COMMUTING FUNCTIONS WITH NO COMMON FIXED POINT(1)

BY
WILLIAM M. BOYCE

Introduction. Let f and g be continuous functions mapping the unit interval I into itself which commute under functional composition, that is, $f(g(x))=g(f(x))$ for all x in I. In 1954 Eldon Dyer asked whether f and g must always have a common fixed point, meaning a point z in I for which $f(z)=z=g(z)$. A. L. Shields posed the same question independently in 1955, as did Lester Dubins in 1956. The problem first appears in the literature in [15] as part of a more general question raised by J. R. Isbell. The purpose of this paper is to answer Dyer's question in the negative by the construction of a pair of commuting functions which have no fixed point in common.

The connection between functions commuting and sharing fixed points appears in several areas of analysis. Perhaps the best-known example is the Markov-Kakutani theorem [11, p. 456], which states that a commuting family of continuous linear mappings of a compact convex subset of a linear topological space into itself has a common fixed point.

The earliest relevant work on commuting functions was done in the 1920's by J. F. Ritt, who published several papers in which he investigated the algebraic properties of functional composition as a binary operation on the set of rational complex functions. His most important result from the modern standpoint was a characterization of commuting (or permutable) rational functions [19]. He proved that if f and g are commuting polynomials, then, within certain homeomorphisms, either they are iterates of the same function ($f=F^n$ and $g=F^m$ for some F, n, m), both powers of x, or both must be Tchebycheff polynomials (defined by the relationship $T_n(\cos x)=\cos nx$). In either case a common fixed point may be shown to exist, so commuting polynomials have a common fixed point.

The subject of commuting functions lay largely dormant until a 1951 paper by Block and Thielman [6] presented some new results on families of commuting polynomials and called attention to Ritt's earlier work. Their paper, together with the connection between commutativity and common fixed points found in other areas of mathematics, seems to have been the inspiration for the questions cited above.

In the last few years a number of papers have been published on commuting

Received by the editors January 9, 1967 and, in revised form, December 1, 1967.

(1) This paper is a condensation of the author's 1967 doctoral dissertation at Tulane University under Professor G. S. Young.

functions, all of them apparently motivated by the conjecture of a common fixed point. These are cited in the list of references. Although the basic question seems topological, the methods employed on the subject have included almost everything but topology, from complex variables to matrices. Some papers have treated related problems, while others have attacked the problem directly but for special cases.

The generation of this counterexample was the result of an extensive investigation of the connection between commuting functions and the permutations defined by Baxter [1] and Baxter and Joichi [3]. These permutations prescribe the manner in which f and g act on the fixed-point sets of fg and gf. The author investigated pairs of functions for which the number N of Type I and II fixed points [1] varied between five and thirteen (N is odd by definition). For N up through nine the conjecture is true, but for three of the cases with $N=11$ the issue is still in doubt. (In unpublished work [2], Baxter and Joichi have also worked through nine and met a "stumbling block" at eleven.) The three unresolved cases are represented by the following permutations:

$$(1\quad 9\quad 5)(2\quad 10\quad 6)(3\quad 11\quad 7)(4\quad 8)$$
$$(1\quad 9\quad 7)(2\quad 10\quad 6)(3\quad 11\quad 5)(4\quad 8)$$
$$(1\quad 11\quad 9\quad 5\quad 3\quad 7)(2\quad 10\quad 8\quad 4\quad 6).$$

The principal difficulty in all these cases is in getting the approximating functions to converge uniformly.

The particular example presented in this paper was developed from a case with $N=13$, represented by the permutation

$$(1\quad 11\quad 13\quad 5)(2\quad 10\quad 12\quad 6)(3\quad 9\quad 7)(4\quad 8).$$

The author used a digital computer to generate the "w-admissible" permutations [3], screen out those with fixed points, and divide them into equivalence classes for further investigation. The theorems and algorithms used are described in [7]. Even after this automatic processing there were 22 cases for $N=11$; for $N=13$ there were 112, of which the counterexample was number 101. The other 111 cases for $N=13$ either must have a common fixed point or cannot be induced by commuting functions.

Independent of and contemporary with the author's work, J. P. Huneke has also obtained examples of pairs of commuting functions with no common fixed point [13], [14]. One of Huneke's examples is identical to the one presented here.

Construction of the functions. The desired functions f and g will be constructed as the limits of a pair of sequences of continuous functions

$$f_1, f_2, f_3, \ldots \to f, \qquad g_1, g_2, g_3, \ldots \to g.$$

The construction associates with each pair of functions $\{f_i, g_i\}$ a set of "stable points" S_i such that $S_i \subset S_{i+1}$, and for x in S_i, $f_{i+j}(x) = f_i(x)$ and $g_{i+j}(x) = g_i(x)$ for all $j \geqq 0$. Thus in the limit $f(x)=f_i(x)$ and $g(x)=g_i(x)$ for x in S_i. The union of the S_i is dense in I, so the limit functions f and g will be determined by the values of f_i and g_i on S_i and be independent of the values on the "(i)-intervals" in between.

In the construction only the first two pairs of functions with their stable points are explicitly stated; the rest of the functions in each sequence are defined inductively. Graphs of the first four functions in each sequence are shown in Figures 2, 3, 4, and 5. Note that from the first three pairs of functions one could easily infer that as i increases the differences between successive functions will be limited to ever-smaller intervals in the center of I. However, using f_4 and g_4 one obtains the correct impression, namely, that with increasing i the differences tend to spread out over more and more of the interval.

The steps of the construction are as follows: first, necessary definitions are stated and the initial functions specified; then a lemma is given which produces the sequences of functions; and finally, a theorem is proven which derives the desired properties of the limit functions.

DEFINITION. Let A be a closed interval. A subset T of A is *the (k)-set of A* if it has k elements, contains the end points of A, and divides A into $k-1$ subintervals of equal lengths.

DEFINITION. Let A and B be closed intervals; let T be the $(2k+2)$-set of A and U be the $(2k)$-set of B for $k \geqq 2$. Let $T=\{t_1, t_2, \ldots, t_{2k+2}\}$ and $U=\{u_1, u_2, \ldots, u_{2k}\}$, where the elements of T and U are numbered in increasing order. Four functions from A onto B will be called the *$(2k+2)$-hook functions from A onto B.* Examples of these are given in Figure 1. The *order* of each of the four functions is $2k+2$. Each has a *direction*—increasing or decreasing—and a *type*—maximum or minimum. Each will map T onto U and be linear on the intervals between points of T; thus it suffices to define them explicitly only on T, as follows:

Type	Direction	Value of the function at t_i			
Minimum	Increasing	u_i,	$i \leqq 2$;	u_{i-2},	$i > 2$.
Maximum	Increasing	u_i,	$i \leqq 2k$;	u_{i-2},	$i > 2k$.
Minimum	Decreasing	u_{2k+1-i},	$i \leqq 2k$;	u_{2k+3-i},	$i > 2k$.
Maximum	Decreasing	u_{2k+1-i},	$i \leqq 2$;	u_{2k+3-i},	$i > 2$.

