$$x' = f(x, y)$$
$$y' = g(x, y)$$

"autonomous system"

~ non-linear

Problem. "sketch its trajectories" → 轨迹形态.

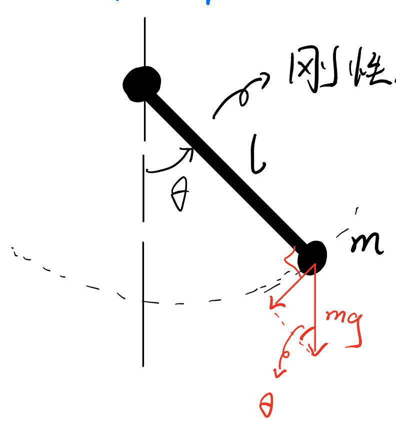
☆: Do it with example:

Non linear pendulum → "非线性摆"

"a lightly damped pendulum"

→ "it illustrates virtually everything."

→ 刚性杆, "rigid rod"



$$m\vec{a} = \vec{F}$$

$$\Rightarrow m \underbrace{l\theta''}_{a} = -mg\sin\theta - c_1 \underbrace{l\theta'}_{v}$$

即: $$\theta'' + \underbrace{\frac{c_1}{m}}_{\gamma}\theta' + \underbrace{\frac{g}{l}}_{\omega^2}\sin\theta = 0$$

c. 阻尼常数 $\quad$ k $\quad$ 非线性

$$\theta'' + c\theta' + k\sin\theta = 0$$

$$\begin{cases} \theta' = \omega \to \text{角速度} \\ \omega' = -k\sin\theta - c\omega \to \text{角加速度} \end{cases}$$

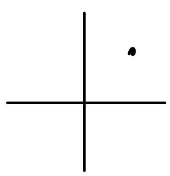
"$c=1$, $k=2$"

$$\Rightarrow \boxed{\begin{cases} \theta' = \omega \\ \omega' = -2\sin\theta - \omega \end{cases}}$$

D. find the simplest possible solutions.

① ⇒ "Look for the critical points of the system.

(x_0, y_0): $\quad f(x_0, y_0) = 0$, $\quad g(x_0, y_0) = 0$.



→ "速度场在这点为0" ⇒ $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ for all time.

↓

解从这里开始.

方法: ① 解联立方程 "solving simultaneously"

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

$$\boxed{\begin{cases} \theta' = \omega \\ \omega' = -2\sin\theta - \omega \end{cases}} \Rightarrow$$

Crit pts.

$\theta = 0, \pm\pi, \pm 2\pi$

$\omega = 0$.

实际. Physically

$\Rightarrow \begin{cases} \theta = 0 \\ \omega = 0 \end{cases}$, $\begin{cases} \theta = \pi \\ \omega = 0 \end{cases}$

↓ stable $\quad$ ↓ unstable

② For each cr. point (x_0, y_0)

→ ① linear system near (x_0, y_0) 找到临近的线性化方程组.

→ ② 画出这个线性方程组的轨迹.

① 如何找出临近的 linear system

接上例

$$\boxed{\begin{cases} \theta' = w \\ w' = -2\sin\theta - w \end{cases}}$$

⇒ crit pts. $\theta = 0, \pm\pi, \pm 2\pi$, $w = 0$.

⇒ 实际 Physically $\begin{cases}\theta=0\\w=0\end{cases}$ → stable, $\begin{cases}\theta=\pi\\w=0\end{cases}$ → unstable

→ 在 (0,0) 线性化这个例子.

