Generalizing Wick's Formula

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1 RE-PROOF OF WICK'S FORMULA

Theorem 1.1 Isserlis' Theorem/Wick's Probability Theorem

If $(x_1, x_2, ..., x_n)$ is a zero-mean (complex) Gaussian random vector, then

$$\mathbb{E}[x_1 x_2 ... x_n] = \sum_{p \in P_n^2} \prod_{(i,j) \in p} \mathbb{E}[x_i x_j]$$

Theorem 1.2 Wick's Formula

Let $A = VV^*$ be a PSD matrix, where v_i's are rows of V, let $x \in \mathbb{C}^n$ be a standard complex Gaussian random vector, *i.e.* $x \sim \mathcal{CN}(0, I)$, then

$$per(VV^*) = \mathbb{E}\left[\prod_{i=1}^{n} |\langle v_i, x\rangle|^2\right] = \frac{1}{n!}\mathbb{E}\left[per(Vxx^*V^*)\right]$$

Proof.

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{n} |\langle v_i, x\rangle|^2\right] &= \mathbb{E}\left[\prod_{i=1}^{n} \langle v_i, x\rangle\overline{\langle v_i, x\rangle}\right] \\ &= \sum_{p \in P_{2n}^2} \prod_{(i,j) \in p} \mathbb{E}[\langle v_i, x\rangle\langle v_j, x\rangle]\end{aligned}$$

Now notice that

$$\mathbb{E}[\langle v_i, x\rangle\langle v_j, x\rangle] = \mathbb{E}[\langle v_i, x\rangle\overline{\langle x, v_j\rangle}] = \mathbb{E}[v_i^* x x^T \overline{v_j}] = v_i^* \mathbb{E}[xx^T]\overline{v_j}$$

However, for any complex Gaussian variable $x = a+bi$, we have that a and b independent follows $\mathcal{N}(0, 1/2)$, thus

$$\mathbb{E}[x^2] = \mathbb{E}[a^2] + 2i\mathbb{E}[ab] - \mathbb{E}[b^2] = 0$$

For any pair of complex Gaussian variables $x_1 = a_1 + b_1 i, x_2 = a_2 + b_2 i$, where $x_1 \neq x_2$, since they are independent, we have $\mathbb{E}[x_1 x_2] = 0$.

Therefore, we have

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{n}|\langle v_i, x\rangle|^2\right] &= \sum_{\sigma\in S_n}\prod_{i=1}^{n}\mathbb{E}[\langle v_i, x\rangle\overline{\langle v_{\sigma(i)}, x\rangle}] \\
&= \sum_{\sigma\in S_n}\prod_{i=1}^{n} v_i^*\mathbb{E}[xx^*]v_{\sigma_i} \\
&= \sum_{\sigma\in S_n}\prod_{i=1}^{n}\langle v_i, v_{\sigma_i}\rangle \\
&= per(VV^*)
\end{aligned}$$

□

2 RANK-2 WICK'S FORMULA

Claim 2.1

Let $A = VV^*$ be a PSD matrix, where v_i's are rows of V, let $x, y \in \mathbb{C}^n$ be independent drawn standard complex Gaussian random vectors, then

$$(n+1)!per(VV^*) = \mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}\left[per(Vxx^*V^* + Vyy^*V^*)\right]$$

Proof. Recall that

$$\begin{aligned}
per(xx^* + yy^*) &= \sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k} per(xx^*[S,T])per(yy^*[\bar{S},\bar{T}]) \\
&= \sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k} k!(n-k)!\prod_{i\in S}x_i\prod_{j\in T}\overline{x_j}\prod_{p\in\bar{S}}y_p\prod_{q\in\bar{T}}\overline{y_q}
\end{aligned}$$

Again, by linearity of expectation and independence of x and y, we have

$$\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}[per(Vxx^*V^* + Vyy^*V^*)] =$$

$$\sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k} k!(n-k)!\mathbb{E}_{x\sim\mathcal{CN}(0,I)}\left[\prod_{i\in S}\langle v_i, x\rangle\prod_{j\in T}\overline{\langle v_j, x\rangle}\right]\mathbb{E}_{y\sim\mathcal{CN}(0,I)}\left[\prod_{p\in\bar{S}}\langle v_p, y\rangle\prod_{q\in\bar{T}}\overline{\langle v_q, y\rangle}\right]$$

