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THE NORMALIZING TRANSFORMATION OF THE IMPLIED VOLATILITY SMILE

MASAAKI FUKASAWA

Center for the Study of Finance and Insurance, Osaka University

We study specific nonlinear transformations of the Black–Scholes implied volatility to show remarkable properties of the volatility surface. No arbitrage bounds on the implied volatility skew are given. Pricing formulas for European payoffs are given in terms of the implied volatility smile.

KEY WORDS: implied volatility, no arbitrage bounds, variance swap, gamma swap.

1. INTRODUCTION

This study is motivated by an elegant formula (11.5) in Gatheral (2006):

$$-2\mathbb{E}\left[\log\frac{S_T}{F}\right]=\int_{-\infty}^{\infty}\sigma(g_2(z))^2\phi(z)\,\mathrm{d}z, \tag{1.1}$$

where F is the forward price of an asset S_T, ϕ is the standard normal density, σ is the Black–Scholes implied volatility as a function of the log-moneyness $k=\log(K/F)$ with strike price K, and g_2 is the inverse function of a specific nonlinear transformation of the log moneyness

$$k\mapsto -d_2(k,\sigma(k)),\quad d_2(k,a)=-\frac{k}{a}-\frac{a}{2}.$$

The essentially same formula can be found in Chriss and Morokoff (1999). See also Carr and Lee (2009). This formula relates the fair strike of a variance swap to the implied volatility smile; see Gatheral (2006) and Carr, Lee, and Wu (2011). Not only to prove it but also to ensure that the formula itself is well defined, the transformation $k\mapsto -d_2(k,\sigma(k))$ has been assumed, implicitly or explicitly, to be an increasing function in the literature. In this paper, we show that the transformation is always increasing and enjoys other nice properties which imply in particular no arbitrage bounds on the implied volatility skew, that is, the first derivative of σ. Gatheral's formula is proved in an extended form. In particular, we obtain a dual formula

$$2\mathbb{E}\left[\frac{S_T}{F}\log\frac{S_T}{F}\right]=\int_{-\infty}^{\infty}\sigma(g_1(z))^2\phi(z)\,\mathrm{d}z, \tag{1.2}$$

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Address correspondence to Masaaki Fukasawa, Center for the Study of Finance and Insurance, Osaka University, 1-3 Machikaneyamacho Toyonaka, Osaka 560-8531, Japan; e-mail: fukasawa@sig.es.osaka-u.ac.jp.

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where g_1 is the inverse function of the transformation

$$k \mapsto -d_1(k, \sigma(k)), \quad d_1(k, a) = -\frac{k}{a} + \frac{a}{2}.$$

The left-hand side expectation coincides with the fair strike of a gamma swap; see, e.g., Davis, Obloj, and Raval (2010). Hence we find a simple dual relation between the two major volatility derivatives, that is, the variance swap and the gamma swap. The following formula for power payoffs is another example; for any q,

$$\mathbb{E}[S_T^q] = F^q \int_{-\infty}^{\infty} \{q e^{(q-1)g_1(z)} - (q-1) e^{q g_2(z)}\} \phi(z)\, \mathrm{d}z. \tag{1.3}$$

The results are model-independent and directly useful in practice.

2. BASIC RESULTS

Let $F > 0$ stand for the T-expiry forward price of an asset S_t at time $t = 0$, which is supposed to be observable as well as S_0. Let $\mathbb{E}$ be the pricing measure at time $t = 0$. We assume the following two conditions throughout this paper.

