Formal verification of systems – a survey of approaches from classical to recent developments

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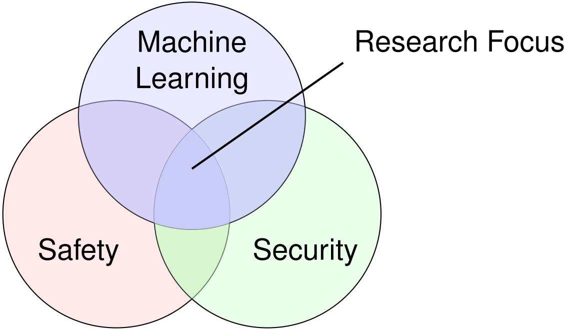
Objectives

- Obtain an initial understanding of formal concepts
- Survey of classical and recent approaches to formal verification
- Also establish the bridge to related work and future research directions I am aiming at

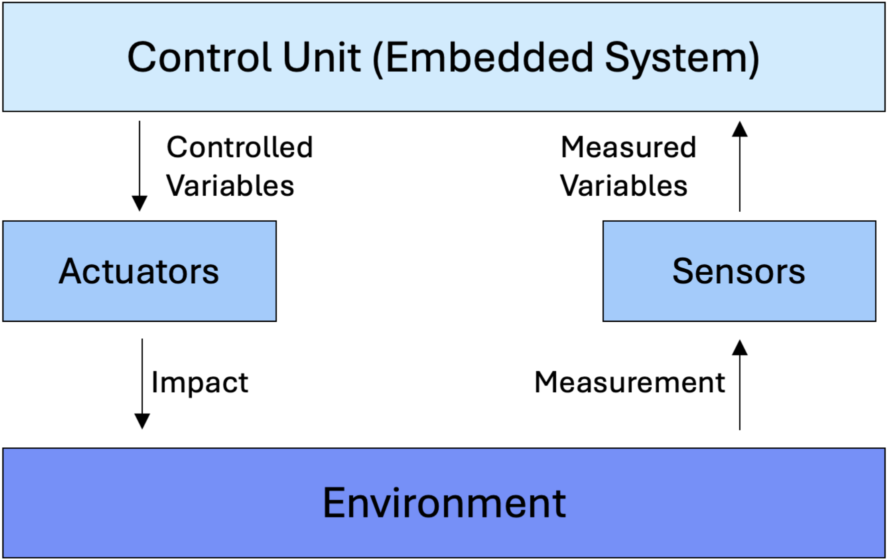
My Research Focus

My background: **formal verification** (particularly model-driven engineering of embedded or cyber-physical systems) and **security.**
Recently, also **machine learning.**
So, in essence, I am interested in **safety** and **security** of **AI-enabled systems** or the application of **Machine Learning** to classical approaches for the verification of safety and security of systems.



Outline

Embedded Systems



Why formal verification?

Language of first-order logic

A language $\mathscr{L}$ of first-order logic consists of the following components:

- Variable symbols: $x_1, x_2, \ldots$
- For each $n \in \mathbb{N}$, a set of n-ary function symbols: $f_0, f_1, \ldots$ The 0-ary function symbols are called constant symbols.
- For each $n \in \mathbb{N}$, a set of n-ary predicate symbols: $p_0, p_1, \ldots$ The 0-ary predicate symbols are the constants $\top$ (for **true**) and $\bot$ (for **false**).
- special symbols: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\forall$ (universal quantification), $\exists$ (existential quantification), and parentheses.

Terms

The set of terms of $\mathscr{L}$ is defined inductively as follows:

- Each variable is a term.
- If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then if $f(t_1, \ldots, t_n)$ is a term.

Variables in terms

We define a function var : Terms $\rightarrow$ Variables that maps each term to the set of variables occurring in it. The function is defined as follows:

- $var(x) = \{x\}$ for each variable x.
- $var(f(t_1, \ldots, t_n)) = var(t_1) \cup \ldots \cup var(t_n)$.

