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A.G.Ramm, Stability of the solution to 3D inverse scattering problems with fixed-energy data, Milan Journ of Mathemat., 70, (2002), 97-161.

Stability of the solutions to 3D inverse scattering problems with fixed-energy data. *†

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Abstract

A review of the author's results is given. Inversion formulas and stability results for the solutions to 3D inverse scattering problems with fixed energy data are obtained. Inversion of exact and noisy data is considered. The inverse potential scattering problem with fixed-energy scattering data is discussed in detail, inversion formulas for the exact and for noisy data are derived, error estimates for the inversion formulas are obtained. The inverse obstacle scattering problem is considered for non-smooth obstacles. Stability estimates are derived for inverse obstacle scattering problem in the class of smooth obstacles. Global estimates for the scttering amplitude are given when the potential grows to infinity in a bounded domain. Inverse geophysical scattering problem is discussed briefly. An algorithm for constructing the Dirichlet-to-Neumann map from the scattering amplitude and vice versa is obtained. An analytical example of non-uniqueness of the solution to a 3D inverse problem of geophysics and a uniqueness theorem for an inverse problem for parabolic equations are given.

*key words: inverse scattering, stability estimates
†Math subject classification:35R30, 35R25, 47H17, 65M30, 65L20, 34C35, 34G20

Contents

1 Introduction

In this paper 3D inversion scattering problems with fixed-energy data are discussed. These problems include inverse problems of potential, obstacle, and geophysical scattering (IPS, IOS, IGS).

Inverse potential scattering problem is discussed in detail: uniqueness of its solution, reconstruction formulas for inversion of the exact data and for inversion of noisy data are given and error estimates for these formulas are obtained. These estimates yield the stability estimates for the solution of the inverse scattering problem.

For the inverse obstacle scattering the uniqueness theorem is proved for rough domains, stability estimates are obtained for $C^{2,\lambda}$ domains, $0 < \lambda < 1$, that is, for domains whose boundary in local coordinates is a graph of $C^{2,\lambda}$ function. Reconstruction formulas are discussed.

For inverse geophysical scattering the inverse scattering problem is reduced to inverse scattering problem for a potential.

Construction of the Dirichlet-to -Neumann map from the scattering data and vice versa is given. Analytical example of nonuniqueness of the solution of an inverse 3D problem of geophysics is given.

The results discussed in this paper were obtained mostly by the author, see [1], [10]-[53], [55]-[56], however the presentation and some of the estimates are improved in this paper. Only some selected results from the cited papers are included in this review.

1.1 The direct potential scattering problem.

We want to study the inverse potential scattering problem of finding $q(x)$ given some scattering data.

Consider the direct scattering problem first and let us formulate some basic results which we need.

Let

$$[\nabla^2 + k^2 - q(x)]u(x, \alpha, k) = 0 \quad \text{in } \mathbb{R}^3, \quad x \in \mathbb{R}^3, \tag{1.1}$$

$$u = e^{ik\alpha\cdot x} + A(\alpha', \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \alpha' := \frac{x}{r} \tag{1.2}$$

Here $u(x, k)$ is the scattering solution, $k = \text{ const } > 0$ is fixed. Without loss of generality we take $k = 1$ in what follows unless other choice is suggested explicitly. A unit vector $\alpha \in S^2$ is given, where S^2 is the unit sphere in $\mathbb{R}^3$. Vector α has a physical meaning of the direction of the incident plane wave, while $\alpha' \in S^2$ is the direction of the scattered wave, k^2 is the fixed energy. The function $A(\alpha', \alpha, k)$ is called the scattering amplitude. It describes the first term of the asymptotics of the scattered field as $r \to \infty$ along the direction $\alpha' = \frac{x}{r}$.

The function $q(x)$ is called the potential. We assume that

$$\begin{aligned} & q \in Q := Q_a \cap L^\infty(\mathbb{R}^3), \\ & \quad Q_a := \{q : q(x) = \overline{q(x)}, \quad q(x) \in L^2(B_a), \quad q(x) = 0 \text{ if } |x| > a\}, \end{aligned} \tag{1.3}$$

where $a > 0$ is an arbitrary large fixed number which we call the range of $q(x)$, and the overbar stands for complex conjugate.

In many results $q \in Q_a$ is sufficient, but $q \in Q$ is used in the proof of a crucial estimate (2.17) below.

1.2 Review of the known results.

Let us formulate some of the known results about the solution to problem (1.1)-(1.2), the scattering solution. These results can be found in many books, for example, in the appendix to [10], where a brief but self-contained presentation of the scattering theory is given.

1.2.1 The scattering problem has a unique solution if $q \in Q_a$.

In fact, the above result is proved for much larger class of q ([9], [6]), but for inverse scattering problem with noisy data it is necessay to assume $q(x)$ compactly supported [11]. Indeed, represent the potential $q(x)$ as $q = q_1 + q_2$, where $q_1 = 0$ for $|x| > a$ and $q_1 = q$ for $|x| \le a$. Call q_2 the tail of the potential q. If one assumes a priori that $q = O(|x|^{-b})$, where $b > 3$, then the contribution of the tail of the potential to the scattering amplitude is of order $O(|a|^{3-b})$ and tends to 0 as $a \to \infty$. At some value of a, say at $a = a_0$, this contribution becomes of the order of the noise in the scattering data. One cannot, in principle, discriminate between the noise and the contribution of the tail of the potential for $a > a_0$. Therefore the tail of q for $a > a_0$ cannot be determined from noisy data.

One has

$$\sup_{x \in \mathbb{R}^3} |u(x,k)| \le c, \quad k = \text{ const } > 0. \tag{1.4}$$

By $c > 0$ we denote various constants. If $q \in Q_a$ then $u(x,k)$ extends as a meromorphic function to the whole complex k-plane. Let $G(x,y,k)$ denote the resolvent kernel of the self-adjoint Schrödinger operator $L = -\nabla^2 + q(x)$ in $L^2(R^3)$:

$$(L - k^2)G(x,y,k) = \delta(x-y) \text{ in } \mathbb{R}^3, \tag{1.5}$$

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial G}{\partial |x|} - ikG \right|^2 ds = 0, \quad y \text{ is fixed }, \quad k > 0. \tag{1.6}$$

The function $u(x,k)$ can be defined by the formula:

$$G(x,y,k) = \frac{e^{ik|y|}}{4\pi|y|} u(x,\alpha,k) + o\left(\frac{1}{|y|}\right), \quad \frac{y}{|y|} = -\alpha, \tag{1.7}$$

where $o\left(\frac{1}{|y|}\right) = O\left(\frac{1}{|y|^2}\right)$ is uniform with respect to x varying in compact sets and formula (1.7) can be differentiated with respect to x [11], [16].

The function $G(x,y,k)$ is a meromorphic function of k on the whole complex k-plane. It has at most finitely many simple poles ik_j, $k_j > 0$, $1 \le j \le J$ in $\mathbb{C}_+ := \{k : Imk > 0\}$ and if $q(x) \not\equiv 0$, $q \in Q_a$, infinitely many poles, possibly not simple, in $\mathbb{C}_- = \mathbb{C} \setminus \overline{\mathbb{C}_+}$. There are no poles on the real line except, possibly at $k = 0$.

The functions $u_j(x) \in L^2(\mathbb{R}^3)$, solving (1.1) with $k = ik_j$, are called eigenfunctions of the discrete spectrum of L, $-k_j^2$ are the negative eigenvalues of L. There are at most finitely many of these if $q \in Q_a$.

The eigenfunction expansion formulas are known:

$$f(x) = \sum_j f_j u_j(x) + \int_{\mathbb{R}^3} \widetilde{f}(\xi) u(x,\xi) d\xi, \quad |\xi| = k, \quad \xi = k\alpha,$$

where

$$f_j := (f, u_j)_{L^2(\mathbb{R}^3)}, \quad \widetilde{f}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(x) \overline{u(x,\alpha,k)} dx,$$

(see e.g. [10]).

If E_λ is the resolution of the identity of the selfadjoint operator L, and $E_\lambda(x,y)$ is its kernel, then

$$\frac{dE_\lambda(x,y)}{d\lambda} = \frac{1}{\pi} ImG(x,y,\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{16\pi^3} \int_{S^2} u(x,\alpha,\sqrt{\lambda}) \overline{u(y,\alpha,\sqrt{\lambda})} d\alpha, \quad \lambda > 0.$$

1.2.2 Properties of the scattering amplitude

The scattering amplitude has the following well-known properties (see e.g. [10]):

$$\begin{aligned} A(\alpha',\alpha,k) &= A(-\alpha,-\alpha',k) \quad \text{(reciprocity)}, \\ \overline{A(\alpha',\alpha,k)} &= A(\alpha',\alpha,-k), \quad k > 0 \quad \text{(reality)}, \\ ImA(\alpha',\alpha,k) &= \frac{k}{4\pi} \int_{S^2} A(\alpha',\beta,k) \overline{A(\alpha,\beta,k)} d\beta, \quad k > 0 \quad \text{(unitarity)}. \end{aligned}$$

In particular,

$$ImA(\alpha,\alpha,k) = \frac{k}{4\pi} \int_{S^2} |A(\alpha,\beta,k)|^2 d\beta \quad \text{(optical theorem)}.$$

If $q \in Q_a$, and $k = 1$, then the scattering amplitude is an analytic function of α' and α on the algebraic variety

$$M := \{\theta : \theta \in \mathbb{C}^3, \theta \cdot \theta = 1\}, \quad \theta \cdot w := \sum_{j=1}^3 \theta_j w_j. \tag{1.8}$$

This variety is non-compact, intersects $\mathbb{R}^3$ over S^2, and, given any $\xi \in \mathbb{R}^3$, there exist (many) $\theta, \theta' \in M$ such that

$$\theta' - \theta = \xi, \quad |\theta| \to \infty, \quad \theta, \theta' \in M. \tag{1.9}$$

In particular, if one chooses the coordinate system in which $\xi = te_3$, $t > 0$, e_3 is the unit vector along the x_3-axis, then the vectors

$$\theta' = \frac{t}{2}e_3 + \zeta_2 e_2 + \zeta_1 e_1, \quad \theta = -\frac{t}{2}e_3 + \zeta_2 e_2 + \zeta_1 e_1, \quad \zeta_1^2 + \zeta_2^2 = 1 - \frac{t^2}{4}, \tag{1.10}$$

satisfy (1.9) for any complex numbers ζ_1 and ζ_2 satisfying the last equation in (1.10) and such that $|\zeta_1|^2 + |\zeta_2|^2 \to \infty$. There are infinitely many such $\zeta_1, \zeta_2 \in \mathbb{C}$. If $q \in Q_a$ then the function $A(\alpha', \alpha, k)$ is a meromorphic function of $k \in \mathbb{C}$ which has poles at the same points as $G(x, y, k)$.

One has

$$-4\pi A(\alpha', \alpha, k) = \int_{B_a} e^{-ik\alpha' \cdot x} q(x) u(x, \alpha, k) dx. \tag{1.11}$$

The S-matrix is defined by the formula

$$S = I + \frac{ik}{2\pi} A, \quad S^* S = S S^* = I, \tag{1.12}$$

and is a unitary operator in $L^2(S^2)$. Thus, A is a normal operator in $L^2(S^2)$.

If $\nabla q \in Q_a$ then

$$\phi(x, \alpha, k) := e^{-ik\alpha \cdot x} u(x, \alpha, k) = 1 + \frac{1}{2ik} \int_0^\infty q(x - r\alpha) dr + o\left(\frac{1}{k}\right), \quad k \to \infty. \tag{1.13}$$

Therefore

$$q(x) = \alpha \cdot \nabla_x \lim_{k \to \infty} \{2ik[\phi(x, \alpha, k) - 1]\}, \tag{1.14}$$

and

$$A(\alpha, \alpha, k) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} q(x) dx + o(1), \quad k \to +\infty. \tag{1.15}$$

1.2.3 The fundamental equation.

Denote $u^+ := u(x, \alpha, k), u^- := u(x, -\alpha, -k)$, $k > 0$. Then $u^+ = Su^-$, that is

$$u^+ = u^- + \frac{ik}{2\pi} \int_{S^2} A(\alpha', \alpha, k) u^-(x, \alpha', k) d\alpha'. \tag{1.16}$$

1.2.4 Completeness properties of the scattering solutions.

a) *If* $h(\alpha) \in L^2(S^2)$ *and*

$$\int_{S^2} h(\alpha)u(x,\alpha,k)d\alpha = 0 \quad \forall x \in B'_R := \{x : |x| > R\}, \quad k > 0 \text{ is fixed} \tag{1.17}$$

then $h(\alpha) = 0$.

Let $N_D(L) = \{w : Lw = 0 \text{ in } D,\ w \in H^2(D)\}$, where $D \subset \mathbb{R}^3$ is a bounded domain, $H^2(D)$ is the Sobolev space.

b) The set $\{u(x,\alpha,k)\}_{\forall\alpha\in S^2}$ is total in $N_D(L-k^2)$, that is, for any $\varepsilon > 0$, however small, and any fixed $w \in N_D(L-k^2)$, there exists $\nu_\varepsilon(\alpha) \in L^2(S^2)$ such that

$$\|w(x) - \int_{S^2} u(x,\alpha,k)\nu_\varepsilon(\alpha)d\alpha\|_{H^2(D)} < \varepsilon. \tag{1.18}$$

The $\nu_\varepsilon(\alpha)$ depends on $w(x)$.

1.2.5 Special solutions

There exists $\psi(x,\theta,k)$, $\psi \in N_D(L-k^2)$, such that

$$[\nabla^2 + k^2 - q(x)]\psi = 0 \text{ in } \mathbb{R}^3, \quad \psi = e^{ik\theta\cdot x}[1 + R(x,\theta,k)], \quad \theta \in M, \tag{1.19}$$

$$\|R\|_{L^\infty(D)} \le c\frac{(\ln|\theta|)^{\frac{1}{2}}}{|\theta|^{\frac{1}{2}}}, \quad |\theta| \to \infty, \quad \theta \in M, \tag{1.20}$$

$$\|R\|_{L^2(D)} \le \frac{c}{|\theta|}, \quad |\theta| \to \infty, \quad \theta \in M, \tag{1.21}$$

where $D \subset \mathbb{R}^3$ is an arbitrary bounded domain.

1.2.6 Property C for the pair $\{L_1 - k^2, L_2 - k^2\}$

Let $L_j w := \sum_{|m|=0}^{M_j} a_{jm}(x)\partial^{|m|}w(x)$, $x \in \mathbb{R}^n$, $n \ge 2$, $j = 1,2$, be linear formal partial differential operators, that is, formal differential expressions.

Let $N_j = N_{jD}(L_j) := \{w : L_j w = 0 \text{ in } D\}$, where $D \subset R^n$ is an arbitrary fixed bounded domain and the equation is understood in the sense of the distribution theory. Consider the subsets of N_j, $j = 1,2$, which form an algebra in the sense that the products $w_1 w_2 \in L^{p'}(D)$, where $w_j \in N_j$, $p' = \frac{p}{p-1}$, and $1 \le p \le \infty$. If $p = 1$ define $p' = \infty$, and if $p = \infty$ define $p' = 1$. We write $\forall w_j \in N_j$ meaning that w_j run through the above subsets of N_j.

Definition 1.1. *We say that the pair of linear partial differential operators* $\{L_1, L_2\}$ *has property* C_p *if and only if the set* $\{w_1 w_2\}$ *is total in* $L^p(D)$*, that is, if* $f(x) \in L^p(D)$ *and*

$$\int_D f(x) w_1(x) w_2(x) dx = 0 \quad \forall w_j \in N_j, j = 1, 2, \tag{1.22}$$

then

$$f(x) = 0. \tag{1.23}$$

If the above holds for any $p \geq 1$*, we say that property* C *holds for the pair* $\{L_1, L_2\}$*.*

Theorem 1.1. *1.1 [11]. Let* $L_j = -\nabla^2 + q_j(x)$*,* $q_j(x) \in Q_a$*,* $k = $ *const* ≥ 0 *is arbitrary fixed. Then the pair* $\{L_1 - k^2, L_2 - k^2\}$ *has property* C*.*

Proof. Note that $\psi_j \in N_j$, $j = 1, 2$, where ψ_j are defined in section 1.2.5 above. Without loss of generality take $k = 1$, let $\psi(x, \theta, 1) := \psi(x, \theta)$. One has

$$\psi_1(x, \theta')\psi_2(x, -\theta) = e^{i(\theta' - \theta)\cdot x}(1 + R_1)(1 + R_2).$$

Choose $\theta', \theta \in M$ such that (1.9) holds with an arbitrary fixed $\xi \in \mathbb{R}^3$. Then

$$\psi_1(x, \theta')\psi_2(x, -\theta) = e^{i\xi\cdot x}(1 + o(1)) \text{ as } |\theta| \to \infty. \tag{1.24}$$

Since the set $\{e^{i\xi\cdot x}\}_{\forall \xi \in \mathbb{R}^3}$ is total in $L^p(D)$, $p \geq 1$, $D \subset \mathbb{R}^n$ is a bounded domain, the conclusion of Theorem 1.1 follows. □

Remark 1.1. *One cannot take unbounded domain* D *in the above argument because* $o(1)$ *in (1.24) holds for bounded domains.*

One can take the space of $f(x)$ larger than $L^1(D)$, for example, the space of distribution of finite order of singulartiy if $q(x)$ is sufficiently smooth [11].

Theorem 1.2. *The set* $\{u_1(x, \alpha, k)u_2(x, \beta, k)\}_{\forall \alpha, \beta \in S^2, k > 0}$ *is complete in* $L^p(D)$*, where* $D \subset \mathbb{R}^3$ *is an arbitrary fixed bounded domain, and* $p \geq 1$ *is fixed.*

Proof. The conclusion of Theorem 1.2 follows from Theorem 1.1 and (1.18). □

1.2.7 Properties of the Fourier coefficients of $A(\alpha', \alpha)$.

We denote $A(\alpha', \alpha, k)|_{k=1} := A(\alpha', \alpha)$, and write

$$A(\alpha', \alpha) = \sum_{\ell=0}^{\infty} A_\ell(\alpha) Y_\ell(\alpha'), \quad A_\ell(\alpha) := \int_{S^2} A(\alpha', \alpha)\overline{Y_l(\alpha')} d\alpha', \tag{1.25}$$

where $Y_\ell(\alpha') = Y_{\ell,m}(\alpha'), -\ell \leq m \leq \ell$, summation over m, is understood in (1.25) and similar formulas below, e.g. (1.31), (1.37), etc,

$$\begin{gathered} Y_{\ell,m}(\alpha) = \frac{(-1)^m i^\ell}{\sqrt{4\pi}} \left[\frac{(2\ell + 1)(\ell - m)!}{(\ell + m)!}\right]^{\frac{1}{2}} e^{im\varphi} P_{\ell,m}(\cos\vartheta), \\ \overline{Y_{\ell,m}(\alpha)} = (-1)^{\ell+m} Y_{\ell,-m}(\alpha), \;\; Y_{\ell,m}(-\alpha) = (-1)^\ell Y_{\ell,m}(\alpha). \end{gathered} \tag{1.26}$$

Here $P_{\ell,m}(\cos\vartheta)=(\sin\vartheta)^m\frac{d^mP_\ell(\cos\vartheta)}{(d\cos\vartheta)^m}$, $0\leq m\leq\ell$, $P_\ell(x)$ is the Legendre polynomial, (ϑ,φ) are the angles corresponding to the point $\alpha\in S^2$, $P_{\ell,-m}(\cos\vartheta)=(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_{\ell,m}(\cos\vartheta)$, $0\leq m\leq\ell$.

Consider a subset $M'\subset M$ consisting of the vectors $\theta=(\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi,\cos\vartheta)$ where ϑ and φ run through the whole complex plane. Clearly $\theta\in M$, but M' is a proper subset of M. Indeed, any $\theta\in M$ with $\theta_3\neq\pm1$ is an element of M'. If $\theta_3=\pm1$, then $\cos\vartheta=\pm1$, so $\sin\vartheta=0$ and one gets $\theta=(0,0,\pm1)\in M'$. However, there are vectors $\theta=(\theta_1,\theta_2,1)\in M$ which do not belong to M'. Such vectors one obtains choosing $\theta_1,\theta_2\in\mathbb{C}$ such that $\theta_1^2+\theta_2^2=0$. There are infinitely many such vectors. The same is true for vectors $(\theta_1,\theta_2,-1)$. Note that in (1.9) one can replace M by M' for any $\xi\in\mathbb{R}^3$, $\xi\neq 2e_3$.

Let us state two estimates ([15]):

$$\max_{\alpha\in S^2}|A_\ell(\alpha)|\leq c\left(\frac{a}{\ell}\right)^{\frac{1}{2}}\left(\frac{ae}{2\ell}\right)^{\ell+1}, \tag{1.27}$$

and

$$|Y_\ell(\theta)|\leq\frac{1}{\sqrt{4\pi}}\frac{e^{r|Im\theta|}}{|j_\ell(r)|},\quad\forall r>0,\quad\theta\in M', \tag{1.28}$$

where

$$j_\ell(r):=\left(\frac{\pi}{2r}\right)^{\frac{1}{2}}J_{\ell+\frac{1}{2}}(r)=\frac{1}{2\sqrt{2}}\frac{1}{\ell}\left(\frac{er}{2\ell}\right)^{\ell}[1+o(1)]\text{ as }\ell\to\infty, \tag{1.29}$$

and $J_\ell(r)$ is the Bessel function regular at $r=0$. Note that $Y_\ell(\alpha')$, defined by (1.26), admits a natural analytic continuation from S^2 to M by taking ϑ and φ in (1.26) to be arbitrary complex numbers. The resulting $\theta'\in M'\subset M$.

1.2.8 A global perturbation formula.

Let $A_j(\alpha',\alpha)$ be the scattering amplitude corresponding to $q_j\in Q_a$, $j=1,2$. Define $A:=A_1-A_2$, $p:=q_1(x)-q_2(x)$. Then [11]

$$-4\pi A(\alpha',\alpha)=\int_{B_a}p(x)u_1(x,\alpha)u_2(x,-\alpha')dx. \tag{1.30}$$

1.2.9 Formula for the scattering solution outside the support of the potential.

Let $\operatorname{supp}(q)\subset B_a$. The fixed-energy scattering data $A(\alpha',\alpha)\forall\alpha',\alpha$, or, equivalently, the data $\{A_\ell(\alpha)\}_{\ell=0,1,2,\dots}\forall\alpha\in S^2$, allow one to write an analytic formula for the scattering solution $u(x,\alpha)$ in the region $B_a':=\mathbb{R}^3\backslash B_a$:

$$u(x,\alpha)=e^{i\alpha\cdot x}+\sum_{\ell=0}^{\infty}A_\ell(\alpha)Y_\ell(\alpha')h_\ell(r),\quad r:=|x|>a,\quad\alpha':=\frac{x}{r}, \tag{1.31}$$

where $A_\ell(\alpha)$ are defined in (1.25), $Y_\ell(\alpha')$ are defined in (1.26),

$$h_\ell(r) := e^{i\frac{\pi}{2}(\ell+1)}\sqrt{\frac{\pi}{2r}}H^{(1)}_{\ell+\frac{1}{2}}(r),$$

$H^{(1)}_\ell(r)$ is the Hankel function, and the normalizing factor is chosen so that

$$h_\ell(r) = \frac{e^{ir}}{r}[1+o(1)] \text{ as } r\to\infty. \tag{1.32}$$

Note that [2, formula (7.1463]:

$$|H^{(1)}_\ell(r)|^2 = \frac{4}{\pi^2}\int_0^\infty K_0(2rsht)(e^{2\ell t}+e^{-2\ell t})dt, \tag{1.33}$$

where $sht := \frac{e^t-e^{-t}}{2}$. This formula implies that $|h_\ell(r)|$ is a monotonically increasing function of ℓ.