The terms *$(2k+2)$-hook function* or simply *hook function* may be used where the order or intervals involved are understood.

DEFINITION. Given a finite subset S_i of I, a nondegenerate closed interval $J \subset I$ is an *(i)-interval* if its end points are in S_i but none of its interior points are.

DEFINITION. The *mesh size* M_i of S_i is the length of the longest (i)-interval.

Next the initial functions with their associated sets of stable points will be specified. We will define f_i, g_i, and S_i for $i=1$ and 2. In each case f_i and g_i will be continuous and linear on each (i)-interval, so they may be defined by specifying only their values at the points of S_i.

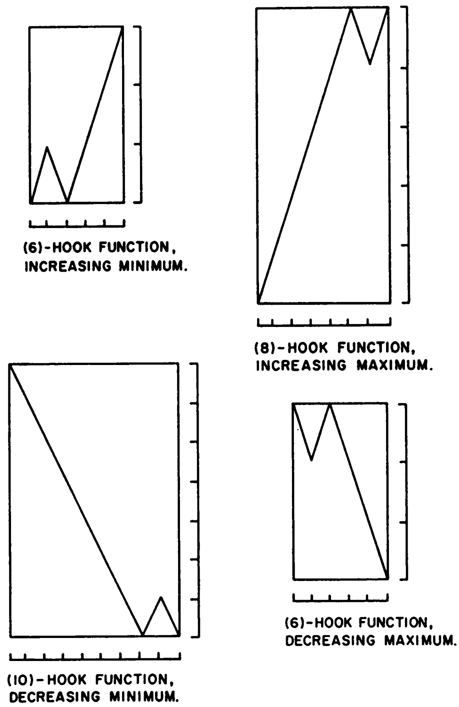

FIGURE 1

First we shall define $S_1=\{0, 1/3, 2/3, 1\}$. Note that S_1 is the (4)-set of I and that $M_1=1/3=(1/3)^1$. On S_1 the values of f_1 and g_1 are as follows:

S_1	0	1/3	2/3	1
f_1	1	0	1	0
g_1	0	1	0	1

f_1 and g_1 are defined to be linear between points of S_1. Their graphs are given in Figure 2. f_1 and g_1 can be seen to commute and have a common fixed point, but no other pair of functions will have either of these properties.

To define S_2 we divide the first and third (1)-intervals into three subintervals, and the middle (1)-interval into five subintervals, so we have

$$S_2 = \{0, 1/9, 2/9, 1/3, 6/15, 7/15, 8/15, 9/15, 2/3, 7/9, 8/9, 1\}.$$

Observe that $S_1 \subset S_2$ and $M_2=1/9=(1/3)^2$. Now f_2 and g_2 shall be defined on S_2:

S_2	0	1/9	2/9	1/3	6/15	7/15	8/15	9/15	2/3	7/9	8/9	1
f_2	1	2/3	1/3	0	1/3	0	1/3	2/3	1	2/3	1/3	0
g_2	0	1/3	2/3	1	2/3	1/3	0	1/3	0	1/3	2/3	1

f_2 and g_2 are linear between points of S_2. Their graphs are shown in Figure 3.

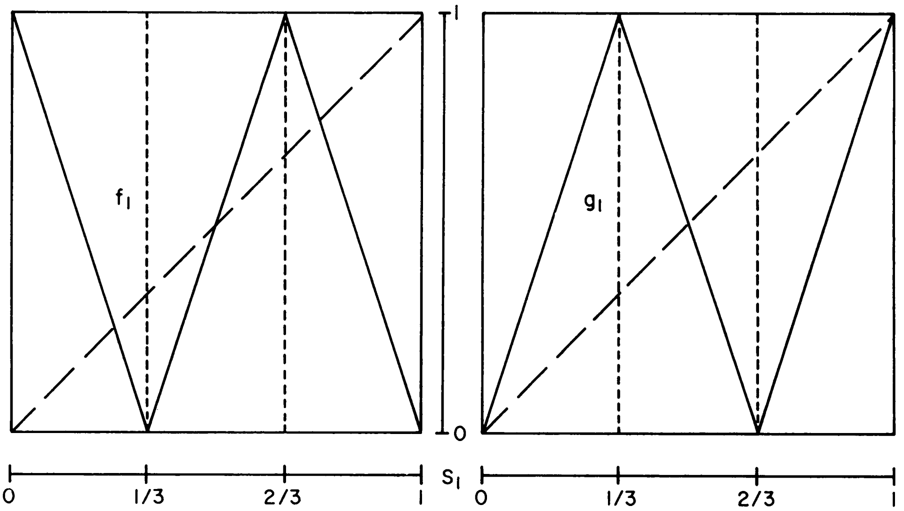

FIGURE 2

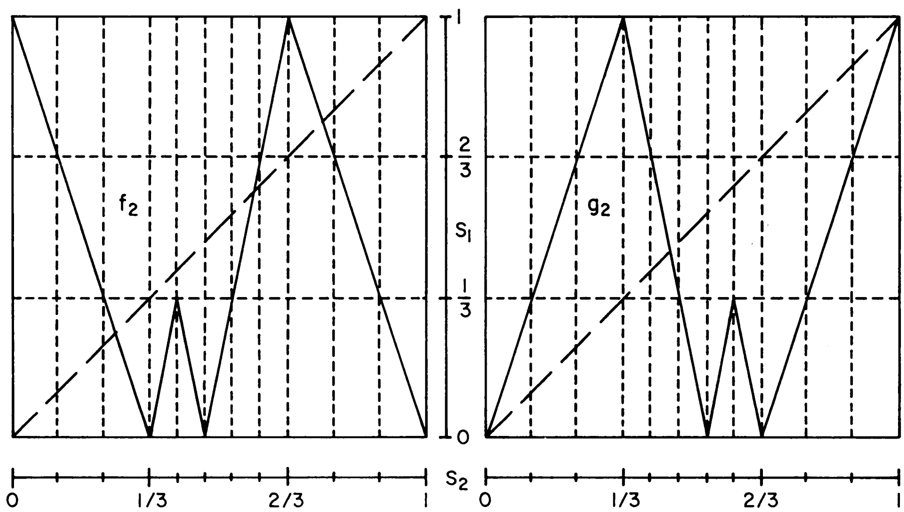

FIGURE 3

Note that $f_2(S_2) = S_1 = g_2(S_2)$ and f_2 and g_2 are (6)-hook functions on the middle (1)-interval.

Figure 4 gives the graphs of f_3 and g_3 and Figure 5 the graphs of f_4 and g_4. These functions are constructed through the inductive process described below.

