Chapter 17 (AIAMA)

Making Complex Decisions

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Sequential Decision Problems

In a sequential decision problem:

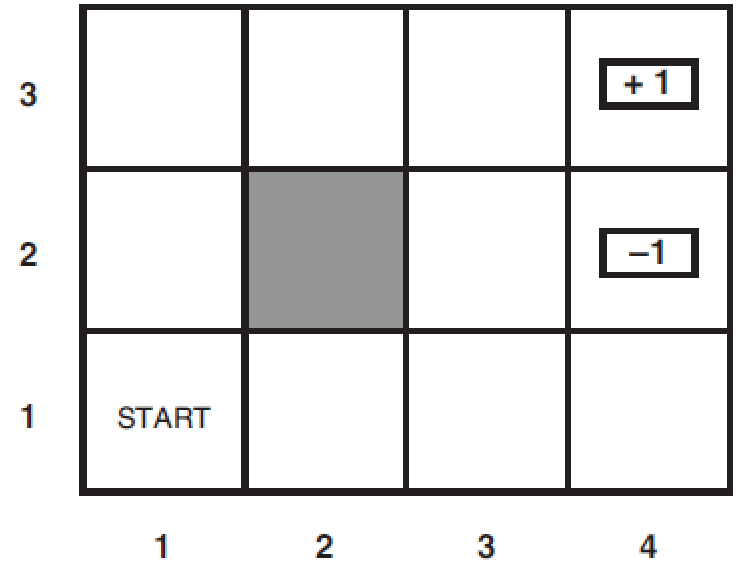
- Utility do not depend on a single decision/action.
  - Example: You have two choices in a game: First choice gives you 10k with probability ½ and 0k otherwise. Second choice gives you 30k with probability ¼ and 5k otherwise. Which action should a rational agent choose?

- Utility depends on a sequence of agent's decisions/actions.
  - What an agent should do at any time depends on what it will do in the future.
  - What an agent does in the future depends on what it did before.

Sequential Decision Problems

Consider an agent's exploration in a 4 × 3 grid environment:

- Agent begins in the START state
- Agent chooses an action in each state
  - **Actions**: UP, LEFT, DOWN, RIGHT
- Exploration ends when agent reaches one of the goal states (+1 or -1)



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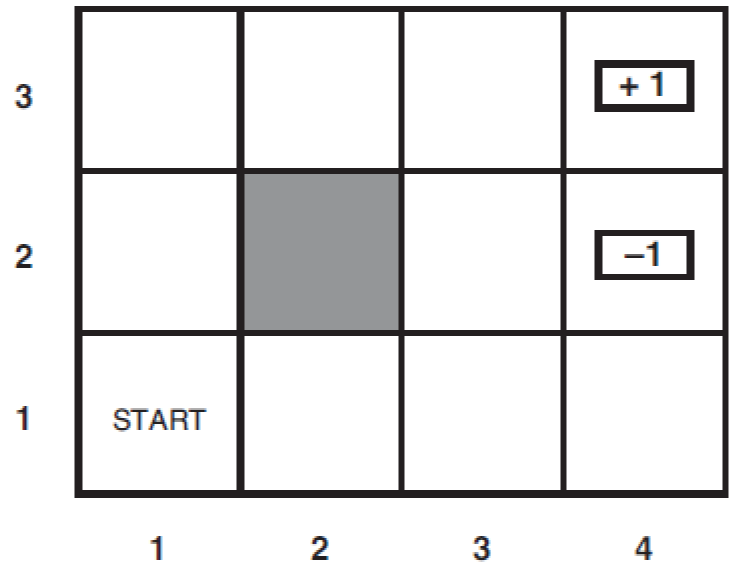
Sequential Decision Problems

Actions of agent:

- Agent chooses an action in each state
  - Actions: UP, LEFT, DOWN, RIGHT
- Action effect is not deterministic rather stochastic
  - Each action takes agent in the right direction with probability 0.8
  - With 0.1 probability, agent can move to the right angles of the intended direction
  - If agent hits a wall, it bumps and stays in the same location



