Parity violation and gauge noninvariance of the effective gauge field action in three dimensions

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The effective gauge field action due to fermions coupled to SU(N) gauge fields in three dimensions is found to change by $\pm\pi|n|$ under a homotopically nontrivial gauge transformation with winding number n. This gauge noninvariance can be eliminated by adding a parity-violating topological term to the action, or by regulating the theory in a way which produces this term automatically in the effective action. The Euler-Heisenberg effective action is calculated in the SU(2) theory and in QED.

I. INTRODUCTION

Gauge invariance of the Euler-Lagrange field equations does not necessarily imply gauge invariance of the action $I[A]$.[1,2] In a functional integral formulation of a field theory, we require $\exp(iI[A])$ to be gauge invariant. If under a gauge transformation the action changes by a number, the parameters of the theory must be quantized so that number is an integral multiple of 2π.[1-4] Another important feature of quantum field theory is symmetry under space-time transformations. If the action is invariant under such a symmetry, but physical quantities which are invariant in the classical field theory are not invariant when the theory is quantized, then we say that the symmetry is spontaneously broken.

In this paper, we discuss a theory in which the spontaneous breakdown of a space-time symmetry, coordinate reflection invariance, and the gauge noninvariance of the action are intimately connected. We consider three-dimensional (2+1 space-time or 3 Euclidean space) SU(N) gauge theories coupled to fermions in the fundamental representation, as well as the U(1) theory, QED—our results, however, may be generalized to higher odd dimensions. The action $I[A,\psi]$ for an odd number of massless fermions coupled to SU(N) gauge fields in three dimensions is invariant under both gauge transformations and space-time reflections (which we shall call parity). We establish here, however, that the effective gauge field action $I_{\text{eff}}[A]$ must violate either one or the other of these symmetries—$I_{\text{eff}}[A]$ is obtained by integrating out the fermionic degrees of freedom. Which symmetry is violated depends on the procedure used to regulate the ultraviolet divergences in $I_{\text{eff}}[A]$: If we regulate in a way which maintains parity as a good symmetry, then $I_{\text{eff}}[A]$ is found to change by an odd multiple of π under a topologically nontrivial gauge transformation. On the other hand, if we introduce a heavy Pauli-Villars regulator field and subtract $\lim_{M\to\infty}I_{\text{eff}}[A,M]$ from $I_{\text{eff}}[A]$, then we cancel the gauge noninvariance in $I_{\text{eff}}[A]$, but introduce parity violation through $I_{\text{eff}}[A]$. A three-dimensional mass term $M\bar{\psi}\psi$ violates parity and for $M\neq 0$ the fermions have parity-violating spin equal to[2] $\frac{1}{2}M/|M|=\pm\frac{1}{2}$. The calculations performed here were discussed in a Letter in which only the results were presented.[5]

Pauli-Villars regularization restores gauge invariance because $\lim_{M\to\infty}I_{\text{eff}}[A,M]$ contains the parity-violating topological term $\pm\pi W[A]$—$W[A]$ is the Chern-Simons secondary characteristic class[2,6]—which changes by $\pm\pi n$ under a homotopically nontrivial gauge transformation U_n with winding number n. The gauge noninvariance of $\pm\pi W[A]$ cancels the gauge noninvariance of $I_{\text{eff}}[A]$. As an alternative to introducing a parity-violating Pauli-Villars regulator to restore gauge invariance at the expense of parity conservation, one may simply add the term $\pm\pi W[A]$ to the gauge field action. Thus the total action consists of the parity-conserving, gauge-noninvariant piece coming from the fermions and the parity-violating, gauge-noninvariant gauge field action. The gauge noninvariance disappears in the sum, but parity violation remains.

Another way to restore gauge invariance is to change the fermion content of the theory: one works with an even number of fermion species, so that the effective action changes by $2\pi n$ under a large gauge transformation. In this case, parity need not be violated, which is not surprising, since an even number of fermions in three dimensions can be paired to form Dirac fermions with parity-conserving mass terms. We shall not concern ourselves here with this well-known method of removing gauge-invariance anomalies, but focus on the novel mechanism which retains the fermion content while modifying the gauge field action.

We shall begin with a demonstration that the effective gauge field action $I_{\text{eff}}[A]$ changes by $\pm\pi|n|$ under a homotopically nontrivial gauge transformation U_n with winding number n. While this was proved in the Letter,[5] the derivation presented here is an alternative one. This calculation does not require any approximations.

The bulk of the paper contains approximate calculations of the effective gauge field action $I_{\text{eff}}[A]$. In Sec. III, we use perturbation theory to demonstrate that

$$I^R_{\text{eff}}[A]\equiv I_{\text{eff}}[A]-\lim_{M\to\infty}I_{\text{eff}}[A,M]$$

contains the Chern-Simons term $\pm\pi W[A]$. We then calculate $I_{\text{eff}}[A]$, exactly, for the special case of gauge fields which produce a constant field strength tensor $F^{\mu\nu}$; $I_{\text{eff}}[A]$ in this case is the three-dimensional analog of the Euler-Heisenberg effective action. In Sec. IV this calculation is performed for the Abelian theory QED, and in Sec. V it is presented for SU(2).

For gauge fields with constant field strength tensor $F^{\mu\nu}$ we find

$$I_{\text{eff}}[A]=\pm\alpha\pi W[A]+I_{\text{NA}}[A]\ ,$$

with

$$\alpha=1 \text{ for SU(2)}\ ,$$

$$\alpha=2 \text{ for QED}\ ,$$

where $W[A]$ is the Chern-Simons term and $I_{\text{NA}}[A]$ is a nonanalytic contribution to the effective action. The lack of analyticity is related to the infrared divergences which are characteristic of three-dimensional gauge theories.[7]

II. GAUGE NONINVARIANCE OF THE ACTION (SU(N))

To show that fermions induce a gauge-noninvariant term in the action, we integrate over the fermion fields in the functional integral:

$$Z=\int d\bar{\psi}\,d\psi\,dA \exp\left[i\int d^3x[\tfrac{1}{2}\operatorname{tr}F_{\mu\nu}F^{\mu\nu}+i\bar{\psi}(\partial\!\!\!/+A\!\!\!/)\psi]\right]$$
$$=\int dA \exp\left[i\left[\int d^3x\,\tfrac{1}{2}\operatorname{tr}F_{\mu\nu}F^{\mu\nu}+I_{\text{eff}}[A]\right]\right]\ , \qquad (2.1)$$

where

$$I_{\text{eff}}[A]=-i\ln\det(\partial\!\!\!/+A\!\!\!/)\ . \qquad (2.2)$$

We use the usual matrix notation $A_\mu=gT^aA_\mu^a$, with T^a anti-Hermitian generators of the group. For definiteness, we work with SU(2) and a doublet of fermions—where $T^a=\sigma^a/2i$, and σ^a are the Pauli matrices—but our results hold in any group for which π_3 is the additive group of integers, Z, and the fermions are in the fundamental representation. The Dirac matrices in three dimensions are Pauli matrices $(\sigma_3,i\sigma_2,i\sigma_1)$.

