Diffusion equation for the financial markets

Damien Benveniste

Introduction

I derive a diffusion equation for the PDF $P_M(s_1, \cdots, s_M, t|s_{01}, \cdots, s_{0M}, t_0)$ to have a M-stocks market with value s_i for the ith stock if we know the value s_{0i} for the ith stock at a initial time t_0. The equation is derived using two approximations: a "mean field" approximation neglecting the fluctuations in the fine grained PDF and a Markovian approximation assuming a rapid decay of the time correlations.

In particular, I prove that the diffusion equation is exact when we have one random Log-normal distributed stock price. The diffusion equation model could therefore lead to corrections of a PDF departing slightly from the Log-normal PDF. I then re-derive a Black Scholes type equation using the stochastic form of the diffusion equation in the 1-stock market case.

1 The model

The first assumption is that a stock price s is only a function of an initial value s_0 at a particular time t_0 and the number of trades $N - N_0$ processed between t_0 and a time t such that $s \equiv s(s_0, N_0|N)$.

Let $\rho(s_0, N_0|N) \equiv \rho(s(s_0, N_0|N), N)$ be a random field which satisfies

$$\frac{d}{dN}s(N) = \rho(s, N),\ s(N_0) = s_0 \tag{1}$$

We define $n(t) = \frac{dN}{dt}$ the number of trades per unit of real time and we have

$$\frac{dt}{dN}\frac{d}{dt}s(t) = \frac{1}{n(t)}\frac{d}{dt}s(t) = \rho(s, t),\ s(t_0) = s_0 \tag{2}$$

$n(t)$ is in practice time dependent and is specific to each stock. It is obvious that $n(t)$ is a quantity strongly related to the common volatility.

We define the "the fine grained" PDF of a market of M stocks

$$P_{M,\rho}(\mathbf{S}, t|\mathbf{S}_0, t_0) = \prod_{m=1}^{M} \delta(s_m - s(s_{0m}, t_0|t)) \tag{3}$$

where $\mathbf{S} = (s_1, \cdots, s_M)$, $\mathbf{S_0} = (s_{01}, \cdots, s_{0M})$ and $\mathbf{R}(t) = (n_1(t)\rho(s_1,t), \cdots, n_M(t)\rho(s_M,t))$. We take the time derivative of (3) and we have using (1)

$$\partial_t P_{M,\rho}(t) = -\nabla_{\mathbf{S}} \cdot [\mathbf{R}(t) P_{M,\rho}(t)] \tag{4}$$

We now integrate this expression and we get

$$P_{M,\rho}(\mathbf{S},t) = P_{M,\rho}(\mathbf{S},t_0) - \nabla_{\mathbf{S}} \cdot [\int_{t_0}^{t} dl \mathbf{R}(l) P_{M,\rho}(l)] \tag{5}$$

We now split the random field $\mathbf{R}(s,t)$ into its mean and fluctuating part

$$\mathbf{R}(\mathbf{S},t) = \bar{\mathbf{R}}(\mathbf{S},t) + \mathbf{R}'(\mathbf{S},t) \tag{6}$$

we substitute into (4)

$$\partial_t P_{M,\rho}(t) = -\nabla_{\mathbf{S}} \cdot [\bar{\mathbf{R}}(t) P_{M,\rho}(t)] - \nabla_{\mathbf{S}} \cdot [\mathbf{R}'(t) P_{M,\rho}(t)] \tag{7}$$

and we substitute (5) only in the fluctuating part of equation (7)

$$\partial_t P_{M,\rho}(t) = -\nabla_{\mathbf{S}} \cdot [\bar{\mathbf{R}}(t) P_{M,\rho}(t)] - \nabla_{\mathbf{S}} \cdot [\mathbf{R}'(t) P_{M,\rho}(\mathbf{S},t_0)] + \partial_{S_i}[R'_i(t)\partial_{S_j}[\int_{t_0}^{t} dl R_j(l) P_{M,\rho}(l)]] \tag{8}$$

