Chapter 14 (AIAMA)

Probabilistic Reasoning

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Bayesian Network: Exact Inference

- **Consider the query**: $P(Burglary|JohnCalls = ture, MaryCalls = true)$
- We need to compute: $P(B|j,m)$ [compute for both b and $\neg b$]
- Express in terms of joint distributions [we already know how to compute joint distribution]

$$P(B|j,m) = \frac{P(B,j,m)}{P(j,m)} = \alpha P(B,j,m)$$

where α is the normalizing constant [we can find the value of α later]

Bayesian Network: Exact Inference

- How to compute $P(B, j, m)$?
- Add all hidden variables to get full joint distribution

$$P(B, j, m) = \sum_a \sum_e P(B, j, m, a, e) = \sum_a \sum_e P(j|a)P(m|a)P(a|B, e)P(B)P(e)$$

  - Query variable: B
  - Evidence variables: J, M
  - Hidden variables: A, E

Bayesian Network: Exact Inference

- Finally, we get the following:

  $P(B|j,m) = \alpha \sum_a \sum_e P(j|a)P(m|a)P(a|B,e)P(B)P(e)$

  - Compute $\mathrm{P}(b|j,m)$ and $\mathrm{P}(\neg b|j,m)$ [expression has α]
  - Find α [Using the eq. $\mathrm{P}(b|j,m) + \mathrm{P}(\neg b|j,m) = 1$]

- **Question**: How many operations (multiplications + additions)?
  - Answer: [For each $P(B|j,m)$] Mult: 16, Sum: 4, Total = 20 operations

Bayesian Network: Exact Inference

- Computing using join distribution naively:

  $P(B|j,m) = \alpha \sum_a \sum_e P(j|a)P(m|a)P(a|B,e)P(B)P(e)$

- Required operations: 20
- **Can we improve?**
  - Yes, move sums inside [*closest to the factors having the hidden variable*]
  - Sums are evaluated early and gets multiplied once [*rather than every time during joint calculation*]

Bayesian Network: Exact Inference

- Move summations inside: $P(B|j,m) = \alpha \sum_a \sum_e P(j|a)P(m|a)P(a|B,e)P(B)P(e)$
- Two alternate ways to do it:
  - Option 1: $P(B|j,m) = \alpha P(B) \sum_a P(j|a)P(m|a) \sum_e P(a|B,e)P\,(e)$
  - Option 2: $P(B|j,m) = \alpha P(B) \sum_e P\,(e) \sum_a P(a|B,e)P(j|a)P(m|a)$
  - Which one is more efficient in terms of operations?

Bayesian Network: Exact Inference

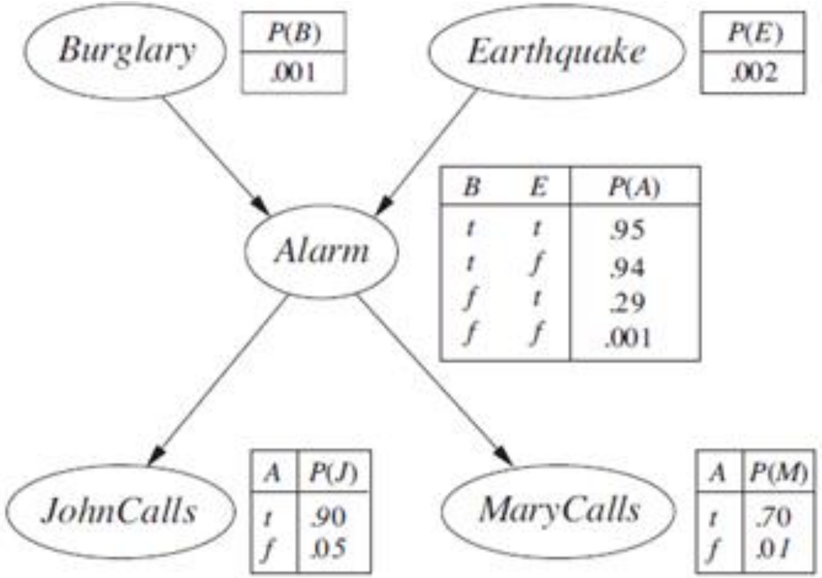
- **Which one is more efficient?** Assume $B = true$ and compute both.
- Equation 1: $P(B|j,m) = \alpha P(B) \sum_a P(j|a)P(m|a) \sum_e P(a|B,e)P\,(e)$

  - Required # of multiplications: 9, # of sums = 3, total = 12
- Equation 2: $P(B|j,m) = \alpha P(B) \sum_e P\,(e) \sum_a P(a|B,e)P(j|a)P(m|a)$

  - Required # of multiplications: 11, # of sums = 3, total = 14
- So order matters! However, finding optimal order is difficult!

