

September 21

1 Momentum + Cautious WD

$$\dot{x} = -m - \mathbb{I}(mx \geq 0)x \quad (1)$$

$$\dot{m} = \nabla f(x) - m. \quad (2)$$

$$H(x, m) = f(x) + \frac{1}{2}m^2 + (mx)_+$$

Let us use $s \in [0, 1]$ to denote the subgradient of $(mx)_+$ within H . Let $p \in [0, 1]$ be the Flippiov variable of $\mathbb{I}(mx \geq 0)$ inside the velocity field. Note that s and p are different. We need to keep track of it in the derivation.

$$\begin{aligned} \dot{H}(s, p) &= g\dot{x} + m\dot{m} + s(m\dot{x} + x\dot{m}) \\ &= -g(m + px) + m(g - m) + s(-m(m + px) + x(g - m)) \\ &= -pxg - m^2 + s(-m^2 + xg - (1 + p)xm) \end{aligned}$$

If α belongs to \dot{H} , we want that for each s , there exists p , such that

$$-pxg - m^2 + s(-m^2 + xg - (1 + p)xm) = -\alpha.$$

So we have

$$\begin{aligned} p(-xg - sxm) - m^2 + s(-m^2 + xg - xm) &= \alpha. \\ p &= \frac{\alpha - (1 + s)m^2 + s(xg - xm)}{xg + sxm}. \end{aligned}$$

At the non-smooth point, we have $xm = 0$, and it gives

$$p = \frac{\alpha - (1 + s)m^2 + s(xg)}{xg}.$$

So p is in range of

$$\frac{\alpha - 2m^2 + xg}{xg}, \quad \frac{\alpha - m^2}{xg}.$$

If $xg > 0$, we have

$$\begin{aligned} m^2 &\leq \alpha \leq m^2 + xg \\ \alpha &\leq 2m^2 \leq \alpha + xg \end{aligned}$$

So

$$2m^2 - xg \leq \alpha \leq 2m^2$$

$$\cap_s \cup_p \dot{H}(s, p).$$

For know the flippiov system, the set valued Lie-derivation is defined as:

$$\mathcal{L}_v H = \{\alpha : \exists p, \quad s.t. \quad \dot{H}(s, p) = \alpha \quad \forall s \in \partial H\}.$$

So we want to find the p such that the $\dot{H}(s, p)$ is independent with the choice of s .

So to make it independent on s , we want

$$-m^2 + xg - (1 + p)xm = 0.$$

This gives

$$\begin{aligned}\dot{H} &= -p(m^2 + (1 + p)xm) - m^2 \\ &= -(1 + p)m^2 - p(1 + p)xm.\end{aligned}$$

If $xm < 0$, we have $p = 0$, and hence $\dot{H} = -m^2$. Invariant set is $\{xm < 0, m = 0\}$.

If $xm > 0$, we have $p = 1$, and hence, $\dot{H} = -(2m^2 + 2(xm)_+)$. Invariant set is $\{xm = 0, m = 0\}$.

If $xm = 0$, we have $\dot{H} = -(1 + p)m^2$. Invariant set is $\{xm = 0, m = 0\}$.

Hence, the invariant set is included in

$$\{xm \leq 0, m = 0\}.$$

But since $m = 0$, we have $xm \leq 0$ anyway.

2 General Fillipov From Scratch

Theorem 1. Assume we have non-smooth function $V(x)$ with clark generalized gradient, and let d be an update direction. We have

Proof.

$$\lim_{\epsilon \rightarrow 0^+} \frac{V(x + \epsilon d) - V(x)}{\epsilon} = \dots$$

□

Theorem 2.1 (Great Theorem). *Yes, it is.*

Proof. Yes.

□

Theorem 2.2 (Great Theorem). *This is the second.*

Proof. This is the best.

□