In our Letter,[5] we demonstrated that $\det(\partial\!\!\!/+A\!\!\!/)$ and, by (2.2), the effective action $I_{\text{eff}}[A]$ are not gauge invariant. More precisely, we showed, following closely the analogous calculation performed by Witten in four dimensions,[3] that

$$\det(\partial\!\!\!/+A\!\!\!/)\rightarrow(-1)^{|n|}\det(\partial\!\!\!/+A\!\!\!/) \qquad (2.3)$$

under a homotopically nontrivial gauge transformation U_n, with winding number n. (We use here a Euclidean formulation of the three-dimensional theory, and we consider gauge transformations which approach the identity at large distances; hence our base manifold is S_3 rather than R_3.) The proof makes use of the observation that $\det i\sigma_\mu(\partial_\mu+A_\mu)$, $\mu=1,2,3$ may be written $\det^{1/2}iD\!\!\!\!/_4$, where $D\!\!\!\!/_4=\gamma_\mu(\partial_\mu+A_\mu)$ and γ_μ are 4×4 matrices. The square root is defined as the product of the positive eigenvalues of $iD\!\!\!\!/_4[A_\mu]$, and we showed that the determinant changes sign under a large gauge transformation. We shall now rederive (2.3), without introducing 4×4 matrices, remaining instead with $\det i\sigma_\mu(\partial_\mu+A_\mu)=\det iD\!\!\!\!/_3[A_\mu]$ in terms of 2×2 Pauli matrices.

To show that $\det iD\!\!\!\!/_3[A_\mu]$ is gauge noninvariant, we vary the gauge field along a continuous path, parametrized by τ from $A_\mu(x^\mu,\tau)=0$ at $\tau=-\infty$ to the pure gauge

$$A_\mu(x^\mu,\tau)=U_n{}^{-1}\partial_\mu U_n$$

at $\tau=+\infty$, where U_n belongs to the nth homotopy class (U_n has winding number n). The value at $\tau=-\infty$ of the determinant $\det iD\!\!\!\!/_3[A]$, which is real because $iD\!\!\!\!/_3[A]$ is Hermitian, can differ from its value at $\tau=+\infty$ only by a sign. This is because two three-dimensional fermions can be combined to form one Dirac fermion with an effective action$=2I_{\text{eff}}[A]$. Therefore, $(\det iD\!\!\!\!/_3[A])^2$ must be invariant under both large and small gauge transformations. As τ is varied from $-\infty$ to $+\infty$, the gauge field must pass through configurations in field space which are not pure gauge, because U_n is not continuously deformable to the identity. Therefore, one or more eigenvalues which are positive (negative) at $\tau=-\infty$ may cross zero and become negative (positive) at $\tau=+\infty$ (see Fig. 1). Since the determinant is defined as the product of all the eigenvalues, it will change sign if the number of eigenvalues which cross from positive to negative minus the number which cross from negative to positive is odd.

We now recognize that the family of vector potentials $A_\mu(x^\mu,\tau)$ is equivalent to an instantonlike four-dimensional gauge field A^i in the gauge $A^4=0$ [the space $x^i=(x^\mu,\tau)$, $i=1,2,3,4$ is the cylinder $S_3\times R$]. The remaining components of $A^i(x^\mu,\tau)$ vary adiabatically as functions of $\tau=x^4$ along the path considered above.

The number of zero crossings of the eigenvalues of $iD\!\!\!\!/_3[A^\mu(\tau)]$ is related to the number of zero modes of the four-dimensional Dirac operator

$$D\!\!\!\!/=\gamma_i(\partial_i+A_i),\quad i=1,2,3,4 \qquad (2.4)$$

with

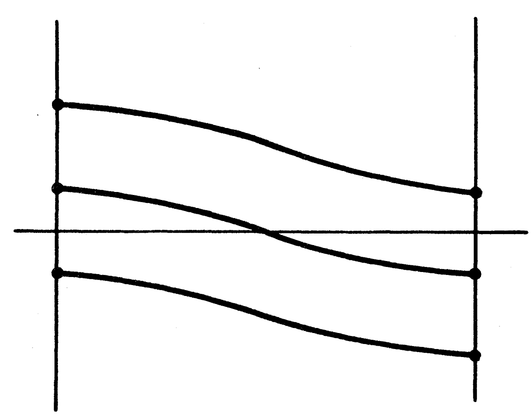

FIG. 1. The eigenvalues of $iD\!\!\!\!/_3[A^\mu(\tau)]$ are plotted along the vertical axis. One eigenvalue is shown to pass from a positive to a negative value as τ varies from $-\infty$ to $+\infty$.

$$\gamma_\mu = i\begin{bmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0 \end{bmatrix},\quad \gamma_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},\quad \gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.5}$$

To see this, we write the Dirac equation $\not{D}\psi=0$ as

$$\frac{d\psi}{d\tau} = -\gamma_4\gamma_\mu(\partial_\mu + A_\mu)\psi , \tag{2.6}$$

where $\mu=1,2,3$. Using the representation (2.5) of the γ matrices,

$$\frac{d\psi}{d\tau} = -\begin{bmatrix} i\sigma_u & 0 \\ 0 & -i\sigma_u \end{bmatrix}(\partial_\mu + A_\mu)\psi . \tag{2.7}$$

We now choose the eigenfunctions of $\not{D}\psi_\pm=0$ to be eigenfunctions of γ_5: $\gamma_5\psi_\pm=\pm\psi_\pm$. With the representation (2.5) of γ_5, Eq. (2.7) becomes

$$\frac{du_+}{d\tau} = -i\sigma_\mu(\partial_\mu + A_\mu)u_+ = -i\not{D}_3[A^\mu]u_+ , \tag{2.8a}$$

$$\frac{du_-}{d\tau} = i\sigma_\mu(\partial_\mu + A_\mu)u_- = i\not{D}_3[A^\mu]u_- . \tag{2.8b}$$

Equations (2.8) are soluble in the adiabatic approximation: let $u_\pm(x^\mu,\tau)=f_\pm\phi^\tau(x^\mu)$, where $\phi^\tau(x^\mu)$ satisfies the eigenvalue equation

$$i\not{D}_3[A^\mu]\phi^\tau(x^\mu) = \lambda(\tau)\phi^\tau(x^\mu) .$$

The eigenvalues $\lambda(\tau)$ vary as a function of τ along the curves of Fig. 1. In the adiabatic approximation, Eqs. (2.8) are

$$\frac{df_+}{d\tau} = -\lambda(\tau)f_+(\tau) , \tag{2.9a}$$

$$\frac{df_-}{d\tau} = \lambda(\tau)f_-(\tau) \tag{2.9b}$$

with solutions

$$f_+(\tau)=f_+(0)\exp\left[-\int_0^\tau d\tau'\,\lambda(\tau')\right] , \tag{2.10a}$$

$$f_-(\tau)=f_-(0)\exp\left[\int_0^\tau d\tau'\lambda(\tau')\right] . \tag{2.10b}$$

$f_+(\tau)$ ($f_-(\tau)$) is normalizable only if λ is positive (negative) for $\tau=+\infty$ and negative (positive) for $\tau=-\infty$.