It is known [[4], formula 8.478] that $|h_\ell(r)|^2$ is a monotonically decreasing function of r if $\ell>0$, and

$$h_\ell(r) = -\frac{i^\ell\sqrt{2}}{r}\left(\frac{2\ell}{er}\right)^\ell[1+o(1)], \quad \ell\to+\infty, \quad r>0. \tag{1.34}$$

The following known estimate can be useful:

$$|Y_{\ell,m}(\alpha)|\le c\ell^{\frac{m}{2}-1}, \quad \alpha\in S^2, \tag{1.35}$$

where $Y_{\ell,m}(\alpha)$ are the normalized in $L^2(S^2)$ spherical harmonics (1.26).

Let us give a formula for the Green's function $G(x,y,k)$ (see (1.5), (1.6)) in the region $|x|>a$, $|y|>a$, where $\operatorname{supp} q(x)\subset B_a$. Let $g(x,y,k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$ and denote by $A_{\ell'\ell}$ the Fourier coefficients of the scattering amplitude:

$$A(\alpha',\alpha) = \sum_{\ell=0}^\infty A_\ell(\alpha)Y_\ell(\alpha') = \sum_{\ell',\ell=0}^\infty A_{\ell'\ell}Y_{\ell'}(\alpha)Y_\ell(\alpha'). \tag{1.36}$$

Then

$$G(x,y,k) = g(x,y,k) + \frac{k^2}{4\pi}\sum_{\ell',\ell=0}^\infty A_{\ell'\ell}Y_{\ell'}(\alpha)Y_\ell(\alpha')h_\ell(k|x|)h_{\ell'}(k|y|), \quad |x|>a, \quad |y|>a, \tag{1.37}$$

where $\alpha' := \frac{x}{|x|}$, $\alpha := -\frac{y}{|y|}$.

Indeed, clearly the function (1.37) solves (1.5) in the region $|x|>a$, $|y|>a$, where $q(x)=0$, it satisfies (1.6), and

$$G(x,y,k) = \frac{e^{ik|y|}}{4\pi|y|}\left[e^{ik\alpha\cdot x} + \sum_{\ell',\ell=0}^\infty A_{\ell'\ell}Y_{\ell'}(\alpha)Y_\ell(\alpha')kh_\ell(k|x|)\right] + o\left(\frac{1}{|y|}\right), \quad \text{as } |y|\to\infty, \ \frac{y}{|y|} = -\alpha. \tag{1.38}$$

By (1.36), (1.31),(1.25), and (1.7), it follows that the function (1.37) has the same main term of asymptotics (1.38) as the Green's function of the Schrödinger operator. Therefore the function (1.37) is identical to the Green's function (1.5)-(1.6) in the region $|x| > a$, $|y| > a$.

2 Inverse potential scattering problem with fixed-energy data

The IPS problem can now be formulated: *given* $A(\alpha', \alpha)$ $\forall \alpha', \alpha \in S^2$, *find* $q(x) \in Q_a$. Throughout this section $k = 1$.

2.1 Uniqueness theorem.

The first result is the uniqueness theorem of Ramm [13], [14].

Theorem 2.1. *If* $q_1, q_2 \in Q_a$ *and* $A_1(\alpha', \alpha) = A_2(\alpha', \alpha)$ $\forall \alpha' \in S_1^2$, $\forall \alpha \in S_2^2$, *where* S_j^2 $j = 1, 2$, *are arbitrary small open subsets of* S^2, *then* $q_1(x) = q_2(x)$.

Proof. The function $A(\alpha', \alpha)$ is analytic with respect to α' and α on the variety (1.8). Therefore its values on $S_1^2 \times S_2^2$ extend uniquely by analyticity to $M \times M$. In particular $A(\alpha', \alpha)$ is uniquely determined in $S^2 \times S^2$. By (1.30) one gets:

$$\int_{B_a} p(x) u_1(x, \alpha) u_2(x, -\alpha') dx = 0 \quad \forall \alpha, \alpha' \in S^2. \tag{2.1}$$

By property C (section 1.2.6, formulas (1.22) -(1.23)) and by (1.18), the orthogonality relation (2.1) implies $p(x) \equiv 0$. □

2.2 Reconstruction formula for exact data.

Fix an arbitrary $\xi \in \mathbb{R}^3$ and choose arbitrary θ', θ satisfying (1.9).

Denote

$$\widetilde{q}(\xi) := \int_{B_a} e^{-i\xi \cdot x} q(x) dx. \tag{2.2}$$

Multiply (1.11) by $\nu(\alpha, \theta) \in L^2(S^2)$, where $\nu(\alpha, \theta)$ will be fixed later, and integrate over S^2 with respect to α:

$$-4\pi \int_{S^2} A(\alpha', \alpha) \nu(\alpha, \theta) d\alpha = \int_{B_a} e^{-i\alpha' \cdot x} \int_{S^2} u(x, \alpha) \nu(\alpha, \theta) d\alpha q(x) dx. \tag{2.3}$$

If $q \in Q_a$, then estimates (1.27) and (1.28) imply that the series (1.25) converges, when α' is replaced by $\theta' \in M$, uniformly and absolutely on $S^2 \times M_c$, where $M_c \subset M$

is an arbitrary compact subset of M. Formula (1.11) implies that α' can be replaced by $\theta' \in M$, since B_a is a compact set in $\mathbb{R}^3$.

Define

$$\rho(x) := \rho(x;\nu) := e^{-i\theta\cdot x}\int_{S^2} u(x,\alpha)\nu(\alpha,\theta)d\alpha - 1, \tag{2.4}$$

and rewrite (2.3), with $\alpha' = \theta'$, as

$$\begin{aligned} -4\pi\int_{S^2} A(\theta',\alpha)\nu(\alpha,\theta)d\alpha &= \int_{B_a} e^{-i\theta'\cdot x + i\theta\cdot x}[\rho(x)+1]q(x)dx \\ &= \widetilde{q}(\xi) + \int_{B_a} e^{-i\xi\cdot x}\rho(x)q(x)dx = \widetilde{q} + \varepsilon, \end{aligned} \tag{2.5}$$

where

$$|\varepsilon| \leq \|q\|_a\|\rho\|_a, \quad \|q\|_a := \|q\|_{L^2(B_a)}. \tag{2.6}$$

The following estimate (see [15], estimate (2.17) and its proof in section 6 below) holds for a suitable choice of $\nu(\alpha,\theta)$:

$$\|\rho\|_a \leq c|\theta|^{-1}, \quad |\theta| \to \infty, \quad \theta \in M. \tag{2.7}$$

From (2.5) and (2.7) one gets the reconstruction formula for inversion of exact, fixed-energy, three-dimensional scattering data:

$$\lim_{\substack{|\theta|\to\infty \\ \theta'-\theta=\xi, \\ \theta,\theta'\in M.}} \left\{-4\pi\int_{S^2} A(\theta',\alpha)\nu(\alpha,\theta)d\alpha\right\} = \widetilde{q}(\xi), \tag{2.8}$$

and the error estimate:

$$-4\pi\int_{S^2} A(\theta',\alpha)\nu(\alpha,\theta)d\alpha = \widetilde{q}(\xi) + O\left(\frac{1}{|\theta|}\right), \quad |\theta| \to \infty, \quad \theta \in M, \tag{2.9}$$

where (1.9) is always assumed.

Let us give an *algorithm for computing the function $\nu(\alpha,\theta)$ for which (2.7), and therefore (2.9), hold, given the scattering data $A(\alpha',\alpha)$ $\forall\alpha',\alpha \in S^2$.*

Fix arbitrarily two numbers a_1 and b such that

$$a < a_1 < b, \tag{2.10}$$

and define the L^2-norm in the annulus:

$$\|\rho\|^2 := \int_{a_1\leq|x|\leq b} |\rho|^2 dx. \tag{2.11}$$

Consider the minimization problem

$$\|\rho\| = \inf := d(\theta), \tag{2.12}$$

where the infimum is taken over all $\nu \in L^2(S^2)$.

It is proved in [15] (see also section (6.3) below) that

$$d(\theta) \leq c|\theta|^{-1} \text{ if } \theta \in M, \quad |\theta| \gg 1. \tag{2.13}$$

The symbol $|\theta| \gg 1$ means that $|\theta|$ is sufficiently large. The constant $c > 0$ in (2.13) depends on the norm $\|q\|_a$ but not on the potential $q(x)$ itself. An algorithm for computing a function $\nu(\alpha, \theta)$, which can be used for inversion of the fixed-energy 3D scattering data by formula (2.9), is as follows:

a) Find an approximate solution to (2.12) in the sense

$$\|\rho(x, \nu)\| < 2d(\theta), \tag{2.14}$$

where in place of 2 in (2.14) one could put any fixed constant greater than 1.

b) Any such $\nu(\alpha, \theta)$ generates an estimate of $\widetilde{q}(\xi)$ with the error $O\left(\frac{1}{|\theta|}\right)$, $|\theta| \to \infty$. This estimate is calculated by the formula

$$\widehat{q} := -4\pi \int_{S^2} A(\theta', \alpha)\nu(\alpha, \theta) d\alpha, \tag{2.15}$$

where $\nu(\alpha, \theta) \in L^2(S^2)$ is any function satisfying (2.14).

We have obtained the following result:

Theorem 2.2. *If (1.3), (1.9) and (2.14) hold, then*

$$\sup_{\xi \in \mathbb{R}^3} |\widehat{q} - \widetilde{q}(\xi)| \leq \frac{c}{|\theta|}, \quad |\theta| \to \infty, \quad \theta \in M. \tag{2.16}$$

Proof. The proof is the same as the proof of (2.5) - (2.7) and is based on the following estimate [11], [15]:

$$\|\rho\|_a \leq c\left(\|\rho\| + |\theta|^{-1}\right), \quad |\theta| \gg 1, \quad \theta \in M. \tag{2.17}$$

The proof of (2.17) is not simple [15]. It is given in section 6. □

2.3 Stability estimate for inversion of the exact data.

Let the potentials $q \in Q$, $j = 1, 2$, generate the scattering amplitudes $A_j(\alpha', \alpha)$.

Let us assume that

$$\sup_{\alpha', \alpha \in S^2} |A_1(\alpha', \alpha) - A_2(\alpha', \alpha)| < \delta. \tag{2.18}$$

We want to estimate $p(x) := q_1(x) - q_2(x)$.

The main tool is formula (1.30).

The result is ([15], [11]):

Theorem 2.3. *If $q_j \in Q$ and (2.18) holds then*

$$\sup_{\xi\in\mathbb{R}^3} |\widetilde{q_1}(\xi) - \widetilde{q_2}(\xi)| \leq c\frac{\ln|\ln\delta|}{|\ln\delta|}, \quad \delta\to 0, \tag{2.19}$$

where the constant $c > 0$ does not depend on $\delta > 0$, $\delta \to 0$.

Proof. Multiply both sides of (1.30) by $\nu_1(\alpha,\theta)\nu_2(-\alpha',\theta_2)$, where $\theta_j \in M$, $j = 1,2$, $\theta_1 + \theta_2 = \xi$, $|\theta_1| \to \infty$, and integrate with respect to α and α' over S^2, to get:

$$\begin{aligned} &-4\pi \int_{S^2}\int_{S^2} A(\alpha',\alpha)\nu_1(\alpha,\theta_1)\nu_2(-\alpha',\theta_2)d\alpha d\alpha' = \\ &\int_{B_a} dx p(x) \int_{S^2} u_1(x,\alpha)\nu_1(\alpha,\theta_1)d\alpha \int_{S^2} u_2(x,\beta)\nu_2(\beta,\theta_2)d\beta, \quad \beta = -\alpha'. \end{aligned} \tag{2.20}$$

Choose ν_1 and ν_2 such that

$$\|\rho_j(\nu_j)\| \leq \frac{c}{|\theta_j|}, \quad |\theta_j| \to \infty, \quad \theta_j \in M, \tag{2.21}$$

where $\rho_j(\nu_j) := \rho_j(x,\nu_j) := e^{-i\theta_j\cdot x}\int_{S^2} u_j(x,\alpha)\nu_j(\alpha,\theta_j)d\alpha - 1$, and note that $\frac{|\theta_1|}{|\theta_2|} \to 1$ as $|\theta_1| \to \infty$, $\theta_1,\theta_2 \in M$, $\theta_1 + \theta_2 = \xi$, $|\xi| \leq \xi_0$, and $c > 0$ in (2.21) does not depend on ξ. From (2.18), (2.20) and (2.21) one gets

$$|\widetilde{p}(\xi)| \leq c\left(|\theta|^{-1} + c\delta\|\nu_1\|_{L^2(S^2)}\|\nu_2\|_{L^2(S^2)}\right), \tag{2.22}$$

where $\theta := \theta_1$. One can choose ν_1 and ν_2 such that ([15], see also section 6.7 below)

$$\|\nu_j(\alpha,\theta_j)\|_{L^2(S^2)} \leq ce^{c|\theta|\ln|\theta|}, \quad \theta = \theta_1, \quad |\theta| \to \infty, \quad \theta_2 = \xi - \theta, \tag{2.23}$$

where $c > 0$ stands for various constants.

Thus (2.22) yields:

$$\sup_{\xi\in\mathbb{R}^3} |\widetilde{p}(\xi)| \leq c\min_{s\gg 1}\left[s^{-1} + c_1\delta e^{c_2 s\ln|s|}\right], \quad 0 < \delta \ll 1, \tag{2.24}$$

where c, c_1 and c_2 are some positive constants, $s := |\theta|$, $\delta \ll 1$ means that $\delta > 0$ is small and $s \gg 1$ means that $s > 0$ is large. However, our argument is valid for $s \geq 1$ and $0 < \delta \leq \frac{1}{2}$.

One gets

$$\min_{s>0}\left[s^{-1} + c_1\delta e^{c_2 s\ln|s|}\right] := \eta(\delta) \leq c_3\frac{\ln|\ln\delta|}{|\ln\delta|}, \quad \delta \to 0, \tag{2.25}$$

and the minimizer is

$$s = s(\delta) = c_2^{-1}\frac{|\ln\delta|}{\ln|\ln\delta|}\left[1 + o(1)\right], \quad \delta \to 0. \tag{2.26}$$

From (2.24)-(2.26) one gets (2.19). □

Remark 2.1. *In the above proof the difficult part is the proof of (2.23). Estimate (2.23) can be derived for $\nu(\alpha,\theta)$ such that*

$$\|\psi(x,\theta)-\int_{S^2}u(x,\alpha)\nu(\alpha,\theta)d\alpha\|_{L^2(B_b)}\leq c\frac{e^{-b\kappa}}{\kappa},\quad \kappa:=|Im\theta|,\ \theta\in M,\ \theta\gg 1,\tag{2.27}$$

and $\|\nu(\alpha,\theta)\|_{L^2(S^2)}=\inf$, where the infimum is taken over all $\nu\in L^2(S^2)$.

In section 6.7 we consider the problem of finding $\nu\in L^2(S^2)$ with minimal norm $\|\nu\|_{L^2(S^2)}:=a(\nu)$ among all $\nu(\alpha,\theta)$ which satisfy the inequality:

$$\|\psi(x)-\int_{S^2}u(x,\alpha)\nu(\alpha,\theta)d\alpha\|_{L^2(B_b)}\leq\varepsilon.\tag{2.28}$$

The necessity to consider the ν with the minimal norm $\|\nu\|_{L^2(S^2)}$ comes from the simple observation: there exists a sequence of $\nu_n\in L^2(S^2)$, $\|\nu_n\|_{L^2(S^2)}=1$, such that

$$\|\int_{S^2}u(x,\alpha)\nu_n(\alpha)d\alpha\|_{L^2(B_b)}\to 0\text{ as }n\to\infty.\tag{2.29}$$

To prove (2.29) note that

$$u(x,\alpha)=e^{i\alpha\cdot x}-\int_{B_b}\frac{e^{i|x-y|}}{4\pi|x-y|}q(y)u(y,\alpha)dy:=e^{i\alpha\cdot x}-Tu,\tag{2.30}$$

where the operator $(I+T)^{-1}:=I+T_1$ is a continuous bijection of $C(B_b)$ onto itself, and $C(B_b)$ is the usual space of continuous in B_b, $b\geq a$, functions equipped with the sup-norm [15]. Since T is compact in $C(B_b)$, the above statement follows from the injectivity of $I+T$, which we now prove:

If $f+Tf=0$, then f is extended to $C(\mathbb{R}^3)$ by the formula $f=-Tf$, and satisfies the following equation $(\nabla^2+1-q(x))f=0$ in $\mathbb{R}^3$ and the radiation condition of the type (1.6) with $k=1$. Therefore $f(x)\equiv 0$ and the injectivity of $I+T$ is proved.

Thus $I+T$ and $I+T_1$ are continuous bijections of $C(B_b)$ into itself for any $b\geq a$.

Writing $u(x,\alpha)=(I+T_1)e^{i\alpha\cdot x}$, one concludes that (2.29) is equivalent to

$$\|\int_{S^2}e^{i\alpha\cdot x}\nu_n(\alpha)d\alpha\|_{L^2(B_b)}\to 0\text{ as }n\to\infty.\tag{2.31}$$

Existence of a normalized sequence $\nu_n(\alpha)$ satifying (2.31) follows from the compactness of the operator

$$Q:L^2(S^2)\to L^2(B_b),\quad Q\nu:=\int_{S^2}e^{i\alpha\cdot x}\nu(\alpha)d\alpha.$$

Of course, the same argument is applicable to the operator $Q_1\nu:=\int_{S^2}u(x,\alpha)\nu(\alpha)d\alpha$, but the bijectivity of $I+T$ in $C(B_b)$, $b\geq a$, is of independent interest.

It follows from (2.29) that, for a given $\varepsilon>0$, one can find ν in (2.29) with an arbitrary large norm $\|\nu\|_{L^2(S^2)}$. By this reason we are interested in ν with minimal norm. Estimate (2.23) gives a bound on the growth of the minimal value of the norm $\|\nu\|_{L^2(S^2)}$, where $\nu=\nu(\alpha,\theta)$ satisfies (2.28) with $\varepsilon=\frac{e^{-b\kappa}}{\kappa}$, $\kappa:=|Im\theta|$, $|\theta|\to\infty$, $\theta\in M$.

2.4 Stability estimate for inversion of noisy data.

Assume now that the scattering data are given with some error: a function $A_\delta(\alpha', \alpha)$ is given such that

$$\sup_{\alpha', \alpha \in S^2} |A(\alpha', \alpha) - A_\delta(\alpha', \alpha)| \leq \delta. \tag{2.32}$$

We emphasize that $A_\delta(\alpha', \alpha)$ is not necessarily a scattering amplitude corresponding to some potential, it is an arbitrary function in $L^\infty(S^2 \times S^2)$ satisfying (2.32). It is assumed that the unknown function $A(\alpha', \alpha)$ is the scattering amplitude corresponding to a $q \in Q$.

The problem is: *Find an algorithm for calculating $\widehat{q}$ such that*

$$\sup_{\xi \in \mathbb{R}^3} |\widehat{q} - \widetilde{q}(\xi)| \leq \eta(\delta), \quad \eta(\delta) \to 0 \text{ as } \delta \to 0, \tag{2.33}$$

and estimate the rate at which $\eta(\delta)$ tends to zero.

An algorithm for inversion of noisy data will now be described.

Let

$$N(\delta) := \left[\frac{|\ln \delta|}{\ln |\ln \delta|} \right], \tag{2.34}$$

where $[x]$ is the integer nearest to $x > 0$,

$$\widehat{A}_\delta(\theta', \alpha) := \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\theta'), \quad A_{\delta\ell}(\alpha) := \int_{S^2} A_\delta(\alpha', \alpha) \overline{Y_\ell(\alpha')} d\alpha', \tag{2.35}$$

$$u_\delta(x, \alpha) := e^{i\alpha \cdot x} + \sum_{\ell=0}^{N(\delta)} A_{\delta\ell}(\alpha) Y_\ell(\alpha') h_\ell(r), \tag{2.36}$$

$$\rho_\delta(x; \nu) := e^{-i\theta \cdot x} \int_{S^2} u_\delta(x, \alpha) \nu(\alpha) d\alpha - 1, \quad \theta \in M, \tag{2.37}$$

$$\mu(\delta) := e^{-\gamma N(\delta)}, \quad \gamma = \text{ const } > 0, \quad \gamma = \ln \frac{a_1}{a} > 0, \tag{2.38}$$

$$a(\nu) := \|\nu\|_{L^2(S^2)}, \quad \kappa := |Im\theta|. \tag{2.39}$$

Consider the variational problem with constraints:

$$|\theta| = \sup := \vartheta(\delta), \tag{2.40}$$

$$|\theta| \left[\|\rho_\delta(\nu)\| + a(\nu) e^{\kappa b} \mu(\delta) \right] \leq c, \quad \theta \in M, \tag{2.41}$$

the norm is defined in (2.11), and it is assumed that

$$\theta' - \theta = \xi, \quad \theta, \theta' \in M, \tag{2.42}$$

where $\xi \in \mathbb{R}^3$ is an arbitrary fixed vector, $c > 0$ is a sufficiently large constant, and the supremum is taken over $\theta \in M$ and $\nu \in L^2(S^2)$ under the constraint (2.41).

Given $\xi \in \mathbb{R}^3$ one can always find θ and θ' such that (2.42) holds.

We prove that $\vartheta(\delta) \to \infty$, in fact

$$\vartheta(\delta) \geq c\frac{|\ln \delta|}{(\ln |\ln \delta|)^2}, \quad \delta \to 0. \tag{2.43}$$

Let $\theta(\delta)$, $\nu_\delta(\alpha)$ be any approximate solution to problem (2.40)-(2.41) in the sense that

$$|\theta(\delta)| \geq \frac{\vartheta(\delta)}{2}. \tag{2.44}$$

Calculate

$$\widehat{q_\delta} := -4\pi \int_{S^2} \widehat{A}_\delta(\theta', \alpha)\nu_\delta(\alpha)d\alpha. \tag{2.45}$$

Theorem 2.4. *If (2.42) and (2.44) hold, then*

$$\sup_{\xi\in\mathbb{R}^3} |\widehat{q_\delta} - \widetilde{q}(\xi)| \leq c\frac{(\ln |\ln \delta|)^2}{|\ln \delta|} \ \textit{as}\ \delta \to 0. \tag{2.46}$$

Proof.

$$\begin{aligned}\widehat{q_\delta} - \widetilde{q}(\xi) =& -4\pi \int_{S^2} \widehat{A}_\delta(\theta', \alpha)\nu_\delta(\alpha)d\alpha - \widetilde{q}(\xi)\\ =& -4\pi \int_{S^2} A(\theta', \alpha)\nu_\delta(\alpha)d\alpha - \widetilde{q}(\xi)\\ &+ 4\pi \int_{S^2} [A(\theta', \alpha) - \widehat{A}_\delta(\theta', \alpha)]\nu_\delta(\alpha)d\alpha.\end{aligned} \tag{2.47}$$

The rest of the proof consists of the following steps two steps.

