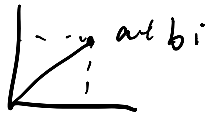
Complex Numbers

$i^2 = -1$

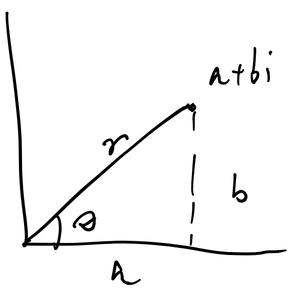


共轭复数 complex conjugate : $z = a+bi$ $\quad \bar{z} = a-bi$ $\quad z\bar{z} = a^2+b^2$

$$\frac{2+i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{2-3+7i}{10} = \frac{-1+7i}{10} = \frac{1}{10}+\frac{7}{10}i$$

① Polar representation. "复数的极坐标"

$$\boxed{a+bi = r \cdot e^{i\theta}}$$



$a+bi = r\cos\theta + i \cdot r\sin\theta.$

$= r(\cos\theta + i \cdot \sin\theta)$

定义 ★

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler's formula 欧拉公式

Exponential Law. ① $e^x e^y = e^{x+y}$

② 求导得到本身. $e^{at} \Rightarrow \frac{dy}{dt} = ay$

$y(0) = 1.$

$$\boxed{① \; e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}}$$

$$\boxed{② \; \frac{de^{i\theta}}{d\theta} = ie^{i\theta}}$$

$D(u+iv) = D(u) + iD(v)$

$e^{a+ib} = e^a \cdot e^{ib}$

③ "infinite series" 无穷级数.

①
$(\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)$

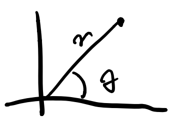
$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$ $\underset{=}{\text{likey}}$ $e^{i\theta_1}\cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

② $\dfrac{d(\cos\theta + i\sin\theta)}{d\theta} = -\sin t + i\cos t = i(\cos t + i\sin t)$

$\hookrightarrow \dfrac{de^{i\theta}}{d\theta} = ie^{i\theta}$

记 Polar form: $\boxed{\alpha = re^{i\theta}}$



r = modulus of α (length)

θ = argument of α.
⇓
angle

★ Advantage of Polar form.

"good for multiplication".

$r_1 e^{i\theta_1}\cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

" $r_1\angle\theta_1 + r_2\angle\theta_2 = r_1\cdot r_2\angle\theta_1+\theta_2$ " ⇒ "mult the length" "add the angle."

example:

$\int e^{-x}\cdot\cos x\,dx = \underbrace{\mathrm{Re}}_{\text{实部}}\int e^{(-1+i)x}\,dx$

⇓

$a+bi = r(\cos\theta + i\sin\theta) = r\cdot e^{i\theta}$

$e^{-x}\cdot\cos x$ = Real part of e^{-x+ix} = $\mathrm{Re}(e^{(-1+i)x})$

∵ $e^{-x}(\cos x + i\sin x) = e^{-x}\cdot e^{ix}$

∴ $e^{(-1+i)x}$的实部 = $e^{-x}(\cos x + i\sin x)$的实部 = $e^{-x}\cdot\cos x$.

$$\int e^{(-1+i)x}\,dx = \frac{e^{(-1+i)x}}{-1+i} = \frac{1}{-1+i}\,e^{-x}(\cos x + i\sin x)$$

$$= \frac{-1-i}{2}\cdot e^{-x}(\cos x + i\sin x)$$

实部 $\frac{-e^{-x}}{2}\cdot(-\cos x + \sin x)$

answer to the question. $\int e^{-x}\cdot\cos x\,dx$

$r\cdot\cos\theta + i\sin\theta$

$e^{(-1+ix)}$

$e^{-x}\cdot e^{ix}$ e^{-x+ix}

$e^{(-1+i)x}$的

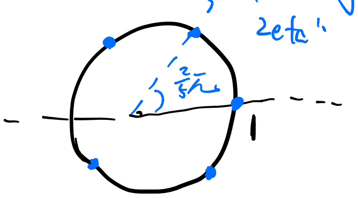
②. $\sqrt[n]{1}$ → 计算1的n次方根.

实数 1 / -1.

虚复数. 1的n次方根 总有 n 个.

n answers as cx. #s.

ξ, $2e\pi i$, 对 $n=5$



$\alpha = \frac{2}{5}\pi$.

$$\xi_1 = e^{i\cdot\frac{2}{5}\pi} = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$$

$$\xi_1^5 = e^{i\cdot\frac{2}{5}\pi\cdot 5} = e^{i2\pi} = 1.$$

ξ_1 → answer

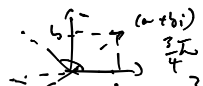
Remind: $\alpha \to$ complex number.

$$\frac{1}{\alpha}\cdot\alpha = 1 \qquad \text{angle}\left(\frac{1}{\alpha}\right) + \text{angle}(\alpha) = 0 \qquad (\text{angle}(1) = 0)$$

$$\left|\frac{1}{\alpha}\right| = \frac{1}{|\alpha|} \qquad \text{ang}\left(\frac{1}{\alpha}\right) = -\text{ang}(\alpha)$$

$$a\cos(\omega t) + b\sin(\omega t) = \overrightarrow{(a, b)} \cdot \overrightarrow{(\cos(\omega t) \cdot \sin(\omega t))} = \sqrt{a^2+b^2}\cos(\omega t - \Phi)$$

