Linear Algebra

Neo Wang

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1 Introduction

Standard Linear Algebra.

2 Week 1

2.1 1/5/2021

- Vector review!
- Two dimensional vector: $\langle v_1, v_2 \rangle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
- The zero vector $= \langle 0, 0 \rangle$ or $\mathbf{0}$
- Linear combinations: linear combinations of v and w refer to any sum of multiples of the vectors.

$$c\mathbf{v} + d\mathbf{w}$$

  is a linear combination of v and w
- In 3-space, a 3 dimensional vector: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \mathbf{v} = (v_1, v_2, v_3)$
- Dot products: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 ... + v_n w_n$
- Dot products are commutative.
- A dot product of zero means the vectors orthogonal.
- Contrived example: v is a vector of weights and w is a vector of distances. $v \cdot w = 0$ means the system is balanced.
- Contrived example 2: Expanding the previous problem to 3 dimensions, we can add more weights, $v = (4, 2, 100), w = (-1, 2, 0), v \cdot w = 0$
- Economics example: Five products with prices $p_1, p_2, ..., p_5$ with quantity $q_1, q_2, ... q_5$. A dot product of zero would mean it breaks even.
- Technically, $\mathbf{0}$ is orthogonal to all vectors.
- $\begin{bmatrix} 4 & 1 \\ 0 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \\ 6 \end{bmatrix}$
- Linear combinations can also be seen as dot products.
- The length or norm of a vector $||v|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}$
- Unit vector trick: if $\vec{v} \cdot \vec{v} = 1$ yes!
- $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$
- If we know the angle θ a vector makes with the x-axis, the unit vector is $\langle cos\theta, sin\theta \rangle$
- Given unit vectors with form $\langle cos\beta, sin\beta \rangle$ then the angle in between can be denoted by $\vec{u_1} \cdot \vec{u_2} = cos(\beta - \alpha)$
- Schwarz Inequality: $|v \cdot w| \leq ||v|| ||w||$

2.2 1/6/2021

- $$n \cdot v = ax + by + cz = d$$
- Planes are also linear combinations. Ex. $x - y - 3z = 0$ is a combination of vectors.
- $x = y + 3z$, set $y = z = 0$, so $\langle 1, 1, 0 \rangle$ is a vector. Then, set $y = 0, z = 1$, so we get $\langle 3, 0, 1 \rangle$. Our combination is $y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$
- In case of a constant, ex. $x - y - 3z = 2$.

- Start as if $d = 0$, in this case giving us the same $y\begin{bmatrix}1\\1\\0\end{bmatrix} + z\begin{bmatrix}3\\0\\1\end{bmatrix}$, then simply add $\begin{bmatrix}2\\0\\0\end{bmatrix}$ for $y\begin{bmatrix}1\\1\\0\end{bmatrix} + z\begin{bmatrix}3\\0\\1\end{bmatrix} + \begin{bmatrix}2\\0\\0\end{bmatrix}$
- Matrix notation. Ex. a_{47}, row 4, col 7
- Identity matrix, denoted by I, is a square matrix with 1's along the main diagonal (where $i = j$)
- $AI = A = IA$ if their multiplication dimensions are valid.
- $$\begin{bmatrix}2 & -1\\3 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\3 & 1 & 3\end{bmatrix} = \begin{bmatrix}-1 & -1 & -5\\18 & 5 & 12\end{bmatrix}$$
- An item A_{ij} = row $i\cdot$ column j
- $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}$
- Linear equations:
$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$
- Coefficient matrix $A = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & -1\end{bmatrix}$
- $$\begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$
- $A\vec{x} = \vec{b}$
- Different options
  - No solutions
  - All planes are parallel
  - 2 are parallel + 3 intercept
  - Infinite solutions (share plane or share line)
  - One solution.

