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Math 151AH: HW3

Problem 1)

Part a)

In [ ]:

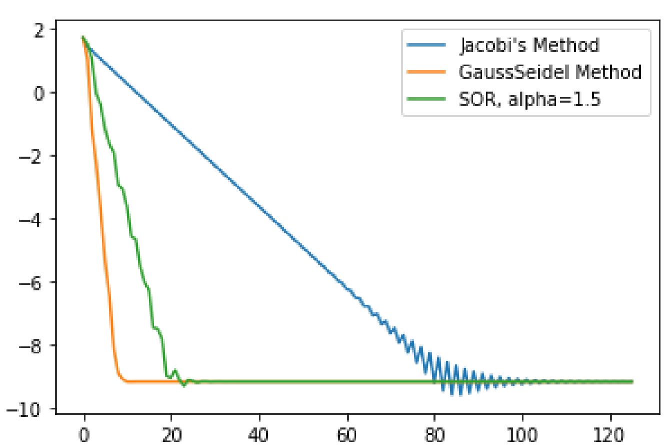
```
import numpy as np
def diagonal(A):
    n = len(A)
    D = np.zeros([n,n])
    D[np.diag_indices(n)] = A[np.diag_indices(n)]
    return D

def error(x):
    return ((x - np.array([1,2,3,4]))**2).sum()**0.5

def Jacobi(A, b, N=100):
    n = len(A)
    A2 = diagonal(A)
    A2inv = np.zeros([n,n])
    A2inv[np.diag_indices(n)] = np.reciprocal(A2[np.diag_indices(n)])
    A1 = A - A2
    x = [np.zeros(n)]
    count = 0
    while error(x[-1]) > 10**(-8) and count < N:
        x.append(A2inv.dot(b-A1.dot(x[-1])))
        count += 1
    return x

a = np.array([[2.4117,0.6557,0.6787,0.6555],
              [0.9157,1.8804,0.7577,0.1712],
              [0.7922,0.8491,3.0905,0.7060],
              [0.9595,0.9340,0.3922,2.3175]])
c = np.array([8.3813,7.6345,14.5862,13.2743])

X = Jacobi(a,c, N=125)
eJ = [np.log(error(x)) for x in X]
```

In [ ]:

```
def SOR(A, b, w, N=100):
    n = len(A)
    x = [np.zeros(n)]
    count = 0
    while error(x[-1]) > 10**(-8) and count < N:
        xk = np.zeros(n)
        xk1 = x[-1]
        for i in range(n):
            s1 = sum([A[i,j]*xk[j] for j in range(i)])
            s2 = sum([A[i,j]*xk1[j] for j in range(i+1,n)])
            xk[i] = w/A[i,i]*(b[i] - s1 - s2) + (1-w)*xk1[i]
        x.append(xk)
        count += 1
    return x

X = SOR(a,c,1.5,N=125)
eS = [np.log(error(x)) for x in X]
```

In [ ]:

```
def GaussSeidel(A, b, N=100):
    n = len(A)
    x = [np.zeros(n)]
    count = 0
    while error(x[-1]) > 10**(-8) and count < N:
        xk = np.zeros(n)
        xk1 = x[-1]
        for i in range(n):
            s1 = sum([A[i,j]*xk[j] for j in range(i)])
            s2 = sum([A[i,j]*xk1[j] for j in range(i+1,n)])
            xk[i] = 1/A[i,i]*(b[i] - s1 - s2)
        x.append(xk)
        count += 1
    return x

X = GaussSeidel(a,c,N=125)
eG = [np.log(error(x)) for x in X]

import matplotlib.pyplot as plt
plt.plot(range(len(eJ)), eJ, label="Jacobi's Method")
plt.plot(range(len(eG)), eG, label="GaussSeidel Method")
plt.plot(range(len(eS)), eS, label="SOR, alpha=1.5")
plt.legend()
plt.show()
```

