TD4

Éléments d'analyse spectrale

Exercice 2

1. Signaux d'énergie finie

1.

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

On pose : $f = 0$ alors,

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} x(t)e^{-2\pi jft}x^*(t)\, dt = TF[x(t)x^*(t)]_{f=0}$$

$$= TF[x(t)]_{f=0} * TF[x^*(t)]_{f=0} = X(f)_{f=0} * X^*(-f)_{f=0}$$

Ainsi,

$$= \int_{-\infty}^{+\infty} X(\tau)X^*(\tau - 0)\, d\tau = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

2.

$$C_{xx}(t) = x(t) * x^*(-t) = TF^{-1}[X(f)] * TF^{-1}[X^*(f)]$$

$$= TF^{-1}[X(f)X^*(f)] = TF^{-1}[S_X(f)]$$

$$C_{xx}(0) = E_x = \int_{-\infty}^{+\infty} S_X(f)\, df$$

3.

$$E_{rect_{2a}(t)} = \int_{-a}^{a} dt = 2a$$

$$C_{rect_{2a}rect_{2a}}(t) = rect_{2a}(t) * rect_{2a}(-t) = rect_{2a}(t) * rect_{2a}(t)$$

$$= Tri_{[-2a,2a]}(t) = \begin{cases} 2a & t \in [-2a, 0] \\ -2a & t \in [0, 2a] \\ 0 & \text{sinon} \end{cases}$$

2. Signaux de puissance finie

1.

On pose : $T = \frac{1}{f_0}$

$$P_s = \frac{2}{T}\int_0^{\frac{T}{2}} \cos(2\pi f_0 t)^2\, dt = \frac{1}{T}\int_0^{\frac{T}{2}} (1 + \cos(4\pi f_0 t))\, dt$$

$$= \frac{1}{T}\left(\frac{T}{2} - (\sin(2\pi) - \sin(0))\right) = \frac{1}{2}$$

$$s(t) = \sum_{n\in\mathbb{Z}} s_n \cos(2\pi n f_0 t) = |\cos(2\pi f_0 t)|$$

$$s_n = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos(2\pi f_0 t)e^{-2\pi jt\frac{2n}{T}}\, dt$$

$$= \frac{1}{2Tj\pi(f_0 - 2nT)}\left[e^{2j\pi t\left(f_0 - \frac{2n}{T}\right)}\right]_{-\frac{T}{4}}^{\frac{T}{4}} + \frac{1}{2j\pi T\left(f_0 + \frac{2n}{T}\right)}\left[e^{2j\pi t\left(f_0 + \frac{2n}{T}\right)}\right]_{-\frac{T}{4}}^{\frac{T}{4}}$$

Faire le calcul
Ainsi,

$$\boxed{s_n = \frac{(-1)^n}{\pi(1-2n)} + \frac{(-1)^n}{\pi(1+2n)} = \frac{2}{\pi}\frac{(-1)^n}{1-4n^2}}$$

$$\boxed{\sum_{n=-3}^{3}|s_n|^2 = \frac{4}{\pi^2}\sum_{n=-3}^{3}\frac{1}{(1-4n^2)^2} = \frac{4}{\pi}+\frac{8}{\pi^2}\left(\frac{1}{3^2}+\frac{1}{15^2}+\frac{1}{35^2}\right)}$$

$$\boxed{s(t) = \frac{2}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{1-4n^2}e^{4\pi jnf_0t}}$$

$$TF[s(t)] = \frac{2}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{1-4n^2}\int_{-\infty}^{+\infty}e^{4\pi jf_0t-2\pi jft}\,dt$$

$$\boxed{TF[s(t)] = \frac{2}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{(1-4n^2)}\delta(f-2nf_0)}$$

$$C_{ss}(t) = \frac{4}{\pi T}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\frac{(-1)^{n+k}}{(1-4n^2)(1-4k^2)}\int_0^{\frac{T}{2}}e^{4\pi jnf_0\tau-4\pi jkf_0(\tau-t)}\,d\tau$$

$$= \frac{4}{nT}\sum_{(n,k)\in\mathbb{Z}^2}s_ns_ke^{4\pi jkf_0t}\int_0^{\frac{T}{2}}e^{4\pi jf_0\tau(n-k)}\,d\tau$$

$$= \frac{4}{nT}\sum_{(n,k)\in\mathbb{Z}^2}s_ns_k\frac{e^{4\pi jkf_0t}}{4\pi jf_0(n+k)}[e^{4\pi jf_0\tau(n-k)}]_0^{\frac{T}{2}}$$

$$= \frac{4}{nT}\sum_{(n,k)\in\mathbb{Z}^2}s_ns_k\frac{e^{4\pi jkf_0t}}{4\pi jf_0(n-k)}(e^{2\pi j(n-k)}-1)$$

$$= \frac{4}{nT}\sum_{(n,k)\in\mathbb{Z}^2}s_ns_k\frac{e^{2\pi j(2tkf_0+n-k)}-e^{4\pi jkf_0t}}{4\pi jf_0(n-k)} = \ldots\mathrm{sinc}(\ldots)$$