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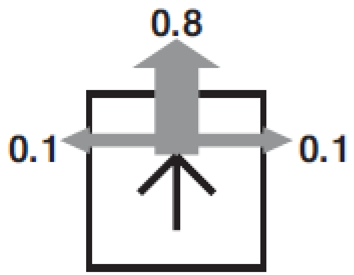
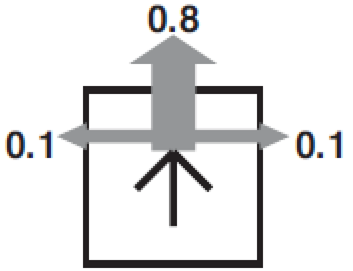
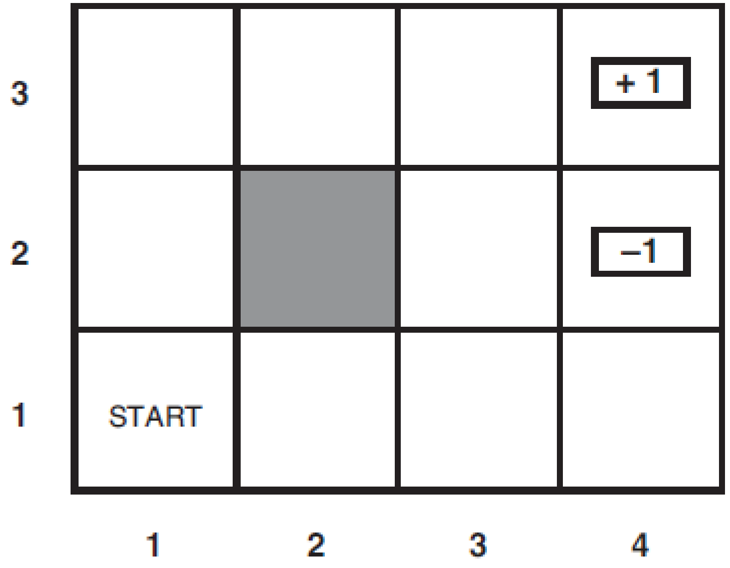


Sequential Decision Problems

Actions of agent: Example

- Agent chooses action UP in (1,1)
  - With 0.8 prob, agent moves to (1,2)
  - With 0.1 prob, agent moves to (2,1)
  - With 0.1 probability, moves to left, bumps to wall and stays at (1,1)





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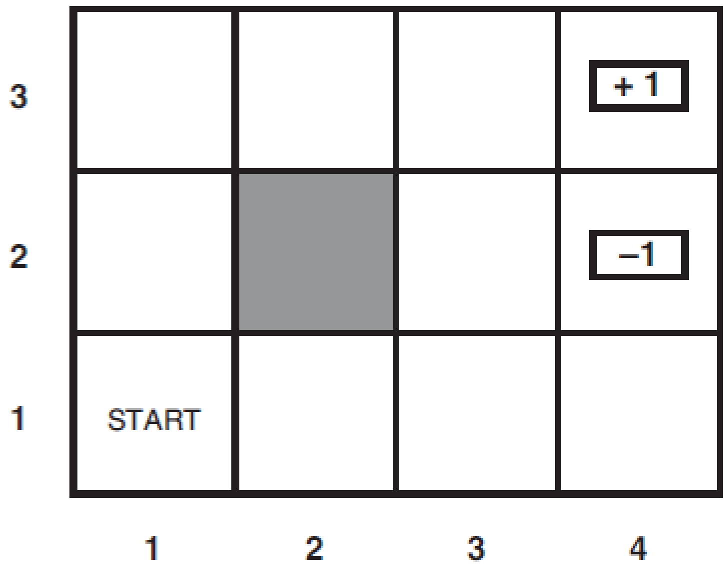
Sequential Decision Problems

What is action effect was deterministic?

- The problem turns into a simple search problem:
  - Find the sequence of moves leading to goal.
  - DFS/BFS can easily find this.

- In sequential decision problem:
  - Action effect is not deterministic
  - Simple search do not work [Why?]



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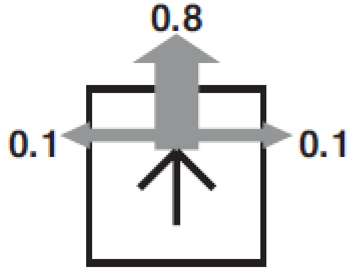
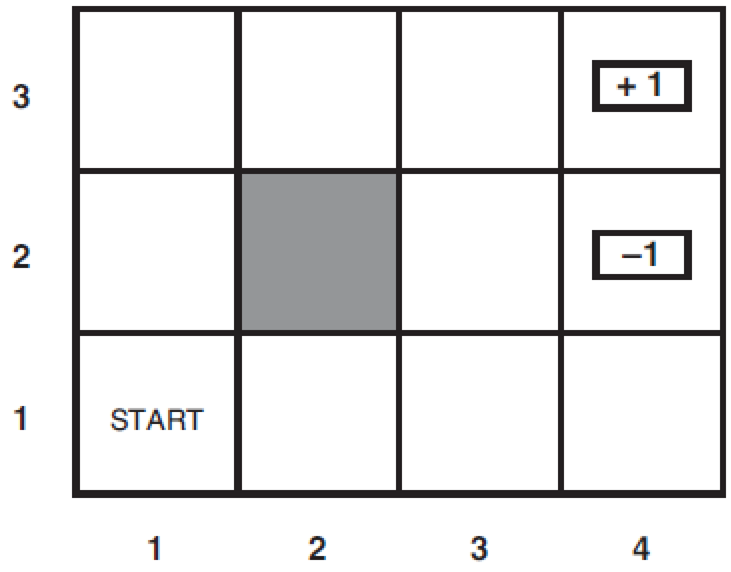


