Probability Models

Sample Space:

flipping a coin, $S = \{H, T\}$

Rolling a die, $S = \{1,2,3,4,5,6\}$

Flipping 2 coins, $S = \{HH, HT, TH, TT\}$

* **Event:** Any subset E of sample space S is called an Event.
* $E \cap F = EF$
* IF $EF = \phi$, then E an F are said to be mutually exclusive.

$$E \cap E^c = \phi \ , \quad S^c = \phi \ , \quad S = E \cup E^c$$

Probability:

(i) $0 \le P(E) \le 1$

(ii) $P(S) = 1$

(iii) $E_1, E_2, \dots$ mutually exclusive events, $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$

$$P(S) = 1 = P(E \cup E^c) = P(E) + P(E^c) \qquad [\because EE^c = \phi]$$

$$\therefore P(E^c) = 1 - P(E)$$

↳ complement of event E

* $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Ex 1·3 $S = \{HH, HT, TH, TT\}$

$E = \{HH, HT\}$, $P(E) = \frac{2}{4} = \frac{1}{2}$

$F = \{HH, TH\}$, $P(F) = \frac{1}{2}$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$

* $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(FG) - P(EG) + P(EFG)$
* $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k) - \cdots + (-1)^{n+1} P(E_1 E_2 \dots E_n)$

Conditional Probability

$$P(E|F) = \frac{P(EF)}{P(F)} \quad , \quad P(F) > 0$$

Ex 1·4: Let,

E be the event that no. is 10

F " " " " " " at least 5

$$\therefore P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}$$

Ex 1·5: $S = \{bb, bg, gb, gg\}$

Let,

E - both boys

F - at least one boy

$$\therefore P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Ex 1·6: Let,

E_i - ith man selects his own hat $1 \le i \le 3$.

$$P(E_i) = \frac{1}{3} \quad , \quad 1 \le i \le 3$$

$$P(E_iE_j) = P(E_j|E_i)\,P(E_i) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \quad , i \ne j$$

$$P(E_1E_2E_3) = \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{6}$$

$$\therefore P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_2E_3) - P(E_3E_1) + P(E_1E_2E_3)$$

$$= \frac{1}{3} + \frac{1}{3} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{6}} - \cancel{\frac{1}{6}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}}$$

$$= \frac{2}{3}$$

$$P(\text{no one selects hat}) = 1 - \frac{2}{3} = \frac{1}{3}$$

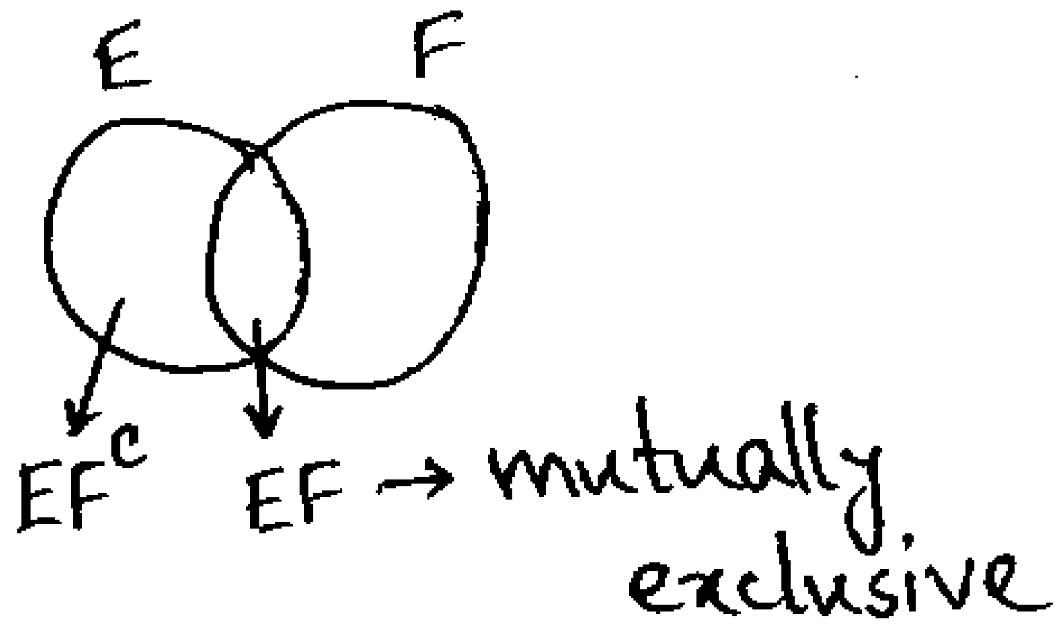
Independent Events:

$$P(EF) = P(E)\,P(F)$$

$P(EF) = P(E|F)\,P(F)$, if independent, $P(E/F) = P(E)$

Bayes' Formula:

Let,
E and F be two events.



$$E = EF \cup EF^c$$

$$\begin{aligned} P(E) &= P(EF) + P(EF^c) \\ &= P(E|F)\,P(F) + P(E|F^c)\,P(F^c) \\ &= P(E|F)P(F) + P(E|F^c)\,(1-P(F)) \end{aligned}$$

$$\begin{aligned} P(F|E) &= \frac{P(EF)}{P(E)} \\ &= \frac{P(E|F)\,P(F)}{P(E|F)P(F) + P(E|F^c)\,P(F^c)} \end{aligned}$$

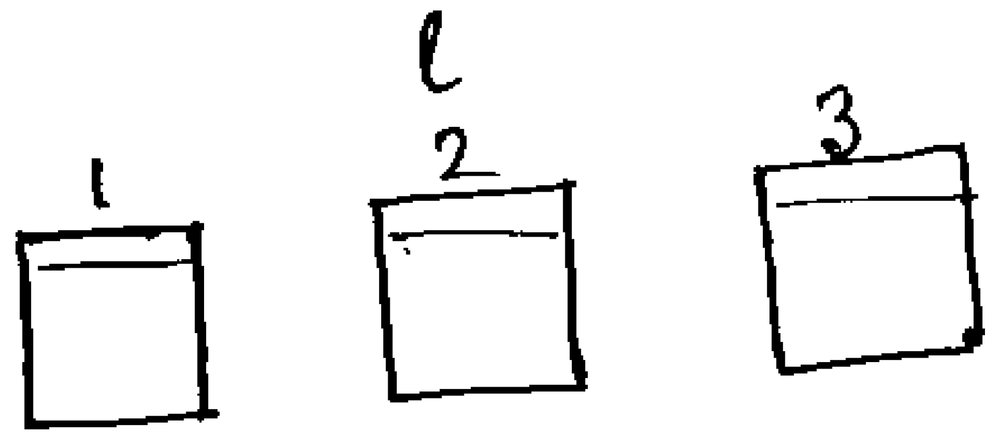
* $F_1, F_2, \ldots, F_n$ mutually exclusive events, $\bigcup_{i=1}^{n} F_i = S$

$$E = \bigcup_{i=1}^{n} EF_i$$

$$P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(E|F_i)\,P(F_i)$$

$$\therefore\ P(F_j|E) = \frac{P(E|F_j)\,P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

Ex 1.5:



$$P(F_1) = P(F_2) = P(F_3) = \frac{1}{3}$$

$F_i \rightarrow l$ is in file i

$\alpha_i \rightarrow$ probability of finding l in i

$E \rightarrow$ searched F_1 & did not find i

Prob. that l is in 1,

$$\begin{aligned} P(F_1|E) &= \frac{P(E|F_1)\,P(F_1)}{\sum_{i=1}^{3} P(E|F_i)\,P(F_i)} \\ &= \frac{(1-\alpha_1)\frac{1}{3}}{(1-\alpha_1)\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \frac{1-\alpha_1}{3-\alpha_1} \end{aligned}$$

CSE 301 (L2)

(Random Variables) (Chap 2)

* We are interested in some functions of the outcome rather than the outcome itself.

