STA 602 Lab 6

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Exercise 1

Suppose $U = (U_1, U_2)$ is jointly Gaussian with parameters

$$\mu = (\mu_1, \mu_2)$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$\Lambda = \Sigma^{-1} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

The conditional is given by the following expression.

$$p(U_1|U_2) = \mathcal{N}(U_1|\mu_{1|2}, \Sigma_{1|2})$$

$$\mu_{1|2} = \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}(U_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \Lambda_{11}^{-1}$$

The conditional for $U_2|U_1$ is just the above equations with flipped signs.

Exercise 2

Applying the above formulas, we arrive at the following full conditionals for X, Y, and Z.

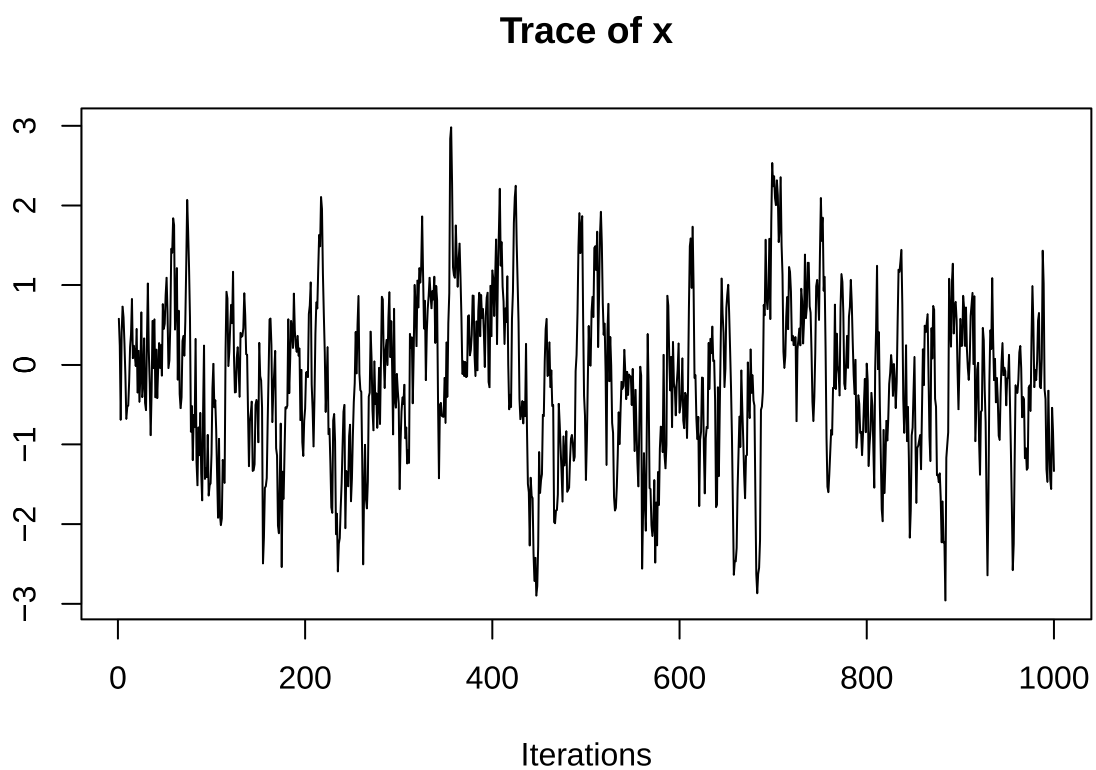
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \sim \mathcal{N}_3 \left[\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 1, 0.9, 0.1 \\ 0.9, 1, 0.1 \\ 0.1, 0.1, 1 \end{bmatrix} \right]$$