Sequential Decision Problems

- Agent cannot reach goal by following the action sequence: UP, UP, RIGHT, RIGHT, RIGHT. [Why?]
  - Because action effects are not deterministic
- What is the probability that agent reaches +1 by by following the path given above as action sequence?
  - Answer: 0.8^5 = 0.32768



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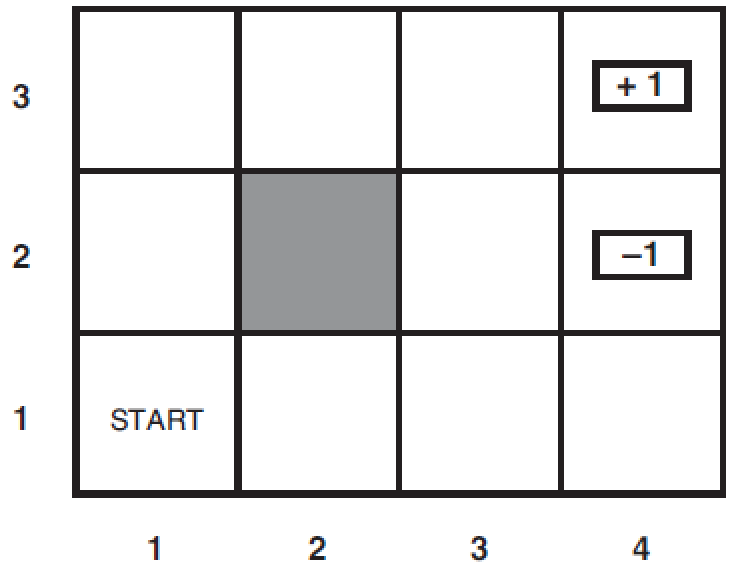
Sequential Decision Problems

Rewards from the environment:

Agent receives a reward from the environment

- For state s, reward is $R(s)$
- $R(s) = -1.0$ and $+1.0$ for terminal states
- $R(s) = -0.04$ in all other states

- The utility of a sequence of steps is the sum of all rewards.
  - Say, agent reaches +1 state after 10 steps
  - total utility $= 10 \times -0.04 + 1.0 = 0.6$



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Sequential Decision Problems

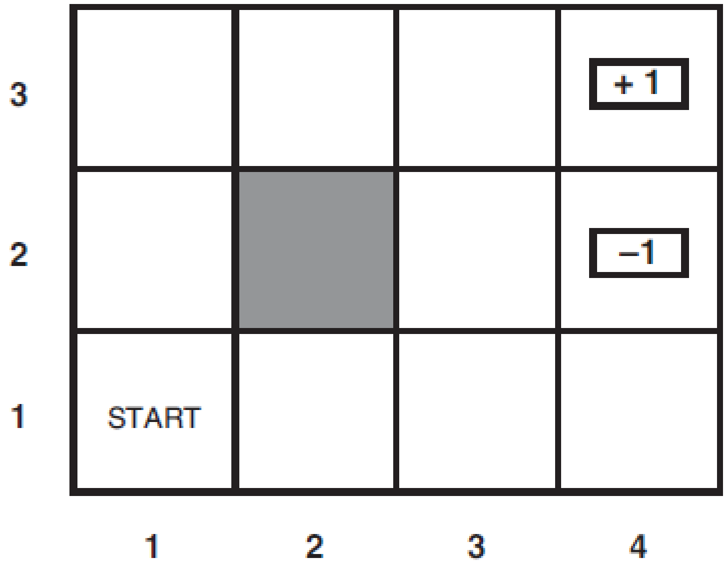
Utility from the environment:

- Agent tries to reach +1 as quickly as possible
  - Otherwise, agent gets negative reward and utility decrease
  - Consider, agent reaches +1 with 10 steps vs 5 steps.

    For 10 steps, total utility received = 0.6

    For 5 steps, total utility received = 0.8

- Utility is gained only from terminal states.
  - However, utility can be decreased/increased by the cost incurred from visiting each state *s* to reach the terminal state.



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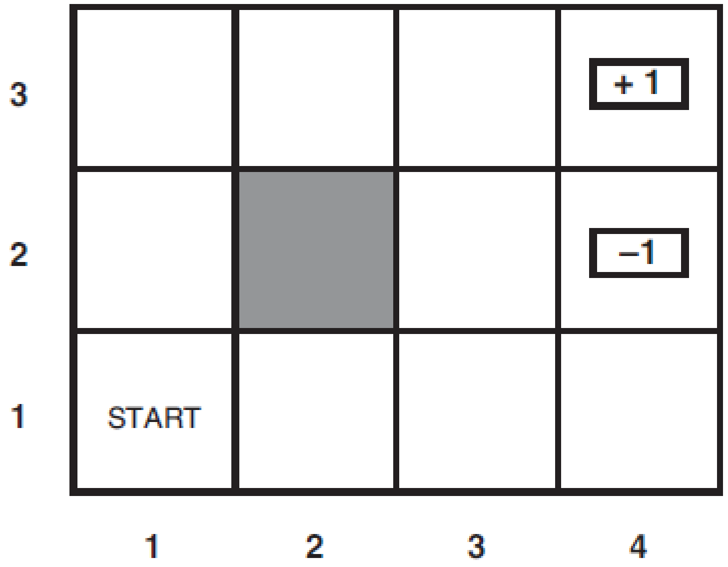
Sequential Decision Problems

