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Digital Linear Filters in Geophysics

Excerpt from Werthmüller et al. (2019); for the full article see github.com/empymod/article-fdesign.

Review

In his Ph.D. thesis, Ghosh (1970) proposed a linear filter method for the numerical evaluation of Hankel transforms that subsequently revolutionized the computation of electromagnetic (EM) responses in the field of geophysical exploration. If you use a code that calculates EM responses in the wavenumber-frequency domain and transforms them to the space-frequency domain, chances are high that it uses the *digital linear filter* (DLF) method. Most practical 1D EM modeling codes rely on the DLF method for rapid computations; these codes are not only used for standalone simulations of EM fields in layered 1D media, but they are also commonly embedded within 3D modeling codes for computing the primary fields in scattered-field formulations or for the background fields required by integral equation methods. Thus, the DLF method is an important component of many commonly used modeling codes for EM geophysical data.

Ghosh (1971a) states that the DLF idea is based on suggestions made four decades earlier by Slichter (1933) and Pekeris (1940), in that "*the kernel function is dependent only on the layer parameters, and an expression relating it to the field measurements can be obtained by mathematical processes.*" However, until the introduction of DLF, these suggestions found no widespread use, likely because of the missing computer power to calculate the filter coefficients. He further states that credit goes to Koefoed (1968, 1970), who retook the task of direct interpretational methods with the introduction of raised kernel functions. DLF is, as such, an improvement of that approach, providing a faster and simpler method.

The DLF technique is sometimes referred to as the *fast Hankel transform* (FHT), popular because of the similarity of the name to the well known *fast Fourier transform* (FFT), although the algorithms for these techniques are completely different. The FHT name was likely inspired by the title of a paper by Johansen and Sørensen (1979). However, the name FHT can be misleading as it has *Hankel* in the name, whereas the DLF approach can more generally be applied to other linear transforms, for example Fourier sine and cosine transforms.

The introduction of linear filters to EM geophysics, in parallel with rising computational power, initiated a wealth of investigations, leading to the development of computer programs that extended and improved the DLF method, and to numerous publications. These publications fall broadly into one or several of three categories: (1) new applications that extend the usage of DLF to other EM measurement techniques; (2) filter improvements that provide either new filters or improved methods for the determination of filter coefficients; and (3) computational tools that compute EM responses using DLF techniques. Here, we briefly review the most relevant publications, without claiming completeness.

(1) New applications

Ghosh used the method originally for the computation of type curves for Schlumberger and Wenner resistivity soundings: Ghosh (1971a) derives a resistivity model from given Schlumberger or Wenner sounding curves; and Ghosh (1971b) provides filters for the inverse operation, deriving resistivity sounding curves from a given resistivity model. The method was next applied to electromagnetic soundings with horizontal and perpendicular coils (Koefoed et al., 1972), to vertical coplanar coil systems (Verma and Koefoed, 1973), to dipoles and other two electrode systems (Das and Ghosh, 1974; Das et al., 1974; Das and Verma, 1980; Sørensen and Christensen, 1994), and to vertical dikes, hence vertical instead of horizontal layers (Niwas, 1975). The first filters were specific to a particular resistivity sounding type and its transform; later publications used the method to get one type curve from another type curve (Kumar and Das, 1977, 1978), or generalized the method to be applicable to a wider set of problems (Davis et al., 1980; Das and Verma, 1981b; Das, 1984; O'Neill and Merrick, 1984). Eventually, it passed from pure layered modeling to primary-secondary field formulations for 3D problems, where DLF is used to compute the spatial Fourier-Hankel transforms in a horizontally layered background medium and to compute transient responses from frequency domain computations (Das and Verma, 1981a, 1982; Anderson, 1984; Newman et al., 1986; Kruglyakov and Bloshanskaya, 2017). Other publications delved into the theory of the method, analyzing the oscillating behaviour of the filters and trying to estimate the error of DLF (Koefoed, 1972, 1976; Johansen and Sørensen, 1979; Christensen, 1990).

(2) Filter improvements

Ghosh (1970) derived the filter coefficients in the spectral domain by dividing the output spectrum by the input spectrum followed by an inverse Fourier transform. Improvements to the determination of filter coefficients were provided by O'Neill (1975); Nyman and Landisman (1977); Das (1982), or specifically for the Fourier transform by Nissen and Enmark (1986). A direct integration method was used by Bichara and Lakshmanan (1976) and Bernabini and Cardarelli (1978). Koefoed and Dirks (1979) proposed a Wiener-Hopf least-squares method, which was further improved by many authors (Guptasarma, 1982; Murakami and Uchida, 1982; Guptasarma and Singh, 1997). Kong (2007) proposed a direct matrix inversion method to solve the convolution equation, which requires only the input and output sample values. To evaluate the filter's effectiveness, he defines a good filter as one that recovers small or weak diffusive EM fields. This method was also used by Key (2009, 2012) to create a suite of filters accurate for marine EM data. Most works have published filters for the Hankel transform with J_0 and J_1 Bessel functions (or $J_{-1/2}$, $J_{1/2}$ if applied to the Fourier sine/cosine transform), as all higher Bessel functions can be rewritten, via recurrence relations, using only these two. Mohsen and Hashish (1994) is one of the rare cases which provides J_2 filter weights.

