Math 151BH - HW3

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May 15, 2023

Problem 1:

(a)

$$\begin{aligned}
< u_i, u_j > &= < \frac{Av_i}{\sigma_i}, \frac{Av_j}{\sigma_j} > \\
&= (\frac{Av_i}{\sigma_i})^*(\frac{Av_j}{\sigma_j}) \\
&= \frac{1}{\sigma_i\sigma_j} v_i^* A^* A v_j \\
&= \frac{\lambda_j}{\sigma_i\sigma_j} v_i^* v_j \\
&= \frac{\lambda_j}{\sigma_i\sigma_j} < v_i, v_j > \\
&= \left\{ \begin{array}{l} 0 \text{ if } i \neq j \text{ by orthogonality} \\ \frac{\lambda_i}{\sigma_i^2} = 1 \text{ if } i = j \text{ by orthonormality} \end{array} \right\}
\end{aligned}$$

(b) Let $\bar{V}, \bar{U}, \bar{\Sigma}$ be the added columns.

$$\begin{aligned}
AV &= A \begin{bmatrix} \hat{V}_r & \bar{V} \end{bmatrix} \\
&= \begin{bmatrix} A\hat{V}_r & A\bar{V} \end{bmatrix} \\
&= \begin{bmatrix} \hat{U}_r\hat{\Sigma}_r & A\bar{V} \end{bmatrix} \\
\Rightarrow A = AVV^* &= \begin{bmatrix} \hat{U}_r\hat{\Sigma}_r & A\bar{V} \end{bmatrix} \begin{bmatrix} \hat{V}_r^* \\ \bar{V}^* \end{bmatrix} \\
&= \hat{U}_r\hat{\Sigma}_r\hat{V}_r^* + A\bar{V}\bar{V}^* \\
&= \hat{U}_r\hat{\Sigma}_r\hat{V}_r^*
\end{aligned}$$

$A\bar{V} = 0$ since $Av_i = 0$ for all $i > r$.

(c)

$$\begin{aligned}
U\Sigma V^* &= \begin{bmatrix} \hat{U}_r & \bar{U} \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{V}_r^* \\ \bar{V}^* \end{bmatrix} \\
&= \begin{bmatrix} \hat{U}_r & \bar{U} \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_r\hat{V}_r^* \\ 0 \end{bmatrix} \\
&= \hat{U}_r\hat{\Sigma}_r\hat{V}_r^* \\
&= A
\end{aligned}$$

Problem 2:

$$\begin{aligned}
A^*Ax = A = A^*b \\
\iff \hat{V}_r\hat{\Sigma}_r^*\hat{U}_r^*\hat{U}_r\hat{\Sigma}_r\hat{V}_r^*x = \hat{V}_r\hat{\Sigma}_r^*\hat{U}_r^*b \\
\iff \hat{V}_r\hat{\Sigma}_r^*\hat{\Sigma}_r^*\hat{V}_r^*x = \hat{V}_r\hat{\Sigma}_r^*\hat{U}_r^*b \\
\iff \hat{V}_r^*x = (\hat{\Sigma}_r^*\hat{\Sigma}_r)^{-1}\hat{\Sigma}_r^*\hat{U}_r^*b \\
\iff \hat{V}_r^*x = \hat{\Sigma}_r^{-1}\hat{U}_r^*b \\
\Leftarrow x = \hat{V}_r\hat{\Sigma}_r^{-1}\hat{U}_r^*b
\end{aligned}$$

The converse of the last line holds as long as $b \neq 0$ since $\hat{V}_r$ has full rank, so only one value of x can produce the nonzero product on the right. Therefore x is the unique minimizer.

Problem 3:

(a)

$$\begin{aligned}
f(x_i) &= -D_h^+D_h^- y(x_i) \text{ for all } 0 < i < n \\
&= -D_h^+ \frac{y(x_i) - y(x_i - h)}{h} \\
&= -\frac{1}{h}\left[\frac{y(x_i + h) - y(x_i + h - h)}{h} - \frac{y(x_i) - y(x_i - h)}{h}\right] \\
&= \frac{2y(x_i) - y(x_{i+1}) - y(x_{i+1})}{h^2}
\end{aligned}$$