Transition model:

- Agent moves a new state by following an action in current state
  - Modeled by a transition probability $P(s'|s, a)$
  - Can be described by a three-dimensional table specifyi all probability values
- The transition model is Markovian
  - $P(s'|s, a)$ depends only on current state s and not on any previous states visited.
  - Example: $P((1,2)|(1,1), UP) = 0.8, P((2,1)|(1,1), UP) = 0.1$, etc.



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Markov Decision Process

- **A Markov Decision Process (MDP) is a sequential decision problem where:**
  - Environment is fully observable [Agent knows where it is]
  - Environment is stochastic [Actions do not always leads to the intended state]
  - Transition model is Markovian
  - Rewards are additive

- **A MDP consists of**
  - A set of states (with initial state s_0)
  - A set of actions ACTIONS(s) in each state s
  - A transition model $P(s'|s, a)$
  - A reward model $R(s)$

.

Markov Decision Process

- A **Partially Observable Markov Decision Process (MDP)** is a sequential decision problem where:
  - Environment is partially observable
    - Agent does not necessarily know where it is in
    - Reward and action do not just depend on *s* but also on how much the agent knows when it is in *s*
  - More complex than ordinary MDPs. [We won’t study in this course]

Markov Decision Process: Solution

- **A solution to an MDP is called a policy that defines**
  - Which action agent should follow for every state
  - Usually denoted by π or $\pi(s)$.
  - If agent is given to follow a policy, the agent always takes the action given in the policy, no matter what the outcome is.

- **A policy $\boldsymbol{\pi}$**
  - Is not an action-planning [which actions leads to goal, simple search problem]

    - In fact, an action-sequence is not guaranteed to take to goal.
  - Is a strategy planning [which action to take at each state, MDP]

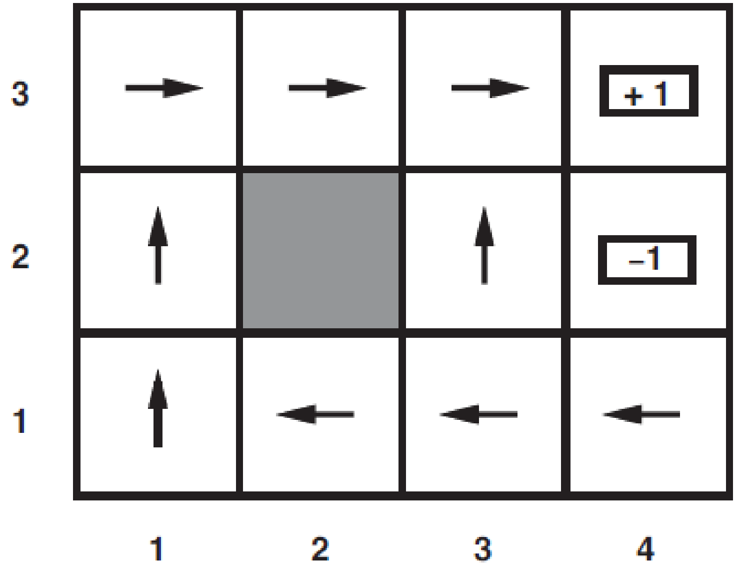
Markov Decision Process: Optimal Policy

- **A policy π**
  - Does not always takes an agent to the same state [stochastic action effects]
  - Each execution of a policy may result in a different state sequence followed by the agent [due to the stochastic nature of the action effect].

- What makes a policy π **optimal**?
  - A policy is optimal if it yields the highest expected utility [not reward]
  - Usually denoted by π^*

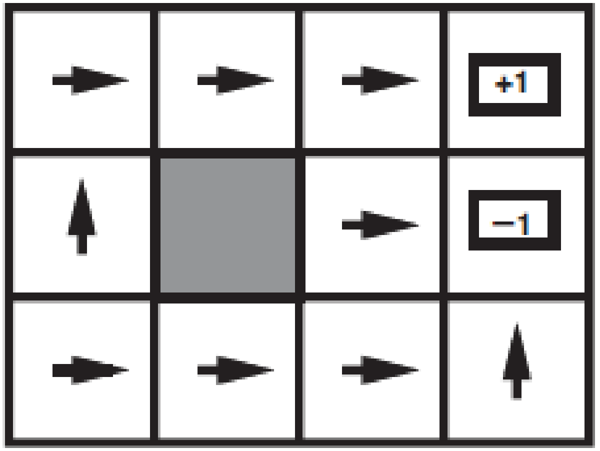
Markov Decision Process: Optimal Policy

- **Example:** An optimal policy for the grid world problem with $R(s) = -0.04$ for nonterminal states.