ASSUMPTION 2.1. *The law of S_T under $\mathbb{E}$ has a density and $\mathbb{E}[S_T] = F$.*

ASSUMPTION 2.2. *The market price $\hat{P}(K)$ of the put option with strike price $K > 0$ and maturity T is given by*

$$\hat{P}(K) = P(K) S_0 / F, \quad P(K) = \mathbb{E}[(K - S_T)_+].$$

DEFINITION 2.3. The (undiscounted) Black–Scholes put price is a function of $k \in \mathbb{R}$ and $\sigma \in (0, \infty)$ defined as

$$P_{\mathrm{BS}}(k, \sigma) = F e^k \Phi(-d_2(k, \sigma)) - F \Phi(-d_1(k, \sigma)),$$

where

$$d_2(k, \sigma) = \frac{-k - \sigma^2/2}{\sigma}, \quad d_1(k, \sigma) = d_2(k, \sigma) + \sigma.$$

DEFINITION 2.4. The Black–Scholes implied volatility is a function of $k \in \mathbb{R}$ defined as

$$\sigma(k) = P_{\mathrm{BS}}(k, \cdot)^{-1}(P(F e^k)),$$

or equivalently,

$$P_{\mathrm{BS}}(k, \sigma(k)) = P(F e^k).$$

As is well known, $P_{\mathrm{BS}}(k, \cdot)$ is an increasing function for each fixed $k \in \mathbb{R}$. Further, note that $(K - F)_+ \leq P(K) < K$ by Jensen's inequality and

$$\lim_{\sigma \to 0} P_{\mathrm{BS}}(k, \sigma) = F(e^k - 1)_+, \quad \lim_{\sigma \to \infty} P_{\mathrm{BS}}(k, \sigma) = Fe^k,$$

so that the Black–Scholes implied volatility is well defined.

DEFINITION 2.5. The first and second normalizing transformations (of log-moneyness) are functions f_1 and f_2 on $\mathbb{R}$ defined as $f_1(k) = -\, d_1(k, \sigma(k))$ and $f_2(k) = -\, d_2(k, \sigma(k))$, respectively, for $k \in \mathbb{R}$.

The purpose of this paper is to analyze these two normalizing transformations. Let $D_{\mathrm{BS}}(K)$ be a function of $K = Fe^k$ defined as

$$D_{\mathrm{BS}}(K) = \frac{1}{K} \frac{\partial P_{\mathrm{BS}}}{\partial k}(k, \sigma)|_{\sigma = \sigma(k)} = \Phi(f_2(k)).$$

Note that $D(K)$ defined as

$$D(K) := \mathbb{E}\left[I(K > S_T)\right] = \frac{\mathrm{d}P}{\mathrm{d}K}(K),$$

where I is the indicator function, is the theoretical (undiscounted) price of a digital option, which is an increasing function of K. By definitions,

$$\begin{aligned} D(K) &= \frac{\mathrm{d}P}{\mathrm{d}K}(K) \\ &= \frac{\mathrm{d}}{\mathrm{d}K} P_{\mathrm{BS}}(\log(K/F), \sigma(\log(K/F))) \\ &= D_{\mathrm{BS}}(K) + \frac{1}{K}\frac{\partial P_{\mathrm{BS}}}{\partial \sigma}(\log(K/F), \sigma(\log(K/F)))\frac{\mathrm{d}\sigma}{\mathrm{d}k}(\log(K/F)) \\ &= D_{\mathrm{BS}}(K) + \phi(-d_2(\log(K/F), \sigma(\log(K/F))))\frac{\mathrm{d}\sigma}{\mathrm{d}k}(\log(K/F)). \end{aligned} \tag{2.1}$$

LEMMA 2.6. *For all* $k \in \mathbb{R}$,

$$f_2(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) < 1.$$

Proof. The inequality is trivial when $f_2(k) = 0$. If $f_2(k) > 0$, it follows from (2.1) that

$$f_2(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) = f_2(k)\frac{D(Fe^k) - D_{\mathrm{BS}}(Fe^k)}{\phi(f_2(k))} \leq f_2(k)\frac{1 - \Phi(f_2(k))}{\phi(f_2(k))} < 1.$$

Here we used the fact that $0 \leq D(K) \leq 1$ and a well-known estimate

$$1 - \Phi(x) < x^{-1}\phi(x), \quad x > 0. \tag{2.2}$$

For the case $f_2(k) < 0$, we have

$$f_2(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) = f_2(k)\frac{D(Fe^k) - D_{\mathrm{BS}}(Fe^k)}{\phi(f_2(k))} \leq -f_2(k)\frac{\Phi(f_2(k))}{\phi(f_2(k))} = -f_2(k)\frac{1 - \Phi(-f_2(k))}{\phi(-f_2(k))} < 1.$$