在 $\theta \to 0$ 处, 用 "$\theta \approx \sin\theta$"

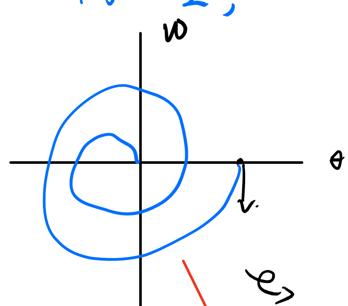
$$\begin{cases} \theta' = w \\ w' = -2\theta - w \end{cases}$$

→ 画它

$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ $\lambda^2 + \lambda + 2 = 0$, $\lambda = \frac{-1 \pm \sqrt{-7}}{2}$ "complex roots"

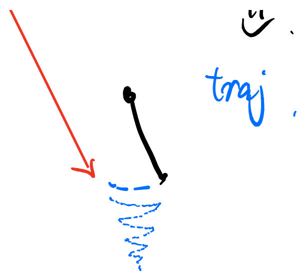
→ Spiral (source or sink)

∵ 实部为 $-\frac{1}{2}$, → traj 会按照 $e^{-\frac{t}{2}}$ 缩小, → sink



e.g. "摆的物理特征"

$\theta \to$ 小的正数.



→ 下一个 Cr. pts. $(\pi, 0)$ → 线性化.

推荐 → 另一种方法: calc. the Jacobian → 雅可比式 M_x.

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_0$$

why use it? → 线性

$\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta f = f(x,y) - f(x_0, y_0)$

$\Rightarrow \Delta f \approx \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y$

令: $x_1 = x - x_0 = \Delta x$, $y_1 = y - y_0 = \Delta y$

则 $\begin{cases} x_1' = \left(\frac{\partial f}{\partial x}\right)_0 x_1 + \left(\frac{\partial f}{\partial y}\right)_0 y_1 \\ y_1' = \left(\frac{\partial g}{\partial x}\right)_0 x_1 + \left(\frac{\partial g}{\partial y}\right)_0 y_1 \end{cases}$ → $x' = f(x, y)$, $y' = g(x, y)$

即 $\Rightarrow \vec{x}_1' = A\vec{x}_1 = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_0, y_0)}$

→ 在 (x_0, y_0) 处计算. $\Rightarrow$ 线性化方程组的矩阵

$\Downarrow$

$$\begin{cases} \theta' = \omega \\ \omega' = -2\sin\theta - \omega \end{cases}$$

$$\Rightarrow \boxed{J = \begin{pmatrix} 0 & 1 \\ -2\cos\theta & -1 \end{pmatrix}}$$

① 在 $(0, 0)$ 处

$$J_0 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} = A.$$

② 在 $(\pi, 0)$

$$J_0 = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = A.$$

$\rightarrow \lambda^2 + \lambda - 2 = 0$, $\Rightarrow (\lambda+2)(\lambda-1) = 0 \Rightarrow \begin{matrix} \lambda_1 = -2 \\ \lambda_2 = 1 \end{matrix}$

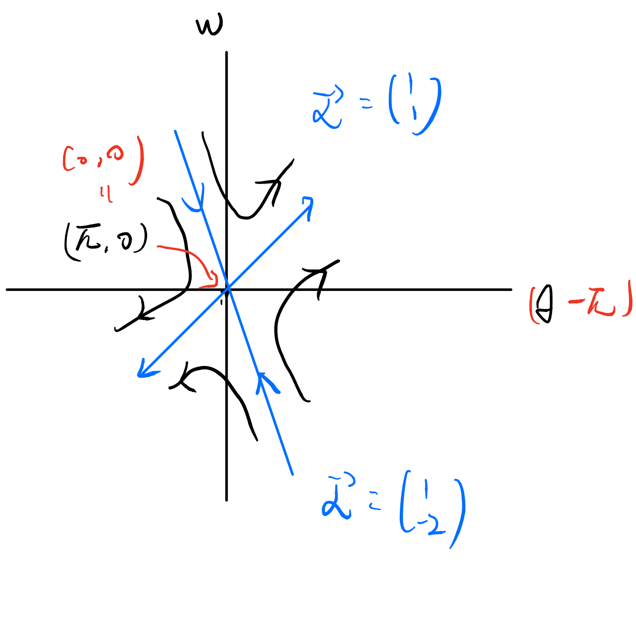
当 $\lambda_1 = 1$. $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \vec{\alpha} = 0 \Rightarrow$

$$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

当 $\lambda_2 = -2$. $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \vec{\alpha} = 0$

