行星运动的求解

先证一个引理:

圆锥曲线在顶点处的曲率半径等于该顶点与（最近的）焦点的距离.

证明:

在极坐标系下，设圆锥曲线的方程为 $r = \dfrac{r_0}{1+\varepsilon\cos\theta}$.

顶点坐标为 $\left(\dfrac{r_0}{1+\varepsilon}, 0\right)$.

任意一点曲率

$$\kappa = \lim_{\Delta s\to 0}\frac{\Delta\phi}{\Delta s} = \frac{\mathrm{d}\theta}{\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta} = \frac{1}{\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}}$$

又 $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{r_0\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}$.

$\therefore$ 在顶点处 $\kappa = \dfrac{1}{\sqrt{\left(\frac{r_0}{1+\varepsilon}\right)^2+0}} = \dfrac{1+\varepsilon}{r_0}$.

$\therefore$ 曲率半径 $\rho = \dfrac{r_0}{1+\varepsilon}$. □

设恒星的质量为 M，万有引力常数为 G.

轨道的极坐标方程可表示为 $r = \dfrac{r_0}{1+\varepsilon\cos(\theta-\theta_0)}$，其中 ε 为轨道的离心率.

令 $\cos(\theta-\theta_0)=1$ 得，近日点与恒星的距离 $r_m = \dfrac{r_0}{1+\varepsilon}$.

近日点速率 v_m 满足 $\dfrac{v_m^2}{r_m} = \dfrac{GM}{r_m^2}$.

$\therefore v_m = \sqrt{\dfrac{GM}{r_m}} = \sqrt{\dfrac{(1+\varepsilon)GM}{r_0}}$

设 $h = v_m r_m = \sqrt{\dfrac{GMr_0}{1+\varepsilon}}$，由角动量守恒，行星在轨道上运动时 $h = vr$ 不变.

单位时间行星与恒星连线扫过的面积

$$\mathrm{d}S = \frac{1}{2}h\mathrm{d}t = \frac{1}{2}\sqrt{\frac{GMr_0}{1+\varepsilon}}\mathrm{d}t = \frac{1}{2}r^2\mathrm{d}\theta$$

若以行星位于近日点的时刻为计时起点，则行星运动到 (r,θ) 处的时刻

$$t = \sqrt{\frac{1+\varepsilon}{GMr_0}}\int_{\theta_0}^{\theta} r^2\mathrm{d}\theta = r_0^2\sqrt{\frac{1+\varepsilon}{GMr_0}}\int_0^{\theta-\theta_0}\frac{\mathrm{d}x}{(1+\varepsilon\cos x)^2}$$

经计算:

$$\int_0^\theta \frac{\mathrm{d}x}{(1+\varepsilon\cos x)^2} = \begin{cases} \theta,\ \varepsilon = 0 \\ -\dfrac{\varepsilon\sin\theta}{(1-\varepsilon^2)(\varepsilon\cos\theta+1)} + \dfrac{2\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\theta}{2}\right)}{(1-\varepsilon^2)^{\frac{3}{2}}} + \left\lfloor\dfrac{x+\pi}{2\pi}\right\rfloor\dfrac{2\pi}{(1-\varepsilon^2)^{\frac{3}{2}}}, \\ \hfill 0<\varepsilon<1 \\ \dfrac{1}{2}\tan\dfrac{\theta}{2} + \dfrac{1}{6}\tan^3\dfrac{\theta}{2},\ \varepsilon = 1 \\ \dfrac{\varepsilon\sin\theta}{(\varepsilon^2-1)(\varepsilon\cos\theta+1)} - \dfrac{2\,\mathrm{arctanh}\left(\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}\tan\frac{\theta}{2}\right)}{(\varepsilon^2-1)^{\frac{3}{2}}},\ \varepsilon > 1 \end{cases}$$

这里，我们省略了对可去间断点的函数值的补充定义.

于是已知 θ、r、t 中的任意一个，可在多项式时间内求出另外两者的有任意数量有效数字的近似值.