  - Tossing 2 dice : sum is seven (1,6) (2,5) (3,4) (6,1) (5,2) (4,3)

* Real-valued fun^c defined on sample space is a random variable.

Ex: X: sum of two fair dice. $P(X=7) = P\{\cdots\} = \frac{6}{36} = \frac{1}{6}$

↳ Random variable always capital.

Cumulative Distribution Function : (CDF)

$$F(b) = P\{X \leq b\} \quad , \ -\infty \leq b \leq \infty$$

Ex: Tossing a dice. $P\{X=i\} = \frac{1}{6}$

$$F(4) = P\{X \leq 4\} = \sum_{i=1}^{4} P\{X=i\} = \frac{4}{6}$$

* CDF is a nondecreasing function → $\sum_i P(X=i) = 1$ & prob. normy

$$P(a < X \leq b) = F(b) - F(a) \quad , \ a<b$$

Discrete Random Variables :

↳ can take at most a countable number of possible values.

- probability mass fun^c $p(a)$ of X, $p(a) = P\{X=a\}$

Bernoulli RV :

success or failure
1 0

$p(0) = P\{X=0\} = 1-p$

$p(1) = P\{X=1\} = p$

Binomial RV:

(n,p) n trial, p success

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} \qquad \binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \left(p + (1-p)\right)^n = 1$$

Ex: X: no. of heads, 4 coins are flipped.

$$P\{X=2\} = \binom{4}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^2 = 6 \cdot \tfrac{1}{4} \cdot \tfrac{1}{4} = \tfrac{3}{8}$$

Geometric RV:

$(n-1)$ trials are failure
nth trial success

X: no. of trials required before success

$$p(n) = P\{X=n\} = p(1-p)^{n-1}, \quad n = 1, 2, \ldots$$

$$\sum_{n=1}^{\infty} p(n) = p \sum_{1}^{\infty} (1-p)^{n-1} = p \cdot \frac{1}{1-1+p} = 1$$

Poission RV: (λ)

$$p(i) = P\{X=i\} = e^{-\lambda} \frac{\lambda^i}{i!}, \quad i = 0, 1, \ldots$$

$$\sum_{i=0}^{\infty} p(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

* BRV when $n \to \infty$, p small $\longrightarrow$ Poission

Proof:

Let X be a binomial RV with parameters $(n p)$

Let, $\lambda = np$

$$P\{X=i\} = \binom{n}{i} p^i (1-p)^{(n-i)}$$

$$= \frac{n!}{i!\,(n-i)!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-i)\cdots(n-i+1)}{i!} \frac{\lambda^i}{n^i} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

Now, For large n, $\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}$, $\frac{n(n-1)\cdots(n-i+1)}{n!} \approx 1$ $\left(1-\frac{\lambda}{n}\right)^i \approx 1$

$$\therefore P\{X=i\} = e^{-\lambda} \frac{\lambda^i}{i!}$$

Continuous RV : $\longrightarrow$ set of possible values is uncountable.

Probability density function,

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx$$

$$P\{X=a\} = \int_a^a f(x)\,dx = 0 \quad , \quad P\{-\infty \le X \le \infty\} = 1$$

* The probability that a continuous random variable will assume any particular value is zero.

$$\cancel{F(a) = \int_{-\infty}^{a} f(x)\,dx}$$

Uniform RV :

$$f(x) = \begin{cases} 1 & , 0<x<1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 dx = 1$$

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx = b-a$$

Exponential RV :

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0 \\ 0 & , x<0 \end{cases}$$

$$F(a) = \int_0^a \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda a} \quad , a \ge 0 \longrightarrow a \to \infty, e^{-\lambda a} = 0$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^a = -(e^{-\lambda a} - 1) =$$

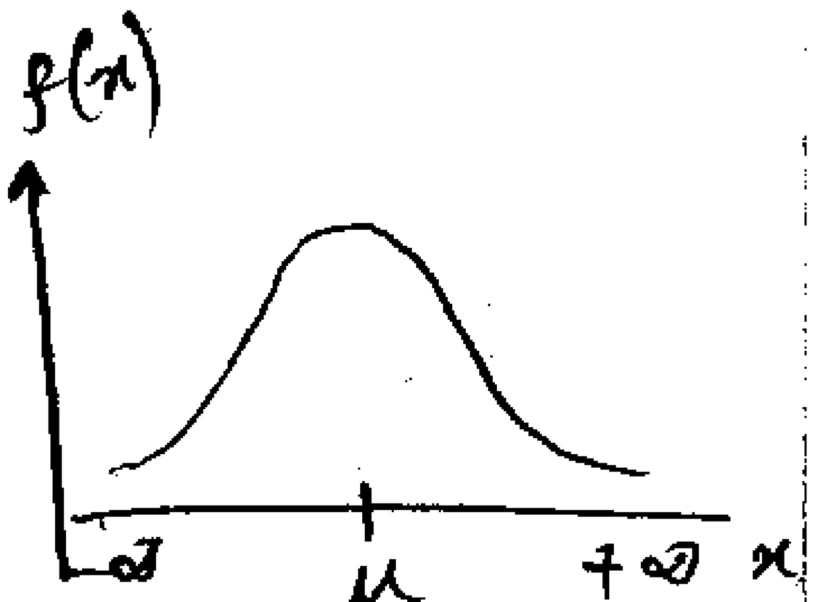
Gamma RV:

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\, dx \qquad \Gamma(n) = (n-1)!$$

Normal RV: (μ, σ)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$



Expectation of a RV:

$$E[X] = \sum_{\substack{x: \\ p(x)>0}} x\, p(x)$$

Ex: Rolling a die, $p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$

$$E[X] = 1.\tfrac{1}{6} + 2.\tfrac{1}{6} + 3.\tfrac{1}{6} + 4.\tfrac{1}{6} + 5.\tfrac{1}{6} + 6.\tfrac{1}{6} = \frac{6.7}{2}.\frac{1}{6} = \frac{7}{2}$$

Expectation of Bernoulli Random Variable:

$$P(1) = p, \quad P(0) = 1-p, \quad E[X] = p$$

Expectation of Binomial RV:

$$E[X] = \sum_{i=0}^{n} i\, p(i)$$

$$= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{i\, n!}{(n-i)!\, i!} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{(n-1)-(i-1)}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$= np[p + (1-p)]^{n-1}$$

$$= np$$

Exp. of G.R.V.

$$E[X] = \sum_{n=1}^{\infty} n p (1-p)^{n-1}$$

$$= p \sum_{n=1}^{\infty} n q^{n-1} \quad [q = 1-p]$$

$$= p \sum_{n=1}^{\infty} \frac{d}{dq}(q^n)$$

$$= p \frac{d}{dq}\left(\sum_{n=1}^{\infty} q^n\right)$$

$$= p \frac{d}{dq} \frac{q}{1-q}$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{1}{p}$$

Exp of P.R.V. :

$$E[X] = \sum_{i=0}^{\infty} \frac{i e^{-\lambda} \lambda^i}{i!}$$

$$= \sum_{i=1}^{\infty} \frac{e^{-\lambda} \lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

Cont^s RV: $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

Exp. of URV:

$$E[X] = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx$$

$$= \frac{1}{\beta - \alpha} \left[\frac{x^2}{2}\right]_{\alpha}^{\beta}$$

$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)}$$

$$= \frac{\alpha + \beta}{2}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x . 1\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

Exp. of E.R.V.

$$E[X] = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \left[\lambda x \frac{e^{-\lambda x}}{-\lambda} - \lambda \frac{e^{-\lambda x}}{(-\lambda)^2}\right]_0^{\infty}$$

$$= \left[-x e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right]_0^{\infty}$$

$$= 0 - 0 + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}$$

CSE 301 (L 3)

