$$F_c\, f(x) = \hat{F}_c(k) = \sqrt{\frac{2}{\pi}} \int_0 f(x) \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos kx \, dx.$$

$$f_c(k) = \sqrt{\frac{2}{\pi}} \left(\frac{1}{k^2+1} \right) \quad \text{—— (4)}$$

Taking inverse fourier cosin transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(k) \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\cos kx}{k^2+1}$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \left(\frac{\cos kx}{k^2+1} \right) dk.$$

Replace by m

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{\cos km}{1+k^2} \, dk$$

By the property definite integral.

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy \Rightarrow$$

$$\int_0^\infty f(k)\,dk = \int_0^\infty f(x)\,dx.$$

2022/12/09

$f(x) = e^{-x}, \quad 0 < x < 2\pi$

$\cancel{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi)}$

$$* \quad f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$* \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx.$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x}\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-\pi}}{1} \right]_0^{2\pi}$$

$$= -\frac{1}{\pi} \left[e^{-2\pi} - e^{-0} \right]$$

$$\int_0^{2\pi} e^{-x}\cos(nx)\,dx = \frac{1}{n^2+1}\left[n e^{-x}\sin(nx) - e^{-x}\cos(nx)\right]_0^{2\pi}$$

$$= \frac{1}{n^2+1}\left[n e^{-2\pi}\sin(n2\pi) - e^{-2\pi}\cos(n2\pi) - n e^{-0}\sin(0) + e^{-0}\cos(0)\right]$$

$$= \frac{1}{n^2+1}\left[0 - e^{-2\pi} - 0 + 1\right]$$

$\cos \to -\sin$

$\sin(nx)$

$-\frac{1}{n}\cos(nx)$

$-\frac{1}{n^2}\sin(nx)$

$$a_n = \frac{1-e^{-2\pi}}{n^2+1}$$

$$\boxed{a_n = \frac{1-e^{-2\pi}}{\pi}\left(\frac{1}{n^2+1}\right)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin(nx)\,dx$$

$$= -\frac{1}{n}e^{-x}\cos(nx) - \frac{1}{n^2}e^{-x}\sin(nx)$$

$$= \frac{1}{n^2}\int e^{-x}\sin(nx)\,dx$$

$$\underbrace{\int e^{-x}\sin(nx)\,dx}_{I} = -\frac{1}{n}e^{-x}\cos(nx) - \frac{1}{n^2}e^{-x}\sin(nx)$$

$$-\frac{1}{n^2}\int e^{-x}\sin(nx)\,dx$$

$$\left(1+\frac{1}{n^2}\right)I = \frac{1}{n^2}\left[n e^{-x}\cos(nx) + e^{-x}\sin(nx)\right]$$

$$I = \frac{n^2}{n^2+1}\times\frac{1}{n^2}\left[n e^{-x}\cos(nx) + e^{-x}\sin(nx)\right]$$

$$\int_0^{2\pi} e^{-x} \cancel{\cos} \sin(nx)\,dx = -\frac{1}{n^2+1}\left[n e^{-x}\cos(nx) + e^{-x}\sin(nx)\right]_0^{2\pi}$$

$$= -\frac{1}{n^2+1}\left[n e^{-2\pi}\cos(n2\pi) + e^{-2\pi}\sin(n2\pi) - n e^{-0}\cos(0) - e^{-0}\sin(0)\right]$$

$$= -\frac{1}{n^2+1}\left[n e^{-2\pi} - n\right]$$

$$= \frac{n}{n^2+1}\left[1 - e^{-2\pi}\right]$$

$$\boxed{b_n = \frac{1-e^{-2\pi}}{\pi}\left(\frac{n}{n^2+1}\right)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

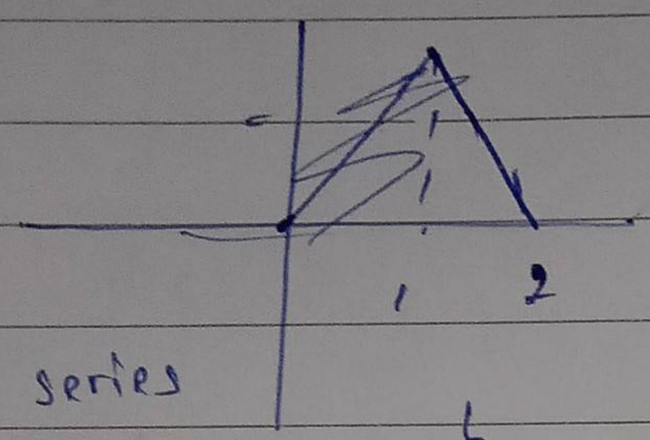
$$= \frac{1}{2}\left[\frac{1-e^{-2\pi}}{\pi}\right] + \sum_{n=1}^{\infty}\left[\left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{n^2+1}\cos(nx) + \left(\frac{1-e^{2\pi}}{\pi}\right)\frac{n}{n^2+1}\sin(nx)\right]$$

