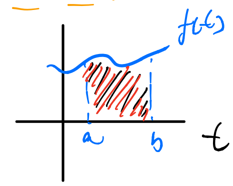
① 引入 → input "unit impulse" (脉冲)

"what an impulse is": 设: 变化的力 $f(t)$ force

impulse of $f(t)$ over $[a, b]$:

$$= \int_a^b f(t)\,dt$$

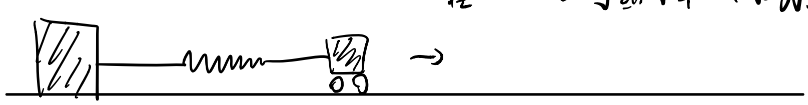


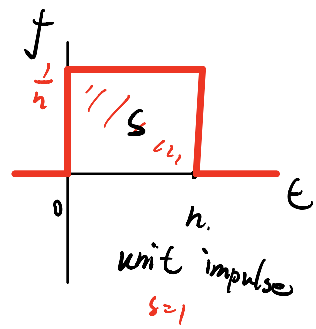
设: $f(t)$ 是 constant F,

impulse $= F \cdot (b-a)$

力乘以力作用的时间.

在 $t=[0, 2]$ 给小车一个拉力.





小车运动 function:

$$y'' + y = \boxed{\frac{1}{h} u_{0h}(t)} = \frac{1}{h}[u(t) - u(t-h)]$$

它的Laplace变换是什么?

$u(t-a)\,g(t-a)$

$\rightsquigarrow e^{-as} G(s)$

$$\rightsquigarrow \frac{1}{h}\left[\frac{1}{s} - \frac{e^{-hs}}{s}\right]$$

当 h 趋于 0 时, Laplace trans 有什么变化

$$\lim_{h\to 0} \frac{1-e^{-hs}}{hs}$$

$u = hs$

$$= \lim_{u\to 0} \frac{1-e^{-u}}{u} \overset{\text{洛必达}}{\underset{\text{法则}}{=}} \lim_{u\to 0} \frac{e^{-u}}{1} = 1$$

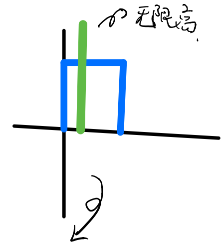
Summary:

$$\frac{1}{h} u_{0,h}(t) \rightsquigarrow \frac{1}{h}\left(\frac{1}{s} - \frac{e^{-hs}}{s}\right)$$

$$\xrightarrow{h\to 0} 1.$$

Paul Dirac's delta function.

狄拉克
δ函数.



$$\delta(t)$$

① 正式开始

$$\delta(t) \rightsquigarrow 1$$

② $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\odot \quad f(t) * \delta(t) \rightsquigarrow F(s)\cdot 1$$

完善地说: $\quad \parallel$

$$u(t)f(t) * \delta(t) \rightsquigarrow F(s)\cdot 1$$

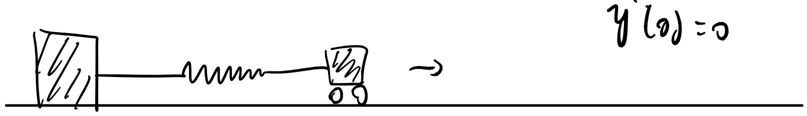
$$u(t)f(t) \rightsquigarrow F(s)$$

$\delta(t)$对$f(t)$的卷积作用就像单位量.

例:

$y(0)=1$

$y'(0)=0$



kicked with impulse A at time $t=\frac{\pi}{2}$.

在很短的时间段里施加这个冲量.

$$y'' + y = A\cdot\delta\left(t-\frac{\pi}{2}\right)$$

$$\rightsquigarrow \quad s^2 Y - s + Y = A\cdot e^{-\frac{\pi}{2}s}\cdot 1$$

$$\Rightarrow \quad Y = \frac{s}{s^2+1} + \frac{Ae^{-\frac{\pi}{2}s}}{s^2+1}$$

$$\frac{A}{s^2+1} \overset{\mathcal{L}^{-1}}{\rightsquigarrow} A\cdot\sin t$$

$$\frac{Ae^{-\frac{\pi}{2}s}}{s^2+1} \overset{\mathcal{L}^{-1}}{\rightsquigarrow}$$