* Expectation of a Fun^c of a R.V. :

$X \rightarrow$ discrete RV with probability mass fun^c $p(x)$

$$E[g(X)] = \sum_{x:p(x)>0} g(x)\,p(x) \longrightarrow \text{discrete}$$

$$= \int_{-\infty}^{\infty} g(x)\, f(x)\, dx \longrightarrow \text{continuous}$$

$$* \; E[aX+b] = \cancel{aE[X]+b} \sum_{x:p(x)>0} (ax+b)\,p(x)$$

$$= a\sum_{x:p(x)>0} x p(x) + b\sum_{x:p(x)>0} p(x)$$

$$= aE[X] + b$$

cont^s case: $E[aX+b] = \int_{-\infty}^{\infty} (ax+b)\, f(x)\, dx$

$$= a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a\cdot E[X] + b$$

Variance:

$$Var(X) = E\left[(X-E[X])^2\right] = E\left[X^2 - 2XE[X] + E^2[X]\right]$$

$$= E[X^2] - 2E^2[X] + E^2[X]$$

$$= E[X^2] - (E[X])^2$$

Ex: X: Rolling a die

$Var(X) = ?$

$$E[X^2] = 1^2\cdot\frac{1}{6} + 2^2\frac{1}{6} + 3^2\cdot\frac{1}{6} + 4^2\cdot\frac{1}{6} + 5^2\cdot\frac{1}{6} + 6^2\cdot\frac{1}{6}$$

$$= \frac{1}{6}\;\frac{6.7.(2.6+1)}{6}$$

$$= \frac{91}{6}$$

$$E[X] = \frac{1}{6}\cdot\frac{6.7}{2} = \frac{7}{2}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Joint Distribution Funᶜ:

$$F(a,b) = P\{X \le a, Y \le b\} \qquad -\infty < a, b < \infty$$

$$F_X(a) = P\{X \le a\} = P\{X \le a, Y \le \infty\} = F(a, \infty)$$

$$F_Y(b) = P\{Y \le b\} = F(\infty, b)$$

$$P(x,y) = P\{X=x, Y=y\}$$

$$P\{X \in A, Y \in B\} = \int_B \int_A f(x,y)\, dx\, dy$$

$$E[aX + bY] = aE[X] + bE[Y]$$

Ex: N player throw their hats. Each randomly select one. ~~Find the expected number of men who select their own~~ hats.

Let, $X \ldots \quad \therefore X = X_1 + X_2 + \cdots + X_N$

where, $X_i = \begin{cases} 1, & \text{if ith man selects his hat} \\ 0, & \text{otherwise} \end{cases}$

$$P\{X_i = 1\} = \frac{1}{N}$$

$$\therefore E[X_i] = 1 \cdot P\{X_i = 1\} + 0 \cdot P\{X_i = 0\} = \frac{1}{N}$$

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = \frac{1}{N} + \frac{1}{N} + \cdots + \frac{1}{N} = 1$$

$\therefore$ on avg only 1 man will select his own hat.

$E[g(X)h(Y)] = E[g(X)]\,E[h(Y)]$, if X and Y are independent.

Jointly Distributed Random Variables :

$$F(a,b) = P\{X \le a, Y \le b\}$$

$$F_X(a) = P\{X \le a, Y \le \infty\} = F(a,\infty) \qquad F_Y(b) = P\{X \le \infty, Y \le b\} = F(\infty, b)$$

$$p(x,y) = P\{X = x, Y = y\}$$

$$P_X(x) = \sum_{y: p(x,y) \geq 0} p(x,y) \qquad P_Y(y) = \sum_{x: p(x,y) \geq 0} p(x,y)$$

$$E[aX + bY] = aE[X] + b\,E[Y]$$

$$\begin{aligned}
Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY - XE[Y] - YE[X] + E[X]E[Y]] \\
&= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

if X & Y are independent, $E[XY] = E[X]E[Y]$

$$\therefore Cov(X,Y) = 0$$

$$\begin{aligned}
Var(X+Y) &= E[((X+Y) - E[X+Y])^2] \\
&= E[(X+Y)^2 - 2(X+Y)E[X+Y] + (E[X+Y])^2] \\
&= E[(X+Y)^2] - 2E[X+Y]E[X+Y] + E[(E[X] + E[Y])^2] \\
&= E[X^2 + 2XY + Y^2] - 2(E[X] + E[Y])(E[X] + E[Y]) + E^2[X] + E^2[Y] + 2E[X]E[Y] \\
&= E[X^2] + 2E[XY] + E[Y^2] - 2E^2[X] - 2E^2[Y] - 4E[X]E[Y] + E^2[X] + E^2[Y] + 2E[X]E[Y] \\
&= E[X^2] - E^2[X] + E[Y^2] - E^2[Y] + 2E[XY] - 2E[X]E[Y] \\
&= \cancel{E[X} \; Var(X) + Var(Y) + 2\,Cov(X,Y)
\end{aligned}$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

Variance of binomial RV :

X - binomial RV

$$X = X_1 + X_2 + \cdots + X_n \qquad X_i = \begin{cases} 1, & \text{if ith trial is a success} \\ 0, & \text{otherwise} \end{cases}$$

$$Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_n)$$

$$Var(X_i) = E[X_i^2] - (E[X_i])^2 \qquad E[X_i^2] = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$= E[X_i] - (E[X_i])^2$$

$$= p - p^2$$

$$\therefore Var(X) = np(1-p)$$

Ex: (Sums of independent poisson RV.)

$$P\{X+Y=n\} = \sum_{k=0}^{n} P\{X=k, Y=n-k\}$$

$$= \sum_{k=0}^{n} P\{X=k\} P\{Y=n-k\}$$

$$= \sum_{k=0}^{n} \frac{e^{-\lambda_1} \lambda_1^k}{k!} \frac{e^{-\lambda_2} \lambda_2^{n-k}}{(n-k)!}$$

$$= \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} \frac{e^{-(\lambda_1+\lambda_2)}}{}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n$$

$\therefore X_1 + X_2$ has a poisson distribution with mean $\lambda_1 + \lambda_2$.

Moment Generating Functions:

$$\phi(t) = E[e^{tX}]$$

$$= \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \end{cases}$$

We call $\phi(t)$ the moment generating func because all of the moments of X can be obtained by successively differentiating $\phi(t)$.

if X is a RV, moments of X are,

$$E[X], E[X^2], \ldots, E[X^n]$$

$$\phi'(t) = \frac{d}{dt} E[e^{tX}] = E\left[\frac{d}{dt} e^{tX}\right] = E[Xe^{tX}]$$

$$\therefore \phi'(0) = E[X]$$

$$\phi''(t) = \frac{d}{dt}\phi'(t) = \frac{d}{dt} E[Xe^{tX}] = E[X^2 e^{tX}]$$

$$\phi''(0) = E[X^2]$$

$$\phi^n(0) = E[X^n]$$

Binomial Distribution:

$$\phi(t) = E[e^{tX}]$$

$$= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$$

$$= (pe^t + 1 - p)^n$$

$$\therefore \phi'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$\therefore E[X] = \phi'(0) = np$$

$$\phi''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

$$E[X^2] = \phi''(0) = n(n-1)p^2 + np$$

$$\therefore Var(X) = E[X^2] - (E[X])^2 = np(1-p)$$

Poisson Distribution:

$$\phi(t) = E[e^{tX}]$$
$$= \sum_{n=0}^{\infty} \frac{e^{tn} e^{-\lambda} \lambda^n}{n!}$$
$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t \lambda)^n}{n!}$$
$$= e^{-\lambda} e^{\lambda e^t}$$
$$= e^{(e^t-1)\lambda}$$

$$\phi'(t) = e^{(e^t-1)\lambda} . \lambda e^t$$
$$\phi''(t) = (\lambda e^t)^2 e^{(e^t-1)\lambda} + \lambda e^{(e^t-1)\lambda}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$
$$E[X] = \phi'(0) = \lambda$$
$$E[X^2] = \phi''(0) = \lambda^2 + \lambda$$
$$\therefore Var[X] = \lambda$$

$$\phi_{X+Y}(t) = E[e^{t(X+Y)}]$$
$$= E[e^{tX} . e^{tY}]$$
$$= E[e^{tX}]\, E[e^{tY}] \quad , X,Y \text{ independent.}$$
$$= \phi_X(t)\, \phi_Y(t)$$

* moment generating func uniquely determines a distribution.

