Hasso Plattner Institute Potsdam

Seminar on Nature Inspired Algorithms
Summer 2017
Potsdam, May 30, 2017

Homework 1

Summer Term 2017

Homework 1 – *Nature Inspired Algorithms*

https://hpi.de/friedrich/teaching/ss17/natinsalg.html

The goal of this homework is to make sure you follow the basic probability theory from Lecture 3.

The homework is submitted on Moodle (https://hpi.de/friedrich/moodle/) by uploading a PDF file with your solutions. You are welcome to write your solutions out by hand and scan them.

Let Ω be a countable set of elementary events and P be a probability measure on Ω. Recall that a *random variable* is a mapping $X\colon \Omega \to \mathbb{R}$. The *expected value* of a random variable X is

$$E(X) = \sum_{\omega \in \Omega} P(\omega) X(\omega),$$

and its variance is

$$\mathrm{Var}(X) = E\left((X - E(X))^2\right).$$

Exercise 1 *Prove the following.*

(a) *For any two arbitrary random variables X and Y, we have $E(X+Y) = E(X) + E(Y)$.* ***Hint: it helps to write*** $P(X = r) = \sum_{\omega\colon X(\omega)=r} P(\omega)$**.**

(b) *For any two independent random variables X and Y, we have $E(XY) = E(X)E(Y)$. Is it necessary to assume independence? Why or why not?*

(c) *For any two independent random variables X and Y, we have $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$. Is it necessary to assume independence? Why or why not?*

a

$$
\begin{aligned}
E(X+Y) &= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} (x+y) P(X=x, Y=y) && (1)\\
&= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} x P(X=x, Y=y) + \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} y P(X=x, Y=y) && (2)\\
&= \sum_{x\in\mathbb{R}} x \sum_{y\in\mathbb{R}} P(X=x, Y=y) + \sum_{y\in\mathbb{R}} y \sum_{x\in\mathbb{R}} P(X=x, Y=y) && (3)\\
&= \sum_{x\in\mathbb{R}} x P(X=x) + \sum_{y\in\mathbb{R}} y P(Y=y) && (4)\\
&= E(X) + E(Y) \blacksquare && (5)
\end{aligned}
$$

b

$$
\begin{aligned}
E(XY) &= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} xy \cdot P(X=x, Y=y) && (6)\\
&= \sum_{x\in\mathbb{R}} x \sum_{y\in\mathbb{R}} y \cdot P(X=x, Y=y) && (7)\\
&= \sum_{x\in\mathbb{R}} x \sum_{y\in\mathbb{R}} y \cdot P(X=x) P(Y=y) && (8)\\
&= \sum_{x\in\mathbb{R}} x P(X=x) \sum_{y\in\mathbb{R}} y P(Y=y) && (9)\\
&= E(X)E(Y) && (10)
\end{aligned}
$$

For arbitrary random variables X and Y the equation $E(XY) = E(X)E(Y)$ only always holds if X $\perp$ Y since $P(X=x, Y=y)$ is always $P(X=x)P(Y=y)$ in that case which is required during the proof. Thus, we have to assume independence. $\blacksquare$

c

$$
\begin{aligned}
Var(X+Y) =& E((X+Y)^2) - E(X+Y)^2 && (11)\\
&= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} (x+y)^2 \cdot P(X=x, Y=y) - (E(X+Y))^2 && (12)\\
&= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} x^2 \cdot P(X=x, Y=y) + \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} 2xy \cdot P(X=x, Y=y) && (13)\\
&+ \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} y^2 \cdot P(X=x, Y=y) - (E(X))^2 - 2E(X)E(Y) - (E(Y))^2 && (14)\\
&= \sum_{x\in\mathbb{R}} x^2 \cdot P(X=x) - (E(X))^2 + \sum_{y\in\mathbb{R}} y^2 \cdot P(Y=y) - (E(Y))^2 && (15)\\
&+ \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} 2xy \cdot P(X=x, Y=y) - 2E(X)E(Y) && (16)\\
&= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 + 2(E(XY) - E(X)E(Y)) && (17)\\
&= Var(X) + Var(Y) + 2Cov(X,Y) && (18)
\end{aligned}
$$

For arbitrary random variables X and Y the equation $Var(X+Y) = Var(X) + Var(Y)$ only always holds if X $\perp$ Y since $Cov(X,Y)$ is always 0 in that case. Thus, we have to assume independence. ■