1. Conversion to 1^{st} order.

2. Borderlines 边界线

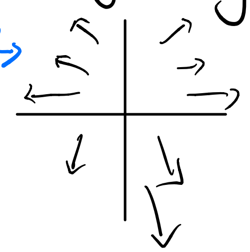
3. Qualitative 定性.

1. Conversion to 1^{st} order

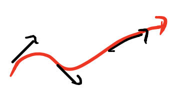
⊙ non-linear

⊙ autonomous

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y) \end{cases} \Rightarrow \vec{F} = f\hat{i} + g\hat{j}$$



Solns: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$



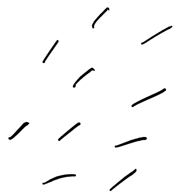
"trajectory"

⊙ Eliminate t. (消除t)

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

一阶常微分方程

Picture:



slope.

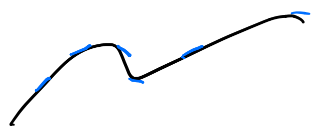
⇝ slope field / direction field

① → solns.

$$y = y(x) \quad \text{or} \quad h(x,y) = 0.$$

⇓

Picture:



⇝ 只有斜率，没有大小或方向.

↓

对上式解
一种粗化.

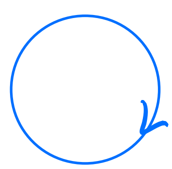
"积分曲线"
integral curve

⇓

例 ①

$$\begin{cases} x' = y \\ y' = -x \end{cases}, \qquad \text{soln}: \begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

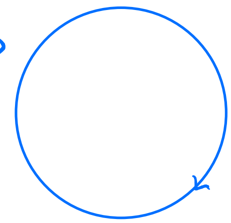
"trajectory" ⇝



② → 消 t.

$$\frac{dy}{dx} = -\frac{x}{y} \rightsquigarrow y\,dy = -x\,dx$$

$\Rightarrow \quad x^2+y^2=C \Rightarrow$



例② predator - prey equation.

捕食者 - 猎物.

$$\begin{cases} x' = -ax + bxy & \text{predator} \\ y' = cy - dxy & \text{prey} \end{cases} \qquad a,b,c,d>0.$$

解 ① → 找 crit pts.

$$x(-a+by)=0$$
$$y(c-dx)=0$$

$$\begin{cases} x=0 \\ y=0 \end{cases}, \quad \begin{cases} x=\frac{c}{d} \\ y=\frac{a}{b} \end{cases}$$

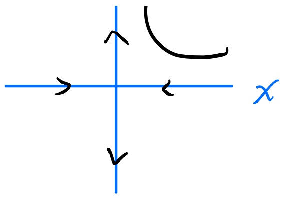
即 ① $(0,0)$, ② $\left(\frac{c}{d}, \frac{a}{b}\right)$

② 在 $(0,0)$ 处线化.

$$\begin{cases} x' = -ax \\ y' = cy \end{cases}$$

$$\Rightarrow A = \begin{bmatrix} -a & 0 \\ 0 & c \end{bmatrix} \Rightarrow \lambda_1=-a,\ \lambda_2=c \Rightarrow \text{saddle}$$

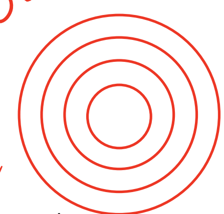
鞍部.



② 设 $a=b=c=d=1$.