$$f(x) = \left(\frac{1-e^{-2\pi}}{\pi}\right)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos(nx) + \frac{n}{n^2+1}\sin(nx)\right]$$

07) Expand $f(x) = x$, $0 < x < 2$ in a half range

a) Sine series (odd)

b) cosine series (even



$L = 2-0 = 2$

~~L = 1~~

Sine series

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

$$= \frac{2}{2}\int_0^2 x\sin\left(\frac{n\pi}{2}x\right)dx$$

D	I
x	$\sin\left(\frac{n\pi x}{2}\right)$
1	$-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$
0	$-\frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)$

$$x\sin\left(\frac{n\pi x}{2}\right)dx = \left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -\frac{2}{n\pi}\left[2\cos\left(\frac{n\pi}{2}\times 2\right) - 0\right] + \frac{4}{(n\pi)^2}\underbrace{\left[\sin\left(\frac{n\pi}{2}\times 2\right) - \sin(0)\right]}_{0}$$

$$= \frac{-4}{n\pi}(-1)^n$$

$\cos n = -1$

$\cos n\pi = (-1)^n$

$$b_n = -\frac{4}{n\pi}(-1)^n$$

$$f(x) = \sum_{n=1}^{\infty} -\frac{4}{\pi}\frac{1}{n}(-1)^n$$

$n=1$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi} \sum_{n=1}^{n} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right)$$

b) cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right)$$

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

$$= \frac{2}{2} \int_0^2 x \cos\left(\frac{n\pi x}{L}\right) dx.$$

$\int$	L	
x	$\cos\left(\frac{n\pi x}{2}\right)$	
1	$\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right)$	
0	$-\frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)$	$+$

$$x \cos\left(\frac{n\pi x}{2}\right) dx = \left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) - \frac{4}{(n\pi)^2} \cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{2}{n\pi}\left[\sin\left(\frac{n\pi \times \cancel{2}}{\cancel{2}}\right) - \sin(0)\right] \quad \sin\pi = 0$$

$$- \frac{4}{(n\pi)^2}\left[\cos\left(\frac{n\pi \times \cancel{2}}{\cancel{2}}\right) - \cos(0)\right]$$

$$= \frac{2}{n\pi}\left[0 - 0\right] - \frac{4}{(n\pi)^2}\left[(-1)^n - 1\right]$$

$$= -\frac{4}{(n\pi)^2}\left[(-1)^n - 1\right]$$

$$f(x) = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2} \right] \cos\left(\frac{n\pi}{2} x \right)$$

Odd fun^{n} and half range sine series

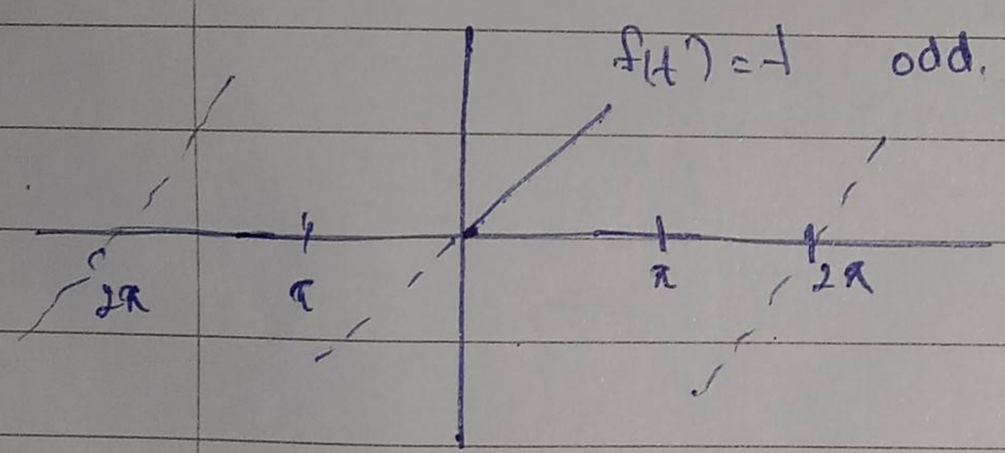
an odd function can be expanded using half it's range from ~~ze~~ 0 to L, that is the range of integration has value L.