Sum of independent RV. :

$$\phi_X(t) = E[e^{tX}]$$
$$= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k}$$
$$= (pe^t + 1 - p)^n$$
$$\phi_Y(t) = (pe^t + 1 - p)^m$$
$$\therefore \phi_{X+Y}(t) = \phi_X(t)\, \phi_Y(t) = (pe^t + 1 - p)^{m+n}$$

$\therefore$ X+Y is binomial RV. having $(m+n)$ & p.

Chap 3
Conditional Probability

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$P_{X|Y}(x|y) = P\{X=x|Y=y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{p(x,y)}{P_Y(y)}$$

$$F_{X|Y}(x|y) = P\{X \le x|Y=y\} = \sum_{a \le x} P_{X|Y}(a|y)$$

$$E[X|Y=y] = \sum_x x\, P\{X=x|Y=y\} = \sum_x x P_{X|Y}(x|y)$$

if X & Y are independent,

$$P_{X|Y}(x|y) = P\{X=x|Y=y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{P(X=x)\, P\{Y=y\}}{P\{Y=y\}} = P(X=x) = P_X(x)$$

__Ex:__ $p(1,1) = 0.5$, $p(1,2) = 0.1$, $p(2,1) = 0.1$, $p(2,2) = 0.3$

$$P_Y(1) = \sum_x p(x,1) = p(1,1) + p(2,1) = 0.5 + 0.1 = 0.6$$

$$P_{X|Y}(1|1) = P\{X=1|Y=1\} = \frac{P\{X=1, Y=1\}}{P\{Y=1\}} = \frac{p(1,1)}{P_Y(1)} = \frac{0.5}{0.6} = \frac{5}{6}$$

__Ex:__

Let,

$$P\{X=k|X+Y=n\} = \frac{P\{X=k, X+Y=n\}}{P\{X+Y=n\}} = \frac{P\{X=k, Y=n-k\}}{P\{X+Y=n\}} = \frac{P(X=k)\, P\{Y=n-k\}}{P\{X+Y=n\}}$$

$$= \frac{e^{-\lambda_1}\lambda_1^{k}}{k!} \cdot \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!} \cdot e^{(\lambda_1+\lambda_2)} \, \frac{n!}{(\lambda_1+\lambda_2)^n}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$, binomial distr.

$$\therefore E[X \mid X+Y=n] = n \cdot \frac{\lambda_1}{\lambda_1+\lambda_2}$$

<u>Continuous RV:</u>

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

<u>Ex:</u>

$$f(x,y) = \begin{cases} 6xy(2-x-y) & , 0<x<1,\ 0<y<1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{6xy(2-x-y)}{\int_0^1 6xy(2-x-y)\,dx}$$

$$= \frac{6xy(2-x-y)}{\int_0^1 (12xy - 6x^2y - 6xy^2)\,dx}$$

$$= \frac{6xy(2-x-y)}{\left[6x^2y - 2x^3y - 3x^2y^2\right]_0^1}$$

$$= \frac{6xy(2-x-y)}{6y - 2y - 3y^2}$$

$$= \frac{6x(2-x-y)}{4-3y}$$

$$E[X|Y=y] = \int x f_{X|Y}(x|y)\, dx$$

$$= \int_0^1 \frac{6x^2(2-x-y)}{4-3y}\, dx$$

$$= \frac{1}{4-3y}\left[4x^3 - \frac{6}{4}x^4 - 2x^3 y\right]_0^1$$

$$= \frac{1}{4-3y}\left[4 - \frac{3}{2} - 2y\right]$$

$$= \frac{5-4y}{8-6y}$$

Ex: $f(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, & 0<x<\infty,\ 0<y<2 \\ 0, & \text{otherwise} \end{cases}$

$$f_{X|Y}(x|y) = \frac{f(x, y=1)}{f_Y(y=1)}$$

$$\therefore f_Y(y=1) = \int_x f(x, y=1)\, dx$$

$$= \int_0^\infty \frac{1}{2} y e^{-xy}\, dx = \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]^\infty = \frac{1}{2}$$

$$f_{X|Y}(x|y=1) = \frac{\frac{1}{2}e^{-x}}{\frac{1}{2}} = e^{-x}$$

$$\therefore E[e^{x/2}|Y=1] = \int_0^\infty e^{x/2} f_{X|Y}(y=1)\, dx$$

$$= \int_0^\infty e^{+x/2} \cdot e^{-x}\, dx$$

$$= \left[\frac{e^{-x/2}}{-\frac{1}{2}}\right]_0^\infty$$

$$= 2$$

Computing Expection by conditioning:

$E[X|Y] \qquad E[X] = E[E[X|Y]]$

* discrete, $\quad E[X] = \sum_y E[X|Y=y]\, P\{Y=y\}$
* cont^s, $\quad E[X] = \int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\, dy$

* Prove that: $\quad E[X] = \sum_y E[X|Y=y]\, P\{Y=y\}$

$$
\begin{aligned}
R.H.S. &= \sum_y E[X|Y=y]\, P\{Y=y\} \\
&= \sum_y \sum_x x\, P\{X=x|Y=y\}\, P\{Y=y\} \\
&= \sum_y \sum_x x \cdot \frac{P\{X=x, Y=y\}}{P\{Y=y\}}\, P\{Y=y\} \\
&= \sum_x x \cdot \sum_y P\{X=x, Y=y\} \\
&= \sum_x x \cdot P\{X=x\} \\
&= E[X]
\end{aligned}
$$

Ex:

Let, X denote no. of misprints.

$$
Y = \begin{cases} 1, & \text{if Sam chooses history book} \\ 2, & \text{if " " probability "} \end{cases}
$$

$$
\begin{aligned}
E[X] &= E[X|Y=1]\, P\{Y=1\} + E[X|Y=2]\, P\{Y=2\} \\
&= 5 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{2} \\
&= \frac{7}{2}
\end{aligned}
$$

Expectation of Sum of Random No. of Random Variables :

Let,

N : no. of accidents

X_i : no. injured in the ith accident.

