"Homogenous" system

decoupling
解耦

$$\vec{x}' = A\vec{x}$$

Coupled.
耦合意味着，右边两个方程都出现了x和y，因此，不能分别解x和y，必须同时解两个

① eigenvalues, e-vectors

② $e^{At}\vec{x}_0$. $e^{At} = \Phi(t) \cdot \Phi(0)^{-1}$

③ Decoupling ~ "解耦" ← 本讲主题.

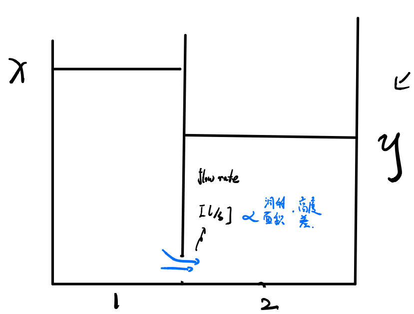
$$\begin{aligned} u &= ax + by \\ v &= cx + dy \end{aligned}$$

find u, v

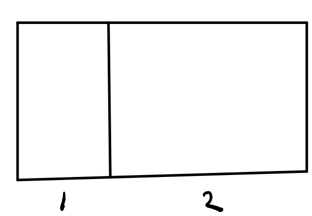
in "uv-coord":

$\rightarrow$ $\left.\begin{aligned} u' &= k_1 u \\ v' &= k_2 v \end{aligned}\right\}$ ~ decoupled

例：



← 例说.



$$\begin{cases} 1x' = c\cdot(y-x) \\ 2y' = c\cdot(x-y) \end{cases}$$ 。 take $c=2$. $\Rightarrow$ $$\begin{cases} x' = -2x + 2y \\ y' = x - y. \end{cases}$$

"decoupling method"

$$\begin{cases} u = x + 2y & \rightsquigarrow \text{容器的总水量,} \\ v = x - y & \rightsquigarrow \text{高度差,} \ \propto \text{pressure.} \end{cases}$$

New system $\to$ in terms of u, v.

$u' = x' + 2y' = 0$

$v' = x' - y' = -3x + 3y$

$$\Rightarrow \begin{cases} u' = 0 \\ v' = -3v. \end{cases}$$ $\rightsquigarrow$ decoupled the system.

soln. $\Rightarrow$ $\boxed{u = c_1 \ , \quad v = c_2 e^{-3t}}$

$\Rightarrow$ In terms of x, y.

$$\begin{cases} x = \frac{1}{3}(u + 2v) = \frac{1}{3}(c_1 + 2c_2 e^{-3t}) \\ y = \frac{1}{3}(u - v) = \frac{1}{3}(c_1 - c_2 e^{-3t}) \end{cases}$$

$$\boxed{\vec{x} = \frac{1}{3} c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{3} c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-3t}}$$

general method:

先决条件：要解耦（decouple），特征值必须都是实数（real）而且是完备的（complete, non-defective.）

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} a_1' & a_2' \\ b_1' & b_2' \end{pmatrix}}_{\hookrightarrow D:\ \text{decoupling matrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

NEED: "inverse of this"

☆☆

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}}_{\hookrightarrow D^{-1} = E = (\vec{\alpha}_1, \vec{\alpha}_2)} \begin{pmatrix} u \\ v \end{pmatrix}$$

cols are the two e-vectors.

☆☆☆：正交化：

$$\boxed{\begin{array}{ccc} \vec{\alpha}_1 & \longleftrightarrow & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \vec{\alpha}_2 & \longleftrightarrow & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ x, y & & u, v \\ \text{system} & & \text{system} \end{array}}$$

$\overset{\text{why}}{\Longrightarrow}$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\qquad\qquad\qquad\qquad\quad \vec{\alpha}_1 \quad \vec{\alpha}_2$

设 $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

1.

decoupling

似 $\vec{x}' = A\vec{x}$，看在 uv-coords 内是否能够解耦

Tips ⊙ 什么是特征值和特征向量。

eiganvalue　　eigan-vecter.

$$(A-\alpha_1 I)\vec{\alpha_1} = \vec{0}$$

$$\Downarrow$$

$$\boxed{A\vec{\alpha_1} = \alpha_1\vec{\alpha_1}}\ :$$

A：对平面做了一个线性变换。

"linear transformation of the plane"

$\vec{\alpha_1}$：对上叙的变换 → 方向保持不变，但是被拉伸了/压缩 → λ.

$$\Downarrow$$

$$AE = A[\vec{\alpha_1}\ \ \vec{\alpha_2}] = [A\vec{\alpha_1}\ \ A\vec{\alpha_2}]$$

$$= [\alpha_1\vec{\alpha_1}\ \ \alpha_2\vec{\alpha_2}]$$

$$= [\alpha_1\ \alpha_2]\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix} \rightarrow E\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}$$

总结： $\vec{x}' = A\vec{x}$ ， $\vec{u} = \binom{u}{v}$

① → $\vec{x} = E\vec{u}$

② $\xrightarrow{代入}$ $\cancel{E}\vec{u}' = AE\vec{u} = \cancel{E}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\vec{u}$

③ → $\vec{u}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\vec{u}$

即 $\begin{cases} u' = \lambda_1 u \\ v' = \lambda_2 v \end{cases}$

解 ⇒ $\begin{cases} u = c_1 e^{\lambda_1 t} \\ v = c_2 e^{\lambda_2 t} \end{cases}$

例：

decoupled:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

① → $\lambda^2 + 3\lambda = 0$. $\lambda_1 = 0$ $\lambda = -3$.

当 $\lambda = 0$. $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \vec{\alpha}_1 = 0 \Rightarrow \alpha_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

当 $\lambda = -3$ $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \vec{\alpha}_2 = 0 \Rightarrow \alpha_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

② → $E = [\alpha_1 \ \alpha_2] = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

$D = E^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{3}$

③ → $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow$ $u = \frac{1}{3}(x + 2y)$ → 总水量.

$v = \frac{1}{3}(-x + y)$ → 高度差