Linear 2^{nd} order ODE

with cons coefficients 常系数.

$$\boxed{y'' + Ay' + By = 0}$$

linear

homogeneous

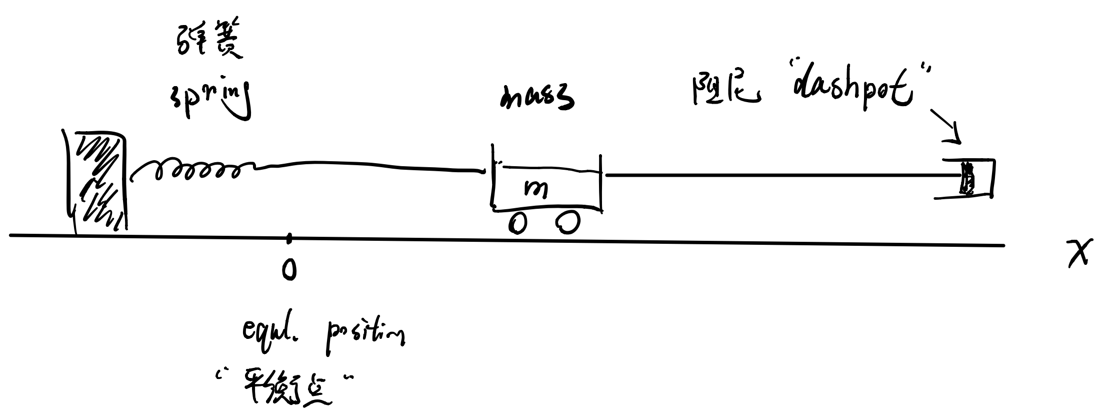
Assume:

gen soln: $\boxed{y = c_1 y_1 + c_2 y_2}$

where y_1, y_2 are solns

initial condition: "通过选择适当的 c_1 和 c_2 来满足"

⊙



$$m x'' = -kx - cx'$$

Newton. spring dashpot

$$\Rightarrow \quad m x'' + cx' + kx = 0$$

$$\Rightarrow \quad x'' + \frac{c}{m} x' + \frac{k}{m} x = 0$$

To solve ODE: $\rightsquigarrow$ · find the soln's (independent)

Basic method:

try: $y = e^{rt}$ $\rightsquigarrow$ indp't

Plug in: $r^2 \cancel{e^{rt}} + Ar\cancel{e^{rt}} + B\cancel{e^{rt}} = 0$

$$\boxed{r^2 + Ar + B = 0}$$ $\rightsquigarrow$ characteristic equation. 特征方程.

$r_1 = \cdots$

$r_2 = \cdots$

Case ①: roots $r_1 \neq r_2$ (Real)

gen soln: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

example:

$$y'' + 4y' + 3y = 0 \quad , (y(0) = 1 \; , \; y'(0) = 0)$$

ca. eq: $r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r_1 = -3 \; , \; r_2 = -1.$

general soln

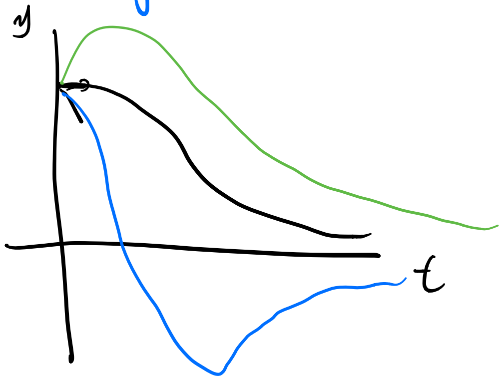
$$\begin{cases} y = c_1 e^{-3t} + c_2 e^{-t} \\ y' = -3c_1 e^{-3t} - c_2 e^{-t} \end{cases}$$

代入 $y(0) = 1$, $y'(0) = 0$

$$\begin{cases} 1 = c_1 + c_2 \\ 0 = -3c_1 - c_2 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{1}{2} \\ c_2 = \frac{3}{2} \end{cases}$$

soln:

$$y = \frac{-1}{2}e^{-3t} + \frac{3}{2}e^{-t}$$



Case 2. Complex roots $r = a \pm bi$

complex soln $y = e^{(a \pm bi)t}$

$$= e^{at + bit} = e^{at}(\cos bt + i\sin bt)$$

$$= \underbrace{e^{at}\cos bt}_{\text{实}} + \underbrace{i \cdot e^{at}\sin bt}_{\text{虚}}$$