LEMMA. *There exist two sequences of functions* $\{f_i\}$ *and* $\{g_i\}$ *and a sequence of sets* $\{S_i\}$ *which satisfy the following requirements for* $i \geqq 2$:

(1) $f_1, f_2, g_1, g_2, S_1, S_2$ *are as defined above.*

(2) [*properties of* S_i] $S_{i-1} \subset S_i$; *if* J *is an* $(i-1)$-*interval, then there is a* $k \geqq 2$ *such that* $S_i \cap J$ *is the* $(2k)$-*set of* J; *and* $M_i \leqq (1/3)^i$.

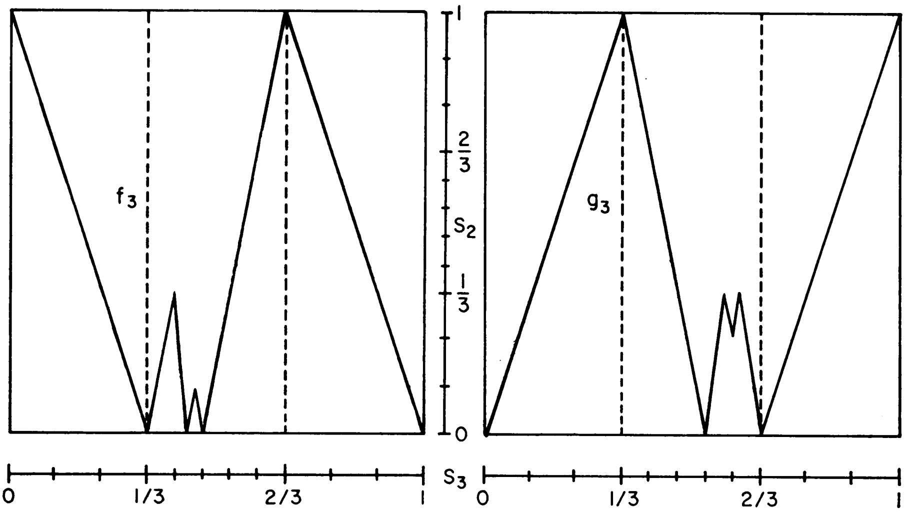

FIGURE 4

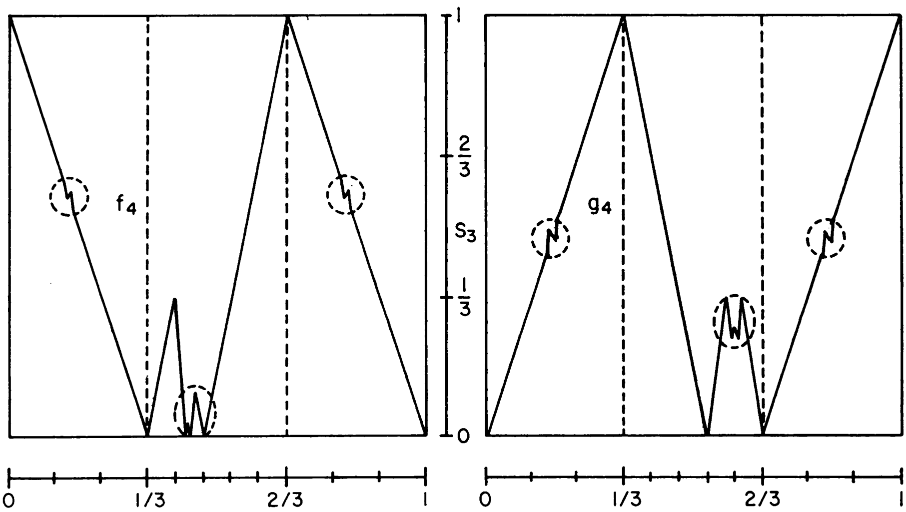

FIGURE 5

(3) [*diagram commutativity*] $f_{i-1}g_i = g_{i-1}f_i$ *on* I.

(4) [f_i *on an* (i)*-interval*] $f_i(S_i) \subset S_{i-1}$; f_i *is linear on each* (i)*-interval and maps it onto an* $(i-1)$*-interval. Further, for* $j \geqq 0$, f_{i+j} *agrees with* f_i *on* S_i, *and if* J *is an* (i)*-interval, then* $f_{i+j}(J) = f_i(J)$; *thus* $\|f_{i+j} - f_i\| \leqq M_{i-1}$, *where* $\|F\|$ *is the* sup norm *of* F *on* I.

(5) [g_i *on an* (i)*-interval*] *same as* 4, *except for* g_{i+j} *and* g_i.

(6) [f_i *on an* $(i-1)$*-interval*] *let* J *be an* $(i-1)$*-interval and define* $J_f = f_{i-1}(J)$. *Then either*

(a) f_i *is linear on* J, $f_i|J = f_{i-1}|J$, *and* $S_i \cap J$ *and* $S_{i-1} \cap J_f$ *have the same cardinality*; *or*

(b) f_i *is a* $(2k+2)$-*hook function from* J *onto* J_f, $S_i \cap J$ *is the* $(2k+2)$-*set of* J, *and* $S_{i-1} \cap J_f$ *is the* $(2k)$-*set of* J_f.

(7) [g_i *on an* $(i-1)$-*interval*] *same as* 6, *except for* g_i, g_{i-1}, *and* J_g.

Proof. The proof will be by induction. Let $L(n)$ designate the proposition that the functions f_i and g_i and sets S_i are well defined for $1 \leqq i \leqq n$ and satisfy requirements 1–7 of the lemma when $2 \leqq i \leqq n$ and $i+j \leqq n$. The proof of the lemma consists of first showing that $L(2)$ is true, then defining a construction by which $L(n)$ implies $L(n+1)$.

