$$\boxed{\vec{x}\,' = A\vec{x}}$$

① 基础矩阵知识

fundamental mx. of A (基础矩阵.)

$$\mathbf{X} = [\vec{x}_1 \ \ \vec{x}_2]$$, $\vec{x}_1, \vec{x}_2$ 线性无关.

→ Basic properties. of $\mathbf{X}$.

① $|\mathbf{X}| \neq 0$, for any t.

② $\mathbf{X}' = A\mathbf{X}$. 带到均为 system 的 soln.

② 写出 system 的通解

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

$$\Rightarrow \underbrace{\vec{x} = \mathbf{X}\cdot\vec{c}}_{\text{gen soln.}} = [\vec{x}_1 \ \ \vec{x}_2]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

→ what do all fundamental mx. look like.

$$[\mathbf{X}\vec{c}_1 \ \ \mathbf{X}\vec{c}_2] = \mathbf{X}[\vec{c}_1 \ \vec{c}_2] \quad C$$

$$= \mathbf{X}\cdot C$$, most general FM. 以知.

② → 进入正题：

formal → method

特例，工程，给予明确的范围.

思想！ 流程！

• → 1×1 case：

$$x' = ax$$

soln： $x = c \cdot e^{at}$

如何推广到 $n \times n$？

“无穷级数的指数函数定义”

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots + \cdots$$

$$\Downarrow$$

$$\frac{de^{at}}{dt} = 0 + a + a^2t + \frac{a^3t^2}{2!} + \cdots$$

$$= a \cdot e^{at}$$

• a Fund Mx for $\vec{x}' = A\vec{x}$

$$e^{At} := \underbrace{I}_{\text{单位矩阵}} + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!}$$

② 证明 e^{At} 为 fundamental Mx. of System.

①satisfy $X' = AX$，($X = e^{At}$)

即.

de^{At}

$$\overline{dt} = 0 + A + A^2 \cdot t \cdots -\cdot$$

$$= A \cdot e^{At} \quad \checkmark \quad \text{Yes.}$$

② $|A| \neq 0$

$$|e^{A(0)}| = 1 \neq 0$$

例: $\begin{matrix} x' = y \\ y' = x \end{matrix} \Rightarrow \vec{x}\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{t^3}{3!}$$

$$= \begin{bmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots & t + \frac{t^3}{3!} + \cdots \\ t + \frac{t^3}{3!} + \cdots & 1 + \frac{t^2}{2!} + \cdots \end{bmatrix}$$

$$\rightarrow e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^t + e^{-t}}{2} & \frac{e^t - e^{-t}}{2} \\ \frac{e^t - e^{-t}}{2} & \frac{e^t + e^{-t}}{2} \end{bmatrix}$$

→ "init. value" $\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$, find $\vec{x}(t)$

→ gen soln: $\vec{x} = e^{At} \cdot \vec{C}$

$$\Rightarrow \vec{x}(0) = e^{A0} \cdot \vec{c}$$

$$\Rightarrow \vec{x}_0 = I_2 \vec{c}$$

★★: 即. $\vec{x} = e^{At} \cdot \vec{x}_0$

$$e^{A+B} \neq e^A \cdot e^B$$

True is special case. "$AB = BA$", 很少见,

三种情况

① $A = c \cdot I = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$

② $B = -A$ $\Rightarrow e^{A-A} = e^A \cdot e^{-A}$

即 $I = e^A \cdot e^{-A}$

即 $\boxed{(e^A)^{-1} = e^{-A}}$

③ $B = A^{-1}$

如何计算 e^{At}

① 无穷级数 (太麻烦)

use exp law. ② $\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$

1). FM still

other ③. $\mathbb{X} \cdot \mathbb{X}(0)^{-1}$

2). value at 0. $\mathbb{X}(0)\mathbb{X}(0)^{-1} = I$

也

则 $\boxed{e^{At} = \mathbb{X}\,\mathbb{X}(0)^{-1}}$

即. $\underbrace{\bar{\Phi}(t)}_{x(t)} = \Phi(t) \cdot \Phi(0)^{-1} \cdot \underbrace{\bar{\Phi}(0)}_{x_0}$.

例 1. $x' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x$, $x(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$e^{At} = e^{Bt} e^{Ct}$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} + \begin{bmatrix} 0 & e^{t} \\ e^{t} & 0 \end{bmatrix}$

Example A: $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$ $e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

Example 3B. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show: $e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

What's the point of the exponential matrix? The answer is given by the theorem below, which says that the exponential matrix provides a royal road to the solution of a square system with constant coefficients: no eigenvectors, no eigenvalues, you just write down the answer!

Theorem 3 *Let A be a square constant matrix. Then*

(1) *(a) $e^{At} = \widetilde{\Phi}_0(t)$, the normalized fundamental matrix at 0;*

(2) *(b) the unique solution to the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x} = e^{At}\mathbf{x}_0$.*

Example 3C. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Solve $\mathbf{x}' = A\,\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, using e^{At}.

Solution. We set $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then (7) is satisfied, and

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

by (8) and Examples 3A and 3B. Therefore, by Theorem 3 (2), we get

$$\mathbf{x} = e^{At}\mathbf{x}_0 = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{2t} \begin{pmatrix} 1+2t \\ 2 \end{pmatrix}.$$

$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{2t} ($