The fourier series of an odd fu^{n}

$a_1 = 0$ $\qquad$ $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$



$f(-x) = -f(x)$

the wave form produce these periodic of period 2π. outside of this range as shown by the dottedline

compute the fourier series of

$$f(x) = \begin{cases} 3 & 0 < x < 1 \\ -3 & -1 < x < 0 \end{cases} \quad \& \quad f(x) = f(x+2)$$

$a_0 = 0$ $\qquad$ $2L = 2 \rightarrow L = 1$

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\, dx =$$

odd. odd

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_{-1}^{+1} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_{-1}^{+1} 3\sin\left(\frac{n\pi x}{L}\right)dx =$$

$$= 2\int_0^1 3\sin(n\pi x)\,dx = 6\int_0^1 \sin(n\pi x)\,dx$$

$$= 6\left[\frac{-\cos n\pi x}{n\pi}\right]_0^1 = \frac{-6}{n\pi}\left|\cos n\pi x\right|_0^1 = \frac{-6}{n\pi}\left(\cos n\pi - \cos 0\right)$$

$$= -\frac{6}{n\pi}\left[(-1)^n - 1\right]$$

$$b_n = -\frac{6}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & n = \text{even} \\ \frac{12}{n\pi}, & n = \text{odd} \end{cases}$$

$$f(x) = \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{12}{n\pi}\sin(n\pi x), \quad n = \text{odd}.$$

$$\int_a^b f(x)\,dx = f_a - f_b$$

Chapter 05

Fourier Transformation

5.1 Fourier Intergral theorem

The fourier intergral of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right] d\lambda$$

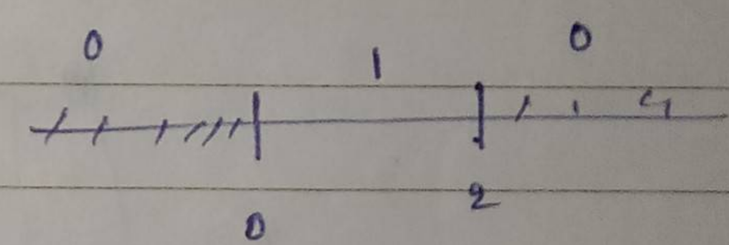
where, $$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt$$

Find the fourier integral representation of the function

$$f(x) = \begin{cases} 0 & ; x<0 \text{ or } x>2 \\ 1 & ; 0<x<2 \end{cases}$$

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt$$



$$= \int_{-\infty}^{0} \overset{\to 0}{f(t)}\cos\lambda t\, dt + \int_0^2 \overset{\to 1}{f(t)}\cos\lambda t\, dt + \int_2^\infty \overset{\to 0}{f(t)}\cos\lambda t\, dt$$

$$= \int_0^2 f(t)\cos\lambda t\, dt = \int_0^2 1\cos\lambda t\, dt$$

$$= \left[\frac{\sin\lambda t}{\lambda}\right]_0^2 = \frac{\sin 2\lambda - 0}{\lambda} = \frac{\sin 2\lambda}{\lambda} = A(\lambda)$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(t) \sin \lambda t \, dt$$

$$= \int_{-\infty}^{0} f(t) \sin \lambda t \, dt + \int_{0}^{2} f t \sin \lambda t \, dt + \int_{2}^{\infty} f(t) \sin \lambda t \, dt$$

$$= \int_{0}^{2} f(t) \sin \lambda t \, dt = \int_{0}^{2} \sin \lambda t \, dt$$

$$= \left[\frac{-\cos \lambda t}{\lambda} \right]_{0}^{2} = -\frac{1}{\lambda} \left(\cos 2\lambda - \cos 0 \right)$$

$$= \frac{-\cos 2\lambda - \cos 0}{\lambda} = \frac{-\cos 2\lambda - 1}{\lambda} = B(\lambda)$$

$$= \frac{1}{\lambda} \left(1 - \cos 2\lambda \right)$$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \left[A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x \right] d\lambda$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \left[\frac{\sin 2\lambda}{\lambda} \cdot \cos \lambda x + \frac{1}{\lambda} \left(1 - \cos 2\lambda \right) \sin \lambda x \right] d\lambda$$

$f(-x) = f(x) \rightarrow$ even

$f(-x) = -f_x \rightarrow$ odd.