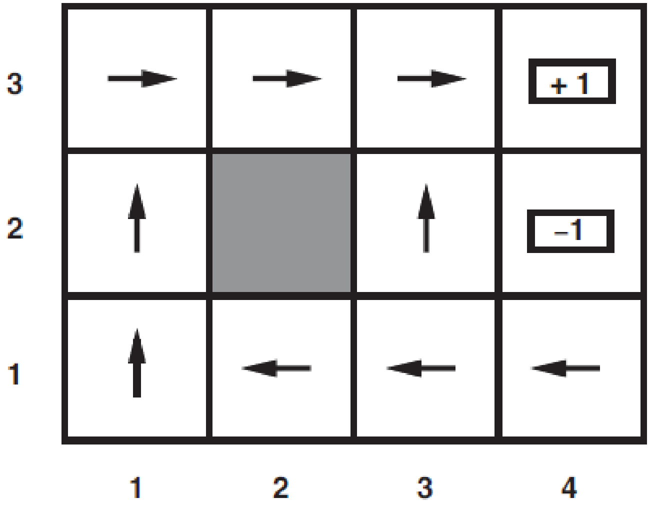


Markov Decision Process: Optimal Policy

- What happens to optimal policy when $\mathbf{R(s)} \leq \mathbf{-1.6284}$?
  - Life is so painful that the agent heads straight for the nearest exit, even if the exit is worth –1.
  - Note the action at (3,2) [Agent chooses -1 to quickly exit]

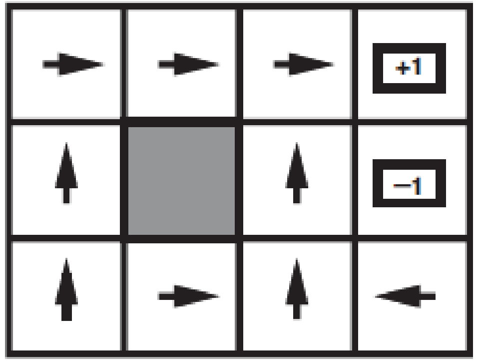


$R(s) < -1.6284$

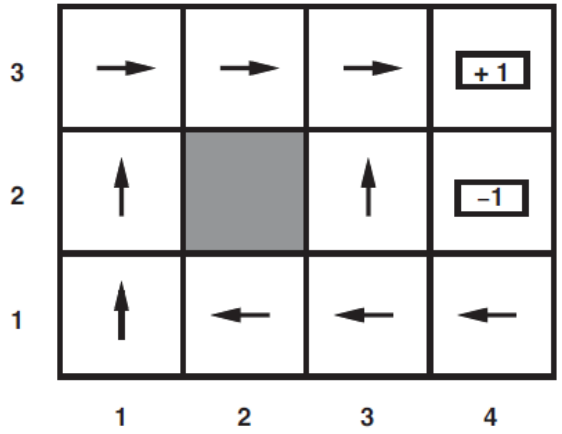


Markov Decision Process: Optimal Policy

- What happens to optimal policy when $-0.4278 \leq R(s) \leq -0.0850$?
  - Life is not so painful but unpleasant; the agent takes the shortest route to +1, and is willing to risk falling to -1 by accident
  - Note the action at (3,1) [Agent chooses UP to quickly exit]

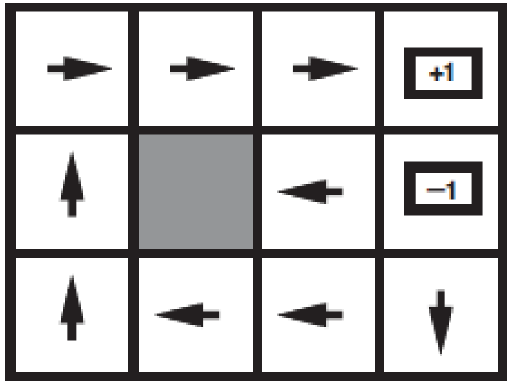


$-0.4278 < R(s) \mu\kappa -0.0850$

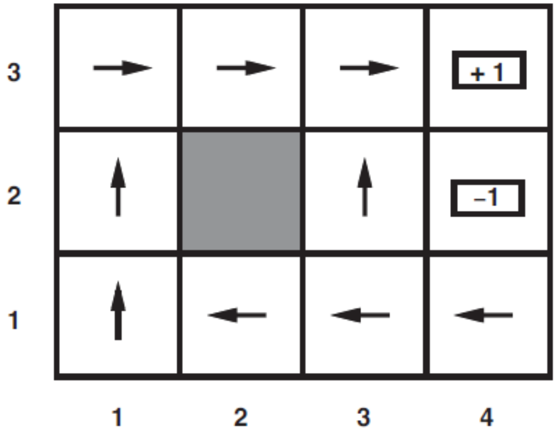


Markov Decision Process: Optimal Policy

- What happens to optimal policy when $-\mathbf{0.0221} < \boldsymbol{R(s)} < \mathbf{0}$?
  - The optimal policy takes *no risks at all*. In (4,1) and (3,2), the agent heads directly away from the –1 state so that it cannot fall in by accident, even though this means banging its head against the wall quite a few times



