Find y_p (particular soln).

$$y'' + Ay' + By = f(x)$$

通解 $y = y_p + c_1 y_1 + c_2 y_2$.

☆ 若 $f(x) = e^{ax}$ (a often <0) / $\begin{cases} \sin wx \\ \cos wx \end{cases}$ / $e^{ax}\begin{cases} \sin wx \\ \cos wx \end{cases}$

↳ $e^{(a+iw)x} \Rightarrow \boxed{e^{\alpha x}}$ input

$$y'' + Ay' + By = e^{\alpha x}$$

$$(D^2 + AD + B)y = f(x)$$

$$p(D)y = f(x)$$

$$\boxed{p(D)y = e^{\alpha x}}$$

Thm.

$$\boxed{p(D)e^{\alpha x} = p(\alpha)e^{\alpha x}}$$

proof

$(D'' + D'A + B)e^{\alpha x}$

$= D''e^{\alpha x} + D'e^{\alpha x} \cdot A + B \cdot e^{\alpha x}$

$= \alpha^2 \cdot e^{\alpha x} + \alpha \cdot e^{\alpha x} \cdot A + B \cdot e^{\alpha x}$

$= (\alpha^2 + A\alpha + B)e^{\alpha x} = p(\alpha) \cdot e^{\alpha x}$

f(x)为指数函数

Exponential input thm: 指数输入理论 → for $p(D)y = e^{\alpha x}$ 解: $y_p = \dfrac{e^{\alpha x}}{p(\alpha)}$

Proof: $p(D)y_p = e^{\alpha x}$?

$$p(D) \cdot \frac{e^{\alpha x}}{p(\alpha)} = \frac{p(\alpha)e^{\alpha x}}{p(\alpha)} = e^{\alpha x} \quad (p(\alpha) \neq 0)$$

but what 对 $p(\alpha) = 0$ 呢?

example!

$$y'' - y' + 2y = 6e^{-x}\sin x$$

①. 找通解. $r^2 - r + 2 = 0$ r

$y_c = \quad \cdots$

② 找特解

$6e^{-x}\sin x$ 为 $6e^{-x+ix}$ 的虚部 $\tilde{y}$

$$(D^2 - D + 2)\tilde{y} = 6e^{(-1+i)x}$$

$$\tilde{y}_p = \frac{6e^{(-1+i)x}}{(-1+i)^2 - (-1+i) + 2} = \frac{6e^{(-1+i)x}}{3 - 3i} = \frac{6}{3}\,\frac{(1+i)}{2}\,e^{-x}(\cos x + i\sin x)$$

image part = ?

$$y_p = Im(\tilde{y}_p) = \frac{5}{3}e^{-x}(\cos x + \sin x) = \frac{5}{3}e^{-x}\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$$

If $p(\alpha)=0$. [$\alpha \to a$ still complex. 复数]

law: $\boxed{\text{exponential-shift rule} \quad p(D)e^{ax}u(x) = e^{ax}p(D+a)u(x)}$
"指数位移法则"

Proof: "let's keep the u general, Suppose we make D simple" 的数学归纳法

①. $p(D)=D$

$$De^{ax}u = e^{ax}\cdot Du + ae^{ax}\cdot u = e^{ax}(Du+au) = e^{ax}(D+a)u.$$

② $p(D)=D^2$

$$D^2e^{ax}u = D(De^{ax}u) = D(e^{ax}(D+a)u) = e^{ax}(D+a)(D+a)u$$

$$= e^{ax}(D+a)^2u$$

③ $(D^2+AD+B)y = e^{ax}$. $(p(a)=0)$

→ 若 a 是 $p(D)$ 的单根, $p'(a)\neq 0$

$$\boxed{☆ \Rightarrow y_p = \frac{xe^{ax}}{p'(a)}}$$

→ 若 a 是重根. $p''(a)\neq 0$

$$\boxed{y_p = \frac{x^2e^{ax}}{p''(a)}}$$

Proof

①. 单根: 假设 $p(D) = (D-b)(D-a) \quad (b \neq a)$

$$p'(D) = (D-a) + (D-b)$$

$$p'(a) = a-b$$

$$p(D)\frac{e^{ax}\cdot x}{p'(a)} \overset{?}{=} e^{ax} \qquad \left[\, y_p = \frac{e^{ax}\cdot x}{p'(a)} \,\right]$$

$$e^{ax}(D+a-b)D\frac{x}{p'(a)} = \frac{e^{ax}\cdot(a-b)}{p'(a)} = e^{ax}\cdot\frac{a-b}{a-b} = e^{ax}$$

example: $y'' - 3y' + 2y = e^{x}$ "1是 D^2-3D+2 的一个单根."

$$p'(D) = 2D-3$$

$$p'(1) = -1$$

$$y_p = \frac{xe^{x}}{-1}$$

$$= -xe^{x}$$