Same story here, we have

$$\mathbb{E}_{x\sim\mathcal{CN}(0,I)}\left[\prod_{i\in S}\langle v_i, x\rangle\prod_{j\in T}\overline{\langle v_j, x\rangle}\right] = \sum_{\sigma\in S_{S\to T}}\prod_{i\in S}\mathbb{E}_{x\sim\mathcal{CN}(0,I)}[\langle v_i, x\rangle\overline{\langle v_j, x\rangle}] = per(A[S,T])$$

Therefore,

$$\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}[per(Vxx^*V^* + Vyy^*V^*)] = \sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k} k!(n-k)!per(A[S,T])per(A[\bar{S},\bar{T}])$$

Claim 2.2

Let $\mathbb{1}$ be a $0,1$-valued n^n-length vector, where $\mathbb{1}_\sigma = 1$ if and only if σ corresponds to a permutation, let $A, B \in \mathbb{C}^{n\times n}$ be two matrices, then

$$\langle \mathbb{1}, A^{\otimes k} \otimes B^{\otimes(n-k)} \mathbb{1}\rangle = \sum_{S,T\in[n],|S|=|T|=k} k!(n-k)! per(A[S,T]) per(B[\bar{S},\bar{T}])$$

Proof.

$$\begin{aligned}
&\langle \mathbb{1}, A^{\otimes k} \otimes B^{\otimes(n-k)} \mathbb{1}\rangle \\
&= \sum_{\sigma\in S_n}\sum_{l\in S_n}\prod_{i=1}^{k} A_{\sigma(i),l(i)} \prod_{j=k+1}^{n} B_{\sigma(j),l(j)} \\
&= \sum_{S,T\subseteq[n],|S|=|T|=k} \sum_{\sigma\in S_k}\sum_{\sigma'\in S_{n-k}}\sum_{l\in S_k}\sum_{l'\in S_{n-k}}\prod_{i=1}^{k} A_{S(\sigma(i)),T(l(i))} \prod_{j=k+1}^{n} B_{\bar{S}(\sigma'(j)),\bar{T}(l'(j))} \\
&= \sum_{S,T\subseteq[n],|S|=|T|=k} \sum_{\sigma\in S_k}\sum_{l\in S_k}\prod_{i=1}^{k} A_{S(\sigma(i)),T(l(i))} \sum_{\sigma'\in S_{n-k}}\sum_{l'\in S_{n-k}}\prod_{j=k+1}^{n} B_{\bar{S}(\sigma'(j)),\bar{T}(l'(j))} \\
&= \sum_{S,T\subseteq[n],|S|=|T|=k} \sum_{\sigma\in S_k}\sum_{l\in S_k}\prod_{i=1}^{k} A[S,T]_{\sigma(i),l(i)} \sum_{\sigma'\in S_{n-k}}\sum_{l'\in S_{n-k}}\prod_{j=k+1}^{n} B[\bar{S},\bar{T}]_{\sigma'(j),l'(j)} \\
&= \sum_{S,T\subseteq[n],|S|=|T|=k} k!(n-k)! per(A[S,T]) per([B[\bar{S},\bar{T}]])
\end{aligned}$$

□

Therefore, we have

$$\begin{aligned}
\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}[per(Vxx^*V^* + Vyy^*V^*)] &= \sum_{k=0}^{n} \langle \mathbb{1}, A^{\otimes k} \otimes A^{\otimes(n-k)} \mathbb{1}\rangle \\
&= \sum_{k=0}^{n} \langle \mathbb{1}, A^{\otimes n} \mathbb{1}\rangle \\
&= (n+1)! per(A)
\end{aligned}$$

□

Claim 2.3

Change of measure between Gaussian and uniform distribution over the unit sphere:

$$\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}\left[per(VxxV^* + VyyV^*)\right] \leq \frac{(n+d-1)!}{(d-1)!}\mathbb{E}_{x,y\in\mathbb{S}_{\mathbb{C}}^{n-1}(1)}\left[per(VxxV^* + VyyV^*)\right]$$

Proof.