$-1+i$



$a+bi = r\cdot\cos\theta + r\cdot\sin\theta\, i$

$= r\cdot e^{i\theta}$

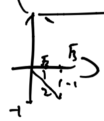
$\theta = \frac{3}{4}\pi$

$\sqrt{2}$

e^{i}

$\boxed{\sqrt{2}\cdot e^{\frac{3}{4}\pi}}$

$e^{i\theta} = \cos\theta + i\sin\theta$



$\boxed{2e^{i\cdot -\frac{\pi}{6}}}$

$$\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-1-2i}{1+1} = -i$$

$$\frac{1-i}{1+i} = \frac{\sqrt{2}e^{i\theta_1}}{\sqrt{2}e^{i\theta_2}} = \frac{i-\frac{\pi}{4}}{\frac{\pi}{4}} = -i$$

$(1-i)^4$

$1-i = (\sqrt{2}\,e^{i\cdot -\frac{\pi}{4}})^4$

$(1-i)(1-i)$

$4\cdot e^{-i\pi}$

$(-2i)^2 = -4$

-4

$(-8-4\sqrt{3}i)(1+\sqrt{3}i)$

$(1+\sqrt{3}i)^3$

$(-2+2\sqrt{3}i)^2(1+\sqrt{3}i) = -8-4\sqrt{3}i-8\sqrt{3}i+$

$(1-3+2\sqrt{3}i)^2$

$\cancel{4-12-4\sqrt{3}i = -8-4\sqrt{3}i}$

rectangular | polar

$z = a+bi$ | $|z|,\ \arg(z)$

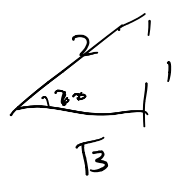
1

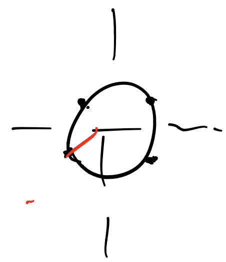
$z: 1-i$ / $\sqrt{2},\ -\frac{\pi}{4}$ / $\sqrt{2}\, e^{i\cdot -\frac{\pi}{4}}$

$\cancel{2-2i}$ | $2,\ \frac{\pi}{6}$ | $2e^{i\frac{\pi}{6}}$

$\cancel{1+\sqrt{3}i}$

$\sqrt{3}+i$





$e^{i\frac{5}{4}\pi}$

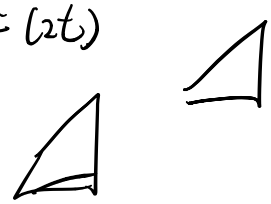
$z^4+4=0$

$z = \sqrt[4]{-4}$ | $-4\, e^{i\frac{\pi}{4}}$

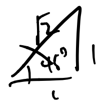
$e^{i\theta} = \cos\theta + i\sin\theta$

$a+bi = r(\cos\theta + i\sin\theta)$

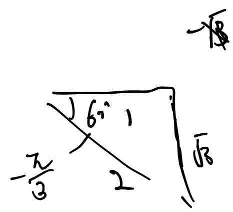
$\cos 2t \qquad \sin(2t)$



$\frac{\pi}{4}$



$\sqrt{2}\cos\left(2t-\frac{\pi}{4}\right)$



$2\cos\left(\sqrt{3}t+\frac{\pi}{3}\right)$

$e^{it} =$

$e^{i}\left(\frac{e^{it}}{2+2i}\right)$

$$\frac{e^{it} = \cos t + \sin t\, i}{2\sqrt{2}\cdot\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4} i\right)}$$

$\frac{\sqrt{2}}{2}$ $\quad 2\sqrt{2}e^{i\frac{\pi}{4}}$

$$\frac{\cos t}{2\sqrt{2}\cos\frac{\pi}{4}} = \frac{\cos t}{2}$$

2, $\frac{\sqrt{2}}{2}$

$\frac{\cos t}{2}$

$$\frac{e^{it}}{2+2i} = (\cos t + i\cdot\sin t)\left(\frac{1}{2+2i}\cdot\frac{2-2i}{2-2i}\right) = \frac{2-2i}{8} = \frac{1-i}{4}$$

$$= (\cos t + i\sin t)\cdot\left(\frac{1-i}{4}\right) \xrightarrow{\text{实部}} \frac{1}{4}\cos t + \frac{1}{4}\sin t$$

$\frac{\sqrt{2}}{4} \qquad = \frac{\sqrt{2}}{4}\cos\left(t-\frac{\pi}{4}\right)$

① $2\cos(3t) + 2\sin(3t)$

$= 2\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right)$ ✓

② $\sqrt{3}\cos(\pi t) - \sin(\pi t)$

$= 2\cos\left(\pi t + \frac{\pi}{6}\right)$ ✓

③ $\cos\left(t-\frac{\pi}{8}\right) + \sin\left(t-\frac{\pi}{8}\right)$

$= \sqrt{2}\cos\left(t-\frac{3\pi}{8}\right)$ ✓

$\mathrm{Im}\, e^{(2+i)x}$

$\int \underline{e^{2x}\sin x}\, dx$

$= \int e^{(2+i)x}\, dx$

$= \frac{1}{2+i} e^{(2+i)x}$

$= \underbrace{\frac{2-i}{5} e^{2x}(\cos x + i\sin x)}$ 取 Im.

$\cancel{\mathrm{Im} \Rightarrow \frac{2-i}{5} e^{2x}\cos x}$

$$\frac{2-i}{5} e^{2x} \cos x + \frac{2-i}{5} e^{2x} \cdot i \sin x$$

$$(2-i) \frac{e^{2x} \cos x}{5} + \frac{e^{2x} \cdot i \sin x}{5} (2-i)$$

$$- \frac{e^{2x} \cos x}{5} i + \frac{2e^{2x} \cdot i \sin x}{5}$$

$$= \underbrace{\frac{e^{2x}}{5} (2 \sin x - \cos x)}$$

✓