Then we have n-1 equations $\frac{2}{h^2}y_i - \frac{1}{h^2}y_{i-1} - \frac{1}{h^2}y_{i+1} = f(x_i)$ for $0 < i < n$. However this uses $n+1$ variables $\{y_i\}_{i=0}^n$. The boundary conditions $y_0 = 1$, $y_n = 2$ give

$$\frac{2}{h^2}y_1 - \frac{1}{h^2}y_2 - \frac{1}{h^2} = f(x_1) \Rightarrow \alpha = \frac{1}{h^2}$$

$$\frac{2}{h^2}y_{n-1} - \frac{1}{h^2}y_{n-2} - \frac{2}{h^2} = f(x_{n-1}) \Rightarrow \beta = \frac{2}{h^2}$$

So $(A_h)_{i,i} = \frac{2}{h^2}$ for $1 \leq i \leq n-1$, $(A_h)_{i,i-1} = -\frac{1}{h^2}$ for $2 \leq i \leq n-1$, and $(A_h)_{i,i+1} = -\frac{1}{h^2}$ for $1 \leq i \leq n-2$. A_h is tridiagonal, so all other entries are zero. Then $A_h y_h = f_h$ enforces all the conditions.

(b) A_h is clearly symmetric from the construction above: each entry on the subdiagonal and superdiagonal is $-\frac{1}{h^2}$, and everything else besides the diagonal is zero.

Let $v \in \mathbb{R}^{n-1}, v \neq 0$.

$$\begin{aligned}
v_i(Av)_i &= \frac{1}{h^2}(2v_i^2 - v_iv_{i-1} - v_iv_{i+1}) \text{ for } 1 < i < n-1 \\
\Rightarrow v^\top Av &= \frac{1}{h^2}\left[\sum_{i=2}^{n-2}(2v_i^2 - v_iv_{i-1} - v_iv_{i+1}) + 2v_1^2 - v_1v_2 + 2v_{n-1}^2 - v_{n-1}v_{n-2}\right] \\
&= \frac{1}{h^2}\left[2\sum_{i=1}^{n-1}v_i^2 - 2\sum_{i=1}^{n-2}v_iv_{i+1}\right] \\
&= \frac{1}{h^2}\left[\sum_{i=1}^{n-2}(v_i - v_{i+1})^2 + v_1^2 + v_{n-1}^2\right] \\
&> 0
\end{aligned}$$

The result is strictly greater than 0 because if $(v_i - v_{i+1}) = 0\ \forall i$ then $v_i = v_j\ \forall i,j$ which must mean that $v_i^2 > 0$ or else $v = 0$.

Problem 4:

(a) From last quarter's midterm, we know that for A hermitian and positive definite, $\|A\|_2 = \lambda_{max}$ and $\|A^{-1}\|_2 = \lambda_{min}^{-1}$. Therefore $\kappa = \frac{\lambda_{max}}{\lambda_{min}} \geq 1$ since SPD implies positive real eigenvalues. Let $a = \frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1} > 1$. Then $a^k > 1$ and $a^{-k} \in (0,1)$, giving

$$2\left[\left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^k + \left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^{-k}\right]^{-1} = 2(a^k + a^{-k})^{-1} \leq 2(a^k)^{-1} = 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k$$

(b)

$$u(x) = \gamma - \frac{2x}{\lambda_{max} - \lambda_{min}} = \frac{\lambda_{max} + \lambda_{min} - 2x}{\lambda_{max} - \lambda_{min}}$$

Since u is linear with respect to x, its values between λ_{min} and λ_{max} will be between $u(\lambda_{min})$ and $u(\lambda_{max})$.

$$u(\lambda_{min}) = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} - \lambda_{min}} = 1$$

$$u(\lambda_{max}) = \frac{\lambda_{min} - \lambda_{max}}{\lambda_{max} - \lambda_{min}} = -1$$

So $-1 \leq u(x) \leq 1$ for $x \in [\lambda_{min}, \lambda_{max}]$.

(c) $T_k(x) = \cos(k \arccos(x))$. $p(x)$ is a k degree polynomial with $p(0) = 1$. Therefore $p \in \tilde{\mathcal{P}}^k$.

$$|p(\lambda)| = \left|\frac{T_k(u(x))}{T_k(\gamma)}\right| \leq \left|\frac{1}{T_k(\gamma)}\right|$$

If the given equality can be shown, it implies that

$$|p(\lambda)| \leq \left|\frac{1}{T_k(\gamma)}\right| = 2\left[\left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^k + \left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^{-k}\right]^{-1}$$

Which means the first inequality will hold.