Formulas

The set of formulas of $\mathscr{L}$ is defined inductively as follows:

- If $t_1, \ldots, t_n$ are terms and p is an n-ary predicate symbol, then if $p(t_1, \ldots, t_n)$ is a formula.
- If φ is a formula, then if $\neg\varphi$ is a formula.
- If φ_1 and φ_2 are formulas, then if $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \rightarrow \varphi_2$, and $\varphi_1 \leftrightarrow \varphi_2$ are formulas.
- If φ is a formula and x is a variable, then if $\forall x.\varphi$ and $\exists x.\varphi$ are formulas.

An example of a formula is $\forall x.\exists y.p(x, y) \rightarrow \neg q(y)$.

Interpretations

An interpretation $\mathcal{M}$ of $\mathscr{L}$ consists of the following components:

- A non-empty set D called the domain of $\mathcal{M}$.
- For each n-ary function symbol f of $\mathscr{L}$, a function $f^{\mathcal{M}} : D^n \to D$.
- For each n-ary predicate symbol p of $\mathscr{L}$, a relation $p^{\mathcal{M}} \subseteq D^n$.

Interpretations of Terms

Let $\mathcal{M}$ be an interpretation for our first-order language. An assignment σ of values to variables, i.e., $\sigma : Variables \to D$.
The value of a term t under σ is denoted by $t^{\mathcal{M}}[\sigma]$ and defined as follows:

- If $t = x$ for a variable x, then $t^{\mathcal{M}}[\sigma] = \sigma(x)$.
- If $t = f(t_1, \ldots, t_n)$, then $t^{\mathcal{M}}[\sigma] = f^{\mathcal{M}}(t_1^{\mathcal{M}}[\sigma], \ldots, t_n^{\mathcal{M}}[\sigma])$.

Validity of Formulas under Interpretations

We say an assignment σ satisfies a formula φ under an interpretation $\mathcal{M}$, denoted by $\mathcal{M}, \sigma \models \varphi$, iff the following conditions hold:

- $\varphi = p(t_1, \dots, t_n)$, then if $(t_1^{\mathcal{M}}[\sigma], \dots, t_n^{\mathcal{M}}[\sigma]) \in p^{\mathcal{M}}$.
- $\varphi = \neg\psi$, then if $\mathcal{M}, \sigma \not\models \psi$.
- $\varphi = \psi_1 \vee \psi_2$, then if $\mathcal{M}, \sigma \models \psi_1$ or $\mathcal{M}, \sigma \models \psi_2$.
- $\varphi = \psi_1 \wedge \psi_2$, then if $\mathcal{M}, \sigma \models \psi_1$ and $\mathcal{M}, \sigma \models \psi_2$.
- $\varphi = \psi_1 \rightarrow \psi_2$, then if $\mathcal{M}, \sigma \models \psi_1$ implies $\mathcal{M}, \sigma \models \psi_2$.
- $\varphi = \psi_1 \leftrightarrow \psi_2$, then if $\mathcal{M}, \sigma \models \psi_1$ if and only if $\mathcal{M}, \sigma \models \psi_2$.
- $\varphi = \forall x.\psi$, then if $\mathcal{M}, \sigma[x \mapsto d] \models \psi$ for all $d \in D$.
- $\varphi = \exists x.\psi$, then if $\mathcal{M}, \sigma[x \mapsto d] \models \psi$ for some $d \in D$.

A formula φ is satisfiable if there exists an interpretation $\mathcal{M}$ and an assignment σ such that $\mathcal{M}, \sigma \models \varphi$.

Models

An interpretation $\mathcal{M}$ is a model of a formula φ, denoted by $\mathcal{M} \models \varphi$, if for all assignments σ, $\mathcal{M}, \sigma \models \varphi$.