or

$$\begin{aligned} \partial_t P_{M,\rho}(t) &= -\nabla_{\mathbf{S}} \cdot [\bar{\mathbf{R}}(t) P_{M,\rho}(t)] - \nabla_{\mathbf{S}} \cdot [\mathbf{R}'(t) P_{M,\rho}(\mathbf{S},t_0)] \\ &+ \partial_{S_i}\partial_{S_j}[\int_{t_0}^{t} dl R'_i(S_i,t) R_j(S_j,l) P_{M,\rho}(l)] - \partial_{S_i}[\int_{t_0}^{t} dl (\partial_{S_i} R'_i)(S_i,t) R_i(S_i,l) P_{M,\rho}(l)]. \end{aligned} \tag{9}$$

Here $S_i = s_i(s_{0i}, t_0|t)$ is understood to be a component of the vector $\mathbf{S}$ and $R_i(S_i,t) = n_i(t)\rho(s_i,t)$ a component of the vector $\mathbf{R}$.

We now average the last equation over the possible realizations of the random field ρ

$$\begin{aligned} \partial_t \langle P_{M,\rho}(t)\rangle_\rho &= -\nabla_{\mathbf{S}} \cdot [\bar{\mathbf{R}}(t) \langle P_{M,\rho}(t)\rangle_\rho] - \nabla_{\mathbf{S}} \cdot [\langle \mathbf{R}'(t) P_{M,\rho}(\mathbf{S},t_0)\rangle_\rho] \\ &+ \partial_{S_i}\partial_{S_j}[\int_{t_0}^{t} dl \langle R'_i(S_i,t) R_j(S_j,l) P_{M,\rho}(l)\rangle_\rho] \\ &- \partial_{S_i}[\int_{t_0}^{t} dl \langle (\partial_{S_i} R'_i)(S_i,t) R_i(S_i,l) P_{m,\rho}(l)\rangle_\rho] \end{aligned} \tag{10}$$

We have the PDF to have a M-stocks market to be

$$P_M(\mathbf{S},t|\mathbf{S}_0,t_0) = \langle P_{M,\rho}(\mathbf{S},t|\mathbf{S}_0,t_0)\rangle_\rho \tag{11}$$

We have an equation but we need some closure assumptions. We first make a "mean field" approximation that ignores the fluctuations in the fine grained PDF

$$\langle R'_i(S_i,t) R_j(S_j,l) P_{M,\rho}(l)\rangle_\rho = \langle R'_i(S_i,t) R'_j(S_j,l)\rangle_\rho \langle P_{M,\rho}(l)\rangle_\rho \tag{12}$$

What do we neglect with the "mean field" approximation? Lets split the time correlation function of the fluctuations and the fine grained PDF into their means and fluctuating parts

$$\begin{aligned} R'_i(S_i,t)R_j(S_j,l) &= \langle R'_i(S_i,t)R_j(S_j,l)\rangle_\rho + \delta(R'_i(S_i,t)R_j(S_j,l)) && (13)\\ P_{M,\rho}(l) &= \langle P_{M,\rho}(l)\rangle_\rho + \delta P_{M,\rho}(l) && (14)\end{aligned}$$

and we have

$$\langle R'_i(S_i,t)R_j(S_j,l)P_{M,\rho}(l)\rangle_\rho = \langle R'_i(S_i,t)R_j(S_j,l)\rangle_\rho\langle P_{M,\rho}(l)\rangle_\rho + \langle\delta(R'_i(S_i,t)R_j(S_j,l))\delta P_{M,\rho}(l)\rangle_\rho \tag{15}$$

We therefore assume that

$$\langle\delta(R'_i(S_i,t)R_j(S_j,l))\delta P_{M,\rho}(l)\rangle_\rho \ll \langle R'_i(S_i,t)R_j(S_j,l)\rangle_\rho\langle P_{M,\rho}(l)\rangle_\rho \tag{16}$$