$$
\begin{aligned}
&\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}\left[per(VxxV^*+VyyV^*)\right]\\
&=\sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k}\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}\left[k!(n-k)!\prod_{i\in S}\langle v_i,x\rangle\prod_{j\in T}\overline{\langle v_j,x\rangle}\prod_{p\in\bar{S}}\langle v_p,y\rangle\prod_{q\in\bar{T}}\overline{\langle v_q,y\rangle}\right]\\
&=\sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k}\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}\left[k!(n-k)!\prod_{i\in S}\langle v_i,\frac{x}{||x||_2}\rangle\prod_{j\in T}\overline{\langle v_j,\tfrac{x}{||x||_2}\rangle}\prod_{p\in\bar{S}}\langle v_p,\frac{y}{||y||_2}\rangle\prod_{q\in\bar{T}}\overline{\langle v_q,\tfrac{y}{||y||_2}\rangle}\right.\\
&\qquad\qquad\left.\cdot||x||_2^{2k}||y||_2^{2(n-k)}\right]\\
&=\sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k}\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}\left[k!(n-k)!\prod_{i\in S}\langle v_i,\frac{x}{||x||_2}\rangle\prod_{j\in T}\overline{\langle v_j,\tfrac{x}{||x||_2}\rangle}\prod_{p\in\bar{S}}\langle v_p,\frac{y}{||y||_2}\rangle\prod_{q\in\bar{T}}\overline{\langle v_q,\tfrac{y}{||y||_2}\rangle}\right]\\
&\qquad\qquad\cdot\mathbb{E}_{x\sim\mathcal{CN}(0,I)}\left[||x||_2^{2k}\right]\mathbb{E}_{y\sim\mathcal{CN}(0,I)}\left[||y||_2^{2(n-k)}\right]\\
&=\sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k}\mathbb{E}_{\tilde{x},\tilde{y}\in\mathbb{S}_{\mathbb{C}}^{n-1}(1)}\left[k!(n-k)!\prod_{i\in S}\langle v_i,\tilde{x}\rangle\prod_{j\in T}\overline{\langle v_j,\tilde{x}\rangle}\prod_{p\in\bar{S}}\langle v_p,\tilde{y}\rangle\prod_{q\in\bar{T}}\overline{\langle v_q,\tilde{y}\rangle}\right]\\
&\qquad\qquad\cdot\mathbb{E}_{x\sim\mathcal{CN}(0,I)}\left[||x||_2^{2k}\right]\mathbb{E}_{y\sim\mathcal{CN}(0,I)}\left[||y||_2^{2(n-k)}\right]
\end{aligned}
$$

Since $||x||_2^2$ is a "scaled" chi-squared random variable with $2d$ dimension of freedom (since we can treat $||x||_2^2$ as $||a||_2^2+||b||_2^2$, where $a=Re(x), b=Im(x)$), the k-th moment of it is $\frac{\Gamma(2d/2+k)}{\Gamma(2d/2)=\frac{(d+k-1)!}{(d-1)!}}$. Therefore, we have

$$
\begin{aligned}
&\mathbb{E}_{x\sim\mathcal{CN}(0,I)}\left[||x||_2^{2k}\right]\mathbb{E}_{y\sim\mathcal{CN}(0,I)}\left[||y||_2^{2(n-k)}\right]\\
&=\frac{(d+k-1)!}{(d-1)!}\cdot\frac{(d+n-k-1)!}{(d-1)!}\\
&=\frac{(d-1)!\prod_{i=1}^{k}(d+i-1)}{(d-1)!}\cdot\frac{(d+n-k-1)!}{(d-1)!}\\
&\leq\frac{(d-1)!\prod_{i=1}^{k}(d+i-1+n-k)}{(d-1)!}\cdot\frac{(d+n-k-1)!}{(d-1)!}\\
&=\frac{(n+d-1)!}{(d-1)!}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}_{x,y\sim\mathcal{CN}(0,I)}\left[per(VxxV^*+VyyV^*)\right]\\
&\leq\sum_{k=0}^{n}\sum_{S,T\in[n],|S|=|T|=k}\mathbb{E}_{x,y\in\mathbb{S}_{\mathbb{C}}^{n-1}(1)}\left[k!(n-k)!\prod_{i\in S}\langle v_i,\frac{x}{||x||_2}\rangle\prod_{j\in T}\overline{\langle v_j,\tfrac{x}{||x||_2}\rangle}\prod_{p\in\bar{S}}\langle v_p,\frac{y}{||y||_2}\rangle\prod_{q\in\bar{T}}\overline{\langle v_q,\tfrac{y}{||y||_2}\rangle}\right]\\
&\qquad\qquad\cdot\frac{(n+d-1)!}{(d-1)!}\\
&=\frac{(n+d-1)!}{(d-1)!}\mathbb{E}_{x,y\in\mathbb{S}_{\mathbb{C}}^{n-1}(1)}\left[per(VxxV^*+VyyV^*)\right]
\end{aligned}
$$

$\square$

Theorem 2.1

The quantity $max_{x,y \in \mathbb{S}_{\mathbb{C}}^{n-1}(1)} per(Vxx^*V^* + Vyy^*V^*)$ gives $\frac{(n+d-1)!)}{(d-1)!(n+1)!}2^n$-approximation to $per(VV^*)$.