□

LEMMA 2.7. *For all k with* $f_1(k) \le 0$,

$$f_1(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) < 1.$$

Proof. By definition, it holds that for all $K > 0$,

$$KD(K) \ge P(K).$$

Combining this and (2.1), we have

$$F\Phi(-d_1(k,\sigma(k))) + K\phi(-d_2(k,\sigma(k)))\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) \ge 0$$

with $k = \log(K/F)$. Since $K\phi(-d_2) = F\phi(-d_1)$, we obtain from (2.2),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) \ge -\frac{1-\Phi(d_1(k,\sigma(k)))}{\phi(d_1(k,\sigma(k)))} > -\frac{1}{d_1(k,\sigma(k))}.$$

□

THEOREM 2.8. *The first and second normalizing transformations* f_1, f_2 *are increasing.*

Proof. By definition,

$$f_1(k) = \frac{k}{\sigma(k)} - \frac{\sigma(k)}{2}, \quad f_2(k) = \frac{k}{\sigma(k)} + \frac{\sigma(k)}{2},$$

so that

$$\frac{\mathrm{d}f_1}{\mathrm{d}k}(k) = \frac{1}{\sigma(k)}\left\{1 - \frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\frac{k}{\sigma(k)}\right\} - \frac{1}{2}\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) = \frac{1}{\sigma(k)}\left\{1 - \frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)f_2(k)\right\}$$

and

$$\frac{\mathrm{d}f_2}{\mathrm{d}k}(k) = \frac{1}{\sigma(k)}\left\{1 - \frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\frac{k}{\sigma(k)}\right\} + \frac{1}{2}\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) = \frac{1}{\sigma(k)}\left\{1 - \frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)f_2(k)\right\} + \frac{\mathrm{d}\sigma}{\mathrm{d}k}(k).$$

(2.3)

Then the monotonicity of f_1 follows from Lemma 2.6. Also, by Lemma 2.6,

$$\frac{\mathrm{d}f_2}{\mathrm{d}k}(k) > \frac{\mathrm{d}\sigma}{\mathrm{d}k}(k).$$

It suffices then to treat the case $\mathrm{d}\sigma/\mathrm{d}k < 0$. By rewriting (2.3), we have

$$\frac{\mathrm{d}f_2}{\mathrm{d}k}(k) = \frac{1}{\sigma(k)}\left\{1 - \frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)f_1(k)\right\}.$$

If $f_1(k) > 0$, we have $\mathrm{d}f_2/\mathrm{d}k > 0$ under $\mathrm{d}\sigma/\mathrm{d}k < 0$. If $f_1(k) \le 0$, we can use Lemma 2.7 to obtain the same inequality. □

3. NO ARBITRAGE BOUNDS

Here we present no arbitrage bounds on the first and second derivatives of the Black–Scholes implied volatility σ.

THEOREM 3.1. *Denote by $p(\cdot)$ the density of $\log(S_T/F)$ under $\mathbb{E}$. Then,*

$$\frac{p(k)}{\phi(f_2(k))} > \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}(k) \geq -\left\{1 - f_2(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right\}\frac{\mathrm{d}f_2}{\mathrm{d}k}(k) \quad \text{a.e. } k \in \mathbb{R}.$$

Proof. The density of S_T is given by $\mathrm{d}\,D/\mathrm{d}\,K$, so we have

$$p(k) = Fe^k\frac{\mathrm{d}D}{\mathrm{d}K}(Fe^k).$$

Now differentiating (2.1) in $k = \log(K/F)$, we obtain

$$p(k) = \phi(f_2(k))\left\{\frac{\mathrm{d}f_2}{\mathrm{d}k}(k) - f_2(k)\frac{\mathrm{d}f_2}{\mathrm{d}k}(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) + \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}(k)\right\}.$$