Step 1. Clearly,

$$\|\rho(x, \nu_\delta)\| \leq \|\rho_\delta(x, \nu_\delta)\| + ||e^{-i\theta\cdot x} \int_{S^2} [u(x, \alpha) - u_\delta(x, \alpha)]\nu_\delta(\alpha)d\alpha||,$$

$$\|\rho_\delta(x, \nu_\delta)\| \leq \|\rho(x, \nu_\delta)\| + ||e^{-i\theta\cdot x} \int_{S^2} [u(x, \alpha) - u_\delta(x, \alpha)]\nu_\delta(\alpha)d\alpha||.$$

One can derive an inequality:

$$||e^{-i\theta\cdot x} \int_{S^2} [u(x, \alpha) - u_\delta(x, \alpha)]\nu_\delta(\alpha)d\alpha|| \leq c||\nu_\delta(\alpha)||_{L^2(S^2)}e^{\kappa b}\mu(\delta),$$

similarly to the derivation of the estimate (2.58) below. Therefore,

$$\|\rho(x, \nu_\delta)\| \leq \|\rho_\delta(x, \nu_\delta)\| + ca(\nu_\delta)e^{\kappa b}\mu(\delta) \leq \frac{c}{\vartheta(\delta)}, \tag{2.48}$$

where the norm is defined in (2.11) and the last inequality in (2.48) holds if (2.41), (2.43) and (2.44) hold.

This estimate and (2.43) imply (see the proof of (2.19) and (2.26) and also the argument at the end of this Section, after the end of the proof of Theorem 2.4) that

$$\left| -4\pi \int_{S^2} A(\theta', \alpha)\nu_\delta(\alpha)d\alpha - \widetilde{q}(\xi) \right| \leq c\frac{(\ln|\ln\delta|)^2}{|\ln\delta|}. \tag{2.49}$$

Step 2. We prove that

$$\left| \int_{S^2} \left[A(\theta', \alpha) - \widehat{A}_\delta(\theta', \alpha \right] \nu_\delta(\alpha) d\alpha \right| \leq ca(\nu_\delta)e^{\kappa b}\mu(\delta) \leq \frac{c}{\vartheta(\delta)} \leq c\frac{(\ln|\ln\delta|)^2}{|\ln\delta|}, \tag{2.50}$$

where $\theta' = \theta'(\delta) = \xi + \theta(\delta)$, and the pair $\{\theta(\delta), \nu_\delta(\alpha)\}$ solves (2.40)-(2.41) approximately in the sense specified above. (See formulas (2.41) and (2.44)).

This estimate follows from (2.41) and from the inequality

$$\|A(\theta', \alpha) - \widehat{A}_\delta(\theta', \alpha)\|_{L^2(S^2)} \leq ce^{\kappa b}\mu(\delta). \tag{2.51}$$

Let us prove (2.51). One has

$$\begin{aligned} \|\widehat{A}_\delta(\theta'(\delta), \alpha) - A(\theta'(\delta), \alpha)\|_{L^2(S^2)} &\leq \| \sum_{\ell=0}^{N(\delta)} \left[\widehat{A}_{\delta\ell}(\alpha) - A_l(\alpha) \right] Y_\ell(\theta')\|_{L^2(S^2)} \\ &+ \| \sum_{\ell=N(\delta)+1}^{\infty} A_\ell(\alpha)Y_\ell(\theta')\|_{L^2(S^2)} := I_1 + I_2, \end{aligned} \tag{2.52}$$

where $N(\delta)$ is given in (2.34) and $\theta' := \theta'(\delta)$. Using (1.28), (1.29) one gets

$$I_1 \leq c\delta N^2(\delta)\frac{e^{\kappa b}(2N)^{N(\delta)+1}}{(eb)^{N(\delta)}}. \tag{2.53}$$

Here we have used the estimate

$$\sup_{\alpha\in S^2} \sum_{\ell=0}^{\infty} \left| A_\ell(\alpha) - \widehat{A}_{\delta\ell}(\alpha) \right|^2 \leq 4\pi\delta^2,$$

which follows from (2.32) and the Parseval equality, and implies

$$\sup_{\alpha\in S^2} \left| A_\ell(\alpha) - \widehat{A}_{\delta\ell}(\alpha) \right| \leq \sqrt{4\pi}\delta. \tag{2.54}$$

We also took into account that there are $(N+1)^2$ spherical harmonics $Y_\ell = Y_{\ell,m}$ with $0 \leq \ell \leq N$, because $\sum_{m=-\ell}^{\ell} 1 = 2\ell + 1$, and $\sum_{\ell=0}^{N}(2\ell+1) = (N+1)^2$. For large N one has $(N+1)^2 = N^2[1+o(1)]$, $N \to \infty$, so we write $(N+1)^2 \leq cN^2$, $c > 1$.

To estimate I_2, use (1.27) -(1.29) and get:

$$I_2 \le c \sum_{\ell=N+1}^{\infty} \left(\frac{ea}{2\ell}\right)^{\ell+1} \frac{e^{\kappa a_1}}{\left(\frac{ea_1}{2\ell}\right)^{\ell+1}} \le ce^{\kappa a_1} \left(\frac{a}{a_1}\right)^{N+1}. \tag{2.55}$$

Minimizing with respect to $N > 1$ the function

$$\delta N^2 \frac{(2N)^{N+1}}{(eb)^N} + s^{N+1}, \quad 0 < s := \frac{a}{a_1} < 1. \tag{2.56}$$

one gets

$$\min_{N>1} \left[\delta N^2 \frac{(2N)^{N+1}}{(eb)^N} + s^{N+1}\right] \le ce^{-\gamma N(\delta)} = c\mu(\delta), \quad \gamma = \ln \frac{a_1}{a} > 0, \tag{2.57}$$

where $N(\delta)$ is given in (2.34) and $\mu(\delta)$ is defined by (2.38). Thus, from (2.52) -(2.56) one gets (2.51). Theorem 2.4 is proved. $\Box$

Alternatively, one can get from (2.47) and (2.23) the following estimate:

$$|\widehat{q_\delta} - \widetilde{q}(\xi)| \le c(\frac{1}{s} + \mu(\delta)e^{s\ln s}), \quad s = |\theta| \gg 1, \ \ \theta \in M.$$

Minimizing the right side of this inequality with respect to s, one gets estimate (2.46).

What is the maximal value of $s = |\theta|$, $\theta \in M$, satisfying inequality (2.41)?

One has $a(\nu) \le ce^{cs\ln s}$ by (2.23), where $c > 0$ stands for various constants independent of δ and θ,

$$||\rho_\delta(\nu)|| \le ||\rho(\nu)|| + ca(\nu)e^{\kappa b}\mu(\delta),$$

and

$$||\rho(\nu)|| \le cs^{-1} \quad s \gg 1,$$

see (2.7), and $\mu = e^{-\gamma N(\delta)}$, $\gamma > 0$.

Thus, inequality (2.41) reduces to

$$ce^{cs\ln s} \le e^{\gamma N(\delta)}.$$

The maximal s, satisfying this inequality, can be calculated asymptotically, as $\delta \to 0$, by solving the following equation:

$$s \ln s = cx(\ln x)^{-1},$$

where $x := |\ln \delta| \gg 1$.

Let $s = xy$, where $y \ll x$ is unknown. Then $y(\ln x + \ln y) = c(\ln x)^{-1}$, or $y(1 + \frac{\ln y}{\ln x}) = \frac{c}{(\ln x)^2}$, $|\ln y| \ll \ln x$. Thus, $y \sim \frac{c}{(\ln x)^2}$, and

$$s = |\theta| \sim c\frac{|\ln \delta|}{(\ln |\ln \delta|)^2}.$$

This implies estimate (2.43). $\Box$

2.5 Stability estimate for the scattering solutions.

Let us assume (2.18) and derive the following estimate:

Theorem 2.5. *If $q_1, q_2 \in Q_a$ and (2.18) holds then*

$$\sup_{\substack{x\in B'_a \\ \alpha\in S^2}} |u_1(x,\alpha) - u_2(x,\alpha)| \le c\mu(\delta), \quad |x| > a, \tag{2.58}$$

where $\mu(\delta)$ is defined by (2.38) and (2.34), and $c > 0$ is a constant.

Proof. Using (1.31), one gets:

$$u_1 - u_2 = \sum_{\ell=0}^{\infty} \left[A_{\ell 1}(\alpha) - A_{\ell 2}(\alpha)\right] Y_\ell(\alpha') h_\ell(r), \quad r > a. \tag{2.59}$$

As stated below formula (1.33), one has

$$|h_\ell(r)| \le |h_{\ell+1}(r)|, \quad r > 0. \tag{2.60}$$

From (2.59) one gets:

$$\|u_1 - u_2\|^2_{L^2(B_b \setminus B_{a_1})} = \sum_{\ell=0}^{\infty} |A_{\ell 1}(\alpha) - A_{\ell 2}(\alpha)|^2 \int_{a_1}^{b} r^2 |h_\ell(r)|^2 dr, \quad a < a_1 < b. \tag{2.61}$$

It follows from (2.60) and (1.34) that

$$\begin{gathered} \sup_{0\le\ell\le N} \int_{a_1}^{b} r^2 |h_\ell(r)|^2 dr = \int_{a_1}^{b} r^2 |h_N(r)|^2 dr \le \\ c\left(\frac{2N}{e}\right)^{2N} \int_{a_1}^{b} \frac{dr}{r^{2N}} \le c\left(\frac{2N}{e}\right)^{2N} a_1^{-2N}, \quad b > a_1. \end{gathered} \tag{2.62}$$

From (2.60)-(2.61) one gets

$$\|u_1 - u_2\|^2_{L^2(B_b \setminus B_a)} \le c\delta^2 \left(\frac{2N}{ea_1}\right)^{2N} + c\sum_{\ell=N+1}^{\infty} \left(|A_{\ell 1}|^2 + |A_{\ell 2}|^2\right) \left(\frac{2\ell}{ea_1}\right)^{2\ell}, \tag{2.63}$$

where we have used monotone decreasing of $r|H_\ell(r)|^2$ as a function of r. Using estimate (1.27) in order to estimate $|A_{\ell j}(\alpha)|$, $j = 1, 2$, one gets:

$$\|u_1 - u_2\|^2_{L^2(B_b \setminus B_{a_1})} \le c\delta^2 \left(\frac{2N}{ea_1}\right)^{2N} + c\left(\frac{a}{a_1}\right)^{2N}, \quad a < a_1. \tag{2.64}$$

Minimization of the right-hand side of (2.63) with respect to $N \ge 1$ yields, as in (2.56), the estimate similar to (2.57):

$$\|u_1 - u_2\|^2_{L^2(B_b \setminus B_{a_1})} \le ce^{-\gamma N(\delta)}, \quad \gamma = \ln\frac{a_1}{a} > 0. \tag{2.65}$$

Since $u_1 - u_2 := w$ solves the equation

$$(\nabla^2 + 1)w = 0 \text{ in } B'_a := \mathbb{R}^3 \backslash B_a, \tag{2.66}$$

one can use the known elliptic estimate:

$$\|w\|_{H^2(D_1)} \leq c\left(\|(\nabla^2 + 1)w\|_{L^2(D_2)} + \|w\|_{L^2(D_2)}\right), \quad D_1 \subset D_2, \tag{2.67}$$

where D_1 is a strictly inner subset of D_2 and $c = c(D_1, D_2)$, and get:

$$\|w\|_{H^2(D_1)} \leq ce^{-\gamma N(\delta)}, \tag{2.68}$$

where D_1 is any annulus $a_1 < a_2 \leq |x| \leq a_3 < b$. By the embedding theorem, (2.68) implies (2.58) in $\mathbb{R}^3$. □

2.6 Spherically symmetric potentials.

If $q(x) = q(r)$, $r := |x|$, then

$$A(\alpha', \alpha) = A(\alpha' \cdot \alpha), \quad A_\ell(\alpha) = A_\ell Y_\ell(\alpha). \tag{2.69}$$

In [23] *the converse is proved: if $q \in Q_a$ and (2.69) holds then $q(x) = q(r)$.*

The scattering data $A(\alpha', \alpha)$ in the case of the spherically symmetric potential is equivalent to the set of the phase shifts δ_ℓ. The phase shifts are defined as follows:

$$1 + \frac{i}{2\pi} A_\ell = e^{2i\delta_\ell}, \quad A_\ell = 4\pi e^{i\delta_\ell} \sin \delta_\ell, \quad k = 1. \tag{2.70}$$

From Theorem 2.1 it follows that is $q = q(r) \in Q_a$ then the set $\{\delta_\ell\}_{\ell=0,1,2,...}$ determines uniquley $q(r)$. A much stronger result is proved by the author in [24]. To formulate this result, denote by $\mathcal{L}$ any subset of nonnegative integers such that

$$\sum_{\ell \in \mathcal{L}, \ell \neq 0} \frac{1}{\ell} = \infty. \tag{2.71}$$

Theorem 2.6. *([24]) If $q(x) = q(r) \in Q_a$ then the data $\{\delta_\ell\}_{\forall \ell \in \mathcal{L}}$ determine $q(r)$ uniquely.*

In [1] and [53] examples are given of quite different potentials $q_1(r)$ and $q_2(r)$, piecewise-constant, $q_j(r) = 0$ for $r > 5$, for which

$$\sup_{\ell \geq 0} \left|\delta_\ell^{(1)} - \delta_\ell^{(2)}\right| < 10^{-5} \text{ and } \sup_{r \geq 0} |q_1(r) - q_2(r)| \geq 1, \text{ where } q_1 \text{ and } q_2$$

are of order of magnitude of 1.

This result shows that the stability estimate (2.19) is accurate.

3 The direct obstacle scattering problem with fixed-frequency data

Let $D \subset \mathbb{R}^3$ be a bounded domain (an obstacle) with boundary S. We want to study the obstacle scattering problem under minimal smoothness assumptions on S.

Recall that if S is $C^{1,\lambda}$, $0 < \lambda \leq 1$, that is, in local coordinates S is a graph of a $C^{1,\lambda}$- function $x_3 = g(x')$, $x' := (x_1, x_2)$, $g \in C^{1,\lambda}$, then the obstacle scattering problem consists of finding the scattering solution $u(x, \alpha)$, which satisfies the equations:

$$(\nabla^2 + 1)u = 0 \text{ in } D' := \mathbb{R}^3 \backslash D, \tag{3.1}$$

$$\Gamma u = 0, \tag{3.2}$$

$$u = e^{i\alpha \cdot x} + A(\alpha', \alpha)\frac{e^{ir}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \frac{x}{r} = \alpha', \tag{3.3}$$

where $\alpha \in S^2$ is given and (3.2) is the Dirichlet condition if $\Gamma u = u$, the Neumann condition if $\Gamma u = u_N$, or the Robin condition if $\Gamma u := u_N + \sigma(s)u$, $\sigma(s) \in L^\infty(S)$, $Im\sigma(s) = 0$, N is the exterior normal to S. We took the wavenumber $k = 1$ without loss of generality.

If S is very rough (non-smooth), N may be not defined on S. The minimal assumptions on the smoothness of S, under which the existence and uniqueness of the solution to the direct scattering problem are established, were introduced in [17] and [18]. These assumptions are:

A_1) If $\Gamma u = u$ then D is an arbitrary bounded domain, that is a bounded open set.

A_2) If $\Gamma u = u_N$ then the assumption on S is:

$$i : H^1(D'_a) \to L^2(D'_a) \text{ is compact}, \quad D'_a := D' \cap B_a. \tag{3.4}$$

Here i is the embedding operator, H^1 is the Sobolev space, $a > 0$ is such a number that the ball $B_a := \{x : |x| \leq a\}$ contains D, and $D' := \mathbb{R}^3 \backslash D$.

A_3) If $\Gamma u = u_N + \sigma(s)u$, then the assumption on S is:

$$i : H^1(D'_a) \to L^2(D'_a) \text{ and } i_1 : H^1(D'_a) \to L^2(S) \text{ are compact.} \tag{3.5}$$

Here the integration on S in the definition of $L^2(S)$ is understood with respect to the two-dimensional Hausdorff measure on S.

The usual classes of domains in the theory of Sobolev spaces are:

1) domains satisfying the cone condition,

2) Lipschitz domains,

and

3) extension domains, (see [7], [8]).

They all are such that the above assumptions A_2) and A_3) hold. Let us recall the definitions of these domains: D satisfies the cone condition if each point of D is the vertex of a cone contained in D along with its closure, the cone in the local coordinates is the region $x'^2 < ax_3^2$, $0 < x_3 < b$, and $a, b > 0$ are fixed positive constants. A domain D is Lipschitz if each point of $S := \partial D$ has a neighborhood $U \subset \mathbb{R}^n$, such that $U \cap D$ can be mapped onto a cube by a quasi-isometric map.

A homeomorphic $f : D_1 \to D_2$ is called quasi-isometric if

$$\lim_{x \to x_0} \sup \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq M, \quad \lim_{y \to y_0} \sup \frac{|f^{-1}(y) - f^{-1}(y_0)|}{|y - y_0|} \leq M$$

for any $x_0 \in D_1$ and any $y_0 \in D_2$ and the Jacobian $\det f'(x)$ preserves its sign in D_1.

A domain D is an extension domain in H^1 if there exists a linear continuous operator $E : H^1(D) \to H^1(\mathbb{R}^3)$, $Eu = u$ on D for all $u \in H^1(D)$.

An extension domain may fail to satisfy the cone condition, but a bounded domain satisfying the cone condition is the union of a finite number of extension domains.

We will use also the domains with finite perimeter. A domain D has finite perimeter if and only if $\|\nabla \chi_D(x)\|_{BV(\mathbb{R}^3)} < \infty$, where $\chi_D(x)$ is the characteristic function of D and BV is the space of functions of bounded variation [7].

Let us state the result from [18].

Theorem 3.1. *Problem (3.1) - (3.3) has a unique weak solution if:*

a) D is an arbitrary bounded domain (open set) in $\mathbb{R}^3$ and $\Gamma u = u$,

b) D is a bounded domain, condition A_2) holds and $\Gamma u = u_N$

c) D is a bounded domain, condition A_3) holds and $\Gamma u = u_N + \sigma(s)u$, $\sigma(s) \in L^\infty(S)$, $Im\sigma(s) = 0$.

The solution to (3.1) - (3.3) for rough boundaries is understood in the weak sense and has the following properties:

a) if $\Gamma u = u$ then $u \in \overset{\circ}{H}{}^1(D'_R) \cap C^\infty_{\ell oc}(D')$, $u \in L^2\left(D', \frac{1}{1+|x|^a}\right)$, $a > 1$, and (3.3) holds with

$$A(\alpha', \alpha) = -\frac{1}{4\pi} \int_S e^{-i\alpha' \cdot s} u_N(s, \alpha) ds, \tag{3.6}$$

where $\overset{\circ}{H}{}^1(D'_R)$ is the Sobolev space of functions which vanish on S,

b) if $\Gamma u = u_N$, then $u \in H^1(D'_R) \cap C^\infty_{\ell oc}(D')$, $u \in L^2\left(D', \frac{1}{1+|x|^a}\right)$, $a > 1$, and (3.3) holds with

$$A(\alpha', \alpha) = \frac{1}{4\pi} \int_S \frac{\partial e^{-i\alpha' \cdot s}}{\partial N} u(s, \alpha) ds, \tag{3.7}$$

c) if $\Gamma u = u_N + \sigma(s)u$, then $u \in H^1(D'_R) \cap C^\infty_{\ell oc}(D')$, $u \in L^2\left(D', \frac{1}{1+|x|^a}\right)$, $a > 1$, and (3.3) holds with

$$A(\alpha', \alpha) = \frac{1}{4\pi} \int_S \left[\frac{\partial e^{-i\alpha \cdot s}}{\partial N_s} + \sigma(s) e^{-i\alpha' \cdot s} \right] u(s, \alpha) ds. \tag{3.8}$$

In [17] and [18] Theorem 3.1 was proved with the operator ∇^2 replaced by a general second-order selfadjoint elliptic operator. The weak solution is defined in the case $\Gamma u = u_N + \sigma(s)u$, as a function $u \in H^1(D'_R) \cap C^\infty_{\ell oc}(D')$ for any $R > R_0$, $B_{R_0} \supset D$, which satisfies (3.3) and satifies the integral relation:

$$\int_{D'} (\nabla u \nabla \phi - u\phi)dx - \int_S \sigma(s)u\phi ds = 0 \quad \forall \phi \in H^1_0(D'). \tag{3.9}$$

Here $H^1_0(D')$ is the set of $H^1(D')$ functions vanishing near infinity and ds is the two-dimensional Hausdorff measure on S. Formula (3.9) makes sense for domains D with finite perimeter.

In [55] conditions are given for (3.4) to hold.

3.1 Uniqueness theorem for inverse obstacle scattering.

The inverse obstacle scattering problem consists of finding S and the boundary condition on S given $A(\alpha', \alpha)\ \forall \alpha' \in S^2$. The scattering amplitude $A(\alpha', \alpha)$ in the obstacle scattering problem satisfies conditions listed in section 1.2.2. In particular, it admits unique analytic continuation from $S^2 \times S^2$ onto $M \times M$.

Let us outline the proof of the uniqueness theorem for inverse obstacle scattering problem with fixed-frequency data. This theorem belongs to the author [16], but we give a new proof [20], see also [19], [56].

Theorem 3.2. *If $A_1(\alpha', \alpha) = A_2(\alpha', \alpha)\ \forall \alpha', \alpha$, running through arbitrary small open subsets of S^2, then $D_1 = D_2 := D$, and the boundary condition on $S := \partial D$ is uniquely determined.*

Proof. As in the proof of Theorem 2.1, the data determine uniquely $A_j(\alpha', \alpha)$ on $S^2 \times S^2$ so that $A_1(\alpha', \alpha) = A_2(\alpha', \alpha)\ \forall \alpha', \alpha \in S^2$. If one has already proved that $D_1 = D_2$, that is, $S_1 = S_2 := S$, then the boundary condition on S is uniquely determined because the scattering solution $u(x, \alpha)$ is uniquely determined by the scattering amplitude in D' (and is analytically determined by formula (1.31) in $B_{a'}$). Thus, the limiting values of $\frac{u_N}{u}$ on S are uniquely determined. If the limit is zero (almost everywhere on S) then $\Gamma u = u_N$, if the limit is infinity (almost everywhere on S) then $\Gamma u = u$, and if the limit is a function $-\sigma(s)$, then $\Gamma u = u_N + \sigma(s)u$. Therefore the main point is to prove that S is uniquely determined by $A(\alpha', \alpha)\ \forall \alpha', \alpha \in S^2$. Let us prove this.

