Simplifying the Expression of Per($xx^* + yy^*$)

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First of all, we can expand the quantity as:

$$Per(xx^* + yy^*) = \sum_{S,T \subseteq [n] | S| = |T|} per(xx^*[S;T]) per(yy^*[\bar{S};\bar{T}])$$

Then since $xx^*[S;T]$ and $yy^*[\bar{S};\bar{T}]$ are rank-1 matrices, we have

$$\begin{aligned}
Per(xx^* + yy^*) &= \sum_{S,T \subseteq [n], |S| = |T|} \prod_{i \in S} x_i \prod_{j \in T} \bar{x_j} \prod_{p \in \bar{S}} y_p \prod_{q \in \bar{T}} \bar{y_q} \\
&= \sum_{k=0}^{n} \sum_{S,T \subseteq [n], |S| = |T| = k} \prod_{i \in S} x_i \prod_{j \in T} \bar{x_j} \prod_{p \in \bar{S}} y_p \prod_{q \in \bar{T}} \bar{y_q} \\
&= \sum_{k=0}^{n} \left(\sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i \prod_{p \in \bar{S}} y_p \right) \left(\sum_{T \subseteq [n], |T| = k} \prod_{j \in T} \bar{x_j} \prod_{q \in \bar{T}} \bar{y_q} \right) \\
&= \sum_{k=0}^{n} \left| \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i \prod_{j \in \bar{S}} y_j \right|^2
\end{aligned}$$

Now we define $f_k := \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} x_i \prod_{j \in \bar{S}} y_j$, then $per(xx^* + yy^*) = \sum_{k=0}^{n} f_k^2$.

Let's consider the generating function for f_k, define

$$G(t) := \prod_{i=1}^{n} (tx_i + y_i)$$

Then it's not hard to see that f_k is exactly the coefficient of t^k in the polynomial $G(t)$. Thus we have:

$$\sum_{k=0}^{n} f_k = G(1) = \prod_{i=1}^{n} (x_i + y_i)$$

But to get to sum of squared f_k, it starts to become difficult. My approach is taking the product of two generating functions. Specifically, let's consider the following polynomial:

$$G(t_1)G(t_2) = \prod_{i=1}^{n} (t_1 x_i + y_i) \prod_{i=1}^{n} (t_2 x_i + y_i) = \prod_{i=1}^{n} (t_1 x_i + y_i)(t_2 x_i + y_i)$$

Then f_k^2 is exactly the coefficient of $t_1^k t_2^k$ in $G(t_1)G(t_2)$. So we need a smart way to set t_1 and t_2 such that if the degree of t_1 and t_2 don't match in a term, then the term cancels out.

Now if we set $t_1 := t$, $t_2 := 1/t$, then

$$G(t_1)G(t_2) = G(t)G(1/t) = \prod_{i=1}^{n}(tx_i + y_i)(1/tx_i + y_i)$$

We can immediately see that every term with a balanced degree of t_1 and t_2 becomes a constant term (coefficient of t^0) in the above polynomial, every imbalanced term would result in a non-constant term (coefficient of non-zero power of t) in the above polynomial. But I couldn't find a way to explicitly write down the constant term.