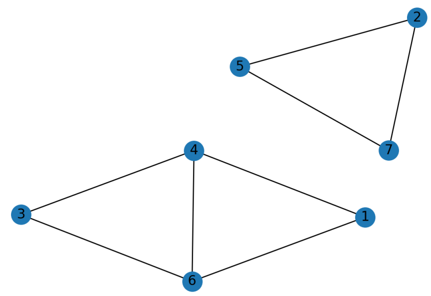
Homework 3

Name:

Student ID:

- Reasoning and work must be shown to gain partial/full credit
- Please include the cover-page on your homework PDF with your name and student ID. Failure of doing so is considered bad citizenship.

1. (1–4 points) **Algebra of graphs**: Considering the undirected graph in the figure, do the following:



   (a) Write the adjacency matrix of the graph

   (b) Then, write both the directed (oriented) and undirected (unoriented) incidence matrix of the graph. (*Hint: For the directed incidence matrix, pick a random direction for each edge.*)

   (c) Verify in networkx that the Laplacian matrix of the graph is $\mathbf{L} = \mathbf{D} - \mathbf{A}$ where $\mathbf{D} = \text{diag}(\mathbf{d})$ and $\mathbf{d}$ is a vector containing the degree of each node, and $\mathbf{A}$ is the adjancency matrix of the graph. Also, verify that $\mathbf{L} = \mathbf{B}\mathbf{B}^T$ where $\mathbf{B}$ is the directed incidence matrix.

   (d) Show that there exist two linearly independent vectors that are in the null-space of the Laplacian and explain why that is the case.

2. (1–4 points) **Graphs with circular symmetry**: Consider an undirected graph with circular symmetry, that is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{0, \ldots, N-1\}$ is the set of nodes and $\mathcal{E} = \{e_0, e_1, \ldots, e_{N-1}\}$ is the set of edges, where $e_0 = (0, N-1)$ and $e_i = (i, i+1), \forall i \in \{0, \ldots, N-2\}$.

   (a) Assume $N = 5$, write the edge set and draw the graph.

   (b) In lecture 5 Slide 27, we show that for a cycle graph, the eigenvalues of the Laplacian may be computed without the need to perform an eigendecomposition of the matrix $\mathbf{L}$. Validate numerically that this statement is correct (*hint: you may use numpy's linalg.eig function to compute the eigenvalues and eigenvectors of the Laplacian. You can also use numpy's fft function* `https://numpy.org/doc/stable/reference/generated/numpy.fft.fft.html`)

   (c) Plot the Fiedler eigenvalue (the second smallest eigenvalue of the Laplacian) for this type of graph as a function of $N = 5$ up to 50 with step 5, and explain the trend.

   (d) Now, fix the size of the graph to $N = 10$. Since you have circular symmetry, you can explore adding new edges connecting node 1 to the i-th node $\forall i \in \{2, \ldots, N-1\}$. Thus, you may add an additional $N - 3$ edges. Explore numerically the trend of the Fiedler eigenvalue, as you test the different possible options for the new edges.