$$\parallel$$

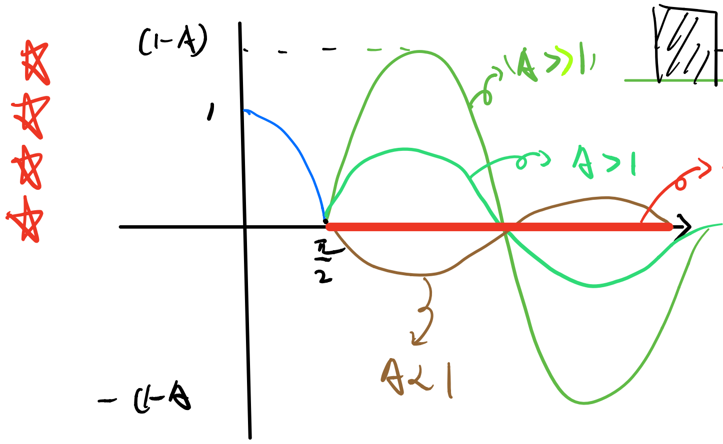
$$u\left(t-\frac{\pi}{2}\right)A\sin\left(t-\frac{\pi}{2}\right)$$

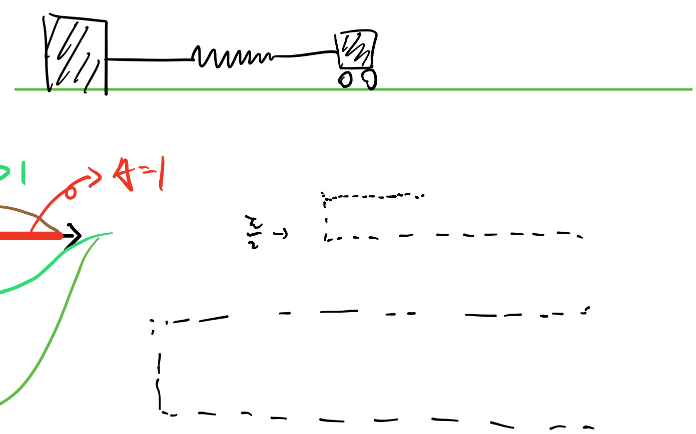
$$-u\left(t-\frac{\pi}{2}\right)A\cos t$$

$$\overset{\mathcal{L}^{-1}}{\rightsquigarrow} \quad y = \cos t + u\left(t-\frac{\pi}{2}\right)A\sin\left(t-\frac{\pi}{2}\right)$$

$$= \quad y = \begin{cases} \cos t & 0 \le t \le \frac{\pi}{2} \\ \cos t - A\cos t & t > \frac{\pi}{2} \end{cases}$$

y 长啥样?





沿伸

A system:

$$y'' + ay' + by = f(t) \xrightarrow{} \text{input}$$

$y = 0$
$y'(0) = 0$

$$\xrightarrow{\mathcal{L}} \quad s^2 Y + asY + bY = F(s)$$

注意：这部分只取决于微分方程的左边(LHS)，意味着只取决于系统，而和输入无关

$$Y = F(s)\,\frac{1}{s^2+as+b}$$

$\frac{1}{s^2+as+b}$ ⟶ 传输函数 transfer function of system $= W(s) \,/\, H(s)$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y(t) = f(t) * w(t)$$

$w(t)$ ⟶ 加权函数 weight function of system

$$= \int_0^t f(u)\,w(t-u)\,du$$

Marvelous 牛逼!!

Tips

⊙. 微分方程的解 ⟶ 用定积分的形式表示了出来.

$$y'' + ay' + by = f(t)$$

解:

$$y(t) = \int_0^t f(u)\,\boxed{w(t-u)}\,du$$

What is $w(t)$ really?

用个简单的例子来 expound.

$$y'' + ay' + b = \delta(t)$$

means → with unit impulse at time $t=0$

kick the mass

开始是静止的

$y(0)=0$
$y'(0)=0$

$$\xrightarrow{\mathcal{L}} \quad s^2 Y + asY + bY = 1$$

$$Y = \frac{1}{s^2+as+b}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y(t) = w(t)$$

所以：

加权函数 weight function 的物理解释：

在 $t=0$ 时刻受到 单位冲量 unit impulse 后的响应.

而 $\int_0^t f(u)\, w(t-u)\, du$.

就相当于

很多短暂冲击的 叠加

Superposition.

kick, kick, ... kick