Inverse scattering with fixed energy data

N. T. Tran*

Department of Mathematics

Kansas State University, Manhattan, KS 66506-2602, USA

*nhantran@ksu.edu

Abstract

In this paper the inverse scattering problem is studied and solved numerically using the method developed in [1]: Given the scattering data, we try to recover the scattering potential.

Key words: inverse scattering; potential, scattering data.

MSC: 35J05; 35J25; 70-08; 68W10; 74Q15.

1 Introduction

In this paper the inverse scattering problem is solved using the method developed in [1]. Consider the scattering problem formulated as below

$$[\nabla^2 + k^2 - q(x)]u(x) = 0 \quad \text{in } \mathbb{R}^3, \quad x \in \mathbb{R}^3, \tag{1.1}$$

$$u(x) = u_0(x) + v(x) = e^{ik\alpha\cdot x} + A(\alpha', \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \tag{1.2}$$

$$r := |x| \to \infty, \quad \alpha' := \frac{x}{r}. \tag{1.3}$$

Here $k > 0$ is a wave number, $q(x)$ is the scattering potential, u is the scattering solution or scattering field, $u_0(x)$ is the incident field, $v(x)$ is the scattered field,

*Mailing address: Mathematics Department, 138 Cardwell Hall, Manhattan, KS 66506

α is a unit vector that indicates the direction of the incident field, $A(\alpha', \alpha, k)$ is the scattering amplitude, $\alpha' \in S^2$ is the direction of the scattered wave.

The main question of the inverse scattering problem is: Given the scattering data A, can one recover the potential q?

In this problem, we assume that

$$q \in Q := Q_a \cap L^\infty(\mathbb{R}^3), \tag{1.4}$$

$$Q_a := \{q : q(x) = \overline{q(x)}, q(x) \in L^2(B_a), q(x) = 0 \text{ if } |x| > a\}, \tag{1.5}$$

where $a > 0$ is an arbitrary constant.

Without loss of generality, we take $k = 1$. If $q \in Q_a$ ans $k = 1$, the scattering amplitude is an analytic function of α' and α on the algebraic variety

$$M := \{\theta : \theta \in \mathscr{C}^3, \theta \cdot \theta = 1\}, \quad \theta \cdot \theta := \sum_{j=1}^{3} \theta_j^2. \tag{1.6}$$

Given any $\zeta \in \mathbb{R}^3$, there exist $\theta, \theta' \in M$ such that

$$\theta' - \theta = \zeta, \quad |\theta| \to \infty. \tag{1.7}$$

We have

$$-4\pi A(\alpha', \alpha, k) = \int_{B_a} e^{-ik\alpha' \cdot x} q(x) u(x, \alpha, k) dx. \tag{1.8}$$

Let $v_\theta(\alpha) \in L^2(S^2)$

$$I := -4\pi \int_{S^2} A(\theta', \alpha) v_\theta(\alpha) d\alpha \tag{1.9}$$

$$= \int_{B_a} dy q(y) e^{-i\theta' \cdot y} \int_{S^2} u(y, \alpha) v_\theta(\alpha) d\alpha \tag{1.10}$$

$$= \int_{B_a} dy q(y) e^{-i\theta' \cdot y} e^{i\theta \cdot y} e^{-i\theta \cdot y} \int_{S^2} u(y, \alpha) v_\theta(\alpha) d\alpha. \tag{1.11}$$

Let $\zeta = \theta' - \theta$ and

$$1 + \rho := e^{-i\theta \cdot y} \int_{S^2} u(y, \alpha) v_\theta(\alpha) d\alpha, \quad ||\rho||_{L^2(B_a)} = O\left(\frac{1}{|\theta|}\right). \tag{1.12}$$

