value of minimum receive SNR of $\gamma_{\min}=6.21$. Therefore, the outage probability will be

$$P_{out}=\left(0.25+0.75\left(1-e^{-\frac{6.21}{20}}\right)\right)\left(0.25+0.75\left(1-e^{-\frac{6.21}{10}}\right)\right)\left(0.25+0.75\left(1-e^{-\frac{6.21}{6.67}}\right)\right)$$
$$\left(0.25+0.75\left(1-e^{-\frac{6.21}{5}}\right)\right)\left(0.25+0.75\left(1-e^{-\frac{6.21}{4}}\right)\right)=12.49\%$$

For the case where there is always a multipath in each bin with average SNR of 20,

$$P_{out}=\left(1-e^{-\frac{6.21}{20}}\right)^5=0.14\%$$

Clearly the outage probability is much smaller in the second case.

15. (a)

$$\begin{aligned}\beta_i &= \frac{1}{T_b}\int_0^{T_b}s_c(t-iT_c)\sum_{j=0}^{N}alpha_j\delta(t-jT_c)s_c(t)dt\\ &= \sum_{j=0}^{N}\alpha_j\frac{1}{T_b}\int_0^{T_b}s_c(t-iT_c)s_c(t-jT_c)dt\\ &= \alpha_j\delta_{ij}=\begin{cases}\alpha_i & \text{for} i^{th}\text{carrier}\\ 0 & \text{for} i\neq j\end{cases}\end{aligned}$$

thus $\beta_i=\alpha_i$ for all i

(b) $\beta_0=a,\quad \beta_1=0.8b,\quad \beta_2=0.2b,\quad \beta_3=0.5c,\quad \beta_4=0.5c$

(c) $P_b=0.5e^{-\gamma_b}\Rightarrow\gamma_0=6.21$
<u>SC:</u>

$$P_out=\left[1-e^{-\gamma_0/\overline{\gamma}}\right]^M=0.099$$

<u>MRC:</u>

$$P_out=1-e^{-\gamma_0/\overline{\gamma}}\left(1+\gamma_0/\overline{\gamma}+\frac{(\gamma_0/\overline{\gamma})^2}{2}\right)=0.025$$

16. Autocorrelation and Cross-correlation for Gold codes, Kasawi codes from the small set
n=8
Kasawi codes ; small set

$$\rho(\tau) = \begin{cases} \frac{-1}{2^n-1} = & -0.0039 \\ \frac{-2^{n/2}+1}{2^n-1} = & -0.0667 \\ \frac{2^{n/2}+1-2}{2^n-1} = & -0.0588 \end{cases}$$

number of sequences $= 2^{n/2} = 16$
large set

$$\rho(\tau) = \begin{cases} \frac{-1}{2^n-1} = & -0.0039 \\ \frac{-1\pm-2^{n/2}}{2^n-1} = & -0.0667, 0.0588 \\ \frac{-1\pm(2^{n/2}+1)}{2^n-1} = & 0.0627, -0.0706 \end{cases}$$

number of sequences $= 2^{3n/2} = 4096$
Gold codes; $t(n) = 2^{(n+2)/2} + 1 = 33$

$$\rho(\tau) = \begin{cases} \frac{-1}{2^n-1} = & -0.0039 \\ \frac{-1t(n)}{2^n-1} = & -0.1294 \\ \frac{t(n)-2}{2^n-1} = & 0.1216 \end{cases}$$

17.

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

To see orthogonality of rows,

$$H_4 H_4^T = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Due to the symmetric nature of H_4, shifts by $T_C/2$ do not change orthogonality properties and cross-correlation is 0 between any two users.

18. $\frac{\alpha_k^2}{\alpha^2} = 0.2512(-6dB)$
$SIR = \frac{\alpha_k^2 3N}{\alpha^2 \xi(K-1)} = 1.9322$
$K = 40$
$N = 100$
$P_b = Q(\sqrt{2r_b}) = 0.0247$
For synchronous users, the situation is worse (13.39)

19. $r(t) = \sum_i \alpha_i b_i s_{c_i}(t) + n(t)$
$r_k = \int_0^{T_b} r(t) s_{c_k}(t)dt = \alpha_k b_k \rho_{kk} + \int_0^{T_b} n(t) s_{c_k}(t)dt + \sum_{j\neq k} \alpha_j b_j \rho_{kj}$

For 2 users, it is easy to see

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

We assume that noise statistics remain unchanged after despreading.