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right]$$

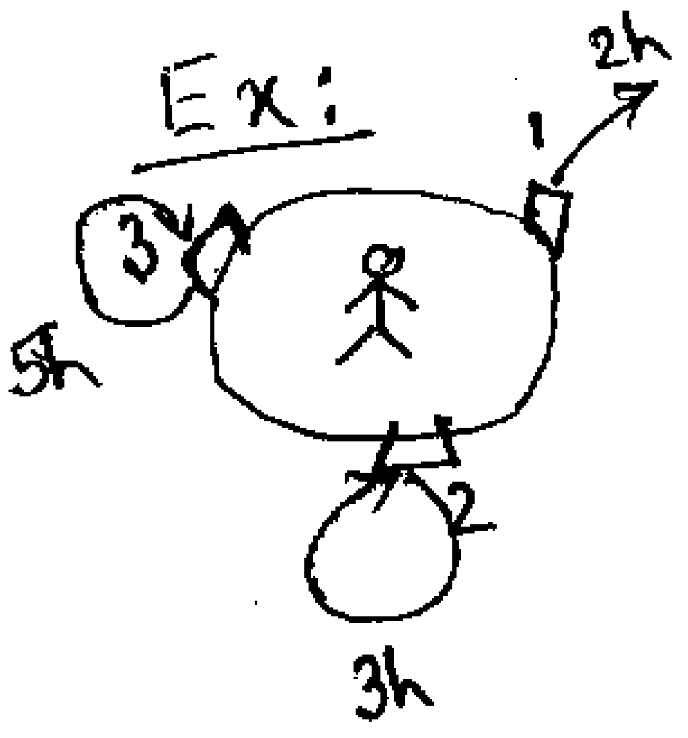
$\sum_{i=1}^{N} X_i$: total no. of injuries

Now,
$$E\left[\sum_{i=1}^{N} X_i \mid N=n\right] = E\left[\sum_{i=1}^{n} X_i \mid N=n\right]$$
$$= E\left[\sum_{i=1}^{n} X_i\right]$$
$$= nE[X]$$

$$E\left[\sum_{i=1}^{N} X_i \mid N\right] = N E[X]$$

$$\therefore E\left[\sum_{i=1}^{N} X_i\right] = E[NE[X]] = E[N]\,E[X] = 4 \times 2 = 8$$

Ex:



Let, X denote the time until the miner reaches safety.

Y : the door he initially chooses.

$$E[X] = E[X|Y=1]\,P\{Y=1\} + E[X|Y=2]\,P\{Y=2\} + E[X|Y=3]\,P\{Y=3\}$$
$$= \frac{1}{3} E[X|Y=1] + \frac{1}{3} E[X|Y=2] + E[X|Y=3] = \frac{1}{3}$$
$$= \frac{1}{3}\left(2+3+E[X] + 5 + E[X]\right)$$

$$3E[X] = 10 + 2E[X]$$

$$\therefore E[X] = 10$$

Variance of RN of RV :

$$Var\left(\sum_{i=1}^{N} X_i\right) = E\left[\left(\sum_{i=1}^{N} X_i\right)^2\right] - \left(E\left[\sum_{i=1}^{N} X_i\right]\right)^2$$

Now, $$E\left[\left(\sum_{i=1}^{N} X_i\right)^2\right] = E\left[E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \mid N\right]\right]$$

$$E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \mid N=n\right] = Var\left(\sum_{i=1}^{n} X_i\right) - \left(E\left[\sum_{i=1}^{n} X_i\right]\right)^2$$

$$= n\,Var(X) - (n\,E[X])^2$$

$$\therefore E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \mid N\right] = N\,Var(X) - (N\,E[X])^2$$

Taking exp. on both sides.

$$E\left[E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \mid N\right]\right] = E[N]\,Var(X) - E[N^2]\,(E[X])^2$$

$$\therefore Var\left(\sum_{i=1}^{N} X_i\right) = E[N]\,Var(X) - E[N^2](E[X])^2 - \left(E\left[\sum_{i=1}^{N} X_i\right]\right)^2$$

$$= E[N]\,Var(X) - E[N^2](E[X])^2 - (E[N]\,E[X])^2$$

$$= E[N]\,Var(X) - (E[X])^2\left\{E[N^2] - (E[N])^2\right\}$$

$$= E[N]\,Var(X) - (E[X])^2\,Var(N)$$

$$E[X] = 3\left[p^3 + (1-p)^3\right] + 4\left[3p^3(1-p) + 3p(1-p)^3\right] + 5\left[6p^3(1-p)^2 + 6p^2(1-p)^3\right]$$

$$E[X] = 2\left[p^2 + (1-p)^2\right] + 3\left[2p^2(1-p) + 2p(1-p)^2\right]$$

Computing Probabilities by Conditioning:

$$X = \begin{cases} 1, & \text{if event } E \text{ occurs} \\ 0, & \text{if " " does not occur} \end{cases}$$

$$E[X] = P(E)$$

$$\therefore E[X|Y=y] = P(E|Y=y)$$

$$\therefore P(E) = \sum_y P[E|Y=y]\, P(Y=y) = \int_{-\infty}^{\infty} P(E|Y=y)\, f_Y(y)\, dy$$

Ex:

$\boxed{X} \leftarrow N$ Let,

X: Number of suits that Rebecca sells

N: Number of customers who enter the store

$$P\{X=0\} = \sum_{n=0}^{\infty} P\{X=0|N=n\}\, P\{N=n\}$$

$$= \sum_{n=0}^{\infty} (1-p)^n \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(1-p)^n \lambda^n}{n!}$$

$$= e^{-\lambda} \cdot e^{\lambda(1-p)}$$

$$= e^{-\lambda p}$$

$$P\{X=k\} = \sum_{n=0}^{\infty} P\{X=k|N=n\}\, P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{n!}{k!\,(n-k)!} \frac{p^k (1-p)^{n-k} e^{-\lambda}\lambda^n}{n!}$$

$$= \frac{e^{-\lambda}(\lambda p)^k}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!}$$

$$= \frac{e^{-\lambda}(\lambda p)^k}{k!} e^{\lambda(1-p)} = e^{-\lambda p}\frac{(\lambda p)^k}{k!}$$

. X has poisson distr. with mean λp.

3.5 Computing Probabilities by Conditioning:

Random Variable X, Event E

$$X = \begin{cases} 1, & \text{if } E \text{ occurs} \\ 0, & \text{if } E \text{ does not occur} \end{cases}$$

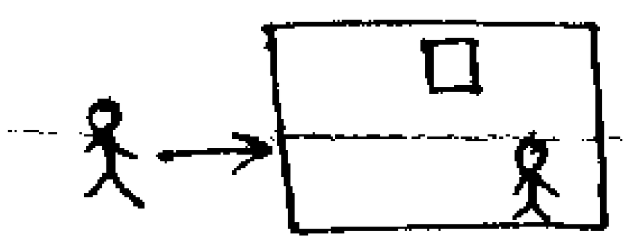
$$\therefore E[X] = \sum_x x\,p(x) = P(E)$$

$$E[X \mid Y=y] = P(E \mid Y=y)$$

$$\therefore P(E) = \sum_y P(E \mid Y=y)\,P(Y=y) \qquad E[X] = \sum_y E[X \mid Y=y]\,P(Y=y)$$

$$= \int_{-\infty}^{\infty} P(E \mid Y=y)\,f_Y(y)\,dy, \quad \text{if } Y \text{ cont}^s$$

Ex:



customer entering store will buy @ watch with p.
customer entering store is poisson distr. with λ

Let,
X : no. of watches sold
N : no. of customers entering store

$$P(X=0) = \sum_{n=0}^{\infty} P(X=0 \mid N=n)\,P(N \neq n)$$

$$= \sum_{n=0}^{\infty} P(X=0 \mid N=n)\,\frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \sum_{n=0}^{\infty} (1-p)^n\,\frac{e^{-\lambda}\lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\{\lambda(1-p)\}^n}{n!}$$

$$= e^{-\lambda}\,e^{\lambda(1-p)}$$

$$= e^{-\lambda p}$$

$$P(X=k) = \sum_{n=k}^{\infty} P(X=k \mid N=n)\,P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k}\,\frac{e^{-\lambda}\lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=k}^{\infty} \frac{(\lambda p)^k \{(1-p)\lambda\}^{n-k}}{(n-k)!\,k!}$$

$$= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{i=0}^{\infty} \frac{((1-p)\lambda)^i}{i!}$$

$$= \frac{e^{-\lambda}(\lambda p)^k}{k!} \cdot e^{\lambda - \lambda p}$$

$$= \frac{(\lambda p)^k\,e^{-\lambda p}}{k!}$$

Ex: (hat problem)

n men take of their hat

match if a man gets his own hat.