Therefore, there exists a one-to-one correspondence between the number of normalizable zero modes $\not{D}\psi_+=0$ ($\not{D}\psi_-=0$), denoted $n_+(n_-)$, and the number of eigenvalues of $i\not{D}_3[A^\mu]$ which pass from negative (positive) to positive (negative) values as τ varies from $-\infty$ to $+\infty$. If $n_- - n_+$ is odd, then the determinant $\det i\not{D}_3[A^\mu]$ will change sign. Moreover, it is well known from instanton studies that

$$n_- - n_+ = n , \tag{2.11}$$

where n is the instanton number (the winding number of U_n). This completes the proof of (2.3): the determinant changes sign under a large gauge transformation with odd winding number.

It is possible to regulate $\det i\sigma_\mu(\partial_\mu + A_\mu)$ in a gauge-invariant way by introducing a massive, parity-violating Pauli-Villars regulator field. The regulated effective action

$$I_{\text{eff}}^R = I_{\text{eff}}(m=0) - I_{\text{eff}}(m\to\infty) \tag{2.12}$$

is gauge invariant, but as we show in Sec. III, a parity-violating term survives in $I_{\text{eff}}(m)$ at $m=\infty$; its finite part is [for SU(2)]

$$I_{\text{eff}}(\text{finite}, m\to\infty) = \pm\pi W[A] , \tag{2.13}$$

where

$$W[A] = \frac{1}{8\pi^2}\int d^3x\, \text{tr}({}^*F_\mu A^\mu - \tfrac{1}{3}A^\mu A^\nu A^\alpha \epsilon_{\mu\nu\alpha}) ,\qquad {}^*F^\mu = \tfrac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \tag{2.14}$$

is the Chern-Simons secondary characteristic class, known to be odd under coordinate reflection and changing by n under a homotopically nontrivial gauge transformation with winding number n.

Alternatively, one may regulate in a parity-preserving but gauge-noninvariant way by defining $\det i\sigma_\mu(\partial_\mu + A_\mu)$ as $\det^{1/2} i\not{D}_4$, as was done in our earlier paper.[5] The regulation is performed by standard Pauli-Villars methods on $\det i\not{D}_4$. These now preserve parity because we are dealing with 4×4 Dirac matrices where the mass term $m\bar{\psi}\psi$ is equal to the sum of a positive plus a negative mass term in the 2×2 γ-matrix formulation.[8]

III. PERTURBATION THEORY

We shall begin by establishing that the Chern-Simons term (2.14) must appear in $I_{\text{eff}}[A]$ when ultraviolet divergences are regulated in a gauge-invariant way. To do so, we use Pauli-Villars regularization and define the regulated action as in Eq. (2.12).

Only the vacuum polarization graph and the triangle graph are ultraviolet divergent, therefore only these two require regularization. For finite m, I_{eff}, in the SU(2) theory is, to order g^3,

$$I_{\text{eff}}[A] = \tfrac{1}{2}\,\text{tr}[T^a,T^b]\int \frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3} A_\nu^a(-p)A_\mu^b(-q)[-i\Pi^{\mu\nu}(p)]$$
$$+\tfrac{1}{3}\,\text{tr}[T^aT^bT^c]\int \frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3} A_\mu^a(-q-k)A_\nu^b(q)A_\alpha^c(k)\Pi^{\mu\nu\alpha}(q,k) + O(g^4) , \tag{3.1}$$

where

$$\Pi^{\mu\nu}(p)=-ig^2\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}[\gamma^\mu S(p+k)\gamma^\nu S(k)]\ , \quad (3.2)$$

$$\Pi^{\mu\nu\alpha}(q,k)=-g^3\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}[\gamma^\mu S(p)\gamma^\nu S(p+q)\times\gamma^\alpha S(p+q+k)] \quad (3.3)$$

with $S(p)=-i/(p-m)$. The vacuum polarization graph is linearly divergent; for $m\to\infty$ it is equal to[2,9]

$$\Pi^{\mu\nu}(m\to\infty)=-\frac{g^2}{3\pi^2}\Lambda g^{\mu\nu}+\frac{m}{|m|}\frac{g^2}{4\pi}(-i\epsilon^{\mu\nu\alpha}p_\alpha)\ , \quad (3.4)$$

where Λ is a momentum cutoff and symmetric integration has removed the logarithmic divergence in both Eqs. (3.2) and (3.3); for $m\to\infty$ we find

$$\Pi^{\mu\nu\alpha}(m\to\infty)=-\frac{m}{|m|}g^3\frac{\epsilon^{\mu\nu\alpha}}{4\pi}\ . \quad (3.5)$$

All higher-order graphs in I_{eff} [Eq. (3.1)] must be proportional to powers of $1/m$ and therefore they vanish when $m\to\infty$. Using Eqs. (3.4) and (3.5) in Eq. (3.1), we obtain the effective action for $m\to\infty$:

$$I_{\mathrm{eff}}(m\to\infty)=-\frac{\Lambda}{6\pi}\int d^3x\,\mathrm{tr}A^2-\frac{m}{|m|}\pi W[A]\ , \quad (3.6)$$

where $W[A]$ is the Chern-Simons term (2.14).

The divergent term in I_{eff} at $m=0$ (which appears in $\pi^{\mu\nu}$) cancels the divergent term in I_{eff} ($m\to\infty$), (3.6), and we obtain the finite effective action

$$I_{\mathrm{eff}}^R=I_{\mathrm{eff}}(\text{finite},\ m=0)\pm\pi W[A]\ , \quad (3.7)$$

where the $\pm$ comes from $m/|m|$ which depends on the sign of m. The effective action in QED differs from I_{eff} [Eq. (3.1)] only by group factors; it follows that the regularized effective action in QED is

$$I_{\mathrm{eff}}^R=I_{\mathrm{eff}}(\text{finite},\ m=0)\pm\frac{e^2}{8\pi}\int d^3x\,A_\mu{}^*F^\mu$$

$$=I_{\mathrm{eff}}(\text{finite},\ m=0)\pm 2\pi W[A]\ . \quad (3.8)$$

Again, the second term is the Chern-Simons secondary characteristic [for a U(1) theory]. Also, the only parity violation in Eqs. (3.7) and (3.8) is due to the Chern-Simons terms, because for $m=0$ both QED and the SU(2) theory are parity conserving and therefore $I_{\mathrm{eff}}(m=0)$ [in both Eqs. (3.7) and (3.8)] conserves parity (coordinate reflection).