Then

$$I = \int_{B_a} dy q(y) e^{-i\zeta \cdot y} + \int_{B_a} dy q(y) e^{-i\theta' \cdot y} \rho(y) \tag{1.13}$$

$$:= I_1 + I_2. \tag{1.14}$$

We have $|I_2| \le ||q||_{L^2(B_a)} ||\rho||_{L^2(B_a)} \le cO\left(\frac{1}{|\theta|}\right)$. Thus

$$\lim_{|\theta|\to\infty} I = I_1 =: \tilde{q}(\zeta), \quad \theta' - \theta = \zeta, \quad \theta', \theta \in M. \tag{1.15}$$

One needs to minimize $||\rho||$ to find $\nu_\theta(\alpha)$

$$\min_{\nu \in L^2(S^2)} ||\rho|| = \min_{\nu \in L^2(S^2)} \left|\left| e^{-i\theta\cdot x} \int_{S^2} u(x,\alpha)\nu_\theta(\alpha) d\alpha - 1 \right|\right|_{L^2(B_b \setminus B_{a_1})}, \tag{1.16}$$

for $\theta \in M$, $|\theta \to \infty|$, $a < a_1 < b$, and

$$\nu(\alpha) := \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \nu_{l,m} Y_{l,m}(\alpha), \tag{1.17}$$

where $Y_{l,m}(\alpha), -l \le m \le l$, is the spherical harmonic,

$$Y_{l,m}(\alpha) = \frac{(-1)^m i^l}{\sqrt{4\pi}} \left[\frac{(2l+1)(l-m)!}{(l+m)!} \right]^{1/2} e^{im\phi} P_{l,m}(\cos\Theta), \tag{1.18}$$

$$\overline{Y_{l,m}(\alpha)} = (-1)^{l+m} Y_{l,-m}(\alpha), \tag{1.19}$$

$$Y_{l,m}(-\alpha) = (-1)^l Y_{l,m}(\alpha) \tag{1.20}$$

$$\int_{S^2} Y_{l',m}(\beta) \overline{Y_{l,m}(\beta)} d\beta = \delta_{l'l}. \tag{1.21}$$

Here

$$P_{l,m}(\cos\Theta) = (\sin\Theta)^m \frac{d^m P(\cos\Theta)}{(d\cos\Theta)^m}, \quad 0 \le m \le l, \tag{1.22}$$

and $P_l(x)$ is the Legendre polynomial, (Θ, ϕ) are the angles corresponding to $\alpha \in S^2$,

$$P_{l,-m}(\cos\Theta) = (-1)^m \frac{(l-m)!}{(l+m)} P_{l,m}(\cos\Theta), \quad 0 \le m \le l. \tag{1.23}$$

If $q \in Q_a$ then $A(\beta,\alpha) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m}(\alpha) Y_{l,m}(\beta)$.

Once we have found ν_θ by minimizing $||\rho||$, the scattering potential can be recovered using

$$-4\pi \int_{S^2} A(\theta', \alpha) \nu_\theta(\alpha) d\alpha = \tilde{q}(\zeta) + O\left(\frac{1}{|\theta|}\right), \tag{1.24}$$

where $\tilde{q}(\zeta)$ is the Fourier transform of q.

2 Numerical Computation

Let $k = 1, \kappa := k^2 - q, r := |x|$, and $x^0 := x/r$. We have

$$u(x) = \begin{cases} e^{i\alpha\cdot x} + \sum_{l=0}^{\infty}\sum_{m=-l}^{l} b_{l,m} h_l(r) Y_{l,m}(x^0), & r \geq a \\ \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{l,m} j_l(\kappa r) Y_{l,m}(x^0), & r \leq a \end{cases} \tag{2.1}$$

$$=: \begin{cases} u_+ \\ u_- \end{cases}, \tag{2.2}$$

where $j_l(r) := \left(\frac{\pi}{2r}\right)^{1/2} J_{l+1/2}(r)$, $J_l(r)$ is the Bessel function of the first kind which is regular at $r = 0$, and

$$h_l(r) := e^{i\frac{\pi}{2}(l+1)}\sqrt{\frac{\pi}{2r}} H^{(1)}_{l+1/2}(r), \tag{2.3}$$

where $H_l^{(1)}(r)$ is the Bessel function of the second kind. We have

$$e^{i\alpha\cdot x} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a^0_{l,m} j_l(r) Y_{l,m}(x^0), \tag{2.4}$$

where $a^0_{l,m} = 4\pi i^l \overline{Y_{l,m}(\alpha)}$.

