Hasso Plattner Institute Potsdam

Seminar on Nature Inspired Algorithms
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Homework 3

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Define the function $g\colon \{0,1\}^n \to \mathbb{R}$ as follows

$$g(x) = \begin{cases} 2 & \text{if } |x|_1 = n, \\ 0 & \text{if } 3n/4 \leq |x|_1 < n, \\ 1 & \text{otherwise;} \end{cases}$$

Exercise 4 *Answer the following.*

(a) What is the expected runtime of RandomSearch *to maximize g?*

(b) Let $\mathcal{E}$ be the event that the initial solution of the (1+1) EA has at most $n/4$ zero bits. Derive an upper bound on $\Pr(\mathcal{E})$.
Hint: use one of the methods we discussed in Lecture 4.

*(c) Suppose $\mathcal{E}$ has **not** occurred. Give a lower bound on the expected time for the (1+1) EA to first generate the all-ones string under this condition.*

(d) Conclude that the expected runtime to maximize g for the (1+1) EA is worse than the runtime of RandomSearch *on g.*
Hint: use the law in Theorem 1.

a

The expected run time of RandomSearch is

$$E(T_{RS}) \in \mathcal{O}(2^n).$$

It does not take g into account but only looks at some random mutation of a bitstring in each iteration. The chance of getting the all-1s-bitstring is $\frac{1}{2^n}$ which gives us the above run time.

b

The probability of the event X of a string with at most $n/4$ zeros is equal to the probability of having more than $3n/4$ ones. We can estimate this probability using Markov's Inequality:

$$Pr(X \geq a) \leq \frac{E(X)}{a}$$

$$\text{let } E(X) = \frac{n}{2} \implies Pr(X \geq (3n/4)) \leq \frac{n/2}{3n/4} = \frac{n}{2} * \frac{4}{3n} = \frac{2}{3}$$

$$\implies Pr(X) \leq \frac{2}{3}$$

c

To reach the optimum from a solution after $\mathcal{E}$ did not occur, we need to flip at least $n/4+1$ zeros without flipping the remaining ones.

$$\begin{aligned}
P(Optimum|\neg\mathcal{E}) &\leq \left(\frac{1}{n}\right)^{n/4+1}\left(1-\frac{1}{n}\right)^{3n/4-1} \\
&= \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{n/4}\left(\frac{n-1}{n}\right)^{3n/4-1} \\
&= \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{n/4}\left(\frac{n-1}{n}\right)^{3n/4}\left(\frac{n-1}{n}\right)^{-1} \\
&= \left(\frac{1}{n-1}\right)\left(\frac{1}{n}\right)^{n/4}\left(\frac{n-1}{n}\right)^{3n/4}
\end{aligned}$$

The expected run time is the inverse of the probability.

$$\begin{aligned}
E(T_{1+1}|\neg\mathcal{E}) &\geq (n-1)\left(\frac{1}{n}\right)^{-n/4}\left(\frac{n-1}{n}\right)^{-3n/4} \\
&= (n-1)\ n^{n/4}\ (n-1)^{-3n/4}\ n^{3n/4} \\
&= (n-1)^{1-(3n/4)}\ n^n
\end{aligned}$$

d

$$
\begin{aligned}
E(T_{RS}) &= \mathcal{O}(2^n) \\
E(T_{1+1}) &= E(T|\mathcal{E})Pr(\mathcal{E}) + E(T|\neg\mathcal{E})Pr(\neg\mathcal{E})
\end{aligned}
$$

We only look at the second part the equation, i.e. $E(T|\neg\mathcal{E})Pr(\neg\mathcal{E})$. The first part cannot become negative, so it will only increase the expected run time. We use $Pr(\neg\mathcal{E})$ from task b and $E(T|\neg\mathcal{E})$ from task c. We follow

$$
E(T|\neg\mathcal{E})Pr(\neg\mathcal{E}) \geq (n-1)^{1-3n/4} n^n \left(1 - \left(\frac{2}{3}\right)\right)
$$

To prove that the run time of 1+1 EA is worse than the one of RandomSearch, we divide the expected run time of 1+1 EA by the one of RandomSearch and show that the quotient goes to infinity when n goes to infinity.

$$
\begin{aligned}
\frac{E(T_{1+1})}{E(T_{RS})} &= \frac{\frac{1}{3}(n-1)^{1-3n/4}\ n^n}{2^n} \\
&= \frac{1}{3}(n-1)^{1-3n/4}\ \left(\frac{n}{2}\right)^n \\
\lim_{n\to\infty}(n-1)^{1-3n/4}\left(\frac{n}{2}\right)^n &= \infty
\end{aligned}
$$

As this goes to infinty, we can see that 1+1 EA has a worse expected runtime than RandomSearch.