IV. $I_{\mathrm{eff}}[A]$ IN THREE-DIMENSIONAL QED

We shall now calculate the total one-loop contribution to the effective action in three-dimensional QED,

$$I_{\mathrm{eff}}[A]=i\,\mathrm{tr}\ln(\not{D}+m)\ ,$$
$$\not{D}(x)=-i\not\partial-e\not{A}(x)\ , \quad (4.1)$$

for the special case of gauge fields which produce a constant field strength tensor, $F^{\mu\nu}=$constant:

$$A^\mu(x)=-\frac{\epsilon^{\mu\alpha\beta}}{2}x_\alpha{}^*F_\beta\ . \quad (4.2)$$

To obtain the complete effective action in this case, it is necessary to calculate both $I_{\mathrm{eff}}[A]$ and $\langle J^\mu\rangle$, the ground-state current in the presence of the background field (4.2). This is because the Chern-Simons term $W[A]$ formally vanishes for A^μ given by (4.2), but its variation with respect to A^μ produces a nonvanishing current. On the other hand, I_{eff} contains terms which are functions of $F^{\mu\nu}F_{\mu\nu}$. These produce terms in $\langle J^\mu\rangle$ which contain derivatives of $F^{\mu\nu}$. For $F^{\mu\nu}=$constant, these terms vanish: a calculation of $\langle J^\mu\rangle$ alone is not sufficient. We shall therefore calculate $I_{\mathrm{eff}}[A]$ and $\langle J^\mu\rangle$ separately, and use

$$\frac{\delta I_{\mathrm{eff}}}{\delta A_\mu}=\langle J^\mu\rangle$$

to deduce that the Chern-Simons term is present in the effective action. The method used here was developed by Schwinger to perform similar calculations in four-dimensional QED.[10] It is a gauge-invariant procedure and therefore gives results which agree with the solutions obtained using Pauli-Villars regularization.

We begin by writing $\langle J^\mu(x)\rangle$ in terms of a Green's function $G(x,x')$ for the Dirac operator,

$$[\not{D}(x)+m]G(x,x')=\delta(x-x')\ , \quad (4.3)$$

$$\langle J^\mu(x)\rangle=ie\,\mathrm{tr}[\gamma^\mu G(x,x')]\,|_{x\to x'}\ . \quad (4.4)$$

The limit $x\to x'$ is performed by taking the average of the terms obtained by letting $x\to x'$ from the future and from the past.[10]

Introducing the operator notation

$$G(x,x')=(x\,|\,G\,|\,x')=(x\,|\,\frac{1}{D+m}\,|\,x')\ ,$$

we write

$$G=i\int_0^\infty ds\{\exp[-i(-\not{D}\not{D}+m^2)s]\}(-\not{D}+m)\ . \quad (4.5)$$

In addition, $\not{D}(-x)=\not{D}(x)$ for $A^\mu(x)$ given by Eq. (4.2), so that

$$I_{\mathrm{eff}}[A]=i\int d^3x\,\mathrm{tr}\ln[\not{D}(x)+m]$$
$$=\frac{i}{2}\int d^3x\,\mathrm{tr}\ln(-\not{D}\not{D}+m^2)\ , \quad (4.6)$$

or

$$\mathscr{L}_{\mathrm{eff}}(x)=\tfrac{1}{2}\,\mathrm{tr}(x\,|\,\ln(-\not{D}\not{D}+m^2)\,|\,x)\ , \quad (4.7)$$

which can be written

$$\mathscr{L}_{\mathrm{eff}}(x)=\frac{i}{2}\,\mathrm{tr}(x\,|\int_0^\infty\frac{ds}{s}\{\exp[-i(-\not{D}\not{D}+m^2)s]-\exp[-i(\Box+m^2)s]\}\,|\,x)\ , \quad (4.8)$$

where the second term has been added so $\mathscr{L}_{\text{eff}}$ will vanish when $A^\mu=0$.

We now define $(x'\,|\,U(s)\,|\,x'')=(x',s\,|\,x'',0)$, where

$$U(s)=e^{-i\mathscr{H}s},\quad \mathscr{H}=-\not{D}\not{D}=-D_\mu D^\mu-\frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}\,,$$

and

$$\sigma_{\mu\nu}\equiv\frac{i}{2}[\gamma_\mu,\gamma_\nu]\,.$$

$U(s)$ may be understood as an operator which describes the development of a quantum-mechanical system in the "proper time" s, governed by the "Hamiltonian" $\mathscr{H}$. To determine $(x',s\,|\,x'',0)$ and $(x',s\,|\,D_\mu\,|\,x'',0)$, we first solve the proper-time dynamical equations

$$\frac{dx_\mu}{ds}=i[\mathscr{H},x_\mu]\,, \tag{4.9}$$

$$\frac{dD_\mu}{ds}=i[\mathscr{H},D_\mu]=-2eF_{\mu\nu}D^\nu-ie\frac{\partial F_{\mu\nu}}{\partial x_\nu}+\frac{e}{2}\sigma_{\lambda\nu}\frac{\partial F^{\lambda\nu}}{\partial x^\mu}\,, \tag{4.10}$$

and then use

$$i\partial_s(x',s\,|\,x'',0)=(x',s\,|\,\mathscr{H}\,|\,x'',0)\,, \tag{4.11}$$

$$\begin{aligned}&[-i\partial'_\mu-eA_\mu(x')](x',s\,|\,x'',0)=(x',s\,|\,D_\mu(s)\,|\,x'',0)\,,\\&[-i\partial''_\mu-eA_\mu(x'')](x',s\,|\,x'',0)=(x',s\,|\,D_\mu(0)\,|\,x'',0)\,,\end{aligned} \tag{4.12}$$

and the boundary condition

$$(x',s\,|\,x'',0)\,|_{s\to0}=\delta(x'-x'')\,. \tag{4.13}$$

$\mathscr{L}_{\text{eff}}$ and $\langle J^\mu\rangle$ can then be written in terms of $(x',s\,|\,x'',0)$ and $(x',s\,|\,D_\mu\,|\,x'',0)$:

$$\mathscr{L}_{\text{eff}}(x)=\frac{i}{2}\int_0^\infty\frac{ds}{s}\,e^{-im^2s}\left[\operatorname{tr}(x,s\,|\,x',0)-\frac{e^{i3\pi/4}}{s^{3/2}}\frac{1}{4\pi^{3/2}}\right]\Bigg|_{x\to x'}\,, \tag{4.14}$$

$$\langle J^\mu(x)\rangle=-e\operatorname{tr}\gamma^\mu\times\int_0^\infty ds\,e^{-im^2s}(x,s\,|\,-\not{D}+m\,|\,x',0)\,|_{x\to x'}\,. \tag{4.15}$$