At $r = a$, we have

$$\begin{cases} u_+ = u_- \\ u'_+ = u'_- \end{cases}. \tag{2.5}$$

This is equivalent to

$$\begin{cases} a^0_{l,m}(\alpha) j_l(a) + b_{l,m} h_l(a) = a_{l,m} j_l(\kappa a) \\ a^0_{l,m}(\alpha) j'_l(a) + b_{l,m} h'_l(a) = a_{l,m}\kappa j'_l(\kappa a) \end{cases}, \quad 0 \leq l < \infty. \tag{2.6}$$

Solving this system yields $a_{l,m}$ and $b_{l,m}$. Then the scattering amplitude can be computed as follows

$$A(\beta,\alpha) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} b_{l,m}(\alpha) Y_{l,m}(\beta). \tag{2.7}$$

For example, take

$$q = \begin{cases} 0, & r > a \\ 50, & r \leq a \end{cases}, \tag{2.8}$$

then from the steps above we can construct sample scattering data $A(\beta,\alpha)$.

We use the following input parameters to solve this inverse scattering problem:

- Wave number: $k = 1$
- The number of terms for approximating the scattering data and solution: $L = 9$. This means

$$A(\beta,\alpha) \simeq \sum_{l=0}^{L} \sum_{m=-l}^{l} b_{l,m}(\alpha) Y_{l,m}(\beta) \tag{2.9}$$

$$u(x) \simeq \begin{cases} e^{i\alpha\cdot x} + \sum_{l=0}^{L} \sum_{m=-l}^{l} b_{l,m} h_l(r) Y_{l,m}(x^0), & r \geq a \\ \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} j_l(\kappa r) Y_{l,m}(x^0), & r \leq a \end{cases} \tag{2.10}$$

- Radius of a ball in $\mathbb{R}^3$: $a = 0.1$
- Radii of the annulus $B_b \setminus B_{a_1}$: $a_1 = a * 1.1, b = 1.2$
- Number of shells to grid the annulus $B_b \setminus B_{a_1}$: $s = 2$
- The potential in Schrödinger operator $(\nabla^2 + k^2 - q)$: $q = 50\chi(B_a)$
- Incident field direction: $\alpha = (0,0,1)$
- A point x in $\mathbb{R}^3$ to get sample scattering data: $x = (1,0,0)$
- Direction of x: $\beta = (1,0,0)$

For example, the scattering amplitude at the point $x = (1,0,0)$ is

$$A = -0.0431312546805 + 0i \tag{2.11}$$

and the scattering solution at this point with the incident direction α is

$$u = 1.02330391636 + 0i \tag{2.12}$$

We use gradient descent method to minimize $||\rho||$ to find $\nu_\theta(\alpha)$

$$\min_{\nu\in L^2(S^2)} ||\rho||^2 = \min_{\nu\in L^2(S^2)} \int_{B_b\setminus B_{a_1}} \left| e^{-i\theta\cdot x} \int_{S^2} u(x,\alpha)\nu_\theta(\alpha)d\alpha - 1 \right|^2 dx. \tag{2.13}$$

In $B_b \setminus B_{a_1}$, $u = u_+$ which can be computed using (2.2).