(3) Computational tools

The most well-known codes are likely Anderson's freely available ones. Anderson (1973) extends the method to transient responses, applying DLF not only to the Hankel transform, but also to the Fourier transform. A transient signal can therefore be obtained be applying twice a digital filter to the wavenumber-frequency domain calculation. Anderson (1975, 1979) presents improved filters for both Fourier and Hankel transforms, introducing measures to significantly speed-up the calculation, such as the lagged convolution or using the same abscissae for J_0 and J_1. Anderson (1982) and the included 801 pt filter became the de facto industry standard, to which subsequent filters were compared. Anderson (1989) presents a hybrid solution that uses either the DLF or a much slower but highly accurate adaptive-quadrature approach, which can also be used to measure the accuracy of the DLF. Key (2012) presented codes for comparing the DLF approach with a speed optimized quadrature method called quadrature-with-extrapolation (QWE); despite its optimizations, QWE is still not as fast as DLF, but it remains valuable as an independent tool for testing the accuracy of particular filters for the DLF method. Other examples include the codes by Johansen (1975), an interactive system for interpretation of resistivity soundings, and a tool to calculate filter coefficients by Christensen (1990). The latter is available upon request and was used, for instance, in all the open-source modeling and inversion routines of CSIRO in the Amira Project 223 (Raiche et al., 2007).

All mentioned publications have in common that they were derived for direct current methods (DC) or low frequency methods, such as time-domain shallow EM methods (TEM), or controlled-source electromagnetics (CSEM), but not for high frequency methods such as ground-penetrating radar (GPR). Generally it was even thought that the filter method works only in the diffusive limit where the quasistatic approximation is valid (e.g., Hunziker et al., 2015). Nevertheless, Werthmüller (2017) tested DLF for modeling 250 MHz center-frequency GPR data using a 401 pt filter that was designed for diffusive EM, and found that it was orders of magnitude faster than quadrature approaches. However, good quality results were only obtained for the first arrivals at short time-offsets and the later arrivals had poor quality. Yet, this promising initial test suggests it might be possible to specifically design a special filter for GPR frequencies where the EM fields propagate as waves rather than by diffusion.

Theory

Most of the articles mentioned in the review have detailed derivations of the digital filter method. In this article we focus on the algorithm, and summarize the theory only very briefly by following Key (2012). In electromagnetics we often have to evaluate integrals of the form

$$F(r) = \int_0^\infty f(l)K(lr)\mathrm{d}l \ , \tag{1}$$

where l and r denote input and output evaluation values, respectively, and K is the kernel function. In the specific case of the Hankel transform l corresponds to wavenumber, r to offset, and K to Bessel functions; in the case of the Fourier transform l corresponds to frequency, r to time, and K to sine or cosine functions. In both cases it is an infinite integral which numerical integration is very time-consuming because of the slow decay of the kernel function and its oscillatory behaviour.

By substituting $r = e^x$ and $l = e^{-y}$ we get

$$e^x F(e^x) = \int_{-\infty}^\infty f(e^{-y})K(e^{x-y})e^{x-y}\mathrm{d}y \ . \tag{2}$$

This can be re-written as a convolution integral and be approximated for an N-point filter by

$$F(r) \approx \sum_{n=1}^N \frac{f(b_n/r)h_n}{r} \ , \tag{3}$$

where h is the digital linear filter, and the logarithmically spaced filter abscissae is a function of the spacing Δ and the shift δ,

$$b_n = \exp\left\{\Delta(-\lfloor(N+1)/2\rfloor + n) + \delta\right\} \ . \tag{4}$$

From equation 3 it can be seen that the filter method requires N evaluations at each r. For example, to calculate the frequency domain result for 100 offsets with a 201 pt filter requires 20,100 evaluations in the wavenumber domain. This is why the DLF often uses interpolation to minimize the required evaluations, either for $F(r)$ in what is referred to as *lagged convolution DLF*, or for $f(l)$, which we here call *splined DLF*.

In the direct matrix inversion method for solving for the digital filter coefficients, equation 3 is cast as a linear system, where Δ and δ are preassigned scalar values and r is a range of M preassigned values r_m. The filter base coefficients b_n are computed using equation 4 and an array of values for l are computed using $l_{mn} = b_n/r_m$. The linear system's left hand side (LHS) matrix $\mathbf{A}$ has dimensions $M \times N$ with coefficients $A_{mn} = f(l_{mn})/r_m$ and the right hand side (RHS) vector $\mathbf{v}$ has elements $v_m = F(r_m)$. The resulting linear system

$$\mathbf{Ah} = \mathbf{v} \tag{5}$$

can be solved by direct matrix inversion, or any other matrix inversion routine, to obtain the vector of filter coefficients $\mathbf{h}$. For a given filter length N, there are many subjective choices that go into the practical implementation of this method, including the choice of the transform pairs $F(r)$ and $f(l)$, as well as the value for M and the particular range and spacing of the values r_m. Once values are chosen for these variables, an optimal filter can be found by a grid search over Δ and δ for values that produce a high quality filter. The choice of metric for what constitutes a high quality filter is also subjective and is further discussed below.

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