Theo: if $u(t) + iv(t)$ 是一个实微分方程的复数解 $y'' + Ay' + By = 0$

Then. $\boxed{u, v \text{ 是它的实解}}$

Proof:

$$(u+iv)'' + A(u+iv)' + B(u+iv) = 0$$

$$\underbrace{u'' + Au' + Bu}_{\text{实部}} + i\underbrace{(v'' + Av' + Bv)}_{\text{虚部}} = 0$$

$$0 + 0 = 0$$

u is sol　　v is sol

solution y.

$$y = e^{at}(c_1 \cdot \cos bt + c_2 \sin bt)$$

纯正弦振荡!

"阻尼相比弹簧常数较小的状态"

example:

$$y'' + 4y' + 5y = 0 \qquad (y(0)=1,\ y'(0)=0)$$

$$\hookrightarrow r^2 + 4r + 5 = 0$$

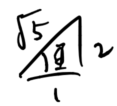
$$r = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$$

$$e^{(-2\pm i)t}$$

$$\hookrightarrow e^{-2t}(\cos t + i\sin t)$$

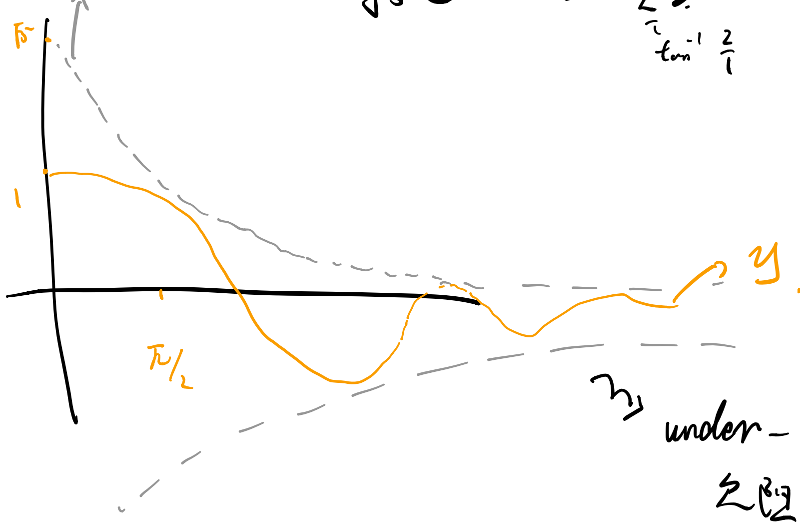
$$y = e^{-2t}(c_1\cos t + c_2\sin t)$$

$y(0)=1$, $y'(0)=0$



$$y = e^{-2t}(\cos t + 2\sin t)$$

$$= \sqrt{5}\,e^{-2t}\cos(t-\underset{\tan^{-1}\frac{2}{1}}{\underbrace{\phi}})$$



“三明治图形”

$\Rightarrow$ under-damped

欠阻尼s.

★ Critically-damped: $r^2 + Ar + B = 0$ has 2 equal roots.
(根相等)

$\Rightarrow (r+a)^2 = 0$,

$= r^2 + 2ar + a^2 = 0$

The ODE: $y'' + 2ay' + a^2 y = 0$

Solution: $y = e^{-at}$

★ 寻找"另一个解", know one soln to $y'' + py' + q = 0$

令: $a^2]\ y = e^{-at} u$.

$2a]\ y' = -ae^{-at} u + e^{-at} u'$

$1]\ y'' = a^2 e^{-at} u - 2ae^{-at} u' + e^{-at} u''$

$0 = 0 + 0 + e^{-at} u'' = 0.$

$\Rightarrow e^{-at} u'' = 0$

$\Rightarrow \boxed{u = c_1 t + c_2.}$ → just t enough. 令 $c_1 = 1$, $c_2 = 0$

$\hookrightarrow y = e^{-at} \cdot t$.

★: 即: $y'' + 2ay' + a^2 y = 0$ → "Critically-damped"

解 $\begin{cases} y_1 = e^{-at} \\ y_2 = e^{-at} \cdot t \end{cases}$