We simplify further the equation by assuming that the correlation functions decay rapidly for $l \ll t$, changing much faster than $P_M(l)$. This allows to replace $P_M(l)$ inside the time integral by $P_M(t)$ giving a "Markovianized" equation. The Markovian approximation assumes that the variables $R'_i(S_i,t)$ time - uncorrelate so fast that

$$\langle R'_i(S_i,t)R'_j(S_j,l)\rangle_\rho P_M(l) \simeq \delta(t-l)P_M(t) \tag{17}$$

We then obtain a diffusion equation

$$\begin{aligned} \partial_t P_M(t) &= -\nabla_{\mathbf{S}}\cdot[\bar{\mathbf{R}}(t)P_M(t)] && (18)\\ &+ \partial_{S_i}\partial_{S_j}[\int_{t_0}^{t} dl\langle R'_i(S_i,t)R'_j(S_j,l)\rangle_\rho P_M(t)]\\ &- \partial_{S_i}[\int_{t_0}^{t} dl\langle(\partial_{S_i}R'_i)(S_i,t)R'_i(S_i,l)\rangle P_M(t)]\end{aligned}$$

We define

$$D_{ij}(S_i,S_j,t,t_0) = \int_{t_0}^{t} dl\langle R'_i(S_i,t)R'_j(S_j,l)\rangle_\rho \tag{19}$$

and

$$\bar{R_i}^*(S_i,t) = \bar{R}_i(S_i,t) + (\partial_{S_i}D_{ij}(S_i,S_j,t,t_0))|_{S_j=S_i} \tag{20}$$

It is important to notice that in theory D_{ij} and $\bar{R_i}^*$ can be computed using real data. then we have

$$\begin{aligned} \partial_t P_M(t) &= -\nabla_{\mathbf{S}}\cdot[\bar{\mathbf{R}}^*(t)P_M(t)] && (21)\\ &+ \partial_{S_i}\partial_{S_j}[D_{ij}(S_i,S_j,t,t_0)P_M(t)]\end{aligned}$$

A interesting quantity could be $P_2(s,s+q,t|s_0,s_0+r_0,t_0)$ giving you the probability of dispersion of 2 stocks from each other.

2 The 1-stock market

In the case where we have only one stock

$$\begin{aligned} \partial_t P_1(t) &= -\partial_s[(n(t)\langle\rho(s,t)\rangle P_M(t)] - \partial_s[\int_{t_0}^t dl \langle \partial_s(n(t)\rho'(s,t))n(l)\rho'(s,t|l)\rangle) P_M(t)] \\ &+ \frac{\partial^2}{\partial s^2}[\int_{t_0}^t dl\langle n(t)\rho'(s,t)n(l)\rho'(s,t|l)\rangle_\rho P_M(t)] \end{aligned} \quad (22)$$

If we define

$$a(s,t) = n(t)\langle\rho(s,t)\rangle + \int_{t_0}^t dl\langle\partial_s(n(t)\rho'(s,t))n(l)\rho'(s,t|l)\rangle \quad (23)$$

and

$$b(s,t) = 2\int_{t_0}^t dl\langle n(t)\rho'(s,t)n(l)\rho'(s,t|l)\rangle_\rho \quad (24)$$

We can cast the equation under its stochastic Langevin equivalent form

$$ds = a(s,t)dt + \sqrt{b(s,t)}dW \quad (25)$$

where W is a brownian motion. This later equation may be compared to the commonly accepted diffusion equation for the stock price

$$ds = \mu s dt + s\sigma dW \quad (26)$$

3 The particular case of the Log-normal distributed stock price

We have $\ln(s) \sim N(\mu t, \sigma^2 t)$ and $\frac{ds}{dt} = r(s,t)$. As a consequence