Let,

E = event that no matches occur

$P_n = P(E)$

M = event that 1st man selects his own hat.

M^c = " " " " does not " " "

$$P_n = P(E|M)\,P(M) + P(E|M^c)\,P(M^c)$$
$$= P(E|M^c)\,P(M^c) \qquad [\because P(E|M) = 0]$$
$$= P(E|M^c)\left(1-\frac{1}{n}\right)$$
$$= P(E|M^c)\,\frac{n-1}{n}$$

Now, $P(E|M^c)$ is the probability of no matches when $(n-1)$ men select from a set of $(n-1)$ hats that does not contain the hat of one of these men.

2 cases: 1st man took ith hat.

Case 1: ith man takes 1st hat $\Rightarrow \frac{1}{n-1} P_{n-2}$

Case 2: ith man does not " " $\Rightarrow P_{n-1}$

$$P(E|M^c) = \frac{1}{n-1} P_{n-2} + P_{n-1}$$

$$\therefore P_n = \frac{n-1}{n} P_{n-1} + \frac{1}{n} P_{n-2}$$

$$\therefore P_n - P_{n-1} = -\frac{1}{n}\left(P_{n-1} - P_{n-2}\right)$$

We have, $P_1 = 0$, $P_2 = \frac{1}{2}$

$$\therefore P_3 - P_2 = -\frac{1}{3}(P_2 - P_1) = -\frac{1}{3!} \Rightarrow P_3 = \frac{1}{2!} - \frac{1}{3!}$$
$$P_4 - P_3 = -\frac{1}{4}(P_3 - P_2) = \frac{1}{4!} \Rightarrow P_4 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

in general, $P_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots + \frac{(-1)^n}{n!} \approx e^{-1}$ as $n \to \infty$

Exactly k matches:

$n-k$ non matches

k matches

$$P(\text{exactly } k \text{ matches}) = \frac{1}{n}\frac{1}{n-1}\frac{1}{n-2}\cdots\frac{1}{n-(k-1)} P_{n-k} = \frac{(n-k)!}{n!} P_{n-k}$$

Since there are $\binom{n}{k}$ choices of a set of k men,

$$P(\text{exactly } k \text{ matches}) = \binom{n}{k}\frac{(n-k)!}{n!} P_{n-k}$$

$$= \frac{P_{n-k}}{k!}$$

$$= \frac{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^{n-k}\frac{1}{(n-k)!}}{k!}$$

$$= \frac{e^{-1}}{k!}, \quad n \to \infty$$

The Ballot Problem:

$A \to n$ votes

$B \to m$ votes

$n > m$, all orderings are equally likely

Let, $P_{n,m}$ = Prob. that A is always ahead of B

$$\therefore P_{n,m} = P\{A \text{ is always ahead} \mid A \text{ receives last vote}\} P\{A \text{ rec. last}\} + P\{A \text{ always ahead} \mid B \text{ receives last vote}\} P(B \text{ rec. last})$$

$$= P_{n-1,m} \cdot \frac{n}{n+m} + P_{n,m-1}\frac{m}{n+m}$$

* We now prove that, $P_{n,m} = \frac{n-m}{n+m}$

Proof: by induction on $n+m$

$\underline{n+m=1:}$

$$P_{1,0}=1 \quad , \text{ true}$$

Assume true for $n+m=k$,

Now for $n+m=k+1$

$$P_{n,m}=\frac{n}{n+m}P_{n-1,m}+\frac{m}{n+m}P_{n,m-1}$$

$$=\frac{n}{n+m}\cdot\frac{n-1-m}{n-1+m}+\frac{m}{n+m}\cdot\frac{n-m+1}{n+m-1}$$

$$=\frac{n^2-n-nm+nm-m^2+m}{(n+m)(n+m-1)}$$

$$=\frac{n^2-m^2-(n-m)}{(n+m)(n+m-1)}$$

$$=\frac{n-m}{n+m}$$

(Proved)

A List Model:

* self organizing file system e_1, e_2, e_3, e_4, e_5
* front-of-the-line rule

E[position of the file to be retrieved]

$$= \sum_{i=1}^{n} E[\text{ position} \mid e_i \text{ is selected}]\, P_i$$

$$= \sum_{i=1}^{n} E[\text{position of } e_i]\, P_i$$

Now, position of $e_i = 1 + \sum_{j \neq i} I_j$

where, $I_j = \begin{cases} 1, & \text{if } e_j \text{ preceeds } e_i \\ 0, & \text{otherwise} \end{cases}$

$$\cancel{E[\text{position of } e_i] = E\left[1 + \sum_{j \neq i} I_j\right]}$$

$$= 1 + \sum_{j \neq i} E[I_j]$$

$$E[I_j] = P\{e_j \text{ preceeds } e_i)$$

$$= P(e_j \mid e_i \text{ or } e_j)$$

$$= \frac{P_j}{P_i + P_j}$$

$\therefore$ Expected position of file ~~estimated~~ request]

$$= \sum_{i=1}^{n} \left(1 + \sum_{j \neq i} \frac{P_j}{P_i + P_j}\right) P_i$$

$$= 1 + \sum_{i=1}^{n} P_i \sum_{j \neq i} \frac{P_j}{P_i + P_j}$$

Chap 4

Stochastic Process

a collection of RVs, $\{\underbrace{x(t)}_{\text{a state of the process at time } t}, t \in T\}$

Markov Chain:

is an stochastic process $\{X_n, n=0,1,2,\dots\}$ such that whenever the process is in state i, there is a fixed probability ~~that~~ P_{ij} that it will next be in state j.

$$P\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \dots X_1=i_1, X_0=i_0\} = P_{ij}$$

for all states $i_0, i_1, \dots, i_{n-1}, i, j$.

* The present state is independent of the past state and depends only on the present state.

$$P_{ij} \geq 0 \quad , i,j \geq 0$$

$$\sum_{j=0}^{\infty} P_{ij} = 1 \quad , i=0,1,\dots$$

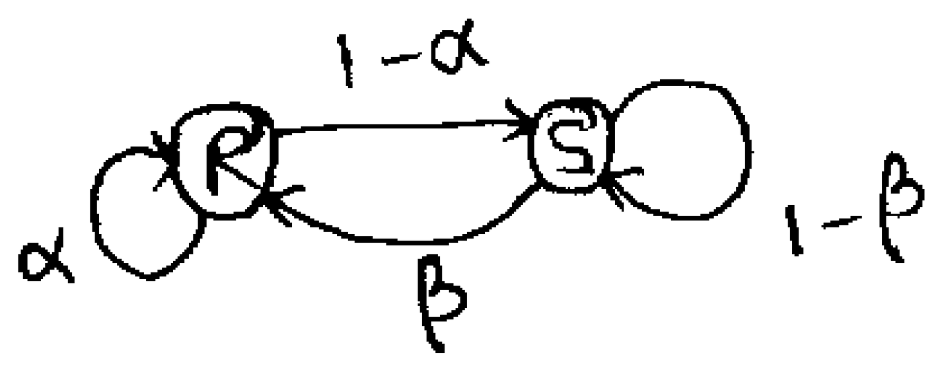
- must make a transition.

