Orthogonal Polynomials

The stochastic discretization of random variables involves global polynomials that are orthogonal with respect to the involved random distribution of the `y_m`. These polynomials can be generated by some recurrence relation with coefficients that depend on the distribution.

Recurrence relations

Orthogonal polynomials H_n with respect to some weight function ω, i.e.,

```
\int_\Gamma \omega(y) H_{n}(y) H_m(y) dy = N^2_{nm}\delta_{nm}
```

where

```
  N_{nn} := \| H_n \|_{\omega}^2 := \int_\Gamma \omega(y) H_{n}(y) H_n(y) dy
```

satisfy the three-term recurrence relation

```
\begin{aligned}
  H_{n+2}(y) & = (a_n y-b_n) H_{n+1}(y) - c_n H_{n}(y)
  \end{aligned}
```

initialized by $H_0 = 0$ and $H_1 = 1$.

Legendre Polynomials (uniform distribution)

For the weight function $\omega(y) = 1/2$ in the interval $[-1,1]$ (uniform distribution), take $a_n = (2n+1)/(n+1)$, $b_n = 0$ and $c_n = n/(n+1)$. The norms of the resulting Legendre polynomials are given by

```
    \| H_n \|^2_\omega = \frac{2}{2n+1}
```

Hermite Polynomials (normal distribution)

For the weight function $\omega(y) = \exp(-y^2/2)/(2\pi)$ (normal distribution), take $a_n = 1$, $b_n = 0$ and $c_n = n$. Then, the first six polynomials read

```
\begin{aligned}
H_0 & = 0\\
H_1 & = 1\\
```

```
H_2 & = y\\
H_3 & = y^2 - 1\\
H_4 & = y^3 - 3y\\
H_5 & = y^4 - 6y^2 +3\\
\end{aligned}
```

and their norms are given by

```
    \| H_n \|^2_\omega = n!
```