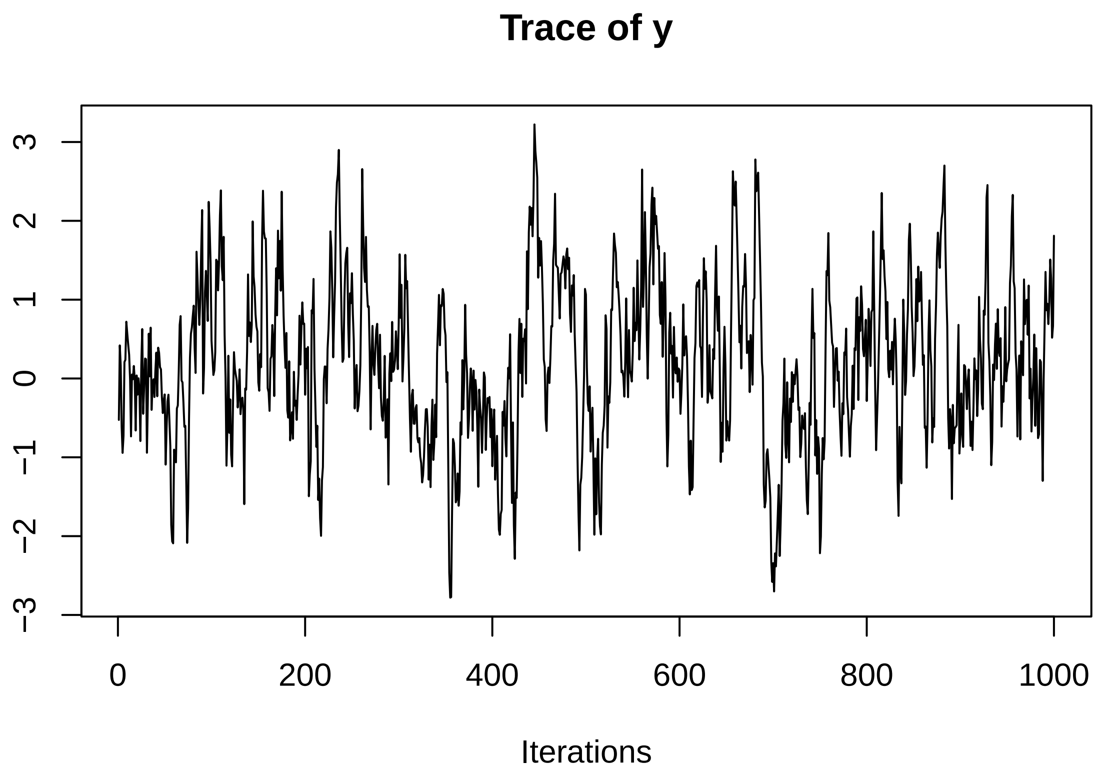
$$X|Y, Z \sim \mathcal{N}(\mu = -0.899Y - 0.01Z,\ \sigma^2 = 0.1899)$$

$$Y|X, Z \sim \mathcal{N}(\mu = -0.899X - 0.01Z,\ \sigma^2 = 0.1899)$$

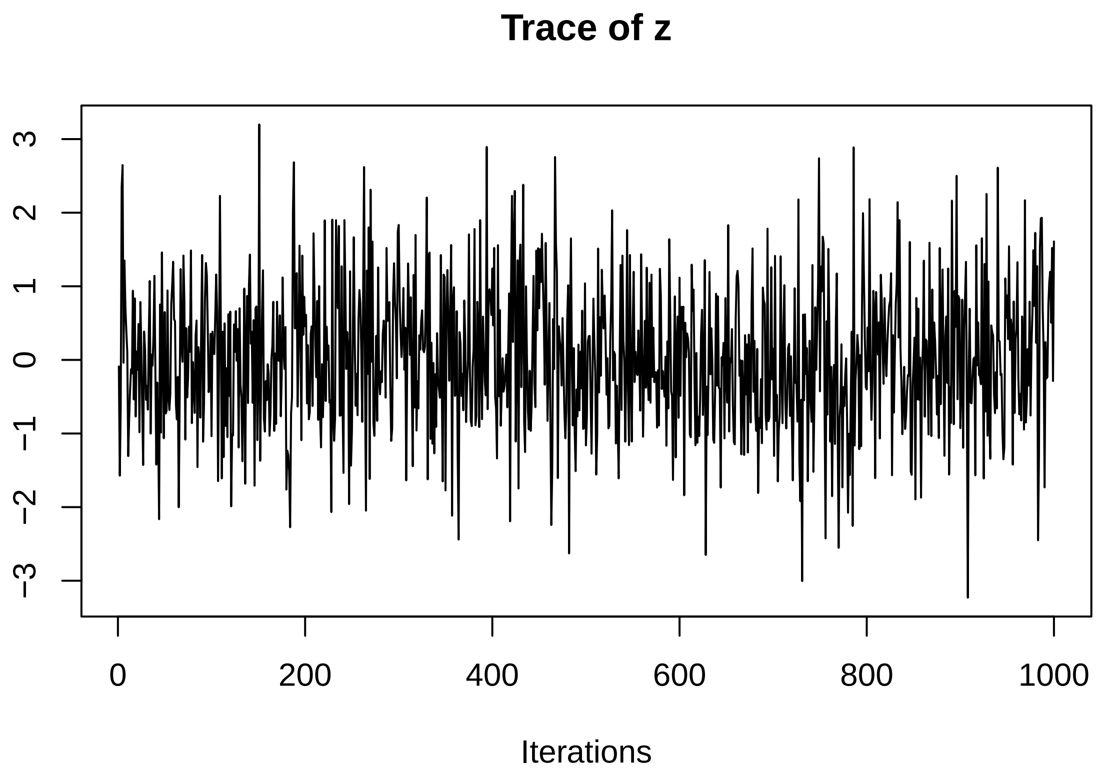
$$Z|X, Y \sim \mathcal{N}(\mu = -0.0526X - 0.0526Y,\ \sigma^2 = 0.98947)$$