Assume the contrary: $S_1 \neq S_2$. Let $D_{12} := D_1 \cup D_2$, $S_{12} = \partial D_{12}$, $D^{12} := D_1 \cap D_2$. Denote by $\widetilde{D}_1$ a connected component of $D_{12} \backslash D_2$. We want to show that $D_{12} \backslash D_2$ is an empty set. An important tool is the formula [20] similar to (1.30):

$$\begin{aligned} &- 4\pi \left[A_1(\alpha', \alpha) - A_2(\alpha', \alpha)\right] = \\ &\int_{S_{12}} \left[u_{1N}(s, -\alpha')u_2(s, \alpha) - u_1(s, -\alpha')u_{2N}(s, \alpha)\right] ds. \end{aligned} \tag{3.10}$$

This formula holds for domains with finite perimeter.

If $A_1 = A_2 \ \forall \alpha', \alpha \in S^2$, then (3.10) yields:

$$\int_{S_{12}} [u_{1N}(s, -\alpha')u_2(s, \alpha) - u_1(s, -\alpha')u_{2N}(s, \alpha)]\, ds = 0 \quad \forall \alpha', \alpha \in S^2. \tag{3.11}$$

From (3.11) and (1.7) one derives:

$$\int_{S_{12}} [G_{1N}(x, s)G_2(s, y) - G_1(x, s)G_{2N}(s, y)]\, ds = 0 \quad \forall x, y \in D'_{12}. \tag{3.12}$$

From (3.12) and Green's formula one gets:

$$G_1(x, y) = G_2(x, y) \quad \forall x, y \in D'_{12}. \tag{3.13}$$

This leads to a contradiction unless $D_1 = D_2$.

Indeed, if $D_1 \neq D_2$, i.e., if $\widetilde{D_1}$ is not empty, take a point s on the boundary of $\widetilde{D_1}$ which belongs to S_1. This point is an interior point for D'_2. Thus

$$G_2(s, y) \to \infty \text{ as } y \to s. \tag{3.14}$$

On the other hand, since $s \in S_1$ one has $\Gamma G_1(s, y) = 0$, i.e., $G_1(s, y) = 0$ if $\Gamma u = u$, $G_{1N}(s, y) = 0$ if $\Gamma u = u_N$, $G_{1N}(s, y) + \sigma(s)G_1(s, y) = 0$ if $\Gamma u = u_N + \sigma(s)u$. In all the three cases

$$\Gamma G_2(s, y) \to \infty, \text{ as } y \to s, \tag{3.15}$$

while

$$\Gamma G_1(s, y) = 0. \tag{3.16}$$

From (3.13), (3.15) and (3.16) one gets a contradiction. Theorem 3.1 is proved. □

Remark 3.1. *In [25] a uniqueness theorem is proved for inverse obstacle problem with the transmission boundary condition: the boundary condition and the boundary and the wavenumber in the interior of the obstacle are uniquely determined by the fixed-frequency scattering data $A(\alpha', \alpha)\ \forall \alpha', \alpha \in S^2$.*

Remark 3.2. *The basic reason to consider the domains with finite perimeter is the validity of the Green's formula in such domains [7].*

In [19] it is proved that if D_1 and D_2 are domains with finite perimeter then $\widetilde{D_1}$ is also such a domain.

3.2 Stability estimate for inverse obstacle scattering.

Consider the scattering problem (3.1) - (3.3) with $\Gamma u = u$, for example. Let D_1 and D_2 be two arbitrary obstacles in the class $\mathcal{O}^\lambda$ which consists of the bounded domains whose boundaries can be covered by finitely many balls B_j, on the patch $S_j = S \cap B_j$ the boundary is described in the local coordinates by the equation $x_3 = g_j(x')$, $x' = (x_1, x_2)$, where $g_j(x') \in C^{2,\lambda}$, $0 < \lambda \leq 1$, and

$$\sup_j \|g_j\|_{C^{2,\lambda}(S_j)} \leq c_0, \tag{3.17}$$

where c_0 does not depend on the choice of $D \in \mathcal{O}^\lambda$.

Let $A_j(\alpha', \alpha)$ be the scattering amplitude corresponding to the obstacle D_j, $j = 1, 2$.

Assume that (2.18) holds. Define the symmetric Hausdorff distance between $S_1 = \partial D_1$ and $S_2 = \partial D_2$:

$$\rho := \max\{\sup_{x\in S_2} \inf_{y\in S_1} |x - y|, \quad \sup_{x\in S_1} \inf_{y\in S_2} |x - y|\}, \tag{3.18}$$

where $\rho = \rho(\delta)$ in our case.

The basic stability estimate [26] can now be formulated. Let $A_m(\alpha', \alpha)$ be the acattering amplitude corresponding to D_m and $\Gamma u = u$ on $S_m := \partial D_m, m = 1, 2$.

Theorem 3.3. *If (2.18) and (3.17) hold then*

$$\rho(\delta) \leq c_1 \left(\frac{\ln|\ln\delta|}{|\ln\delta|}\right)^{c_2}, \quad \delta \to 0, \tag{3.19}$$

where c_1, c_2 are positive constants independent of $\delta > 0$.

Proof. Let us sketch the steps of the proof.

Step 1.

$$\rho(\delta) \to 0 \text{ as } \delta \to 0. \tag{3.20}$$

This follows from the uniquness theorem 3.2 and from the compactness of the set $\mathcal{O}^\lambda$ in the space $C^{2,\mu}$, $\mu < \lambda$. For simplicity of the presentation we assume that $j = 1$, that is, there is just one patch in the covering of S, for example, S is star-shaped.

Step 2. There exists an integer m such that

$$cd^m(x) \leq |w(x)| \leq c\varepsilon^{cd^c(x)}, \quad w(x) := u_1(x, \alpha) - u_2(x, \alpha), \tag{3.21}$$

where $d(x)$ is the distance from a point $x \in D'_{12}$ to S_{12}, we assume that $d(x) \sim \rho$, $\varepsilon = ce^{-\gamma N(\delta)}$, $\gamma =$ const > 0, $N(\delta) = \frac{|\ln\delta|}{\ln|\ln\delta|}$, and $c > 0$ here and below stand for *various* constants indepent of δ and x. Symbol $d(x) \sim \rho$ means that $c_1 d(x) \leq \rho \leq c_2 d(x)$ with some constants $c_1, c_2 > 0$.

Let us show that (3.21) implies (3.19). From (3.21) one gets, replacing d by ρ, dropping w and taking log:

$$\ln\rho \leq c + c\rho^c \ln\varepsilon. \tag{3.22}$$

Recall that c stands for different constants.

From (3.22) one gets

$$\frac{\rho^c}{\ln\frac{1}{\rho}} \leq \frac{c}{\ln\frac{1}{\varepsilon}}. \tag{3.23}$$

Since $\rho^\beta < \frac{1}{\ln \frac{1}{\rho}}$ as $\rho \to 0$ and $\beta > 0$, estimate (3.23) implies

$$\rho \leq c \left(\frac{1}{\ln \frac{1}{\varepsilon}} \right)^{\frac{1}{c}} = c_1 \left(\frac{\ln |\ln \delta|}{|\ln \delta|} \right)^{c_2},$$

where we have used the definition of ε, namely $\varepsilon = ce^{-\gamma N(\delta)}$, which implies $\frac{1}{\ln \frac{1}{\varepsilon}} \sim \frac{\ln |\ln \delta|}{|\ln \delta|}$.

Let us give the details of the proof.

Step 1. Assume that (3.20) is false. Then $\rho(\delta) \geq c > 0$ as $\delta \to 0$. Since $\mathcal{O}^\lambda$ is a compact set in $C^{2,\mu}$, $\mu < \lambda$, one can select sequences $S_{1n(\delta)}$ and $S_{2n(\delta)}$ which converge in $C^{2,\mu}$ to S_1 and S_2 correspondingly, as $\delta \to 0$. Since $A(\alpha', \alpha)$ depends continuously on S (see [27], [16]) in the sense

$$\lim_{n\to\infty} \sup_{\alpha',\alpha \in S^2} |A_n(\alpha', \alpha) - A(\alpha', \alpha)| = 0, \tag{3.24}$$

where the limit is taken in the process $S_n \to S$ in $C^{2,\mu}$, $0 < \mu < \lambda$, one concludes that $A_1(\alpha', \alpha) = A_2(\alpha', \alpha)$ for the limiting surfaces S_1 and S_2. By the uniqueness theorem 3.1 it follows that $S_1 = S_2$. Therefore $\rho := \rho(S_1, S_2) = 0$. However, $\rho = \lim_{\delta\to 0} \rho(\delta) \geq c > 0$. This is a contradiction which proves (3.20).

Step 2. The function $w(x) := u_1(x, \alpha) - u_2(x, \alpha)$, where $u_j(x, \alpha)$ is the scattering solution corresponding to the obstacle D_j, $j = 1, 2$, solves the equation

$$(\nabla^2 + 1)w = 0 \text{ in } D'_{12}, \quad D_{12} := D_1 \cup D_2, \tag{3.25}$$

satifies the radiation condition

$$\lim_{r\to\infty} \int_{|s|=r} \left| \frac{\partial w}{\partial |x|} - iw \right|^2 ds = 0, \tag{3.26}$$

and

$$\|w\|_{C^{2,\lambda}(D'_{12})} \leq c. \tag{3.27}$$

It is proved in [6],vol.3, p.14, that solutions to elliptic second order equations with smooth coefficients cannot have zeros of infinite order up to the boundary without vanishing identically. This implies existence of an integer $m > 0$ for which the left inequality (3.21) holds.

The proof of the right inequality requires some preparations.

Let us sketch the steps of this proof.

Step 3. By the result of Theorem 2.5 one gets estimate (2.58) in the region $|x| > a$, $B_a \supset D$.

Although in Theorem 2.5 the functions u_1 and u_2 were the solutions to the Schrödinger equations with potentials vanishing in B'_a, the estimate (2.58) is proved for any solutions to equation (3.25) whose difference satisfy (3.26) and (2.18). In particular, estimate (2.58) holds for our w. Let us define $\varepsilon = c\mu(\delta) = ce^{-\gamma N(\delta)}$, $N(\delta) = \frac{|\ln \delta|}{\ln |\ln \delta|}$.

Step 4. Let us prove the right inequality (3.21). Extend w from D'_{12} into D_{12} so that the estimate similar to (3.27) holds:

$$\|w\|_{C^{2,\lambda}(\mathbb{R}^3)} \le c_1. \tag{3.28}$$

This is possible: using, for example, the known Stein's theorem one can extend u_j into D_j since S_j is $C^{2,\lambda}$ - smooth, $j = 1, 2$, and then $w := u_1 - u_2$ will be the $C^{2,\lambda}$- smooth extension of w from D'_{12} into D_{12}. Define

$$f(x) := -(\nabla^2 + 1)w \text{ in } \mathbb{R}^3, \tag{3.29}$$

where we denoted by w the extended to $\mathbb{R}^3$ function $w(x)$. Then $f(x) = 0$ in D'_{12}, $f(x) \in C^\lambda(\mathbb{R}^3)$, and

$$w(x) = \int_{D_{12}} \frac{e^{i|x-y|}}{4\pi|x-y|} f(y)dy, \quad x \in \mathbb{R}^3. \tag{3.30}$$

Denote $|x| = r$, $|y| = \rho$. Set $z := re^{i\varphi}$. Choose any point $x \in D'_{12}$, in a neighborhood of $\widetilde{D_1}$, a connected component of $D_{12}\backslash D_2$, such that $d(x) := dist(x, S_1) \sim \rho := \rho(\delta)$. By step 1, $\rho(\delta) \to 0$ as $\delta \to 0$, so that $\rho(\delta)$ is small for small δ.

Consider analytic continuation of $w(x)$, defined by (3.30), on the complex z-plane (as in [11]). Let ω be the angle between vectors x and y, $x \in D'_{12}$. Since

$$|x - y| = \sqrt{(r - \rho e^{i\omega})(r - \rho e^{-i\omega})},$$

this analytic continuation, that is, replacement $r \to z$, in the expression $|x-y|$, is possible if $re^{i\varphi} - \rho e^{i\omega} \neq 0$ and $re^{i\varphi} - \rho e^{-i\omega} \neq 0$. Choose a point O on S_1 closest to x and the coordinate system in which the origin is at O, and the x_1x_2- plane is tangent to S_1 at the point O. Since S_1 is sufficiently smooth, there exists a cone K with an opening $\theta_0 > 0$ and vertex at O which belongs to D'_{12} and its axis passes through the point x. The function $|x - y|$ admits analytic continuation in the z-plane from the ray $r > 0$ to the sector $|\varphi| < \theta_0$. Since there are no points of D_{12} inside the cone K, the expression

$$\zeta := \left[(re^{i\varphi} - \rho e^{i\omega})(re^{i\varphi} - \rho e^{-i\omega})\right]^{\frac{1}{2}}, \quad Im\zeta \ge 0, \quad r > 0, \quad |\arg z| = |\varphi| < \theta_0,$$

does not vanish in the region $r > 0$, $|\arg z| := |\varphi| < \theta_0$. Therefore the function (3.30), which we denote $W(z)$, considered as a function of z, admits analytic continuation in the sector $|\arg \varphi| < \theta_0, r > 0$, and satisfies the following inequalities there:

$$|W(z)| \le c, \quad |\arg z| < \theta_0, \quad r \ge 0, \tag{3.31}$$

$$|W(z)| \le \varepsilon, \quad \arg z = 0, \quad r \ge a, \tag{3.32}$$

where

$$\varepsilon = ce^{-\gamma\frac{|\ln\delta|}{\ln|\ln\delta|}}. \tag{3.33}$$

One can map the sector $|\arg z| < \theta_0$, conformally onto the half-plane, $z \to t = z^{\frac{\pi}{2\theta_0}}$, $|\arg t| < \frac{\pi}{2}$. Then $W(z) := v(t)$, where $v(t)$ is analytic in the half-plane $|\arg t| < \frac{\pi}{2}$, and satisfies there the inequalities

$$|v(t)| \le c, \quad |\arg t| < \frac{\pi}{2}; \quad |v(t)| < \varepsilon, \quad t \ge a_1. \tag{3.34}$$

The known two-constants theorem (see [3]) and (3.34) imply:

$$|v(t)| \le c_1 \varepsilon^{h(t)}, \tag{3.35}$$

where $h(t)$ is the harmonic measure corresponding to the domain R on the complex plane t with the boundary consisting of the lines $\{Ret = 0, \ -\infty < Imt < \infty\}$ and $\{Imt = 0, Ret > a_1 > 0\}$. Recall that $h = h(t_1, t_2)$, the harmonic measure, is a harmonic function which solves the problem:

$$\Delta h := h_{t_1 t_1} + h_{t_2 t_2} = 0 \text{ in } R, \tag{3.36}$$

$$h = 0 \text{ at } t_1 = 0, \quad -\infty < t_2 < \infty; \quad h = 1 \text{ at } t_2 = 0, \quad t_1 \ge a;$$

and $h(t_1, t_2)$ is bounded at infinity, $t = t_1 + it_2$.

By the maximum principle, $1 > h(t_1, t_2) > 0$ in R, and, by the Hopf lemma, (see [5] p.34),

$$\frac{\partial h}{\partial t_1} > 0 \text{ at } t_1 = 0. \tag{3.37}$$

Thus

$$h(t_1, 0) \ge ct_1, \quad 0 < t_1 \le t_0, \quad c = \text{ const } > 0, \tag{3.38}$$

where $t_0 > 0$ is a sufficiently small number.

From (3.35) and (3.38) it follows that in a sufficiently small neighborhood of the origin one has $|v(t_1, 0)| \le c\varepsilon^{ct_1}$. Returning to the z-variable one gets

$$|W(z)| \le c\varepsilon^{c|z|^c}, \quad z = r = |x| > 0, \tag{3.39}$$

where $c > 0$ stands for various constants.

Since $W(r) = w(x)$, inequality (3.39) is identical to the right inequaltiy (3.21).

Theorem 3.2 is proved. □

Remark 3.3. *An interesting open problem is to construct S in the inverse obstacle scattering problem analytically from noisy data $A_\delta(\alpha', \alpha)$ in the way it is done in section (2.4), formula (2.45), for the reconstruction of the potential in the inverse potential scattering problem, or even from exact data in the way it was done in section 2.2, formula (2.8). In the next section we prove that such a reconstruction formula does exist for exact data for the inverse obstacle scattering problem.*

3.3 Existence of a reconstruction formula.

Let $A(\alpha', \alpha)$ be the scattering amplitude corresponding to an obstacle D, and assume, for example, that $\Gamma u = u$, so that the Dirichlet condition holds on S. Take $\alpha' = \theta' \in M$ in (3.6), multiply (3.6) by a $\nu(\alpha, \theta)$ and integrate with respect to α over S^2 to get:

$$-4\pi \int_{S^2} A(\theta', \alpha)\nu(\alpha, \theta)d\alpha = \int_S e^{-i\theta' \cdot s} \int_{S^2} u_N(s, \alpha)\nu(\alpha, \theta)d\alpha\, ds. \tag{3.40}$$

Here $\nu(\alpha, \theta) \in L^2(S^2)$ is some function which is chosen so that (3.41) holds. We prove below the following lemma:

Lemma 3.1. *The set $\{u_N(s, \alpha)\}_{\forall \alpha \in S^2}$ is total in $L^2(S)$.*

This implies that, given an arbitrary small number $\eta > 0$ and $\theta \in M$, there exists a $\nu_\eta(\alpha, \theta) \in L^2(S^2)$ such that

$$\| \int_{S^2} u_N(s, \alpha)\nu_\eta(\alpha, \theta)d\alpha - \frac{\partial e^{i\theta \cdot s}}{\partial N} \|_{L^2(S)} \leq \eta. \tag{3.41}$$

From (3.40), with $\nu = \nu_\eta$, and (3.41) one gets:

$$-4\pi \lim_{\eta \to 0} \int_{S^2} A(\theta', \alpha)\nu_\eta(\alpha, \theta)d\alpha = \int_S e^{-i\theta' \cdot s} \frac{\partial e^{i\theta \cdot s}}{\partial N} ds \tag{3.42}$$

Let us assume $\theta, \theta' \in M$, $\theta - \theta' = \xi$, $\xi \in \mathbb{R}^3$ is an arbitrary fixed vector.

By Green's formula one gets:

$$\int_S e^{-i\theta' \cdot s} \frac{\partial e^{i\theta \cdot s}}{\partial N} ds = \frac{1}{2} \int_S \frac{\partial e^{i(\theta - \theta') \cdot s}}{\partial N} ds = -\frac{|\xi|^2}{2} \int_D e^{i\xi \cdot x} dx = -\frac{|\xi|^2}{2} \widetilde{\chi_D}(\xi), \tag{3.43}$$

where $\chi_D(x) = 1$ in D, $\chi_D(x) = 0$ in D'.

From (3.42) and (3.43) one gets

An inversion formula (see [26]) for finding D from the data $A(\alpha', \alpha)$:

$$\widetilde{\chi_D}(\xi) = 8\pi |\xi|^{-2} \lim_{\eta \to 0} \int_{S^2} A(\theta', \alpha)\nu_\eta(\alpha, \theta)d\alpha. \tag{3.44}$$

Before proving the basic Lemma 3.1, a remark is in order: in contrast to our theory for inverse potential scattering problem (see the inversion formula (2.8) and the algorithm for calculating the function $\nu(\alpha, \theta)$ in (2.15)) we do not give an algorithm for calculating the function $\nu_\eta(\alpha, \theta)$ in (3.44).

Finding such an algorithm is an open problem.

We now prove Lemma 3.1:

Assume the contrary. Then there exists a function $f \in L^2(S)$ such that

$$\int_S f(s)u_N(s, \alpha)ds = 0 \quad \forall \alpha \in S^2. \tag{3.45}$$

Define

$$v(y) := \int_S f(s)G_N(s,y)ds, \tag{3.46}$$

where $G(x,y)$ is the Green's function:

$$(\nabla^2 + 1)G(x,y) = -\delta(x-y) \text{ in } D', \tag{3.47}$$

$$G(x,y) = 0, \quad x \in S, \tag{3.48}$$

G satisfies the radiation condition (1.6).

We claim that (3.45) implies

$$v(y) = 0 \text{ in } D', \tag{3.49}$$

where $v(y)$ is defined in (3.46).

Indeed,

$$(\nabla^2 + 1)v = 0 \text{ in } D', \tag{3.50}$$

and

$$v(y) = o\left(\frac{1}{|y|}\right) \text{ as } |y| \to \infty. \tag{3.51}$$

The relation (3.51) follows from (3.45) and (1.7).

It is well known [16], p. 25, that (3.50) and (3.51) imply (3.49).

If (3.49) holds, then, taking $y \to \sigma \in S$ (along the normal to S at the point σ) in (3.46), one gets

$$f(s) = 0. \tag{3.52}$$

Indeed, if $\sigma \in S$ then

$$G_N(s,y) \to \delta_S(s-\sigma) \text{ as } y \to \sigma, \tag{3.53}$$

where $\delta_S(s-\sigma)$ is the delta-function on the surface S (see formula (3.58) below). From (3.53), and (3.49) and (3.46) formula (3.52) follows. □

Let us give a proof of the relation (3.53).

Consider the problem:

$$(\nabla^2 + 1)v = 0 \text{ in } D', \tag{3.54}$$

$$v = f(s) \text{ on } S, \tag{3.55}$$

$$v \text{ satisfies the radiation condition (3.26).} \tag{3.56}$$

This problem has a unique solution representable by the Green's formula:

$$v(x) = \int_S f(s)G_N(s,x)ds. \tag{3.57}$$

Since (3.55) holds for this unique solution (3.57) taking $x \to \sigma \in S$ one gets (3.53), where $\delta_S(s-\sigma)$ is the distribution which acts by the formula

$$\int_S f(s)\delta_S(s-\sigma)ds = f(\sigma). \tag{3.58}$$

Relation (3.53) is proved. $\square$

This argument requires $f(s)$ to be continuous if one understands the delta-function in the usual sense. However, if $\delta_S(s-\sigma)$ is understood as the kernel (in the distributional sense) of the identity operator in some space of functions $f(s)$, for which problem (3.54)-(3.56) has a unique solution, then (3.58) makes sense in the space for example, in $L^2(S)$, and (3.58) is understood in this case as equality of the elements of this space. In the case of $L^2(S)$ this means that the equality holds almost everywhere on S with respect to two-dimensional Hausdorff measure on S. Lemma 3.1 is proved.