* transition matrix,

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \dots \\ P_{10} & P_{11} & P_{12} & \dots \\ \vdots & & & \\ P_{i0} & P_{i1} & P_{i2} & \dots \end{bmatrix}$$

Ex: weather,

$$P = \begin{array}{c c} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix} \end{array}$$



Ex: Communication System $\quad i \longrightarrow \boxed{\qquad} \longrightarrow j$

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \end{array}$$

Chapman - Kolmogorov Eq^n:

P_{ij} - one step transition probability

P_{ij}^n - n step transition probability that a process in state i will be in state j after n additional transitions.

$$P_{ij}^n = P\{X_{n+m} = j \mid X_m = i\}, \quad n \geq 0, \; i,j \geq 0$$

$$\boxed{P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m}, \quad \text{for all } n,m \geq 0, \text{ all } i,j$$

Proof:

$$\begin{aligned} P_{ij}^{n+m} &= P\{X_{n+m} = j \mid X_0 = i\} \\ &= \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k \mid X_0 = i\} \\ &= \sum_{k=0}^{\infty} P\{X_{n+m} = j \mid X_n = k, X_0 = i\} \, P\{X_n = k \mid X_0 = i\} \\ &= \sum_{k=0}^{\infty} P_{kj}^m P_{ik}^n \end{aligned}$$

* the n step transition matrix may be obtained by multiplying the matrix by itself n times.

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

Ex: $\alpha = 0.7$, $\beta = 0.4$, $P = \begin{array}{c|cc} & R & S \\ \hline R & 0.7 & 0.3 \\ S & 0.4 & 0.6 \end{array}$

$$P^{(2)} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$$P^{(4)} = (P^2)^2 = \begin{bmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{bmatrix}$$

$$P_{00}^{(4)} = 0.5749$$

* Markov chain is irreducible if there is only one class that is, if all states communicate each other.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$\hookrightarrow$ absorbing state

* class of markov chain are $\{0,1\}$, $\{2\}$, $\{3\}$

state:

recurrent state

transient state

Markov Chain

Ex

$$P = \begin{array}{c} \\ MT \\ RR \\ RS \\ SR \\ SS \end{array} \begin{array}{cccc} & TW & & \\ RR & RS & SR & SS \\ \end{array}$$

$$P = \begin{array}{c|cccc} & RR & RS & SR & SS \\ \hline RR & 0.7 & 0.3 & 0 & 0 \\ RS & 0 & 0 & 0.4 & 0.6 \\ SR & 0.5 & 0.5 & 0 & 0 \\ SS & 0 & 0 & 0.2 & 0.8 \end{array}$$

what is the probability that it will rain on Thursday.

$$P^{(2)} = \begin{bmatrix} 0.49 & 0.21 & 0.12 & 0.18 \\ 0.20 & 0.20 & 0.12 & 0.48 \\ 0.35 & 0.15 & 0.20 & 0.30 \\ 0.1 & 0.10 & 0.16 & 0.64 \end{bmatrix}$$

$$\therefore \; P_{00}^{(2)} + P_{02}^{(2)} = 0.49 + 0.12 = 0.61$$

4.4 Limiting Probability:

* 2 properties of the states of a Markov chain,

- state i is said to have period d if $P_{ii}^{n} = 0$ whenever n is not divisible by d and d is the largest integer with this property.

- ~~A state with period 1 is said to be~~ aperiodic.

Th^m: If $\lim_{n\to\infty} P_{ij}^{n}$ exists and is independent of i

and $\pi_j = \lim_{n\to\infty} P_{ij}^{n}$, $j \geq 0$

then π_j is the unique nonnegative sol^n of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad , \; j \geq 0$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$

limiting probability of a column.

Ex:

$$P = \begin{array}{c} \\ R \\ S \end{array}\!\!\begin{array}{cc} R & S \\ \end{array}$$

$$P = \begin{array}{c|cc} & R & S \\ \hline R & \alpha & 1-\alpha \\ S & \beta & 1-\beta \end{array}$$

$$\pi_0 = \alpha \pi_0 + \beta \pi_1$$

$$\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$$

$$\pi_0 + \pi_1 = 1$$

$$\pi_0 = \frac{\beta}{1-\alpha+\beta}$$

$$\pi_1 = \frac{1-\alpha}{1-\alpha+\beta}$$

Hardy - Weinberg Law and Markov Chain in Genetics :

- Each individual gene pair AA, Aa, aa

$$P\{AA\} = p_0 \ , \ P\{aa\} = q_0 \quad P\{Aa\} = r_0$$

$$p_0 + q_0 + r_0 = 1$$

Let, next generation probability,

$$P\{AA\} = p, \quad P\{aa\} = q, \quad P\{Aa\} = r$$

$$P\{A\} = P\{A|AA\} p_0 + P\{A|aa\} q_0 + P\{A|Aa\} r_0 = p_0 + 0 + \frac{1}{2} r_0$$

$$\therefore p = P\{AA\} = P\{A\}P\{A\} = \left(p_0 + \frac{r_0}{2}\right)\left(p_0 + \frac{r_0}{2}\right) = \left(p_0 + \frac{r_0}{2}\right)^2$$

Similarly, $q = P\{aa\} = \left(q_0 + \frac{r_0}{2}\right)^2$

$$r = P\{Aa\} = 2\left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right) = P\{A\}P\{a\} + P\{a\}P\{A\}$$

now,

$$p + q + r = \left(p_0 + \frac{r_0}{2}\right)^2 + \left(q_0 + \frac{r_0}{2}\right)^2 + 2\left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$$

$$= (p_0 + q_0 + r_0)^2$$

$$= 1$$

$$p + \frac{r}{2} = \left(p_0 + \frac{r_0}{2}\right)^2 + \left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$$

$$= \left(p_0 + \frac{r_0}{2}\right) \cdot (p_0 + q_0 + r_0)$$

$$= p_0 + \frac{r_0}{2}$$

$$= P\{A\}$$

$\therefore$ % remains unchanged.

The Gambler's Ruin Problem:

Let,

X_n denote the players fortune at time n, the process $\{X_n, n=0,1,2\dots\}$ is a Markov chain with transition prob.

$$P_{00} = P_{NN} = 1$$

$$P_{i,i+1} = p = 1 - P_{i,i-1} \quad , i=1,2,\dots,N-1$$

$\therefore$ The markov chain has 3 classes $\{0\}, \{1,2,\dots,N-1\} \{N\}$

transient

recurrent

Let,

P_i , $i=0,1,2,\dots,N$ denote the prob. that starting with i the gambler will eventually reach N.

$$P_i = pP_{i+1} + qP_{i-1} \quad , i=1,2,\dots,N-1$$

$$pP_i + qP_i = pP_{i+1} + qP_{i-1}$$

$$\therefore P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}) \quad , i=1,2,\dots N-1$$

Since $P_0 = 0$,

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$

$$\vdots$$

$$P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$\vdots$$

$$P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = \left(\frac{q}{p}\right)^{N-1} P_1$$

Adding first i-1 eq^n's: $P_i - P_1 = P_1\left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \dots + \left(\frac{q}{p}\right)^{i-1}\right]$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} P_1 & , \text{if } \frac{q}{p} \neq 1 \\ iP_1 & , \text{if } \frac{q}{p} = 1 \end{cases}$$

Now, $P_N = 1 = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^N}{1-\frac{q}{p}} P_1 & \text{, if } p \neq \frac{1}{2} \\ NP_1 & \text{, if } p = \frac{1}{2} \end{cases}$

$$P_1 = \begin{cases} \dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} & \text{, if } p \neq \frac{1}{2} \\ \dfrac{1}{N} & \text{, if } p = \frac{1}{2} \end{cases}$$

Putting the value of P_1,

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & \text{, if } p \neq \frac{1}{2} \\ \dfrac{i}{N} & \text{, if } p = \frac{1}{2} \end{cases}$$

As $N \to \infty$,

$$P_i \to \begin{cases} 1-\left(\frac{q}{p}\right)^i & , p > \frac{1}{2} \\ 0 & , p \leq \frac{1}{2} \end{cases}$$

