Statistics Assignment 1

WD-Leong

1 Sub-Problem 1

Find the proportion of permutations whose sum is even for a binary sequence of K bits, including all subsets. For example, for a binary sequence of 3 bits, there are a total of 8 permutations whose sum is even:

$$\{\}, \{0\}, \{0, 0\}, \{1, 1\}, \{0, 0, 0\}, \{0, 1, 1\}, \{1, 0, 1\} \text{ and } \{1, 1, 0\}.$$

Since there are a total of 15 permutations for the binary sequence of 3 bits such that

$$\begin{aligned} &\{\}, \{0\}, \{1\}, \{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}, \{0, 0, 0\}, \{0, 0, 1\}, \\ &\{0, 1, 0\}, \{0, 1, 1\}, \{1, 0, 0\}, \{1, 0, 1\}, \{1, 1, 0\} \text{ and } \{1, 1, 1\}, \end{aligned}$$

the proportion of terms whose sum is even is

$$\frac{8}{15} = 0.533.$$

Similarly, there are 4 permutations whose sum is even in a binary sequence of 2 bits

$$\{\}, \{0\}, \{0, 0\}, \{1, 1\},$$

and since there are a total of 7 possible permutations

$$\{\}, \{0\}, \{1\}, \{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\},$$

the proportion of terms whose sum is even is $4/7 = 0.571$.

1.1 Solution 1

Let $\{0, 1\}^K$ denote the binary sequence with K bits, e.g.

$$\{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}$$

for $\{0,1\}^2$. It is straightforward to note that there are 2^K permutations in $\{0,1\}^K$, excluding the subsets, for $K \geq 1$. Since 2^K is divisible by 2 for $K \geq 1$, there are

$$\frac{2^K}{2} = 2^{K-1}$$

permutations whose sum is even. Now consider $\{0,1\}^{K+1}$.

Let S_0 and S_1 be the set of permutations whose sum is, respectively, even and odd in $\{0,1\}^K$. If a zero is placed at the outermost bit of the $K+1$ sequence whose permutations is in S_0, the sum would still be even since adding zero to an even number yields an even number again. Similarly if a one is placed at the outermost bit of the $K+1$ sequence whose permutations is in S_1, the sum would be odd as adding one to an odd number yields an even number. To illustrate this point, consider $\{0,1\}^2$.

Bit 0	Bit 1	Sum
0	0	0
0	1	1
1	0	1
1	1	2

As can be observed from the table above, there are 2 permutations with an even sum. For $\{0,1\}^3$, placing a zero outermost bit on the 2 permutations whose sum is even yields an even sum again, while placing a one on the outermost bit on the remaining 2 permutations whose sum is odd yields an odd sum, leading to a total of 4 permutations whose sum is even.

Bit 0	Bit 1	Bit 2	Sum
0	0	0	0
0	1	1	2
1	0	1	2
1	1	0	2

By the same logic, placing a zero or one on the outermost bit whose permutaions are, respectively, in S_0 and S_1 will yield the permutations whose sum is odd in $\{0,1\}^{K+1}$ as seen in the table below.

As a result, the following result can be established. Given that there are c_K permutations in $\{0,1\}^K$ whose sum is even, there are $c_{K+1} = 2c_K$ permutations in $\{0,1\}^{K+1}$ whose sum is even. The recursive relation is given by

$$c_K = 2c_{K-1} = 4c_{K-2} = \ldots = 2^{K-1}c_1 = 2^{K-1}$$

Bit 0	Bit 1	Bit 2	Sum
0	0	1	1
0	1	0	1
1	0	0	1
1	1	1	3

with $c_1 = 1 = 2^0$. Hence, the total number of permutations in the binary sequence of K bits whose sum is even is

$$m'_K = \sum_{i=1}^{K} c_i = \sum_{i=1}^{K} 2^{i-1} = \sum_{i=0}^{K-1} 2^i = \frac{2^K - 1}{2 - 1} = 2^K - 1.$$

Accounting for the null set, the total number of permutations in the binary sequence of K bits whose sum is even is

$$m_K = m'_K + 1 = 2^K - 1 + 1 = 2^K.$$

Similarly, there are a total of n_K permutations in the binary sequence of K bits, including all subsets, given by

$$n_K = \sum_{i=0}^{K} 2^i = \frac{2^{K+1} - 1}{2 - 1} = 2^{K+1} - 1.$$

This leads to the result that the proportion of permutations in the binary sequence of K bits, including all subsets, whose sum is even is given by

$$p_K = \frac{m_K}{n_K} = \frac{2^K}{2^{K+1} - 1}$$

which is asymptotically 0.5 as $K \to \infty$.