20. ML decision minimizes $r = RAb + n$
$(r - RAb)^2$
$= rr^T - 2b^T A^T R^T r + b^T A^T R^T RAb$
first term is same for all b , A is symmetric, R is symmetric
$\therefore maximize - 2b^T ARr + b^T AR^2 Ab$ or minimize $2b^T Ar - b^T ARAb$

21. (a) If he transmits only one spreading sequence, there will be no other interference than noise. So

$$BER = Q\left(\sqrt{\frac{2P(\gamma)\gamma}{1}}\right) = Q\left(\sqrt{2P(\gamma)\gamma}\right)$$

R bits/sec is achieved in this case.
If he transmits both with equal energy

$$BER = Q\left(\sqrt{\frac{P(\gamma)\gamma}{\frac{1}{2N}P(\gamma)\gamma + 1}}\right)$$

2R bits/sec is achieved in this case

(b) Assume that SNR_0 is such that
$P_b^\star = Q(\sqrt{SNR_0})$ since we want to keep $P_b \leq P_b\star$
$S(\gamma)\gamma = SNR_0$ for $\gamma_0 \leq \gamma \leq \gamma_1$
$\frac{P(\gamma)\gamma}{1+\frac{1}{2N}P(\gamma)\gamma} = SNR_0$ for $\gamma_1 \leq \gamma$
$P(\gamma) = 0$ for $\gamma \leq \gamma_0$
therefore

$$P(\gamma) = \begin{cases} 0 & \gamma \leq \gamma_0 \\ \frac{SNR_0}{\gamma} & \gamma_0 \leq \gamma \leq \gamma_1 \\ \frac{1}{\gamma}\frac{SNR_0}{1-\frac{SNR_0}{2N}} & \gamma_1 \leq \gamma \end{cases}$$

According to power constraint

$$\int_0^\infty P(\gamma)p(\gamma)d\gamma = 1$$

Define $K_1 = SNR_0 \quad K_2 = \frac{SNR_0}{1-\frac{SNR_0}{2N}}$
power constraint will be

$$K_1 \int_{\gamma_0}^{\gamma_1} \frac{1}{\gamma}p(\gamma)d\gamma + K_2 \int_{\gamma_1}^{\infty} \frac{1}{\gamma}p(\gamma)d\gamma = 1$$

and

$$P(\gamma) = \begin{cases} 0 & \gamma \leq \gamma_0 \\ \frac{K_1}{\gamma} & \gamma_0 \leq \gamma \leq \gamma_1 \\ \frac{K_2}{\gamma} & \gamma_1 \leq \gamma \leq \infty \end{cases}$$

(c)

$$J = Rp\{\gamma_0 \leq \gamma \leq \gamma_1\} + 2Rp\{\gamma_1 \leq \gamma\} + \lambda\left\{K_1 \int_{\gamma_0}^{\gamma_1} \frac{p(\gamma}{\gamma}d\gamma + K_2 \int_{\gamma_1}^{\infty} \frac{p(\gamma}{\gamma}d\gamma - 1\right\}$$

$$\frac{\partial J}{\partial \gamma_0} = 0 \quad \frac{\partial J}{\partial \gamma_1} = 0$$

$$\Rightarrow \gamma_0 = -\frac{\lambda K_1}{R}$$

$$\Rightarrow \gamma_1 = -\frac{\lambda(K_2 - K_1)}{R}$$

Chapter 15

1. City has 10 macro-cells
each cell has 100 users
$\therefore$ total number of users = 1000
Cells are of size 1 sqkm
maximum distance traveled to traverse = $\sqrt{2}$km
$\therefore$ time $=\frac{\sqrt{2}}{30} = 169.7s$
In the new setup
number of cells = 10^5 microcells
total number of users = 1000×100^2users
time = $\frac{\sqrt{2}\times 10}{30\times 10^3}$ =1.69s
$\therefore$ number of users increases by 10000 and handoff time reduces by 1/100

2. See Fig 1

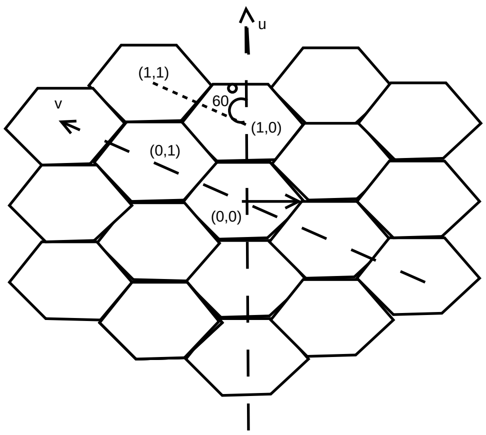

Figure 1: Problem 2

$$D^2 = (j20)^2 + (i20)^2 - 2(j20)(i20)\cos(2\pi/3)$$
$$\Rightarrow D = 2a\sqrt{i^2+j^2+ij} = \sqrt{3}R\sqrt{i^2+j^2+ij}$$

3. diamond shaped cells, R= 100m
$D_{min} = 600m$
$D = 2KR$
$K = \frac{D}{2R} = \frac{600}{2\times 100} = 3$
$N = K^2 = 9$

   (a) number of cells per cluster = N = 9
   (b) number of channels per cell = total number/N = 450/9 = 50