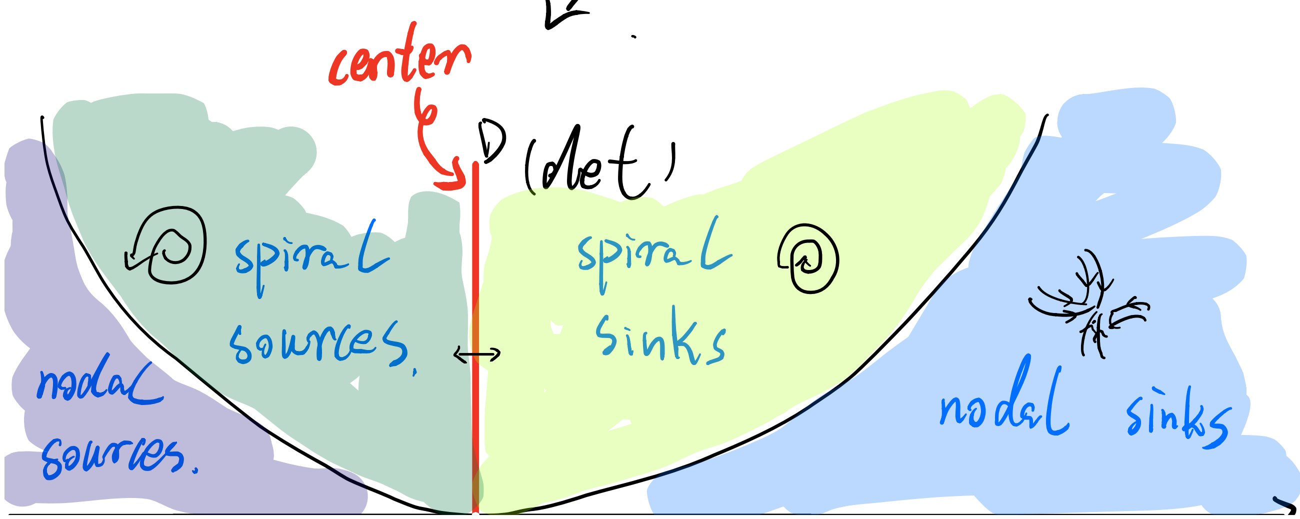
$$\begin{cases} x' = -x + xy \\ y' = y - xy \end{cases} \qquad J = \begin{pmatrix} -1+y & x \\ -y & 1-x \end{pmatrix}$$

$$J_{(1,1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xRightarrow{\text{linear system.}} \begin{cases} x' = y \\ y' = -x \end{cases} \sim \text{center}$$



∴ 引出第二个话题，"边界线情形" borderline cases

GST



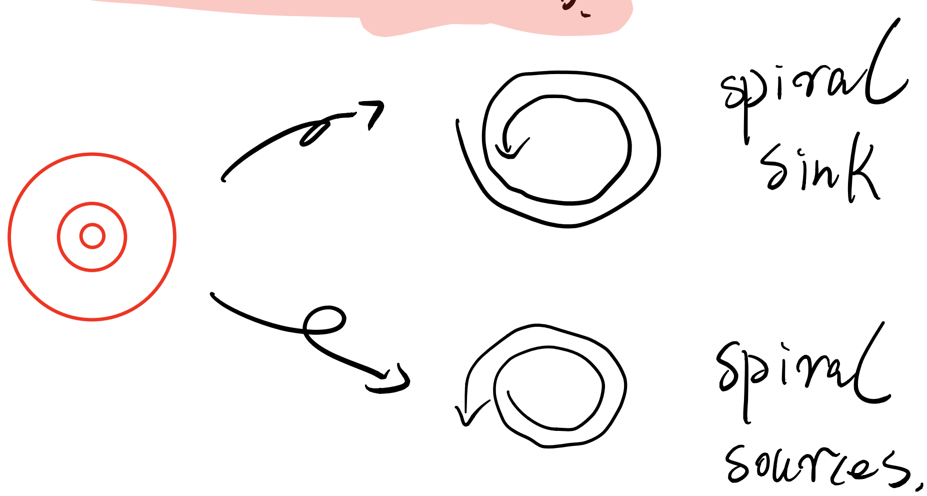
T (traj)

SADDLES

$$\lambda^2 - T\lambda + D = 0.$$

$$\Downarrow$$

在边界线上，一个参数的微小变化可以改变它的变化方式。



即 nonlinear system at (1,1) 可能是 Center ①, spiral sink ②, spiral sources ③ 三种之一.

$$\begin{cases} x' = -x + xy \\ y' = y - xy \end{cases}$$

$\longrightarrow$ elim t.

$$\frac{dy}{dx} = \frac{y(1-x)}{x(-1+y)}$$

$$\Rightarrow \frac{y-1}{y}\, dy = \frac{1-x}{x}\, dx$$

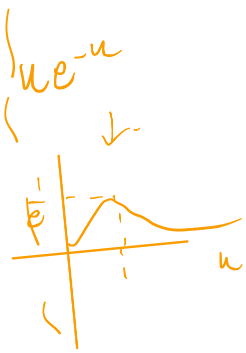
$$\Rightarrow \left(1-\frac{1}{y}\right) dy = \left(\frac{1}{x}-1\right) dx$$

