Research Log from 2024.1.4-2024.1.10

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Work: Reading paper ”Maximizing Products of Linear Forms, and The Permanent of Positive Semidefinite Matrices”

1 QUESTIONS

1.1 Question 1

Lemma 1.1 3.2

Any $A \succeq 0$ can be factorized as $A = V^\dagger V$, where v_i are the columns of V. Consider the following pair of convex programs:

$$\mu^*(A) \equiv \min \lambda^n \text{ s.t. } \begin{cases} V\,\mathrm{Diag}(\alpha)V^\dagger \preceq \lambda I_n \\ \prod_{i=1}^n \alpha_i \geq 1 \\ \alpha_i > 0 \end{cases}$$

$$\nu^*(A) \equiv \max \prod_{i=1}^n v_i^\dagger P v_i \text{ s.t. } \begin{cases} \mathrm{Tr}(P) = n \\ P^\dagger = P \\ P \succeq 0 \end{cases}$$

Then $r(A) \leq \nu^*(A) = \mu^*(A)$.

The proof involves arguing weak duality of the two programs:

$$\prod_{i=1}^n v_i^\dagger P v_i = \prod_{i=1}^n \alpha_i v_i^\dagger P v_i \leq \left(\frac{1}{n}\sum_{i=1}^n \alpha_i v_i^\dagger P v_i\right)^n \leq \left(\frac{Tr(P)\lambda}{n}\right)^n = \lambda^n$$

I don’t quite understand why the first $=$ isn’t $\leq$ by the second constraint of $\mu^*(A)$. Also, why does the third inequality hold? I don’t see the transformation.

1.2 Question 2

Why can’t we directly solve $r(A)$ and $rel(A)$, but can exactly solve in polynomial time $\nu^*(A)$

1.3 Question 3

Does the following two objective functions equivalent?

$$\min_x \sum_i \min_{y_i} f(x, y_i) = \min_x \min_{y_i} \sum_i f(x, y_i)$$

$$s.t. \quad (x, y_1, y_2, ..., y_n) \in P$$

where P is some polytope.

1.4 Question 4

Does the following correct?

$$\underset{x}{min}\,\underset{y}{max}\, f(x,y) = -\underset{x}{min}\,\underset{y}{min} -f(x,y) = \underset{x}{max}\,\underset{y}{min} -f(x,y)$$

1.5 Question 5

In the scheduling model problem, I'm trying to schedule n jobs in a day, each job i has a start time s_i and an end time t_i, any two jobs cannot overlap. The objective function is to make the job schedule as dispersed (discrete) as possible: that is, to minimize the sum of squares of intervals. $i.e.$ $(s_j - t_i)^2$, where j is the successor of i. so the natural programming is as follows:

$$\underset{s,t}{min} \sum_{i=1}^{n} \underset{\substack{j\in[n] \\ s_j > t_i}}{min} (s_j - t_i)^2$$

$$s.t. \begin{cases} b_{ij} s_i \geq b_{ij} t_j \\ (1-b_{ij}) s_j \geq (1-b_{ij}) t_i \\ s, t \in [0, 24] \\ b_{ij} \in \{0, 1\} \end{cases}$$

One way to represent the min function is:

$$\underset{s,t}{min} \sum_{i=1}^{n} \underset{p_i}{max}\, p_i$$

$$s.t. \begin{cases} b_{ij} s_i \geq b_{ij} t_j \\ (1-b_{ij}) s_j \geq (1-b_{ij}) t_i \\ b_{ij} p_i \leq b_{ij} (s_j - t_i)^2 \ \forall j \in [n], i \in [n] \\ s, t \in [0, 24] \\ b_{ij} \in \{0, 1\} \end{cases}$$

This induces question 3 and 4.

Another way to substitute the min function is:

$$\underset{s,t}{min} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} b_{ij} (s_j - t_i)^2$$

$$s.t. \begin{cases} b_{ij} s_i \geq b_{ij} t_j \\ (1-b_{ij}) s_j \geq (1-b_{ij}) t_i \\ \displaystyle\sum_{j=1}^{n} \alpha_{ij} b_{ij} = 1 \ \forall i \in [n] \\ s, t \in [0, 24] \\ b_{ij} \in \{0, 1\} \end{cases}$$