5.2. Fourier cosine intergral

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x \right] d\lambda$$

$$A(\lambda) = \int_{-\infty}^{\infty} \overbrace{f(t)}^{\text{even}} \underbrace{\cos \lambda t}_{\text{even}} \, dt \quad \text{and} \quad B(\lambda) = \int_{-\infty}^{\infty} \overbrace{f(t)}^{\text{even}} \underbrace{\sin \lambda t}_{\text{odd}} \, dt$$

If $f(t)$ is even $B(\lambda) = 0$

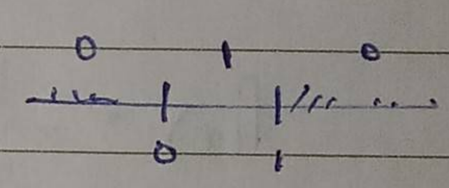
$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(t) \cos \lambda t \, dt \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t \, . \, dt \, . \, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t \, . \, dt \, . \, d\lambda$$

Q. Find the fourier intergral representation of the fun'

$$f(x) = \begin{cases} 1 & ; |x| < 1 \rightarrow -1 < x < 1 \\ 0 & ; |x| > 1 \quad -1 > x > 1 \end{cases}$$



$x = -x \Rightarrow f(-x) = f(x)$

Therefore fun is even

fourier cosine integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, . \, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^1 \cos\lambda x \int_0^1 1.\cos\lambda t.dt.d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^1 \cos\lambda x\left[\int_0^1 \cos\lambda t\, dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^1 \cos\lambda x \left|\frac{\sin\lambda t}{\lambda}\right|_0^1 d\lambda$$

$$= \frac{2}{\pi}\int_0^1 \cos\lambda x\left(\frac{\sin\lambda - \sin 0}{\lambda}\right)d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^1 \frac{\cos\lambda x.\sin\lambda}{\lambda}\, d\lambda$$

$$\Rightarrow \int_0^1 \frac{\cos\lambda x.\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & ; |x|<1 \\ 0 & ; |x|>1 \end{cases}$$

5.3 Fourier sine integral

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right]d\lambda$$

$$A(\lambda) = \int_{-\infty}^{\infty} \underbrace{f(t)}_{\text{odd}}\,\underbrace{\cos\lambda t}_{\text{even}}\,dt \quad \text{and} \quad B(\lambda) = \int_{-\infty}^{\infty} \underbrace{f(t)}_{\text{odd}}\,\underbrace{\sin\lambda t}_{\text{odd}}\,dt$$

If $f(t)$ is an odd function $A(\lambda) = 0$

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt\, \sin\lambda x\, d\lambda$$

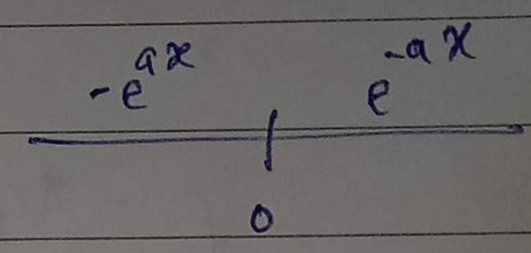
$$= \frac{1}{\pi} \int_0^\infty \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t \, dt \, d\lambda$$

$$\boxed{f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t \, dt \, d\lambda}$$

The fourier sine intergral of $f(x)$ is given by,

Find the fourier integral representation of the function

$$f(x) = \begin{cases} -e^{ax} & ; x < 0 \\ e^{-ax} & ; x > 0 \end{cases}$$



$x = -x \Rightarrow \cancel{-e^{ax}} \; f(-x) = -f(x)$

Therefore the fun is odd.

$$f(x) = \begin{cases} -e^{-ax} & ; -x < 0 \\ e^{ax} & ; -x > 0 \end{cases} = \begin{cases} -e^{-ax} & ; x > 0 \\ e^{ax} & ; x < 0 \end{cases}$$

$$f(-x) = -f(x)$$

$\therefore f(x)$ is an odd fun^n

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty e^{-at} \sin \lambda t \, dt \, d\lambda$$

using integration by parts

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \left| \frac{e^{-at}}{a^2 + \lambda^2} (-a \sin \lambda x - \lambda \cos \lambda x) \right|_0^\infty \lambda d \Big).$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \left[0 - \frac{1}{a^2+\lambda^2} (0-\lambda) \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \, \frac{\lambda}{a^2+\lambda^2} \, d\lambda = f(x)$$

$$\Rightarrow \int_0^\infty \sin \lambda x \left(\frac{\lambda}{a^2+\lambda^2} \right) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 \left(-e^{ax} \right) ; & x < 0 \\ \pi/2 \left(e^{-ax} \right) ; & x > 0 \end{cases}$$

$$= \begin{cases} -\pi/2 \, e^{ax} ; & x < 0 \\ \pi/2 \, e^{-ax} ; & x > 0 \end{cases}$$

