Six-band

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In the basics $|p_z,\uparrow\rangle, |p_z,\downarrow\rangle, |d_{xz+iyz},\downarrow\rangle, |d_{xz-iyz},\uparrow\rangle, |d_{xz+iyz},\uparrow\rangle, |d_{xz-iyz,\downarrow}\rangle$，考虑 $k_z = 0$

$$H = \begin{pmatrix} M(k) & 0 & 0 & -iA_1k_- & -iA_2k_+ & 0 \\ 0 & M(k) & -iA_1k_+ & 0 & 0 & -iA_2k_- \\ 0 & iA_1 & -M(k) & 0 & 0 & 0 \\ iA_1k_+ & 0 & 0 & -M(k) & 0 & 0 \\ iA_2k_- & 0 & 0 & 0 & -M(k)+\delta & 0 \\ 0 & iA_2k_+ & 0 & 0 & 0 & -M(k)+\delta \end{pmatrix} \tag{1}$$

做一个基矢变换

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

可以得到

$$H^{'} = \begin{pmatrix} M(k) & -iA_1k_- & -iA_2k_+ & & & \\ iA_1k_+ & -M(k) & 0 & & & \\ iA_2k_- & 0 & -M(k)+\delta & & & \\ & & & M(k) & -iA_1K_+ & 0 \\ & & & iA_1k_- & -M(k) & 0 \\ & & & iA_2k_+ & 0 & -M(k)+\delta \end{pmatrix} \tag{3}$$

先考虑其中一个块的内容，其能量本征值满足

$$E^3 + (M(k)-\delta)E^2 - [M(k)^2 + A_1^2k^2 + A_2^2k^2]E - M(k)^3 + \delta M(k)^2 - M(k)[A_1k^2 + A_2^2k^2] + \delta A_1^2k^2 = 0 \tag{4}$$

考虑 vortex，并写到动量空间的极坐标下之后可以得到

$$H_{BdG}=\begin{pmatrix} H_k-\mu & i\Delta_e e^{-i\theta}(\partial_k-\frac{i}{k}\partial_\theta) \\ i\Delta_e e^{i\theta}(\partial_k+\frac{i}{k}\partial_\theta) & \mu-H_k \end{pmatrix} \tag{5}$$

将其投影到 H_k 的本征空间可以得到

$$\begin{pmatrix} E_1-\mu & & & \Delta_{11} & C_{12} & C_{13} \\ & E_2-\mu & & C_{21} & \Delta_{22} & C_{23} \\ & & E_3-\mu & C_{31} & C_{32} & \Delta_{33} \\ \Delta_{11}^\dagger & C_{21}^\dagger & C_{31}^\dagger & \mu-E_1 & & \\ C_{12}^\dagger & \Delta_{22}^\dagger & C_{32}^\dagger & & \mu-E_2 & \\ C_{13}^\dagger & C_{23}^\dagger & \Delta_{33}^\dagger & & & \mu-E_3 \end{pmatrix} \tag{6}$$

其中 $\Delta_{ii}=i\Delta_e e^{-i\theta}(\partial_k-iA_k^{11}-\frac{i\partial_\theta+A_\theta^{11}}{k})$,$C_{ij}=\Delta_e e^{-i\theta}(A_k^{ij}-\frac{i}{k}A_\theta^{ij})$ 将其变换到 basics$|\psi_{1e}\rangle,|\psi_{1h}\rangle,|\psi_{2e}\rangle,|\psi_{2h}\rangle,|\psi_{2e}\rangle,|\psi_{2h}\rangle$ 可以得到

$$\begin{pmatrix} E_1-\mu & \Delta_{11} & 0 & C_{12} & 0 & C_{13} \\ \Delta_{11}^\dagger & \mu-E_1 & C_{21}^\dagger & 0 & C_{31}^\dagger & 0 \\ 0 & C_{21} & E_2-\mu & \Delta_{22} & 0 & C_{23} \\ C_{12}^\dagger & 0 & \Delta_{22}^\dagger & \mu-E_2 & C_{32}^\dagger & 0 \\ 0 & C_{31} & 0 & C_{32} & E_3-\mu & \Delta_{33} \\ C_{13}^\dagger & 0 & C_{23}^\dagger & 0 & \Delta_{33}^\dagger & \mu-E_3 \end{pmatrix} \tag{7}$$