Use Lemma 2.6 and Theorem 2.8 to obtain the result. □

LEMMA 3.2. *For all $k \geq 0$, $f_2(k) \geq \sqrt{2k}$.*

Proof. This is because the arithmetic mean exceeds the geometric mean. □

THEOREM 3.3. *For all $k > 0$,*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) < \frac{1}{\sqrt{2k}}.$$

Proof. This follows from Lemmas 2.7 and 3.2. □

REMARK 3.4. Theorem 3.3 should be compared with Lemma 3.1 of Lee (2004); it was shown that $\sigma(k) < \sqrt{2k}$ for sufficiently large $k > 0$. Also, Rogers and Tehranchi (2010) gave bounds on the derivative of σ^2.

LEMMA 3.5. *For all $k \leq 0$, $f_1(k) \leq -\sqrt{2|k|}$.*

Proof. This is because the arithmetic mean exceeds the geometric mean. □

THEOREM 3.6. *For all $k < 0$,*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) > -\frac{1}{\sqrt{2|k|}}.$$

Proof. This follows from Lemmas 2.7 and 3.2. □

REMARK 3.7. In Lemma 3.3 of Lee (2004), it was shown that $\sigma(k) < \sqrt{2|k|}$ for sufficiently small $k < 0$ if and only if $\mathbb{E}[I(S_T = 0)] < 1/2$. Also, Rogers and Tehranchi (2010) gave bounds on the derivative of σ^2.

4. PRICING FORMULAS FOR EUROPEAN PAYOFFS

Here we present pricing formulas for a general European payoff which extend Gatheral's formula for the fair price of log contract.

LEMMA 4.1. *If there exists $q > 0$ such that $\mathbb{E}[S_T^{1+q}] < \infty$, then there exists $q' > 0$ such that $f_1(k) > \sqrt{q'k}$ for all $k > 1/q'$.*

Proof. By theorem 3.2 of Lee (2004), there exists $\alpha \in (0, 2)$ such that $\sigma(k) < \sqrt{\alpha|k|}$ for all $k > 1/(2-\alpha)$. Since $f_1(k) = f_2(k) - \sigma(k) \geq \sqrt{2k} - \sigma(k)$ by Lemma 3.2, we obtain the result. □

LEMMA 4.2. *If there exists $q > 0$ such that $\mathbb{E}[S_T^{-q}] < \infty$, then there exists $q' > 0$ such that $f_2(k) < -\sqrt{q'|k|}$ for all $k < -1/q'$.*

Proof. By theorem 3.4 of Lee (2004), there exists $\alpha \in (0, 2)$ such that $\sigma(k) < \sqrt{\alpha|k|}$ for all $k < -1/(2-\alpha)$. Since $f_2(k) = -d_1(k, \sigma(k)) + \sigma(k) \leq -\sqrt{2|k|} + \sigma(k)$ by Lemma 3.5, we obtain the result. □

LEMMA 4.3. *For any function Ψ of polynomial growth,*

$$\lim_{k\to-\infty} \Psi(k)\sigma(k)\phi(f_1(k)) = 0, \quad \lim_{k\to-\infty} \Psi(k)\left|\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right|\phi(f_1(k)) = 0,$$

$$\lim_{k\to\infty} \Psi(k)\sigma(k)\phi(f_2(k)) = 0, \quad \lim_{k\to\infty} \Psi(k)\left|\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right|\phi(f_2(k)) = 0.$$

Furthermore, we have the following convergences.

