Project 2 – Nature-Inspired Algorithms

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1 (1+1)-EA *LeadingOnes*

Theorem 1. *The runtime of the (1+1) EA on LeadingOnes is* $\mathcal{O}(n^2)$

Proof. We use fitness levels to find an upper bound for the runtime of (1+1) EA on *LeadingOnes*. We partition $\{0,1\}^n$ into disjoint sets $A_0, A_1, \dots, A_n$ where $x \in A_i$ iff it has i leading ones.

To escape A_i the first i bits must leave unchanged and at least the bit $i+1$ must flip. The remaining bits can flip arbitrarily, because we can only get more leading ones and so cannot get worse.

Thus we get the probability to leave A_i, $s_i = \frac{1}{n}(1-\frac{1}{n})^i$ and $\frac{1}{s_i} = n(1-\frac{1}{n})^{-i}$

So we conclude:

$$\begin{aligned}
E(T) &= \sum_{i=1}^{n-1} \frac{1}{s_i} \leq \sum_{i=1}^{n} \frac{1}{s_i} \\
&= \sum_{i=1}^{n} n\left(1-\frac{1}{n}\right)^{-i} = n\sum_{i=1}^{n}\left(1-\frac{1}{n}\right)^{-i} \\
&\leq n\sum_{i=1}^{n}\left(1-\frac{1}{n}\right)^{-n} \leq n\sum_{i=1}^{n} e \qquad\qquad \text{as } \lim_{n\to\infty}\left(1+\frac{1}{-n}\right)^{-n} = e \\
&= n^2 e = \mathcal{O}(n^2)
\end{aligned}$$

□

2 (1+1)-EA *Jump*$_k$

Theorem 2. *The runtime of the (1+1) EA on* $Jump_k$ *is* $\mathcal{O}(n^k)$.

Proof. We use fitness levels to find an upper bound for the runtime of (1+1) EA on $Jump_k$.

We partition the data as for the *OneMax* function, where $x \in A_i$ iff it has i 1s, but we merge the partitions $n-k$ until $n-1$, because they all receive the same value from $Jump_k$. The probability to leave a partition s_i is defined as for *OneMax*. Only the second last parition gets a different probability. Thus, we have two cases that we retrieved from the definition of $Jump_k$: 1) $|x| < n-k$ (all partitions but the last two) and 2) $n-k \leq |x| < n$ (the second last partition) which we consider separately in the following and afterwards sum the results, what we can do because $E(T) = \sum_{i=1}^{m-1} \frac{1}{s_i} = \sum_{i=1}^{m-2} \frac{1}{s_i} + \frac{1}{s_{m-1}}$.

Case 1: $|x| < n-k$
For this we partition $\{0,1\}^n$ into disjoint sets $A_0, A_1, \dots, A_j$ where $a \in A_i$ iff $|x|_a = i$. We do this as we have analysed the runtime of this in Lecture 4 as (1+1) EA with *OneMax*. Until case 2, the score of *Jump* behaves identically to that of *OneMax* $\implies$ runtime until $|x| = n-k$ is in $\mathcal{O}(n\ \log(n))$.

Case 2: $n-k \leq |x| < n$
This case is just one partiton A_k, which contains all a with $n-k \leq |x|_a < n$. So we cover the part after case 1 until the optimal solution containing only 1s. For us to leave A_k and reach the optimal solution A_{k+1}, we need to flip exacly the remaining k bits and leave the other $n-k$ bits unflipped. Thus, $s_k = (\frac{1}{n})^k\ (1-\frac{1}{n})^{n-k}$.

We conclude

$$
\begin{aligned}
E(T_{s_k}) &= \frac{1}{\left(\frac{1}{n}\right)^k \ \left(1-\frac{1}{n}\right)^{n-k}} \\
&= \left(\frac{1}{n}\right)^{-k} \ \left(1-\frac{1}{n}\right)^{k-n} \\
&= \frac{1}{\left(\frac{1}{n}\right)^k} \ \left(1-\frac{1}{n}\right)^{k-n} \\
&= n^k \ \left(1-\frac{1}{n}\right)^{k-n} \\
&= n^k \ \left(1-\frac{1}{n}\right)^{k} \ \left(1-\frac{1}{n}\right)^{-n} && \text{if we now look at } \lim_{n\to\infty} \\
&\leq n^k \ 1e && \text{as } 1-\frac{1}{n} \text{ goes to 1 and } 1^k = 1 \\
&\text{and } \left(1-\frac{1}{n}\right)^{-n} \text{ goes to } e \text{ as shown in Section 1.} \\
&= \mathcal{O}(n^k)
\end{aligned}
$$

From this we can follow that 1+1 (EA) with $Jump_k$ has a runtime of $\mathcal{O}(n\log(n) + n^k)$. For $k < 2$ we have a runtime of $\mathcal{O}(n\log(n))$, because n^0 and n^1 are dominated by $\mathcal{O}(n\log(n))$. For $k \geq 2$ we get $\mathcal{O}(n^k)$, so the Theorem only holds for $k \geq 2$.

$\square$