Validity

A formula φ is valid if for all interpretations $\mathcal{M}$ and all assignments σ, $\mathcal{M}, \sigma \models \varphi$.
We write $\models \varphi$ to denote that φ is valid.

Free Variables in Fomulas

The set of free variables of a formula φ, denoted by $FV(\varphi)$, is defined inductively as follows:

- $FV(p(t_1, \ldots, t_n)) = var(t_1) \cup \ldots \cup var(t_n)$.
- $FV(\neg\psi) = FV(\psi)$.
- $FV(\psi_1 \wedge \psi_2) = FV(\psi_1) \cup FV(\psi_2)$.
- $FV(\psi_1 \vee \psi_2) = FV(\psi_1) \cup FV(\psi_2)$.
- $FV(\psi_1 \rightarrow \psi_2) = FV(\psi_1) \cup FV(\psi_2)$.
- $FV(\forall x.\psi) = FV(\psi) \setminus \{x\}$.
- $FV(\exists x.\psi) = FV(\psi) \setminus \{x\}$.

Term Substitution

Let φ be a formula, x a variable, and t a term. The formula $\varphi[t/x]$ is obtained by replacing all occurrences of x in φ by t. The substitution is defined inductively as follows:

- $(p(t_1, \ldots, t_n))[t/x] = p(t_1[t/x], \ldots, t_n[t/x])$.
- $(\neg\psi)[t/x] = \neg\psi[t/x]$.
- $(\psi_1 \wedge \psi_2)[t/x] = \psi_1[t.x] \wedge \psi_2[t/x]$.
- $(\psi_1 \vee \psi_2)[t/x] = \psi_1[t/x] \vee \psi_2[t/x]$.
- $(\psi_1 \rightarrow \psi_2)[t/x] = \psi_1[t/x] \rightarrow \psi_2[t/x]$.
- $(\forall y.\psi)[t/x] = \forall y.\psi[t/x]$ if $x \in FV(t)$.
- $(\exists y.\psi)[t/x] = \exists y.\psi[t/x]$ if $x \in FV(t)$.
- $(\forall x.\psi)[t/x] = \forall x.\psi$.
- $(\exists x.\psi)[t/x] = \exists x.\psi$.

So, $\varphi[t/x]$ represents the formular obtained by substituting every **free** occurrence of the variable x in φ by the term t.

Calculus

A calculus is a mechanism to prove formulas by applying rules.
A rule of a calculus has the form $\frac{\varphi_1,\ldots,\varphi_n}{\psi}$, where $\varphi_1, \ldots, \varphi_n$ are premises and ψ is the conclusion. The rule states that if $\varphi_1, \ldots, \varphi_n$ are derivable, then ψ is derivable.
We denote that a formula can be proved by a calculus by $\vdash \varphi$.

Sequent Calculus

In sequent calculus, we have sequences $\Gamma \vdash \Delta$, where Γ and Δ are sets of formulas.

The interpretation is that if all formulas in Γ are true, then at least one formula in Δ is true.

Sequent Calculus Rules

$$\frac{-}{\Gamma, \varphi \Rightarrow \varphi, \Delta} \text{ Taut} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{ Cut}$$

$$\frac{-}{\Gamma, \bot \Rightarrow \Delta} \bot \Rightarrow \qquad \frac{-}{\Gamma \Rightarrow \Delta, \top} \Rightarrow \top$$

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{ Weakening left} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \text{ Weakening right}$$

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{ Contraction left} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \text{ Contraction right}$$

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{ Exchange left} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda} \text{ Exchange right}$$

Sequent Calculus Rules

$$\frac{-}{\Gamma, \bot \Rightarrow \Delta} \bot \Rightarrow \qquad \frac{-}{\Gamma \Rightarrow \Delta, \top} \Rightarrow \top$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta} \neg \Rightarrow \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \Rightarrow \neg$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \vee \Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \Rightarrow \vee$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \wedge \Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \Rightarrow \wedge$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Lambda}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \to\Rightarrow \qquad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} \Rightarrow\to$$