2.3 1/7/2021

- In Algebra I, we solved 2x2 examples such as
$$x - 2y = 1$$
$$3x + 2y = 11$$
- We would solve by adding together to get $4x = 12$, $x = 3$
- We want our solution to form a triangle.
- Elimination fails when there are no solutions.
- Elimination also fails when there are infinitely many solutions.
$$x - 2y = 1$$
$$3x - 6y = 3$$
- Take the equation
$$\begin{bmatrix}2 & 4 & -2\\4 & 9 & -3\\-2 & -3 & 7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\8\\10\end{bmatrix}$$

- We first solve to get
$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 3 & 11 \\ 0 & 1 & 5 \end{bmatrix}$$
- Then we get the triangular system
$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 3 & 11 \\ 0 & 0 & 4 \end{bmatrix}$$
- We have to multiply matrices. Each element is a dot product of a row of the first matrix with a column of the second ex. b_{23} would come from taking the dot product of row 2 of the first matrix and column 3 of the second.
- Associative $(AB)C = A(BC)$
- **NOT COMMUTATIVE** $AB \neq BA$
- $A\mathbf{x} = \mathbf{b}$
- $A \longrightarrow$ Coefficient matrix
- $x \longrightarrow$ Column vector of unknowns
- $b \longrightarrow$ Column of scalars
- What if we had $b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ and wanted $b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$
- To form our matrices, we will start with the identity matrix.
- In the example above, we would have
$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -2a+b \\ c \end{bmatrix}$$
- What would we use to add 5 times the second row to the third? E_{32} do $i+j$ to the position to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$
- To subtract a multiple k of row j from row i is the identify matrix with a $-k$ in the i, j position.
- $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$
- Row exchange matrix (exchanges i and j) exchange them through the identity matrix.
- Ex. $i = 1$ $j = 3$, then we get matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
- Augmented Matrix includes the two sides of the equation.
- Our example 4 images above would be $\begin{bmatrix} 1 & 0 & 0 & a \\ -2 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{bmatrix}$

3 Week 2

3.1 1/12/2020

- For multiplication, the entry in row i column j is the dot of row i of A and column j of B.
- Distributed from the left and right
- Associative
- Suppose A is a square matrix. A^{-1} is a matrix of the same such that:
- $AI = IA = A$
- $A^{-1}A = AA^{-1} = I$
- An inverse matrix may not exist if
  - not square
  - if any row is a linear combination of other rows
- If an inverse matrix exists, it is really helpful to solve a system of equations.
- A 2×2 matrix is invertible if and only if $ad - bc$, the determinant, is not zero. In this case, we find the inverse of a 2x2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- A diagonal matrix of any size is invertible when none of the diagonal entries are zero.
- If they are invertible $(AB)^{-1} = B^{-1}A^{-1}$
- Any matrix that causes a known change, should be easy to invert. We have to think of a matrix that would undo what is done. Ex.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  subtracts 5 times row1 from row2

- So we would want something that adds 5 times row1 to row2 Result:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

-

$$P^{-1} = P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- When the inverse is not immediately obvious, one method of calculating it is called the Gauss-Jordan method. This is done by.
  1. Putting the matrix you wish to invert next to the identity matrix
  2. Use row transformations (elimination) to transform your matrix into the identity.
  3. The identity will have been transformed into the inverse matrix.
- Ex.

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

- $2R_2 + R_1$

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

- $3R_3 + R_2$

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{bmatrix}$$

- $2R_2 + R_3$

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 3 & 6 & 3 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{bmatrix}$$

- $6R_1 + R_2$

$$A = \begin{bmatrix} 12 & 0 & 0 & 9 & 6 & 3 \\ 0 & 6 & 0 & 3 & 6 & 3 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 12 & 0 & 0 & 9 & 6 & 3 \\ 0 & 6 & 0 & 3 & 6 & 3 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{12} \\ \frac{1}{6} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{bmatrix}$$

3.2 1/13/2021

- We are trying to factorize for the lower and upper triangular matrices.
- Ex. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$
- every E_{ij}^{-1} matrix is lower triangular, with 1's on the main diagonal. The off diagonla entry is +k to undo the elimination done by the - k when multiplied from the left by E_{ij}
- The product of several elimination matrices is still lower triangular.