(d)

$$\begin{aligned}
w &= \frac{1}{2}(z + \frac{1}{z}) \\
\Rightarrow 2zw &= z^2 + 1 \\
\Rightarrow 0 &= z^2 - 2zw + 1 \\
\Rightarrow z &= w \pm \sqrt{w^2 - 1} \text{ by the quadratic formula.}
\end{aligned}$$

Let $z_1 = w + \sqrt{w^2 - 1},\ z_2 = w - \sqrt{w^2 - 1}$. Then

$$\begin{aligned}
z_1 z_2 &= w^2 - w^2 + 1 = 1 \Rightarrow z_1 = z_2^{-1} \\
\Rightarrow J(z_1^k) &= \frac{1}{2}(z_1^k + z_1^{-k}) \\
&= \frac{1}{2}(z_2^{-k} + z_2^k) \\
&= J(z_2^k)
\end{aligned}$$

Finally, for $w \in [-1, 1]$,

$$\begin{aligned}
\cos(k \arccos(w)) &= \cos(\frac{k}{i}\mathrm{Log}(w + \sqrt{w^2 - 1})) \\
&= \cos(\frac{k}{i}\mathrm{Log}(z_1)) \\
&= \cos(\frac{1}{i}\mathrm{Log}(z_1^k)) \\
&= \frac{1}{2}[e^{\mathrm{Log}(z_1^k)} + e^{-\mathrm{Log}(z_1^k)}] \\
&= \frac{1}{2}[z_1^k + z_1^{-k}] \\
&= J(z_1^k)
\end{aligned}$$

(e)

$$\begin{aligned}
w = \gamma \Rightarrow z_1 &= \gamma + \sqrt{\gamma^2 - 1} \\
&= \frac{\kappa + 1}{\kappa - 1} + \sqrt{\left(\frac{\kappa + 1}{\kappa - 1}\right)^2 - 1} \\
&= \frac{\kappa + 1}{\kappa - 1} + \sqrt{\frac{4\kappa}{(\kappa - 1)^2}} \\
&= \frac{\kappa + 1}{\kappa - 1} + \frac{2\sqrt{\kappa}}{\kappa - 1} \\
&= \frac{(\sqrt{\kappa} + 1)^2}{(\sqrt{\kappa} + 1)(\sqrt{\kappa} - 1)} \\
&= \frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1} \\
\Rightarrow T_k(\gamma) &= J\left(\left(\frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1}\right)^k\right) \\
&= \frac{1}{2}\left[\left(\frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1}\right)^k + \left(\frac{\sqrt{\kappa} + 1}{\sqrt{\kappa} - 1}\right)^{-k}\right]
\end{aligned}$$

Problem 5:

(a)

$$f(x) = \sum_{k\in\mathbb{Z}} \frac{c_k}{\sqrt{2\pi}} e^{ikx}$$
$$\Rightarrow f'(x) = \sum_{k\in\mathbb{Z}} \frac{ikc_k}{\sqrt{2\pi}} e^{ikx}$$
$$\Rightarrow \mathcal{F}(f') = (ikc_k)_{k\in\mathbb{Z}}\text{Suppose } f^{(p)}(x) = \sum_{k\in\mathbb{Z}} \frac{(ik)^p c_k}{\sqrt{2\pi}} e^{ikx}$$
$$\Rightarrow f^{(p+1)}(x) = \sum_{k\in\mathbb{Z}} \frac{(ik)^{p+1} c_k}{\sqrt{2\pi}} e^{ikx}$$

Thus the general statement is proved by induction.