To prove $L(2)$, we first observe that requirement 1 only specifies the functions and sets we must consider. For requirement 2, we have earlier noted that $S_1 \subset S_2$ and that $M_2 = 1/9 = (1/3)^2$. S_2 divides the three (1)-intervals evenly into three, five, and three subintervals, respectively, so that the corresponding values of k are two, three, and two. Let us momentarily defer requirement 3, the "diagram commutativity". For requirements 4 and 5 we have noted that $f_2(S_2) = S_1 = g_2(S_2)$, so containment is implied. By definition, f_2 and g_2 are linear on (2)-intervals, and from the definitions we see that consecutive points of S_2 are mapped onto consecutive points of S_1 by both f_2 and g_2; thus the image of a (2)-interval is a single (1)-interval. The rest of 4 and 5 is satisfied trivially, since for $L(2)$ we need only consider $i+j = 2 = i$. Now we may return to "diagram commutativity". Since f_2 and g_2 are linear on (2)-intervals and map (2)-intervals onto (1)-intervals, and f_1 and g_1 are linear on (1)-intervals, it follows that $f_1 g_2$ and $g_1 f_2$ are linear on (2)-intervals. Thus to prove that the two composites agree on I it suffices to show that they agree on S_2. Their values on S_2 can be computed easily from the definitions and are found to agree, as follows:

S_2	0	1/9	2/9	1/3	6/15	7/15	8/15	9/15	2/3	7/9	8/9	1
$f_1 g_2 = g_1 f_2$	1	0	1	0	1	0	1	0	1	0	1	0

For requirements 6 and 7 we note that $J_f = I = J_g$ for all three (1)-intervals. For the first and third (1)-intervals we can see from Figure 3 that case (a) applies, since on both intervals f_2 and g_2 are linear. The cardinalities of $S_2 \cap J$ and $S_1 \cap I$ are both four. On the middle (1)-interval both f_2 and g_2 are (6)-hook functions from J onto I, so that $k=2$. f_2/J is minimum increasing and g_2/J is minimum decreasing. $S_2 \cap J$ is the (6)-set of J, dividing the interval into five subintervals, and $S_1 \cap I$ is the (4)-set of I. Thus requirements 1–7 are all met when $i = i+j = 2$, and $L(2)$ is true.

Now we must devise a construction by which $L(n)$ will imply $L(n+1)$. We will obtain the functions and sets needed for $L(n+1)$ by simply adjoining f_{n+1}, g_{n+1}, and S_{n+1} to the partial sequences of f_i, g_i, and S_i for $i \leqq n$ which are assumed to exist as part of $L(n)$. The functions and set for $n+1$ must be defined in such a way as to satisfy requirements 1–7 of the lemma when $i = n+1$ and $i+j = n+1$.

The need to satisfy the requirements goes a long way toward defining f_{n+1}, g_{n+1}, and S_{n+1}. For requirement 2 we must have $S_n \subset S_{n+1}$, and for requirements 4 and 5 f_{n+1} must agree with f_n and g_{n+1} agree with g_n on S_n. Let J be an (n)-interval and define $J_f=f_n(J)$ and $J_g=g_n(J)$. We know from requirements 4 and 5 of $L(n)$ that J_f and J_g are $(n-1)$-intervals, and in order to satisfy these same requirements for $L(n+1)$ we must define f_{n+1} and g_{n+1} so that $f_{n+1}(J)=J_f$ and $g_{n+1}(J)=J_g$. Since from requirements 4 and 5 f_n and g_n are linear on J, we have the situation depicted in Figure 6. f_{n+1} must agree with f_n on the end points of J, since the end points are

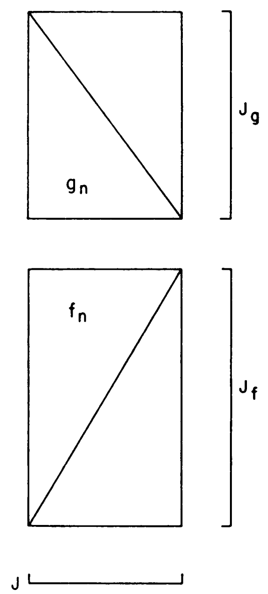

FIGURE 6

in S_n, and the image of J under f_{n+1} must be the same as its image under f_n. Thus the problem of defining f_{n+1} on J is that of deforming the graph of f_n (if necessary) to get the graph of f_{n+1} while staying within $J\times J_f$ and holding the values at the end points fixed. Similarly, we must define g_{n+1} on J so that the graph of g_{n+1} remains in $J\times J_g$ and agrees with g_n at the end points.

The need to satisfy requirement 3, $f_ng_{n+1}=g_nf_{n+1}$, compels us to consider f_n on J_g and g_n on J_f. We have previously noted that J_f and J_g are $(n-1)$-intervals. Using requirements 4 and 5 from $L(n)$ with $i=n-1$, we see that g_{n-1} maps J_f onto an $(n-2)$-interval $(J_f)_g$, which we will call J_h, and f_{n-1} maps J_g onto an $(n-2)$-interval $(J_g)_f$. But from the "commutativity" of $L(n)$ we know that $f_{n-1}g_n=g_{n-1}f_n$, so $(J_g)_f=f_{n-1}(J_g)=f_{n-1}g_n(J)=g_{n-1}f_n(J)=g_{n-1}(J_f)=(J_f)_g=J_h$. Also, since $n\geqq n-1$ and J_f and J_g are $(n-1)$-intervals, we can use 4 and 5 again to obtain $f_n(J_g)=f_{n-1}(J_g)=J_h$ and $g_n(J_f)=g_{n-1}(J_f)=J_h$. Thus we have the diagram in Figure 7, with all functions onto. Furthermore, by 4 and 5, if x is one of the end points of J then $f_n(x)$ and $g_n(x)$ are in S_{n-1}; but f_n and g_n agree with f_{n-1} and g_{n-1} on S_{n-1}, so $f_ng_n(x)=f_{n-1}g_n(x)$ $=g_{n-1}f_n(x)=g_nf_n(x)$. Thus f_ng_n and g_nf_n agree on the end points of J.

Let us now describe how to define f_{n+1}, g_{n+1}, and S_{n+1}. Since I is covered by

$$\begin{tikzcd} J \arrow[r, "f_n"] \arrow[d, "g_n"'] & J_f \arrow[d, "g_n"] \\ J_g \arrow[r, "f_n"] & J_h \end{tikzcd}$$

FIGURE 7

(n)-intervals, and we know that f_{n+1} must agree with f_n and g_{n+1} agree with g_n on S_n, the end points of the (n)-intervals, it suffices to give a construction for f_{n+1}, g_{n+1}, and S_{n+1} only on the arbitrary (n)-interval J. So the problem reduces to (a) finding functions f_{n+1} and g_{n+1} to make the diagram in Figure 8 commute while f_{n+1} and

$$\begin{tikzcd} J \arrow[r, "f_{n+1}"] \arrow[d, "g_{n+1}"'] & J_f \arrow[d, "g_n"] \\ J_g \arrow[r, "f_n"] & J_h \end{tikzcd}$$

FIGURE 8

g_{n+1} agree on the end points of J with f_n and g_n and (b) defining $S_{n+1} \cap J$; so all meet requirements 1–7 of $L(n+1)$.

Since J_f and J_g are $(n-1)$-intervals, we may apply requirements 6 and 7 of $L(n)$ to get information on f_n/J_g, g_n/J_f, $S_n \cap J_f$, and $S_n \cap J_g$ which will aid the construction. J_h is an $(n-2)$-interval, so from requirement 2 we know that there is a $k \geqq 2$ such that $S_{n-1} \cap J_h$ is the $(2k)$-set of J_h. Then from 6 we know that either (a) f_n is linear on J_g and $S_n \cap J_g$ is the $(2k)$-set of J_g; or (b) f_n is a $(2k+2)$-hook function from J_g onto J_h and $S_n \cap J_g$ is the $(2k+2)$-set of J_g. A similar conclusion for g_n/J_f and $S_n \cap J_f$ follows from requirement 7. Note in particular that if f_n/J_g and g_n/J_f are both hook functions, then they have the same order, $2k+2$.

