Class Test 1 **Date: June 21, 2023**

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY

Department of Computer Science and Engineering

L-3/T-II CSE 301: Mathematical Analysis for Computer Science

Time: 30 minutes **Marks: 20**

Student Name: ______________________________ Student No: ____________

1. We are already acquainted with the Josephus problem. We are now presented with a new variation of the problem. As usual, we begin with a group of n individuals numbered from 1 to n arranged in a circular formation and eliminate every second remaining person until only one individual remains. However, in this variant, we perform the eliminations in reverse order. For instance, when $n = 10$, the sequence of eliminations is $9, 7, 5, 3, 1, 8, 4, 10, 2$, resulting in the survival of individual 6.

   Now, answer the following questions:

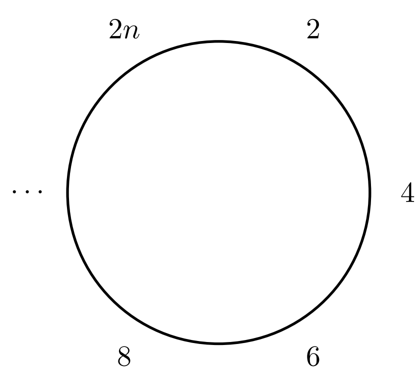
   (a) Formulate the recurrence relations for determining the survivor's number in this new variation of the Josephus problem. Remember to specify the base case too. (7)

   (b) Show that the survivor's number can be expressed as $n - 2l$, where $n = 2^m + l$ and $0 \le l < 2^m$. (3)

Solution:

(a) Let the survivor's number for the new variation be denoted by $J(n)$. Base case: $J(1) = 1$.

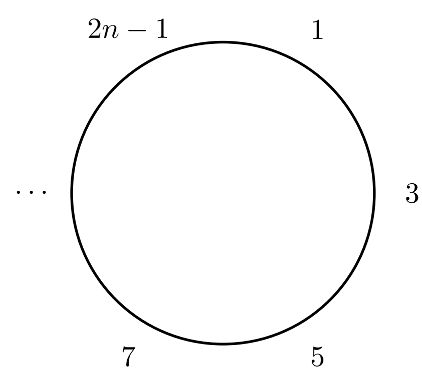
Let's suppose that we have $2n$ individuals originally. After the 1st go-round all odd individuals are eliminated, and we're left with



This is just like starting out with n individuals, except that each individual's number has been doubled. That is,

$$J(2n) = 2J(n), \text{ for } n \ge 1$$

With $2n + 1$ individuals, all even individuals are eliminated at first and individual $2n + 1$ is wiped out just after individual 2, and so we're left with



Again we almost have the original situation with n individuals, but this time their numbers are doubled and *decreased* by 1. Thus

$$J(2n + 1) = 2J(n) - 1, \text{ for } n \ge 1$$

Combining these equations with $J(1) = 1$ gives us a recurrence that defines J in all cases.

(b) We know that the general recurrence

$$\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n) + \beta, \text{ for } n \ge 1 \\ f(2n+1) &= 2f(n) + \gamma, \text{ for } n \ge 1 \end{aligned}$$

has solution in the form $f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$, where

$$\begin{aligned} A(n) &= 2^m \\ B(n) &= 2^m - l - 1 \\ C(n) &= l \end{aligned}$$

In this variation, $\alpha = 1, \beta = 0, \gamma = -1$. Hence $J(n) = 2^m \times 1 + l \times (-1) = 2^m - l = (2^m + l) - 2l = n - 2l$.[1]

[1] Can also be solved using induction on m using the fact that $n - 2l$ can be expressed as $2^m - l$.

2. We have previously studied the formulation for the minimum number of moves required in the Tower of Hanoi problem. Now, we are presented with a new variation where the tower comprises $2n$ disks having n distinct sizes and exactly two disks of each size. As usual, we can only move one disk at a time and cannot place a larger disk onto a smaller one at any time. For this particular variation, we have devised the following steps to solve the problem:

   1. We 'miraculously' transfer the top $2 \times (n-1)$ disks to the intermediary peg.
   2. Next, we move the two largest disks to the destination peg.
   3. Finally, we transfer the $2 \times (n-1)$ disks, once again 'miraculously', from the intermediary peg to the destination peg.

   Now, answer the following questions:

   (a) Formulate a recurrence relation to determine the number of moves required based on the aforementioned steps. Remember to specify the base case too. (3)

   (b) Show that the number of moves required for the given steps is equal to $2^{n+1} - 2$. (3)

   (c) Prove that this solution preserves the original order of the top $2 \times (n-1)$ disks while reversing the order of the two largest disks. (4)

   Solution:

   (a) Let A_n denote the number of moves for the $2n$ disks. Then the three aforementioned steps incur $A_{n-1}, 2, A_{n-1}$ moves respectively. So, $A_n = 2A_{n-1} + 2$ and $A_0 = 0$.

   (b) We prove using induction on n.

   - *Base case*: For $n = 0$, $A_0 = 2^{0+1} - 2 = 0$. Hence, the base case holds.
   - *Induction hypothesis:* Let $A_m = 2^{m+1} - 2, \forall 0 \leq m \leq n-1$.
   - *Induction step:* Now we prove the assertion true for $m = n$.

   $$A_n = 2A_{n-1} + 2 = 2(2^{(n-1)+1} - 2) + 2 = 2^{n+1} - 2$$

   (c) We prove the statement "this solution preserves the original order of the top $2 \times (n-1)$ disks while reversing the order of the two largest disks." using induction on n.

   - *Base case*: For $n = 1$, we have only two disks of equal size; moving them one after another to the destination peg will reverse their order as this is a LIFO operation. And for $n = 1$, top $2 \times (n-1)$ disks basically means an empty stack of disks. Hence, the base case holds.[2]
   - *Inductive hypothesis:* Let the statement be true $\forall 1 \leq m \leq n-1$.
   - *Induction step*: Now we prove the statement true for $m = n$. Transferring the top $2 \times (n-1)$ disks to the intermediary peg (Step 1.) will result in the reversal of their two largest disks (i.e., the second-largest disks overall) by our induction hypothesis. Then moving the two largest disks (Step 2.) to the destination peg will reverse their order as this is again a LIFO operation. Finally, transferring the top $2 \times (n-1)$ disks from the intermediary peg to the destination peg (Step 3.) will result in another reversal of the the second-largest disks (by our induction hypothesis), reinstating their original order. The above-mentioned arguments imply that the original order of the top $2 \times (n-1)$ disks will be preserved but the two largest disks will be reversed, completing our proof.

[2] For $n = 0$, we do not have any physical notion of top $2 \times (n-1) = -2$ disks. Therefore, $n = 0$ cannot be a valid base case as per the problem statement.