Thus, formula (3.44) is proved. $\square$

4 Limiting procedure and stability estimates

Let $D \subset \mathbb{R}^3$ be a bounded domain, $\chi(x)$ be the characteristic function of D, $t > 0$ be a parameter, $q(x) = t\chi(x)$ be the potential. Consider the potential scattering problem (1.1)–(1.2). We prove that the scattering solution $u(x, \alpha; t)$ converges, as $t \to +\infty$, to $u(x, \alpha)$, the scattering solution corresponding to the obstacle D, and $u(x, \alpha)$ satisfies the Dirichlet boundary condition on $S := \partial D$. This result is old [29]. We also prove the following, more recent estimates [21]:

$$\|u(x, \alpha; t)\|_{L^2(D)} \leq \frac{c}{t^{\frac{1}{2}}}, \quad \|u(x, \alpha; t) - u(x, \alpha)\|_{L^2(\widetilde{D}')} \leq \frac{c}{t^{\frac{1}{2}}}, \tag{4.1}$$

$$\|\nabla u(x, \alpha; t)\|_{L^2(B_R)} \leq c(R), \quad B_R \supset D, \tag{4.2}$$

$$\|u(x, \alpha; t)\|_{L^2(S)} \leq \frac{c}{t^{\frac{1}{2}}}, \tag{4.3}$$

where $\widetilde{D}'$ is a strictly inner compact subdomain of $D' := \mathbb{R}^3 \backslash D$.

Assume there are two obstacles D_1 and D_2, $D_j \subset \mathcal{O}^{2,\lambda}$, $0 < \lambda \leq 1$. Let

$$A(\alpha', \alpha; t_1, t_2) := A_1(\alpha', \alpha; t_1) - A_2(\alpha', \alpha; t_2).$$

Then we prove the following stability estimate:

$$\sup_{\alpha', \alpha \in S^2} |A(\alpha', \alpha; t_1, t_2)| \leq c \left(\frac{1}{t_{12}^{\frac{1}{2}}} + \rho \right), \tag{4.4}$$

where $t_{12} = \min(t_1, t_2)$ and ρ is the symmetric Hausdorff distance between S_1 and S_2, defined by (3.18).

Let us prove the above estimates.

One has

$$\left[\nabla^2 + 1 - t\chi(x)\right] u(x, \alpha; t) = 0 \text{ in } \mathbb{R}^3. \tag{4.5}$$

Define

$$\|u\|^2 := \int_{\mathbb{R}^3} \frac{|u(x)|^2 dx}{(1 + |x|^2)^{\frac{\sigma}{2}}}, \quad \sigma > 1. \tag{4.6}$$

We drop the α-dependence in $u(x, \alpha; t)$.

Assume that

$$\|u(x; t)\| \leq c, \tag{4.7}$$

where $c =$ const > 0 does not depend on t.

As $t \to +\infty$, one can select, using (4.7), a weakly convergent in the norm (4.6) sequence, denoted again u. Thus

$$u(x, t) \rightharpoonup u, \tag{4.8}$$

where the weak convergence holds in the Hilbert space corresponding to the norm (4.6). By the elliptic estimates (see formula (6.1) below), formulas (4.8) and (4.5) imply

$$\|u(x; t)\|_{H^2(\widetilde{D}')} \leq c, \quad \|u(x, t) - u(x)\|_{H^2(\widetilde{D}')} \to 0 \text{ as } t \to \infty. \tag{4.9}$$

Multiply (4.5) by $\overline{u}(x, t)$ and integrate over a ball $B_R \supset D$, to get

$$\int_{B_R} \left[|\nabla u(x; t)|^2 + t|u(x; t)|^2\right] dx = \int_{B_R} |u(x; t)|^2 dx + \int_{\partial B_R} \frac{\partial u}{\partial N} \overline{u}\, ds, \tag{4.10}$$

where the bar stands for complex conjugate.

From (4.10), (4.7) and (4.9) one gets

$$\int_{B_R} |\nabla u(x; t)|^2 dx \leq c(R), \quad \int_{B_R} |u(x; t)|^2 dt \leq \frac{c(R)}{t}. \tag{4.11}$$

This yields (4.2) and the first inequality in (4.1).

Let us prove (4.3). The embedding theorem yields (see, e.g.,[7], p. 66):

$$\|u(x; t)\|_{L^2(S)} \leq c \left(\varepsilon \|\nabla u(x; t)\|_{L^2(D)} + \varepsilon^{-1} \|u(x; t)\|_{L^2(D)}\right), \quad 0 < \varepsilon < 1. \tag{4.12}$$

Take $\varepsilon = t^{-\frac{1}{2}}$ and use (4.11). Then (4.12) yields (4.3). Let us prove the second inequality (4.1).

Denote $v := u(x; t) - u(x)$. Then

$$\nabla^2 v + v = 0 \text{ in } D'; \quad v \text{ satisfies (3.26)}, \tag{4.13}$$

$$v = u(x; t) \text{ on } S. \tag{4.14}$$

By Green's formula, one gets

$$v(x) = \int_S u(s; t) \frac{\partial G(s, x)}{\partial N_s} ds, \tag{4.15}$$

where $G(x, y)$ is the Green's function for the Dirichlet operator $\nabla^2 + 1$ in D'.

$$|v(x)| \leq \|u(s; t)\|_{L^2(S)} \|\frac{\partial G(s, x)}{\partial N_s}\|_{L^2(S)}. \tag{4.16}$$

From (4.16) and (4.3) the second estimate (4.1) follows.

It is clear that the function $u = u(x)$ in (4.8) is the scattering solution $u(x, \alpha)$ corresponding to the obstacle scattering problem with the Dirichlet condition on S. Indeed, $u = 0$ on S (see(4.3)), u solves equation (4.13) in D', $u - e^{i\alpha \cdot x}$ satisfies the radiation condition. These three conditions determine u uniquely, and since $u(x, \alpha)$ satisfies these conditions, it follows that $u(x, \alpha)$.

Let us prove (4.4).

If $D_1 = D_2 = D$, then formula (1.30) yields

$$|A(\alpha', \alpha; t_1, t_2)| \leq |t_2 - t_1| \, \|u_1(x, \alpha; t_1)\|_{L^2(D)} \|u_2(x, -\alpha'; t_2)\|_{L^2(D)} \leq c \frac{|t_2 - t_1|}{(t_1 t_2)^{\frac{1}{2}}}. \quad (4.17)$$

If $t_1 = t_2 = +\infty$, and $D_1 \neq D_2$, then formula (3.10) yields:

$$|A(\alpha', \alpha)| \leq c\rho(S_1, S_2). \quad (4.18)$$

In the general case a combination of the above estimates yields (4.4) (see [21]) and also the estimate

$$|A(\alpha', \alpha; t_1, t_2)| \leq c \left[|t_1 - t_2| + \rho(S_1, S_2) \right] \quad (4.19)$$

useful when $|t_1 - t_2|$ is small and $t_1, t_2 \in [1, t_0]$, where $t_0 > 1$ is fixed.

If $D_1 = D_2$, then (4.19) yields

$$|A(\alpha', \alpha; t_1, t_2)| \leq c|t_1 - t_2|. \quad (4.20)$$

5 Inverse geophysical scattering with fixed-frequency data

Consider the problem

$$\left[\nabla^2 + 1 + v(x) \right] w = -\delta(x - y) \text{ in } \mathbb{R}^3, \quad (5.1)$$

$$\lim_{r \to \infty} \int_{|S|=r} \left| \frac{\partial w}{\partial |x|} - iw \right|^2 ds = 0, \quad (5.2)$$

where $v(x) \in L^2(\mathbb{R}^3)$ is a compactly supported real-valued function with support in $\mathbb{R}^3_-$, the lower half-space. In acoustics u has the physical meaning of the pressure, $v(x)$ is the inhomogeneity in the velocity profile. We took the fixed wavenumber $k = 1$ without loss of generality. The source y is on the plane $P := \{x : x_3 = 0\}$, i.e., on the surface of the Earth, the receiver $x \in P$.

The data are the values $\{w(x, y)\}_{\forall x, y \in P}$.

The inverse geophysical scattering problem is: *given the above data, find* $v(x)$.

The uniqueness theorem for the solution to this problem is obtained in [30], [11].

Problem (5.1) -(5.2) differs from the inverse potential scattering by the source: it is a point source in (5.1) and a plane wave in (1.2). Let us show how to reduce the inverse geophysical scattering problem to inverse potential scattering problem using the "lifting" [11], [35]. Suppose the data $w(x, y)$, $x \in P$, $y \in P$, are given. Fix y and solve the problem:

$$(\nabla^2 + 1)w = 0 \text{ in } \mathbb{R}^3_+ = \{x : x_3 > 0\}, \tag{5.3}$$

$$w = w(x, y), \quad x \in P, \tag{5.4}$$

$$w \text{ satifies } (5.2). \tag{5.5}$$

This problem has a unique solution and there is a Poisson-type analytical formula for the solution to (5.3)-(5.5), since the Green's function of the Dirichlet operator $\nabla^2 + 1$ in the half-space $\mathbb{R}^3_+$ is known explicitly, analytically:

$$G_1(x, y) = \frac{e^{i|x-y|}}{4\pi|x - y|} - \frac{e^{ik|x-\overline{y}|}}{4\pi|x - \overline{y}|}, \quad \overline{y} := (y_1, y_2, -y_3). \tag{5.6}$$

Therefore the data $w(x, y)\forall x \in P$ determine uniquely and explicitly (analytically) the data $w(x, y)\forall x \in \mathbb{R}^3_+$, $y \in P$. We have lifted the data from P to $\mathbb{R}^3_+$ as far as x-dependence is concerned and similarly we can get $w(x, y)\forall x, y \in \mathbb{R}^3_+$ given $w(x, y)\forall x, y \in P$.

If $w(x, y)$ is known for all $x, y \in \mathbb{R}^3_+$, then one uses formula (1.7) and calculates analytically the scattering solution $u(x, \alpha)$ corresponding to the potential $q(x) := -v(x)$ and $k = 1$, where $\alpha \in S^2_- := \{\alpha : \alpha \in S^2, \alpha_3 \leq 0\}$. Given $u(x, \alpha)$ for all $x \in \mathbb{R}^3_+$ and $\alpha \in S^2_-$, one can calculate the scattering amplitude $A(\alpha', \alpha)$ $\forall \alpha' \in S^2_+ := \{\alpha : \alpha \in S^2, \alpha_3 \geq 0\}$.

If the scattering amplitude $A(\alpha', \alpha)$, corresponding to the compactly supported $q(x) = -v(x) \in L^2(\mathbb{R}^3)$ is known $\forall \alpha' \in S^2_+, \forall \alpha \in S^2_-$, then the uniqueness of the solution to inverse geophysical problem follows from theorem 2.1.

Stability estimates obtained for the solution to inverse potential scattering problem with fixed-energy data remain valid for the inverse geophysical problem: via the lifting process one gets the scattering amplitude $A(\alpha', \alpha)$ corresponding to the potential $q(x) = -v(x)$, and the stability estimates for $\widetilde{q}(\xi)$, obtained in sections 2.3 and 2.4, yield stability estimates for $\widetilde{v}(\xi) = -\widetilde{q}(\xi)$.

Practically, however, there are two points to have in mind. The first point is: if the noisy data $u_\delta(x, y)$ are given, where $\sup_{x,y\in P} |u_\delta(x, y) - u(x, y)| < \delta$, then one has to overcome the following difficuly in the lifting process: data $\varphi(x, y), x, y \in P$, such that $|\varphi(x, y)| < \delta$, may not decay as $|x| \to \infty$, $|y| \to \infty$ on P, and this brings the main difficulty.

The second point is: if one uses the inversion algorithms presented in sections 2.2-2.4, then one uses the data $A(\alpha', \alpha)\forall \alpha', \alpha \in S^2$. Of course, the exact data $A(\alpha', \alpha)\forall \alpha' \in S^2_+, \alpha \in S^2$, determine uniquely the data $A(\alpha', \alpha)$ $\forall \alpha', \alpha \in S^2$, but practically finding the full data from the partial data is an ill-posed problem.

6 Proofs of some estimates.

Here we prove some technical results used above: estimates (1.18), (1.19), (1.20), (1.30), (2.13), and (2.17).

6.1 Proof of (1.18).

It is sufficient to prove (1.18) with $L^2(D)$ in place of $H^2(D)$: since $w(x)$ and

$$\int_{S^2} u(x,\alpha)\nu_\varepsilon(\alpha)d\alpha$$

solve equation (1.1), the elliptic estimate (see [5])

$$\|\varphi\|_{H^2(D_1)} \leq c\left[\|L\varphi\|_{L^2(D_2)} + \|\varphi\|_{L^2(D_2)}\right], \quad D_1 \subset D_2, \tag{6.1}$$

where $L = \nabla^2 + 1 - q(x)$, D_1 is strictly inner subdomain of D_2 and $c = c(D_1, D_2) =$ const > 0, implies that $\|\varphi\|_{H^2(D_1)} \leq c\|\varphi\|_{L^2(D_2)}$ if $L\varphi = 0$. If (1.18), with $k = 1$ and $L^2(D)$ in place of $H^2(D)$, is false then

$$\int_D w(x) \int_{S^2} u(x,\alpha)\nu(\alpha)d\alpha = 0 \quad \forall \nu(\alpha) \in L^2(S^2). \tag{6.2}$$

Therefore

$$\int_D w(x)u(x,\alpha)dx = 0 \quad \forall \alpha \in S^2. \tag{6.3}$$

This implies

$$\int_D w(x)G(x,y)dx = 0 \quad \forall y \in D', \tag{6.4}$$

where $G(x,y)$ is the Green's function of the operator L satisfying the radiation condition (1.6).

Indeed, denote the integral on the left-hand side of (6.4) by $\varphi(y)$. Then

$$L\varphi = 0 \text{ in } D'; \quad \varphi = o\left(\frac{1}{|y|}\right) \text{ as } |y| \to \infty. \tag{6.5}$$

The second relation (6.5) follows from (6.3) and (1.7). From (6.5) one gets (6.4) by lemma 1 on p.25 in [16]. From (6.4) it follows that

$$L\varphi = -w(x) \text{ in } D, \quad \varphi = 0 \text{ in } D', \quad \varphi \in H^2_{loc}(\mathbb{R}^3). \tag{6.6}$$

Thus

$$L\varphi = -w \text{ in } D, \ \varphi = \varphi_N = 0 \text{ on } S. \tag{6.7}$$

Multiply (6.7) by $\overline{w}$, integrate over D, then by parts using the boundary conditions (6.7), use the equation $Lw = 0$ and get

$$\int_D |w(x)|^2 dx = 0. \tag{6.8}$$

Thus $w(x) = 0$. Estimate (1.18) is proved. □

6.2 Proof of (1.20) and (1.21).

From (1.19) one gets

$$\nabla^2 R + 2i\theta\cdot\nabla R - q(x)R = q(x) \text{ in } \mathbb{R}^3. \tag{6.9}$$

Denote $L = \nabla^2 + 2i\theta\cdot\nabla$, and define

$$w(x) := L^{-1}f = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\widetilde{f}(\xi)e^{i\xi\cdot x}}{\xi^2 + 2\xi\cdot\theta}d\xi. \tag{6.10}$$

Note that $Lw = -f(x)$. We will prove below that (see [11], [15]):

$$\|L^{-1}f\|_{L^\infty(D_1)} \le c\left(\frac{\ln|\theta|}{|\theta|}\right)^{\frac{1}{2}}\|f\|_{L^2(D)}, \quad \theta\in M, \quad |\theta|\to\infty, \tag{6.11}$$

where D_1 is an arbitrary compact domain $c = c(D_1, \|q\|_{L^2(B_a)})$, $D\subset B_a$.

We will also prove that

$$\|L^{-1}f\|_{L^2(D_1)} \le \frac{c}{|\theta|}\|f\|_{L^2(D)}, \quad |\theta|\to\infty, \quad \theta\in M. \tag{6.12}$$

Let us show that (6.11) implies existence of the special solutions (1.19).

If (6.11) and (6.12) hold, then (1.20) and (1.21) are easily derived.

Indeed, rewrite (6.9) as

$$R = L^{-1}qR + L^{-1}q. \tag{6.13}$$

From (6.11) and (6.13) it follows that

$$\|L^{-1}qR\|_{L^\infty(D)} \le c\left(\frac{\log|\theta|}{|\theta|}\right)^{\frac{1}{2}}\|qR\|_{L^2(D)} \le c\left(\frac{\log|\theta|}{|\theta|}\right)^{\frac{1}{2}}\|q\|_{L^2(D)}\|R\|_{L^\infty(D)}. \tag{6.14}$$

Therefore the operator $L^{-1}q : L^\infty(D)\to L^\infty(D)$ has the norm going to zero as $|\theta|\to\infty$, $\theta\in M$. Thus equation (6.13) is uniquely solvable in $L^\infty(D)$ if $|\theta|\gg 1$, $\theta\in M$. Moreover, the following estimate holds:

$$\|R\|_{L^\infty(D)} \le c\|L^{-1}q\|_{L^\infty(D)} \le c\left(\frac{\ln|\theta|}{|\theta|}\right)^{\frac{1}{2}}\|q\|_{L^2(D)}. \tag{6.15}$$

Estimate (1.20) follows.

To derive (1.21) from (6.12) one writes

$$\|L^{-1}qR\|_{L^2(D)} \le \frac{c}{|\theta|}\|qR\|_{L^2(D)} \le \frac{c}{|\theta|}\|q\|_{L^2(D)}\|R\|_{L^\infty(D)}. \tag{6.16}$$

Therefore (6.13), (6.15) and (6.16) yield (1.21):

$$\|R\|_{L^2(D)} \le \frac{c}{|\theta|}\|q\|^2_{L^2(D)}\left(\frac{\log|\theta|}{|\theta|}\right)^{\frac{1}{2}} + \|L^{-1}q\|_{L^2(D)} \le \frac{c}{|\theta|}. \tag{6.17}$$

Proof of (6.11). If $\theta \in M$ then $\theta = a + ib$, $a, b \in \mathbb{R}^3$, $a \cdot b = 0$, $a^2 - b^2 = 1$. Choose the coordinate system such that $a = \tau e_2$, $b = te_1$, $\tau = (1+t^2)^{\frac{1}{2}}$, e_j, $1 \le j \le 3$, are the orthonormal basis vectors. Then

$$\xi^2 + 2\theta \cdot \xi = \xi_1^2 + \xi_2^2 + \xi_3^2 + 2\tau\xi_2 + 2it\xi_1 = \xi_1^2 + (\xi_2 + \tau)^2 + \xi_3^2 - \tau^2 + 2it\xi_1. \tag{6.18}$$

This function vanishes if and only if

$$\xi_1 = 0, \quad (\xi_2 + \tau)^2 + \xi_3^2 = \tau^2. \tag{6.19}$$

Equation (6.19) defines a circle C_τ or radius τ in the plane $\xi_1 = 0$ centered at $(0, -\tau, 0)$. Let T_δ be a toroidal neighborhood of C_τ, where the section of the torus by a plane orthogonal to C_τ is a square with size 2δ and the center at C_τ.

Denote $u(x) := L^{-1}f$, where $L^{-1}f$ is defined in (6.10). One has

$$|u(x)| \le \frac{1}{(2\pi)^3}\left|\int_{T_\delta} \frac{\widetilde{f}(\xi)e^{i\xi\cdot x}d\xi}{\xi^2 + 2\xi\cdot\theta}\right| + \frac{1}{(2\pi)^3}\left|\int_{\mathbb{R}^3\setminus T_\delta} \frac{\widetilde{f}(\xi)e^{i\xi\cdot x}d\xi}{\xi^2 + 2\xi\cdot\theta}\right| := I_1 + I_2, \tag{6.20}$$

$$\begin{aligned}
I_1 &\le c\|\widetilde{f}\|_{L^\infty(\mathbb{R}^3)} \int_{T_\delta} \frac{d\xi}{|\xi^2 + 2\xi\cdot\theta|} \\
&\le c\|f\|_{L^1(\mathbb{R}^3)} \int_{-\delta}^{\delta} d\xi_1 \int_0^{2\pi} d\varphi \int_{\tau-\delta}^{\tau+\delta} \frac{\rho d\rho}{\sqrt{4t^2\xi_1^2 + (\xi_1^2 + \rho^2 - \tau^2)^2}} \\
&= c\|f\|_{L^1(\mathbb{R}^3)} \int_0^{\delta} d\xi_1 \int_{\xi_1^2 - 2\tau\delta + \delta^2}^{\xi_1^2 + 2\tau\delta + \delta^2} \frac{d\mu}{\sqrt{4t^2\xi_1^2 + \mu^2}} \\
&\le c(D)\|f\|_{L^2(D)} \int_0^{\delta} d\xi_1 \int_0^{3\tau\delta} \frac{d\mu}{\sqrt{4t^2\xi_1^2 + \mu^2}}, \quad 0 < \delta < \frac{\tau}{2},
\end{aligned} \tag{6.21}$$

where $\rho^2 = (\xi_2 + \tau)^2 + \xi_3^2$, $\mu^2 = (\xi_1^2 + \rho^2 - \tau^2)^2$, we have used the Cauchy inequality $\|f\|_{L^1(\mathbb{R}^3)} = \|f\|_{L^1(D)} \le c(D)\|f\|_{L^2(D)}$ and an elementary inequality $\xi_1^2 + 2\tau\delta + \delta^2 \le 3\tau\delta$, which holds if $\xi_1^2 \le \delta^2$ and $\tau > 2\delta$.

Let $\beta := 2t\xi_1$. Then

$$\frac{1}{2t}\int_0^{2t\delta} d\beta \int_0^{3\tau\delta} \frac{d\mu}{\sqrt{\beta^2 + \mu^2}} \le \frac{1}{2t}\int_0^{3(t+\tau)\delta} d\rho\rho \int_0^{\frac{\pi}{2}} d\varphi \frac{1}{\rho} = \frac{\pi}{4t} 3(t+\tau)\delta \le c\delta, \tag{6.22}$$

where we have used the relations $\frac{\tau}{t} \to 1$ as $t \to \infty$ and took into account that $t \to \infty$ if $|\theta| \to \infty$. From (6.21) and (6.22) one gets

$$I_1 \le c\|f\|_{L^2(D)}\delta. \tag{6.23}$$

By $c > 0$ we denote various constants independent of δ and t.