If we now let functions be graphed "sideways", with the axis of the independent variable vertical and the dependent variable's axis horizontal, we can represent a typical situation as in Figure 9. In this figure g_n/J_f and f_n/J_g have been graphed sideways, and both are hook functions. The length of J is somewhat exaggerated; J and the subintervals of J_f and J_g are all (n)-intervals.

The problem divides naturally into four cases, depending on the properties of the functions f_n/J_g and g_n/J_f:

(I) Both functions are linear.

(II) One function is linear, the other is a hook function.

(III) Both are hook functions of the same type.

(IV) Both are hook functions, but of opposite type.

Recalling that k is such that $S_{n-1} \cap J_h$ is the $(2k)$-set of J_h, for each of the four cases

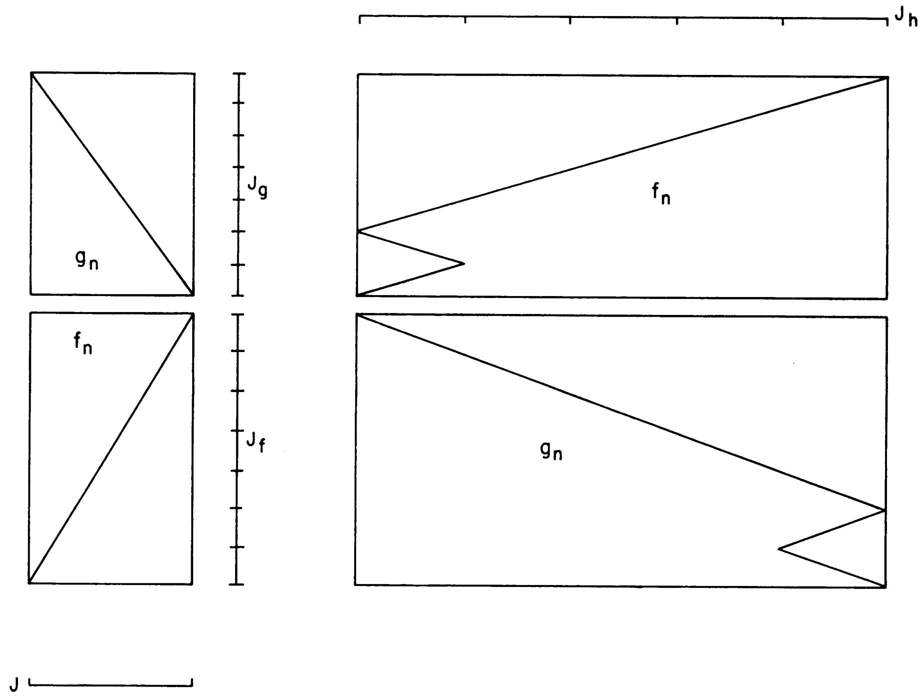

FIGURE 9

we define f_{n+1}, g_{n+1}, and S_{n+1} as follows:

(I) Define $f_{n+1}=f_n$ and $g_{n+1}=g_n$ on J, and define $S_{n+1} \cap J$ to be the $(2k)$-set of J.

(II) Define $S_{n+1} \cap J$ to be the $(2k+2)$-set of J. If f_n/J_g is linear, then $f_{n+1}=f_n$ on J and g_{n+1} is a $(2k+2)$-hook function from J onto J_g with the same direction as g_n/J; g_{n+1}/J has the same type as g_n/J_f if and only if f_n/J_g is increasing. (See Figure 10.) If g_n/J_f is linear, then the same definition holds except with f and g reversed.

(III) Define $f_{n+1}=f_n$ and $g_{n+1}=g_n$ on J, and define $S_{n+1} \cap J$ to be the $(2k+2)$-set of J.

(IV) Define $S_{n+1} \cap J$ to be the $(2k+4)$-set of J. f_{n+1}/J is a $(2k+4)$-hook function with the same direction as f_n/J, and g_{n+1}/J is a $(2k+4)$-hook function with the same direction as g_n/J. f_{n+1}/J has the same type as f_n/J_g if and only if g_n/J_f is increasing, and g_{n+1}/J has the same type as g_n/J_f if and only if f_n/J_g is increasing. (See Figure 11.)

Now it must be shown that requirements 1–7 are satisfied by f_{n+1}, g_{n+1}, and S_{n+1} as defined. Requirement 1 is inherited from $L(n)$. For requirement 2, in each case $S_{n+1} \cap J$ is the $(2m)$-set of J for some m, so $S_n \subset S_{n+1}$. Since in each case the cardinality of $S_{n+1} \cap J$ is no less than that of $S_n \cap J_f$, which is at least four by $L(n)$, we must have $m \geqq 2$. Thus each (n)-interval is split into at least three equal subintervals by S_{n+1}, so we have $M_{n+1} \leqq (1/3)M_n \leqq (1/3)^{n+1}$.

Requirements 6 and 7 will be considered next. In case I, f_n/J_g and g_n/J_f are linear and so $S_n \cap J_f$ and $S_n \cap J_g$ are both $(2k)$-sets. We defined $S_{n+1} \cap J$ to be the