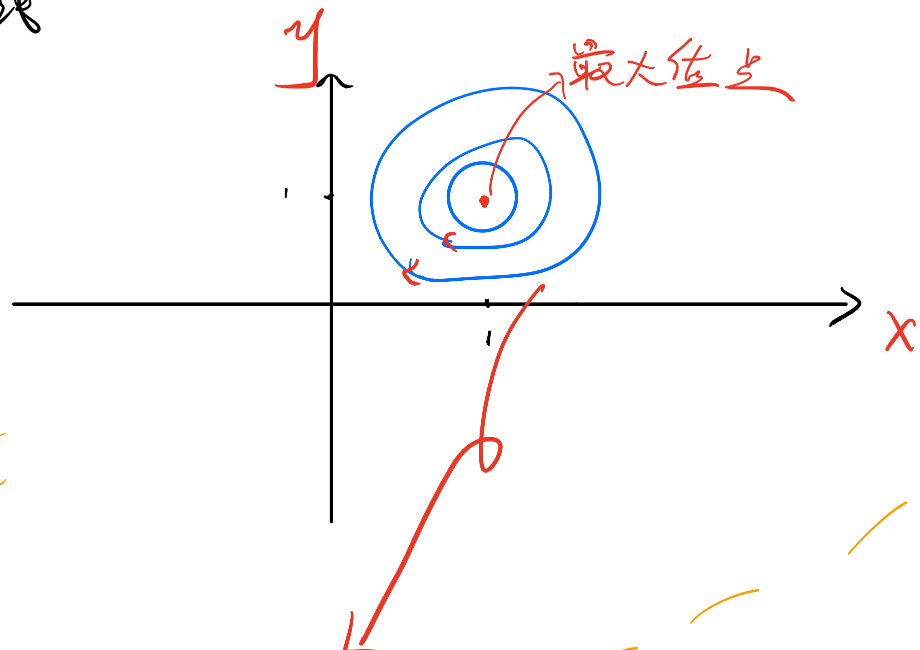
$$\Rightarrow y - \ln y = \ln x - x + C_1$$

$$\Rightarrow e^{y} \cdot \frac{1}{y} = x \frac{1}{e^{x}} \cdot C_2$$

$$\Rightarrow \frac{x}{e^x}\cdot\frac{y}{e^y} = C \qquad \Rightarrow x\cdot e^{-x}\cdot y e^{-y} = h(x,y)$$

积分曲线





3. 定性行为 ☆☆☆

以恒定速率 k 捕鱼的影响 "effect"

$$\begin{cases} x' = -ax + bxy \boxed{-kx} \\ y' = cy - dxy \boxed{-ky} \end{cases}$$

$\Downarrow$.

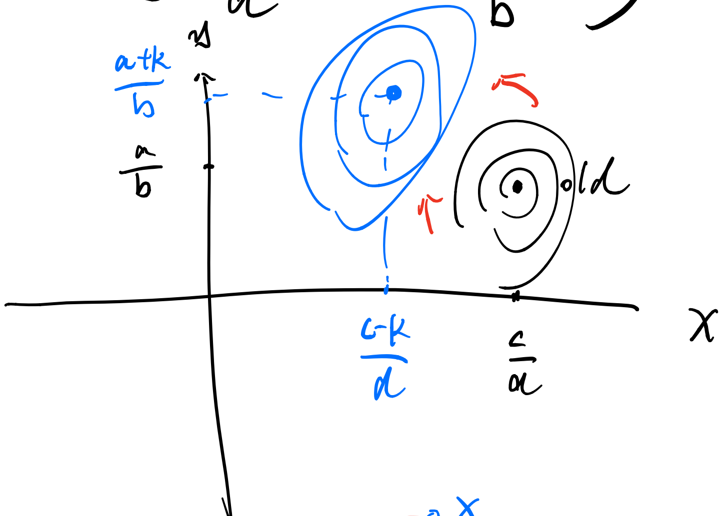
$$\begin{cases} x' = -(a+k)\,x + bxy \\ y' = (c-k)\,y - dxy. \end{cases}$$

old crit. point:

$\rightarrow \left(\frac{c}{d}, \frac{a}{b}\right)$

Now ---

$\rightarrow \left(\frac{c-k}{d}, \frac{a+k}{b}\right)$



即: 捕鱼降低了鲨鱼的数量. $\rightarrow x$ $x\downarrow$

增加了鱼的数量. $\rightarrow y$ $y\uparrow$

Volterra's principle

"沃尔泰拉法则"