In order to compute the integral over S^2, we create a grid of S^2 as follows. Let S^2 be partitioned into P non-intersecting subdomains $S_{ij}, 1 \leq i \leq m_\Theta, 1 \leq j \leq m_\Phi$, using spherical coordinates, where m_Θ is the number of intervals of Θ

between 0 and 2π and m_Φ defines the number of intervals of Φ between 0 and π. Then $P = m_\Theta m_\Phi + 2$, which includes the two poles of the sphere. m_Θ is defined in this way: $m_\Theta = m_\Phi + |\Phi - \frac{\pi}{2}| m_\Phi$. This means the closer it is to the poles of the sphere, the more intervals for Θ are used. Then the point (Θ_i, Φ_j) in S_{ij} is chosen as follows

$$\Theta_i = i\frac{2\pi}{m_\Theta}, \quad 1 \le i \le m_\Theta, \tag{2.14}$$

$$\Phi_j = j\frac{\pi}{m_\Phi + 1}, \quad 1 \le j \le m_\Phi. \tag{2.15}$$

One should be careful when choosing the distribution of collocation points on a sphere. If one chooses $\Phi_j = j\frac{\pi}{m_\Phi}, 1 \le j \le m_\Phi$, then when $j = m_\Phi$, $\Phi_j = \pi$ and thus there is only one point for this Φ regardless of the value of Θ as shown in (2.16). The position of a point in each S_{ij} can be computed by

$$(x, y, z)_{ij} = (\cos\Theta_i \sin\Phi_j, \sin\Theta_i \sin\Phi_j, \cos\Phi_j). \tag{2.16}$$

The annulus $B_b \setminus B_{a_1}$ can be gridded by generating the same grid structure used for S^2 at various places inside the annulus.

θ', θ, and ζ in (1.7) are chosen to minimize $||\rho||$ and recover the potential as follows

$$\theta', \theta \in M, \quad |\theta| \to \infty \tag{2.17}$$

$$\theta' - \theta = \zeta, \quad \zeta \in \mathbb{R}^3. \tag{2.18}$$

In particular,

$$\theta' = (100.00 + 0i, 0.00 + 99.99781248i, 0.75 + 0i), \tag{2.19}$$

$$\zeta = (0, 0, 1.5) \tag{2.20}$$

$$\theta = \theta' - \zeta \tag{2.21}$$

After minimizing $||\rho||$, we get $v_\theta(\alpha)$, where $\alpha \in S^2$:

$$\begin{array}{cc}
0.02024818 + 0.00000000e+00i & -0.00420416 + 0.00000000e+00i \\
0.00355223 - 3.46944695e-18i & 0.06400551 + 0.00000000e+00i \\
0.38655623 + 0.00000000e+00i & 0.15310050 + 0.00000000e+00i \\
-0.06139328 + 0.00000000e+00i & -0.77578066 + 0.00000000e+00i \\
0.02024818 + 0.00000000e+00i & -0.00420416 + 0.00000000e+00i \\
0.00355223 + 0.00000000e+00i & 0.06400551 + 3.46944695e-18i \\
0.38655623 - 6.93889390e-18i & 0.09403160 + 0.00000000e+00i \\
0.09403160 + 0.00000000e+00i &
\end{array} \tag{2.22}$$

When q is constant, the Fourier transform of q can be computed analytically and it is

$$\tilde{q}_{exact} = \int_{B_a} q e^{-i\zeta \cdot y} dy = \frac{4\pi q}{|\zeta|^3}(\sin(|\zeta|a) - |\zeta|a\cos(|\zeta|a) = 0.208968649858. \tag{2.23}$$

After minimizing $||\rho||$, we get

$$\tilde{q}_{recovered} \simeq -4\pi \int_{S^2} A(\theta', \alpha)\nu_\theta(\alpha) d\alpha = 0.20174486552 - 3.21136652018e - 05i \tag{2.24}$$

and the relative error is: 3.5%.

References

[1] Ramm, A. G. (2002). Stability of the solutions to 3D inverse scattering problems with fixed-energy data. *Milan Journal of Mathematics, 70(1), 97-161.*