Exercise 3

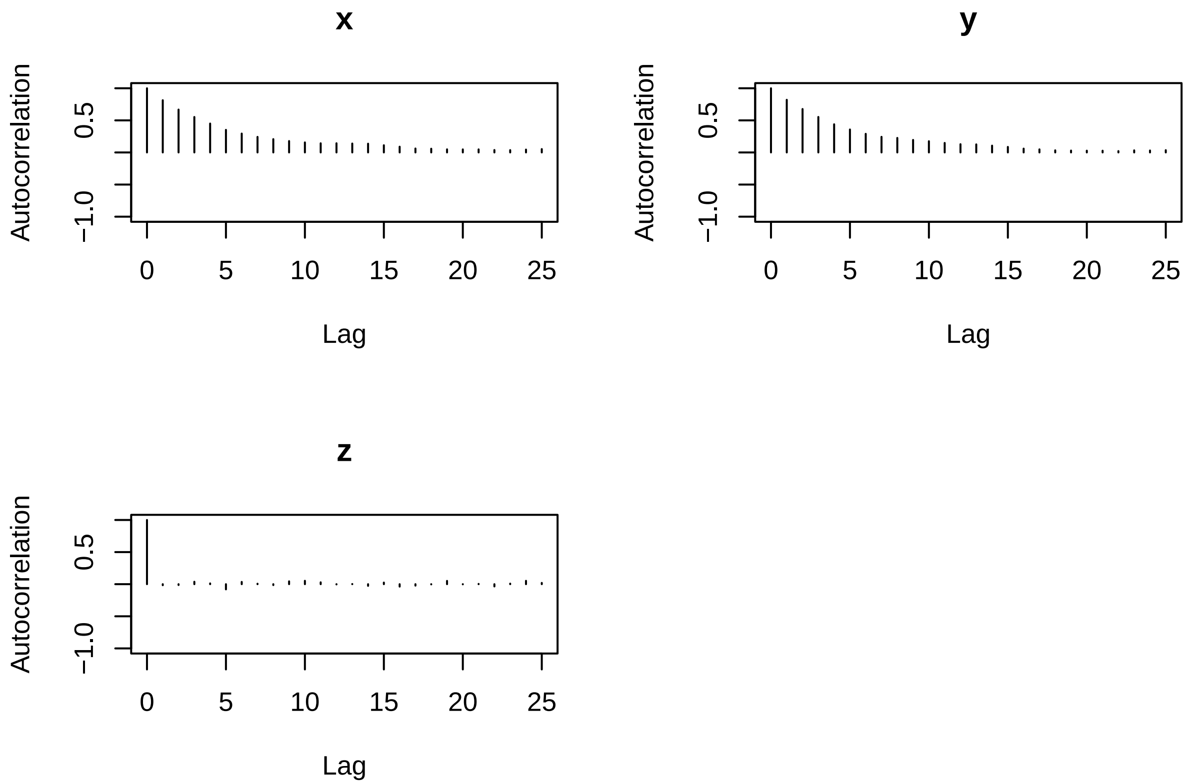




Trace of y
3
2
1
0
−1
−2
−3
0
200
400
600
800
1000
Iterations



Trace of z
3
2
1
0
−1
−2
−3
0
200
400
600
