Wavefield Imaging EE4595
Programming Assignment

Deadline: 24-07-2020 (23:59)
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- Do not forget your **name**, **study number**, and **email address**!

Problem Statement In this assignment, we consider a two-dimensional linearized inverse scattering problem (inverse medium problem). The goal is to obtain an image of an object by solving the data equation with respect to the contrast function.

First, we consider a two-dimensional homogeneous background configuration that is characterized by a constant background wave speed c_b. The position vector in this 2D configuration is given by $\boldsymbol{\rho} = x\mathbf{i}_x + y\mathbf{i}_y$ and a line source with unit amplitude is located at a point with position vector $\boldsymbol{\rho}_\mathrm{s} = x_\mathrm{s}\mathbf{i}_x + y_\mathrm{s}\mathbf{i}_y$. The field generated by this source satisfies the Helmholtz equation

$$\left(\partial_x^2 + \partial_y^2 + k_\mathrm{b}^2\right)\hat{u}^\mathrm{inc} = -\delta(x - x_\mathrm{s})\delta(y - y_\mathrm{s}),$$

where $k_\mathrm{b} = \omega/c_\mathrm{b}$ is the wave number of the background medium and the delta functions on the right-hand side are Dirac distributions. Using an $\exp(\mathrm{j}\omega t)$ time convention, the incident field that radiates away from the line source is given by

$$\hat{u}^\mathrm{inc}(\boldsymbol{\rho}, \mathrm{j}\omega) = -\frac{\mathrm{j}}{4}H_0^{(2)}(k_\mathrm{b}|\boldsymbol{\rho} - \boldsymbol{\rho}_\mathrm{s}|). \tag{1}$$

In this equation, $H_0^{(2)}$ is the so-called Hankel function of the second kind and order zero.

An object, occupying the bounded domain $\mathbb{D}$, is embedded in this background material and is characterized by the contrast function

$$\chi(\boldsymbol{\rho}) = \left[\frac{k(\boldsymbol{\rho})}{k_\mathrm{b}}\right]^2 - 1,$$

where $k(\boldsymbol{\rho}) = \omega/c(\boldsymbol{\rho})$ is the position-dependent wave number of the object. The Green's function for this 2D configuration has the same form as the background field, that is,

$$\hat{G}(\boldsymbol{\rho} - \boldsymbol{\rho}', \mathrm{j}\omega) = -\frac{\mathrm{j}}{4}H_0^{(2)}(k_\mathrm{b}|\boldsymbol{\rho} - \boldsymbol{\rho}'|) \tag{2}$$

and in the Born approximation, the scattered field at location $\boldsymbol{\rho} \notin \mathbb{D}$ is given by

$$\hat{u}^{\mathrm{sc}}(\boldsymbol{\rho}, \mathrm{j}\omega) = k_{\mathrm{b}}^2 \int_{\boldsymbol{\rho}' \in \mathbb{D}} \hat{G}(\boldsymbol{\rho} - \boldsymbol{\rho}', \mathrm{j}\omega)\, \chi(\boldsymbol{\rho}')\, \hat{u}^{\mathrm{inc}}(\boldsymbol{\rho}', \mathrm{j}\omega)\, \mathrm{d}V. \tag{3}$$

The goal is to reconstruct $\chi(\boldsymbol{\rho})$ for $\boldsymbol{\rho} \in \mathbb{D}$ from knowledge of $u^{\mathrm{sc}}(\boldsymbol{\rho}, \mathrm{j}\omega)$ at a number of receiver locations $\boldsymbol{\rho} = \boldsymbol{\rho}_m^{\mathrm{R}} \notin \mathbb{D}$, $m = 1, 2, ..., M$.

Solution

1. First, find out what the Hankel function $H_0^{(2)}(z)$ really is. Locate the corresponding function in Matlab and learn how it works.
2. Substitute Eqs. (1) and (2) in Eq. (3) and write down the resulting integral representation explicitly.
3. For simplicity, take $k_{\mathrm{b}} = 1$ and introduce a Cartesian coordinate system with x-axis horizontal and pointing to the right, y-axis vertical and pointing down. Take the object domain $\mathbb{D}$ to be a rectangle such that its upper left corner is at the origin of the coordinate system and its lower right corner is located at the point with coordinates (λ, λ), where $\lambda = 2\pi/k_{\mathrm{b}}$ is the wavelength of the background field. The source is located at the point $\boldsymbol{\rho}_{\mathrm{s}} = (\lambda/2)\mathbf{i}_x + 10\lambda\mathbf{i}_y$. Make a sketch of the configuration (this will be Figure 1 in your report). Consider using the `line` command of Matlab for this sketch.
4. Create a Matlab script file to store your work.
5. Introduce a uniform grid (same increment in the x- and y-directions) on $\mathbb{D}$ with a step size $h = \lambda/20$. Compute the total number of grid points N.
6. Compute the incident field on the grid and store it as a 2D array in Matlab. Make a 2D image of the real and imaginary parts and the absolute value of $\hat{u}^{\mathrm{inc}}$. Make sure that the orientation of the image corresponds to the orientation of your coordinate system. Use `axis equal tight` command to keep the aspect ratio.
7. Change the location of the source (move it closer to the object domain $\mathbb{D}$, for example) and repeat the calculations. Does the background field distribution change? Change the wave number to $k_{\mathrm{b}} = 2$ and repeat the calculations. Explain the changes in the background

field. When you are done, return to the case where the source is located at $\boldsymbol{\rho}_{\rm s} = (\lambda/2)\mathbf{i}_x + 10\lambda\mathbf{i}_y$ and $k_{\rm b} = 1$.

8. Introduce a contrast function $\chi(\boldsymbol{\rho}) \geq 0$ of your object (your choice!) with $\boldsymbol{\rho}$ on the grid and store it as a 2D array in Matlab.

9. Make a 2D image of your contrast using the `imagesc` command. Add a colorbar to it.

10. Define the receiver domain $\mathbb{D}^{\rm rec}$ as a line segment of length $L = 3\lambda$ with end points at $(-\lambda, 1.5\lambda)$ and $(2\lambda, 1.5\lambda)$. Draw it in Figure 1.

11. Introduce a grid on $\mathbb{D}^{\rm rec}$ using a uniform step size so that there are M uniformly spaced grid points on the line segment, where $M > 0$ is an integer parameter of your problem that you will investigate later.

12. Discretize the data equation (3) by approximating the integral via a sum (use the composite midpoint rule, for example). Write down the discretized equation.

13. Study the `reshape` command of Matlab. Reshape the 2D array of the contrast function into a column vector $\mathsf{x} \in \mathbb{C}^{N\times 1}$.

14. Build the system matrix $\mathsf{A} \in \mathbb{C}^{M\times N}$ of your discretized data equation.

15. Compute the singular value decomposition of the system matrix using the `svd` command. Plot the singular values. Change M and investigate the dependence of the singular values on M.

16. Compute the scattered field as $\mathsf{u}^{\rm sc} = \mathsf{A}\mathsf{x}$ in Matlab.

17. Consider now $\mathsf{u}^{\rm sc}$ as given and determine the minimum norm solution $\mathsf{x}_{\rm mn}$ using the singular value decomposition or the `pinv` command of Matlab.

18. Reshape the solution vector $\mathsf{x}_{\rm mn}$ into a 2D array and make an image with `imagesc`. Compare this image with the image of the original contrast.

19. Repeat the previous step with a different number of receivers M. Make an image.

20. Add noise to the data vector $\mathsf{u}^{\rm sc}$ using the `rand` command. Investigate the influence of noise on the reconstruction results.