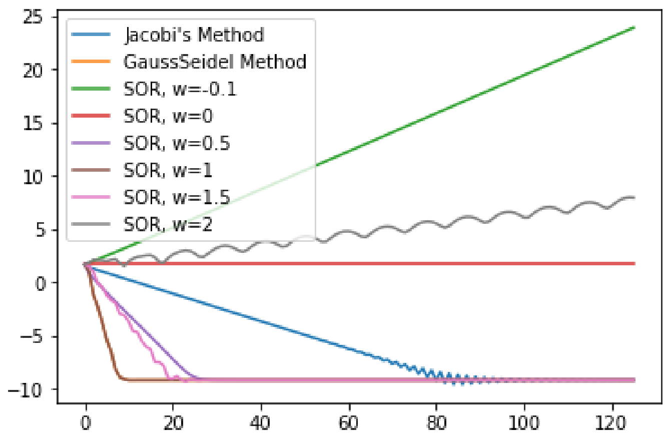


Part b)

As seen below, SOR doesn't converge for values of $\omega \notin (0, 2)$

In [ ]:

```
plt.plot(range(len(eJ)), eJ, label="Jacobi's Method")
plt.plot(range(len(eG)), eG, label="GaussSeidel Method")
results = {}
for w in [-0.1,0,0.5,1,1.5,2]:
    res = SOR(a,c,w,N=125)
    res = [np.log(error(x)) for x in res]
    plt.plot(range(len(res)), res, label="SOR, w="+str(w))

plt.legend()
plt.show()
```



Problem 2)

$\det D = \prod_i^n \lambda_i \Rightarrow \det D^{-1} = (\prod_i^n \lambda_i)^{-1}$

$\Rightarrow \det(D^{-1}L) = \det(D^{-1})\det(L) = (\prod_i^n \lambda_i)^{-1}(0) = 0$

since L is strictly lower triangular. Therefore $D^{-1}L$ has no nonzero eigenvalues and is also strictly lower triangular (diagonal times strictly lower triangular). This implies that $I + \omega D^{-1}L$ is lower triangular and thus its diagonal entries (all ones) are its eigenvalues. Then

$\det(I + \omega D^{-1}L) = \prod_i^n 1 = 1 \Rightarrow \det((I + \omega D^{-1}L)^{-1}) = 1$

By the same logic, we find that

$\det((1-\omega)I - \omega D^{-1}U) = \prod_i^n (1-\omega) = (1-\omega)^n$

$\Rightarrow |\det \mathcal{D}| = |(1)(1-\omega)^n| = |1-\omega|^n$

Since convergence requires $|\det \mathcal{D}| < 1$, this implies that $|1-\omega| < 1 \Rightarrow \omega \in (0,2)$ is required.

Problem 3)

Let $S_n = \sum_{k=0}^n M^k$

Then $S_n M = \sum_{k=0}^n M^{k+1} = \sum_{k=1}^{n+1} M^k = S_n - I + M^{n+1}$

$\Rightarrow S_n(I - M) = I - M^{n+1} \Rightarrow S_n = (I - M^{n+1})(I-M)^{-1}$

$\Rightarrow \lim_{n\to\infty} S_n = (I-M)^{-1} - \lim_{n\to\infty} M^{n+1}(I-M)^{-1}$
$= (I-M)^{-1} - (0)(I-M)^{-1} = (I-M)^{-1}$

Problem 4)

Part a)

$x^{(k+1)} = x^{(k)} + \alpha b - \alpha A x^{(k)} = \alpha b - (\alpha A - I)x^{(k)}$

$\Rightarrow A_2^{-1} = \alpha I \Rightarrow A_2 = \alpha^{-1} I$

$\Rightarrow A_1 = A - \alpha^{-1} I$

Part b)

