FOURIER SERIES.

$$y'' + ay' + by = f(t)$$

input

soln: $y(t) \to$ response.

当 $f(t) =$ exp'l , sin t , cos t.

傅里叶级数的作用 "从这些特殊的 $f(t)$ 变得普遍"

"They are not as special as they look."

periodic $= 2\pi$.

只有周期性的 $\to$ $$\boxed{f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt}$$

Input　　Response

$$b_n \sin nt \rightsquigarrow b_n y_n^{(s)}(t)$$

$$a_n \cos nt \rightsquigarrow a_n y_n^{(c)}(t)$$

$+$

$$f(t) \rightsquigarrow \sum a_n y_n^{(c)}(t) + b_n y_n^{(s)}(t) + c_1.$$

叠加原理

ODE is linear

☆ calculate the Fourier series

given $f(t)$ periodic. period $= 2\pi$, find F-series

"正交关系" ⇝ orthogonality relations

$u(t)$, $v(t)$ fns on $\mathbb{R}$. (say 2π is a period.)

Orthogonal on $[-\pi, \pi]$ if $\boxed{\int_{-\pi}^{\pi} u(t)\, v(t)\, dt = 0}$

Thm:

collection of functions $\begin{cases} \sin nt & n = 1, \cdots, \infty \\ \cos mt & m = 0, \cdots, \infty \end{cases}$, 任意两个不同的元素在$[-\pi, \pi]$这个区间内正交

若相同 ⋯ 不正交, $\int_{-\pi}^{\pi} u(t)\, v(t)\, dt \neq 0$.

$$\begin{cases} \int_{-\pi}^{\pi} \sin(2t) \cos(60t)\, dt = 0 \\ \begin{cases} \int_{-\pi}^{\pi} \sin^2(nt)\, dt = \pi \\ \int_{-\pi}^{\pi} \cos^2(nt)\, dt = \pi. \end{cases} \end{cases}$$

Proof:

① Trig identities "三角恒等式"

② Complex expls. "复指数"

☆☆ ③. Use ODE

假设 $m \neq n$. satisfy $\sin nt, \cos nt$

$u'' + n^2 u = 0$. $\Rightarrow u_n'' = -n^2 u_n$

U_n, V_m 为任意两个函数.

$$\int_{-\pi}^{\pi} u_n'' v_m \, dt \overset{\text{分部积分}}{=} \underbrace{u_n' v_m \Big]_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} u_n' v_m' \, dt$$

$$\Rightarrow \int_{-\pi}^{\pi} u_n'' v_m \, dt = -n^2 \int_{-\pi}^{\pi} u_n v_m \, dt$$

∵ $\int_{-\pi}^{\pi} u_n'' v_m \, dt$ 对于 u和v是对称的吗?

$\int_{-\pi}^{\pi} u_n'' v_m \, dt = -\int_{-\pi}^{\pi} v_m' u_n' \, dt$ 是对称

$\int_{-\pi}^{\pi} u_n'' v_m \, dt = -n^2 \int_{-\pi}^{\pi} u_n v_m \, dt$ 不对称.

既对称又不对称 的 唯一可能.

$$\int_{-\pi}^{\pi} u_n v_m \, dt = 0. \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} u_n'' v_m \, dt = -\int_{-\pi}^{\pi} v_m' u_n' \, dt$$

$$\int_{-\pi}^{\pi} u_m'' v_n \, dt = -\int_{-\pi}^{\pi} v_n' u_m' \, dt$$

$\because \int_{-\pi}^{\pi} v_m' u_n' \, dt$ 为对称函数 $= \int_{-\pi}^{\pi} v_n' u_m' \, dt$

则 $\int_{-\pi}^{\pi} u_n'' v_m \, dt = \int_{-\pi}^{\pi} u_m'' v_n \, dt$

$\Rightarrow -n^2 \int_{-\pi}^{\pi} u_n v_m \, dt = -m^2 \int_{-\pi}^{\pi} u_m v_n \, dt$

$\because n \neq m$

$\therefore \int_{-\pi}^{\pi} u_n v_m \, dt = \int_{-\pi}^{\pi} u_m v_n \, dt = 0.$

Prom:

$$f(t) = C_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

given $f(t)$, 2π is period. find a_n, b_n?

$$f(t) = \cdots a_n \cos nt \cdots + a_k \cos kt$$

$\times \cos nt$ — We want — some other term.

$$f(t) \cdot \cos nt = \cdots a_n \cdot \cos^2 nt + \cdots a_k \cdot \cos kt \cdot \cos nt$$

$$\int_{-\pi}^{\pi} f(t) \cos nt \, dt = \cdots \underbrace{\int_{-\pi}^{\pi} a_n \cos^2 nt}_{= a_n \pi} \cdots + \underbrace{\int_{-\pi}^{\pi} a_k \cos kt \cos nt \, dt}_{= 0}$$

$$\Rightarrow \boxed{a_n = \frac{\int_{-\pi}^{\pi} f(t) \cos nt \, dt}{\pi}} \qquad n = 1, 2, 3 \cdots$$

$$b_n = \frac{\int_{-\pi}^{\pi} f(t) \sin nt \, dt}{\pi} \qquad n = 1, 2, 3 \cdots$$

$$f(t) = \underline{C_0} + \cdots a_n \cos nt \cdots$$

$$\int_{-\pi}^{\pi} f(t) \, dt = 2\pi C_0 + \cdots \underbrace{\int_{-\pi}^{\pi} a_n \cos nt}_{= 0} \cdots$$

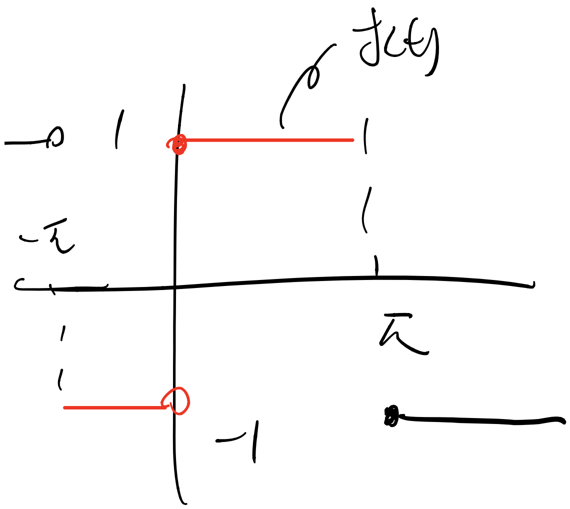
$$\Rightarrow c_0 = \frac{\int_{\pi}^{\pi} f(t)\,dt}{2\pi}.$$

☆☆☆

$$f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \cos nt$$

其中

$$\begin{cases} \boxed{a_n = \dfrac{\int_{\pi}^{\pi} f(t) \cos nt \, dt}{\pi}.} \\[2ex] b_n = \dfrac{\int_{\pi}^{\pi} f(t) \sin nt \, dt}{\pi} \\[2ex] c_0 = \dfrac{\int_{\pi}^{\pi} f(t)\,dt}{2\pi} = \dfrac{a_0}{2} \end{cases}$$



$$a_n = 0.$$

$$b_n = -1\int_{-\pi}^{0} \sin nt\,dt + \int_{0}^{\pi} \sin nt\,dt$$

$$= \frac{1-\cos n\pi}{n}\pi + \frac{1-\cos n\pi}{n}\pi$$

$$= \frac{2}{n}(1-\cos n\pi)\pi$$

$$= \frac{2}{n} \cdot \begin{cases} 2 & n=\text{奇} \\ 0 & n=\text{偶} \end{cases}$$