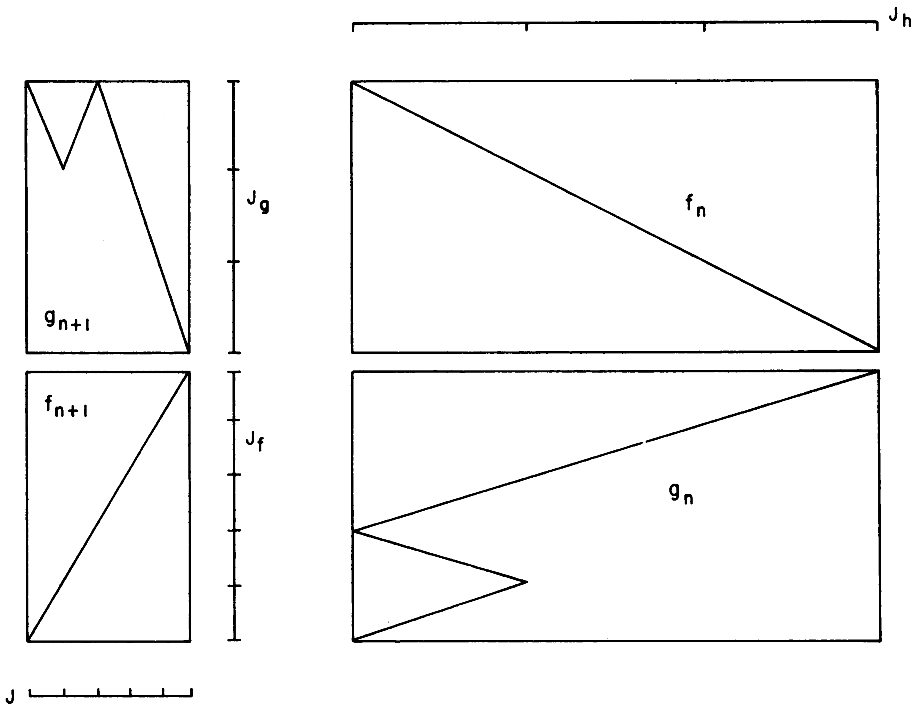

FIGURE 10

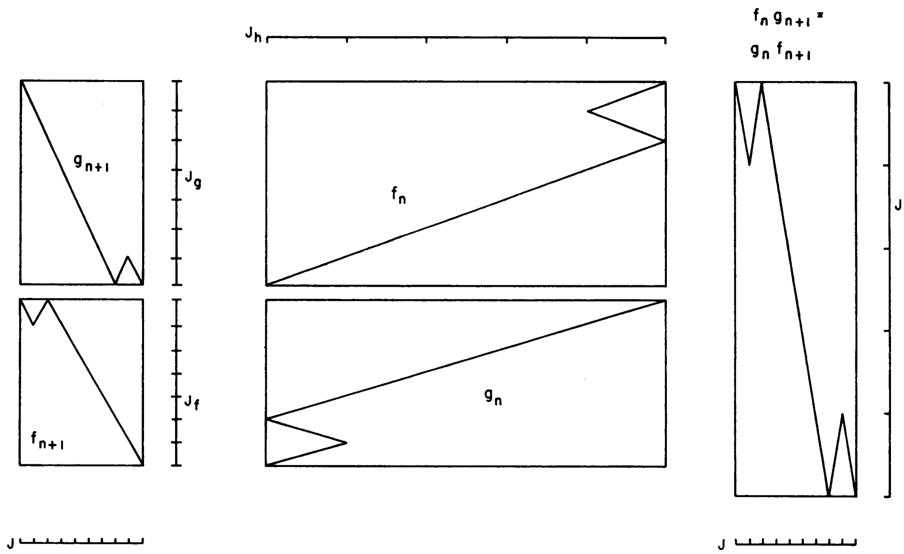

FIGURE 11

$(2k)$-set of J, and $f_{n+1}=f_n$ and $g_{n+1}=g_n$ on J, so case (a) is satisfied for both 6 and 7. For case II, assume that f_n/J_g is linear and g_n/J_f is a hook function; then $S_n \cap J_g$ is the $(2k)$-set of J_g, g_n/J_f has order $2k+2$, and $S_n \cap J_f$ is the $(2k+2)$-set of J_f.

We defined $S_{n+1} \cap J$ to be the $(2k+2)$-set of J and $f_{n+1}=f_n$ on J; so f_{n+1} is linear from J onto J_f, $S_{n+1} \cap J$ is the $(2k+2)$-set of J, and $S_n \cap J_f$ is the $(2k+2)$-set of J_f. Thus case (a) holds for 6. g_{n+1}/J was defined to be a $(2k+2)$-hook function from J onto J_g, with $S_{n+1} \cap J$ being the $(2k+2)$-set of J and $S_n \cap J_g$ the $(2k)$-set of J_g, so case (b) holds for 7. If on the other hand g_n/J_f is linear instead of f_n/J_g, then case (a) holds for 7 and case (b) for 6. In cases III and IV both f_n/J_g and g_n/J_f are $(2k+2)$-hook functions, and $S_n \cap J_g$ and $S_n \cap J_f$ are $(2k+2)$-sets. In case III we defined f_{n+1}/J and g_{n+1}/J to be linear, and $S_{n+1} \cap J$ to be the $(2k+2)$-set of J, so case (a) holds for 6 and 7. In case IV both f_{n+1}/J and g_{n+1}/J are defined as $(2k+4)$-hook functions and $S_{n+1} \cap J$ as the $(2k+4)$-set of J, so case (b) holds for 6 and 7. Thus requirements 6 and 7 of $L(n+1)$ are satisfied in all four cases.

For requirement 4 we note that in both cases (a) and (b) of 6 and 7 we have $f_{n+1}(S_{n+1} \cap J) = S_n \cap J_f$, f_{n+1} linear on $(n+1)$-intervals, and $(n+1)$-intervals mapped onto (n)-intervals. Thus on all of I, $f_{n+1}(S_{n+1}) \subset S_n$ and f_{n+1} maps $(n+1)$-intervals linearly onto (n)-intervals. Since f_{n+1}/J was defined to agree in direction with f_n/J, f_{n+1}/J agrees with f_n/J on $S_n \cap J$, the end points of J; thus f_{n+1} agrees with f_n on S_n. For $i \leqq n$, by $L(n)$ f_n agrees with f_i on S_i, and $S_i \subset S_n$, so f_{n+1} agrees with f_i on S_i. In all cases $f_{n+1}(J) = f_n(J) = J_f$, where J is an (n)-interval. If J_i is an (i)-interval and $i \leqq n$, then by $L(n)$ $f_n(J_i) = f_i(J_i)$; J_i is the union of (n)-intervals, and the images of (n)-intervals under f_{n+1} and f_n coincide, so $f_{n+1}(J_i) = f_n(J_i) = f_i(J_i)$. Since $f_i(J_i)$ is an $(i-1)$-interval, for x in J_i $f_{n+1}(x)$ and $f_i(x)$ lie in the same $(i-1)$-interval, so $|f_{n+1}(x) - f_i(x)| \leqq M_{i-1}$ and $\|f_{n+1} - f_i\| \leqq M_{i-1}$. Thus requirement 4 of $L(n+1)$ is satisfied. Requirement 5 is satisfied by exactly the same arguments for g instead of f.

The only remaining requirement is number 3, the "diagram commutativity," or showing that f_{n+1} and g_{n+1} as defined make the diagram in Figure 8 commutative. For case I all four functions are linear, so the two composites $f_n g_{n+1}$ and $g_n f_{n+1}$ are linear; but the composites agree on the end points of J, so they must agree on all of J.

For cases II and III the following observation is helpful. Let A, B, C, and D be intervals, V and W linear onto functions, and H a hook function from B onto C, so that we have

$$A \xrightarrow{V} B \xrightarrow{H} C \xrightarrow{W} D.$$

Then HV and WH are hook functions of the same order as H, HV is the same type as H, and WH is the same type as H if and only if W is sense-preserving (increasing).