$-0.0221 < R(s)\mu< 0$



Markov Decision Process: Reward Types

- Assume an agent visits an state sequence $[s_0, s_1, ..,]$. The utility obtained is $U_h([s_0, s_1, ..,])$ can be considered in two ways:

- **Additive rewards**: $U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) \ldots$
  - Already considered in our previous grid world example

- **Discounted rewards**: $U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots$
  - Future rewards are discounted by a factor γ where $0 < \gamma \leq 1$.
  - γ close 1: Agent considers future rewards as highly as current rewards
  - γ close 0: Agent considers distant rewards insignificant

Markov Decision Process: Expected Utility is Bounded

- With discounted rewards, the utility obtained from visiting an infinite sequence of states is still finite as shown below:

  - Assume rewards are bounded by $\pm R_{max}$ and $\gamma < 1$

  - The total utility after infinite number of steps is:

$$U_h = \sum_{i=0}^{\infty} \gamma^i R(s_i) \leq \sum_{i=0}^{\infty} \gamma^i R_{max} = R_{max} \left(\frac{1}{1-\gamma} \right)$$

Markov Decision Process: Expected Utility of a Policy

- Assume agent starts at initial state s.
- Agent reaches state S_t at time t when executing a policy π [Note S_t is not a deterministic state; it is a random variable. Why?]
- Agent acts in an infinite-horizon [n*o limit on the # of steps, but exploration ends as soon as terminal state is reached*]
- The probability distribution over the state sequences $S_1, S_2, \ldots.$ visited by the agent is determined by:
  - The initial state s
  - The policy π
  - The transition model $P(s'|s, a)$

Markov Decision Process: Expected Utility for a Policy

- The expected utility obtained by executing a given policy is computed as:

$$U^{\pi}(s) = E[\sum_{t=0}^{\infty} \gamma^{t} R(S_t)] \text{ where } S_0 = s$$

[We can't just sum $R(S_t)$ as $R(S_t)$ is a random variable having a probability distribution over states, so we need to take the expected value]

- Can we compute $U^{\pi}(s)$ using the above equation?
  - Not easy!
  - As t increases, computing $R(S_t)$ becomes extremely difficult [with exponential number of values for the random variable S_t, WHY?]

Markov Decision Process: Optimal Policy

- **Optimal Policy**: The optimal policy is the one that gives the maximum expected utility:

$$\pi_s^* = argmax_\pi U^\pi(s)$$

  - π_s^* is the optimal policy starting with state s.
  - π_s^* is a policy; hence it recommends an action for every state.

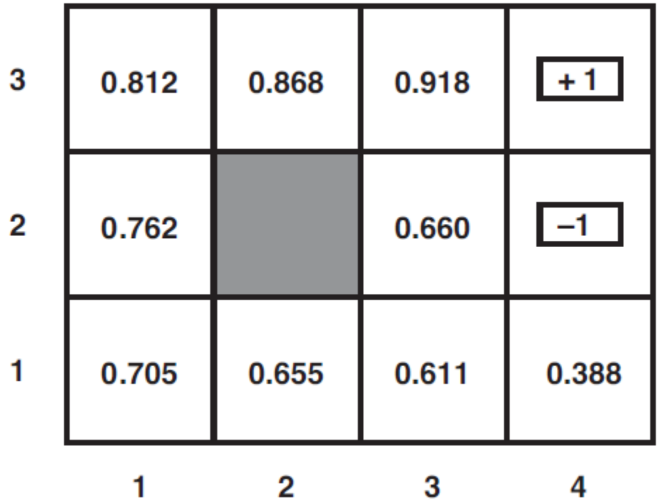
- Optimal policy is independent of start state s. [WHY?]
  - Remember policy is like a function specifying an action for every state; hence it does not matter where you start; you always have an action-sequence as per the optimal policy.
- We can simply write the optimal policy as π^*.

Markov Decision Process: Utility Function

- The maximum achievable utility starting from any state s is given by: $U^{\pi^*}(s)$
  - The sum of the discounted rewards if agent executed optimal policy starting from s.
  - We simply express it $U(s)[= U^{\pi^*}(s)]$
    - Note $U(s)$ and $R(s)$ are quite different quantities!
    - Note $U(s)$ and $U^{\pi}(s)$ are two different quantities!
    - $U(s)$ depends on s [WHY?]