4. (a) R=1km
D=6km
N = $\frac{A_{cluster}}{A_{cell}} = \frac{\sqrt{3}D^2/2}{3\sqrt{3}R^2/2} = \frac{1}{3}(D/R)^2 = \frac{1}{3}6^2 = 12$
number of cells per cluster = N = 12

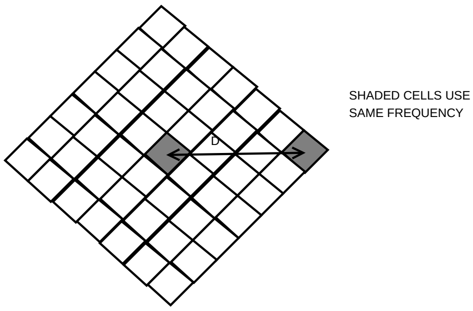

Figure 2: Problem 3

(b) number of channels in each cell = 1200/12 = 100

(c) $\sqrt{i^2+j^2+ij} = 2\sqrt{3} \quad \Rightarrow \quad i=2, \quad j=2$

5. R=10m
D=60m
$\gamma_I = 2$
$\gamma_0 = 4$
$M = 4$ for diamond shaped cells

$$SIR_a = \frac{R^{-\gamma_I}}{MD^{-\gamma_0}} = \frac{R^{-2}}{4D^{-4}} = 32400$$

$$SIR_b = \frac{R^{-4}}{4D^{-4}} = 324$$

$$SIR_c = \frac{R^{-2}}{4D^{-2}} = 9$$

$$SIR_a > SIR_b > SIR_c$$

6. $\gamma = 2$
$BPSK$
$P_b = 10^{-6} \rightarrow P_b = Q(\sqrt{2\gamma_b}) \Rightarrow \gamma_b = SIR_0 = 4.7534$
$B = 50MHz$
each user$100KHz = B_s$
$SIR = \frac{1}{M}\left(\frac{D}{R}\right)^\gamma$ M=6 for hexagonal cells
$a_1 = 0.167$
$a_2 = 3$
$N > \frac{1}{a_2}\left(\frac{SIR_0}{a_1}\right)^{2/\gamma} \Rightarrow N \geq 9.4879 \therefore N = 10$
$C_u = 50$

7. $G = 100$
$\xi = 1$
$\lambda = 1.5$
With no sectorization

$$SIR = \frac{1}{\frac{\xi}{3G}(N_c-1)(1+\lambda)} = 4.7534$$

$$N_c = \lfloor 26.2450 \rfloor = 26$$

With sectorization, interference is reduced by a factor of 3

$$N_c = \lfloor 76.7349 \rfloor = 76$$

8. SINR $= \frac{G}{\sum_{i=1}^{N_c-1} X_i + N}$
$\alpha = p(X_i = 1)$
$N \sim G(0.247N_c, 0.078N_c)$
$P_out = p(SIR < SIR_0)$

   (a) $P_{out} = p\left(\frac{G}{\sum_{i=1}^{N_c-1} X_i+N} < SIR_0\right) = p\left(\sum_{i=1}^{N_c-1} X_i + N > \frac{G}{SIR_0}\right)$

   (b) $X = \sum_{i=1}^{N_c-1} X_i$ then X $\sim Bin(\alpha, N_c - 1)$
   $p(x + N > G/SIR_0) = \sum_{N=0}^{N_c-1} p(n + N > G/SIR_0|x = n)p(x = n)$

$$p(x = n) = \binom{N_c - 1}{n} \alpha^n (1 - \alpha)^{N_c-1-n}$$

$$\begin{aligned} p(x + N > G/SIR_0) &= \sum_{N=0}^{N_c-1} p(N > G/SIR_0 - n|x = n)p(x = n) \\ &= \sum_{N=0}^{N_c-1} p\left(\frac{N - 0.247N_c}{\sqrt{0.078N_c}} > \frac{\frac{G}{SIR_0} - n - 0.247N_c}{\sqrt{0.078N_c}}|x = n\right) \\ &= \sum_{N=0}^{N_c-1} Q\left(\frac{\frac{G}{SIR_0} - n - 0.247N_c}{\sqrt{0.078N_c}}\right) p(x = n) \end{aligned}$$

   (c) $N_c = 35$
   $\alpha = 0.5$
   $SIR_0 = 5$
   $G = 150$
   $p = 0.0973$

   MATLAB

```
for i = 1:length(n)
    pn(i) = (factorial(Nc-1)./(factorial(n(i)).*factorial(Nc-1-n(i))))...
            *alpha.^n(i)*(1-alpha).^(Nc-1-n(i));
end

sump = 0; for i = 1:length(n)
    f = ((G/sir0)-n(i)-.247*Nc)/(sqrt(.078*Nc));
    sump = sump + .5*erfc((f)/sqrt(2))*pn(i);
end
```

   (d) If x can be approximated as Gaussian then
   x $\sim G((N_c - 1)\alpha, (N_c - 1)\alpha(1 - \alpha))$
   $x + N \sim G(0.247N_c + (N_c - 1)\alpha, 0.078N_c + (N_c - 1)\alpha(1 - \alpha))$

$$p(x + N > G/SIR_0) = Q\left(\frac{\frac{G}{SIR_0} - (0.247N_c + (N_c - 1)\alpha)}{\sqrt{0.078N_c + (N_c - 1)\alpha(1 - \alpha)}}\right)$$

   (e) p= 0.0969 (very accurate approximation!)

