行星运动的求解

先证一个引理:

圆锥曲线 $r=\dfrac{r_0}{1+\varepsilon\cos\theta}$ 在顶点处的曲率半径等于 r_0.

证明:

顶点坐标为 $\left(\dfrac{r_0}{1+\varepsilon},0\right)$.

设 $x=r\cos\theta$, $y=r\sin\theta$, 则

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}\,(r\sin\theta)}{\mathrm{d}\,(r\cos\theta)}=\frac{\mathrm{d}\dfrac{r_0\sin\theta}{1+\varepsilon\cos\theta}}{\mathrm{d}\dfrac{r_0\cos\theta}{1+\varepsilon\cos\theta}}=\frac{\cos\theta(1+\varepsilon\cos\theta)+\varepsilon\sin^2\theta}{-\sin\theta(1+\varepsilon\cos\theta)+\varepsilon\sin\theta\cos\theta}=-\frac{\cos\theta+\varepsilon}{\sin\theta}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-\mathrm{d}\dfrac{\cos\theta+\varepsilon}{\sin\theta}}{\mathrm{d}\dfrac{r_0\cos\theta}{1+\varepsilon\cos\theta}}=-\frac{1}{r_0}\cdot\frac{\mathrm{d}\dfrac{\cos\theta+\varepsilon}{\sin\theta}}{\mathrm{d}\dfrac{\cos\theta}{1+\varepsilon\cos\theta}}=-\frac{1}{r_0}\cdot\frac{\dfrac{-1-\varepsilon\cos\theta}{\sin^2\theta}}{\dfrac{-\sin\theta}{(1+\varepsilon\cos\theta)^2}}=-\frac{1}{r_0}\cdot\frac{(1+\varepsilon\cos\theta)^3}{\sin^3\theta}$$

任意一点曲率

$$\kappa=\frac{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}}=-\frac{1}{r_0}\cdot\frac{\left(\dfrac{1+\varepsilon\cos\theta}{\sin\theta}\right)^3}{\left[1+\left(\dfrac{\cos\theta+\varepsilon}{\sin\theta}\right)^2\right]^{\frac{3}{2}}}=-\frac{1}{r_0}\cdot\frac{(1+\varepsilon\cos\theta)^3}{\left[\sin^2\theta+(\cos\theta+\varepsilon)^2\right]^{\frac{3}{2}}}$$

$\therefore$ 在顶点处 $\kappa=-\dfrac{1}{r_0}\cdot\dfrac{(1+\varepsilon)^3}{(1+\varepsilon)^{2\times\frac{3}{2}}}=-\dfrac{1}{r_0}$.

$\therefore$ 曲率半径 $\rho=\left|\dfrac{1}{\kappa}\right|=r_0$. □

设恒星的质量为 M，万有引力常数为 G.

轨道的极坐标方程可表示为 $r=\dfrac{r_0}{1+\varepsilon\cos(\theta-\theta_0)}$，其中 ε 为轨道的离心率.

令 $\cos(\theta-\theta_0)=1$ 得，近日点与恒星的距离 $r_m=\dfrac{r_0}{1+\varepsilon}$.

近日点速率 v_m 满足 $\dfrac{v_m^2}{r_0}=\dfrac{GM}{r_m^2}$.

$\therefore v_m=\sqrt{\dfrac{GMr_0}{r_m^2}}=(1+\varepsilon)\sqrt{\dfrac{GM}{r_0}}$

设 $h=v_mr_m=\sqrt{GMr_0}$，由角动量守恒，行星在轨道上运动时 $h=vr$ 不变.

单位时间行星与恒星连线扫过的面积

$$\mathrm{d}S=\frac{1}{2}h\mathrm{d}t=\frac{1}{2}\sqrt{GMr_0}\mathrm{d}t=\frac{1}{2}r^2\mathrm{d}\theta$$

若以行星位于近日点的时刻为计时起点，则行星运动到 (r,θ) 处的时刻

$$t=\sqrt{\frac{1}{GMr_0}}\int_{\theta_0}^{\theta}r^2\mathrm{d}\theta=r_0\sqrt{\frac{r_0}{GM}}\int_0^{\theta-\theta_0}\frac{\mathrm{d}x}{(1+\varepsilon\cos x)^2}$$

经计算:

$$\int_0^\theta \frac{\mathrm{d}x}{(1+\varepsilon\cos x)^2} = \begin{cases} \theta,\ \varepsilon=0 \\ -\dfrac{\varepsilon\sin\theta}{(1-\varepsilon^2)(\varepsilon\cos\theta+1)} + \dfrac{2\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\theta}{2}\right)}{(1-\varepsilon^2)^{\frac{3}{2}}} + \left\lfloor\dfrac{x+\pi}{2\pi}\right\rfloor\dfrac{2\pi}{(1-\varepsilon^2)^{\frac{3}{2}}}, \\ \hfill 0<\varepsilon<1 \\ \dfrac{1}{2}\tan\dfrac{\theta}{2}+\dfrac{1}{6}\tan^3\dfrac{\theta}{2},\ \varepsilon=1 \\ \dfrac{\varepsilon\sin\theta}{(\varepsilon^2-1)(\varepsilon\cos\theta+1)} - \dfrac{2\,\mathrm{arctanh}\left(\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}\tan\frac{\theta}{2}\right)}{(\varepsilon^2-1)^{\frac{3}{2}}},\ \varepsilon>1 \end{cases}$$

这里，我们省略了对可去间断点的函数值的补充定义.

于是已知 θ、r、t 中的任意一个，可在多项式时间内求出另外两者的有任意数量有效数字的近似值.