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Work: Finishing the EPTAS on disk graphs [1] and getting intuition about the procedure of approximation-preserving reduction while realizing bounded ply, $O_\epsilon(1)$-far from independent, and finally bounded local radius.

1 APPROXIMATION PRESERVING REDUCTION STEP 3

Let G_2 be the resulting subgraph we obtained after *Reduction 2*, let $\mathcal{D}_2 \subseteq \mathcal{D}$ be the subset of disks representing the vertices of G_2. Then we have for every vertex $v \in V$, $N_{G_2}(v) = S(v) \cup I(v)$, where $I(v)$ is a set of independent vertices, and $S(v)$ is small, *i.e.* $|S(v)| \leq O_\epsilon(1)$. Based on this G_2 graph, we can give the third step of the $(1+\epsilon)$-approximation preserving reduction which outputs a subgraph $G_3 \subseteq G_2$ with bounded local radius $O_\epsilon(1)$.

Definition 1.1

Given a graph $G = (V, E))$, two vertices $u, v \in V(G)$ are called *false twins* if they have the same neighbors, *i.e.* $N_G(u) = N_G(v)$.

Note that *false twins* induces an equivalent relation on $V(G_2)$.
Denote $N_{G_2}(X)$ be the set of neighbors of X in G_2, and $d_X := |N_{G_2}(X)|$

Algorithm 4 Reduce3(G_2)

$\mathcal{X} \leftarrow \{FT_{G_2}(v) : v \in V(G_2)\}$
for $X \in \mathcal{X}$ **do**
  $d_X \leftarrow |N_{G_2}(X)|$
  $V_X \leftarrow$ an arbitrary subset of X of size $\min\{c(k,\epsilon) \cdot d_X, |X|\}$
end for
$G_3 \leftarrow G_2[\cup_{X \in \mathcal{X}} V_X]$

Lemma 1.1

G_3 satisfies the following two properties:

1. The local radius of G_3 is $\frac{1}{\epsilon}^{O(1)}$
2. Given any $(1+\frac{\epsilon}{2})$-approximation solution for $P_\mathcal{G}$ on G_3, one can compute in $n^{O(1)}$ time a $(1+\frac{\epsilon}{4})$-approximation solution for $P_\mathcal{G}$ on G_2

Proof of the second part of the lemma. Let S be any $(1+\frac{\epsilon}{2})$-approximation solution on G_3, we can create a set $S' \subset V(G_2)$ as follows: for each $X \in \mathcal{X}$, if

$|X \cap S| \le (c(k,\epsilon) - k) \cdot d_X$, let $U_X = X \cap S$, otherwise, let $U_X = (X \cap S) \cup N_{G_2}(X)$. Let $S' = \cup_{X \in \mathcal{X}} U_X$.

Claim 1.1

S' is a feasible solution: $G_2 - S'$ does not contain any cycle.

Since $G_3 - S$ is acyclic, any cycle in $G_2 - S$ must contain some vertices in $V(G_2) \setminus V(G_3)$. So it suffices to guarantee that every vertex in $(V(G_2) \setminus V(G_3)) \setminus S'$ is not involved in any cycle.

For any $X \in \mathcal{X}$, if $X \subseteq V(G_3)$, then $(V(G_2) \setminus V(G_3)) \cap X = \emptyset$, so we are fine. Assume $X \nsubseteq V(G_3)$. Let $\bar{S} := V(G_3) \setminus S$. We can first make two observations, which would be useful:

1. $N_{G_2}(X) \subseteq V(G_3)$: Since $X \nsubseteq V(G_3)$, then $|X| \ge c(k,\epsilon) \cdot d_X$. Let $X' \subset N_{G_2}(v)$ be a false-twin class in $\mathcal{X}$. Then, $d_{X'} \ge |X| \ge d_X \ge |X'|$. Thus X' would be preserved in G_3.

2. Either $|X \cap \bar{S}| \le 1$ or $|N_{G_2}(X) \cap \bar{S}| \le 1$. Otherwise, X and $N_{G_2}(X)$ would induce at least a 4-cycle in $G_3 - S$, contradicts the fact that S is a feasible solution on G_3.

By Observation 1, we have $N_{G_2}(X) \setminus S = N_{G_2}(X) \cap \bar{S}$. Now if $N_{G_2}(X) \cap \bar{S} \le 1$, then each vertex in X has degree 0 or 1 in $G_2 - S$, thus is not involved in any cycle. So we can only consider the case when $N_{G_2}(X) \cap \bar{S} > 1$.

By Observation 2, we have $|X \cap \bar{S}| \le 1$. Since $X \nsubseteq V(G_3)$, then $|X \cap V(G_3)| = c(k,\epsilon) \cdot d_X$. Therefore, $|X \cap S| = |X \cap V(G_3)| - |X \cap \bar{S}| \ge c(k,\epsilon) \cdot d_X - 1 > (c(k,\epsilon) - 1) \cdot d_X \ge (c(k,\epsilon) - k) \cdot d_X$. In this case, all of the neighbors of X are included in U_X and deleted, *i.e.* $N_{G_2}(X) \subseteq U_X \subseteq S'$. (which is basically why we delete all the neighbors of X when constructing S')

So we conclude that S' is indeed feasible for G_2.

Claim 1.2

S' is a $(1 + \frac{\epsilon}{2})$-approximation solution for G_2.