WTS $\frac{\|g(x)-g(y)\|}{\|x-y\|} < 1 \; \forall x, y \in \mathbb{R}^n$

$$\begin{aligned}
\|g(x) - g(y)\| &= \|x - y - \alpha A(x-y)\| \\
&= \|(I - \alpha A)(x-y)\| \\
&\leq \|I - \alpha A\| \|x - y\| \\
\Rightarrow \frac{\|g(x)-g(y)\|_2}{\|x-y\|_2} &\leq \|I - \alpha A\|_2 \\
&= \rho(I - \alpha A) \\
&= \max_{\lambda}\{|1 - \alpha\lambda|\} \text{ s.t. } \lambda \text{ is an eigenvalue of A}
\end{aligned}$$

Therefore for convergence we need that

$$\begin{aligned}
\max_{\lambda}\{|1-\alpha\lambda|\} < 1 \\
\Rightarrow |1 - \alpha\lambda| < 1 \; \forall \lambda \\
\Rightarrow 0 < \alpha\lambda < 2 \; \forall\lambda \\
\Rightarrow 0 < \alpha < \frac{2}{\lambda} \; \forall\lambda \\
\Rightarrow \alpha \in (0, \frac{2}{\lambda_1})
\end{aligned}$$

Part c)

From part b) we have that $\frac{e^{(k+1)}}{e^{(k)}} \leq \rho(I - \alpha A) \Rightarrow e^{(k+1)} \leq \rho(I - \alpha A)^k e^{(0)}$

To optimize convergence, minimize $\rho(I - \alpha A) = max_{1 \leq i \leq n}\{|1 - \alpha\lambda_i|\}$

For $\alpha \geq 0, \;\; 1 - \alpha\lambda_1 \leq 1 - \alpha\lambda_2 \leq \ldots \leq 1 - \alpha\lambda_n$

Thus the greatest in absolute value will be either $(1 - \alpha\lambda_1)$ or $(1 - \alpha\lambda_n)$. If $\alpha \leq 0$, then the ordering will reverse, but the extremes will be the same. At the optimal α, it will hold that $|1 - \alpha\lambda_1| = |1 - \alpha\lambda_n|$ or else α could be adjusted to lower the greater of the two at the expense of the other.

It cannot be that $1 - \alpha\lambda_1 = 1 - \alpha\lambda_n$ since this implies that $\lambda_1 = \lambda_n$.

Therefore it must be that $1 - \alpha\lambda_1 = -1 + \alpha\lambda_n \Rightarrow \alpha = \frac{2}{\lambda_1 + \lambda_n}$

Part d)

Since A is symmetric and positive definite, we have that $\|A^{-1}\|_2 = \frac{1}{\lambda_{min}}$ and $\|A\|_2 = \lambda_{max}$. Therefore $\kappa_2(A) = \frac{\lambda_1}{\lambda_n}$.

$$\begin{aligned}
\rho(M) &= \rho(-A_2^{-1}A_1) \\
&= \rho(-\alpha I(A - \alpha^{-1}I)) \\
&= \rho(I - \alpha A) \\
&= |1 - \alpha\lambda_1| \\
&= |1 - \frac{2\lambda_1}{\lambda_1 + \lambda_n}| \\
&= |\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}| \\
&= \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n} \\
&= \frac{\frac{\lambda_1}{\lambda_n} - 1}{\frac{\lambda_1}{\lambda_n} + 1} \\
&= \frac{\kappa_2(A) - 1}{\kappa_2(A) + 1}
\end{aligned}$$

Problem 5)

Part a)

$$G_\gamma(1,1) = \begin{bmatrix} \gamma(1-1)\cos(\frac{\pi}{10}) - \gamma(1-1)\sin(\frac{\pi}{10}) + 1 \\ \gamma(1-1)\sin(\frac{\pi}{10}) + \gamma(1-1)\cos(\frac{\pi}{10}) + 5(1-1)^3 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Part b)

$$J_{G|x} = \begin{bmatrix} \gamma\cos(\frac{\pi}{10}) & -\gamma\sin(\frac{\pi}{10}) \\ \gamma\sin(\frac{\pi}{10}) + 15(x_1 - 1)^2 & \gamma\cos(\frac{\pi}{10}) \end{bmatrix}$$

$$J_{G|x^*} = \begin{bmatrix} \gamma\cos(\frac{\pi}{10}) & -\gamma\sin(\frac{\pi}{10}) \\ \gamma\sin(\frac{\pi}{10}) & \gamma\cos(\frac{\pi}{10}) \end{bmatrix}$$