Sequent Calculus Rules

$$\frac{\Gamma, \varphi[t/x] \Rightarrow \Delta}{\Gamma, \forall x.\varphi(x) \Rightarrow \Delta} \forall \Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \varphi[y/x]}{\Gamma \Rightarrow \Delta, \forall x.\varphi(x)} \Rightarrow \forall$$

$$\frac{\Gamma, \varphi[y/x] \Rightarrow \Delta}{\Gamma, \exists x.\varphi(x) \Rightarrow \Delta} \exists \Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta, \exists x.\varphi(x), \varphi[t/x]}{\Gamma \Rightarrow \Delta, \exists x.\varphi(x)} \Rightarrow \exists$$

In the quantifier rules, t is a term, and y is a 'fresh' variable, i.e., a variable that does not occur in Γ, Δ, or φ.
Alternatively, the rules can also be stated in the form

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \forall x.\varphi(x)} \Rightarrow \forall$$

Here, it must be guaranteed that x is not free in any formula in Γ or Δ.
The existential formula can be handled similarly.

Example Deduction: $\forall x.(P(x) \wedge Q \Rightarrow \forall x.P(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), Q \Rightarrow P(x)}\ \text{Taut}}{P(x) \wedge Q \Rightarrow P(x)}\ \wedge\Rightarrow}{\forall x.(P(x \wedge Q) \Rightarrow P(x))}\ \forall\Rightarrow}{\forall x.(P(x) \wedge Q) \Rightarrow \forall x.P(x)}\ \Rightarrow\forall$$

Here, $\forall\Rightarrow$ uses $[x/x]$ as replacement, i.e., just the same free variable is taken.

Example Deduction: $\forall x.(A \to B) \Rightarrow A \to \forall x.B$

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    \dfrac{
      \dfrac{}{A \Rightarrow A, B}\,\text{Taut}
      \qquad
      \dfrac{}{A, B \Rightarrow B}\,\text{Taut}
    }{A, A \to B \Rightarrow B}\,{\to}{\Rightarrow}
  }{A, \forall x.(A \to B) \Rightarrow B}\,\forall \Rightarrow
}{
  \dfrac{A, \forall x.(A \to B) \Rightarrow \forall x.B}{\forall x.(A \to B) \Rightarrow A \to \forall x.B}\,{\Rightarrow}{\to}
}\,\Rightarrow \forall
$$

Here, the application of $\Rightarrow \forall$ requires that x is not free in A.

Example of a Failing Deduction: $\exists x.P(x) \wedge \exists x.Q(x) \Rightarrow \exists x.(P(x) \wedge Q(x))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(y) \Rightarrow P(x) \wedge Q(x)}{P(x), Q(y) \Rightarrow \exists x.(P(x) \wedge Q(x))}\Rightarrow\exists}{P(x), \exists x.Q(x) \Rightarrow \exists x.(P(x) \wedge Q(x))}\exists\Rightarrow}{\exists x.P(x), \exists x.Q(x) \Rightarrow \exists x.(P(x) \wedge Q(x))}\exists\Rightarrow}{\exists x.P(x) \wedge \exists x.Q(x) \Rightarrow \exists x.(P(x) \wedge Q(x))}\wedge\Rightarrow
$$

Here, the deduction fails because the variable x is not fresh in the application of $\exists \Rightarrow$ and therefore the new variable y is introduced. However, then the deduction cannot be completed.

Soundness and Completeness of Sequent Calculus

- A calculus is sound if all provable formulas are valid, denoted by $\vdash \varphi \Rightarrow \models \varphi$.
- A calculus is complete if all valid formulas are provable, denoted by $\models \varphi \Rightarrow \vdash \varphi$.
- The sequent calculus is sound and complete for first-order logic.