How to solve inhomogeneous systems

以矩阵方式来思考 "matrices"

"Using Matrices."

Theory: $\vec{x}\,' = A\vec{x}$ — A is constant (2×2)

定理A: Gen soln to $\vec{x}\,' = A\vec{x}$ is

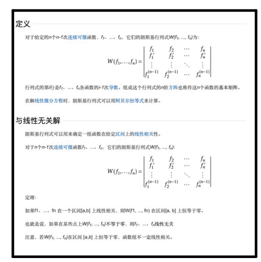
$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

其中，x_1, x_2 是线性无关 解
linearly indepent

定理B: Wronskian of two solns.

$$W(\vec{x}_1, \vec{x}_2) := \left| \vec{x}_1 \quad \vec{x}_2 \right|$$

(行列式) determinant det.


定义

对于给定的n个n-1次连续可微函数，f_1、…、f_n，它们的朗斯基行列式$W(f_1, \ldots, f_n)$为：

$$W(f_1,\ldots,f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

行列式的第i行是f_1、…、f_n各函数的i-1次导数。组成这个行列式的n阶方阵也称作这n个函数的基本矩阵。

在解线性微分方程时，朗斯基行列式可以用阿贝尔恒等式来计算。

与线性无关解

朗斯基行列式可以用来确定一组函数在给定区间上的线性相关性。

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定理：

如果f1、…、fn 在一个区间[a,b] 上线性相关，则W(f1, ..., fn) 在区间[a, b] 上恒等于零。

也就是说，如果在某些点上$W(f_1, \ldots, f_n)$不等于零，则f_1、…、f_n线性无关

注意，若$W(f_1, \ldots, f_n)$在区间 [a,b] 上恒等于零，函数组不一定线性相关。

Thm: Either. $W(t)$ $\equiv 0$ (if $\vec{x}_1, \vec{x}_2$ 相关) not linearly indedpendent

or never 0 (for any t value)

($\vec{x}_1, \vec{x}_2$ 线性相关)

Tips：不单独的看待每一个解，而是把它们一起放在一个矩阵里。

And. It is the properties of that matrix.

$\Downarrow$

方程组的
基本矩阵
fundamental matrix of the system $\vec{x}' = AX$

$$X := [\vec{x}_1 \ \ \vec{x}_2], \quad \vec{x}_1, \vec{x}_2 \text{ ind. solns.}$$

Properties: ① $|X| \neq 0$ for any t.

② $X' = AX \iff [\vec{x}_1{}' \ \ \vec{x}_2{}'] = A[\vec{x}_1 \ \ \vec{x}_2]$
$= [A\vec{x}_1 \ \ A\vec{x}_2]$

$\vec{x}_1{}' = A\vec{x}_1$
$\vec{x}_2{}' = A\vec{x}_2$

● → 进入主题

Inhomog. systems.

$$x' = ax + by + r_1(t)$$
$$y' = cx + dy + r_2(t)$$

$\downarrow$ 矩阵化

$$\vec{x}\,' = A\vec{x} + \vec{r}(t)$$

Find $\vec{x}_p$

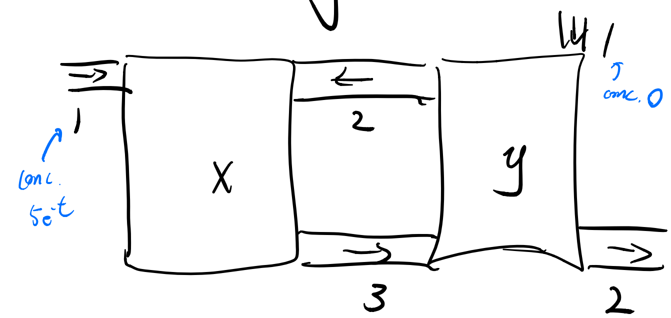
定理C: $\boxed{\vec{x}_{gen} = \vec{x}_c + \vec{x}_p}$ (linearity + superposition)

$\vec{x}_c$: gen soln to $\vec{x}\,' = A\vec{x}$

$\vec{x}_p$: Part. Soln.

例: Mixing Problem.