Bayesian Network: Exact Inference

- Using the CPT data, we calculate:



$P(b|j,m) = \alpha \times 0.00059224$
$P(\neg b|j,m) = \alpha \times 0.0014919$

Hence,

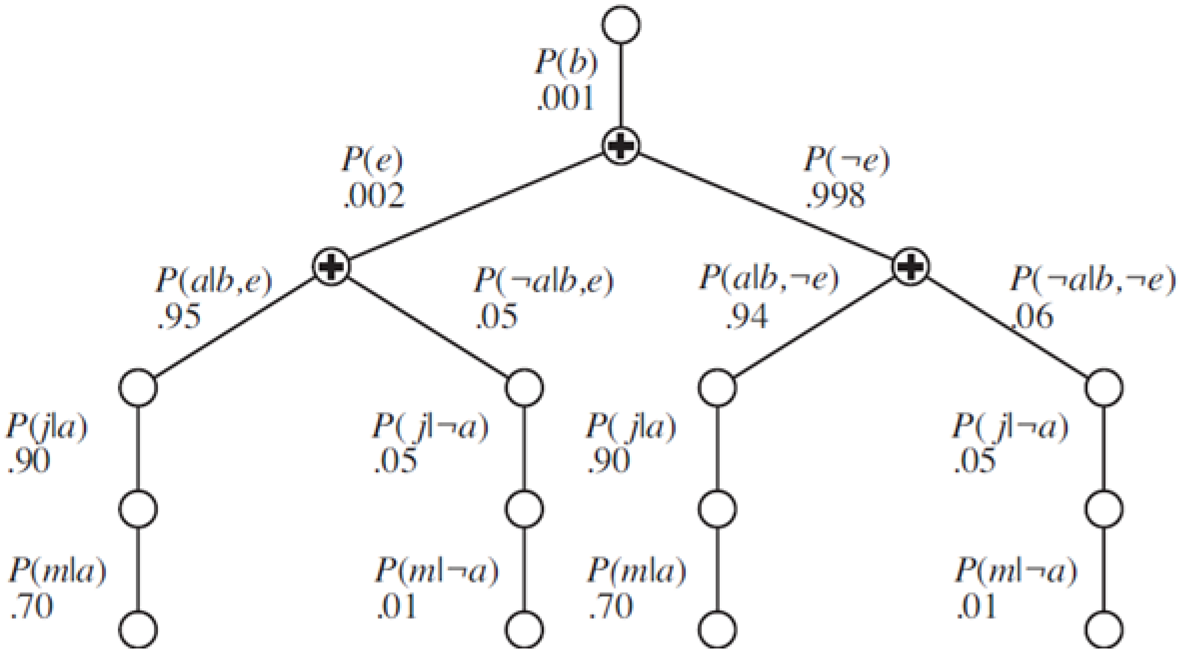
$P(B|j,m) = \alpha \times (0.00059224,\ 0.0014919)$
$= (0.284, 0.716)$ [After normalizing as probabilities sum to 1]

Bayesian Network: Exact Inference

- An evaluation tree is shown for the expression:

$$\alpha P(B) \sum_e P\,(e) \sum_a P(a|B,e)P(j|a)P(m|a)$$

  - Blank circled nodes represent multiplication [*Notice the repetition of sub-paths*]