800
1000
Iterations

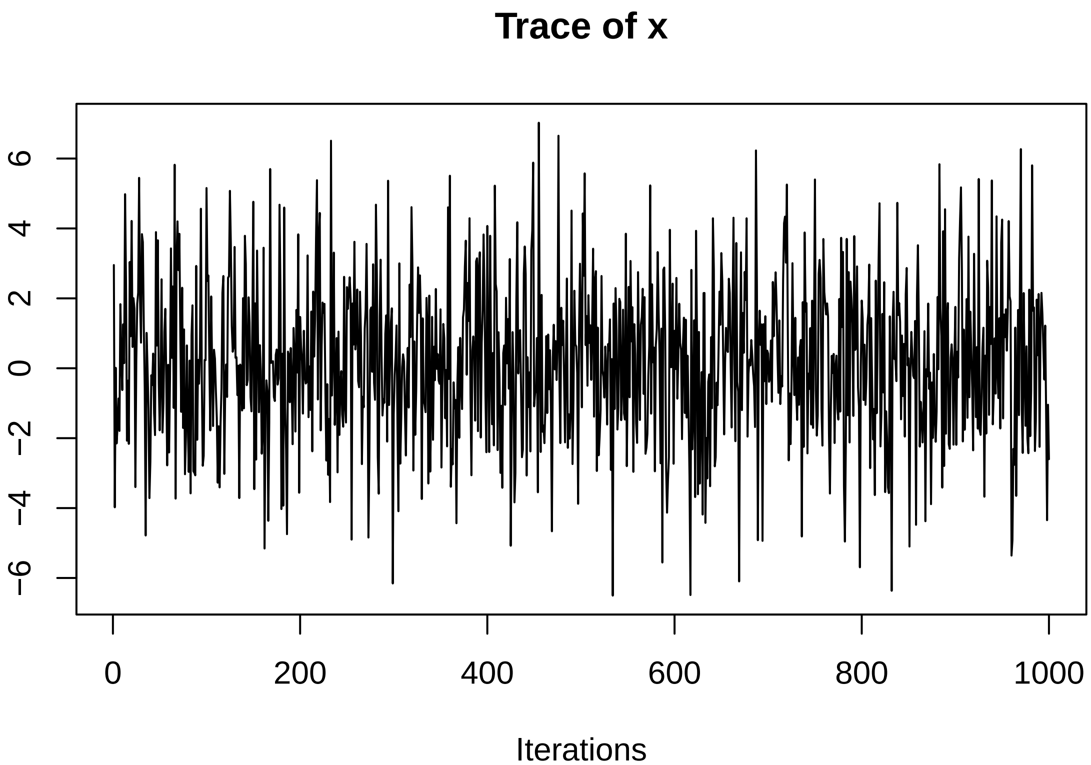


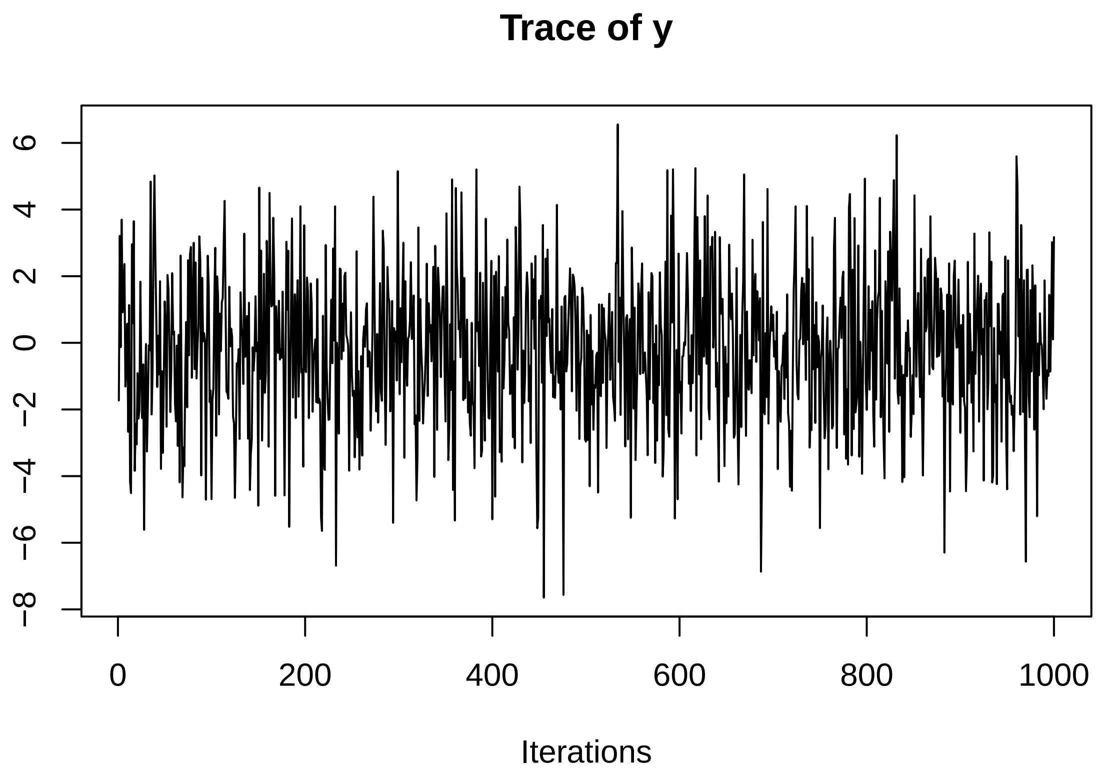
Exercise 4

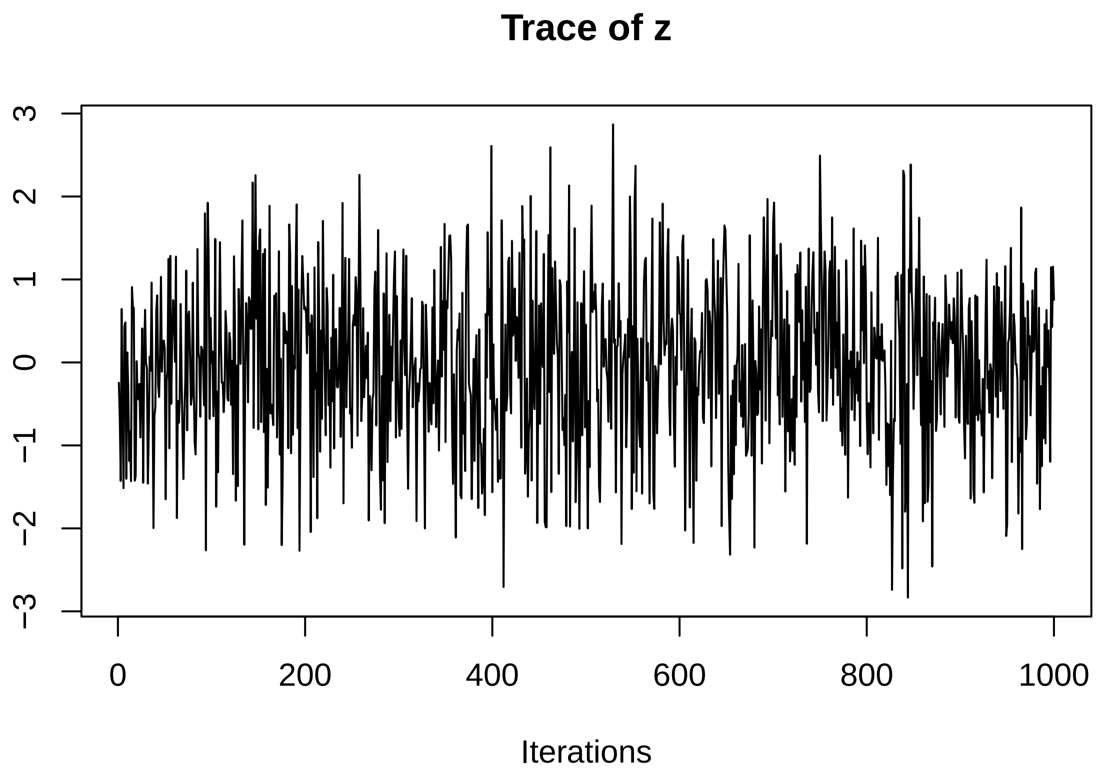
$$X, Y|Z \sim \mathcal{N}\left(\mu = \begin{bmatrix} -0.1Z \\ -0.1Z \end{bmatrix}, \sigma^2 = \begin{bmatrix} 5.266, -4.734 \\ -4.734, 5.266 \end{bmatrix}\right)$$

