Computational Homework Assignment 4

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1 Problem Setting Up

Problem 5.43:

Consider the 1-D steady-state conduction in a cylindrical aluminium rod of $20cm$ diameter and $50cm$ length. The two ends of the rods are maintained at $100°C$ and the rod loses energy at the lateral, curved surace by convection to ambient air at $15°C$, with a heat transfer coefficient of $25W/m^2K$. Solve this problem by means of a finite element formulation and compare the results with the corresponding analytical solution. Study the effect of the number of elements used on the numerical solution.

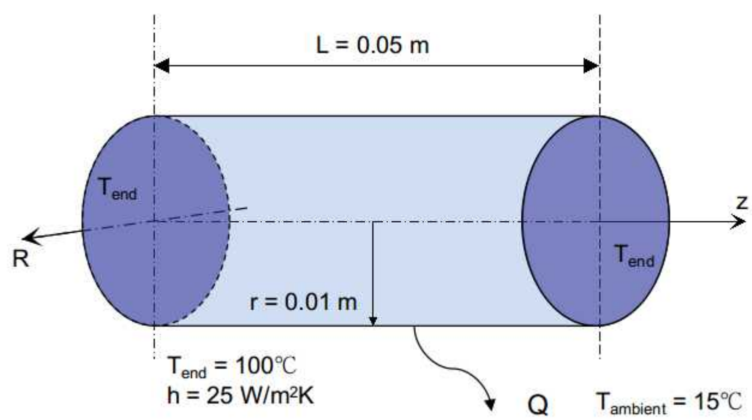

Figure 1: Problem layout

2 Mathematical Description and Analytical Solution

Deriving the governing eqation for this problem under cylindrical coordinate. Note that it is stated in problem that this is a 1-D steady state problem. Since the physical domain is aximetrical and BCs at two ends of the rod are identical, it's reasonable to take only half of the rod as our compuational domain.

Basic layout of this problem is shown in figure down below:

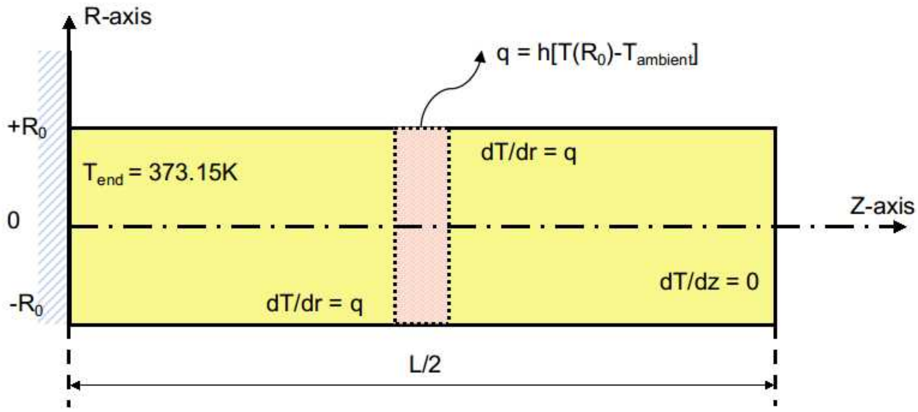

Figure 2: 2-D Scheme of the problem

Thus the governing equation is as shown below:

$$k\frac{\partial^2 T}{\partial z^2} = \frac{hP}{A}\left(T - T_{ambient}\right) \tag{1}$$

where $k = 237W/mK$. BCs along longtitute (symetric axis) direction:

$$T\left(0\right) = 100^{\circ}C = 373.15K \tag{2}$$

$$\frac{\partial T}{\partial z}\left(\frac{L}{2}\right) = 0 \tag{3}$$

Thus this equation could be solved analytically by assuming solution is of format:

$$T\left(z\right) = C + Ae^{\lambda z} + Be^{-\lambda z} \tag{4}$$

Thus applying BCs and solve for coefficients of the solution:

$$C = Ta \tag{5}$$

$$\lambda = \sqrt{\frac{hP}{kA}} \tag{6}$$

$$A = -\frac{T_a}{1 + e^{\lambda L}} \tag{7}$$

$$B = -Ta - A \tag{8}$$

3 Solution of the Problem Using FEM

Since in this case there's no source term, governing equation of our problem could be converge into FEM format using Galerking method. In our case, for simplicity, linear 2-node element (linear shape function) is applied.