Now for case II assume again that f_n/J_g is linear. Then f_{n+1}/J is linear, and g_n/J_f and g_{n+1}/J are hook functions of the same order. By the preceding remark $g_n f_{n+1}$ is a hook function of the same order and type as g_n/J_f, and $f_n g_{n+1}$ is a hook function of the same order as g_n/J_f. If f_n/J_g is increasing, then g_{n+1}/J is defined to be the same type as g_n/J_f, and by the remark $f_n g_{n+1}$ has the same type as g_{n+1}/J which is the same as the type of g_n/J_f. If f_n/J_g is decreasing, then g_{n+1}/J is the opposite type as g_n/J_f, but $f_n g_{n+1}$ is the opposite type as g_{n+1}/J; so again $f_n g_{n+1}$ has the same type as g_n/J_f. Thus f_ng_{n+1} and g_nf_{n+1} are both hook functions of the same order and type as g_n/J_f. Since they agree on the end points of J, they also have the same direction. Order, type, and direction determine a hook function, so f_ng_{n+1} and g_nf_{n+1} must be the same function on J. By symmetry this conclusion is valid when it is g_n/J_f which is the linear function.

For case III f_n/J_g and g_n/J_f are hook functions of the same order and type, and f_{n+1}/J and g_{n+1}/J are linear. Then f_ng_{n+1} and g_nf_{n+1} are hook functions of the same order and type as f_n/J_g and g_n/J_f. Since they agree on the end points of J, they are hook functions of the same order, type, and direction; hence they are the same function and agree everywhere.

Case IV is more complicated than the other three. For each combination of directions of f_n/J_g, g_n/J_f, f_n/J, and g_n/J and types of f_n/J_g and g_n/J_f the definition gives us a different set of four functions. This would appear to be six binary options, or $2^6=64$ different combinations, for a given value of k. But the fact that f_ng_n and g_nf_n agree on the end points of J, hence have the same direction, requires that of the four functions an even number must be increasing and an even number decreasing. Thus there are only three independent choices of direction. Let us assume these are f_n/J_g, g_n/J_f, and f_n/J, with the direction of g_n/J depending on the other three. Recall that the directions of f_{n+1}/J and g_{n+1}/J were defined to be the same as f_n/J and g_n/J. Now since for case IV f_n/J_g and g_n/J_f have opposite types, we know that one of them is maximum and the other minimum. Then the types of f_{n+1}/J and g_{n+1}/J, according to the definition, are determined by the types and directions of f_n/J_g and g_n/J_f. Thus the only free choices are whether f_n/J_g is maximum or minimum, and the choice of the direction of three of the functions, a total of $2^4=16$ possible combinations.

If we observe that the problem is symmetric in f and g, we can reduce the number of combinations that must be considered to eight, for without loss of generality we can assume that, say, f_n/J_g is maximum and g_n/J_f is minimum. When we make this assumption, the type of f_{n+1}/J depends only on the direction of g_n/J_f, and the type of g_{n+1}/J depends only on the direction of f_n/J_g. Thus we have the eight combinations given on the following table. Here the quantities in parentheses are those either fixed or dependent on other choices.

	f_n/J_g		g_n/J_f		f_{n+1}/J		g_{n+1}/J	
	dir	(type)	dir	(type)	dir	(type)	(dir)	(type)
1.	incr	max	incr	min	incr	max	incr	min
2.	incr	max	incr	min	decr	max	decr	min
3.	incr	max	decr	min	incr	min	decr	min
4.	incr	max	decr	min	decr	min	incr	min
5.	decr	max	incr	min	incr	max	decr	max
6.	decr	max	incr	min	decr	max	incr	max
7.	decr	max	decr	min	incr	min	incr	max
8.	decr	max	decr	min	decr	min	decr	max

Rather than presenting the computations for all eight cases, which are similar, one will be worked out as an example. Case 2 is a good illustration since it includes all four kinds of hook functions. It is depicted in Figure 11 with $k=3$.

For the purpose of the proof we will denote the elements of $S_{n+1} \cap J$ as $\{a_1, a_2, \ldots, a_{2k+4}\}$, the elements of $S_n \cap J_f$ as $\{b_1, b_2, \ldots, b_{2k+2}\}$, the elements of $S_n \cap J_g$ as $\{c_1, \ldots, c_{2k+2}\}$, and the elements of $S_{n-1} \cap J_h$ as $\{d_1, \ldots, d_{2k}\}$. Then the four functions are defined as follows, observing that they are continuous and linear where not otherwise defined.

$$f_{n+1}(a_1) = b_{2k+2}; \quad f_{n+1}(a_2) = b_{2k+1}; \quad f_{n+1}(a_i) = b_{2k+5-i}, \quad i \geqq 3.$$
$$g_{n+1}(a_i) = c_{2k+3-i}, \quad i \leqq 2k+2; \quad g_{n+1}(a_{2k+3}) = c_2; \quad g_{n+1}(a_{2k+4}) = c_1.$$
$$f_n(c_i) = d_i, \quad i \leqq 2k; \quad f_n(c_{2k+1}) = d_{2k-1}; \quad f_n(c_{2k+2}) = d_{2k}.$$
$$g_n(b_1) = d_1; \quad g_n(b_2) = d_2; \quad g_n(b_i) = d_{i-2}, \quad i \geqq 3.$$

Let us then check the commutativity. For a_1 and a_2 we have

$$f_n g_{n+1}(a_1) = f_n(c_{2k+2}) = d_{2k}; \quad g_n f_{n+1}(a_1) = g_n(b_{2k+2}) = d_{2k}.$$
$$f_n g_{n+1}(a_2) = f_n(c_{2k+1}) = d_{2k-1}; \quad g_n f_{n+1}(a_2) = g_n(b_{2k+1}) = d_{2k-1}.$$

For $3 \leqq i \leqq 2k+2$ we have

$$f_n g_{n+1}(a_i) = f_n(c_{2k+3-i}) = d_{2k+3-i}; \quad g_n f_{n+1}(a_i) = g_n(b_{2k+5-i}) = d_{2k+3-i}.$$

And at the other end, for a_{2k+3} and a_{2k+4},

$$f_n g_{n+1}(a_{2k+3}) = f_n(c_2) = d_2; \quad g_n f_{n+1}(a_{2k+3}) = g_n(b_2) = d_2.$$
$$f_n g_{n+1}(a_{2k+4}) = f_n(c_1) = d_1; \quad g_n f_{n+1}(a_{2k+4}) = g_n(b_1) = d_1.$$

Thus the diagram is commutative on $S_{n+1} \cap J$, and since the composites are linear in between, the diagram in Figure 8 must be commutative on all of J. Note that the composites are both a "double hook" function, which in this case is decreasing on J.