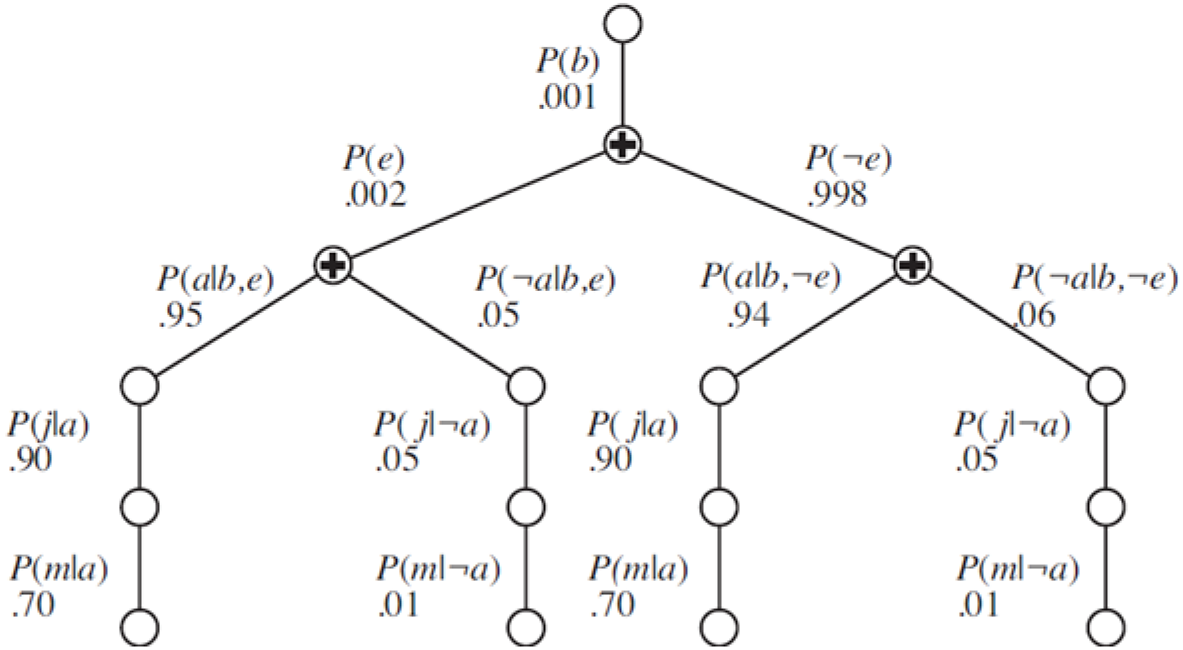


Bayesian Network: Exact Inference

- **Problem:** Repeated subexpression evaluation
- The following products are computed twice in two paths of the tree!

$$P(j \mid a)P(m \mid a)$$

$$P(j \mid \neg a)P(m \mid \neg a)$$



Bayesian Network: Variable Elimination Algorithm

- Steps of variable elimination algorithm:
  - Step 1: Compute factors for each node of Bayesian Network and instantiate/restrict evidence variables.
  - Step 2: For some order of variables $(V_1, V_2, .., V_N)$:
    - (2a) If variable V is a hidden variable, then multiply all the factors of V
    - (2b) Sum-out V
  - Step 3: Multiply remaining factors.
  - Step 4: Normalize the final factor to eliminate α.

Bayesian Network: Variable Elimination Algorithm

- Consider the following expression: $P(B|j, m)$.
    - Query variable: B
    - Evidence variable: J, M
    - Hidden variable: A, E

- We will use variable elimination algorithm to calculate

Bayesian Network: Variable Elimination Algorithm

- Query: $P(B|j, m)$.
- Joint probability expression of BN:

$$P(B, E, A, J, M) = P(B)P(E)P(A|B, E)P(J|A)P(M|A)$$

- **Step 1**: Compute all the factors of BN.