Let us estimate I_2:

$$I_2^2 \le c\|\widetilde{f}\|_{L^2(\mathbb{R}^3)} \int_{\mathbb{R}^3\setminus T_\delta} \frac{d\xi}{|\xi^2 + 2\theta\cdot\xi|^2} = c\|f\|^2_{L^2(D)}\mathcal{J}, \tag{6.24}$$

where the Parseval equality was used and by $\mathcal{J}$ the integral in (6.24) is denoted. One has

$$\mathcal{J} \le \int_{|\xi_1|>\delta} \frac{d\xi}{|\xi^2 + 2\theta \cdot \xi|^2} + \int_{|\xi_1|<\delta, |\rho-\tau|\ge\delta} \frac{d\xi}{|\xi^2 + 2\theta \cdot \xi|^2} := j_1 + j_2. \tag{6.25}$$

Let us estimate j_1:

$$\begin{aligned} j_1 &\le c \int_\delta^\infty d\xi_1 \int_0^\infty \frac{\rho d\rho}{4\xi_1^2 t^2 + (\xi_1^2 + \rho^2 - \tau^2)^2} \\ &\le c \int_\delta^\infty d\xi_1 \int_{\xi_1^2-\tau^2}^\infty \frac{d\mu}{4\xi_1^2 t^2 + \mu^2} \le c \int_\delta^\infty \frac{d\xi_1}{\xi_1 t} \left(\frac{\pi}{2} - \operatorname{arctg} \frac{\xi_1^2 - \tau^2}{2\xi_1 t} \right) \end{aligned} \tag{6.26}$$

Let $\frac{\xi_1}{2t} = x$. Then the integral on the right-hand side of (6.26) can be written as:

$$j_1 \le \frac{c}{t} \int_{\frac{\delta}{2t}}^\infty \frac{dx}{x} \left[\frac{\pi}{2} - \operatorname{arctg} \left(x - \frac{\tau^2}{4t^2} \frac{1}{x} \right) \right]. \tag{6.27}$$

If $t \to \infty$, then $\frac{\tau^2}{4t^2} \to \frac{1}{4}$. Let us use the elementary inequalities:

$$\frac{\pi}{2} - x \le \operatorname{arctg} \frac{1}{x}, \quad 0 < x \le \frac{\pi}{2}; \tag{6.28}$$

$$\operatorname{arctg} \frac{1}{x} \le \frac{\pi}{2} - \frac{x}{2}, \quad x \to +0. \tag{6.29}$$

Then

$$\frac{\pi}{2} - \frac{1}{y} \le \operatorname{arctg} y \le \frac{\pi}{2} - \frac{1}{2y}, \quad y \to +\infty. \tag{6.30}$$

Thus, with $A := \frac{\tau^2}{4t^2}$, one gets

$$\frac{1}{x} \left[\frac{\pi}{2} - \operatorname{arctg} \left(x - \frac{A}{x} \right) \right] \le \frac{1}{x} \left(x - \frac{A}{x} \right)^{-1} \le \frac{c}{x^2}, \quad x \to +\infty, \tag{6.31}$$

and

$$\frac{1}{x} \left[\frac{\pi}{2} - \operatorname{arctg} \left(x - \frac{A}{x} \right) \right] \le \frac{c}{x}, \quad x \to +0. \tag{6.32}$$

From (6.3), (6.4) and (6.27) one gets

$$j_1 \le \frac{c}{t} \left(\int_{\frac{\delta}{2t}}^1 \frac{dx}{x} + c \right) \le \frac{c}{t} |\ln \frac{\delta}{2t}|, \quad t \to +\infty,$$

so

$$j_1 \le c \frac{|\ln \frac{\delta}{2t}|}{t}, \quad t \to +\infty. \tag{6.33}$$

Let us estimate j_2:

$$j_2 = \int_{-\delta}^{\delta} d\xi_1 \int_{\substack{\rho>\tau+\delta\\ 0<\rho\le\tau-\delta}} d\rho\rho \int_0^{2\pi} d\varphi \frac{1}{4\xi_1^2t^2 + (\xi_1^2+\rho^2-\tau^2)^2}$$
$$\le c\int_0^{\delta} d\xi_1 \left\{ \int_{\xi_1^2+(\tau+\delta)^2-\tau^2}^{\infty} \frac{d\mu}{4\xi_1^2t^2+\mu^2} + \int_{2\tau\delta-\delta^2-\xi_1^2}^{\tau^2-\xi_1^2} \frac{d\nu}{4\xi_1^2t^2+\nu^2} \right\} \tag{6.34}$$

where $\mu = \xi_1^2 + \rho^2 - \tau^2$ and $\nu = \tau^2 - \rho^2 - \xi_1^2$.
One has:

$$j_2 \le \frac{c}{t}\int_0^{\delta} \frac{d\xi_1}{\xi_1} \left(\frac{\pi}{2} - \operatorname{arctg}\frac{\xi_1^2+2\delta\tau+\delta^2}{2\xi_1 t} + \operatorname{arctg}\frac{\tau^2-\xi_1^2}{2\xi_1 t} - \operatorname{arctg}\frac{2\tau\delta-\delta^2-\xi_1^2}{2\xi_1 t} \right)$$
$$\le \frac{c}{t}\int_0^{\delta} \frac{d\xi_1}{\xi_1} \left(\frac{\pi}{2} - \operatorname{arctg}\frac{\delta}{\xi_1} + \operatorname{arctg}\frac{t}{\xi_1} - \operatorname{arctg}\frac{\delta}{2\xi_1} \right), \tag{6.35}$$

where we have used the monotonicity of $\operatorname{arctg} x$, for example,

$$\operatorname{arctg}\frac{\xi_1^2+2\delta\tau+\delta^2}{2\xi_1 t} \ge \operatorname{arctg}\frac{\delta}{\xi_1},$$

etc., and the relation $\frac{\tau}{t} \to 1$ as $t \to +\infty$, $\tau > t$.
By (6.28),

$$\frac{\pi}{2} - \operatorname{arctg}\frac{\delta}{\xi_1} \le \frac{\xi_1}{\delta}, \quad \operatorname{arctg}\frac{t}{\xi_1} \le \frac{\pi}{2} - \frac{\xi_1}{2t}, \quad \operatorname{arctg}\frac{\delta}{2\xi_1} \ge \frac{\pi}{2} - \frac{2\xi_1}{\delta}. \tag{6.36}$$

From (6.35) and (6.36) one gets:

$$j_2 \le \frac{c}{t}\int_0^{\delta} \frac{d\xi_1}{\xi_1} \left(\frac{\xi_1}{\delta} - \frac{\xi_1}{2t} + \frac{2\xi_1}{t} \right) = \frac{c}{t}\left(1 - \frac{\delta}{2t} + \frac{2\delta}{t}\right) = \frac{c}{t}\left(1 + \frac{3\delta}{2t}\right). \tag{6.37}$$

Thus

$$j_2 \le \frac{c}{t}, \quad t \to +\infty. \tag{6.38}$$

From (6.20), (6.23), (6.25), (6.33) and (6.37) one gets:

$$|u(x)| \le c\|f\|_{L^2(D)} \left[\delta + \left(\frac{|\ln\frac{\delta}{2t}|}{t} \right)^{\frac{1}{2}} + \frac{1}{t^{\frac{1}{2}}} \right]. \tag{6.39}$$

Choose $\delta = \frac{1}{t}$. Then (6.39) yields

$$|u(x)| \le c\|f\|_{L^2(D)} \left(\frac{\ln|\theta|}{|\theta|} \right)^{\frac{1}{2}}, \quad |\theta| \to \infty, \quad \theta \in M. \tag{6.40}$$

Estimate (6.11) is proved. $\square$

Let us prove (6.12).

Let $L(\xi) = \xi^2 + 2\theta \cdot \xi$, $\partial = -i\nabla$.

Define $\mathcal{L}(\xi) := \left(\sum_{|j|\geq 0} \left|L^{(j)}(\xi)\right|^2\right)^{\frac{1}{2}}$. Then

$$\mathcal{L}(\xi) = (|\xi^2 + 2\theta \cdot \xi|^2 + 4|\xi + \theta|^2 + 36)^{\frac{1}{2}} \geq |Im\theta| + 3.$$

In [6], vol. 2, p. 31, it is proved that $\|L^{-1}f\|_{L^2(D_1)} \leq \frac{1}{\min_\xi |\mathcal{L}(\xi)|}\|f\|_{L^2(D)}$. Therefore

$$\begin{aligned}\|L^{-1}f\|_{L^2(D_1)} \leq & \frac{c}{\min_{\xi\in\mathbb{R}^3} |\mathcal{L}(\xi)|}\|f\|_{L^2(D)} \\ \leq & \frac{c}{|\theta|}\|f\|_{L^2(D)}, D \subset D_1, \quad \theta \in M, \quad |\theta| \to \infty,\end{aligned} \tag{6.41}$$

where $c = c(D_1, D) > 0$ is a constant and we have used the relation

$$c_1|\theta| \leq |Im\theta| \leq |\theta|, \quad c_1 > 0, \text{ if } \theta \in M, \quad |\theta| \to \infty.$$

Estimate (6.41) is identical to (6.12). $\square$

6.3 Proof of (2.17).

If ρ is defined by (2.4), where $u(x, \alpha)$ solves (1.1) then ρ solves the equation

$$\nabla^2\rho + 2i\theta \cdot \nabla\rho - q(x)\rho = q(x) \text{ in } \mathbb{R}^3, \quad \theta \in M. \tag{6.42}$$

Let $h = |\theta|^{-1}$, $h \to 0$, $\delta(\xi) := \xi^2 + 2\beta \cdot \xi$, $\beta = h\theta$, $\beta \cdot \beta = h^2$, $|\beta| = 1$,

$$N := \{\xi : \delta(\xi) = 0, \quad \xi \in \mathbb{R}^3\},\ N_h := \{\xi : dist(\xi, N) \leq h, \xi \in \mathbb{R}^3\},\ N_h' := \mathbb{R}^3 \backslash N_h,$$

$P = P_1 + iP_2$, $P_1 = ReP$. Note that $dP_1 \neq 0$ on N, where dP_1 is the differential of P_1.

Define

$$F_h u := \widehat{u} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} u(x)e^{-i\xi\cdot xh^{-1}}dx. \tag{6.43}$$

Then

$$F_h(-ih\partial_j u(x)) = \xi_j\widehat{u}(\xi); \quad ih\partial_{\xi_j}\widehat{u}(\xi) = \widehat{x_j u}. \tag{6.44}$$

Denote

$$\|\rho\|_a := \|\rho\|_{L^2(B_a)}, \quad \|\rho\| := \|\rho\|_{L^2(\mathbb{R}^3)}, \quad \|\rho\|_{a,b} = \|\rho\|_{L^2(B_a \backslash B_a)}, \quad b > a, \tag{6.45}$$

$$\|g(< hD >)\rho\| := \|g(\sqrt{1 + \xi^2})\widehat{\rho}(\xi)\|, \quad D = -i\nabla. \tag{6.46}$$

The following Hardy-type inequality will be useful:

If $f(t) \in C^1(-h,h)$, $f(0)=0$, then

$$\int_{-h}^{h} t^{-2}|f(t)|^2 dt \le 4\int_{-h}^{h} |f'(t)|^2 dt, \quad h>0. \tag{6.47}$$

Let us sketch the basic steps of the proof of (2.17)

Step1. If $\rho \in C_0^2(B_r)$ and

$$P(hD)\rho := (hD)^2\rho + 2\beta \cdot hD\rho = -h^2 v, \quad v \in L_0^2(B_r), \tag{6.48}$$

where $L_0^2(B_r)$ is the set of $L^2(B_r)$ functions with compact support in the ball B_r, then

$$h\| < hD >^2 \rho\| \le c\|P(hD)\rho\|, \qquad \forall h \in (0,h_0), \tag{6.49}$$

where $h_0 > 0$ is a sufficiently small number.

Step 2. Let A_1 be a bounded domain with a smooth boundary and $A \subset A_1$, $\eta \in C_0^\infty(A_1)$, $0 \le \eta \le 1$, $\eta(x) = 1$ in A, A is a strictly inner subdomain of A_1.

Claim. *If*

$$P(hD)\rho = 0 \quad \text{in} \quad A_1, \tag{6.50}$$

then

$$h\|D\rho\|_{L^2(A)} \le c\|\rho\|_{L^2(A_1)}. \tag{6.51}$$

Step 3. Write (6.42) as

$$P(hD)\rho = -h^2(q\rho + q). \tag{6.52}$$

Let

$$\eta \in C_0^\infty(B_b), \quad 0 \le \eta(x) \le 1,$$
$$\eta(x) = 1 \quad x \in B_{a_1}, \quad a < a_1 < b.$$

Then

$$P(\eta\rho) = (P\eta - \eta P)\rho - h^2\eta(q\rho + q), \quad P = P(hD). \tag{6.53}$$

Apply (6.49) to (6.53) and get

$$\begin{gathered} h\| < hD >^2 (\rho\eta)\| \le c\|(P\eta - \eta P)\rho\| + \\ ch^2\|q\|_{L^\infty(B_a)}\|\rho\|_{L^2(B_a)} + ch^2\|q\|_{L^2(B_a)}. \end{gathered} \tag{6.54}$$

Since $\eta = 1$ in B_a, one gets:

$$h\|\rho\|_a \le h\| < hD >^2 (\eta\rho)\| \le ch^2\|\rho\|_a + ch^2 + c\|(P\eta - \eta P)\rho\|. \tag{6.55}$$

So

$$\|\rho\|_a \le ch + ch^{-1}\|(P\eta - \eta P)\rho\|. \tag{6.56}$$

Since $D\eta = 0$ in B_a one gets:

$$\begin{aligned} \| \left(P\eta - \eta P\right) \rho\| &= \|\rho(hD)^2\eta + 2h^2 D\eta \cdot D\rho + 2h\rho\beta \cdot D\eta\| \\ &\leq c(h^2 + h)\|\rho\|_{a_1,b} + ch^2\|D\rho\|_{a_1,b}. \end{aligned} \tag{6.57}$$

Using (6.57), one gets

$$h\|D\rho\|_{a_1,b} \leq c\|\rho\|_{a_1-\varepsilon,b+\varepsilon}. \tag{6.58}$$

From (6.58), (6.57) and (6.55) one obtains:

$$\|\rho\|_a \leq c\left(h + \|\rho\|_{a_1-\varepsilon,b+\varepsilon}\right). \tag{6.59}$$

Since $\varepsilon > 0$ is arbitrarily small, the desired inequality (2.17) follows. □

To complete the proof one has to prove (6.49) and (6.51).

6.4 Proof of (6.49).

Write (6.49), using Parseval's equality, as

$$h\|(1 + |\xi|^2)\widehat{\rho}\| \leq c\|P(\xi)\widehat{\rho}\|. \tag{6.60}$$

If $\xi \in N'_h$, then $h(1 + |\xi|^2) \leq c|P(\xi)|$, so

$$\begin{aligned} &h^2 \int_{N'_h} (1 + |\xi|^2)^2 |\widehat{\rho}(\xi)|^2 d\xi \leq c^2 \int_{N'_h} |P(\xi)|^2 |\widehat{\rho}(\xi)|^2 d\xi \\ &\leq c^2 \int_{\mathbb{R}^3} |P(\xi)\widehat{\rho}(\xi)|^2 d\xi = c^2 \int_{\mathbb{R}^3} |P(hD)\rho|^2 dx. \end{aligned} \tag{6.61}$$

If $\xi \in N_h$, then use the local coordinates in which the set N is defined by the equations:

$$t = 0, \quad \xi_1 = 0, \quad t = P_1(\xi), \tag{6.62}$$

and the ξ_1-axis is along vector μ defined by the equation $\beta = m + i\mu$. Since $dP_1 \neq 0$ on N, these local coordinates can be defined.

Put $f := P_1(\xi)\widehat{\rho}(\xi)$. Then $f = 0$ at $t = 0$, $f \in C^\infty(\mathbb{R}^3)$ since $\rho(x)$ was assumed in **Step 1** to have compact support, and (6.47) yields:

$$\int_{-h}^{h} |\widehat{\rho}(\xi)|^2 dt \leq 4 \int_{-h}^{h} |f'_t|^2 dt. \tag{6.63}$$

Integrating (6.63) over the remaining variables, one gets:

$$\int_{N_h} |\widehat{\rho}(\xi)|^2 d\xi \leq c \int_{N_h} \left|\nabla_\xi \left(P_1(\xi)\widehat{\rho}(\xi)\right)\right|^2 d\xi \leq c \int_{\mathbb{R}^3} \left|\nabla_\xi \left(P_1(\xi)\widehat{\rho}(\xi)\right)\right|^2 d\xi. \tag{6.64}$$

Since N_h is compact, one has

$$h^2 \int_{N_h} (1+|\xi|^2)^2 |\widehat{\rho}(\xi)|^2 d\xi \leq ch^2 \int_{N_h} |\widehat{\rho}(\xi)|^2 d\xi. \tag{6.65}$$

Using Parseval's equality, S. Bernstein's inequality for the derivative of entire functions of exponential type, and the condition $\operatorname{supp} \rho(x) \subset B_r$, one gets:

$$\begin{aligned} h^2 \int_{\mathbb{R}^3} \left| \nabla_\xi \left(P_1(\xi) \widehat{\rho}(\xi) \right) \right|^2 d\xi = \int_{\mathbb{R}^3} |x|^2 |P_1(hD)\rho(x)|^2 dx = r^2 \int_{B_r} \left| P_1(hD)\rho(x) \right|^2 dx \\ \leq r^2 \int_{\mathbb{R}^3} \left| P(hD)\rho(x) \right|^2 dx. \end{aligned} \tag{6.66}$$

From (6.64)-(6.66) it follows that

$$h^2 \int_{N_h} (1+|\xi|^2)^2 |\widehat{\rho}(\xi)|^2 d\xi \leq c \int_{\mathbb{R}^3} |P(hD)\rho(x)|^2 dx. \tag{6.67}$$

From inequalities (6.61) and (6.67) it follows that inequality (6.49) is proved. □

6.5 Proof of (6.51).

Multiply (6.50) by $\eta\overline{\rho}$, take the real part and integrate by parts to get:

$$\begin{aligned} h \int_{A_1} \eta |\nabla \rho|^2 dx &= -\frac{h}{2} \int_{A_1} \left(\overline{\rho} \nabla \rho + \rho \nabla \overline{\rho} \right) \nabla \eta dx + 2Re \left(i\beta_j \int_{A_1} \rho_j \overline{\rho} \eta dx \right) \\ &= \frac{h}{2} \int_{A_1} |\rho|^2 \nabla^2 \eta dx + 2Re \left(i\beta_j \int_{A_1} \rho_j \overline{\rho} \eta dx \right), \end{aligned} \tag{6.68}$$

where $\rho_j := \frac{\partial \rho}{\partial x_j}$ and summation is done over the repeated indices.

One has

$$|\nabla^2 \eta| \leq c, \quad |\beta_j| \leq 1, \quad |2\rho_j \overline{\rho}| \leq \frac{h}{2} |\rho_j|^2 + \frac{2}{h} |\rho|^2. \tag{6.69}$$

From (6.69) and (6.68) one gets:

$$h \int_{A_1} \eta |\nabla \rho|^2 dx \leq ch \int_{A_1} |\rho|^2 dx + \frac{h}{2} \int_{A_1} \eta |\nabla \rho|^2 dx + \frac{2}{h} \int_{A_1} \eta |\rho|^2 dx. \tag{6.70}$$

Thus

$$h^2 \int_{A} |\nabla \rho|^2 dx \leq h^2 \int_{A_1} \eta |\nabla \rho|^2 dx \leq c \int_{A_1} |\rho|^2 dx. \tag{6.71}$$

Inequality (6.51) is proved. □

Let us prove that

$$\|\psi(x,\theta) - \int_{S^2} u(x,\alpha) \nu_\varepsilon(\alpha) d\alpha \|_{L^2(D)} \leq \varepsilon, \quad \theta \in M, \quad |\theta| \to \infty, \tag{6.72}$$

implies

$$\|\nu_\varepsilon\|_{L^2(S^2)} \geq ce^{\frac{\kappa d}{2}}, \quad \kappa = |Im\theta|, \quad d = diamD, \quad |\theta| \to +\infty. \tag{6.73}$$

Indeed, (6.72), (1.19) and (1.20) imply:

$$\|\int_{S^2} u(x,\alpha)\nu_\varepsilon(\alpha)d\alpha\|_{L^2(D)} \geq \|\psi(x,\theta)\|_{L^2(D)} - \varepsilon \geq ce^{\frac{\kappa d}{2}}, \quad \theta \in M, \quad |\theta| \gg 1. \tag{6.74}$$

If (6.73) is false for some $\varepsilon > 0$, then there is a sequence $\theta_n \in M$, $|\theta_n| \to \infty$, such that

$$\|\nu_\varepsilon\|_{L^2(S^2)} e^{-\frac{\kappa_n d}{2}} \to 0, \quad n \to \infty. \tag{6.75}$$

This contradicts (6.74) since (6.75) implies

$$\|\int_{S^2} u(x,\alpha)\nu_\varepsilon(\alpha)d\alpha\|_{L^2(D)} \leq c\|\nu_\varepsilon(\alpha)\|_{L^2(S^2)} = o\left(e^{\frac{\kappa_n d}{2}}\right) \text{ as } n \to \infty. \tag{6.76}$$

Therefore estimate (6.73) is proved. □

6.6 Proof of (2.13).

One has

$$\int_{S^2} u(x,\alpha)\nu(\alpha)d\alpha = e^{i\theta\cdot x}(1+\rho),$$

where

$$\rho := e^{-i\theta\cdot x}\int_{S^2} u(x,\alpha)\nu(\alpha)d\alpha - 1,$$

$$\psi(x,\theta) = e^{i\theta\cdot x}(1+R), \quad \|R\|_{L^2(B_{b_1})} \leq \frac{c}{|\theta|}, \quad \theta \in M, |\theta| \gg 1,$$

where $b_1 > b$.

By (1.18), there exist a $\nu(\alpha)$ such that

$$\|e^{i\theta\cdot x}(1+\rho) - e^{i\theta\cdot x}(1+R)\|_{L^2(B_{b_1})} \leq \frac{e^{-\kappa b_1}}{\kappa}, \quad \kappa = |Im\theta|.$$

Therefore

$$\|(\rho - R)e^{i\theta\cdot x}\|_{L^2(B_{b_1})} \leq \frac{e^{-\kappa b_1}}{\kappa},$$

so that

$$e^{-\kappa b_1}\|\rho - R\|_{L^2(B_{b_1})} \leq \frac{e^{-\kappa b_1}}{\kappa},$$

and

$$\|\rho - R\|_{L^2(B_{b_1})} \leq \frac{1}{\kappa}.$$

This implies
$$\|\rho\|_{L^2(B_{b_1})} \le \|\rho - R\|_{L^2(B_{b_1})} + \|R\|_{L^2(B_{b_1})} \le \frac{c}{|\theta|}.$$
Thus, inequality (2.13) follows. $\Box$

We claim that $\|\rho\|_{L^2(B_b)}$ is of the order $O(\frac{1}{|\theta|})$.

