1 a为何值时，下述线性方程组有唯一解？a为何值时，此方程组无解？

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 + 2x_2 - ax_3 = 9 \\ 2x_1 - x_2 + 3x_3 = 6 \end{cases}$$

$a = -\frac{2}{3}$ 无解

$a \neq -\frac{2}{3}$ 有唯一解

Proof. 该方程组对应的矩阵为 $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -a & 9 \\ 2 & -1 & 3 & 6 \end{array}\right) \xrightarrow[③+(-2)①]{②+(-1)①} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -a-1 & 6 \\ 0 & -3 & 1 & 0 \end{array}\right)$

$$\xrightarrow{③+3②} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -a-1 & 6 \\ 0 & 0 & -3a-2 & 18 \end{array}\right)$$

故当 $-3a-2 \neq 0$，即 $a \neq -\frac{2}{3}$ 时，该方程组有唯一解

当 $a = -\frac{2}{3}$ 时，该方程组无解

2. 求三次多项式 $f(x) = ax^3 + bx^2 + cx + d$ 满足 $f(0)=1$，$f(1)=2$，$f'(0)=1$，$f'(1)=-1$

Proof $f'(x) = 3ax^2 + 2bx + c$，题条件就等价于

$f(x) = -2x^3 + 2x^2 + x + 1$

$$\begin{cases} d = 1 \\ a+b+c+d = 2 \\ c = 1 \\ 3a+2b+c = -1 \end{cases}$$

对应的矩阵为 $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 3 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{②+(-3)①}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -3 & -7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[\substack{①+(-1)③ \\ ①+(-1)④}]{\substack{②+2\cdot③ \\ ②+3\cdot④}} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{①+1②}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{cases} a = -2 \\ -b = -2 \\ c = 1 \\ d = 1 \end{cases} \quad 即 \quad \begin{cases} a = -2 \\ b = 2 \\ c = 1 \\ d = 1 \end{cases}$$

则 $f(x) = -2x^3 + 2x^2 + x + 1$

3 给定平面直角坐标系中的5个点 $A=(-2,-2)$，$B=(1,-1)$，$C=(2,0)$，$D=(0,2)$，$E=(-1,1)$，求一条二次代数曲线 $G(x,y) = ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ 通过5个点。

$G(x,y) = -7x^2 + 10xy - 7y^2 + 2x + 2y + 24$

Proof $\{ 4a + 8b + 4c - 2d - 2e + f = 0$

Proof
$$\begin{cases} a - 2b + c + d - e + f = 0 \\ 4a + 2d + f = 0 \\ 4c + 2e + f = 0 \\ a - 2b + c - d + e + f = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & -2 & 1 & -1 & 1 & 1 \\ 4 & 8 & 4 & -2 & -2 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 4 & 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{④}+(-4)\text{①}]{\text{②}+(-1)\text{①},\ \text{③}+(-4)\text{①}} \begin{pmatrix} 1 & -2 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 2 & 0 \\ 0 & 16 & 0 & -6 & 2 & -3 \\ 0 & 8 & -4 & -2 & 4 & -3 \\ 0 & 0 & 4 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{\text{互换}} \begin{pmatrix} 1 & -2 & 1 & 1 & -1 & 1 \\ 0 & 8 & -4 & -2 & 4 & -3 \\ 0 & 16 & 0 & -6 & 2 & -3 \\ 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{③}+(-2)\text{②}} \begin{pmatrix} 1 & -2 & 1 & 1 & -1 & 1 \\ 0 & 8 & -4 & -2 & 4 & -3 \\ 0 & 0 & 8 & -2 & -6 & 3 \\ 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{③}+(-2)\text{④}} \begin{pmatrix} 1 & -2 & 1 & 1 & -1 & 1 \\ 0 & 8 & -4 & -2 & 4 & -3 \\ 0 & 0 & 0 & -2 & -10 & 1 \\ 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{换位}]{\text{③}+(-2)\text{⑤}} \begin{pmatrix} 1 & -2 & 1 & 1 & -1 & 1 \\ 0 & 8 & -4 & -2 & 4 & -3 \\ 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -12 & 1 \end{pmatrix} \xrightarrow[\text{①}+1\text{④}]{\text{②}+(-2)\cdot\text{④}} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 8 & -4 & 0 & 2 & -3 \\ 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -12 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{②}+1\text{③}} \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 & 4 & -2 \\ 0 & 0 & 4 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -12 & 1 \end{pmatrix} \Rightarrow \begin{cases} a - 2b + c = -f \\ 8b + 4e = 2f \\ 4c + 2e = -f \\ -d + e = 0 \\ -12e = -f \end{cases}$$

