2^{nd} linear homogeneous ODE.

← theoretical

$$y'' + p(x)\,y' + q(x)\,y = 0$$

→ 所有的解是什么?

linear in y, y', y''

sol'n method:

$$\begin{cases} y_2 \neq c\,y_1 \\ y_1 \neq c'\,y_2 \end{cases}$$

y_1, y_2 ~ independent: "y_2 不是 y_1 乘以一个常数"

所有的解是什么? Then, all sol'ns?

$\begin{bmatrix} c_1y_1 \\ c_2y_2 \end{bmatrix}$ → "两个解的常系数线性组合" → "但是,为什么?" "Why"?

ans: $y = c_1y_1 + c_2y_2$ WHY?

Two statement:

Q1 ①: Why are $c_1y_1 + c_2y_2$ solutions? ← "answer it elegantly"

②. Why all the sol'n's? 为什么这些解就是所有的解.

解决 Q1. → Superposition. "叠加法"

superposition principle

if y_1, y_2 soln's to lin homogen ODE.

$\Rightarrow c_1 y_1 + c_2 y_2$ is a soln.

Proof

$y'' + py' + qy = 0$

$\rightarrow D^2 y + pDy + qy = 0.$

简写为

$(D^2 + pD + q)y$ 不是乘法.

★ "linear operator"

$L = D^2 + pD + q$

$\Rightarrow Ly = 0.$

$u(x) \rightarrow \boxed{L} \rightarrow v(x)$

★ linear combination

law ① $L(u_1 + u_2) = L(u_1) + L(u_2)$

law ② $L(cu) = cL(u)$ (constant, function)

ex: ① D is linear.

$(u_1 + u_2)' = u_1' + u_2'$

$(cu)' = cu'$

② $L = D^2 + pD + q.$

$L(y_1 + y_2) = (y_1 + y_2)'' + p(y_1 + y_2)' + q \cdot c(y_1 + y_2)$

$= \cdots + py_1' + py_2' + qcy_1 + qcy_2$

$= Ly_1 + Ly_2$

$Lcy = (cy)'' + p \cdot (cy)' + q \cdot cy$

$= c \cdot y'' + pc \cdot y' + cqy$

$= c \cdot Ly$

Proof of superposition:

ODE: $Ly = 0$

$$L(c_1 y_1 + c_2 y_2) = L(c_1 y_1) + L(c_2 y_2)$$

$$= c_1 \underbrace{L(y_1)}_{=0} + c_2 \underbrace{L(y_2)}_{=0}$$

L is linear operator

Solving the initial value problem

→ (fit init. values)

$\{c_1 y_1 + c_2 y_2\}$ is enough to satisfy any initial value.

Proof:

$y(x_0) = a$

$y'(x_0) = b$

代入 x_0

$$y = c_1 y_1 + c_2 y_2 = c_1 y_1(x_0) + c_2 y_2(x_0) = a$$

$$y' = c_1 y_1' + c_2 y_2' = c_1 y_1'(x_0) + c_2 y_2'(x_0) = b$$

c_1, c_2 是未知数/变量 "variables" → to find $y = a$, $y' = b$

→ 如果，矩阵可解，(solvable for c_1, c_2) → $\boxed{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x_0} \neq 0 \text{ (值不为0)}}$

朗斯基行列式 "Wronskian"

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

(func of x) $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

显然. if $y_2 = c y_1$, $W(y_1, y_2) = 0$.

Theo. if y_1, y_2 sol'n's to ODE

要么 $W(y_1, y_2) \equiv 0$ (for all x) 或者 $W(y_1, y_2)$ is never 0. (Problem 5 Part II)

2^{nd}

ODE 的通解:

$$\{c_1 y_1 + c_2 y_2\} = \{c_1' u_1 + c_2' u_2\}$$

其中 u_1, u_2 相互线性独立.

$$u_1 = \bar{c}_1 y_1 + \bar{c}_2 y_2$$

$$u_2 = \bar{\bar{c}}_1 y_1 + \bar{\bar{c}}_2 y_2.$$

找正交解 or normalized. sol'n's

at 0 or x_0

$$Y_1 : \begin{cases} Y_1(0) = 1 \\ Y_1'(0) = 0 \end{cases}$$

$$Y_2 : \begin{cases} Y_1(0) = 0 \\ Y_1'(0) = 1 \end{cases}$$

ex: $y'' + y = 0$

$y_1 = \cos x = Y_1$

$y_2 = \sin x = Y_2$

$y'' - y = 0$

$y_1 = e^x$

$y_2 = e^{-x}$

$y = C_1 e^x + C_2 e^{-x}$

$y' = C_1 e^x - C_2 e^{-x}$

$Y_1 \begin{cases} C_1 + C_2 = 1 \\ C_1 - C_2 = 0 \end{cases} \Rightarrow C_1 = C_2 = \frac{1}{2}$

$Y_2 \begin{cases} C_1 + C_2 = 0 \\ C_1 - C_2 = 1 \end{cases} \Rightarrow C_1 = \frac{1}{2}, \ C_2 = -\frac{1}{2}$

即:

$\begin{cases} Y_1 = \dfrac{e^x + e^{-x}}{2} \\ Y_2 = \dfrac{e^x - e^{-x}}{2} \end{cases}$

通解中的最优解.

好在哪 why?

Y_1, Y_2 在 $x=0$ 处 正交化.

soln to IVP : ODE + $\begin{cases} y(0) = a = y_0 \\ y'(0) = b = y_0' \end{cases}$

初值问题.

$y \Rightarrow \boxed{y_0 Y_1 + y_0' Y_2}$

立刻得到初值问题的解.

解决Q2前所需理论知识:

① 存在和唯一性定理. Ex + Un Thorm.

$y'' + py' + qy = 0$. p, q 对所有x都连续.

Then, 有且仅有一个满足给定的初值条件, $\begin{cases} y(0) = A \\ y'(0) = B \end{cases}$

Q2：我们要ODE的所有解.

声明：claim：$\{C_1Y_1 + C_2Y_2\}$ are all solutions.

Proof：取一个解 $u(x)$，$\begin{cases} u(0) = u_0 \\ u'(0) = u_0' \end{cases}$

then，$u_0Y_1 + u_0'Y_2$ 满足同一个初始条件.

则：$u = u_0Y_1 + u_0'Y_2$.