Using the above inequalities, one gets:
$$e^{-b\kappa}\|\rho - R\|_{L^2(B_b)} \le \|(\rho - R)e^{i\theta\cdot x}\|_{L^2(B_b)} \le \|(\rho - R)e^{i\theta\cdot x}\|_{L^2(B_{b_1})} \le \frac{e^{-\kappa b_1}}{\kappa}.$$
Thus
$$\|\rho - R\|_{L^2(B_b)} \le \frac{e^{-(b_1-b)\kappa}}{\kappa}.$$
Recall that $c_1|\theta| \le \kappa \le |\theta|$, $0 < c_1 < \frac{1}{2}$, as $|\theta| \to \infty$, $\theta \in M$. Therefore,
$$\|\rho\|_{L^2(B_b)} \ge \|R\|_{L^2(B_b)} - \|\rho - R\|_{L^2(B_b)} \ge \frac{c}{|\theta|} - \frac{e^{-\gamma\kappa}}{\kappa}, \quad \gamma = b_1 - b > 0.$$

Thus, the above claim is verified, since, as $|\theta| \to \infty$, $\theta \in M$, one has $\frac{|\theta|}{\kappa} \to \sqrt{2}$ and $\frac{e^{-\gamma\kappa}}{\kappa} = o\left(\frac{1}{|\theta|}\right)$. $\Box$

Uniqueness class for the solution to the equation $L\rho = 0$.
$$L\rho := (\nabla^2 + 2i\theta\cdot\nabla)\rho = 0 \text{ in } \mathbb{R}^3, \quad \int_{\mathbb{R}^3} |\rho(x)|^2(1+|x|^2)^\ell dx < \infty, \quad -1 < \ell < 0. \tag{6.77}$$
Taking the distributional Fourier transform of (6.77) one gets:
$$L(\xi)\widetilde{\rho} = (\xi^2 + 2\theta\cdot\xi)\widetilde{\rho} = 0. \tag{6.78}$$
Thus $\operatorname{supp}\widetilde{\rho} = C_\tau := \{\xi : \xi \in \mathbb{R}^3, \quad L(\xi) = 0\}$, and C_τ is the circle (6.19). By theorem 7.1.27 in [[6], vol 1, p.174] one has:
$$\int_{C_\tau} |\widetilde{\rho}|^2 ds \le c \lim_{R\to\infty}\sup\left(\frac{1}{R^2}\int_{|x|\le R} |\rho(x)|^2 dx\right). \tag{6.79}$$
Using (6.77) we derive for $-1 < \ell < 0$ and large R the following estimates:
$$\begin{aligned}\infty > c > \int_{\mathbb{R}^3} |u|^2(1+|x|^2)^\ell dx &\ge \int_{|x|\le R} \frac{|u|^2 dx}{(1+|x|^2)^{|\ell|}} \\ &\ge \frac{1}{(1+R^2)^{|\ell|}}\int_{|x|\le R} |u|^2 dx \ge \frac{c}{R^{2|\ell|}}\int_{|x|\le R}|u|^2 dx.\end{aligned} \tag{6.80}$$
Combining (6.79) and (6.80) one gets
$$\int_{C_\tau} |\widetilde{\rho}|^2 ds \le c \lim_{R\to\infty}\sup\frac{R^{2|\ell|}}{R^2} = 0, \quad |\ell| < 1.$$
Thus $\widetilde{\rho}(\xi) = 0$, as claimed. $\Box$

Estimate (6.80) is valid in $\mathbb{R}^n$, $n \ge 2$. It was used in [54] and [11].

6.7 Proof of (2.23)

Let $\|\nu_\varepsilon\| := \|\nu_\varepsilon\|_{L^2(S^2)}$ and $m(\varepsilon, \theta) := \inf \|\nu\|$ where the infimum is taken over all $\nu \in L^2(S^2)$ such that (6.72) holds with $\varepsilon = \frac{e^{-b\kappa}}{\kappa}$.

We prove the following estimate:

$$m(\varepsilon, \theta) \leq ce^{c|\theta| \ln |\theta|} \text{ as } |\theta| \to \infty, \quad \theta \in M, \quad \varepsilon = \frac{e^{-b\kappa}}{\kappa}, \quad b > a, \tag{6.81}$$

where $\theta \in M$, $|\theta| \to \infty$, $\kappa = |Im\theta|$, and $c > 0$ stands for various constants.

Let us describe the steps of the proof.

Step 1. Prove the estimate

$$m(\varepsilon, \theta) \leq ce^{\kappa r} \left(\frac{2n(\varepsilon)}{er} \right)^{n(\varepsilon)} n^2(\varepsilon), \quad r \geq b, \quad \theta \in M, \quad \varepsilon > 0, \tag{6.82}$$

where

$$\ln(n(\varepsilon)) = \ln(|\ln \varepsilon|)[1 + o(1)], \quad \varepsilon \to +0. \tag{6.83}$$

The choice of $n(\varepsilon)$ in (6.83) is justified below (see (6.97)) and estimate (6.82) is proved also below.

Step 2. Minimize the right-hand side of (6.82) with respect to $r \geq b$ to get

$$m(\varepsilon, \theta) \leq c(2\kappa)^{n(\varepsilon)} n^2(\varepsilon). \tag{6.84}$$

The minimizer is $r = \frac{n(\varepsilon)}{\kappa}$.

Step 3. Take $\varepsilon = \varepsilon(\theta) = \frac{e^{-b\kappa}}{\kappa}$, $\kappa \to \infty$, in (6.84). Then

$$\ln n = \ln(\kappa b + \ln \kappa)[1 + o(1)] = (\ln \kappa) \left[1 + O\left(\frac{1}{\ln \kappa} \right) \right], \quad \kappa \to +\infty, \tag{6.85}$$

so, for $\varepsilon = \frac{e^{-b\kappa}}{\kappa}$ one has:

$$c_1 \kappa \leq n \leq c_2 \kappa, \quad \kappa \to +\infty, \quad c_1 > 0. \tag{6.86}$$

From (6.84) and (6.85) one gets:

$$m(\theta) = m(\varepsilon(\theta), \theta) \leq ce^{c|\theta| \ln |\theta|}, \quad |\theta| \to \infty, \quad \theta \in M. \tag{6.87}$$

Estimate (6.81) is obtained. $\square$

Proof of (6.82). Since $u(x, \alpha) = (I + T_1)e^{i\alpha \cdot x}$ where $I + T_1 := (I + T)^{-1}$ is a bijection of $C(B_b)$ onto $C(B_b)$, inequality (6.72) with $D = B_b$ is equivalent to

$$\|(I + T_1)^{-1}\psi - \int_{S^2} e^{i\alpha \cdot x} \nu_\varepsilon(\alpha) d\alpha\|_{L^2(B_b)} \leq c\varepsilon, \tag{6.88}$$

where $c =$ const > 0 does not depend on ε and θ,

$$(I+T_1)^{-1}\psi = (I+T)\psi, \qquad T\psi = \int_{B_a} \frac{e^{i|x-y|}}{4\pi|x-y|} q(y)\psi(y)dy.$$

We take $b > a$, therefore the function

$$\varphi := \psi + T\psi,$$

has the maximal values, as $|\theta| \to \infty$, of the same order of magnitude as the function ψ, that is, as the function $e^{i\theta\cdot x}$. The function φ solves the equation

$$\left(\nabla^2 + 1\right)\varphi = 0 \text{ in } \mathbb{R}^3. \tag{6.89}$$

Indeed,

$$(\nabla^2+1)\varphi = (\nabla^2+1)\psi - q\psi = q\psi - q\psi = 0,$$

as claimed.

Therefore, one can write:

$$\varphi := \varphi(x,\theta) = \sum_{\ell=0}^{\infty} 4\pi i^\ell \varphi_\ell Y_\ell(\alpha')j_\ell(r), \quad r=|x|, \quad \alpha' = \frac{x}{|x|}, \tag{6.90}$$

where Y_ℓ are defined in (1.26), $j_\ell(r)$ are defined in (1.29), and $\varphi_\ell = \varphi_\ell(\theta)$ are some coefficients.

It is known that

$$e^{i\alpha\cdot x} = \sum_{\ell=o}^{\infty} 4\pi i^\ell \overline{Y_\ell(\alpha)} Y_\ell(\alpha')j_\ell(r), \quad \alpha' = \frac{x}{r}, \tag{6.91}$$

so

$$\int_{S^2} e^{i\alpha\cdot x}\nu_\varepsilon(\alpha)d\alpha = \sum_{\ell=0}^{\infty} 4\pi i^\ell \nu_{\varepsilon\ell} Y_\ell(\alpha')j_\ell(r), \tag{6.92}$$

where $\nu_{\varepsilon\ell} = (\nu_\varepsilon, Y_\ell)_{L^2(S^2)}$.

Choose

$$\nu_{\varepsilon\ell} = \varphi_\ell \quad \text{for} \quad \ell \le n(\varepsilon), \quad \nu_{\varepsilon\ell} = 0 \quad \text{for} \quad \ell > n(\varepsilon), \tag{6.93}$$

where $n(\varepsilon)$ is the same as in (6.83).

Then (6.88) implies:

$$\begin{aligned} &\|\varphi - \int_{S^2} e^{i\alpha\cdot x}\nu_\varepsilon(\alpha)d\alpha\|^2_{L^2(B_b)} \\ &= \sum_{\ell=n(\varepsilon)+1}^{\infty} 16\pi^2 \int_0^b r^2|j_\ell(r)|^2dr|\varphi_\ell|^2 \approx c \sum_{\ell=n(\varepsilon)+1}^{\infty} \frac{1}{\ell^2}|\varphi_\ell|^2 \left(\frac{eb}{2\ell}\right)^{2\ell} < c\varepsilon, \end{aligned} \tag{6.94}$$

where formula (1.29) was used and it is assumed that $n(\varepsilon) \gg 1$.

From (6.93) and formulas (6.90) and (6.91) with $\alpha = \theta \in M$, one gets:

$$\|\nu_\varepsilon\|^2 = \sum_{\ell=0}^{n(\varepsilon)} |\varphi_\ell|^2 \le c \sum_{\ell=0}^{n(\varepsilon)} \sum_{m=-\ell}^{\ell} |Y_\ell(\theta)|^2 \le cn^2(\varepsilon) \frac{e^{2\kappa r}}{|j_{n(\varepsilon)}(r)|^2}, \quad \forall r > 0, \tag{6.95}$$

where we have used the formula

$$\sum_{\ell=0}^{n} \sum_{m=-\ell}^{\ell} 1 = (n+1)^2,$$

we estimated $|\varphi_\ell|$ as the coefficient

$$|(e^{i\theta\cdot x}, Y_\ell)_{L^2(S^2)}|^2 = 16\pi^2 |Y_\ell(\theta)|^2$$

of the main term of φ, that is, the function $e^{i\theta\cdot x}$, we have used estimate (1.28), which gives

$$|\varphi_\ell|^2 \le c \frac{e^{2\kappa r}}{|j_\ell(r)|^2}, \qquad \forall r > 0,$$

and we replaced $|j_\ell(r)|$ by $|j_{n(\varepsilon)}(r)|$, the smaller quantity.

Choose $r > b$ and use (1.29) to get the inequality:

$$\sum_{\ell=n(\varepsilon)+1}^{\infty} |\varphi_\ell|^2 \left(\frac{eb}{2\ell}\right)^{2\ell} \le \sum_{\ell=n(\varepsilon)+1}^{\infty} e^{2\kappa r} \left(\frac{b}{r}\right)^{2\ell} \le c_1 e^{2\kappa r} \left(\frac{b}{r}\right)^{2n(\varepsilon)} < c\varepsilon, \quad r > b, \tag{6.96}$$

which implies (6.94). Thus (6.94) holds if

$$e^{\kappa r} \left(\frac{b}{r}\right)^{n(\varepsilon)} \le c\sqrt{\varepsilon},$$

where c stands for various constants. One has

$$\min_{r>b} e^{\kappa r} \left(\frac{b}{r}\right)^n = e^n \left(\frac{b\kappa}{n}\right)^n$$

and the minimizer is

$$r = \frac{n}{\kappa}.$$

Consider therefore the equation

$$e^n \left(\frac{b\kappa}{n}\right)^n = c\sqrt{\varepsilon}$$

and solve it asymptotically for $n = n(\varepsilon)$ as $\varepsilon \to 0$, where $\kappa > 1$ is arbitrary large but fixed. Taking logarithm, one gets

$$\ln c - \frac{1}{2}\ln\frac{1}{\varepsilon} = n - n\ln n + n\ln(b\kappa).$$

Thus

$$|\ln\varepsilon| = \ln\frac{1}{\varepsilon} = 2n\ln n[1 + o(1)],$$

and

$$\ln|\ln\varepsilon| = (\ln n)(1 + o(1)), \quad \varepsilon \to +0. \tag{6.97}$$

Hence, we have justified formula (6.83).

From (6.94), (6.96) and (1.29), one gets

$$\|\nu_\varepsilon\| \le cn^2(\varepsilon)\frac{e^{\kappa r}(2n(\varepsilon))^{n(\varepsilon)}}{(er)^{n(\varepsilon)}} \quad \forall r > b, \quad \kappa = |Im\theta|, \quad \theta \in M. \tag{6.98}$$

Estimate (6.82) is established. □

6.8 Proof of (1.30).

Let G_j be the Green's function corresponding to $q_j(x)$, $j = 1, 2$. By Green's formula one gets

$$G_2(x, y) - G_1(x, y) = \int_{B_a} p(z)G_1(x, z)G_2(z, y)dz, \quad p := q_1(x) - q_2(x). \tag{6.99}$$

Take $|y| \to \infty$, $\frac{y}{|y|} = -\alpha$ and use (1.7) to get:

$$u_2(x, \alpha) - u_1(x, \alpha) = \int_{B_a} p(z)G_1(x, z)u_2(z, \alpha)dz. \tag{6.100}$$

Take $|x| \to \infty$, $\frac{x}{|x|} = \alpha'$, use (1.7) and (1.2) and get:

$$A_2(\alpha', \alpha) - A_1(\alpha', \alpha) = \frac{1}{4\pi}\int_{B_a} u_1(z, -\alpha')u_2(z, \alpha)p(z)dz. \tag{6.101}$$

Since $A(\alpha', \alpha) = A(-\alpha, -\alpha')$, formula (6.101) is equivalent to (1.30). □

7 Construction of the Dirichlet-to-Neumann map from the scattering data and vice versa.

Consider a ball $B_a \supset D = \operatorname{supp} q(x)$ and assume that the problem

$$\left[\nabla^2 + 1 - q(x)\right] w = 0 \text{ in } B_a, \quad w = f \text{ on } S_a := \partial B_a, \tag{7.1}$$

is uniquely sovable for any $f \in H^{\frac{3}{2}}(S_a)$, where $H^{\ell}(S_a)$ is the Sobolev space.

Then the $D-N$ map is defined as

$$\Lambda : f \to w_N \tag{7.2}$$

where w_N is the normal derivative of w on S_a, N is the normal to S_a pointing into $B'_a := \mathbb{R}^3 \backslash B_a$.

If Λ is known, then $q(x)$ can be found as follows.

The special solution (1.19)-(1.22) satisfies the equation:

$$\psi(x) = e^{i\theta\cdot x} - \int_{B_a} G(x-y)q(y)\psi(y)dy, \tag{7.3}$$

where

$$G(x) := e^{i\theta\cdot x} G_0(x)$$

and

$$\nabla^2 G(x) + G(x) = -\delta(x), \quad in \quad \mathbb{R}^3.$$

Thus

$$\nabla^2 G_0 + 2i\theta \cdot \nabla G_0 = -\delta(x), \tag{7.4}$$

so that $G_0(x-y)$ is the Green's function of the operator L, see (6.10), that is

$$G_0(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\xi\cdot x} d\xi}{\xi^2 + 2\xi \cdot \theta}. \tag{7.5}$$

The function $G(x)$ can be considered known.

Since $q\psi = (\nabla^2 + 1)\psi$, one can write, for $x \in B'_a$,

$$\begin{aligned}
&\int_{B_a} G(x-y)q(y)\psi(y)dy = \int_{B_a} G(\nabla^2+1)\psi dy \\
&= \int_{S_a} \left[G(x-s)\psi_N(s) - G_N(x-s)\psi(s)\right] ds = \int_{S_a} G(x-s)(\Lambda - \Lambda_0)\psi(s)ds \\
&+ \int_{S_a} \left[G(x-s)\Lambda_0\psi - G_N(x-s)\psi\right] ds = \int_{S_a} G(x-s)(\Lambda-\Lambda_0)\psi(s)ds.
\end{aligned} \tag{7.6}$$

Here Λ_0 is Λ for $q(x) = 0$, we have used Green's formula and took into account that

$$\int_{S_a} \left[G(x-s)\Lambda_0\psi - G_N(x-s)\psi\right] ds = \int_{B_a} \left[G(\Delta+1)\varphi - \varphi(\Delta+1)G\right] dy = 0,$$

where φ solves problem (7.1) with $q(x) = 0$ and $\varphi = f$ on S_a.

From (7.3) and (7.6) taking $x \to s \in S_a$ one gets a linear Fredholm- type equation for $\psi|_{S_a}$:

$$\psi(s) = e^{i\theta\cdot s} - \int_{S_a} G(s-s')(\Lambda - \Lambda_0)\psi(s')ds'. \tag{7.7}$$

If Λ is known, one can find from (7.7) $\psi|_{S_a}$ and then find $q(x)$ using the following calculation.

Define

$$t(\theta', \theta) := \int_{B_a} e^{-i\theta'\cdot y} q(y)\psi(y, \theta)dy. \tag{7.8}$$

By Green's formula, as in (7.6), one gets

$$t(\theta', \theta) = \int_{S_a} e^{-i\theta'\cdot s}(\Lambda - \Lambda_0)\psi(s, \theta)ds. \tag{7.9}$$

From (7.8) one gets, using (1.19), (1.20) and (1.9):

$$\lim_{\substack{|\theta| \to \infty \\ \theta' - \theta = \xi \\ \theta \in M}} t(\theta', \theta) = \int_{B_a} e^{-i\xi\cdot x} q(x)dx := \widetilde{q}(\xi). \tag{7.10}$$

Therefore the knowledge of Λ allows one to recover $\widetilde{q}(\xi)$ by formula (7.10), but first one has to solve equation (7.7). We leave to the reader to check that the homogeneous equation (7.7) has only the trivial solution so that Fredholm-type equation (7.7) is uniquely solvable in $L^2(S_a)$ (see a proof in [33]).

Practically, however, there are essential difficulties: a) the function $G(x, y)$ is not known, analytically and it is difficult to solve equation (7.7) by this reason, b) the $D-N$ map is not given analytically as well.

Let us show how to construct Λ *from the scattering amplitude* $A(\alpha', \alpha)$ *and vice versa.*

If Λ is given then we have shown how to find $q(x)$ and if $q(x)$ is found then $A(\alpha', \alpha)$, the scattering amplitude, can be found.

Conversely, suppose $A(\alpha', \alpha)$ *is known and show how* Λ *can be found..*

If $A(\alpha', \alpha)$ is given, then the scattering solution can be calculated in B'_a by formula (1.31).

Let $f \in H^{\frac{3}{2}}(S_a)$ be given, $\mathbb{G}(x, y)$ be the Green's function of the operator $-\nabla^2 + q(x) - 1$ in $\mathbb{R}^3$ which satisfies *the radiation condition (1.6)*, and define

$$w(x) = \int_{S_a} \mathbb{G}(x, s)\sigma(s)ds, \tag{7.11}$$

such that

$$w = f \text{ on } S_a. \tag{7.12}$$

Since $(\nabla^2 + 1)w = 0$ in B'_a, $w = f$ on S_a and w satisfies (1.6), one can find w in B'_a explicitly:

$$w(x) = \sum_{\ell=0}^{\infty} \frac{f_\ell}{h_\ell(a)} Y_\ell(\alpha')h_\ell(r), \quad r \geq a, \quad z = |x|, \quad \alpha' = \frac{x}{r}, \tag{7.13}$$

where f_ℓ are the Fourier coefficients of f:

$$f(s) = \sum_{\ell=0}^{\infty} f_\ell Y_\ell(\alpha'), \quad s \in S_a. \tag{7.14}$$

Therefore the function

$$w_N^- = \lim_{|x|\to a, x\in B_a'} \frac{\partial w(x)}{\partial r}$$

is known. By the jump formula for single-layer potentials one has ([16], p. 14)

$$w_N^+ = w_N^- + \sigma. \tag{7.15}$$

The map $\Lambda : f \to w_N^+$ is constructed as soon as we find $\sigma(s)$, because w_N^- is already found.

To find σ, consider the asymptotics of $w(x)$ as $|x| \to \infty$, $\frac{x}{|x|} = \beta$. Using (1.7) and (7.11), one gets:

$$\frac{1}{4\pi}\int_{S_a} u(s,-\beta)\sigma(s)ds = \eta(\beta) := \sum_{\ell=0}^{\infty} \frac{f_\ell Y_\ell(\beta)}{h_\ell(a)}, \tag{7.16}$$

where we have used (7.13) and the asymptotics $h_\ell(r) \sim \frac{e^{ir}}{r}$ as $r \to +\infty$. As we have already mentioned, the function $u(s,\alpha')$ is known explicitly (see formula (1.31)), and equation (7.16) is uniquely solvable for $\sigma(s)$. Analytical solution of equation (7.16) for $\sigma(s)$ can be obtained as a series

$$\sigma(s) = \sum_{\ell=0}^{\infty} \sigma_\ell Y_\ell(\alpha'), \quad \alpha' = \frac{s}{|s|}. \tag{7.17}$$

Substitute (6.91) with $\alpha = -\beta$ into (1.31), take $r = a$ in (1.31) and $\alpha' = \frac{s}{a}$, and substitute (1.31) into (7.16). By our choice of the spherical harmonics (1.26) both systems $\{Y_\ell\}_{\ell=0,1,2,\dots}$ and $\{\overline{Y_\ell}\}_{\ell=0,1,2,\dots}$ form orthonormal bases of $L^2(S^2)$. Therefore one gets:

$$\begin{aligned}
&\frac{1}{4\pi}\sum_{\ell=0}^{\infty} 4\pi i^\ell \overline{Y_\ell}(-\beta) j_\ell(a) a^2 \int_{S^2} Y_\ell(\alpha')\sigma(a\alpha')d\alpha' \\
&+ \frac{1}{4\pi}\sum_{\ell=0}^{\infty} A_\ell(-\beta) h_\ell(a) a^2 \int_{S^2} Y_\ell(\alpha')\sigma(a\alpha')d\alpha' \\
&= \sum_{\ell=0}^{\infty} \frac{f_\ell Y_\ell(\beta)}{h_\ell(a)}.
\end{aligned} \tag{7.18}$$

Denote

$$\int_{S^2} \sigma(a\alpha') Y_{\ell,m}(\alpha')d\alpha' := \sigma_{\ell m}. \tag{7.19}$$

Using (1.26) one gets:

$$Y_{\ell,m}(-\beta)=(-1)^{\ell}Y_{\ell,m}(\beta),\quad \overline{Y_{\ell,m}(-\beta)}=(-1)^{\ell}\overline{Y_{\ell,m}(\beta)}\\ =(-1)^{\ell+\ell+m}Y_{\ell,-m}(\beta)=(-1)^{m}Y_{\ell,-m}(\beta). \tag{7.20}$$

Also define $A_{\ell m,\ell' m'}$ by the formula:

$$A_{\ell,m}(-\beta)=\sum_{\ell',m'}A_{\ell m,\ell' m'}Y_{\ell',-m'}(\beta). \tag{7.21}$$

The above definition differs from (1.36) and is used for convenience in this section.