$$s\frac{d\ln(s)}{dt} = r(s,t) \quad (27)$$

We have

$$a(s,t) = \langle r(s,t)\rangle + (\partial_s \int_{t_0}^t dl\langle r'(s,t)r'(s_2,t|l)\rangle)|_{s_2=s} \quad (28)$$

because we have the property that

$$s(s_0,t_0|t)P_{1,\rho}(t) = sP_{1,\rho}(l) \quad (29)$$

we can write

$$\langle r(s,t)\rangle = s\frac{d}{dt}\langle\ln(s)\rangle = s\mu \quad (30)$$

Because the two random variables $\ln s$ and $\ln s_2$ are independent variables we have

$$\langle r'(s,t)r'(s_2,t|l)\rangle = ss_2\frac{d^2}{dtdl}\langle(\ln s(t))'(\ln s_2(t|l))'\rangle = ss_2\frac{d^2}{dtdl}\langle(\ln s(t))'\rangle\langle(\ln s_2(t|l))'\rangle = 0 \quad (31)$$

We have also

$$\begin{aligned} b(s,t) &= 2\int_{t_0}^{t} dl \langle r'(s,t) r'(s,t|l)\rangle \\ &= 2s^2 \int_{t_0}^{t} dl \frac{d^2}{dtdl} \langle (\ln s(t))'(\ln s(t|l))'\rangle \\ &= 2s^2 \int_{t_0}^{t} dl \frac{d^2}{dtdl} \sigma^2 \min(s,t) \\ &= 2s^2\sigma^2 \int_{t_0}^{t} dl \delta(t-l) \end{aligned} \tag{32}$$

For $t' > t$ we have $\int_{t_0}^{t'} dl\delta(t-l) = 1$, therefore

$$\int_{t_0}^{t'} dl\delta(t-l) = \int_{t_0}^{t} dl\delta(t-l) + \int_{t}^{t'} dl\delta(t-s) = 2\int_{t_0}^{t} dl\delta(t-l) \tag{33}$$

or $\int_{t_0}^{t} dl\delta(t-l) = \frac{1}{2}$ and finally $b(s,t) = s^2\sigma^2$ and

$$ds = \mu s dt + s\sigma dW \tag{34}$$

We can conclude that for the Log-normal distributed case the mean field approximation and the Markov approximation lead to the exact form of the diffusion equation. Because the Log-normal distribution is usually consider as a "good guess" for the stock price, the different approximations giving equation (25) seem quite justified and could lead to a relevant correction of the distribution.

4 The Black-Scholes type equation

If $C(t, s(t))$ is the price of a derivative of the stock s, using Ito lemma and the equation (25), we get

$$\begin{aligned} dC(t,s(t)) &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial s}ds + \frac{1}{2}\frac{\partial^2 C}{\partial s^2} b(s,t)dt \\ &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial s}a(s,t)dt + \frac{\partial C}{\partial s}\sqrt{b(s,t)}dW + \frac{1}{2}\frac{\partial^2 C}{\partial s^2} b(s,t)dt \end{aligned} \tag{35}$$

We have a portfolio P with x_t share of the stock and y_t share of a bond of value B

$$P = x_t S + y_t B \tag{36}$$

We assume the evolution of the bond to driven by processes such as interest rates and in general

$$dB = \frac{dB}{dt}(B,t)dt \tag{37}$$

We have

$$\begin{aligned} dP &= x_t dS + y_t dB \\ &= x_t a(s,t)dt + x_t \sqrt{b(s,t)}dW + y_t \frac{dB}{dt} dt \end{aligned} \tag{38}$$

We want to match the evolution of P to the evolution of C. We get

$$x_t = \frac{\partial C}{\partial s} \tag{39}$$

and requiring $P = C$ we have

$$y_t = \frac{1}{B}(C - s\frac{\partial C}{\partial s}) \tag{40}$$

Using $dP = dC$ we get

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial s^2} b(s,t) + \frac{d(\ln B)}{dt}(s\frac{\partial C}{\partial s} - C) = 0 \tag{41}$$