Solving the set of equations (4.9)—(4.13), one finds

$$(x',s\,|\,x'',0)=C(x',x'')\exp\left[-\frac{i}{4}(x'-x'')_\mu[eF\coth(eFs)]^{\mu\nu}(x'-x'')_\nu\right]\frac{e^{-L(s)}}{s^{3/2}}\exp\left[\frac{ies}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]\,, \tag{4.16}$$

$$\begin{aligned}&(x',s\,|\,D_\mu(0)\,|\,x'',0)=\tfrac{1}{2}[eF\coth(eFs)-1]_{\mu\nu}(x'-x'')^\nu(x',s\,|\,x'',0)\,,\\&C(x',x'')=C\exp\left[ie\int_{x'}^{x''}A^\mu(x)dx_\mu\right]\,,\quad C=\frac{e^{3\pi i/4}}{8\pi^{3/2}}\,,\end{aligned} \tag{4.17}$$

$$L(s)=\tfrac{1}{2}\operatorname{tr}\ln[(eFs)^{-1}\sinh(eFs)]\,, \tag{4.18}$$

where the trace (4.18) extends over space-time indices only. These solutions differ only slightly from their four-dimensional counterparts: the factor $1/s^{3/2}$ is $1/s^2$ in four dimensions; also, the constant C is different.

The quantities $\mathscr{L}_{\text{eff}}$ and $\langle J^\mu\rangle$, as they were defined above, are not gauge invariant: they contain linearly divergent, gauge-noninvariant terms $\sim A^\mu A_\mu/(x-x')\,|_{x\to x'}$ and $A^\mu/(x-x')\,|_{x'\to x}$, respectively. However, the corresponding gauge-invariant quantities

$$\begin{aligned}&\langle J^\mu(x)\rangle_{\text{GI}}=ie\operatorname{tr}[\gamma^\mu\tilde{G}(x,x')]\,|_{x\to x'}\,,\\&(\mathscr{L}_{\text{eff}})_{\text{GI}}=i\operatorname{tr}\ln[\tilde{G}^{-1}(x,x')]\,,\end{aligned} \tag{4.19}$$

with

$$\tilde{G}(x,x')=\exp\left[-ie\int_x^{x'}A(x'')dx''\right]G(x,x')\,,$$

are free of divergences. Taking the limit $x\to x'$ in Eqs. (4.14) and (4.15), and dropping the terms which are not gauge invariant, or equivalently defining $\langle J^\mu\rangle$ and $\mathscr{L}_{\text{eff}}$ to be the gauge-invariant quantities (4.19), we obtain

$$\langle J^\mu(x)\rangle=-emC\int_0^\infty\frac{ds}{s^{3/2}}e^{-im^2s}\operatorname{tr}\left[\gamma^\mu e^{-L(s)}\exp\left[i\frac{es}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]\right]\,, \tag{4.20a}$$

$$\mathscr{L}_{\text{eff}}(x)=iC\int_0^\infty\frac{ds}{s^{5/2}}e^{-im^2s}\left\{\tfrac{1}{2}\operatorname{tr}\left[e^{-L(s)}\exp\left[i\frac{es}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]\right]-1\right\}\,. \tag{4.20b}$$

Using $\gamma^\mu=(\sigma_3,i\sigma_1,i\sigma_2)$, we find

$$\exp\left[i\frac{es}{2}\sigma_{\mu\nu}F^{\mu\nu}\right]=\exp\left[i\frac{es}{2}\gamma^{\alpha\,*}F_\alpha\right]=\mathbb{1}\cos(es\,|{}^*F\,|)+i\gamma^\alpha\frac{{}^*F_\alpha}{|{}^*F\,|}\sin(es\,|{}^*F\,|)\,, \tag{4.21}$$

where $|{}^*F\,|=(B^2-E^2)^{1/2}$.

Since $L(s)$ is determined by a determinant, we can evaluate it from the eigenvalues of F, which can be found with the assistance of the relation (in three dimensions)

$$F^{\mu\nu}F_{\nu\alpha}F^{\alpha\beta}=-\frac{F^2}{2}F^{\mu\beta}=-({}^*F^2)F^{\mu\beta}\,. \tag{4.22}$$

We iterate the eigenvalue equation $F\psi=\lambda\psi$ and find $\lambda^3=-\,|{}^*F\,|^2\lambda$: $\lambda=0,\ \pm i\,|{}^*F\,|$. Then, by Eq. (4.18),

$$\begin{aligned}e^{-L(s)}&=\left[\frac{es\lambda_0}{\sinh(es\lambda_0)}\right]^{1/2}\left[\frac{es\lambda_1}{\sinh(es\lambda_1)}\right]^{1/2}\\&\quad\times\left[\frac{es\lambda_2}{\sinh(es\lambda_2)}\right]^{1/2}\\&=\frac{es\,|{}^*F\,|}{\sin(es\,|{}^*F\,|)}\,.\end{aligned} \tag{4.23}$$

Inserting Eqs. (4.21) and (4.23) into Eq. (4.20), we obtain the solutions

$$\langle J^\mu(x)\rangle=-2ie^2mC\int_0^\infty\frac{ds}{s^{1/2}}e^{-im^2s\,*}F^\mu=\frac{m}{|\,m\,|}\frac{e^2}{4\pi}{}^*F^\mu\,, \tag{4.24}$$

$$\begin{aligned}\mathscr{L}_{\rm eff}(x)&=iC\int_0^\infty\frac{ds}{s^{5/2}}\,e^{-im^2s}[es\,|{}^*F\,|\cot(es\,|{}^*F\,|)-1]\\&=\frac{1}{8\pi^{3/2}}\int_0^\infty\frac{ds}{s^{5/2}}\\&\qquad\times e^{-m^2s}[es\,|{}^*F\,|\coth(es\,|{}^*F\,|)-1]\,,\end{aligned}$$

where in the last expression we have deformed the path of integration: $s\to -is$.