9. define

$$\gamma_k = \frac{g_k P_k}{n_k + \rho \sum_{k \neq j} g_{kj} p_j} \qquad k, j \in \{1, \dots K\}$$

where,
g_k is channel power gain from user k to his base station n_k is thermal noise power at user k's base station
ρ is interference reduction factor ($\rho \sim 1/G$)
g_{kj} is channel power gain from j^{th} interfering transmitter to user k's base station
p_k is user k's Tx power
p_j is user j's Tx power
define a matrix F such that

$$F_{kj} = \begin{cases} 0 & k = j \\ \frac{\gamma_k^\star g_{kj} \rho}{g_k} & k \neq j \end{cases}$$

$k, j \in \{1, \dots K\}$

$$u = \left(\frac{\gamma_1^\star n_1}{g_1}, \frac{\gamma_2^\star n_2}{g_2}, \dots, \frac{\gamma_K^\star n_K}{g_K} \right)$$

If Perron Ferbinius eigenvalue of F is less than 1 , then a power control policy exists. The optimal power control policy is given to be $P^\star = (I - F)^{-1} u$

10. Matlab

```
D = 2:.01:10;
R = 1;
gamma = 2;
Pdes = R^(-gamma);

for i = 1:length(D)
    Pint = 6*(.2*(D(i)-R)^(-gamma)+.2*(D(i)-R/2)^(-gamma)+.2*(D(i))^(-gamma)...
           +.2*(D(i)+R/2)^(-gamma)+.2*(D(i)+R)^(-gamma));
    Pintbest = 6*((D(i)+R)^(-gamma));
    Pintworst = 6*((D(i)-R)^(-gamma));
    ASE(i) = log(1+Pdes/Pint)/(pi*(.5*D(i))^2);
    ASEbest(i) = log(1+Pdes/Pintbest)/(pi*(.5*D(i))^2);
    ASEworst(i) = log(1+Pdes/Pintworst)/(pi*(.5*D(i))^2);
end
```

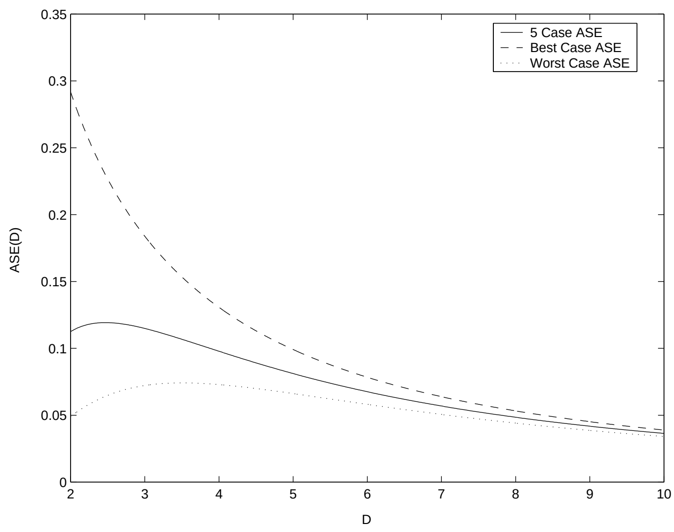

Figure 3: Problem 10

11. $P_t = 5W$
$B = 100KHz$
$N_0 = 10^{-16} W/Hz$
$P_r = P_t K \left(\frac{d_0}{d}\right)^3 \qquad d_0 = 1, K = 100$

 (a) D=2R
 2 users share the band available
 Each user gets 50KHz

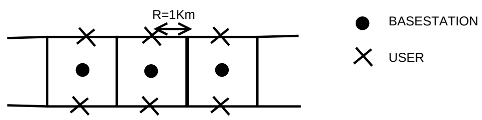

Figure 4: Problem 11a

 (b) $\overline{P}_b = \frac{1}{4\overline{\gamma}_b} \Rightarrow 10^{-3} = \frac{1}{4\overline{\gamma}_b} \Rightarrow \gamma_b = 250$
 If D(n) = 2nR, number of users that share band = 2(2(n-1)+1)
 $\therefore$ each user gets $\frac{100KHz}{2(2n-1)} = B_u(n)$
 interference is only from first tier

$$SIR(n) = \frac{P_t K \left(\frac{d_0}{R}\right)^3}{\frac{N_0}{2} B_u(n) + 2\left(P_t K \left(\frac{d_0}{\sqrt{R^2 + D(n)^2}}\right)^3\right)} > 25$$