取令 $f=24$ (为了都是整数), 则 $e=2$, $d=2$, $c=-7$, $b=5$, $a=-7$

则 $G(x,y) = -7x^2 + 10xy - 7y^2 + 2x + 2y + 24$ (差一个倍数都对)

4. 懒得抄题了.. 不可以配备

Proof 设 A. B. C. D 分别配备 a. b. c. d 千克. 则

$$\begin{cases} 5a + 4b + 7c + 10d = 100 \\ \dots \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 5 & 3 & 25 \\ 5 & 4 & 7 & 10 & 100 \end{array}\right)$$

$$\begin{cases} \cdots \\ 20a + 25b + 10c + 5d = 200 \\ 2a + 2b + 10c + 6d = 50 \end{cases} \rightarrow \left(\begin{array}{cccc|c} 5 & 4 & 1 & 10 & 100 \\ 4 & 5 & 2 & 1 & 40 \end{array}\right)$$

$$\xrightarrow[③+(-4)①]{②+(-5)\cdot ①} \left(\begin{array}{cccc|c} 1 & 1 & 5 & 3 & 25 \\ 0 & -1 & -18 & -5 & -25 \\ 0 & 1 & -18 & -11 & -60 \end{array}\right) \xrightarrow{③+1\cdot ②} \left(\begin{array}{cccc|c} 1 & 1 & 5 & 3 & 25 \\ 0 & 1 & 18 & 5 & 25 \\ 0 & 0 & -36 & -16 & -85 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 2 & 2 & 10 & 6 & 50 \\ 0 & 2 & 36 & 10 & 50 \\ 0 & 0 & 36 & 16 & 85 \end{array}\right) \xrightarrow[①+(-1)\cdot ②]{①+(-\frac{5}{18})③ \quad ②+(-1)③ \quad 然后} \left(\begin{array}{cccc|c} 2 & 0 & 0 & \frac{68}{9} & \frac{1105}{18} \\ 0 & 2 & 0 & -6 & -35 \\ 0 & 0 & 36 & 16 & 85 \end{array}\right)$$

$$\Rightarrow \begin{cases} 2a = -\frac{68}{9}d + \frac{1105}{18} \\ 2b = 6d - 35 \\ 36c = -16d + 85 \end{cases}$$

由于 $c \geq 0$ 则 $0 \leq d \leq \frac{85}{16}$

$b \geq 0$，则 $d \geq \frac{35}{6}$

而 $\frac{35}{6} > \frac{85}{16}$，则该方程组无非负解 无法配备.

5 ① 利用外积，把 (a,b) 与 (c,d) 看作 $\mathbb{R}^3$ 中的 $\vec{\alpha}=(a,b,0)$，$\vec{\beta}=(c,d,0)$

$\vec{\alpha}\times\vec{\beta} = (ad-bc)\vec{e_3}$ $(\vec{e_3}=(0,0,1))$，则 $S = |\vec{\alpha}\times\vec{\beta}| = |ad-bc|$

② 记 $\vec{m}=(a,b)$，$\vec{n}=(c,d)$.

$$|\cos\langle\vec{m},\vec{n}\rangle| = \left|\frac{\vec{m}\cdot\vec{n}}{|\vec{m}|\cdot|\vec{n}|}\right| = \frac{|ac+bd|}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}} \qquad 记 \theta = \langle\vec{m},\vec{n}\rangle$$

(此角 sin 值为正)

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}} = \frac{|ad-bc|}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$$

$$S = |\vec{m}|\cdot|\vec{n}|\sin\theta = |ad-bc|$$

$\mathbb{R}^n$ 中 n 个向量构成的图形的体积为 n 个向量的行列式的绝对值