September 21

1 Momentum + Cautious WD

$$\dot{x} = -m - \mathbb{I}(mx \geq 0)x \tag{1}$$
$$\dot{m} = \nabla f(x) - m. \tag{2}$$

$$H(x, m) = f(x) + \frac{1}{2}m^2 + (mx)_+$$

Let us use $s \in [0, 1]$ to denote the subgradient of $(mx)_+$ within H. Let $p \in [0, 1]$ be the Flippiov variable of $\mathbb{I}(mx \geq 0)$ inside the velocity field. Note that s and p are different. We need to keep track of it in the derivation.

$$\begin{aligned}
\dot{H}(s, p) &= g\dot{x} + m\dot{m} + s(m\dot{x} + x\dot{m}) \\
&= -g(m + px) + m(g - m) + s(-m(m + px) + x(g - m)) \\
&= -pxg - m^2 + s(-m^2 + xg - (1 + p)xm)
\end{aligned}$$

If α belongs to $\dot{H}$, we want that for each s, there exists p, such that

$$-pxg - m^2 + s(-m^2 + xg - (1 + p)xm) = -\alpha.$$

So we have

$$p(-xg - sxm) - m^2 + s(-m^2 + xg - xm) = \alpha.$$

$$p = \frac{\alpha - (1 + s)m^2 + s(xg - xm)}{xg + sxm}.$$

At the non-smooth point, we have $xm = 0$, and it gives

$$p = \frac{\alpha - (1 + s)m^2 + s(xg)}{xg}.$$

So p is in range of

$$\frac{\alpha - 2m^2 + xg}{xg}, \qquad \frac{\alpha - m^2}{xg}.$$

If $xg > 0$, we have

$$m^2 \leq \alpha \leq m^2 + xg$$
$$\alpha \leq 2m^2 \leq \alpha + xg$$

So

$$2m^2 - xg \leq \alpha \leq 2m^2$$

$$\cap_s \cup_p \dot{H}(s, p).$$

For know the flippov system, the set valued Lie-derivation is defined as:

$$\mathcal{L}_v H = \{\alpha \colon \exists p, \quad s.t. \quad \dot{H}(s, p) = \alpha \quad \forall s \in \partial H\}.$$

So we want to find the p such that the $\dot{H}(s,p)$ is independent with the choice of s.

So to make it independent on s, we want

$$-m^2 + xg - (1+p)xm = 0.$$

This gives

$$\begin{aligned}\dot{H} &= -p(m^2 + (1+p)xm) - m^2 \\ &= -(1+p)m^2 - p(1+p)xm.\end{aligned}$$

If $xm < 0$, we have $p = 0$, and hence $\dot{H} = -m^2$. Invariant set is $\{xm < 0, \quad m = 0\}$.

If $xm > 0$, we have $p = 1$, and hence, $\dot{H} = -(2m^2 + 2(xm)_+)$. Invariant set is $\{xm = 0, \quad m = 0\}$.

If $xm = 0$, we have $\dot{H} = -(1+p)m^2$. Invariant set is $\{xm = 0, m = 0\}$.

Hence, the invariant set is included in

$$\{xm \leq 0, \quad m = 0\}.$$

But since $m = 0$, we have $xm \leq 0$ anyway.

2 General Fillipov From Scratch

Theorem 1. Assume we have non-smooth function $V(x)$ with clark generalized gradient, and let d be an update direction. We have

Proof.

$$\lim_{\epsilon \to 0^+} \frac{V(x + \epsilon d) - V(x)}{\epsilon} = \ldots$$

□

Theorem 2.1 (Great Theorem)**.** *Yes, it is.*

Proof. Yes. □

Theorem 2.2 (Great Theorem)**.** *This is the second.*

Proof. This is the best. □

References