Probabilistic Modes of Convergence

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1 Introduction

In probability theory, we use several notions of 'convergence' for a sequence of random variables. Some of these notions are stronger than others, so it is natural to ask when one convergence implies another. In the following definitions and results, we assume that X and $(X_n)_{n\geq 1}$ are all real-valued random variables on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2 Definitions

Below are definitions for a few of the more commonly-used modes of convergence.

Almost sure convergence ($a.s.$)

$X_n \xrightarrow{a.s.} X$ if

$$\mathbb{P}(X_n \to X \text{ as } n \to \infty) = 1$$

Convergence in probability ($\mathbb{P}$)

$X_n \xrightarrow{\mathbb{P}} X$ if for all $\epsilon > 0$, as $n \to \infty$,

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0$$

Convergence in distribution (d)

$X_n \xrightarrow{d} X$ if whenever $\mathbb{P}(X \leq \cdot)$ is continuous at x, then as $n \to \infty$,

$$\mathbb{P}(X_n \leq x) \to \mathbb{P}(X \leq x)$$

Convergence in L^p, for $1 \leq p < \infty$

$X_n \xrightarrow{L^p} X$ if as $n \to \infty$,

$$\mathbb{E}[|X_n - X|^p] \to 0$$

Convergence in L^∞

First define $\|X\|_\infty = \inf\{M : |X| \leq M \text{ almost surely}\}$. Then $X_n \xrightarrow{L^\infty} X$ if as $n \to \infty$,

$$\|X_n - X\|_\infty \to 0$$

Notes on definitions

Almost sure convergence is the natural measure-theoretic extension of the notion of pointwise convergence of functions. We simply require the convergence to occur at almost every point rather than at every point.

Convergence in probability is (as we will see) weaker than this, and means that with high probability, (X_n) will not make large deviations from X.

Convergence in distribution is even weaker, and depends only on the distributions of the random variables. The random variables do not even need to be defined on the same probability space.

The L^p spaces define a whole family of modes of convergence, with a larger value of p giving stronger convergence. Convergence in L^∞ is the same as uniform convergence almost everywhere.

3 Results

All of the results are summarised in Figure 1. Arrows indicate strength of convergence.

$$\begin{tikzcd}
 & L^p \arrow[r, "2"] & L^1 \arrow[dr, bend left, "3"] & & \\
L^\infty \arrow[ur, bend left, "1"] \arrow[dr, bend right, "5"'] & & & \mathbb{P} \arrow[ul, dashed, bend left, "4"] \arrow[r, "7"] & d \\
 & a.s. \arrow[urr, bend right, "6"'] & & &
\end{tikzcd}$$

Figure 1: Modes of convergence

4 Proofs

A proof is given for each arrow in Figure 1.

1. L^∞ convergence implies L^p convergence

$$
\begin{aligned}
\mathbb{E}[|X_n - X|^p] &= \int_\Omega |X_n - X|^p \, \mathrm{d}\mathbb{P} \\
&\leq \|X_n - X\|_\infty^p \int_\Omega \mathrm{d}\mathbb{P} \\
&\to 0
\end{aligned}
$$

2. L^p convergence implies L^1 convergence

$$
\begin{aligned}
\mathbb{E}[|X_n - X|]^p &\leq \mathbb{E}[|X_n - X|^p] && \text{(Jensen's inequality for } p \geq 1) \\
&\to 0
\end{aligned}
$$

3. L^1 convergence implies convergence in probability

$$
\begin{aligned}
\mathbb{P}(|X_n - X| > \epsilon) &= \int_\Omega \mathbb{I}\{|X_n - X| > \epsilon\} \, \mathrm{d}\mathbb{P} \\
&\leq \int_\Omega \frac{1}{\epsilon} |X_n - X| \, \mathrm{d}\mathbb{P} && \text{(Markov's inequality)} \\
&\leq \frac{1}{\epsilon} \mathbb{E}[|X_n - X|] \\
&\to 0
\end{aligned}
$$

4. Convergence in probability implies convergence in L^1, in a uniformly integrable family

Assuming (X_n) is uniformly integrable; for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $B \in \mathcal{F}$,

$$
\mathbb{P}(B) < \delta \implies \int_\Omega |X_n - X| \, \mathbb{I}_B \, \mathrm{d}\mathbb{P} < \frac{\epsilon}{2}
$$

Also, $X_n \xrightarrow{\mathbb{P}} X$ so there is an $N \in \mathbb{N}$ such that

$$
n \geq N \implies \mathbb{P}\left(|X_n - X| \geq \frac{\epsilon}{2}\right) < \delta
$$

Putting these together:

$$
\begin{aligned}
\mathbb{E}[|X_n - X|] &= \int_\Omega |X_n - X| \, \mathbb{I}\left\{|X_n - X| < \frac{\epsilon}{2}\right\} \, \mathrm{d}\mathbb{P} \\
&\quad + \int_\Omega |X_n - X| \, \mathbb{I}\left\{|X_n - X| \geq \frac{\epsilon}{2}\right\} \, \mathrm{d}\mathbb{P} \\
&< \epsilon
\end{aligned}
$$

5. L^∞ convergence implies almost sure convergence

Since $\|X_n - X\|_\infty \to 0$; for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies \|X_n - X\|_\infty < \epsilon$$

So by the definition of $\|\cdot\|_\infty$,

$$\begin{aligned}
&\mathbb{P}(|X_n - X| > \epsilon \text{ for some } n \geq N) \\
&\leq \mathbb{P}(|X_n - X| > \|X_n - X\|_\infty \text{ for some } n \geq N) \\
&= \mathbb{P}\Big(\bigcup_{n=1}^{\infty} \{|X_n - X| > \|X_n - X\|_\infty\} \Big) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}\big(|X_n - X| > \|X_n - X\|_\infty\big) \\
&= 0
\end{aligned}$$

6. Almost sure convergence implies convergence in probability

Fix ϵ and let $A_N = \{|X_n - X| < \epsilon \text{ for all } n \geq N\}$. Note that A_N increases to

$$\bigcup_{N=1}^{\infty} A_N = \{|X_n - X| < \epsilon \text{ eventually}\} \supseteq \{X_n \to X\}$$

Therefore

$$\begin{aligned}
\mathbb{P}(|X_N - X| < \epsilon) &\geq \mathbb{P}(A_N) \\
&\to \mathbb{P}\Big(\bigcup_{N=1}^{\infty} A_N \Big) \\
&\geq \mathbb{P}(X_n \to X) \\
&= 1
\end{aligned}$$

7. Convergence in probability implies convergence in distribution

Take x a continuity point of $\mathbb{P}(X \leq \cdot)$, and fix $\epsilon > 0$.

$$\begin{aligned}
\mathbb{P}(X_n \leq x) &\leq \mathbb{P}(X \leq x + \epsilon) + \mathbb{P}(|X_n - X| > \epsilon) \\
&\to \mathbb{P}(X \leq x + \epsilon) \qquad \text{as } n \to \infty
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{P}(X_n \leq x) &\geq \mathbb{P}(X \leq x - \epsilon) - \mathbb{P}(|X_n - X| > \epsilon) \\
&\to \mathbb{P}(X \leq x - \epsilon) \qquad \text{as } n \to \infty
\end{aligned}$$

So $\mathbb{P}(X_n \leq x) \to \mathbb{P}(X \leq x)$, by continuity at x.