 using Matlab , $\qquad n = 4, \quad SIR = 261.9253, \quad D = 8R$

 Matlab

```
Pt = 5;
R = 1000;
sigma_2 = 1e-16;
n = 1;
```

```
D = 2*n*R;

Bu = (100/(2*(2*n-1)))*1e3;
K = 100;
d0 = 1;
Pdes = Pt*K*(d0/R)^3;
Pint = 2*(Pt*K*(d0/sqrt(R^2+D^2))^3);
Npower = sigma_2*Bu;

sir = Pdes/(Npower+Pint);
while sir < 250
    n = n+1;
    D = 2*n*R;
    Bu = (100/(2*(2*n-1)))*1e3;
    K = 100;
    d0 = 1;
    Pdes = Pt*K*(d0/R)^3;
    Pint = 2*(Pt*K*(d0/sqrt(R^2+D^2))^3);
    Npower = sigma_2*Bu;
    sir = Pdes/(Npower+Pint);
end
```

(c) $ASE = \frac{(R_1+R_2)/B}{2km \times 2km}$
$R_1 = R_2 = B_u(1)\log(1 + SIR(1))$
$B_u(1) = 50KHz, \quad SIR(1) = 5.5899$
$ASE = 0.6801 bps/Hz/km^2$

12. B=100KHz
$N_0 = 10^{-9} W/Hz$
$K = 10$
$P = 10mW$ per user

(a) $0 \leq \alpha \leq 1$ α is channel gain between cells.
See Matlab
If α is large, interference can be decoded and subtracted easily so capacity grows with α as high SNR's (beyond an α value) .
For low SNR values (α less than a value) c decreases with increase in α as interference is increased which cannot be easily decoded due to low SNR.

MATLAB CODE:

```
B = 100e3;
sigma_2 = 1e-9;
P = 10e-3;
K = 10;
ss = .001;

alpha = 0:.01:1;
theta = 0:ss:1;
for i = 1:length(alpha)
    capvec = log2(1+(K*P*(1+2*alpha(i)*cos(2*pi*theta)).^2)/(sigma_2*B));
    C(i) = (1/K)*sum(capvec)*ss;
```

```
end
```

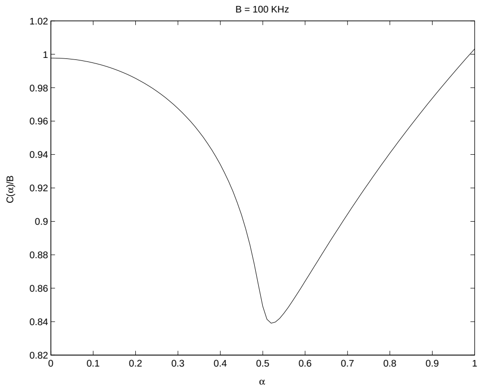

Figure 5: Problem 12a

(b) C(K)↓ as K↑ because as the number of mobile per cell increases system resources get shared more and so per user capacity C(K) has to fall.

```
MATLAB CODE:

BB = 100e3;
sigma_2 = 1e-9;
P = 10e-3;
K = 1:.1:30;
ss = .001;
alpha = .5;
theta = 0:ss:1;
for i = 1:length(K)
    capvec = log2(1+(K(i)*P*(1+2*alpha*cos(2*pi*theta)).^2)/(sigma_2*B));
    C(i) = (1/K(i))*sum(capvec)*ss;
end
```

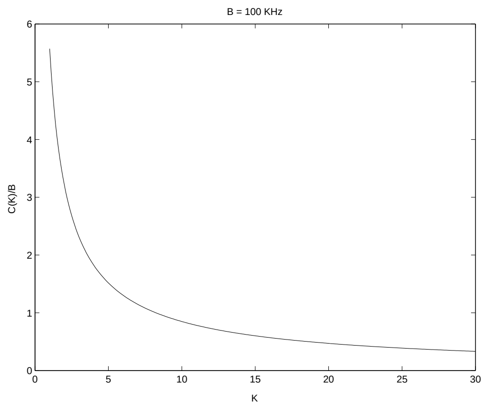

Figure 6: Problem 12b

(c) as transmit power P ↑, capacity C ↑ but gets saturated after a while as the system becomes interference limited.

```
MATLAB CODE:
```

```
B = 100e3;
sigma_2 = 1e-9;
P = [0:.1:100]*1e-3;
K = 10;
ss = .001;
alpha = .5;
theta = 0:ss:1;
for i = 1:length(P)
    capvec = log2(1+(K*P(i)*(1+2*alpha*cos(2*pi*theta)).^2)/(sigma_2*B));
    C(i) = (1/K)*sum(capvec)*ss;
end
```

