Research Questions 4.5

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1 PREVIOUS PROGRESS

Given PSD matrices A, B, we can expand $per(A+B)$ as:

$$per(A+B) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\langle \mathbb{1}, (A^{\otimes k} \otimes B^{\otimes n-k})\mathbb{1}\rangle$$

We can further expand the bilinear term by:

$$\langle \mathbb{1}, (A^{\otimes k} \otimes B^{\otimes n-k})\mathbb{1}\rangle = k!(n-k)! \sum_{S,T \in \binom{n}{k}} per(A[S;T])per(B[\bar{S};\bar{T}])$$

Therefore, if we can utilize the fact that $\begin{pmatrix} A[S;T] & 0 \\ 0 & B[\bar{S};\bar{T}] \end{pmatrix}$ is low rank and come up with an SDP relaxation to better approximate each bilinear term, we get a better approximation for $per(A+B)$.

Now let's consider the SDP relaxation for $\sum_{S,T\in\binom{n}{k}} per(A[S;T])per(B[\bar{S};\bar{T}])$:

$$\begin{aligned} \min \sum_{S\in\binom{n}{k}} \prod_{i=1}^{k} D_{S_i,S_i} \prod_{j=1}^{n-k} Q_{\bar{S}_j,\bar{T}_j} \\ s.t.\ D \succeq A \\ Q \succeq B \end{aligned} \tag{1}$$

We have proven that this SDP relaxation is an upper bound of $\sum_{S,T\in\binom{n}{k}} per(A[S;T])per(B[\bar{S};\bar{T}])$. Now we are trying to find the connection between it and the maximum product of linear forms.

2 THIS WEEK'S WORK

First, we want to get an equivalence form of (1) as the same step in paper:

$$\begin{aligned} \min \sum_{S\in\binom{n}{k}} \lambda^k \mu^{n-k} \frac{1}{\prod_{i=1}^{k} a_{S_i} \prod_{j=1}^{n-k} b_{\bar{S}_j}} \\ s.t.\ V^* Diag(a) V \preceq \lambda I_n \\ U^* Diag(b) U \preceq \mu I_n \\ a_i, b_i > 0 \end{aligned} \tag{2}$$

Note that we have to add constraints $\prod_{i=1}^k a_{S_i} \prod_{j=1}^{n-k} b_{\bar{S}_j} = 1$ for $\forall S \in \binom{n}{k}$ to get the natural form:

$$
\begin{aligned}
\min \sum_{S\in\binom{n}{k}} \lambda^k \mu^{n-k} \\
s.t.\ V^* Diag(a) V \preceq \lambda I_n \\
U^* Diag(b) U \preceq \mu I_n \\
\prod_{i=1}^k a_{S_i} \prod_{j=1}^{n-k} b_{\bar{S}_j} = 1 \text{ for } \forall S \in \binom{n}{k} \\
a_i, b_i > 0
\end{aligned}
\tag{3}
$$

But is this still an equivalent form as (1)? Are the additional constraints even satisfiable?

Next, let's try to do a metaphor for dual as well, let $A = V^*V$, $B = U^*U$, then a reasonable guess would be:

$$
\begin{aligned}
\max \sum_{S\in\binom{n}{k}} \prod_{i=1}^k v_{S_i}^* P v_{S_i} \prod_{j=1}^{n-k} u_{\bar{S}_j}^* H u_{\bar{S}_j} \\
s.t.\ tr(P) + tr(H) = n \\
P \succeq 0, Q \succeq 0
\end{aligned}
\tag{4}
$$

Now we can attempt to show week duality, for any fixed $S \in \binom{n}{k}$:

$$
\begin{aligned}
\prod_{i=1}^k v_{S_i}^* P v_{S_i} \prod_{j=1}^{n-k} u_{\bar{S}_j}^* H u_{\bar{S}_j} &= \prod_{i=1}^k a_{S_i} v_{S_i}^* P v_{S_i} \prod_{j=1}^{n-k} b_{\bar{S}_j} u_{\bar{S}_j}^* H u_{\bar{S}_j} \quad \text{(by the additional constraints in (3))} \\
&\le \left(\frac{1}{k}\sum_{i=1}^k a_{S_i} v_{S_i}^* P v_{S_i}\right)^k \left(\frac{1}{n-k}\sum_{j=1}^{n-k} b_{\bar{S}_j} u_{\bar{S}_j}^* H u_{\bar{S}_j}\right)^{n-k} \quad \text{(Cauchy-Schwarz)} \\
&= \left(\frac{1}{k}\langle P, \sum_{i=1}^k a_{S_i} v_{S_i} v_{S_i}^*\rangle\right)^k \left(\frac{1}{n-k}\langle H, \sum_{j=1}^{n-k} b_{\bar{S}_j} u_{\bar{S}_j} u_{\bar{S}_j}^*\rangle\right)^{n-k} \\
&\le \left(\frac{1}{k}\langle P, \lambda I_n\rangle\right)^k \left(\frac{1}{n-k}\langle H, \mu I_n\rangle\right)^{n-k} \\
&= \left(\frac{\lambda tr(P)}{k}\right)^k \left(\frac{\mu tr(H)}{n-k}\right)^{n-k} \\
&\le ? \quad \textbf{(How to proceed after here, or are above ineqs too loose?)}
\end{aligned}
$$

Also, if we take a step back, (4) is actually SDP relaxation of the following product of linear forms:

$$
\max_{x=x_1\oplus x_2\in\mathbb{S}_{2n}} \sum_{S\in\binom{n}{k}} \prod_{i=1}^k |\langle v_{S_i}, x_1\rangle|^2 \prod_{j=1}^{n-k} |\langle u_{\bar{S}_j}, x_2\rangle|^2
$$

But, note that $A+B=(V;U)^*(V;U)$, the natural metaphor of the product of linear forms should be:

$$\begin{aligned}\max_{x\in\mathbb{S}_{2n}}\prod_{i=1}^{n}|\langle v_i\oplus u_i,x\rangle|^2 &= \max_{x=x_1\oplus x_2\in\mathbb{S}_{2n}}\prod_{i=1}^{n}|\langle v_i,x_1\rangle+\langle u_i,x_2\rangle|^2\\ &\geq \max_{x=x_1\oplus x_2\in\mathbb{S}_{2n}}\sum_{k=0}^{n}\sum_{S\in\binom{n}{k}}\prod_{i=1}^{k}|\langle v_{S_i},x_1\rangle|^2\prod_{j=1}^{n-k}|\langle u_{\bar{S}_j},x_2\rangle|^2\end{aligned}$$

There are several $|\langle v_i,x_1\rangle\langle u_i,x_2\rangle|$ missed counting

Which one is that we are seeking for to correspond to our SDP relaxation?