If we now let $m\to 0$, $\langle J^\mu(x)\rangle$ remains unchanged, and $\mathscr{L}_{\rm eff}$ becomes

$$\begin{aligned}L^0_{\rm eff}(x)&=\tfrac{1}{8}\left[\frac{e\,|{}^*F\,|}{\pi}\right]^{3/2}\int_0^\infty\frac{ds}{s^{5/2}}(s\coth s-1)\\&=\frac{1}{2\pi^2}\zeta(\tfrac{3}{2})\left[\frac{e\,|{}^*F\,|}{2}\right]^{3/2}\,,\end{aligned} \tag{4.25}$$

where $\zeta(\frac{3}{2})$ is the Riemann ζ function

$$\zeta(\tfrac{3}{2})=\sum_{n=1}^\infty\frac{1}{n^{3/2}}\,.$$

Using Eq. (4.25) and functionally integrating Eq. (4.24), we deduce that $I_{\rm eff}[A]$, for $m=0$ and $F^{\mu\nu}=$constant, is

$$\begin{aligned}I_{\rm eff}[A]&=\pm\frac{e^2}{8\pi}\int d^3x\,A^{\mu\,*}F_\mu\\&\quad+\frac{1}{2\pi^2}\zeta(\tfrac{3}{2})\int d^3x\left[\frac{e\,|{}^*F\,|}{2}\right]^{3/2}\\&=\pm 2\pi W[A]+\frac{1}{2\pi^2}\zeta(\tfrac{3}{2})\int d^3x\left[\frac{e\,|{}^*F\,|}{2}\right]^{3/2}\,,\end{aligned} \tag{4.26}$$

where the sign of the first term depends on the sign of m.

V. $I_{\rm eff}[A]$ IN NON-ABELIAN GAUGE THEORIES

We now extend our calculation of $I_{\rm eff}$ to non-Abelian gauge fields. For definiteness we consider an SU(2) gauge theory coupled to fermions in the fundamental representation, but these results are easily generalized to other gauge groups. In non-Abelian theories, there are two types of gauge fields which produce a constant field-strength tensor $F_a^{\mu\nu}=$constant.[11] The first is an "Abelian" gauge field,

$$A_a^\mu=-\eta_a\frac{\epsilon^{\mu\alpha\beta}}{2}x_\alpha{}^*F_\beta\,, \tag{5.1}$$

where $F_a^{\mu\nu}=\eta_aF^{\mu\nu}$, η^a is a constant unit vector in isospin space, and $F^{\mu\nu}$ is a constant. The second is a constant gauge field $A_a^\mu=$constant, which is truly non-Abelian: $[A^\mu,A^\nu]\neq 0$, where $A^\mu=gA_a^\mu t_a$ and $t_a=\tau_a/2i$ (τ^a are the Pauli matrices). We shall not consider here the special case $F_a^{\mu\nu}=0$, which corresponds to a pure gauge

$$A^\mu(x)=U^{-1}(x)\gamma^\mu U(x)\,.$$

In the non-Abelian theory, $I_{\rm eff}$ and $\langle J_a^\mu(x)\rangle$ are

$$I_{\rm eff}[A]=-i\,\mathrm{tr}\ln(\not{D}+m)\,, \tag{5.2}$$

$$\begin{aligned}\langle J_a^\mu(x)\rangle&=-g\,\mathrm{tr}\left[\gamma^\mu t_a\frac{1}{\not{D}+m}\right]\,,\\ \not{D}(x)&=-i(\not\partial+\not{A})\,.\end{aligned} \tag{5.3}$$

The calculation of $\langle J_a^\mu(x)\rangle$ and $\mathscr{L}_{\rm eff}(x)$ for an Abelian gauge field (5.1) follows directly from the calculation of these quantities in QED (Sec. IV). Keeping track of factors of 2 which comes from group-theoretical terms—$\mathrm{tr}[t_at_b]$—we find

$$\langle J_a^\mu(x)\rangle=\frac{m}{|\,m\,|}\frac{g^2}{8\pi}{}^*F_a^\mu\,, \tag{5.4a}$$

$$I_{\rm eff}[A]=\pm W[A]+\frac{1}{\pi^2}\zeta(\tfrac{3}{2})\int d^3x\left[\tfrac{1}{4}\,\mathrm{tr}\,{}^*F^2\right]^{3/2}\,. \tag{5.4b}$$

We shall now calculate $I_{\rm eff}[A]$ [Eq. (5.2)], for a constant non-Abelian gauge field $A^\mu=$constant. In this case, it is easy to demonstrate that $\langle J_a^\mu(x)\rangle$ is indeed given by $\delta I_{\rm eff}[A]/\delta A_a^\mu$ so that a separate calculation of $\langle J_a^\mu(x)\rangle$ is unnecessary.

For $A^\mu=$constant Eq. (5.2) becomes, with $\int d^3x=$vol,

$$\begin{aligned}I_{\rm eff}[A]&=i(\mathrm{vol})\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}\ln(\not{p}-i\not{A}+m)\\&=i(\mathrm{vol})\int\frac{d^3p}{(2\pi)^3}\ln\det(\not{p}-i\not{A}+m)\,,\\ \mathscr{L}_{\rm eff}[A]&=i\int\frac{d^3p}{(2\pi)^3}\ln\det(\not{p}-i\not{A}+m)\,.\end{aligned} \tag{5.5}$$

It is more convenient to calculate the effective action in

Euclidean space. We therefore Wick rotate and introduce $(-iA^0,A^1,A^2)\to(A^3,A^1,A^2)$:

$$\mathscr{L}_{\text{eff}}=\int\frac{d^3p}{(2\pi)^3}\ln\det[\sigma^\mu(p^\mu-iA^\mu)+im]\ . \tag{5.6}$$

The 3×3 matrix A_a^μ can be diagonalized by performing rotations in space and in isospin space

$$A_a^\mu\to R_I^{ab}R_s^{\mu\nu}A_b^\nu=U^{-1}(R_I)U^{-1}(R_s)AU(R_I(U(R_s)\ ,$$

where $U(R_I)=\exp(i\theta_R\cdot\tau)$ and $U(R_s)=\exp(i\theta_s\cdot\sigma)$. If we define the diagonal matrix

$$\frac{g}{2}A_a^\mu\equiv A=\begin{bmatrix}A_1&0&0\\0&A_2&0\\0&0&A_3\end{bmatrix}, \tag{5.7}$$

then the determinant in Eq. (5.6) can be calculated simply:

$$\begin{aligned}\det\left[\sigma^\mu\left[p^\mu-\frac{g}{2}A_a^\mu\tau_a\right]+im\right]&=(p^2+m^2)^2+2p^2\mathrm{tr}A^2\\&\quad-4(p_1{}^2A_1{}^2+p_2{}^2A_2{}^2+p_3{}^2A_3{}^2)+\Delta\ ,\\ \Delta=2m^2\,\mathrm{tr}A^2+(\mathrm{tr}A^2)^2-\mathrm{tr}\,{}^*F^2+8i\det A\ ,\end{aligned} \tag{5.8}$$