- *If there exists $q > 0$ such that $\mathbb{E}[S_T^{1+q}] < \infty$, then*

$$\lim_{k\to\infty} \Psi(k)\sigma(k)\phi(f_1(k)) = 0, \quad \lim_{k\to\infty} \Psi(k)\left|\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right|\phi(f_1(k)) = 0.$$

- *If there exists $q > 0$ such that $\mathbb{E}[S_T^{-q}] < \infty$, then*

$$\lim_{k\to-\infty} \Psi(k)\sigma(k)\phi(f_2(k)) = 0, \quad \lim_{k\to-\infty} \Psi(k)\left|\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right|\phi(f_2(k)) = 0.$$

Proof. By Theorems 3.1 and 3.3, $\lim\ sup_{|k|\to\infty}\ \sigma(k)^2/|k| \leq 2$. Then the convergence with σ follows immediately from Lemmas 3.2, 3.5, 4.1, and 4.2. To deal with the derivative of σ, notice that by (2.1),

$$D(Fe^k) - D_{\mathrm{BS}}(Fe^k) = \phi(f_2(k))\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) = e^{-k}\phi(f_1(k))\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k),$$

so that it suffices to show

$$\lim_{k\to-\infty}\hat\Psi(k)D(Fe^k)=0,\quad \lim_{k\to\infty}\hat\Psi(k)(1-D(Fe^k))=0,$$

$$\lim_{k\to-\infty}\hat\Psi(k)D_{\mathrm{BS}}(Fe^k)=0,\quad \lim_{k\to\infty}\hat\Psi(k)(1-D_{\mathrm{BS}}(Fe^k))=0$$

for $\hat\Psi(k)=\Psi(k)$ and $\Psi(k)e^k$. The first two follow from the integrability condition of S_T. Observing $D_{\mathrm{BS}}(Fe^k)=\Phi(f_2(k))$ and $\phi(f_1(k))=e^k\phi(f_2(k))$, the last two follow from Lemmas 3.2, 3.5, 4.1, and 4.2 and (2.2). □

THEOREM 4.4. *Let Ψ be a twice differentiable function such that Ψ' is of polynomial growth. Let g_1 and g_2 be the inverse functions of f_1 and f_2, respectively.*

- *If there exists $q>0$ such that $\mathbb{E}[S_T^{1+q}]<\infty$, then*

$$\mathbb{E}\left[\frac{S_T}{F}\Psi\left(\log\frac{S_T}{F}\right)\right]=\int_{-\infty}^{\infty}\left\{\Psi(g_1(z))-\Psi'(g_1(z))\left\{g_1(z)-\frac{1}{2}\sigma(g_1(z))^2\right\}\right\}\phi(z)\,\mathrm{d}z$$
$$+\int_{-\infty}^{\infty}\Psi''(k)\sigma(k)\phi(f_1(k))\,\mathrm{d}k.$$

- *If there exists $q>0$ such that $\mathbb{E}[S_T^{-q}]<\infty$, then*

$$\mathbb{E}\left[\Psi\left(\log\frac{S_T}{F}\right)\right]=\int_{-\infty}^{\infty}\left\{\Psi(g_2(z))-\Psi'(g_2(z))\left\{g_2(z)+\frac{1}{2}\sigma(g_2(z))^2\right\}\right\}\phi(z)\,\mathrm{d}z$$
$$+\int_{-\infty}^{\infty}\Psi''(k)\sigma(k)\phi(f_2(k))\,\mathrm{d}k.$$

Proof. Note that the existence of g_1 and g_2 follows from Theorem 2.8. First, we prove the second formula. Since the density of S_T is given by $\mathrm{d}D/\mathrm{d}K$,

$$\mathbb{E}[\Psi(\log(S_T/F))]=\int_0^{\infty}\Psi(\log(K/F))\frac{\mathrm{d}^2P}{\mathrm{d}K^2}(K)\,\mathrm{d}K=\int_{-\infty}^{\infty}\Psi(k)\frac{\mathrm{d}^2P}{\mathrm{d}K^2}(Fe^k)Fe^k\mathrm{d}k.$$