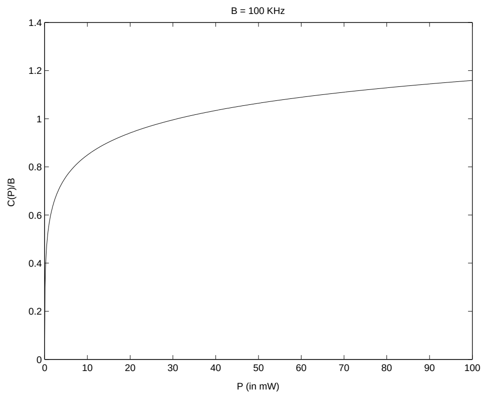

Figure 7: Problem 12c

Chapter 16

1. d=1Km , $P_\gamma = P_t d^{-\gamma}$

   (a) $P_t = P_\gamma d^\gamma$
   $P_t(\gamma = 2) = 10KW$
   $P_t(\gamma = 2) = 1 \times 10^{10} W$

   (b) d= 500m
   $P_t(\gamma = 2) = 2.5KW$
   Total power is simply twice.
   $P_{total} = 5KW$
   $P_t(\gamma = 4) = 6.25 \times 10^8 W$
   Total power $P_{total} = 1.25 \times 10^9 KW$

   (c)

$$P_t(\gamma = 2) = 10 \times 10^{-3} \left(\frac{1}{N+1} \times 10^3 \right)^2 = \frac{1 \times 10^4}{(N+1)^2}$$

$$P_t(\gamma = 4) = 10 \times 10^{-3} \left(\frac{1}{N+1} \times 10^3 \right)^4 = \frac{1 \times 10^{10}}{(N+1)^4}$$

$$P_t(\gamma = 2) = (N+1) \frac{1 \times 10^4}{(N+1)^2} = \frac{1 \times 10^4}{(N+1)}$$

$$P_t(\gamma = 4) = (N+1) \frac{1 \times 10^{10}}{(N+1)^4} = \frac{1 \times 10^{10}}{(N+1)^3}$$

2. $\gamma_1 = \gamma_2 = 7dB$
   $\gamma_3 = 10dB$
   $\rho = 1$
   $n_i = 1 \forall i$
   $G = \begin{bmatrix} 1 & 0.06 & 0.04 \\ 0.09 & 0.9 & 0.126 \\ 0.064 & 0.024 & 0.8 \end{bmatrix}$

   (a) We expand

$$(I - F)P \quad \geq \quad u$$

   where

$$u = \left(\frac{\gamma_1^\star n_1}{g_{11}}, \frac{\gamma_2^\star n_2}{g_{22}}, \ldots, \frac{\gamma_N^\star n_N}{g_{NN}} \right)^T$$

$$F_{kj} = \begin{cases} 0 & k = j \\ \frac{\gamma_k^\star g_{kj} \rho}{g_{kk}} & k \neq j \end{cases}$$

$$P = (P_1 P_2 \ldots P_N)^T$$

first row of (I-F)P

$$\begin{aligned}
= P_1 - \frac{\gamma_1^\star g_{12} P_2}{g_{11}} - \frac{\gamma_1^\star g_{13} P_3}{g_{11}} &\geq \frac{\gamma_1^\star n_1}{g_{11}} \\
g_{11} P_1 &\geq \gamma_1^\star n_1 + \gamma_1^\star g_{12} P_2 + \gamma_1^\star g_{13} P_3 \\
\frac{g_{11} P_1}{n_1 + g_{12} P_2 + g_{13} P_3} &\geq \gamma_1^\star
\end{aligned}$$

which is precisely the SNR constraint for user 1.
We can similarly show this for users 2 and 3.

(b) $F = \begin{bmatrix} 0 & 0.3007 & 0.2005 \\ 0.5012 & 0 & 0.7017 \\ 0.8 & 0.3 & 0 \end{bmatrix}$ From Matlab:

$$\lambda = abs(eig(F)) = (0.8667\ 0.4791\ 0.4791)$$

As $max(|\lambda|) = 0.8667 < 1$ a feasible power vector for the system exists.

(c) $u = [5.0119\ 5.5687\ 12.5]$
$P^\star = (I - F)^{-1} u = [39.7536\ 71.6634\ 65.8019]$

3. P(0) = [50 50 50]

```
MATLAB CODE

P = [50 50 50];
k = 1;
while k<50
    for i = 1:3
        sum_int = 0;
        for j = 1:3
            if i~=j
                sum_int = sum_int + rho*(G(i,j)*P(j));
            end
        end
        gamma(i) = (G(i,i)*P(i))/(n(i)+sum_int);
    end
    P_plot(:,k) = P(:);
    Gamma_plot(:,k) = gamma(:);
    k = k+1;
    for i = 1:3
        P(i) = (gamma_des(i)/gamma(i))*P(i);
    end
end
```

