Math 151BH - HW1

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Problem 1:

$$\begin{aligned}\kappa_\infty(x) &= \lim_{\delta\to 0^+} \sup_{\|\delta_x\|\le\delta} \frac{|x_1+\delta_{x1}-x_2-\delta_{x2}-x_1+x_2|/|x_1-x_2|}{\|\delta_x\|_\infty/\|x\|_\infty} \\ &= \lim_{\delta\to 0^+} \sup_{\|\delta_x\|\le\delta} \frac{|\delta_{x1}-\delta_{x2}|/|x_1-x_2|}{\|\delta_x\|_\infty/\|x\|_\infty} \\ &= \lim_{\delta\to 0^+} \sup_{\|\delta_x\|\le\delta} \frac{\|x\|_\infty}{|x_1-x_2|}\cdot\frac{|\delta_{x1}-\delta_{x2}|}{\|\delta_x\|_\infty}\end{aligned}$$

the supremum is achieved for $\delta_x = (\delta, -\delta)$

$$\begin{aligned}&= \lim_{\delta\to 0^+} \frac{\|x\|_\infty}{|x_1-x_2|}\cdot\frac{2\delta}{\delta} \\ &= 2\frac{\|x\|_\infty}{|x_1-x_2|}\end{aligned}$$

then

$$\lim_{|x_1-x_2|\to 0} \kappa_\infty(x) = \infty$$

Problem 2:

$$\begin{aligned}0 \le \kappa(x) &= \lim_{\delta\to 0^+} \sup_{|\delta_x|\le\delta} \frac{|\sqrt{x+\delta_x}-\sqrt{x}|/\sqrt{x}}{|\delta_x|/x} \\ &= \lim_{\delta\to 0^+} \sup_{|\delta_x|\le\delta} \frac{|\sqrt{x+\delta_x}-\sqrt{x}|}{|\delta_x|}\sqrt{x}\end{aligned}$$

by MVT, $\frac{|\sqrt{x+\delta_x}-\sqrt{x}|}{|\delta_x|} \le \frac{1}{2}(x-\delta)^{-\frac{1}{2}}$, giving

$$\begin{aligned}&= \lim_{\delta\to 0^+} \frac{1}{2}(x-\delta)^{-\frac{1}{2}}x^{\frac{1}{2}} \\ &= \frac{1}{2}\end{aligned}$$

So this problem is well-conditioned.

Problem 3:
Let A, B be the first and second matrices respectively.
Since A is upper triangular, its eigenvalues are on the diagonals and thus equal 1.
For B, we can solve the characteristic polynomial:

$$(1-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda = 0, 2$$

For generalized a,

$$(1-\lambda)^2 - 1000a = 0 \Rightarrow \lambda^2 - 2\lambda + (1 - 1000a) = 0$$
$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4 - 4 + 4000a}}{2} = 1 \pm \sqrt{1000a} \Rightarrow \|\lambda\|_\infty = 1 + \sqrt{1000a}$$
$$\Rightarrow \hat{\kappa(0)} = \frac{d}{da}[1 + \sqrt{1000a}] = \frac{1}{2}\sqrt{1000}a^{-\frac{1}{2}}$$
$$\Rightarrow \lim_{x \to 0} \hat{\kappa}(x) = \infty$$

Problem 4: Let $P(c, x) = \sum_{i=0}^{n} c_i x^i$. Since $p'(r_j) = \frac{\delta}{\delta x} P(a, r_j) \neq 0$, by the implicit function theorem there exists a neighborhood around (a, r_j) such that $P(c, x) = 0$ can be solved for x in terms of coefficients c. Let this continuously differentiable function be $g(c)$. Then on this neighborhood, $P(c, g(c)) = 0$ for all c. Differentiating,

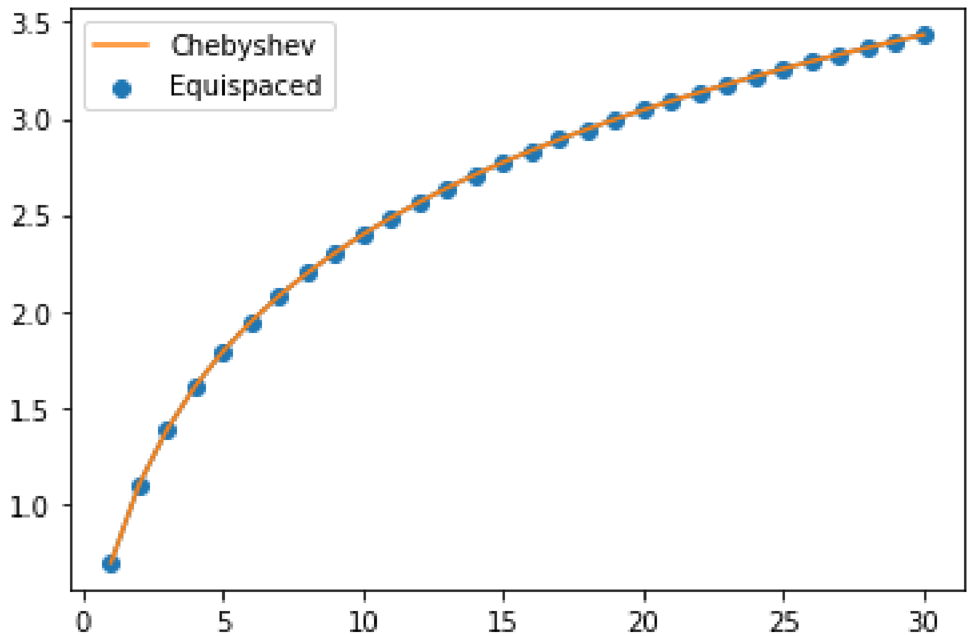
$$0 = \sum_{i=0}^{n} c_i g(c)^i$$
$$\Rightarrow 0 = \sum_{i=0}^{n} i c_i g(c)^{i-1} \frac{\delta}{\delta c_k} g(c) + g(c)^k$$
$$\Rightarrow 0 = \sum_{i=0}^{n} i a_i r_j^{i-1} \frac{\delta}{\delta c_k} g(a) + r_j^k$$
$$\Rightarrow \hat{\kappa} = |\frac{\delta}{\delta c_k} g(a)| = \frac{|r_j^k|}{|p'(r_j)|}$$
$$\Rightarrow \kappa = \frac{|r_j^k|}{|p'(r_j)|} \cdot \frac{|a_k|}{|r_j|}$$
$$= \frac{|a_k r_j^{k-1}|}{|p'(r_j)|}$$

