Hasso Plattner Institute Potsdam

Seminar on Nature Inspired Algorithms
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Homework 2

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Homework 2 – *Nature Inspired Algorithms*

https://hpi.de/friedrich/teaching/ss17/natinsalg.html

The goal of this homework is to make sure you understand some basic tools from runtime analysis of Nature-inspired algorithms.

The homework is submitted on Moodle (`https://hpi.de/friedrich/moodle/`) by uploading a PDF file with your solutions. You are welcome to write your solutions out by hand and scan them.

Recall the MacLaurin series expansion of the exponential function that I wrote on the board:

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

We let ln denote the *natural logarithm* (logarithm with base e).

Exercise 2 *Show the following claim by induction.*

(a) For all $x \geq -1$ and all $k \in \mathbb{N}$, $(1+x)^k \geq 1 + kx$.

Show the following two inequalities.

(b) For all $x > -1$, $x \neq 0$: $\ln(1+x) < x$.

(c) For all $x > -1$, $x \neq 0$ and all $r > 0$: $(1+x)^r < e^{rx}$.

Exercise 3 *Consider the (1+1) EA and suppose the current solution has exactly k 0s (and, thus, $n-k$ 1s). Recall that mutation considers the current solution and flips each bit independently with probability $1/n$.*

(a) Suppose that k is a constant with respect to n. Show that the probability that the mutation creates the all-1s individual in the next step is $\Theta(n^{-k})$.

(b) Show that there is a constant c (independent of k and n) such that the probability that the mutation creates an individual with strictly less than k 0s is at least ck/n.

2 Exercise

2.a

Claim 1. *For all* $x \geq -1$ *and all* $k \in \mathbb{N} : (1+x)^k \geq 1 + kx$

Proof. Base case $k = 1$: $(1+x)^1 \geq 1 + 1 \cdot x$
Inductive hypothesis: Suppose the theorem holds for all values of k up to some n, $n \geq 1$.
Inductive step: Let $k = n + 1$

$$
\begin{align}
(1+x)^n &\geq 1 + nx && \cdot (1+x) \tag{1}\\
(1+x)^{n+1} &\geq (1+nx)(1+x) \tag{2}\\
(1+x)^{n+1} &\geq 1 + x + nx + nx^2 \tag{3}\\
(1+x)^{n+1} &\geq 1 + (n+1)x + nx^2 \tag{4}\\
(1+x)^{n+1} &\geq 1 + (n+1)x + nx^2 \geq 1 + (n+1)x && \text{since } nx^2 \geq 0 \tag{5}\\
(1+x)^{n+1} &\geq 1 + (n+1)x \tag{6}\\
& \tag{7}
\end{align}
$$

So the theorem holds for $k = n + 1$. By the principle of mathematical induction, the theorem holds for all $k \in \mathbb{N}$. □

2.b

Claim 2. *For all* $x > -1,\ x \neq 0 : \ln(1+x) < x$

Proof. The idea is to proof that $f(x) = \ln(1+x) - x$ is negative for all x > -1, x $\neq 0$.

$$
\begin{align}
f'(x) &= \frac{1}{1+x} - 1 \tag{8}\\
0 &= \frac{1}{1+x_E} - 1 && \cdot (x_E + 1) \tag{9}\\
0 &= x_E \tag{10}\\
f''(x_E) &= -\frac{1}{(1+x_E)^2} < 0 \to \text{P}(0|0) \text{ is local maximum} \tag{11}\\
& \tag{12}
\end{align}
$$

This shows that $f(x)$ is only non-negative at its single global maximum P(0|0) but negative otherwise. Since the theorem is not supposed to hold for $x = 0$ it is proven. □

2.c

Claim 3. *For all* $x > -1, x \neq 0$ *and all* $r > 0 : (1+x)^r < e^{rx}$

Proof.

$$\begin{aligned}
(1+x)^r &< e^{rx} && \ln \text{ since } x \neq 0 && (13)\\
\ln((1+x)^r) &< \ln(e^{rx}) && \text{Logarithm power rule} && (14)\\
r \cdot \ln(1+x) &< rx && : r \text{ since } r \neq 0 && (15)\\
\ln(1+x) &< x && \text{See Claim 2} && (16)
\end{aligned}$$

□

3 Exercise

3.a

Proof.

$$\begin{aligned}
P(X=n) &= \left(\frac{1}{n}\right)^k \left(1-\frac{1}{n}\right)^{n-k} && && (17)\\
&= n^{-k}\left(1-\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^{-k} && \lim_{n\to\infty} && (18)\\
&= n^{-k} && \text{since } \left(1-\frac{1}{n}\right)^n = 1 \text{ and } \left(1-\frac{1}{n}\right)^{-k} = 1 && (19)\\
&= \Theta(n^{-k}) && && (20)
\end{aligned}$$

□

3.b

Proof.

$$\begin{aligned}
P(\text{"Flip 1...k 0's"}) \geq P(\text{"Flip k 0's"}) &= \frac{k}{n}\left(1-\frac{1}{n}\right)^{n-k} \geq \frac{k}{en} && (21)\\
&\geq c\frac{k}{n} \text{ for } c = \frac{1}{e} && (22)
\end{aligned}$$

□