Similarly, in the other seven cases the diagram is commutative, so requirement 3 of $L(n+1)$ is satisfied in case IV. This completes the proof that $L(n)$ implies $L(n+1)$, and hence the proof of the lemma.

THEOREM. *There exist continuous functions f and g which map the unit interval I onto itself and commute under functional composition but have no common fixed point.*

Proof. From the lemma we have two sequences of functions $\{f_i\}$ and $\{g_i\}$ which satisfy requirements 1–7. To prove the theorem it must be shown that the sequences converge uniformly to continuous functions f and g, that f and g commute, and that f and g have no comon fixed point.

To prove uniform convergence of $\{f_i\}$ it must be shown that $\{f_i\}$ is a Cauchy sequence relative to the sup norm $\|\cdots\|$ on I. So let an $\varepsilon > 0$ be chosen. There is an N such that $(1/3)^{N-1} < \varepsilon$. Then when $n \geqq m > N$, by requirements 2 and 4 we have

$$\|f_n - f_m\| \leqq M_{m-1} \leqq (1/3)^{m-1} < (1/3)^{N-1} < \varepsilon.$$

Thus $\{f_i\}$ converges uniformly on I, so that $\{f_i\}$ has a unique continuous limit function f. Similarly $\{g_i\}$ converges uniformly to a continuous limit function g.

To get commutativity we use the inequality

$$\begin{aligned}|fg(x)-gf(x)| \leqq\ & |f(g(x))-f_n(g(x))|+|f_n(g(x))-f_n(g_{n+1}(x))| \\ & +|f_ng_{n+1}(x)-g_nf_{n+1}(x)|+|g_n(f_{n+1}(x))-g_n(f(x))| \\ & +|g_n(f(x))-g(f(x))|.\end{aligned}$$

Now let $\varepsilon>0$ be given. We can choose N_1 such that $n>N_1$ implies $\|f-f_n\|<\varepsilon/4$; N_2 such that $n>N_2$ implies $\|g-g_n\|<\varepsilon/4$; N_3 and δ_3 such that $n, m>N_3$ and $|x-y|<\delta_3$ imply $|f_n(x)-f_m(y)|<\varepsilon/4$; N_4 such that $n>N_4$ implies $\|g-g_n\|<\delta_3$; N_5 and δ_5, and then N_6 defined for g as N_3, δ_3, and N_4 were for f. Then since $\|f_ng_{n+1}-g_nf_{n+1}\|=0$ for all n, when $n>\max(N_1, N_2, N_3, N_4, N_5, N_6)$ we have

$$|fg(x)-gf(x)| < \varepsilon/4+\varepsilon/4+0+\varepsilon/4+\varepsilon/4 = \varepsilon.$$

Thus the limit functions commute.

For the final portion of the proof of the theorem it must be shown that f and g have no common fixed point. This will be done by using the characteristics of f_2 and g_2 and requirements 4 and 5 of the lemma. Referring back to Figure 3, we can see the manner in which f_2 and g_2 map (2)-intervals onto (1)-intervals. Now if J_2 is a (2)-interval and $f_2(J_2)=J_1$, a (1)-interval, by requirement 4 whenever $i\geqq 2$, we must have $f_i(J_2)=f_2(J_2)=J_1$. Thus in the limit, $f(J_2)=J_1$. This means that the graph of f is in one of the rectangles of Figure 3 if and only if the graph of f_2 is. Comparing the rectangles through which the graph of f_2 passes with the rectangles through which the diagonal passes, it is clear that all fixed points of f must lie in the third, seventh, and ninth (2)-intervals. In a similar manner, comparing the graph of g_2 with the rectangles and the diagonal shows that all the fixed points of g must lie in the first, fifth, and eleventh (2)-intervals. Since no (2)-interval can contain fixed points of both f and g, they can have no common fixed point.

References

1. Glen Baxter, *On fixed points of the composite of commuting functions*, Proc. Amer. Math. Soc. **15** (1964), 851–855.

2. ———, personal communication, November 19, 1965.

3. Glen Baxter and J. T. Joichi, *On permutations induced by commuting functions, and an imbedding question*, Math. Scand. **13** (1963), 140–150.

4. ———, *On functions that commute with full functions*, Nieuw Arch. Wisk. (3) **12** (1964), 12–18.

5. ———, *On a class of commuting functions*, Nieuw Arch. Wisk. (3) **12** (1964), 126–136.

6. H. D. Block and H. P. Thielman, *Commutative polynomials*, Quart. J. Math. Oxford Ser. (2) **2** (1951), 241–243.

7. William M. Boyce, *Generation of a class of permutations associated with commuting functions*, Mathematical Algorithms **2** (1967), 19–26.

8. S. C. Chu and R. D. Moyer, *On continuous functions, commuting functions, and fixed points*, Fund. Math. **59** (1966), 91–95.

9. Haskell Cohen, *On fixed points of commuting functions*, Proc. Amer. Math. Soc. **15** (1964), 293–296.

10. Ralph DeMarr, *A common fixed point theorem for commuting mappings*, Amer. Math. Monthly **70** (1963), 535–537.

11. Nelson Dunford and Jacob T. Schwartz, *Linear Operators*, Part I, Interscience, New York, 1958.

12. J. H. Folkman, *On functions that commute with full functions*, Proc. Amer. Math. Soc. **17** (1966), 383–386.

13. J. P. Huneke, *Two counterexamples to a conjecture on commuting continuous functions of the closed unit interval*, Abstract 67T-231, Notices Amer. Math. Soc. **14** (1967), 284.

14. ———, *On common fixed points of commuting continuous functions on an interval*, Doctoral Dissertation, Wesleyan Univ., Middletown, Conn., 1967.

15. J. R. Isbell, *Commuting mappings of trees*, Research Problem #7, Bull. Amer. Math. Soc. **63** (1957), 419.

16. J. T. Joichi, *On functions that commute with full functions and common fixed points*, Nieuw Arch. Wisk. (3) **14** (1966), 247–251.

17. Gerald Jungck, *Commuting mappings and common fixed points*, Amer. Math. Monthly **73** (1966), 735–738.

18. J. E. Maxfield and W. J. Mourant, *Common fixed points of commuting continuous functions on the unit interval*, Nederl. Akad. Wetensch. Proc. Ser. A **68**; Indag. Math. **27** (1965), 668–670.

19. J. F. Ritt, *Permutable rational functions*, Trans. Amer. Math. Soc. **25** (1923), 399–448.

20. A. J. Schwartz, *Common periodic points of commuting functions*, Michigan Math. J. **12** (1965), 353–355.

21. A. L. Shields, *On fixed points of commuting analytic functions*, Proc. Amer. Math. Soc. **15** (1964), 703–706.

BELL TELEPHONE LABORATORIES, INC.,
MURRAY HILL, NEW JERSEY
TULANE UNIVERSITY,
NEW ORLEANS, LOUISIANA