根据中岛变换，将其变换成上面 $4\cdot 4$ 和一个 $2\cdot 2$ 的块对角之中，首先将 H 分解为 $H=H_0+H^1+H^2$, 其中 H_0 表示对角部分，H^1 表示非对角但是在对角块的部分，H^2 表示非对角块部分。由 $H^{'}=e^S He^{-S}$，考虑一阶近似，可以得到变换矩阵为

$$S_{ml}^{(1)}=\frac{-H_{ml}^2}{E_m-E_l} \tag{8}$$

同样去一阶近似可以得到

$$H^{'}=H_0+H^1+[H^2,S^1] \tag{9}$$

首先考虑上面 $4*4$ 的部分，

$$H_{04}=\begin{pmatrix} E_1-\mu & \Delta_{11} & 0 & C_{12} \\ \Delta_{11}^\dagger & \mu-E_1 & C_{21}^\dagger & 0 \\ 0 & C_{21} & E_2-\mu & \Delta_{22} \\ C_{12}^\dagger & 0 & \Delta_{22}^\dagger & \mu-E_2 \end{pmatrix} \tag{10}$$

一阶微扰项为

$$
H_{04}^{'} = \begin{pmatrix} & & \frac{C_{13}C_{23}^{\dagger}(E_2 - E_1)}{(E_1+E_3)(E_2+E_3)} & \\ & & & \frac{C_{31}^{\dagger}C_{32}(E_2 - E_1)}{(E_1+E_3)(E_2+E_3)} \\ \frac{C_{13}^{\dagger}C_{23}(E_2 - E_1)}{(E_1+E_3)(E_2+E_3)} & & & \\ & \frac{C_{32}^{\dagger}C_{31}(E_2 - E_1)}{(E_1+E_3)(E_2+E_3)} & & \end{pmatrix} \tag{11}
$$

首先求解 H_{04} 的本征态

$$
\begin{pmatrix} E_1 - \mu & i\Delta_e e^{-i\theta}(\partial_k - iA_k^{11} - \frac{i\partial_\theta + A_\theta^{11}}{k}) & 0 & \Delta_e e^{-i\theta}(A_k^{12} - \frac{i}{k}A_\theta^{12}) \\ i\Delta_e e^{i\theta}(\partial_k - iA_k^{11} + \frac{i\partial_\theta + A_\theta^{11}}{k}) & \mu - E_1 & \Delta_e e^{i\theta}(A_k^{12} + \frac{i}{k}A_\theta^{12}) & 0 \\ 0 & \Delta_e e^{-i\theta}(A_k^{21} - \frac{i}{k}A_\theta^{21}) & E_2 - \mu & i\Delta_e e^{-i\theta}(\partial_k - iA_k^{22} - \frac{i\partial_\theta + A_\theta^{22}}{k}) \\ \Delta_e e^{i\theta}(A_k^{21} + \frac{i}{k}A_\theta^{21}) & 0 & i\Delta_e e^{i\theta}(\partial_k - iA_k^{22} + \frac{i\partial_\theta + A_\theta^{22}}{k}) & \mu - E_2 \end{pmatrix} \tag{12}
$$

做一个变换

$$
U = \frac{1}{\sqrt{k}} \begin{pmatrix} e^{i(j-1)\theta} & & & \\ & -ie^{ij\theta} & & \\ & & e^{i(j-1)\theta} & \\ & & & -ie^{ij\theta} \end{pmatrix} \tag{13}
$$

变换后可以得到

$$
U^{\dagger}HU = \begin{pmatrix} E_1 - \mu & \Delta_e(\partial_k - iA_k^{11} + \frac{j - \frac{1}{2} - A_\theta^{11}}{k}) & 0 & -i\Delta_e(A_k^{12} - \frac{i}{k}A_\theta^{12}) \\ \Delta_e(\frac{j - \frac{1}{2} - A_\theta^{11}}{k} + iA_k^{11} - \partial_k) & \mu - E_1 & i\Delta_e(A_k^{12} + \frac{i}{k}A_\theta^{12}) & 0 \\ 0 & -i\Delta_e(A_k^{21} - \frac{i}{k}A_\theta^{21}) & E_2 - \mu & \Delta_e(\partial_k - iA_k^{22} + \frac{j - \frac{1}{2} - A_\theta^{22}}{k}) \\ i\Delta_e(A_k^{21} + \frac{i}{k}A_\theta^{21}) & 0 & \Delta_e(\frac{j - \frac{1}{2} - A_\theta^{22}}{k} + iA_k^{22} - \partial_k) & \mu - E_2 \end{pmatrix} \tag{14}
$$

$$
= i\Delta_e\partial_k s_0\tau_y + \frac{1}{2}(E_1 + E_2 - 2\mu)s_0\tau_z + \frac{1}{2}(E_1 - E_2)s_z\tau_z + \left(\quad \right) \tag{15}
$$

其中前两项似乎可以看成四维的 JRmodel

$$
H = i\Delta_e\partial_k s_0\tau_y + \frac{1}{2}(E_1 + E_2 - 2\mu)s_0\tau_z + \frac{1}{2}(E_1 - E_2)s_z\tau_z \tag{16}
$$

设其本征态为 $\psi(k)$, 考虑零能解，可以得到

$$
i\Delta_e\partial_k s_0\tau_y\psi(k) = [\frac{1}{2}(E_1 + E_2 - 2\mu)s_0\tau_z + \frac{1}{2}(E_1 - E_2)s_z\tau_z]\psi(k) \tag{17}
$$