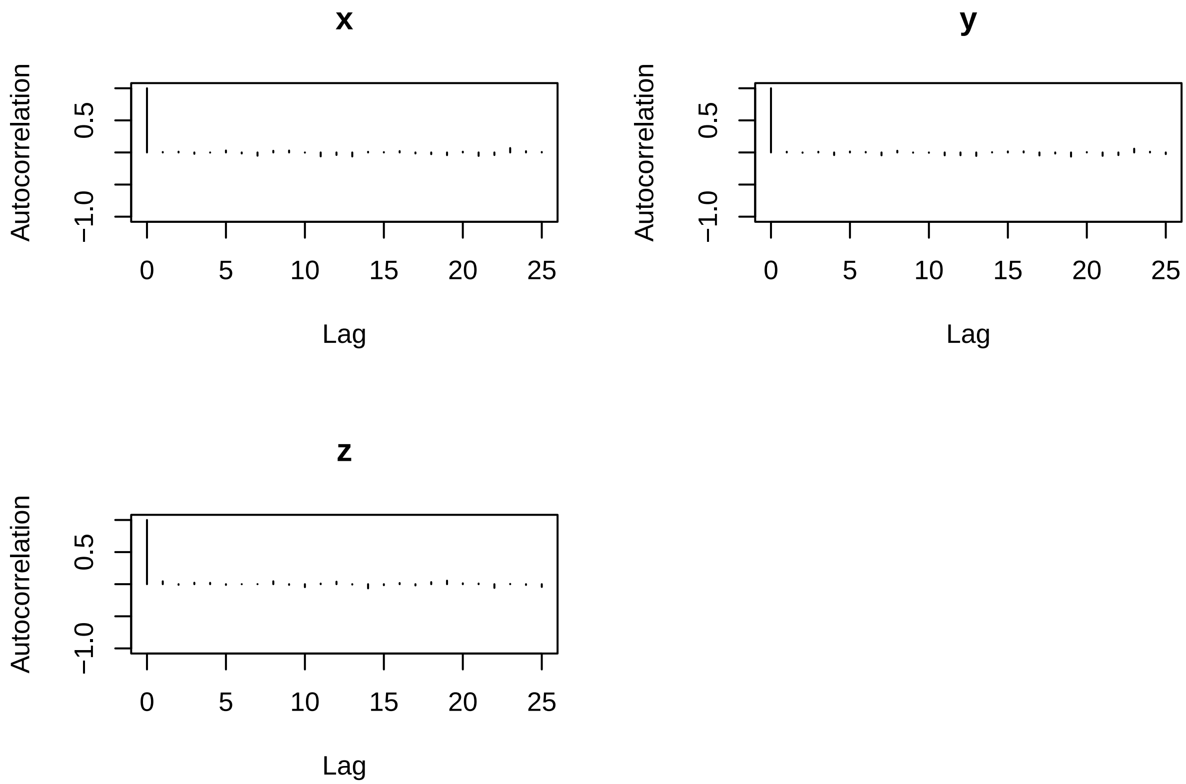
$$Z|X, Y \sim \mathcal{N}(\mu = -0.0526X - 0.0526Y, \sigma^2 = 0.98947)$$

Exercise 5




Trace of y
6
4
2
0
−2
−4
−6
−8
0
200
400
600
800
1000
Iterations


Trace of z
3
2
1
0
−1
−2
−3
0
200
400
600
800
1000
Iterations



Exercise 6

With a high correlation between X and Y, we see that if we use the Gibbs Sampler on each full conditional, we get a sticky chain. The result was the first example that we saw a high autocorrelation among X and Y samples. After we use block updates on X and Y together, the chain we obtained suddenly gets much better, and the autocorrelation also disappears. The high correlation between X and Y limits the sampler's exploration ability, making it less efficient. Treating it as a block speeds up because the relationship between Z covers a much wider area.