Markov Decision Process: Utility Function

- Following figure shows the maximum utility $U(s)$ for different states in the grid world problem for $\gamma = 1$ and $R(s) = -0.04$
- Note that states closer to +1 have higher utilities compared to states far from it.
  - Why? [Because starts far away needs more steps to reach +1, thus additive rewards (-0.04 per state) reduces the utility]



Markov Decision Process: Optimal Action

- Agent can use the utility function $U(s)$ to choose the best action at each state:
  - Choose the action a in state s which gives you maximum utility from the next possible states; such action-sequence defines an optimal policy.

$$\pi^*(s) = argmax_{a \in A(s)} \sum_{s'} U(s')P(s'|s,a)$$

  - The above is the optimal policy equation.

Markov Decision Process: Value Iteration

- **Value iteration**: An algorithm to calculate the optimal policy
- Basic idea of the algorithm:
  - Step 1: Calculate utility $U(s)$ for every state s [Remember $U(s)$ is the maximum utility achievable from state s by following optimal policy]
  - Step 2: Calculate optimal action for each state based on calculated $U(s)$ in Step 1 using optimal policy equation [*see previous slide*].
  - Seems chicken and egg problem? [*As in Step 1, we don't have an optimal policy!*]
    - No! $U(s)$ can be calculated without knowing optimal policy!

Markov Decision Process: Bellman Equation

- **Utility function**: Let's define utility function in a different way.
- The optimal utility $U(s)$ of a state s satisfies the following equation:

$$U(s) = R(s) + \gamma \max_{a} \sum_{s'} P(s'|s,a)U(s')$$

  - $U(s)$ = immediate reward + expected discounted utility of next state [*assume agent follows optimal action as per optimal policy*]
  - This is optimal substructure property for the optimal utility!

- This is called **Bellman equation**.

Markov Decision Process: Bellman Equation Example

3	0.812	0.868	0.918	+1
2	0.762		0.660	−1
1	0.705	0.655	0.611	0.388
	1	2	3	4

- Let’s verify Bellman Equation for grid world problem.
- Bellman equation for state (1,1) is:

$$
\begin{aligned}
U(1,1) = -0.04 + \gamma \max[\ & 0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1), && (Up) \\
& 0.9U(1,1) + 0.1U(1,2), && (Left) \\
& 0.9U(1,1) + 0.1U(2,1), && (Down) \\
& 0.8U(2,1) + 0.1U(1,2) + 0.1U(1,1)\ ]. && (Right)
\end{aligned}
$$

- Plugging in numbers from the figure:
  - We can show UP is the best action
  - Choose any other action (i.e., RIGHT) and verify Bellman equation won’t be satisfied.

Markov Decision Process: Value Iteration Algorithm

- Based on Bellman equations
- Number of states $= n$, Number of Bellman equations $= n$
- Number of unknowns $= n$ [$U(s)$ for every state s]
- If equations were linear, we could easily use Gauss Elimination to solve.
    - Bellman equations are non-linear due to the max operation!
    - Hence, easier approaches won't help
- Solution: iterative optimization known as **Value Iteration** algorithm.

Markov Decision Process: Value Iteration Algorithm

- Iterative approach to solve Bellman equations [**Value Iteration Algorithm**]
  - Start with random values for Us [*usual practice: start with all zeros*]
  - Repeat until equilibrium:
    - Calculate right hand side of Bellman equations using current values of U's and use it to update left-hand side (new updated values for $U's$):

$$U_{i+1}(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) U_i(s')$$

[The above step is known as **Bellman update**]

Markov Decision Process: Value Iteration Algorithm

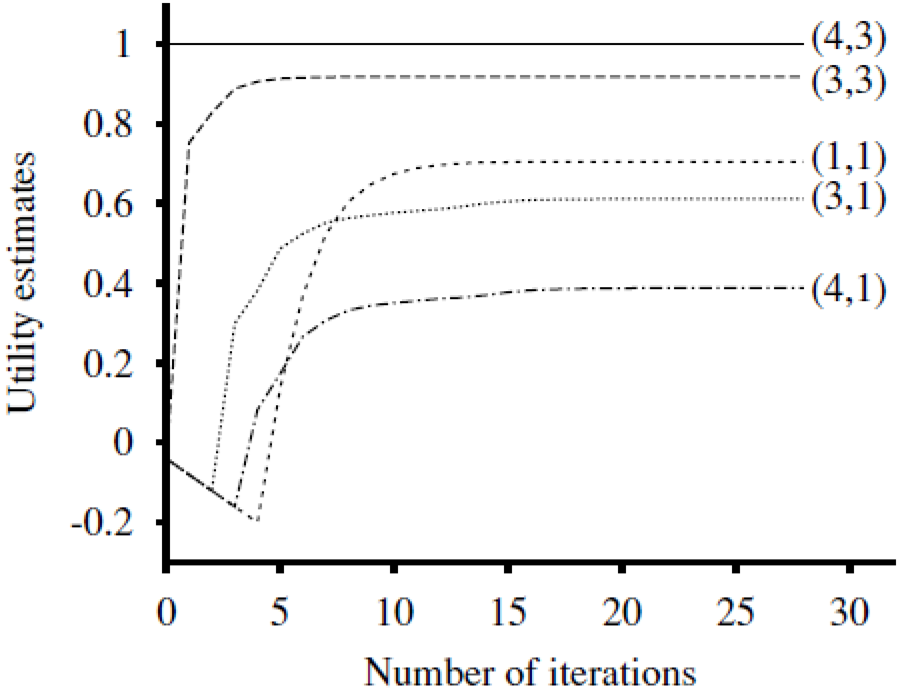
- Pseudocode of value iteration algorithm:

function VALUE-ITERATION(mdp, ϵ) **returns** a utility function
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$,
    rewards $R(s)$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', vectors of utilities for states in S, initially zero
    δ, the maximum change in the utility of any state in an iteration

  repeat
    $U \leftarrow U'$; $\delta \leftarrow 0$
    for each state s **in** S **do**
      $U'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\, U[s']$
      if $|U'[s] - U[s]| > \delta$ **then** $\delta \leftarrow |U'[s] - U[s]|$
  until $\delta < \epsilon(1-\gamma)/\gamma$
  return U

Markov Decision Process: Value Iteration Example

- Value iteration applied to grid world problem with initial utilities as 0s.
  - Note the number of iterations required for convergence of utilities for different states.



Markov Decision Process: Value Iteration Algorithm

- Does the value iteration converge to a solution?
- If apply Bellman update infinitely often, the equilibrium is guaranteed.
  - Final utility values must be the solutions to Bellman equations.
  - The solution is also unique.
  - *Proof of convergence:* We won't study in this course.
- Once utilities are obtained, we can also find the optimal policy! [*best action from each state satisfying Bellman equation*]

Markov Decision Process: Policy Iteration Algorithm

- Is it possible to get an optimal policy even when the utility function estimate is inaccurate?

    - *Answer: Yes*. If one action is clearly better than all others, then the exact magnitude of the utilities on the states involved need not be precise.

- An alterative approach can be designed using the above idea. This is known as **Policy Iteration.**

Markov Decision Process: Policy Iteration Algorithm

- **Steps of Policy Iteration Algorithm**
  - Start with a random policy π_0
  - Repeat until convergence:
    - Policy Evaluation: Using the current policy π_i, calculate utilities U_is as:

$$U_i(s) = U^{\pi_i}(s)$$

    - Policy improvement: Using the utilities U_i's, calculate a new udpdated policy π_{i+1} as:

$$\pi_{i+1}(s) = argmax_{a \in A(s)} \sum_{s'} U_i(s') P(s'|s, a)$$

  - *Termination criteria*: The above algorithm terminates when policy improvement yields no change.

Markov Decision Process: Policy Iteration Algorithm

- How to evaluate the two steps?
  - *Policy improvement*: Pretty straightforward.
  - *Policy evaluation*: Not easy thought! [we discussed it previously]
    - *Solution?* Use an approach similar to the value iteration algorithm.
    - *One important difference:* Bellman equations for unities will be linear now! [why?]

Markov Decision Process: Policy Iteration Algorithm

- *Policy evaluation*:
  - Utilities $U_i(s)$ of each state will satisfy the following simpler version of Bellman equation:

$$U_i(s) = R(s) + \gamma \sum_{s'} P(s'|s, \pi_i(s)) U_i(s')$$

  - The equations are linear [*no max operation over actions as the policy is given and agent just follows the action given in the policy*]
  - We have n equations with n unknowns; all equations are linear.
  - Gaussian Elimination (or similar methods) will solve this.

Markov Decision Process: Policy Iteration Algorithm

- *Policy evaluation*:
  - Small state space: Gaussian elimination with $O(n^3)$ run-time.
  - For large space: Use modified policy iteration to find U's
    - Same as policy iteration algorithm
    - Use small number of iterations as exact solution is not needed

Markov Decision Process: Policy Iteration Algorithm

- **Pseudocode:**

function POLICY-ITERATION(mdp) **returns** a policy
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$
  local variables: U, a vector of utilities for states in S, initially zero
    π, a policy vector indexed by state, initially random

  repeat
    $U \leftarrow$ POLICY-EVALUATION(π, U, mdp)
    $unchanged? \leftarrow$ true
    for each state s **in** S **do**
      if $\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\ U[s'] > \sum_{s'} P(s' \mid s, \pi[s])\ U[s']$ **then do**
        $\pi[s] \leftarrow \operatorname{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\ U[s']$
        $unchanged? \leftarrow$ false
  until $unchanged?$
  return π