Part c)

$$J_{G|x^*} = \gamma U \text{ where } U = \begin{bmatrix} \cos(\frac{\pi}{10}) & -\sin(\frac{\pi}{10}) \\ \sin(\frac{\pi}{10}) & \cos(\frac{\pi}{10}) \end{bmatrix}$$

$$\begin{gathered} U^\intercal U = I \Rightarrow U \text{ is orthogonal, so} \\ \|J_{G|x^*}\|_2 = \|\gamma U\|_2 = \gamma\|U\|_2 = \gamma \end{gathered}$$

Part d)

FPI converges when $\|J_{G|x^*}\|_2 < 1$, which in this case is equivalent to $\gamma < 1$.

Problem 6)

Part a)

$F(\mathbf{0}) = (0^2 + 0^2 + 5(0), 2(0)(0) + 3(0)^2 + 0)^\intercal = \mathbf{0}$

Part b)

$$J_{F|x} = \begin{bmatrix} 2x_1 + 5 & 2x_2 \\ 2x_2 & 2x_1 + 6x_2 + 1 \end{bmatrix}$$

$$J_{F|x^*} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_{F_1|x^*} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$H_{F_2|x^*} = \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix}$$

Part c)

$$J_{F|x^*} = \begin{bmatrix} 1 - 5\alpha & 0 \\ 0 & 1 - \alpha \end{bmatrix}$$

$$\|J_{G|x^*}\| = \rho(J_{G|x^*}) = \max\{|1 - 5\alpha|, |1 - \alpha|\} < 1 \text{ for } \alpha \in (0, \frac{2}{5})$$

Part d)

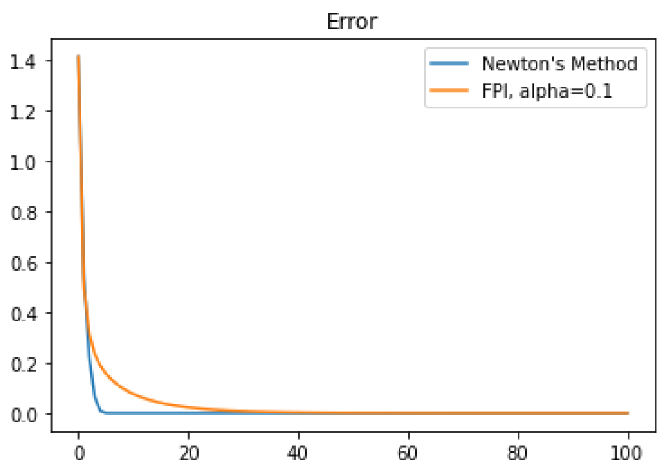
We know that $F \in C^2(\mathbb{R}^2)$ since each component is a polynomial of x_1 and x_2. It then immediately follows that $G \in C^2(\mathbb{R}^2)$ since G is a linear combination of two twice continuous functions x and F. It was shown earlier that for $x^* = \mathbf{0}$, $F(x^*) = \mathbf{0}$. This then implies that $G(x^*) = x^* - \alpha F(x^*) = x^*$. It was also shown that there exists sufficiently small α such that $\rho(J_{G|x^*}) = \|J_{G|x^*}\| < 1$. These results satisfy the conditions of the Local Convergence Theorem of Vector FPI, so we can guarentee that $\{x^{(k)}\}$ converges to x^* for sufficiently close $x^{(0)}$ to x^*.

