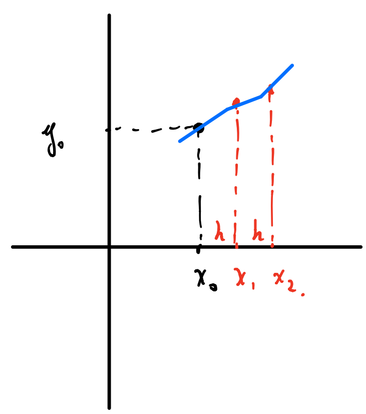
Numerical SOL'Ns

$$y' = f(x, y(x))$$

⊙ initial condition $\left\{ y(x_0) = y_0 \right.$ (初值) IVP (initial value problem)

1. base method: Euler's method "欧拉方法"

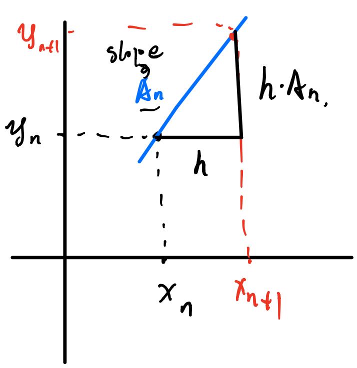
hypotenuse



h → 步长. step size

$x_1 = x_0 + h$

loop.



$$y_{n+1} - y_n = h \cdot A_n \rightarrow y_{n+1} = y_n + hA_n$$

⊡ Euler eqns

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h \cdot A_n$$

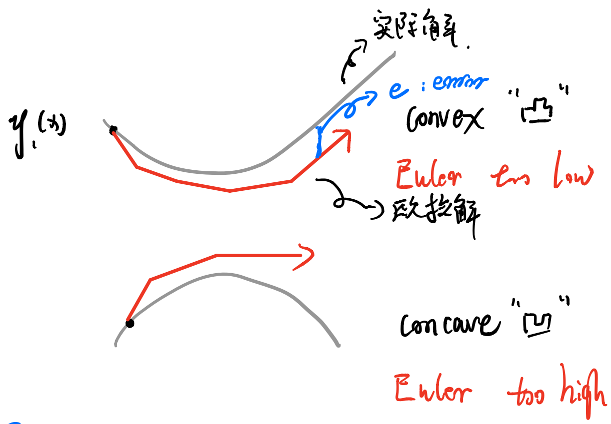
$$A_n = f(x_n, y(x_n))$$

example: $y' = x^2 - y^2$

$y(0) = 1$

$h = 0.1$

n	x_n	y_n	A_n	$h \cdot A_n$
0	0	1	-1	-.1
1	0.1	0.9	-0.8	-0.08
2	0.2	0.82	-0.63	-0.063.



"How to know about $y(x)$ is convex / concave

solution 解

"Calculus" → 微积分.

$y'' > 0 \to$ convex

$y'' < 0 \to$ concave

$y' = x^2 - y^2$. $y(0) = 1$

$\Rightarrow y'' = 2x - 2yy'$ $y'(0) = -1$

$y''(0) = 2 \cdot 0 - 2 \cdot 1 \cdot -1 = 2 > 0$

$\Rightarrow$ convex, 初始点的解为凸函数.

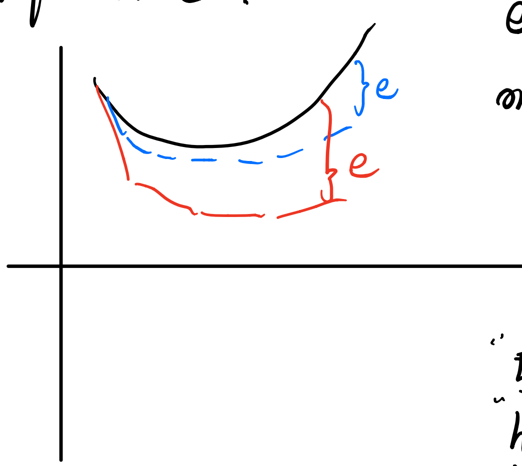
则 它对应的 Euler too low

"欧拉解过低".

结论: 欧拉法存在系统误差.

2. Better method.

2.1. smaller step size.



error e depends on step size h.

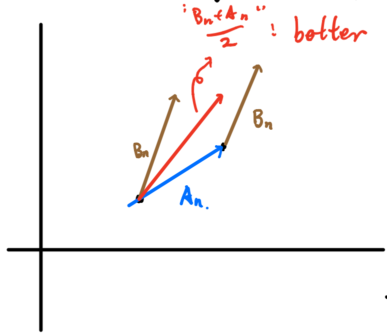
$\boxed{e \sim C \cdot h.}$

"Euler first-order method"

have the step size, have the error.

"more better"

2.2 Find a better slope than A_n.



RK2

Heun's method

$$x_{n+1} = x_n + h.$$

$$y_{n+1} = y_n + h\cdot\left(\frac{A_n+B_n}{2}\right)$$

"temporary"

Euler. 中间值. 《差的 y_{n+1}》

$$\rightarrow \tilde{y}_{n+1} = y_n + h\cdot A_n.$$

$$B_n = f(x_{n+1}, \tilde{y}_{n+1})$$

$\boxed{e \sim C_2 h^2} \rightarrow$ second-order method.

二阶误差方法.

好处：降低步长，以使误差 e 只有原来的 $\frac{1}{4}$.

$h \rightarrow \frac{1}{2}h$.

RK4: "$\frac{A_n + 2B + 2C_n + D_n}{6}$" super-slope

圆: some pitfalls (陷阱) of numericals computation in general

数值计算中的常见陷阱.

#1: you'll find.

#2: $y' = y^2$.

$y' = y^2$

separate variables.

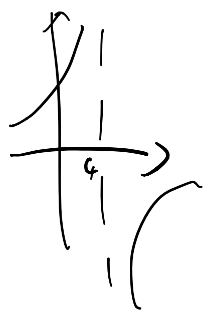
$$\frac{dy}{dx} = y^2$$

$$\frac{dy}{y^2} = dx$$

solusion: $y = \frac{1}{c-x}$

singilarity

奇点.



数值法.

got lost in eternity, in infinity

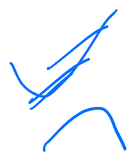
$y' = y^2 - x^2$

n	x_n	y_n	m_n	$m_n h$
0	0	-1	1	0.5
1	0.5	-0.5	0	0

$y''_0 = 2y \cdot y' - 2x$

$= 2 \cdot -1 \cdot 1 - 0$

$= -2$



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