5.4 Fourier Transformation

The generalization of complex fourier series is known as the fourier transform. The term fourier transform can be used in a mathematical function and this is also used in the representation of the frequency domain. The fourier transform helps to extend the fourier series to the non periodic functions. The fourier transform of a function $f(x)$ is given by

$$f(f(x)) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

Q. $f(x) = e^{-ax}$, $x > 0$ and a is a (+). constant. Find FT of $f(x)$

$$F(f(x)) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax}\, e^{-ikx}\, dx. = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x(a+ik)}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left| \frac{e^{-(a+ik)}}{-(a+ik)} \right|_{0}^{\infty}$$

$$= \frac{(a-ik)}{(a-ik)} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{(a+ik)} = \frac{(a-ik)}{\sqrt{2\pi}\,(a^2+k^2)}$$

Q $f(x) = \begin{cases} 1 & ; |x| < a/2 \\ 0 & ; |x| > a/2 \end{cases}$ a (+) constant find the FT of $f(x)$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh(ix) = \cos x.$$

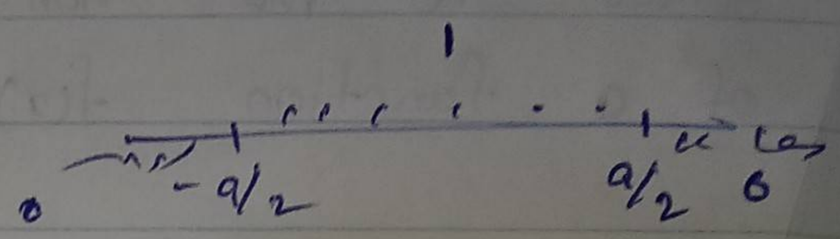
$$\sinh(ix) = i \sin x.$$

$$\cosh(x) = \cos(ix)$$

$$\sinh(x) = i \sin(ix)$$

$$-\frac{a}{2} < x < \frac{a}{2}$$

$$-\frac{a}{2} > x > \frac{a}{2}$$



$$F(f(x)) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a/2} 0 \cdot e^{-ikx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} 1 \cdot e^{-ikx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{a/2}^{\infty} 0 \cdot e^{ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} e^{-ikx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ikx}}{-ik} \right]_{-a/2}^{a/2}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ika/2}}{-ik} - \frac{e^{-ik\cdot -a/2}}{-ik} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ika/2} - e^{ika/2}}{-ik} \right] = \frac{1}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-1}{ik} \left[e^{-ika/2} - e^{ika/2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-1}{ik} \left[e^{ika/2} - e^{-ika/2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-1}{ik} \left[e^{ika/2} - e^{-ika/2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{ik} \left[2 \sinh \left(\frac{ika}{2} \right) \right]$$

$$= \frac{-1}{\sqrt{2\pi}} \frac{i}{k} \left[2 \sinh \left(\frac{ika}{2} \right) \right]$$

$$= \frac{-i}{\sqrt{2\pi}\, k} \, 2i \sin \left(\frac{ka}{2} \right) = \frac{2 \sin (ka/2)}{k\sqrt{2\pi}}$$

5.5 Inverse Transform

Inverse F.T of $f(k)$ gives $f(x)$ as,

$$f^{-1}(\hat{f}(k)) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k)\, e^{ikx}\, dk.$$

Find the F.T of the fun $f(x) = e^{-a|x|}$; $-\infty < x < \infty$

$$f(x) \begin{cases} e^{-a(-x)} ; & x<0 \\ e^{-ax} ; & x \geq 0 \end{cases}$$

$$f^{-1}(\hat{f}(k)) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k)\, e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ax} \cdot e^{ikx}\, dk + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} \cdot e^{ikx}\, dk$$