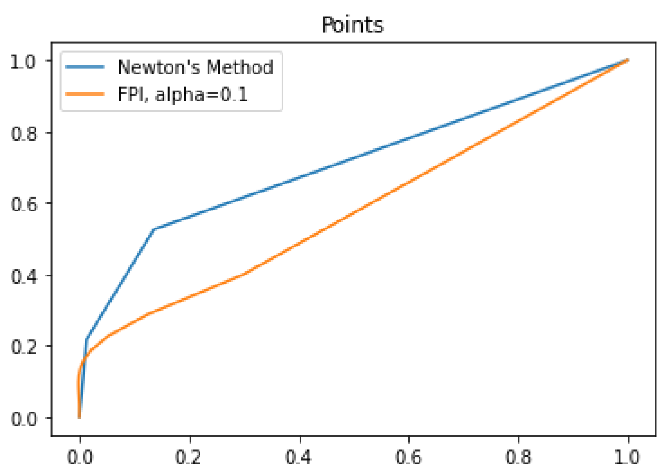
Part e)

In [ ]:

```
F = lambda x: np.array([x[0]**2 + x[1]**2 + 5*x[0], 2*x[0]*x[1] + 3*x[1]**2 + x[1]])
J = lambda x: np.array([[2*x[0] + 5, 2*x[1]],
                        [2*x[1], 2*x[0] + 6*x[1] + 1]])

def Newton(F, J, n, N=100, x0=None):
    if x0 == None:
        x = [np.zeros(n)]
    else:
        x = [x0]
    count = 0
    while count < N:
        xk1 = x[-1]
        xk = xk1 - np.linalg.solve(J(xk1),F(xk1))
        x.append(xk)
        count += 1
    return x

def FPI(F, n, alpha, N=100, x0=None):
    if x0 == None:
        x = [np.zeros(n)]
```

```
    else:
        x = [x0]
    count = 0
    while count < N:
        xk1 = x[-1]
        xk = xk1 - alpha*F(xk1)
        x.append(xk)
        count += 1
    return x

xN = np.array(Newton(F, J, 2, x0=(1,1)))
xF = np.array(FPI(F, 2, 0.1,x0=(1,1)))
def error(x):
    x = x**2
    return (x[:,0]+x[:,1])**0.5

plt.plot(range(len(xN)), error(xN), label="Newton's Method")
plt.plot(range(len(xF)), error(xF), label="FPI, alpha=0.1")
plt.legend()
plt.title("Error")
plt.show()

plt.plot(xN[:,0],xN[:,1], label="Newton's Method")
plt.plot(xF[:,0],xF[:,1], label="FPI, alpha=0.1")
plt.legend()
plt.title("Points")
plt.show()
```





Problem 7)

As shown below, 0 iterations of Jacobi's method fails to converge, but 1 iteration successfully converges.

In [ ]:

```python
F = lambda x: np.array([x[0]**2 + x[1]**2 + 5*x[0], 2*x[0]*x[1] + 3*x[1]**2 + x[1]])
J = lambda x: np.array([[2*x[0] + 5, 2*x[1]],
                        [2*x[1], 2*x[0] + 6*x[1] + 1]])

def Jacobi(A, b, N=100):
    n = len(A)
    A2 = diagonal(A)
    A2inv = np.zeros([n,n])
    A2inv[np.diag_indices(n)] = np.reciprocal(A2[np.diag_indices(n)])
    A1 = A - A2
    x = [np.zeros(n)]
    count = 0
    while count < N:
        x.append(A2inv.dot(b-A1.dot(x[-1])))
        count += 1
    return x

def Newton(F, J, n, N=100, x0=None, N2=100):
    if x0 == None:
        x = [np.zeros(n)]
    else:
        x = [x0]
    count = 0
    while count < N:
        xk1 = x[-1]
        xk = xk1 - Jacobi(J(xk1),F(xk1),N2)[-1]
        x.append(xk)
        count += 1
    return x



def error(x):
    x = x**2
    return (x[:,0]+x[:,1])**0.5
for i in [0,1,10,100]:
    xN = np.array(Newton(F, J, 2, x0=(1,1), N2=i))
    plt.plot(range(len(xN)), error(xN), label="Newton's Method + Jacobi, "+str(i)+" iterations")
plt.legend()
plt.title("Error")
plt.show()
```