2 Sub-Problem 2

Find the proportion of permutations whose sum is even for a sequence $X \in \{0, 1, 2\}$ of length K, including all subsets. For example, for a $\{0, 1, 2\}$ sequence of length 2, there are a total of 7 permutations whose sum is even:

$$\{\}, \{0\}, \{2\}, \{0, 0\}, \{0, 2\}, \{1, 1\} \text{ and } \{2, 0\}.$$

Since there are a total of 13 permutations for the sequence of length 2 such that

$$\{\}, \{0\}, \{1\}, \{2\}, \{0, 0\}, \{0, 1\}, \{0, 2\}, \{1, 0\},$$
$$\{1, 1\}, \{1, 2\}, \{2, 0\}, \{2, 1\} \text{ and } \{2, 2\},$$

the proportion of terms whose sum is even is $7/13 = 0.538$.

2.1 Solution 2

Let $\{0,1,2\}^K$ denote the sequence of $\{0,1,2\}$ of length K, e.g.

$$\{0,0\},\{0,1\},\{0,2\},\{1,0\},\{1,1\},\{1,2\},\{2,0\},\{2,1\},\{2,2\}$$

for $\{0,1,2\}^2$. Hence, $\{0,1,2\}^K$ has a total of 3^K permutations.

To facilitate the solution, consider the table below for $\{0,1,2\}^2$ with $3^2 = 9$ permutations. There are a total of 4 out of 9 permutations whose sum is even.

Bit 0	Bit 1	Sum
0	0	0
0	1	1
0	2	2
1	0	1
1	1	2
1	2	3
2	0	2
2	1	3
2	2	4

In the same manner as Solution 1, let S_0 and S_1 denote the sets of permutations in $\{0,1,2\}^K$, respectively, with an even and odd sum. Adding a zero or two to the outermost bit of S_0 will yield an even sum as the adding two even numbers will yield an even number while adding a one to the outermost bit of S_1 will yield an even sum since the addition of two odd numbers will be even.

Let c_K be the number of permutations in $\{0,1,2\}^K$ for which the sum is even. Then since the number of odd permutations = Total number of permutations − number of even permutations,

$$\begin{aligned} c_{K+1} &= 2 \times \text{No. of even permutations} + \text{No. of odd permutations} \\ &= 2c_K + (3^K - c_K) = 3^K + c_K, \text{ with } c_0 = 1 \text{ (the null set).} \end{aligned}$$

Expanding the above result leads to

$$
\begin{aligned}
c_K &= 3^{K-1} + c_{K-1} \\
&= 3^{K-1} + 3^{K-2} + c_{K-2} \\
&= 3^{K-1} + 3^{K-2} + \ldots + 3^0 + c_0 \\
&= \sum_{i=0}^{K-1} 3^i + c_0 = \frac{3^K - 1}{2} + 1 = \frac{3^K + 1}{2}.
\end{aligned}
$$

The cumulative number of even permutations is then given by

$$
\begin{aligned}
m_K &= \sum_{i=0}^{K} c_i = \sum_{i=0}^{K} \frac{3^i + 1}{2} \\
&= \frac{1}{2} \sum_{i=0}^{K} 3^i + \frac{1}{2} \sum_{i=0}^{K} 1 \\
&= \frac{3^{K+1} - 1}{4} + \frac{K+1}{2} = \frac{3^{K+1} + 2K + 1}{4}.
\end{aligned}
$$

Finally, the total number of permutations in $\{0, 1, 2\}^K$ including all subsets is given by

$$
n_K = \sum_{i=0}^{K} 3^i = \frac{3^{K+1} - 1}{2}.
$$

Hence, the proportion of permutations whose sum is even can be found to be

$$
p_K = \frac{m_K}{n_K} = \frac{3^{K+1} + 2K + 1}{4} \bigg/ \frac{3^{K+1} - 1}{2} = \frac{3^{K+1} + 2K + 1}{2\big(3^{K+1} - 1\big)},
$$

and $p_K \to \dfrac{3^{K+1}}{2 \times 3^{K+1}} = 0.5$ as $K \to \infty$.