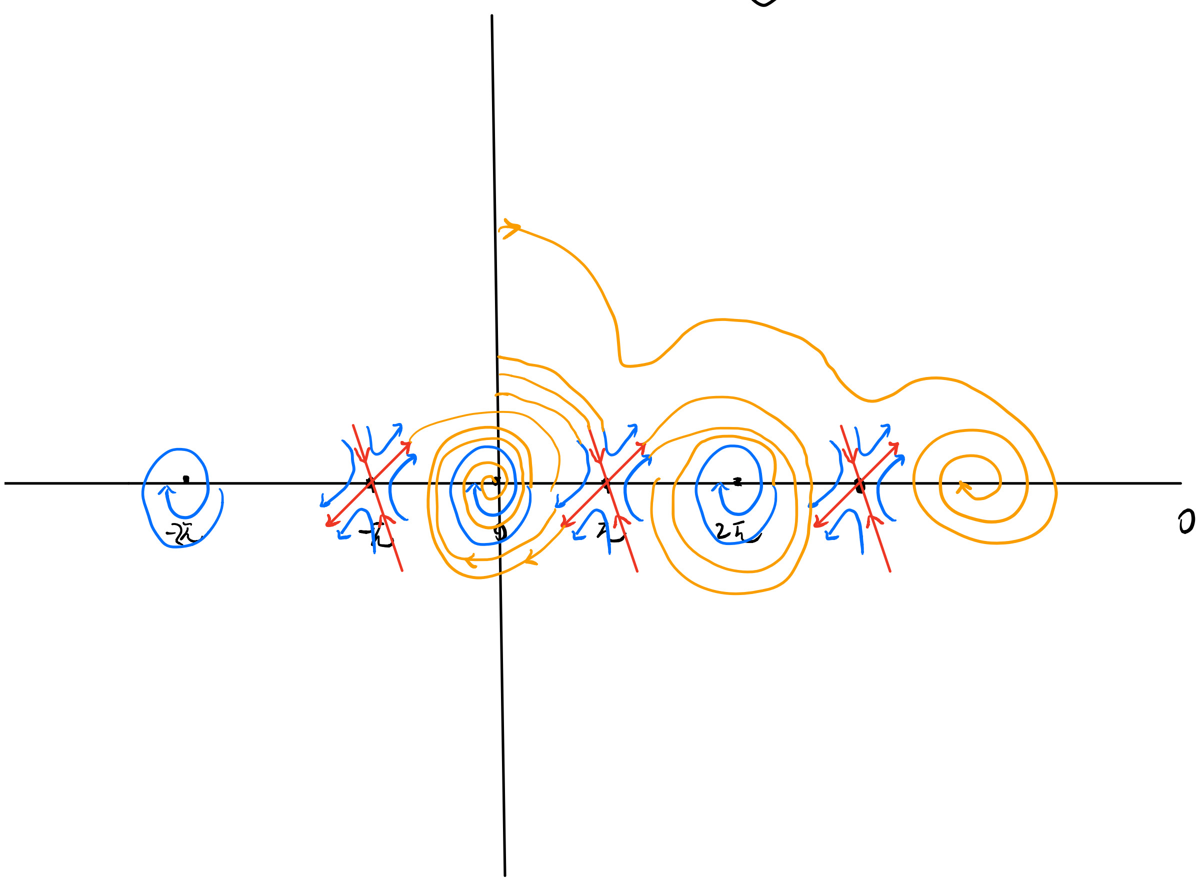
$$\vec{\alpha}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\rightarrow$ 画 $(\pi, 0)$ 图.



→ 大图

画出每个临界点周围的轨迹，并加上什么.



总结：

$$\begin{cases} \theta' = w \\ w' = -2\sin\theta - w \end{cases}$$

① → 找临界点

(x_0, y_0) ……

② → 线性化

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}。$$

③ → 画小图

④ → 画大图。