Using (2.1), we have

$$\begin{aligned}\frac{\mathrm{d}^2P}{\mathrm{d}K^2}(Fe^k)&=\frac{\mathrm{d}D}{\mathrm{d}K}(Fe^k)\\&=\frac{1}{Fe^k}\phi(f_2(k))\left\{\frac{\mathrm{d}f_2}{\mathrm{d}k}(k)\left(1-f_2(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right)+\frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}(k)\right\}.\end{aligned}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}k}\phi(f_2(k))=-\phi(f_2(k))f_2(k)\frac{\mathrm{d}f_2}{\mathrm{d}k}(k),$$

we have

$$-\int_{-\infty}^{\infty}\Psi(k)\phi(f_2(k))f_2(k)\frac{\mathrm{d}f_2}{\mathrm{d}k}(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\,\mathrm{d}k=\left[\Psi(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\phi(f_2(k))\right]_{-\infty}^{\infty}$$
$$-\int_{-\infty}^{\infty}\left\{\Psi'(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)+\Psi(k)\frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}(k)\right\}\phi(f_2(k))\,\mathrm{d}k.$$

Hence, by Lemma 4.3,

$$\mathbb{E}[\Psi(\log(S_T/F))] = \int_{-\infty}^{\infty} \phi(f_2(k)) \left\{ \Psi(k)\frac{\mathrm{d}f_2}{\mathrm{d}k}(k) - \Psi'(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) \right\} \mathrm{d}k. \tag{4.1}$$

Since

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi(f_2(k))\Psi'(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\,\mathrm{d}k \\
&= \left[\phi(f_2(k))\Psi'(k)\sigma(k)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\mathrm{d}(\phi\Psi' \circ g_2)}{\mathrm{d}z}(f_2(k))\frac{\mathrm{d}f_2}{\mathrm{d}k}(k)\sigma(k)\,\mathrm{d}k \\
&= -\int_{-\infty}^{\infty} \left\{ \Psi''(k)\frac{\mathrm{d}g_2}{\mathrm{d}z}(f_2(k))\phi(f_2(k)) - \Psi'(k)f_2(k)\phi(f_2(k)) \right\} \frac{\mathrm{d}f_2}{\mathrm{d}k}(k)\sigma(k)\,\mathrm{d}k,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\mathbb{E}[\Psi(\log(S_T/F))] \\
&= \int_{-\infty}^{\infty} \left\{ \Psi(g_2(z)) - \Psi'(g_2(z))z\sigma(g_2(z)) + \Psi''(g_2(z))\sigma(g_2(z))\frac{\mathrm{d}g_2}{\mathrm{d}z}(z) \right\} \phi(z)\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} \left\{ \Psi(g_2(z)) - \Psi'(g_2(z))z\sigma(g_2(z)) \right\} \phi(z)\,\mathrm{d}z + \int_{-\infty}^{\infty} \Psi''(k)\sigma(k)\phi(f_2(k))\,\mathrm{d}k.
\end{aligned}$$

By definition,

$$g_2(z) - z\sigma(g_2(z)) = k - f_2(k)\sigma(k) = k + d_2(k, \sigma(k))\sigma(k) = -\frac{\sigma(k)^2}{2},$$

which completes the proof for the second formula. To prove the first one, repeat the same argument by replacing Ψ by $k \mapsto e^k\Psi(k)$. For example, use $\phi(f_1(k)) = e^k\phi(f_2(k))$ to obtain

$$\mathbb{E}[\Psi(\log(S_T/F))S_T/F] = \int_{-\infty}^{\infty} \phi(f_1(k)) \left\{ \Psi(k)\frac{\mathrm{d}f_1}{\mathrm{d}k}(k) - \Psi'(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k) \right\} \mathrm{d}k, \tag{4.2}$$

instead of (4.1). □

REMARK 4.5. By letting $\Psi(k) = k$ in Theorem 4.4, we obtain (1.1) and (1.2).