$e^{ax} = ; x<0$

$e^{-ax} ; x \geq 0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x(a+ik)}\, dk + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{x(-a+ik)}\, dk.$$

$$= \frac{1}{\sqrt{2\pi}} \left| \frac{e^{(a+ik)}}{(a+ik)} \right|_{\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left| \frac{e^{-a+ik}}{-a+ik} \right|_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(a+ik)}{(a+ik)} \cdot \frac{1}{(a-ik)} + \frac{1}{\sqrt{2\pi}} \frac{(a-ik)}{(a-ik)} \cdot \frac{1}{(a+ik)}$$

$$=$$

$$= \frac{2a}{a^2+k^2}$$

$$\Downarrow$$

$$\mathcal{F}\left(e^{-a|x|}\right) = \frac{2a}{a^2+k^2} = \mathcal{F}^{-1}\left[\frac{2a}{a^2+k^2}\right] = e^{-a|x|}$$

$$=$$

Q. Find $F^{-1}\left[\frac{1}{(9+k^2)(4+k^2)}\right]$

Using partial fractions

$$F^{-1}\left[\frac{1}{(9+k^2)(4+k^2)}\right] = \frac{1}{5} F^{-1}\left[\frac{-1}{(3^2+k^2)} + \frac{1}{2^2+k^2}\right]$$

$\frac{-1}{5(9+k^2)} + \frac{1}{5(4+k^2)}$

$$= \frac{1}{5.6} F^{-1}\left[\frac{-3.2}{3^2+k^2}\right] + \frac{1}{5.4} F^{-1}\left[\frac{2.2}{2^2+k^2}\right]$$

$$= \frac{-1}{30} F^{-1}\left(\frac{2.3}{3^2+k^2}\right) + \frac{1}{20}\left(\frac{2.2}{2^2+k^2}\right)$$

$$= \frac{-1}{30} e^{-3|x|} + \frac{1}{20} e^{-2|x|}$$

5.6 Fourier Sine / Cosine Transform

The fourier sine tranform of $f(x)$ $0 < x < \infty$ denoted by $\hat{f}_s(k)$,

$$F_s(f(x)) = \hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx$$

The fourier cosine transform of $f(x)$ $0 < x < \infty$ denoted by $\hat{f}_c(k)$

$$F_c(f(x)) = \hat{f}_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx$$

Inverse F.S.T of $\hat{f}_s(k)$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(k) \sin kx \, dk.$$

$\int u\,dv = uv - \int v\,du$

Inverse of F.C.T of $\hat{f}_c(k)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(k) \cos kx \, dk$$

Q. Find fourier sine and cosine transform of $f(x) = e^{-x}$, and hence show that

$$\int_0^\infty \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m} \cdot \int_0^\infty \frac{x \sin mx}{1+x^2} dx.$$

Sine transform

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx.$$

$u = \sin kx$

$dv = e^{-x}$

$du = k.\cos kx$

$v = -e^{-x}$

$$= \sqrt{\frac{2}{\pi}} \underbrace{\int_0^\infty e^{-x} \sin kx \, dx}_{I}$$

$$I = -e^{-x}.\sin kx - \int -e^{-x}.k.\cos kx$$

$u = k\cos kx$

$dv = -e^{-x}$

$v = e^{-x}$

$du = -\sin kx\, k$

$$= -e^{-x}.\sin kx \left[k\cos kx . e^{-x} - \int_0^\infty e^{-x}.-k^2 \sin kx \, dx. \right]$$

$$I = -e^{-x} \sin kx - \left[e^{-x} k \cos kx + k^2 \underbrace{\int_0^\infty e^{-x} \sin kx \, dx.}_{I} \right]$$

$$I + k^2 I = -e^{-x} \sin kx - e^{-x} k \cos kx$$

$$I = \frac{-e^{-x} \sin kx - e^{-x} k \cos kx}{1+k^2}$$

Using definite intergral

$$\hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \left[\frac{-e^{-x} \sin kx - e^{-x} k \cos kx}{1+k^2} \right]$$

$$\hat{f}_s(k) = \sqrt{\frac{2}{\pi}} \left(\frac{k}{1+k^2} \right) \text{———} (1)$$

Taking inverse fourier transform of (1).

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(k) \sin kx \, dk = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{k}{k^2+1} \right) \sin kx \, dk$$

$$= \frac{2}{\pi} \int_0^\infty \frac{k \sin kx}{k^2+1} \, dk$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{k \sin kx}{k^2+1} \, dk$$

Replace x by m

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{k \sin km}{k^2+1} \, dk$$

$$\int_a^b f(x) \, dx = \int_a^b f(y) \, dy$$

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{x^2+1} \, dx.$$

$$\frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin mx}{x^2+1} \, dx.$$