两边同时乘以 $s_0\tau_y$ 可以得到

$$
\partial_k\psi(k) = \frac{1}{\Delta_e}[\frac{1}{2}(E_1 + E_2 - 2\mu)s_0\tau_x + \frac{1}{2}(E_1 - E_2)s_z\tau_x]\psi(k) \tag{18}
$$

$\psi(k)$ 必定是 $\frac{1}{2}(E_1+E_2-2\mu)s_0\tau_x+\frac{1}{2}(E_1-E_2)s_z\tau_x$ 的本征态，可以求得其本征态为

$$\psi_1^+=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\qquad \psi_1^-=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\psi_2^+=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\qquad \psi_2^-=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix} \tag{19}$$

由此可得

$$\partial_k\psi_1(k)=\eta\frac{E_1-\mu}{\Delta_e}\psi_1(k) \tag{20}$$

$$\partial_k\psi_2(k)=\eta\frac{E_2-\mu}{\Delta_e}\psi_2(k) \tag{21}$$

其中 $\eta=\pm$, 由此可以得到

$$\psi_1(k)=Ce^{\int^k\eta\frac{E_1-\mu}{\Delta_e}dk'} \tag{22}$$

$$\psi_2(k)=Ce^{\int^k\eta\frac{E_2-\mu}{\Delta_e}dk'} \tag{23}$$

其零能解出现在 $E_1-\mu$ 或 $E_2-\mu$ 改变符号的地方。此时再来求解这个矩阵的本征值

$$\begin{pmatrix}0 & \Delta_e(\frac{j-\frac{1}{2}-A_\theta^{11}}{k}-iA_k^{11}) & 0 & -i\Delta_e(A_k^{12}-\frac{i}{k}A_\theta^{12})\\ \Delta_e(\frac{j-\frac{1}{2}-A_\theta^{11}}{k}+iA_k^{11}) & 0 & i\Delta_e(A_k^{12}+\frac{i}{k}A_\theta^{12}) & 0\\ 0 & -i\Delta_e(A_k^{21}-\frac{i}{k}A_\theta^{21}) & 0 & \Delta_e(\frac{j-\frac{1}{2}-A_\theta^{22}}{k}-iA_k^{22})\\ i\Delta_e(A_k^{21}+\frac{i}{k}A_\theta^{21}) & 0 & \Delta_e(\frac{j-\frac{1}{2}-A_\theta^{22}}{k}+iA_k^{22}) & \end{pmatrix} \tag{24}$$

将其分块对角，即求解

$$\begin{pmatrix}\Delta_e(\frac{j-\frac{1}{2}-A_\theta^{11}}{k}-iA_k^{11}) & -i\Delta_e(A_k^{12}-\frac{i}{k}A_\theta^{12})\\ -i\Delta_e(A_k^{21}-\frac{i}{k}A_\theta^{21}) & \Delta_e(\frac{j-\frac{1}{2}-A_\theta^{22}}{k}-iA_k^{22})\end{pmatrix} \tag{25}$$

$$=\frac{\Delta_e}{k}(j-\frac{1}{2})\sigma_0+\Delta_e\begin{pmatrix}-\frac{A_\theta^{11}}{k}-iA_k^{11} & -\frac{A_\theta^{12}}{k}-iA_k^{12}\\ -\frac{A_\theta^{21}}{k}-iA_k^{21} & -\frac{A_\theta^{22}}{k}-iA_k^{22}\end{pmatrix} \tag{26}$$

各向同性的时候费米面处 SU(2) 的 Berry phase 为

$$\phi_{FS}=\oint_{FS}\vec{A_{ij}}d\vec{k} \tag{27}$$

$$=\int_0^{2\pi}kd\theta\,\langle\psi_i|\,\frac{\partial_\theta}{k}\,|\psi_j\rangle \tag{28}$$

$$=2\pi A_\theta^{ij} \tag{29}$$

其中 A_k 无法通过规范变换全部消除，即除了 Berry phase 似乎还有其他项，但这些项是否为 0 我还无法确定。

1 Conclusion

在 $k_z = 0$ 的时候，将电子部分哈密顿量化成了 3*3 分块对角，然后考虑超导以及 vortex，将其写到 particle-hole space，并且投影到电子哈密顿量的本征态上，然后将 6*6 的哈密顿量用中岛变换将非对角块的部分变换到块对角部分，目前只考虑到一阶近似。先考虑 4*4 这一块部分，首先不考虑中岛变换过来的微扰项，这一部分似乎可以化成 4*4 的 Jackiw-Rebbi，然后按照能量可以变成两套 2*2 的 Jackiw-Rebbi。然后进一步考虑其他项的时候，发现除了 SU(2) 的 Berry phase 还有一些 A_k 项，感觉无法消掉。