Clearly, $|U_X| \le (1 + \frac{1}{c(k,\epsilon)} - k) \cdot |X \cap S|$. Therefore:

$$\begin{aligned}
|S'| = \sum_{X \in \mathcal{X}} U_X &\le (1 + \frac{1}{c(k,\epsilon)} - k) \cdot |S| \\
&\le (1 + \frac{1}{c(k,\epsilon)} - k) \cdot (1 + \frac{\epsilon}{4}) \cdot opt_{\mathcal{G}}(G_3) \\
&\le (1 + \frac{1}{c(k,\epsilon)} - k) \cdot (1 + \frac{\epsilon}{4}) \cdot opt_{\mathcal{G}}(G_2)
\end{aligned}$$

Take an appropriate $c(k,\epsilon)$, then $|S'| \le (1 + \frac{\epsilon}{2}) \cdot opt_{\mathcal{G}}(G_2)$.

□

Proof. **Proof of the first part of the lemma**

Observation 1: For each $X \in \mathcal{X}$, $|v_X| = O_\epsilon(1)$. To see this, first note that any two vertices $v, v' \in V(G_3)$ are false twins if.f. they are false twins in G_2. This follows from the bounded ply of G_2. One simple fact can be shown that a disk graph H of ply p has at most $O(p \cdot |V(H)|)$ edges. Consider the induced subgraph $G_2[X \cup N_{G_2}(X)]$ contains at least $|X| \cdot d_X$. By the fact, this induced subgraph can contain at most $O_\epsilon(|X| + d_X)$ edges. Therefore, either $|X| = O_\epsilon(1)$ or $d_X = O_\epsilon(1)$. If $|X| = O_\epsilon(1)$, then trivially $|V_X| = O_{G_2}(1)$. If $d_X = O_\epsilon(1)$, then $|V_X| \leq (1 + \frac{1}{\epsilon}) \cdot d_X = O_\epsilon(1)$.

Now we only need a little extra effort to bound the local radius of G_3. Recall that for each $v \in V(G_3)$, the neighbor of v can be partitioned in to $S(v)$ and $I(v)$. Now we slightly modify the partition, then we can have local radius $O_\epsilon(1)$. Notice that in previous counterexample, the local radius is unbounded mainly because the disks in $I(v)$ is blocking the faces in $D(v)$. We can prevent this situation happens by creating a new partition $S^*(v)$ and $I^*(v)$ and guaranteeing for each $u \in I^*(v)$, $D(u) \not\subseteq \cup_{w \in \{v\} \cup S*(v)} D_w$. We create $(S^*(v), I^*(v))$ as follows: a vertex $u \in N_{G_3}(v)$ is included in S^* if $u \in S(v) \cup S^2(v)$ or $N_{G_3}(u) \subseteq \{v\} \cup S(v) \cup S^2(v)$, then $I^*(v)$ is simply $N_{G_3}(v) \setminus S^*(v)$. This justification should be simple. We can still have $|S^*(v)| = O_\epsilon(1)$ and $|I^*(v)| = O_\epsilon(1)$.

To see why this gives us bounded radius, let's consider $E_S = \cap_{w \in \{v\} \cup S} D_w$ for $S \subset S^*$. A geometric observation is that if a disk D is not contained in the union of a set of disks, then the boundary of D crosses the boundary of the intersection of the disks in the set at most twice. So the intersection pattern of E_S and the disks D_u for $u \in I^*$ should be a star. Therefore, within this induced arrangement subgraph, any two faces have a distance 3-path (cross the boundary of $I^*(v)$ twice).

Let $\mathcal{S} = \{S \subseteq S^* : E_S \neq \emptyset\}$, let $A[E_S]$ denote the induced subgraph of the arrangement grap of $\mathcal{D}_\ni$ consisting of the faces contained in E_S. Then we can prove the following statement for any d: if the radius of $A[E_S]$ is at most r for any $S \in \mathcal{S}$ with $|S^*| - |S| = d$, then the radius of $A[E_S]$ is at most $f(\epsilon, r)$ for any $S \in \mathcal{S}$ with $|S^*| - |S| = d + 1$. Then we can apply induction.

2 HANDLING BOUNDED RADIUS

Definition 2.1 SQGM Property

A graph class $\mathcal{G}$ has the subquadratic grid minor property if there exist constants $\alpha > 0$ and $1 \leq c \leq 2$ such that, for any $t > 0$, every graph $G \in \mathcal{G}$, excluding the $t \times t$-grid as a minor, has treewidth at most $\alpha \cdot t^c$.

Proposition 2.1

Let $\prod$ be an η-modulated and reducible graph optimization problem, then $\prod$ has an EPTAS on every induced-subgraph-closed graph class with the *SQGM* property.

Proposition 2.2

Let G be a planar graph with treewidth w, then G contains $\lfloor w/5 \rfloor \times \lfloor w/5 \rfloor$ minor.

Lemma 2.1

Given a disk graph G with local radius r. Let $\mathcal{D}$ be some realization of G, and let $t' \in \mathbb{N}$. If $A_{\mathcal{D}}$ contains the grid of size $t' \times t'$ as a minor, then G contains a grid of size $t \times t$ as a minor for $t = \Omega(t'/r)$

Proposition 2.3

Let G be a geometric graph that has a realization of ply p whose arrangement graph has treewidth w. Then, the treewidth of G is $O(w \cdot p)$.

Thus bounded local radius disk graph has the SQGM property and thus admits EPTAS on a bunch of graph optimization problem.

$\square$

REFERENCES

[1] Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, Jie Xue, and Meirav Zehavi. A framework for approximation schemes on disk graphs, 2022.