where ${}^*F^\mu=gt_a{}^*F_a^\mu$. All Lorentz- and gauge-invariant quantities can be written in terms of A^μ,[11]

$$\begin{aligned}&\tfrac{1}{2}\,\mathrm{tr}\,{}^*F^2=4\sum_{a\neq b}A_a{}^2A_b{}^2\ ,\\&-(D^\lambda{}^*F^\mu)(D_\lambda{}^*F_\mu)=\mathrm{tr}A^2(\tfrac{1}{2}\,\mathrm{tr}\,{}^*F^2)-\det A^2\ ,\\&\tfrac{1}{3}\epsilon_{abc}F_{a\nu}^\mu F_{b\lambda}^\nu F_{c\mu}^\lambda=64\det A^2\ ,\end{aligned} \tag{5.9}$$

where $D^\lambda{}^*F^\mu=\epsilon_{abc}A_b^\lambda{}^*F_c^\mu$. The Chern-Simons term (2.14) can also be written in terms of A:

$$\pi W[A]=\frac{i}{\pi}(\det A)(\mathrm{vol})\ . \tag{5.10}$$

I_{eff}, as given by Eq. (5.5), contains a linear ultraviolet divergence. We shall renormalize this expression using Pauli-Villars regularization. We define

$$I_{\text{eff}}^R=I_{\text{eff}}(m=0)-I_{\text{eff}}(m\to\infty)\ . \tag{5.11}$$

Adding a second term so that $I_{\text{eff}}[A]$ vanishes for $A=0$, we may write Eq. (5.5) as

$$\mathscr{L}_{\text{eff}}=-2\int\frac{d^3p}{(2\pi)}\int_0^\infty\frac{ds}{s}(\exp\{-is^2[(p^2+m^2)^2+2p^2\mathrm{tr}A^2-4(p_1{}^2A_1{}^2+p_2{}^2A_2{}^2+p_3{}^2A_3{}^2)+\Delta]\}-\exp[-is^2(p^2+m^2)^2])\ . \tag{5.12}$$

If we let $q_i=\sqrt{s}\,p_i$ and change the order of integration, we obtain

$$\mathscr{L}_{\text{eff}}=-2\int_0^\infty\frac{ds}{s^{5/2}}\int\frac{d^3q}{(2\pi)^3}(\exp\{-i[(q^2+m^2s)^2+2s(q_1{}^2a+q_2{}^2b+q_3{}^2c)+s^2\Delta]\}-\exp[-i(q^2+m^2s)^2]) \tag{5.13}$$

with $a=-A_1{}^2+A_2{}^2+A_3{}^3$, $b=A_1{}^2-A_2{}^2+A_3{}^2$, $c=A_1{}^2+A_2{}^2-A_3{}^2$. The ultraviolet divergence in $\mathscr{L}_{\text{eff}}$ has become a divergence in the s integral at $s=0$. We can regulate this divergence by introducing a cutoff s and integrating by parts. The resulting infinite and finite parts of $\mathscr{L}_{\text{eff}}$ are

$$\mathscr{L}_{\text{eff}}(\infty)=\frac{1}{s^{3/2}}I_1(s)\Bigg|_\epsilon^\infty+\frac{2}{s^{1/2}}I_2(s)\Bigg|_\epsilon^\infty\ , \tag{5.14}$$

$$\mathscr{L}_{\text{eff}}(\text{finite})=-\int_0^\infty\frac{ds}{s^{3/2}}\left[\frac{d}{ds}I_1(s)-I_2(s)+2s\frac{d}{ds}I_2(s)\right], \tag{5.15}$$

where

$$I_1(s)=\tfrac{4}{3}\int\frac{d^3q}{(2\pi)^3}(\exp\{-i[(q^2+m^2s)^2+2s(q_1{}^2a+q_2{}^2b+q_3{}^2c)+s\Delta^2]\}-\exp[-i(q^2+m^2s)^2])\ , \tag{5.16}$$

$$I_2(s)=-i\tfrac{4}{3}\int\frac{d^3q}{(2\pi)^3}[2(q_1{}^2a+q_2{}^2b+q_3{}^2c)]\exp\{-i[(q^2+m^2s)^2+2s(q_1{}^2a+q_2{}^2b+q_3{}^2c)+s^2\Delta]\}\ . \tag{5.17}$$

Finally, if we introduce the identity

$$e^{-i(q^2+m^2s)^2}=(i\pi)^{-1/2}\int_{-\infty}^\infty dx\,e^{ix^2}e^{2i(q^2+m^2s)x}\ , \tag{5.18}$$

then the resulting momentum integrations can be performed easily, and we find

$$I_1(s)=\frac{-i\pi}{12\sqrt{2}}\int_{-\infty}^\infty dx\,e^{ix^2}e^{2im^2sx}\left[\frac{e^{-is^2\Delta}}{[x-as]^{1/2}[x-bs]^{1/2}[x-cs]^{1/2}}-\frac{1}{x^{3/2}}\right], \tag{5.19}$$

$$I_2(s)=\frac{i\pi}{12\sqrt{2}}\int_{-\infty}^{\infty}dx\, e^{ix^2}e^{2im^2sx}e^{-is^2\Delta}\left[\frac{a}{[x-as]^{3/2}[x-bs]^{1/2}[x-cs]^{1/2}}+\frac{b}{[x-bs]^{3/2}[x-as]^{1/2}[x-cs]^{1/2}}+\frac{c}{[x-cs]^{3/2}[x-as]^{1/2}[x-bs]^{1/2}}\right]. \tag{5.20}$$

The infinite part of $\mathscr{L}_{\text{eff}}$ [Eq. (5.14)] is independent of the fermion mass m. In addition, the finite part of $\mathscr{L}_{\text{eff}}$ [Eq. (5.15)] for $m\neq 0$ is

$$\begin{aligned}\mathscr{L}_{\text{eff}}(\text{finite})&=\frac{m}{|m|}\frac{i}{\pi}\det A+O(1/m)\,,\\ I_{\text{eff}}(\text{finite})&=\frac{m}{|m|}\pi W[A]+O(1/m)\,,\end{aligned} \tag{5.21}$$

and

$$\mathscr{L}_{\text{eff}}(m=0)=-2\int_0^{\infty}\frac{ds}{s^{1/2}}\left[I_3(s)+\frac{d}{ds}I_4(s)\right] \tag{5.22}$$

with

$$I_3(s)=-\Delta\, e^{-is^2\Delta}\frac{\pi}{12\sqrt{2}}\int_{-\infty}^{\infty}dx\, e^{ix^2}\left[\frac{1}{[x-as]^{1/2}[x-bs]^{1/2}[x-cs]^{1/2}}\right], \tag{5.23}$$

$$I_4(s)=\frac{i\pi}{12\sqrt{2}}e^{-is^2\Delta}\int_{-\infty}^{\infty}dx\, e^{ix^2}\left[\frac{a}{[x-as]^{3/2}[x-bs]^{1/2}[x-cs]^{1/2}}+\frac{b}{[x-bs]^{3/2}[x-as]^{1/2}[x-cs]^{1/2}}+\frac{c}{[x-cs]^{3/2}[x-as]^{1/2}[x-bs]^{1/2}}\right]. \tag{5.24}$$

