SUBSTITUTIONS. “变量代换法”

1. Scaling “尺度变换”

$$y' = f(x, y) \xrightarrow[\boxed{y_1 = \frac{y}{b}}]{\boxed{x_1 = \frac{x}{a}}}$$

① change units “改变单位”

② make the variables dimensionless. → “无量纲” 不带单位的 纯数字.

③ reduce or simplify the constants.
“减少常数”

example:

Big Temperture

$$\frac{dT}{dt} = k(M^4 - T^4)$$

T: internal temperature

M: external temperature
constant

① $T_1 = \frac{T}{M}$. → dimensionless $\Rightarrow T = T_1 M$.

⇓

$$M\frac{dT_1}{dt} = k \cdot M^4(1 - T_1^4) \Rightarrow \frac{dT_1}{dt} = k_1(1 - T_1^4)$$

k_1 ↓ kM^3 → “lumping constants”

1.1 DIRECT $\rightsquigarrow$ New Variable = old Vars = $T_1 = \frac{T}{m}$.

example: $y' = p(x)\cdot y + q(x)\cdot y^n \quad (n \neq 0)$ $\rightsquigarrow$ "Bernoulli equation" 伯努力方程.

"Everything multiplied by y or a power of y, two terms"
这两项必须包含y或者y的幂,

$$\Downarrow$$

$$\frac{y'}{y^n} = p(x)\frac{1}{y^{n-1}} + q(x)$$

$$\Downarrow \; v = \frac{1}{y^{n-1}} = y^{1-n}, \quad v' = (1-n)\frac{1}{y^n}\cdot y'$$

$$\boxed{\frac{v'}{1-n} = p(x)\cdot v + q(x)} \rightsquigarrow \text{"linear"}$$

★: "Learn methods, not final formulas"

$$y' = \frac{y}{x} - y^2$$

$$\downarrow$$

①. $\frac{y'}{y^2} = \frac{1}{x}\cdot\frac{1}{y} - 1$

$\downarrow v = \frac{1}{y},\ v' = \frac{-1}{y^2}\cdot y'$

② $-v' = \frac{1}{x}\cdot v - 1 \Rightarrow \boxed{v' + \frac{v}{x} = 1} \rightsquigarrow$ "linear"

$\downarrow e^{\int \frac{1}{x}dx} = x$

③. $xv' + v = x \Leftrightarrow (xv)' = x \Rightarrow xv = \frac{1}{2}x^2 + C \Rightarrow v = \frac{1}{2}x + \frac{C}{x}$

$\downarrow v = \frac{1}{y}$

④ $\frac{1}{y} = \frac{x}{2} + \frac{C}{x} \Rightarrow \boxed{y = \cancel{\frac{2}{x}} \cdot \frac{2x}{x^2 + \underbrace{2C}_{C_1}}} \rightsquigarrow$ solution.

1.2 INVERSE

old Var = New Var. = $T = M \cdot T_1$

"Homogenerous" → 齐次式

$$y' = F\left(\frac{y}{x}\right)$$

$$\begin{cases} y' = \dfrac{x^2 y}{x^3+y^3} = \dfrac{y/x}{1+\left(\frac{y}{x}\right)^3} \\ xy' = \sqrt{x^2+y^2} \Rightarrow y' = \sqrt{1+\left(\frac{y}{x}\right)^2} \end{cases}$$

具有不变性："invariant under the operation zoom"

Invariant 缩放变换不变.

↳ $x = ax_1$, $y = ay_1$, $\dfrac{dy_1}{dx_1} = F\left(\dfrac{y_1}{x_1}\right)$

solve it:

$$y' = F\left(\frac{y}{x}\right)$$

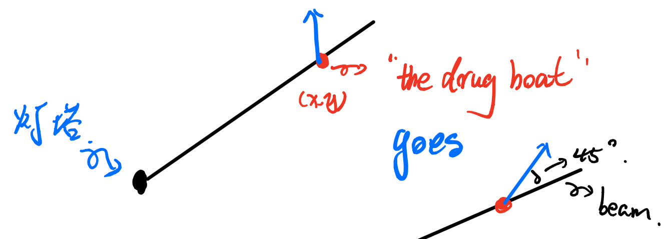
$z = \frac{y}{x}$, → $y = zx$ → $y' = z'x + z$

$$z'x + z = F(z) \rightarrow x \cdot \frac{dz}{dx} = F(z) - z \rightarrow \frac{dz}{F(z)-z} = \frac{dx}{x}$$

solve it ✓.

☆: differential equations model.

the escape strategy.



slope constants 45° to beam.

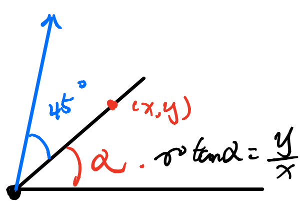
Q: what's the boats path?

☆ "PATH => Curve"

a function.

解:

设 $y = y(x)$ is unknown graph.



$$y' = \tan(\alpha + 45^\circ) = \frac{\tan\alpha + \tan 45^\circ}{1 - \tan\alpha \cdot \tan 45^\circ}$$

$y' =$ slope

$$\Rightarrow y' = \frac{y/x + 1}{1 - y/x}$$ "homogeneous" ODE

$z = y/x \to y' = z'x + z$

$$z'x + z = \frac{z+1}{1-z} \to \cancel{(zx)' = \frac{z+1}{1-z}}$$

$$\frac{z+1 - z + z^2}{1-z}$$

$$\frac{dz}{dx} \cdot x = \frac{z+1}{1-z} - z = \frac{1+z^2}{1-z}$$

$$dz \cdot \frac{1-z}{1+z^2} = \frac{dx}{x}$$

$$\cancel{zx = \int \frac{z+1}{1-z} dz} =$$

$$\tan^{-1} z - \frac{\ln(1+z^2)}{2} = \ln x + C.$$

$$\downarrow$$

$$\tan^{-1} z = \ln\sqrt{1+z^2} + \ln x + C$$

$$\downarrow$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln x\sqrt{1+\left(\frac{y}{x}\right)^2} + C = \ln\sqrt{x^2+y^2} + C.$$

$\downarrow$ polar coordinates "极坐标"

$$\theta = \ln r + C \rightarrow e^{\theta} = r \cdot C$$

$$\downarrow$$

$$r = C_1 \cdot e^{\theta}$$