Homework 2

Name:

Student ID:

- Reasoning and work must be shown to gain partial/full credit
- Please include the cover-page on your homework PDF with your name and student ID. Failure of doing so is considered bad citizenship.

1. (1–4 points) **On walks and the adjacency matrix, and the incidence matrix**:
   (a) Consider a directed graph with no cycles or self-loops: explain why for any $k > 0$ the matrix $\boldsymbol{A}^k$ has its diagonal equal to zero.
   (b) an undirected graph with no self-loop. Why is the diagonal of $\boldsymbol{A}^2$ equal to the degree sequence?
   (c) Suppose you have a tree with 20 nodes. What is the dimension of its incidence matrix $\boldsymbol{B}$?

2. (1–4 points) **Max-Flow**:*Python/NetworkX question* Consider a group of N friends, indexed by $i \in [0, 1, \ldots, N-1]$, on a road trip. Suppose that each person paid for various group expenditures throughout the course of the trip. Some of the friends spent more than their share of the total (denoted by T), and naturally, some paid less than their share. At the end of the trip, those who are in the deficit need to settle the balance with the others, so that each of them pay exactly T/N.
   (a) Choose $N = 10$ and set the total expenditure over the course of the trip to $T = 1000 \cdot N$.
   (b) Generate a vector of individual expenditures according to a multinomial distribution with N using the following script:
   `amounts_spent = numpy.random.multinomial(T, pvals = np.ones(N) / N)`
   Verify that the sum of all the elements in the `amounts_spent` array is T.
   (c) Generate a complete graph among the friends with edge capacity set to ∞ for all edges.
   (d) Identify the nodes that (1) need to pay (i.e., $T/N - \text{amounts\_spent}[i] > 0$) and (2) those that need to receive money (i.e., $T/N - \text{amounts\_spent}[i] \leq 0$) to balance the books.
   (e) Build the following flow network:
      1. Introduce a source node s and connect it to the nodes in the former set with edge capacity equal to their individual balance (equal to $T/N - \text{amounts\_spent}[i]$).
      2. Introduce a target node t and connect each node in the latter group to t with edge capacity equal to their individual balance (equal to $\text{amounts\_spent}[i] - T/N$).

      Plot the flow network. You can use the `flow_layout` the method in the attached script file to get a flow layout for this graph.
   (f) If you run the Max-Flow algorithm, it can be shown that each flow represents the set of transactions that need to take place to balance the books. Use the max-flow algorithm to find the transactions required to settle up the balances, and verify whether the books were balanced using the flow.
   Useful methods: `networkx.maximum_flow`.
   Useful methods: Use `draw_flow` from the attached script file to draw the flow network with the flows highlighted.