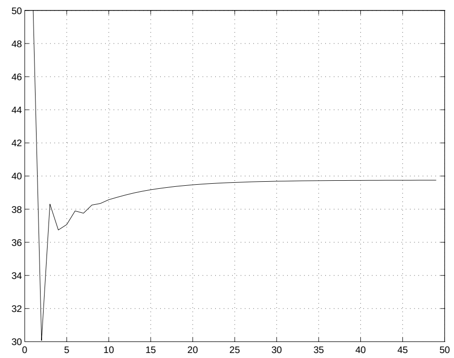

Figure 1: Problem 3:Power, User 1

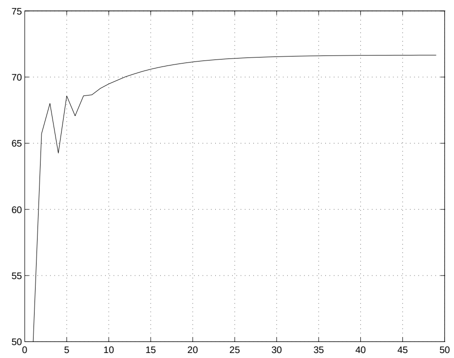

Figure 2: Problem 3:Power, User 2

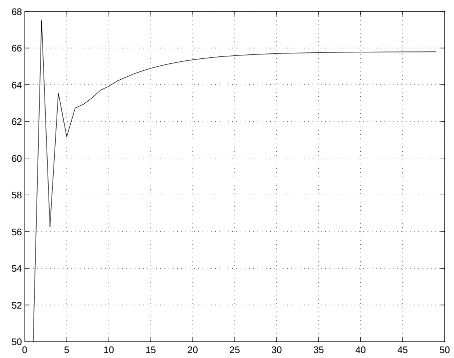

Figure 3: Problem 3:Power, User 3

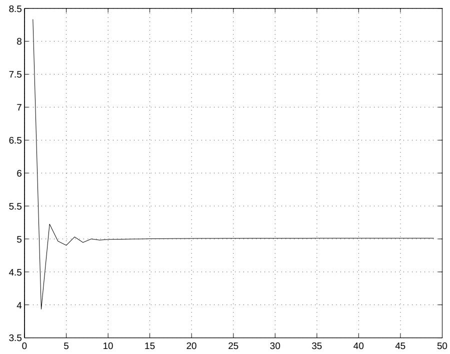

Figure 4: Problem 3:SNR, User 1

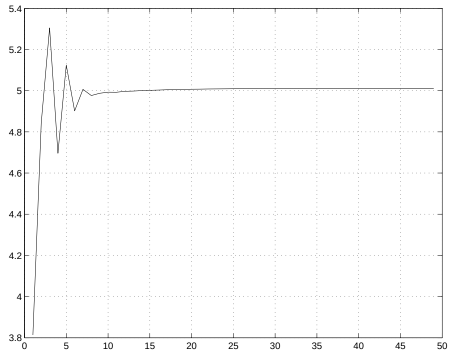

Figure 5: Problem 3:SNR, User 2

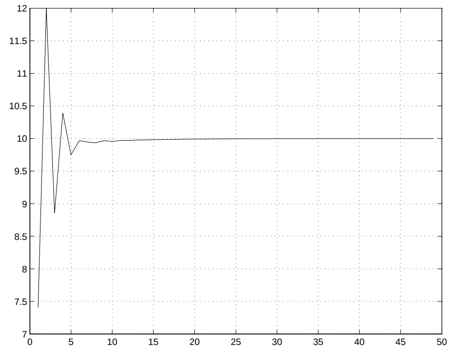

Figure 6: Problem 3:SNR, User 3

4. (a) $\gamma = 2$, for N= 2 , we want $P_\gamma(10m) = 10mW$ which will ensure that $P_\gamma(d > 10m) < 10mW$
    $P_{max}(10)^{-2} = 10 \times 10^{-3}$
    $P_{max} = 1W$
   (b) $\gamma = 2$, N= 4
    $P_\gamma(20m) = 10mW$
    $P_{max}(20)^{-2} = 10 \times 10^{-3}$
    $P_{max} = 4W$
   (c) $\gamma = 4$, N= 4
    $P_\gamma(20m) = 10mW$
    $P_{max}(20)^{-4} = 10 \times 10^{-3}$
    $P_{max} = 1600W$

5. MATLAB CODE:

```
clear;
Ra = 20:-1:1;
for j = 1:length(Ra)
    for k = 1:100
        no_of_Cluster = 1;
        node(1,1,no_of_Cluster) = unifrnd(0,100);
        node(1,2,no_of_Cluster) = unifrnd(0,100);
        R = Ra(j);
        no_of_nodes = size(node,1);
        while (no_of_nodes == 1) | (no_of_Cluster > 1)
            no_of_Cluster = no_of_Cluster + 1;
            no_of_nodes = no_of_nodes + 1;
            node(1,1,no_of_Cluster) = unifrnd(0,100);
            node(1,2,no_of_Cluster) = unifrnd(0,100);
            new_node = reduce_clusters(node,R);
            node = new_node;
            no_of_Cluster = size(node,3);
        end
        no_of_nodes_this_run(k) = no_of_nodes;
        clear node;
    end
    des_ans(j) = mean(no_of_nodes_this_run);
end
```