THEOREM 4.6. *Let Ψ be an absolutely continuous function with derivative Ψ' of polynomial growth. Let g_1 and g_2 be the inverse functions of f_1 and f_2, respectively.*

- *If there exists $q > 0$ such that $\mathbb{E}[S_T^{1+q}] < \infty$, then*

$$\mathbb{E}\left[\frac{S_T}{F}\Psi\left(\log\frac{S_T}{F}\right)\right] = \int_{-\infty}^{\infty} \left\{ \Psi(g_1(z)) + \Psi'(g_1(z)) - \Psi'(g_2(z))e^{g_2(z)} \right\} \phi(z)\,\mathrm{d}z.$$

- *If there exists $q > 0$ such that $\mathbb{E}[S_T^{-q}] < \infty$, then*

$$\mathbb{E}\left[\Psi\left(\log\frac{S_T}{F}\right)\right] = \int_{-\infty}^{\infty} \left\{ \Psi(g_2(z)) - \Psi'(g_2(z)) + \Psi'(g_1(z))e^{-g_1(z)} \right\} \phi(z)\,\mathrm{d}z.$$

Proof. To prove the first formula, start from (4.2) and observe that

$$\begin{aligned}\int \Psi'(g_2(z))e^{g_2(z)}\phi(z)\,\mathrm{d}z &= \int \Psi'(k)e^k\phi(f_2(k))\frac{\mathrm{d}f_2}{\mathrm{d}k}(k)\,\mathrm{d}k \\ &= \int \Psi'(k)\phi(f_1(k))\left\{\frac{\mathrm{d}f_1}{\mathrm{d}k}(k)+\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right\}\mathrm{d}k \\ &= \int \Psi'(g_1(z))\phi(z)\,\mathrm{d}z + \int \phi(f_1(k))\Psi'(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\,\mathrm{d}k.\end{aligned}$$

To prove the second formula, start from (4.1) and observe that

$$\begin{aligned}\int \Psi'(g_1(z))e^{-g_1(z)}\phi(z)\,\mathrm{d}z &= \int \Psi'(k)e^{-k}\phi(f_1(k))\frac{\mathrm{d}f_1}{\mathrm{d}k}(k)\,\mathrm{d}k \\ &= \int \Psi'(k)\phi(f_2(k))\left\{\frac{\mathrm{d}f_2}{\mathrm{d}k}(k)-\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\right\}\mathrm{d}k \\ &= \int \Psi'(g_2(z))\phi(z)\,\mathrm{d}z - \int \phi(f_2(k))\Psi'(k)\frac{\mathrm{d}\sigma}{\mathrm{d}k}(k)\,\mathrm{d}k.\end{aligned}$$

□

REMARK 4.7. When considering the put payoff functions as Ψ in Theorem 4.6, we recover $P(Fe^k) = P_{\mathrm{BS}}(k, \sigma(k))$.

REMARK 4.8. The assumption of polynomial growth in Theorems 4.4 and 4.6 can be relaxed by assuming instead the existence of higher order moments of S_T and S_T^{-1} in the light of theorems 3.2 and 3.4 of Lee (2004). Then, we have, for example (1.3) by taking $\Psi(k) = \exp(qk)$.

REMARK 4.9. The formulas in Theorems 4.4 and 4.6 enable us to derive the fair price of an European option directly from the implied volatility surface. The point is that no derivative of σ is appeared in those formulas. This is quite important in practice because the implied volatility $\sigma(k)$ is discretely observed. The integrations involve with the inverse functions g_i, $i = 1, 2$; they can be, however, treated efficiently as follows. In order to approximate an integral of the form

$$\int \psi(g_i(z))\phi(z)\,\mathrm{d}z$$

by using finite data $f_i(k_j)$, $j = 1, \ldots, n$, we interpolate $(f_i(k_j), \psi(k_j))$ by a C^1 and piecewise C^2 function h so that $\psi(k_j) = h(f_i(k_j))$. It suffices then to compute

$$\int h(z)\phi(z)\,\mathrm{d}z.$$

Further, by using a piecewise polynomial function as h, we can avoid numerical integrations due to a well-known property of the Hermite polynomial system. In terms of static hedging, the standard model-free formula (11.1) in Gatheral (2006) will be more convenient. On the other hand, our formulas are more compatible with the interpolation and extrapolation of the implied volatility. Moreover, Theorem 4.6 has an advantage that it admits functions which are not twice differentiable.

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