- The BN has five factors [*note the joint probability expression*]:
  - $f_1(B) = P(B)$
  - $f_2(E) = P(E)$
  - $f_3(A, B, E) = P(A|B, E)$
  - $f_4(J, A) = P(J|A)$
  - $f_5(M, A) = P(M|A)$

Variable Elimination Algorithm: Compute Factors

- Query: $P(B|j,m)$.
- **Step 1a**: Compute all the factors of BN. [*don't restrict evidences now*]

$f_1(B) = P(B)$

T	0.001
F	0.999

$f_2(E) = P(E)$

T	0.002
F	0.998

$f_3(A,B,E) = P(A|B,E)$

T	T	T	0.95
T	T	F	0.94
T	F	T	0.29
T	F	F	0.001
F	T	T	0.05
F	T	F	0.06
F	F	T	0.71
F	F	F	0.999

$f_4(J,A) = P(J|A)$

T	T	0.90
T	F	0.05
F	T	0.10
F	F	0.95

$f_5(M,A) = P(M|A)$

T	T	0.70
T	F	0.01
F	T	0.30
F	F	0.99

Variable Elimination Algorithm: Restrict Factors

- Query: $P(B|j,m)$.
- **Step 1b**: Instantiate/restrict evidence variables in all factors [j *and* m].
  - Restrict J *to* true *in* $f_4(J,A)$. [*Keep only rows where* $J = true$]
  - Note that J is removed from factor expression.

$f_4(J,A) = P(J|A)$

T	T	0.90
T	F	0.05
F	T	0.10
F	F	0.95

⇨

$f_4(A)$

T	0.90
F	0.05

Variable Elimination Algorithm: Restrict Factors

- Query: $P(B|j, m)$.
- **Step 1b**: Instantiate/restrict evidence variables in all factors. [j *and* m].
  - Restrict M *to* true *in* $f_5(M, A)$. [*Keep only rows where* $M = true$]

$f_5(M, A) = P(M|A)$

T	T	0.70
T	F	0.01
F	T	0.30
F	F	0.99

$\Rightarrow$

$f_5(A)$

T	0.70
F	0.01

Variable Elimination Algorithm: Multiply Factors

- Query: $P(B|j,m)$.
- **Step 2**: Select every variable V (in any order) which is hidden [Here, A and E]
  - Multiply all factors of V and then sum-out the variable.
  - Suppose we select in this order: E, A.
    - First we select E
    - Multiple all factors having E and get a new factor: $f_6(A,B,E) = f_2(E) \times f_3(A,B,E)$
    - Sum-out E and get a new factor: $f_7(A,B) = \sum_E f_6(A,B,E)$
    - Note that after summing-out, E is removed from factor-argument.

Bayesian Network: Multiply Factors

- **Step 2a**: Multiple all factors having E and get a new factor:

$$f_6(A, B, E) = f_2(E) \times f_3(A, B, E)$$

- Factor multiplication is a point-wise multiplication of elements in two factors:
  - Multiply rows which have matching values of the variables.
  - Number of rows in the new factor 2^N where N is number of total variables in the new factor:
    - Multiplying $f_1(A, B)$ and $f_2(B, C)$ will give us $f_3(A, B, C)$ which will have 8 rows.

Bayesian Network: Multiply Factors

- For variable E:
  - **Step 2a**: Multiple all factors having E and get a new factor:
  - **Multiply**: $f_6(A, B, E) = f_2(E) \times f_3(A, B, E)$

$f_2(E) = P(E)$

T	0.002
F	0.998

$\times$

$f_3(A, B, E) = P(A|B, E)$

T	T	T	0.95
T	T	F	0.94
T	F	T	0.29
T	F	F	0.001
F	T	T	0.05
F	T	F	0.06
F	F	T	0.71
F	F	F	0.999

$\Rightarrow$

$f_6(A, B, E)$

T	T	T	0.0019
T	T	F	0.93812
T	F	T	0.00058
T	F	F	0.000998
F	T	T	0.0001
F	T	F	0.05988
F	F	T	0.00142
F	F	F	0.997002

Bayesian Network: Multiply Factors

- For variable E:
  - **Step 2b**: Sum-out resulting factor for E and get a new factor:
  - **Sum-out *E***: $f_7(A,B) = \sum_E f_6(A,B,E)$ [$sum\ corresponding\ rows\ that\ have\ T\ and\ F\ for\ E$]

$f_6(A,B,E)$

T	T	T	0.0019
T	T	F	0.93812
T	F	T	0.00058
T	F	F	0.000998
F	T	T	0.0001
F	T	F	0.05988
F	F	T	0.00142
F	F	F	0.997002

$\Rightarrow$

$f_7(A,B)$

T	T	0.94002
T	F	0.001578
F	T	0.05998
F	F	0.998422

Bayesian Network: Multiply Factors

- For variable A:
  - **Step 2a**: Multiple all factors having A and get a new factor:
  - **Multiply**: $f_8(A,B) = f_4(A) \times f_5(A) \times f_7(A,B)$

