Research Log from 2024.1.4-2024.1.10

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Work: Reading paper "Maximizing Products of Linear Forms, and The Permanent of Positive Semidefinite Matrices"

1 QUESTIONS

1.1 Question 1

Lemma 1.1 3.2

Any $A \succeq 0$ can be factorized as $A = V^\dagger V$, where v_i are the columns of V. Consider the following pair of convex programs:

$$\mu^*(A) \equiv \min \lambda^n \text{ s.t. } \begin{cases} V \operatorname{Diag}(\alpha) V^\dagger \preceq \lambda I_n \\ \prod_{i=1}^n \alpha_i \geq 1 \\ \alpha_i > 0 \end{cases}$$

$$\nu^*(A) \equiv \max \prod_{i=1}^n v_i^\dagger P v_i \text{ s.t. } \begin{cases} \operatorname{Tr}(P) = n \\ P^\dagger = P \\ P \succeq 0 \end{cases}$$

Then $r(A) \leq \nu^*(A) = \mu^*(A)$.

The proof involves arguing weak duality of the two programs:

$$\prod_{i=1}^n v_i^\dagger P v_i = \prod_{i=1}^n \alpha_i v_i^\dagger P v_i \leq \left(\frac{1}{n}\sum_{i=1}^n \alpha_i v_i^\dagger P v_i\right)^n \leq \left(\frac{Tr(P)\lambda}{n}\right)^n = \lambda^n$$

I don't quite understand why the first $=$ isn't $\leq$ by the second constraint of $\mu^*(A)$. Also, why does the third inequality hold? I don't see the transformation.