$$(A_k)(\phi) = (F_{S_1}) + (F_{S_2}) + (F_{S_3}) \tag{9}$$

Rewrite in to 1-D steady state problem format:

$$k\frac{\phi_0 - \phi_1}{\Delta x} = h(\phi_1 - T_a) \tag{10}$$

$$k\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{\Delta x^2} = \frac{hP}{A}(\phi_i - T_a) \tag{11}$$

$$k\frac{\phi_{N-1} - \phi_N}{\Delta x} = 0 \tag{12}$$

Rewrite these equations into matrix format, rigidity matrix and bias vectors could be acquired.

MATLAB code for FEM method is shown down below.(In this code I still used full length since it's easier)

Listing 1: FEM1D

```
function [T,M,G] = FEM_1D(dx,L,Ta,h,T0,k,r)

N=L/dx;

M=zeros(N+1);

% Initializing output vector
T=zeros(N+1,1);
%  T(1)=T0; T(N+1)=T0;

% Cross section coefficients
A=pi*r^2; P=2*pi*r;

%% Initializing rigidity matrix
for i=1:N+1

    if i==1
        M(i,i)=dx/k;M(i,i+1)=1;
    elseif i~=N+1
        M(i,i-1)=k/(dx^2);
        M(i,i)=-2*k/(dx^2);
        M(i,i+1)=k/(dx^2);
    else
```

```
        M(i,i)=-dx/k;M(i,i-1)=1;
    end
end

% Eliminating nodes on Dirichlet BCs
%   M2=M(2:N,2:N);

%% Initializing convection terms

I=eye(N+1);

I=I*(h*P/A);
  I(1,1)=0;I(N+1,N+1)=0;

%% Combine into final rigidity matrix

G=M-I;

%% Bias vector

b=ones(N+1,1);
b=b*(h*P*Ta/A);
b(1)=T0;b(N+1)=T0;

%% Solve

T=G\b;%(2:N)
T(1,1)=T0;T(N+1,N+1)=T0;
```

Running Result of my code:

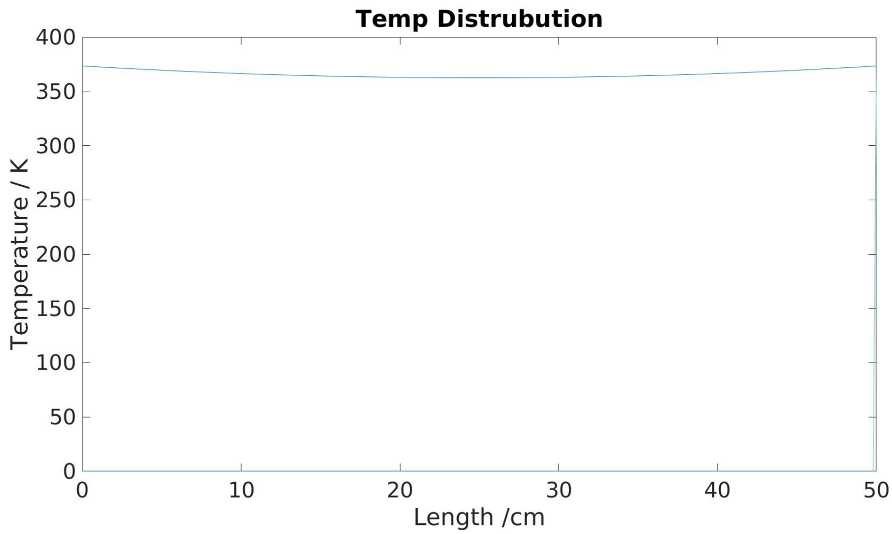

Figure 3: Result at grid step of 0.001 m