Therefore, the renormalized effective action (5.11) is

$$I_{\text{eff}}^R=I_{\text{eff}}(m=0)\pm\pi W[A]\,, \tag{5.25}$$

where $I_{\text{eff}}(m=0)=(\text{vol})\mathscr{L}_{\text{eff}}(m=0)$ is given by (5.22), and $W[A]$ is the Chern-Simons term (2.14). The first term is nonanalytic in A, as can be verified by calculating Eq. (5.6) for special cases such as $A_1=A_2=A_3$. More importantly, $I_{\text{eff}}(m=0)$ is manifestly invariant under coordinate inversion and therefore the only parity violation in I_{eff}^R is contained in the Chern-Simons term $W[A]$.

CONCLUSIONS

The action $I[A,\psi]$ for an odd number of massless fermions coupled to SU(N) gauge fields in three dimensions is invariant under parity. We have found, however, that the ground-state current $\langle J_a^\mu\rangle=\delta I_{\text{eff}}/\delta A_{\mu a}$, a physical quantity, does not possess a well-defined parity transformation, i.e., parity is spontaneously broken. This is explicitly seen for the very special case of a constant, Abelian field-strength tensor: $\langle J_a^\mu\rangle=g^2/8\pi\,{}^*F_a^\mu$, the vector current, is given by the pseudovector ${}^*F^\mu$.

It is easy to generalize these results to higher odd dimensions. The calculation of I_{eff} has been performed in five dimensions, where it again contains the parity-violating term $\pm\pi W_5[A]$, with $W_5[A]$ the five-dimensional Chern-Simons secondary characteristic class.[12]

In three dimensions, there exists a "physical" reason for expecting the ground-state current $\langle J^\mu\rangle$ to be proportional to ${}^*F^\mu$—we consider first QED. If we introduce a heavy Pauli-Villars regulator field with mass M, then *all* regulator fermions have spin $\frac{1}{2}M/|M|$. This alignment of the fermion spins "polarizes" the vacuum, and the ground-state expectation value of the spin-density operator $\langle 0|\psi^\dagger\frac{1}{2}\sigma_z\psi|0\rangle$ does not vanish. We observe that $\psi^\dagger\sigma_z\psi$ is obtained from the current associated with the spin part of the Lorentz transformation

$$\delta^{\alpha\beta}=[\gamma^\alpha,\gamma^\beta]\psi\,,$$

that is,

$$\psi^\dagger\sigma_z\psi=\frac{\epsilon_{\mu\alpha\beta}}{2}J_F^{\mu\alpha\beta}\,,$$

where

$$J_F^{\mu\alpha\beta}=\bar\psi\gamma^\mu[\gamma^\alpha,\gamma^\beta]\psi\,.$$

The heavy fermions induce an electromagnetic current

$$\begin{aligned}J_{\text{em}}^{\mu\alpha\beta}&=F^{\mu\nu}\delta^{\alpha\beta}A_\nu\\ &=F^{\mu\alpha}A^\beta-F^{\mu\beta}A^\alpha\,,\end{aligned}$$

so that,

$$\langle 0|\tfrac{1}{2}\epsilon_{\mu\alpha\beta}J_F^{\mu\alpha\beta}|0\rangle\approx\tfrac{1}{2}\epsilon_{\mu\alpha\beta}J_{\text{em}}^{\mu\alpha\beta}={}^*F^\mu A_\mu\,.$$

On dimensional grounds we therefore obtain, to lowest order in e and $1/M$—without calculations—

$$\langle 0|\psi^\dagger\sigma_z\psi|0\rangle=\alpha\frac{e^2}{|M|}{}^*F^\mu A_\mu\,,$$

where α is a dimensionless constant, while a lowest-order calculation gives $\alpha=1/4\pi$. In three dimensions $\psi^\dagger\sigma_z\psi=\bar\psi\psi$, so the self-energy of the regulator field is

$$\langle M\bar{\psi}\psi\rangle = \frac{e^2 M}{4\pi |M|} {}^*F^\mu A_\mu \; .$$

This self-energy is due to the self-interaction of the fermion current with the induced electromagnetic field:

$$\langle M\bar{\psi}\psi\rangle = \langle J^\mu\rangle_M A_\mu$$

or

$$\langle J^\mu\rangle_M = \frac{Me^2}{|M|\, 4\pi} {}^*F^\mu$$

($\langle J^\mu\rangle_M$ is the contribution of the regulator field to $\langle J^\mu\rangle$). While we are unable to determine the constant α in QED without a direct calculation, in the non-Abelian theory α is determined by the quantization condition necessary to maintain gauge invariance. Therefore in the non-Abelian theory, we may use the above argument to deduce—without any calculations—that

$$\langle J_a^\mu\rangle_M = \frac{N}{8\pi}\frac{M}{|M|} g^2 {}^*F_a^\mu \; ,$$

where N is an integer.

The violation of parity in odd dimensions is analogous to the nonconservation of the axial-vector current in two and four dimensions where Pauli-Villars regularization introduces a mass which violates axial symmetry. We therefore complete the program of generalizing the axial anomaly to higher dimensions, supplementing the even-dimensional generalizations of the standard result,[13] by establishing the existence of appropriate anomalies in odd dimensions. The "anomaly" in odd dimensions appears as a parity-violating topological term in the ground-state current $\langle J_a^\mu\rangle$, rather than as a topological term ($\sim {}^*FF$) in $\partial_\mu\langle J_5{}^\mu\rangle$ (there exists no axial-vector current in odd dimensions, i.e., no γ_5). In both cases, the anomaly causes a physical ground-state current to violate a symmetry of the original action $I[A,\psi]$.

Results similar to those presented in this paper have been obtained independently by L. Avarez-Gaume and E. Witten [Nucl. Phys. B234, 269 (1984)]. More recently, these results have been derived by a method which makes use of the axial anomaly in two dimensions.[12]

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