"$x'' + x = \tan t$, 找 x_p. Prob 2"



容量1L的tank.

flow rate: (L/s)

x: salt in tank 1

y: . . . . . 2.

$$\begin{cases} x' = -3x + 2y + 5e^{-t} \\ y' = 3x - 4y + 0 \end{cases}$$

$$\vec{x}\,' = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} \vec{x} + \begin{pmatrix} 5e^{-t} \\ 0 \end{pmatrix}$$

Method to solve $\vec{x}\,' = A\vec{x} + \vec{r}$, Find $\vec{x}_p$.

参数变分: Variation of Parameters.

第一步 $\Rightarrow \vec{x}_p = v_1(t)\cdot\vec{x}_1 + v_2(t)\,\vec{x}_2$

$\Rightarrow \vec{x}_p = \mathbb{X}\,\vec{v}$ 必须放在右边 $= \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\longrightarrow$ 代入方程组, find $\vec{v}$.

$\vec{x}_p{}' = A\vec{x}_p + \vec{r}$

微. $\rightarrow \cancel{\mathbb{X}'\vec{v}} + \mathbb{X}\vec{v}\,' = \cancel{A\mathbb{X}\vec{v}} + \vec{r}$

$\because \mathbb{X}' = A\mathbb{X}$.

$\Rightarrow \mathbb{X}\,\vec{v}\,' = \vec{r}$

$\Rightarrow \vec{v}\,' = \mathbb{X}^{-1}\vec{r}$

$\Rightarrow \vec{v} = \int \mathbb{X}^{-1}\vec{r}\,dt$ (积分每个元素 integrate each entry)

$\vec{x}_p = \mathbb{X}\int \mathbb{X}^{-1}\vec{r}\,dt$. (找到一个特解就行.)

不是本组的例。

例： $\vec{x}' = Ax$ ， $A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 9 + 8 = 0 \quad \lambda^2 = 1$

$\lambda_1 = 1 \quad , \quad \lambda_2 = -1$

当 $\lambda_1 = 1$

$\begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} \vec{\alpha} = 0 \Rightarrow \vec{\alpha} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

当 $\lambda = -1$

$\begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} \vec{\alpha} = 0 \Rightarrow \vec{\alpha} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

(2) $x' = Ax$ ， $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 25 = 0$

$\lambda = 3 \pm 4i$

取 $\lambda = 3 + 4i$

$\begin{pmatrix} -4i & -4 \\ 4 & -4i \end{pmatrix} \vec{\alpha} = 0 \Rightarrow \vec{\alpha} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$z(t) = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(3+4i)t}$$

$$= e^{3t} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{4ti}$$

$$= e^{3t} \cdot \begin{pmatrix} i \\ 1 \end{pmatrix} \cdot [\cos(4t) + \sin(4t) i]$$

$$= e^{3t} \begin{pmatrix} -\sin(4t) \\ \cos t \end{pmatrix} + i e^{3t} \cdot \begin{pmatrix} \cos(4t) \\ \sin(4t) \end{pmatrix}$$

$$x_1 = \mathrm{Re}(z) = e^{3t} \begin{pmatrix} -\sin(4t) \\ \cos 4t \end{pmatrix}$$

$$x_2 = \mathrm{Im}(z) = e^{3t} \begin{pmatrix} \cos(4t) \\ \sin(4t) \end{pmatrix}$$

$$x = c_1 \cdot x_1 + c_2 x_2.$$

2.

Inhomogeneous Case: Variation of Parameters Formula

The fundamental matrix $\Phi(t)$ also provides a very compact and efficient integral formula for a particular solution to the inhomogeneous equation $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t)$. (presupposing of course that one can solve the homogeneous equation $\mathbf{x}' = A(t)\mathbf{x}$ first to get Φ.) In this short note we give the formula (with proof!) and one example.

Variation of parameters: (solving inhomegeneous systems)

(H) $\mathbf{x}' = A(t)\mathbf{x} \rightsquigarrow \Phi(t)$ = fundamental matrix

(I) $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t)$

Variation of parameters formula for solution to (I) (just like order 1 DE's):

$$\mathbf{x} = \Phi \cdot \left(\int \Phi^{-1} \cdot \mathbf{F}\, dt + \mathbf{C} \right).$$