$f_4(A)$

T	0.90
F	0.05

$\times$

$f_5(A)$

T	0.70
F	0.01

$\Rightarrow$

$f_4(A) \times f_5(A)$

T	0.63
F	0.0005

$\times$

$f_7(A,B)$

T	T	0.94002
T	F	0.001578
F	T	0.05998
F	F	0.998422

$\Rightarrow$

$f_8(A,B)$

T	T	0.592213
T	F	0.00099414
F	T	0.00002999
F	F	0.000499211

Bayesian Network: Multiply Factors

- For variable A:
  - **Step 2b**: Sum-out resulting factor for E and get a new factor:
  - **Sum-out *A***: $f_9(B) = \sum_A f_8(A,B)$ [$sum\ corresponding\ rows\ that\ have\ T\ and\ F\ for\ E$]

$f_8(A,B)$

T	T	0.592213
T	F	0.00099414
F	T	0.00002999
F	F	0.000499211

$\Rightarrow$

$f_9(B)$

T	0.59224299
F	0.001493351

Bayesian Network: Multiply Remaining Factors

- Query: $P(B|j,m)$.
- **Step 3:** Multiply remaining factors: $f_{10}(B) = f_1(B) \times f_9(B)$

$f_1(B) = P(B)$

T	0.001
F	0.999

$\times$

$f_9(B)$

T	0.59224299
F	0.001493351

$\Rightarrow$

$f_{10}(B)$

T	0.000592
F	0.001492

Bayesian Network: Normalize Result

- Query: $P(B|j,m)$.
- **Step 4**: Normalize the final factor to eliminate α.

$f_{10}(B)$

T	0.000592
F	0.001492

$\Rightarrow$

$f_{11}(B)$

T	0.284172
F	0.715828

- **Answer**: $P(b|m,j) = 0.284, P(\neg b|m,j) = 0.716$

Bayesian Network: Variable Elimination Algorithm

- Algorithm pseudocode [RN book]
  - Note: Step 1 (initialization) is missing in the pseudocode [it is implied in the MAKE-FACTOR function which is not a good approach!]

function ELIMINATION-ASK(X, **e**, bn) **returns** a distribution over X
  inputs: X, the query variable
    e, observed values for variables **E**
    bn, a Bayesian network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $factors \leftarrow [\,]$
  for each var **in** ORDER(bn.VARS) **do**
    $factors \leftarrow$ [MAKE-FACTOR(var, **e**)|$factors$]
    if var is a hidden variable **then** $factors \leftarrow$ SUM-OUT(var, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

Bayesian Network: Irrelevant Variable

- Consider the query: $P(JohnCalls|Burglary = true)$

- Incorporating all hidden variables of the Bayesian Network, we get:

$$P(J|b) = \alpha P(J,b) = \alpha \sum_{e} \sum_{a} \sum_{m} P(j,m,a,b,e) = \sum_{e} \sum_{a} \sum_{m} P(J|a)P(m|a)P(a|b,e)P(b)P(e)$$

Bayesian Network: Irrelevant Variable

- A simple rearrangement of the expression would yield us:

$$\mathbf{P}(J \mid b) = \alpha\, P(b) \sum_{e} P(e) \sum_{a} P(a \mid b, e) \mathbf{P}(J \mid a) \sum_{m} P(m \mid a)\ .$$

- Note that the inner term $\sum_m P(m|a)$ will evaluate to 1. So it is eliminated from the expression entirely [Irrelevant variable]

- *Variable elimination algorithm will automatically sum-out m from the expression!*
  - Why? [At some point, m will be considered as a hidden variable and the summation over m will eliminate it]

Bayesian Network: Irrelevant Variable

- In general, we can remove any leaf node that is not a query variable or an evidence variable.
- After removal of initial leaves, there may be some more leaf nodes, and these too may be irrelevant and removed [continue this process].
- *Every variable that is not an ancestor of a query variable or evidence variable is irrelevant to the query.*