6. $p(d) = \frac{e^{-d/D}}{D}$ $d > 0$
   Prob that one of the copies arrives after D = prob(d¿D)

$$= \int_D^\infty \frac{1}{D} e^{-x/D} dx = e^{-1}$$

   Prob that all N copies arrive after D assuming independence = e^{-N}
   N=1, Prob = 0.3679
   N=5, Prob = 0.0067
   Since the paths were identical(had similar delay profile), throughput goes down by a facto of 5 as we are sending same information on all paths.
   As we try to decrease delay, the throughput also goes down.
   Delay is proportional to throughput. This is the trade-off

7. $D_{ij} = \frac{f_{ij}}{c_{ij}}$
The function is linear in f_{ij}, so it is both convex and concave. Specifically, it is convex.

$$\frac{\partial D_{ij}}{\partial c_{ij}} = -\frac{f_{ij}}{c_{ij}^2}$$

$$\frac{\partial^2 D_{ij}}{\partial c_{ij}^2} = 2\frac{f_{ij}}{c_{ij}^3} > 0 \text{ for } f_{ij}, c_{ij} > 0$$

Hence it is convex in c_{ij} too.

8. C=10Mbps

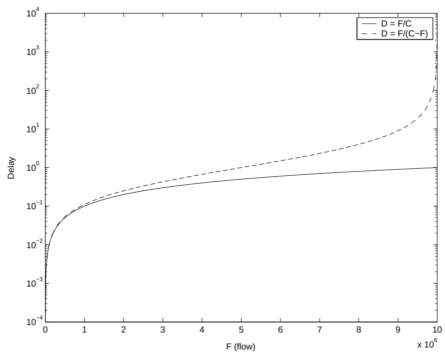

Figure 7: Problem 8

9. $\lambda = \frac{f_{ij}/c_{ij} - f_{ij}}{f_{ij}/c_{ij}} = \frac{c_{ij}}{c_{ij}-f_{ij}}$

   (a) $x = \frac{f_{ij}}{c_{ij}} \quad 0 \leq x < 1$
   $\lambda = \frac{1}{1-x}$
   (b) $1 \leq \lambda < \infty$
   (c) $\lambda > 10 \Rightarrow \frac{1}{1-x} > 10 \Rightarrow x > 9/10 \Rightarrow f_{ij} > 0.9c_{ij}$
   (d) Network for which delay is calculated based on (16.6) will be more congested as the metric will tell us that the delay less (say below a threshold fixed apriori) and so we will keep on placing more traffic.

10. (a) Since there is full cooperation, we can adapt R based on the values of C.
    Using Matlab

$$R_{opt} = [29.322.336.3]Kbps$$

$$\overline{D}_{min} = 0.993$$

   (b) $\overline{D}_{min} = 1.3028$
   (c) Notice that even with a lower data rate, we get 31 % increase in average distortion without cross-layer design

MATLAB

```
clear D0 = .38;
R0 = 18.3e3;
```

```
Theta = 2537;
K = 1;
L = 3040;

T = 350e-3;
C = [45 24 60]*1e3;
p = [.5 .25 .25];

count = 1;
ss = .1;
Rpos1 = [(R0/1e3)+ss:ss:(C(1)/1e3)-ss]*1e3;
Rpos2 = [(R0/1e3)+ss:ss:(C(2)/1e3)-ss]*1e3;
Rpos3 =[(R0/1e3)+ss:ss:(C(3)/1e3)-ss]*1e3;
for i1 = 1:length(Rpos1)
    R(1) = Rpos1(i1);
    for i2 = 1:length(Rpos2)
        R(2) = Rpos2(i2);
        for i3 = 1:length(Rpos3)
            R(3) = Rpos3(i3);
            Rv(count,:) = R;
            AvgDist(count) = 0;
            for i = 1:length(C)
                Dist = D0+(Theta/(R(i)-R0))+K*exp(-(C(i)-R(i))*(T/L));
                AvgDist(count) = AvgDist(count) + Dist*p(i);
            end
            count = count+1;
        end
    end
end
[AvgDistmin,d] = min(AvgDist) Ropt = Rv(d,:)
```

11.

$$\frac{E_b}{N_0}(C_B) = \frac{2^{C/B}-1}{C/B}$$

As C $\uparrow$, numerator $\uparrow$ exponentially whereas denominator $\uparrow$ only linearly. $\therefore \frac{E_b}{N_0}(C_B) \uparrow$ as C $\uparrow$ for fixed B

$$\frac{E_b}{N_0}(B) = B\frac{2^{C/B}-1}{C}$$

$$\frac{\partial \frac{E_B}{N_0}(B)}{\partial B} = \frac{-2^{C/B}ln2}{BC} - \frac{2^{C/B}}{C} + \frac{1}{C}$$

$$2^{C/B} > 1 \text{for} C, B > 0$$

$$\therefore \frac{E_b}{N_0}(B) \downarrow \text{ as B} \uparrow \text{ for C fixed}$$