Equating the coefficients in front of $Y_{\ell,-m}(\beta)$ in (7.18) one gets

$$i^{\ell}(-1)^{m}j_{\ell}(a)a^{2}\sigma_{\ell m}+\frac{a^{2}}{4\pi}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}A_{\ell' m',\ell m}h_{\ell'}(a)\sigma_{\ell' m'}=\frac{f_{\ell,-m}}{h_{\ell}(a)}, \tag{7.22}$$

or

$$\sigma_{\ell m}+\frac{(-1)^{m}(-i)^{\ell}}{4\pi j_{\ell}(a)}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}A_{\ell' m',\ell m}h_{\ell'}(a)\sigma_{\ell' m'}=\frac{f_{\ell,-m}(-1)^{m}(-i)^{\ell}}{a^{2}j_{\ell}(a)h_{\ell}(a)}. \tag{7.23}$$

The matrix of the linear system (7.23) is ill-conditioned. In [33] estimates of the entries of the matrix of (7.19) are obtained and the case of the noisy data is discussed. □

Finally let us show (see [34]) that *it is impossible to get an estimate*

$$\|Qf\|\le\varepsilon(|\theta|)\|f\|,\quad \theta\in M,\quad \|f\|:=\|f\|_{L^{2}(D)},\quad \varepsilon(t)\to 0\text{ as }t\to+\infty, \tag{7.24}$$

if

$$Qf=\int_{D}\Gamma(x,y,\theta)f(y)dy, \tag{7.25}$$

where

$$L\Gamma:=(\nabla^{2}+2i\theta\cdot\nabla)\Gamma=-\delta(x-y)\text{ in }D,\quad \theta\in M, \tag{7.26}$$

$$\Gamma=0\text{ on }S:=\partial D, \tag{7.27}$$

and we assume that

$$\text{the problem } L\rho=0,\quad \rho=0\quad\text{on}\quad S\quad\text{has only the trivial solution}. \tag{7.28}$$

Indeed, choose a $q(x)\in L^{\infty}(D)$ such that the problem

$$\left[\nabla^{2}+1-q(x)\right]w=0\text{ in }D,\quad w=0\text{ on }S, \tag{7.29}$$

has a non-trival solution.

Define $\rho=e^{-i\theta\cdot x}w$. Then $\rho\not\equiv 0$, and

$$L\rho-q\rho=0\text{ in }D,\quad \rho=0\text{ on }S. \tag{7.30}$$

Because of our assumption (7.24), one gets:

$$\rho = \int_D \Gamma(x,y)q(y)\rho(y)dy := T\rho. \tag{7.31}$$

Were (7.24) true, it would imply for $|\theta| \gg 1$, $\theta \in M$, that the operator $T : L^2(D) \to L^2(D)$ in (7.31) has small norm, so $\rho = 0$, contrary to our assumption. $\square$

8 Examples of nonuniqueness for an inverse problem of geophysics.

8.1 Statement of the problem.

In this section the result from [28] is presented.

Let $D \subset \mathbb{R}^n_+ := \{x : x \in \mathbb{R}^n, x_n \geq 0\}$ be a bounded domain, part S of the boundary Γ of D is on the plane $x_n = 0$, $f(x,t)$ is a source of the wavefield $u(x,t)$, and $c(x) > 0$ is a velocity profile. The wavefield, e.g., the acoustic pressure, solves the problem:

$$c^{-2}(x)u_{tt} - \Delta u = f(x,t) \quad \text{in} \quad D \times [0,\infty), \quad f(x,t) \not\equiv 0, \tag{8.1}$$

$$u_N = 0 \quad \text{on} \quad \Gamma \tag{8.2}$$

$$u = u_t = 0 \quad \text{at} \quad t = 0. \tag{8.3}$$

Here N is the unit outer normal to Γ, u_N is the normal derivative of u on Γ. If $c^2(x)$ is known, then the direct problem (8.1)-(8.3) is uniquely solvable. The inverse problem (IP) we are interested in is the following one:

(IP) *Given the data* $u(x,t) \quad \forall x \in S$, $\forall t > 0$, *can one recover* $c^2(x)$ *uniquely?*

The basic result is: *the answer to the above question is no.*

An analytical construction is presented of two constant velocities $c_j > 0$, $j = 1,2$, $c_1 \neq c_2$, which can be chosen arbitrary, and a source, which is constructed after $c_j > 0$ are chosen, such that the solutions to problems (8.1)-(8.3) with $c^2(x) = c_j^2$, $j = 1,2$, produce the same surface data on S for all times:

$$u_1(x,t) = u_2(x,t) \quad \forall x \in S, \quad \forall t > 0. \tag{8.4}$$

The domain D we use is a box: $D = \{x : a_j \leq x_j \leq b_j, 1 \leq j \leq n\}$.

This construction is given in the next section. At the end of section 8.2 the data on S are suggested, which allow one to uniquely determine $c^2(x)$.

8.2 Example of nonuniqueness of the solution to IP.

Our construction is valid for any $n \geq 2$. For simplicity we take $n = 2$, $D = \{x : 0 \leq x_1 \leq \pi, 0 \leq x_2 \leq \pi\}$. Let $c^2(x) = c^2 = \text{const} > 0$. The solution to (8.1)-(8.3) with $c^2(x) = c^2 = \text{const}$ can be found analytically

$$u(x,t) = \sum_{m=0}^{\infty} u_m(t)\phi_m(x), \quad m = (m_1, m_2) \tag{8.5}$$

where

$$\begin{aligned}
\phi_m(x) &= \gamma_{m_1m_2}\cos(m_1x_1)\cos(m_2x_2), \\
\int_D \phi_m^2(x)dx &= 1, \quad \Delta\phi_m + \lambda_m\phi_m = 0, \\
\phi_{mN} &= 0 \quad \text{on} \quad \Gamma, \quad \lambda_m := m_1^2 + m_2^2, \\
\gamma_{00} &= \frac{1}{\pi}, \quad \gamma_{m_10} = \gamma_{0m_2} = \frac{\sqrt{2}}{\pi}, \\
\gamma_{m_1m_2} &= 2/\pi \text{ if } \quad m_1 > 0 \quad \text{and} \quad m_2 > 0,
\end{aligned} \tag{8.6}$$

$$\begin{aligned}
u_m(t) := u_m(t,c) &= \frac{c}{\sqrt{\lambda_m}}\int_0^t \sin[c\sqrt{\lambda_m}(t-\tau)]f_m(\tau)d\tau, \\
f_m(t) &:= \int_D f(x,t)\phi_m(x)dx.
\end{aligned} \tag{8.6$'$}$$

The data are

$$u(x_1, 0, t) = \sum_{m=0}^{\infty} u_m(t,c)\gamma_{m_1m_2}\cos(m_1x_1). \tag{8.7}$$

For these data to be the same for $c = c_1$ and $c = c_2$, it is necessary and sufficient that

$$\sum_{m_2=0}^{\infty} \gamma_{m_1m_2}u_m(t,c_1) = \sum_{m_2=0}^{\infty} \gamma_{m_1m_2}u_m(t,c_2), \quad \forall t > 0, \quad \forall m_1. \tag{8.8}$$

Taking Laplace transform of (8) and using (6$'$) one gets an equation, equivalent to (8.8),

$$\sum_{m_2=0}^{\infty} \gamma_{m_1m_2}\overline{f}_m(p)\left[\frac{c_1^2}{p^2 + c_1^2\lambda_m} - \frac{c_2^2}{p^2 + c_2^2\lambda_m}\right] = 0, \quad \forall p > 0, \quad \forall m_1. \tag{8.9}$$

Take $c_1 \neq c_2$, $c_1, c_2 > 0$, arbitrary and find $\overline{f}_m(p)$ for which (8.9) holds. This can be done by infinitely many ways. Since (8.9) is equivalent to (8.8), the desired example of nonuniqueness of the solution to IP is constructed.

Let us give a specific choice: $c_1 = 1$, $c_2 = 2$, $\overline{f}_{m_1m_2} = 0$ for $m_1 \neq 0$, $m_2 \neq 1$ or $m_2 \neq 2$, $\overline{f}_{02}(p) = \frac{1}{p+1}$, $\overline{f}_{01}(p) = -\frac{p^2+1}{(p+1)(p^2+16)}$. Then (8.9) holds. Therefore, if

$$f(x,t) = \frac{\sqrt{2}}{\pi}\left[f_{01}(t)\cos(x_2) + f_{02}(t)\cos(2x_2)\right], \quad c_1 = 1, \quad c_2 = 2, \tag{8.10}$$

then the data $u_1(x,t) = u_2(x,t) \quad \forall x \in S$, $\forall t > 0$. In (8.10) the values of the coefficients are

$$f_{01}(t) = -\frac{2}{17}\exp(-t) - \frac{15}{17}\left[\cos(4t)\frac{1}{4}\sin(4t)\right], \quad f_{02}(t) = \exp(-t). \tag{8.11}$$

Remark 8.1. *The above example brings out the question:*

What data on S are sufficient for the unique identifiability of $c^2(x)$?

The answer to this question one can find in [16] and [11].

In particular, if one takes $f(x,t) = \delta(t)\delta(x-y)$, and allows x and y run through S, then the data $u(x,y,t) \quad \forall x,y \in S$, $\forall t > 0$, determine $c^2(x)$ uniquely. In fact, the low frequency surface data $\tilde{u}(x,y,k)$, $\forall x,y \in S \quad \forall k \in (0,k_0)$, where $k_0 > 0$ is an arbitrary small fixed number, determine $c^2(x)$ uniquely under mild assumptions on D and $c^2(x)$. By $\tilde{u}(x,y,k)$ is meant the Fourier transform of $u(x,y,t)$ with respect to t.

Remark 8.2. *One can check that the non-uniqueness example with constant velocities is not possible to construct, as was done above, if the sources are concentrated on S, that is, if $f(x_1,x_2,t) = \delta(x_2)f_1(x_1,t)$.*

9 A uniqueness theorem for inverse boundary value problem for parabolic equations

Consider the problem:

$$u_t + Lu = 0, \quad x \in D, \quad t \in [0,T], \tag{9.1}$$

$$u = 0 \quad \text{at} \quad t = 0 \tag{9.2}$$

$$u = f(s)\delta(t) \quad \text{on} \quad S. \tag{9.3}$$

Here $\delta(t)$ is the delta-function, D is a bounded domain in $\mathbb{R}^n, n \geq 3$, with a smooth boundary S, $f \in H^{3/2}(S)$, $Lu := -div[a(x)gradu] + q(x)u$, $a(x)$ and $q(x)$ are real-valued functions, $q \in L^2(D)$, $0 < a_0 \leq a(x) \leq a_1$, where a_0 and a_1 are positive constants, and $a(x) \in C^2(\bar{D})$, where $\bar{D}$ is the closure of D. Let $h(s,t) := a(s)u_N$, where N is the unit exterior normal to S.

The IP (inverse problem) is: given the set of ordered pairs $\{f(s), h(s,t)\}$ for all $t \in [0,T]$ and all $f \in H^{3/2}(S)$, find $a(x)$ and $q(x)$.

We prove that IP has at most one solution by reducing the uniqueness of the solution to IP to the Ramm's uniqueness theorem for the solution to elliptic boundary value problem [11].

This theorem says:

Let

$$Lu + \lambda u = 0 \text{ in } D, \quad u = f(s) \text{ on } S, \tag{9.4}$$

and assume that the above problem is uniquely solvable for two distinct real values of λ. Suppose that the set of ordered pairs $\{f, h\}$ is known at these values of λ for all $f \in H^{3/2}(S)$, where $h := a(s)u_N$, and u_N is the normal derivative on S of the solution to (9.5). Then the operator L is uniquely determined, that is, the functions $a(x)$ and $q(x)$ are uniquely determined.

We apply this theorem as follows.

First, we claim that the data $h(s,t)$, known for $t \in [0,T]$ are uniquely determined for all $t > 0$. If $\delta(t)$ is replaced by a function $\eta(t) \in C_0^\infty(0,T)$, $\int_0^T \eta(t)dt = 1$, then the data $h(s,t)$ known for $t \in [0,T]$ are uniquely determined for $t > T$.

Secondly, if this claim is established, then Laplace-transform problem (9.1)-(9.3) to get the elliptic problem studied in [11]:

$$Lv + \lambda v = 0 \text{ in } D, \quad u = f(s) \text{ on } S, \tag{9.5}$$

and the data $H(s,\lambda)$, where $v := \int_0^\infty e^{-\lambda t}u(x,t)dt$.

The data $H(s,\lambda) := \int_0^\infty e^{-\lambda t}h(s,t)dt$ are known for all $\lambda > 0$.

Thus, Ramm's theorem yields uniqueness of the determination of L, and the proof is completed.

We now sketch the proof of the claim:

The solution to the time-dependent problem can be written as:

$$u(x,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} c_j \phi_j(x), \tag{9.6}$$

where $L\phi_j(x) = \lambda_j\phi_j(x)$ in $D, \phi_j(x) = 0$ on $S, ||\phi_j(x)||_{L^2(D)} = 1$. The coefficients $c_j := -\int_S f(s)a(s)\phi_{jN}(s)ds$.

Note that the series for $u(x,t)$ and the series obtained by the termwise differentiation of it with respect to t converge absolutely and uniformly in $D \times (0,\infty)$, each of the terms is analytic with respect to t in the region $\Re t > 0$, and consequently so are these series.

Therefore the functions $u(x,t)$ and $h(s,t) := u_N(s,t)$ are analytic with respect to t in the region $\Re t > 0$, so the data are uniquely determined for $t > T$ as claimed. □

At $t = 0$ the series (9.6) is singular: it does not converge uniformly or even in $L^2(D)$. By this reason the above argument is formal. One can make it rigorous if one replaces the delta-function in (9.3) by a $C_0^\infty(0,T)$ function $\eta(t)$, $\int_0^T \eta(t)dt = 1$, and uses the argument similar to the one in [52].

References

[1] Airapetyan, R.G., Ramm, A.G., Smirnova, A.B., Example of two different potentials which have practically the same fixed-energy phase shifts, *Phys. Lett A*, 254, N3-4, (1999), 141-148.

[2] Erdelyi, A., Higher transcedental functions, *McGraw-Hill*, New York, 1953.

[3] Evgrafov, M., Analytic functions, *Saunders*, Philadelphia, 1966.

[4] Gradshteyn, I., Ryzhik, I., Table of intergals, series and products, *Acad. Press*, Boston, 1994.

[5] Gilbarg, D., Trudinger, N., Elliptic partial differential equations of second order, Springer-Verlag, New York, 1983.

[6] Hörmander, L., The analysis of linear partial differential operators, vol. 1-4, *Springer-Vetlag*, New York, 1983-1985.

[7] Maz'ya, V., Sobolev spaces, *Springer Verlag*, New York, 1985.

[8] Maz'ya, V., Poborchi, S, Differentiable functions on bad domains, *World Scientific*, Singapore, 1997.

[9] Pearson, D., Quantum scattering and spectral theory, *Acad. Press*, New York, 1988.

[10] Ramm, A.G., Random fields estimation theory, *Longman Scientific & Wiley*, New York, 1990, pp.1-281. Expanded Russian edition: *Mir*, Moscow, 1996, pp. 1-352.

[11] Ramm, A.G., Multidimensional inverse scattering problems, *Longman/Wiley*, New York, 1992, pp.1-385. Expanded Russian edition: *Mir*, Moscow, 1994, pp.1-496;

[12] Ramm, A.G., Absence of the discrete positive spectrum of the Dirichlet Laplacian in some infinite domains, Vestnik Leningrad. Gosudarstv. Universiteta, ser math., mech. and astronom., 13, (1964), 153-156. (in Russian)

[13] Ramm, A.G., Completeness of the products of solutions to PDE and uniqueness theorems in inverse scattering, *Inverse problems*, 3, (1987), L77-L82.

[14] Ramm, A.G., Recovery of the potential from fixed energy scattering data. *Inverse Problems*, 4, (1988), 877-886; 5, (1989) 255.

[15] Ramm, A.G., Stability estimates in inverse scattering, *Acta Appl. Math.*, 28, N1, (1992), 1-42.

[16] Ramm, A.G., Scattering by obstacles, D.Reidel, *Dordrecht*, 1986, pp.1-442.

[17] Ramm, A.G., Ruiz, A., Existence and uniqueness of scattering solutions in non-smooth domains, *J. Math. Anal. Appl.*, 201, (1996), 329-338.

[18] Ramm, A.G., Sammartino, M., Existence and uniqueness of the scattering solutions in the exterior of rough domains, in the book "Operator Theory and Its Applications", *Amer. Math. Soc.*, Fields Institute Communications vol.25, pp.457-472, Providence, RI, 2000. (editors A.G.Ramm, P.N.Shivakumar, A.V.Strauss),

[19] Ramm, A.G. Uniqueness theorems for inverse obstacle scattering problems in Lipschitz domains, *Applic. Analysis*, 59, (1995), 377-383.

[20] Ramm, A.G., New method for proving uniqueness theorems for obstacle inverse scattering problems, *Appl. Math. Lett.*, 6, N6, (1993), 19-22.

[21] Ramm, A.G., Stability estimates for obstacle scattering, *J. Math. Anal. Appl.* 188, N3, (1994), 743-751.

[22] Ramm, A.G., Necessary and sufficient condition for a PDE to have property C, *J. Math. Anal.*, Appl.156, (1991), 505-509.

[23] Ramm, A.G., Symmetry properties for scattering amplitudes and applications to inverse problems, *J. Math. Anal.* Appl., 156, (1991), 333-340.

[24] Ramm, A.G., Inverse scattering problem with part of the fixed-energy phase shifts, *Comm. Math. Phys.* 207, N1, (1999), 231-247.

[25] Ramm, A.G., Pang, P., Yan, G., A uniqueness result for the inverse transmission problem, *Internat. Jour. of Appl. Math.*, 2, N5, (2000), 625-634.

[26] Ramm, A.G., Stability of the solution to inverse obstacle scattering problem, *J.Inverse and Ill-Posed Problems*, 2, N3, (1994), 269-275.

[27] Ramm, A.G., Continuous dependence of the scattering amplitude on the surface of an obstacle, *Math. Methods in the Appl. Sci.*, 18, (1995), 121-126.

[28] Ramm, A.G., Examples of nonuniqueness for an inverse problems of geophysics, *Appl. Math. Lett.*, 8, N4, (1995), 87-90.

[29] Ramm, A.G., On a method of solving the Dirichlet problem in some infinite domains. *Mathematika, Izvestiya VUZOV*, 5, (1965), 124-127. (In Russian) *Math. Rev. 32:7993.*

[30] Ramm, A.G., Completeness of the products of solutions to PDE and inverse problems, *Inverse problems*, 6, (1990), 643-664.

[31] Ramm, A.G., Multidimensional inverse problems and completeness of the products of solutions to PDE, *J. Math. Anal. Appl.,* 134, 1, (1988), 211-253; 139, (1989) 302.

[32] Ramm, A.G., A uniqueness theorem for an inverse problem, *Appl. Math. Lett.,* 1, (1988), 185-187.

[33] Ramm, A.G., Finding potential from the fixed-energy scattering data via D-N map, *J. of Inverse and Ill-Posed Problems*, 4, N2,(1996), 145-152.

[34] Ramm, A.G., A formula for inversion of boundary data, *J. of Inverse and Ill-Posed Problems*, 3, N5, (1995), 411-415.

[35] Ramm, A.G., Uniqueness theorems for multidimensional inverse problems with unbounded coefficients, *J. Math. Anal. Appl.* 136, (1988), 568-574.

[36] Ramm, A.G., A simple proof of uniqueness theorem in impedance tomography.*Appl. Math. Lett.* 1, N3, (1988), 287-290.

[37] Ramm, A.G., Multidimensional inverse problems: Uniqueness theorems, *Appl. Math. Lett.* 1, N4, (1988), 377-380.

[38] Ramm, A.G., Multidimensional inverse scattering problems and completeness of the products of solutions to homogeneous PDE, *Zeitschr. f. angew. Math. u. Mech.*, 69, (1989) N4, T13-T22.

[39] Ramm, A.G., Stability of the numerical method for solving the 3D inverse scattering problem with fixed energy data,*Inverse problems* 6, (1990), L7-12.

[40] Ramm, A.G., Rakesh, Property C and an inverse problem for a hyperbolic equation, *J. Math. Anal. Appl.*, 156, (1991), 209-219.

[41] Ramm, A.G., Stability of the numerical method for solving 3D inverse scattering problem with fixed energy data, *J.f.die reine und angew. Math*, 414, (1991), 1-21.

[42] Ramm, A.G., Property C and inverse problems, *ICM-90 Satellite Conference Proceedings, Inverse Problems in Engineering Sciences,* Proc. of a conference held in Osaka, Japan, Aug. 1990, Springer Verlag, New York, 1991, pp. 139-144.

[43] Ramm, A.G., Stability of the solution to inverse scattering problem with exact data, *Appl.Math.Lett.*, 5, 1, (1992), 91-94.

[44] Ramm, A.G., Stability of the inversion of 3D fixed-frequency data, *J.Math.Anal.Appl.*, 169, N2(1992), 329-349.

[45] Ramm, A.G., Stability of the solution to 3D fixed-energy inverse scattering problem, *J.Math.Anal.Appl.*, 170, N1 (1992), 1-15.

[46] Ramm, A.G., Algorithm for solving 3D inverse scattering problems with noisy discrete fixed-energy data, *Proceedings of ICES-92 conference on inverse problems.* In the book: Inverse Problems, Atlanta Technology Publications, Atlanta, Georgia, (1993), pp.70-74.

[47] Ramm, A.G., Roy, D., Inverse geophysical problems for some non-compactly supported inhomogeneities, *Appl. Math. Lett.*, 6, N6, (1993), 15-18.

[48] Ramm, A.G., Multidimensional inverse scattering: solved and unsolved problems, *Proc.Intern.Confer. on Dynamical Syst.and Applic., Vol.1, Atlanta,* (1994), pp.287- 296. (Eds. G.Ladde and M.Sambandham)

[49] Ramm, A.G., Berman, G., Stability estimate in scattering theory and its application to mesoscopic systems and quantum chaos, *Physics A*, 27, N18, (1994), 6157-6166.

[50] Ramm, A.G., Stability of the solution to 3D inverse scattering problems with fixed-energy data, *Proc. ASME Nov.6-11, 1994, meeting. Inverse problems in mechanics*, AMD-Vol 186, pp.99-102.

[51] Ramm, A.G., Finding conductivity from boundary measurements, *Comp.& Math.with Appl.*, 21, N8, (1991), 85-91.

[52] Ramm, A.G., A non-overdetermined inverse problem of finding the potential from the spectral function, Intern. Jour. of Diff. Eqs and Appl., 3, N1, (2001), 15-29.

[53] Ramm, A.G., Gutman, S., Piecewise-constant positive potentials with practically the same fixed-energy phase shifts, Applic. Analysis, 78, N1-2, (2001), 207-217.

[54] Sylvester, J., Uhlmann, G., A global uniquness theorem for inverse boundary value problem, *Ann. Math*, 125, (1987), 153-169.

[55] Gol'dshtein, V., Ramm, A.G., Embedding operators for rough domains, *Mathem. Inequalities and Applications*, 4, N1, (2001), 127-141.

[56] Ramm, A.G., Inverse prolems, *Springer*, New York, 2005.