Problem 5:

(a)

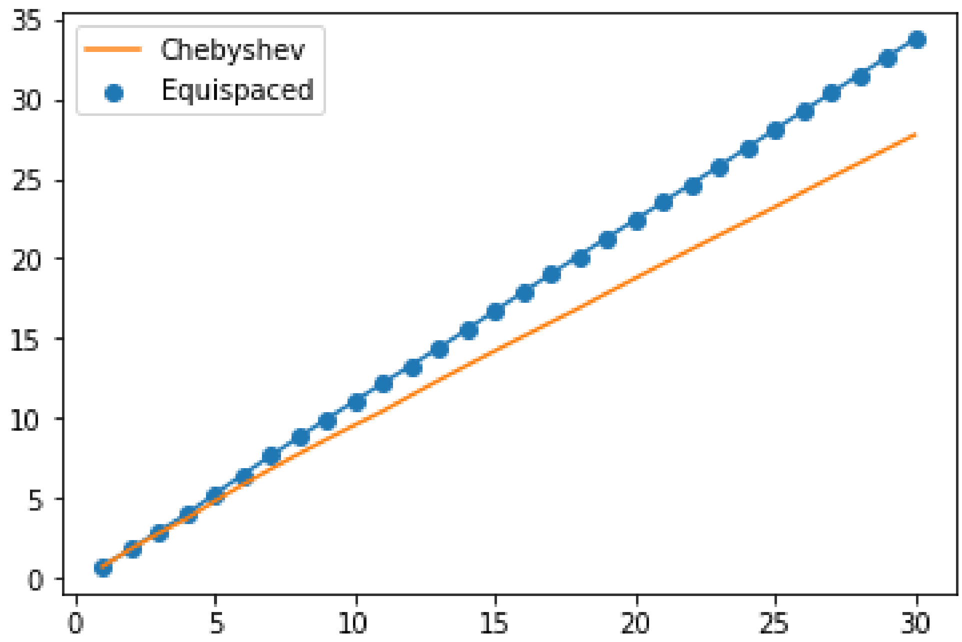
```
import numpy as np
import matplotlib.pyplot as plt

def Vandermonde(x):
    M = np.eye(len(x))
    for i in range(len(x)):
        M[i,:] = [x[i]**j for j in range(len(x))]
    return M

def Equispaced(n):
    return np.linspace(-1,1,n+1)

def Chebyshev(n):
    return np.cos(np.pi*np.linspace(0,1,n+1))
```

```
domain = list(range(1,31))
equi = [np.linalg.norm(Vandermonde(Equispaced(n)),np.inf) for n in domain]
cheby = [np.linalg.norm(Vandermonde(Chebyshev(n)),np.inf) for n in domain]

plt.plot(domain, np.log(equi))
plt.scatter(domain, np.log(equi), label="Equispaced", marker="o")
plt.plot(domain, np.log(cheby), label="Chebyshev")
plt.legend()
plt.show()
```



(b)

```
equi = [np.linalg.cond(Vandermonde(Equispaced(n)),np.inf) for n in domain]
cheby = [np.linalg.cond(Vandermonde(Chebyshev(n)),np.inf) for n in domain]

plt.plot(domain, np.log(equi))
plt.scatter(domain, np.log(equi), label="Equispaced", marker="o")
plt.plot(domain, np.log(cheby), label="Chebyshev")
plt.legend()
plt.show()
```



Problem 6:

(a)
$$\|x_1 + x_2\|_2^2 = < x_1 + x_2, x_1 + x_2 > = < x_1, x_1 > +2 < x_1, x_2 > + < x_2, x_2 > = \|x_1\|_2^2 + \|x_2\|_2^2$$

(b) Let the above be the base case ($n = 2$). Suppose it is true for $n - 1$. Then

$$\sum_{i=1}^{n} \|x_i\|_2^2 = \|\sum_{i=1}^{n-1} x_i\|_2^2 + \|x_n\|_2^2$$

Since x_i are mutually orthogonal,

$$< \sum_{i=0}^{n-1} x_i, x_n > = \sum_{i=1}^{n-1} < x_i, x_n > = 0$$

Therefore by the base case,

$$\|\sum_{i=1}^{n-1} x_i\|_2^2 + \|x_n\|_2^2 = \|\sum_{i=1}^{n} x_i\|_2^2$$