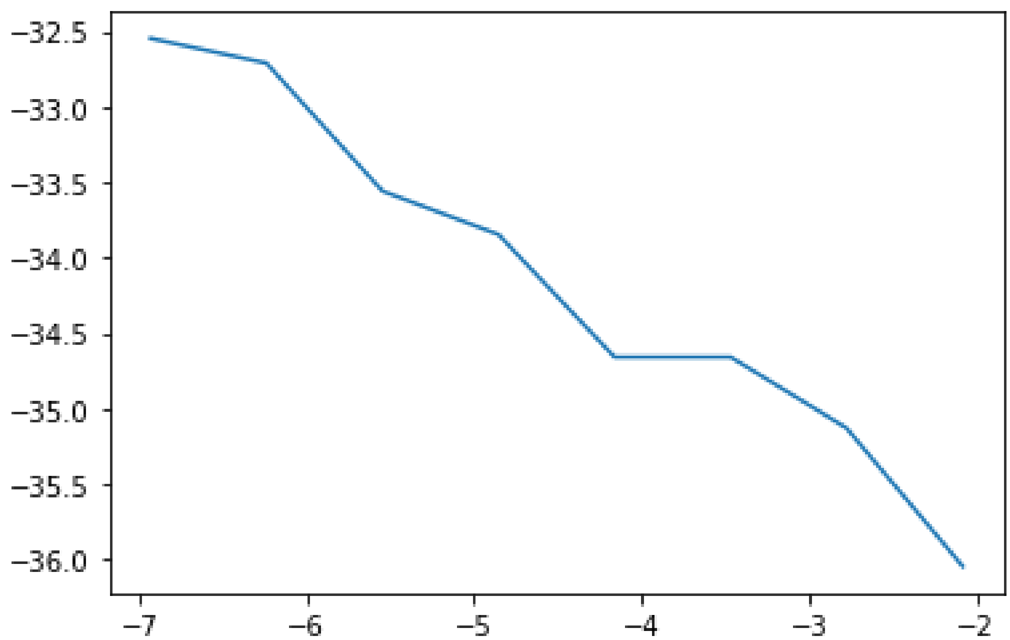
(b) Let $c_k^{(p)} = (ik)^p c_k$. By Parseval's relation,

$$\sum_k |c_k^{(p)}|^2 < \infty$$
$$\Rightarrow 0 = \lim_{k\to\pm\infty} |c_k^{(p)}|^2$$
$$\Rightarrow 0 = \lim_{k\to\pm\infty} |c_k^{(p)}|$$
$$= \lim_{k\to\pm\infty} |(ik)^p c_k|$$
$$= \lim_{k\to\pm\infty} |k^p c_k|$$
$$= \lim_{k\to\pm\infty} \left|\frac{c_k}{k^{-p}}\right|$$
$$\Rightarrow |c_k| = o(|k|^{-p})$$

Problem 6:

(a) Let $y_{\text{exact}} = 1 - x + x^2 + x^3$ and $f(x) = -2 - 6x$

(b)
```
import numpy as np
import math
import matplotlib.pyplot as plt

yexact = lambda x: 1 -x + x**2 + x**3
f = lambda x: -2 -6*x

ns = [2**i for i in range(3,11)]

def CG(n,b):
    m = len(b)
    bnorm = np.linalg.norm(b,2)
    Aprod = lambda v: (n)**2 *(((v[:-1]-v[1:])**2).sum() + v[0]**2 + v[-1]**2)
    def Amult(v):
        w = np.zeros(len(v))
        w[1:-1] = (n)**2 *(2*v[1:-1]-v[:-2]-v[2:])
        w[0] = (n)**2 *(2*v[0]-v[1])
        w[-1] = (n)**2 *(2*v[-1]-v[-2])
        return w
    x = np.zeros(m)
    r = b
    p = r
    count = 0
    while(count <= n and np.linalg.norm(r,2)/bnorm >= 10**(-10)):
        alpha = np.linalg.norm(r,2)**2/Aprod(p)
        x = x + alpha*p
        r1 = r
        r = r - alpha*Amult(p)
        beta = np.linalg.norm(r,2)**2/np.linalg.norm(r1,2)**2
        p = r + beta*p
        count += 1
    return x


results = []
for N in ns:
    X = np.linspace(0,1,N+1)[1:-1]
    fh = np.array([f(x) for x in X])
    fh[0] += N**2
    fh[-1] += 2*N**2
    yexac = np.array([yexact(x) for x in X])
    CG(N,fh)
    results.append(np.linalg.norm(CG(N,fh)-yexac,np.Inf))


plt.plot(np.log([1/n for n in ns]),np.log(results))
```


−32.5
−33.0
−33.5
−34.0
−34.5
−35.0
−35.5
−36.0
−7
−6
−5
−4
−3
−2