$a(n) \rightsquigarrow A(x)$

$\sum 1 + x + x^2 + x^3 + \cdots$

$1 \rightsquigarrow \frac{1}{1-x}, \ |x|<1$

$\frac{1}{n!} \rightsquigarrow e^x$

power series 幂级数 : $\sum_0^\infty a_n x^n = A(x)$

$\downarrow n=1,2,3\cdots$

$$\sum_0^\infty a(n) x^n = A(x)$$

↳ 离散版.

将离散变连续的:

↳ continuous analog:

$n = 0, 1, 2, \ldots$

$\downarrow$

$t: \ 0 \le t < \infty$

$$\Rightarrow \int_0^\infty a(t) x^t \, dt = A(x) \quad A(e^{-s})$$

把 x^t 变成以e为底, 这样好算.

$x = e^{\ln x}$

$x^t = (e^{\ln x})^t$

$0<x<1$, (不然不收敛)

$\ln x < 0 \Rightarrow s = -\ln x$

Laplace transform

连续版

$$\Rightarrow \boxed{\int_0^\infty f(t)(e^{-s})^t \, dt = F(s)}$$

变换和算子的区别

$f(t) \longrightarrow$ transform $\longrightarrow F(s)$

$f(t) \longrightarrow$ operator. $\longrightarrow g(t)$

线性变换

Linear transform.

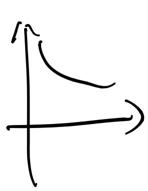
$$\mathcal{L}(f(t)) = F(s) \quad \begin{cases} \mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g) \\ \mathcal{L}(cf) = c \cdot \mathcal{L}(f) \end{cases}$$

$$f(t) \rightsquigarrow F(s)$$

例

① $1 \rightsquigarrow ?$

$$\int_0^\infty e^{-st}\,dt = \lim_{R\to\infty} \int_0^R e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^R$$



$$= \lim_{R\to\infty} \frac{e^{-sR} - 1}{-s}$$

$$\boxed{= \frac{1}{s}}$$ 当 $s>0$ 时 $\to \frac{1}{s}$.

$$1 \rightsquigarrow \frac{1}{s}, \quad s>0$$

What is it if s is negative? 没有意义 meaningless.

② $e^{at} \rightsquigarrow$?

f(t) 的 Laplace 变换已知。

$e^{at} f(t) \rightsquigarrow$

$$\int_0^\infty e^{at} f(t) e^{-st} dt$$

$$= \int_0^\infty f(t) e^{-(s-a)t} dt$$

$$= F(s-a)$$

$$\boxed{e^{at} \rightsquigarrow \frac{1}{s-a}}$$

即 $\boxed{e^{at} f(t) \rightsquigarrow F(s-a) \quad (F(s),\ s>0)}$

$\boxed{\text{指数位移}}$

exponential-shift

$e^{(a+bi)t} \rightsquigarrow$ 也适用。

③ $\sin at \rightsquigarrow$

$\cos at \rightsquigarrow$

hack

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\frac{s+ia+s-ia}{(s-a)^2}.$$

$$\mathcal{L}(\cos at) = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$$

$$= \frac{1}{2} \cdot \frac{2s}{s^2+a^2} = \frac{s}{s^2+a^2}.$$

"当改变虚部 + - 号，函数不变，代表它为实数。

eg. $\cos at \rightsquigarrow \frac{s}{s^2+a^2},\ s>0$ $(s=-\ln x)$

$\sin at \rightsquigarrow \frac{a}{s^2+a^2},\ s>0$

Laplace 逆变换!

$$\frac{1}{s(s+3)} \overset{\mathcal{L}^{-1}}{\rightsquigarrow}$$

$$\frac{1}{s(s+3)} = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right) \rightsquigarrow \frac{1}{3}(1-e^{-3t})$$

find $\lim_{t\to\infty} \frac{t^n e^{-st}}{-s}$ $(s>0)$

$= \frac{1}{s}\lim_{t\to\infty} \frac{t^n}{e^{st}} \xrightarrow[\text{入则}]{\text{洛必达}} = 0$.

④. $t^n \rightsquigarrow$?

$$④\ \int_0^\infty \underset{\text{Diff}}{t^n}\, \underset{\text{int}}{e^{-st}}\,dt = t^n \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty n t^{n-1}\cdot \frac{e^{-st}}{-s}dt$$

$$= 0 - 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$= \frac{n}{s}\mathcal{L}(t^{n-1})$$

*. $\mathcal{L}(t^n) = \frac{n}{s}\mathcal{L}(t^{n-1}) = \frac{n}{s}\cdot\frac{n-1}{s}\mathcal{L}(t^{n-2})$

$$= \frac{n(n-1)\cdots\cdots 1}{s^n}\cdot\overset{=\frac{1}{s}}{\mathcal{L}(t^0)} = \frac{n!}{s^{n+1}}$$

$$t^n \rightsquigarrow \frac{n!}{s^{n+1}}.$$

$$\int_0^\infty f(t)\,e^{-st}\,dt$$