Chap 5

Exponential Distribution

PDF, $f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$

CDF, $F(x) = \int_{-\infty}^{x} f(y)\, dy = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$

$$= \int_{-\infty}^{x} \lambda e^{-\lambda y}\, dy$$

$$= \lambda \left[\frac{e^{-\lambda y}}{-\lambda} \right]_{-\infty}^{x}$$

$$= -\left(e^{-\lambda x} - 1\right)$$

$P(X \leq x) = 1 - e^{-\lambda x} = F(x)$

$P(X > x) = e^{-\lambda x}$

mean, $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{0}^{\infty} \lambda x e^{-\lambda x}\, dx = \left[\lambda x \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{(-\lambda)^2} \lambda \right]_0^{\infty} = \left[\cancel{-x e^{-\lambda x}} \right]_0^{\infty} = \frac{1}{\lambda}$$

$$\phi(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx = \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^{\infty} = \frac{\lambda}{\lambda - t}$$

$(t < \lambda)$

~~$E[X^2] = \phi''(0) =$~~

$$\phi'(t) = \frac{\lambda}{(\lambda - t)^2}$$

$$\phi''(t) = \frac{2\lambda(\lambda - t)}{(\lambda - t)^4} = \frac{2\lambda}{(\lambda - t)^3}$$

$$\therefore E[X^2] = \phi''(0) = \frac{2}{\lambda^2}$$

$$\therefore Var[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Properties of Exp. distr.:

A random variable X is said to be without memory or memory less if

$$P\{X > s+t \mid X > t\} = P\{X > s\} \quad , s,t \geq 0$$

$$\Rightarrow \frac{P\{X > s+t,\ X > t\}}{P\{X > t\}} = P(X > s)$$

$$\Rightarrow P\{X > s+t\} = P\{X > s\}\, P\{X > t\}$$

Exp. distr. is memoryless, since,

$$e^{-\lambda(s+t)} = e^{-\lambda s} \cdot e^{-\lambda t}$$

Ex:

X: amount of time a customer spend in the bank.

$$\lambda = \frac{1}{10}$$

~~(i) $P\{X > 15\} = e^{-15\lambda} = e^{-3/2} \approx 0.220$~~

(ii) $P\{X > 15 \mid X > 10\} = P\{X > 5\} = e^{-5\lambda} = e^{-1/2} \approx 0.607$

* Exp. distr. is the only memory less distribution:

Proof:

Let, X is memoryless and $\bar{F}(x) = P\{X > x\}$

$$\bar{F}(s+t) = P\{X > s+t\} = P\{X > s\}\, P\{X > t\} = \bar{F}(s)\,\bar{F}(t)$$

$\therefore \bar{F}(x)$ satisfies the equ^n,

$$g(s+t) = g(s)\, g(t)$$

The only solu^n of this equ^n is,

$$g(x) = e^{-\lambda x}$$

$$\therefore g(s+t) = e^{-\lambda(s+t)} = e^{-\lambda s} \cdot e^{-\lambda t} = g(s)\, g(t)$$

$\therefore$ We must have, $\bar{F}(x) = e^{-\lambda x}$

$\therefore F(x) = P\{X \leq x\} = 1 - e^{-\lambda x}$ $\quad \therefore$ (proved)

Ex:

lifetime of a lightbulb is exp. distr. with mean 10h.

$$P\{\text{lifetime} > 5\} = 1 - F(5) = 1 - (1 - e^{-5\lambda}) = e^{-5/10} = e^{-1/2}.$$

if lifetime is not exponential,

$$P\{\text{lifetime} > t+5 \mid \text{lifetime} > t\} = \frac{1 - F(t+5)}{1 - F(t)}$$

Further properties of exp. distr. :

* X_1 & X_2 indep. exp. random variable with means $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$

$$\therefore P\{X_1 < X_2\} = \int_0^\infty P\{X_1 < X_2 \mid X_1 = x\}\, \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_0^\infty P\{x < X_2\}\, \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_0^\infty e^{-\lambda_2 x}\, \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2) x}\, dx$$

$$= \lambda_1 \left[\frac{e^{-(\lambda_1+\lambda_2)x}}{-(\lambda_1+\lambda_2)} \right]_0^\infty$$

$$= \frac{\lambda_1}{-(\lambda_1+\lambda_2)} (-1)$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Ex:

$$\lambda_1 = \frac{1}{1000}, \quad \lambda_2 = \frac{1}{500}$$

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{\frac{1}{1000}}{\frac{1}{1000} + \frac{1}{500}} = \frac{1}{1+2} = \frac{1}{3}$$

Counting Process:

- A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total no. of events that have occured up to time t.

- ex: # of persons entered store
  # of birth, # of goals

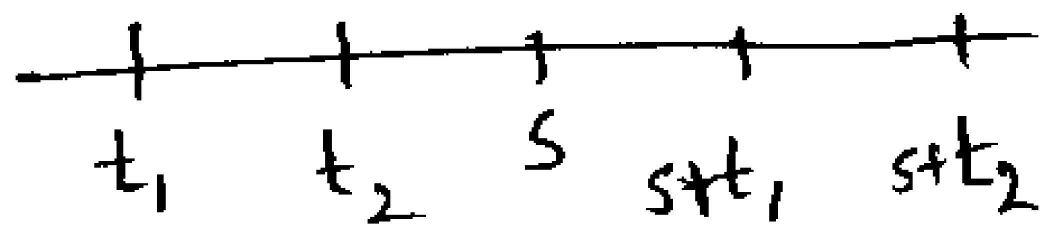
- Properties:

  (i) $N(t) \geq 0$
  (ii) $N(t)$ is integer valued.
  (iii) if $s<t$ then $N(s) \leq N(t)$.
  (iv) For $s<t$, $N(t) - N(s) =$ # of events during interval $(s,t]$

∗ A ~~counting process~~ is said to ~~have~~ possess independent increments if the number of events that occur in disjoint time intervals are independent.

∗ A counting process is said to possess stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval.

i.e. $N(t_2+s) - N(t_1+s)$
$= N(t_2) - N(t_1)$



Poisson Process :

A counting process $\{N(t), t \geq 0\}$ is said to be a poisson process having rate λ, $\lambda > 0$ if

(i) $N(0) = 0$

(ii) The process has independent increments.

(iii) The # of events in any interval of length t is poisson distributed with mean λt, i.e. for all $s, t \geq 0$

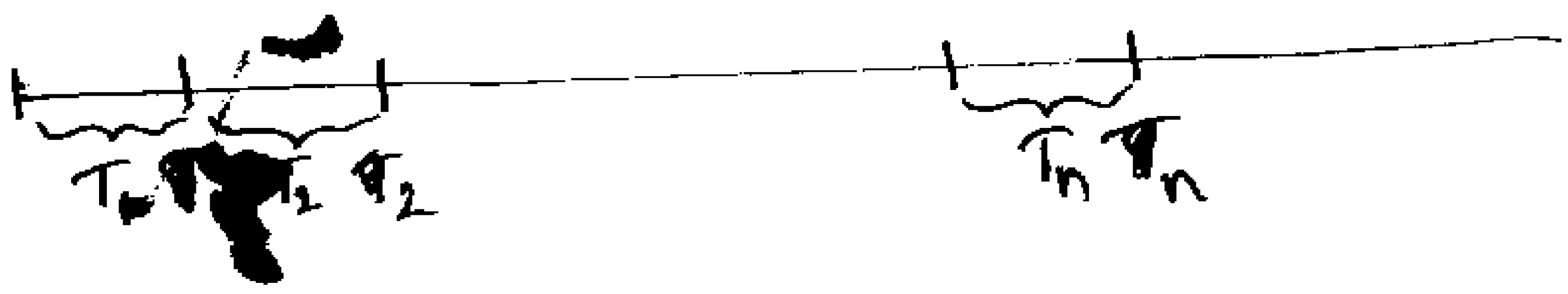
$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

* From cond (iii) it follows that poisson process has ~~stationary increments also,~~

$$E[N(t)] = \lambda t$$

So, λ is called the rate of the process.

Inter-arrival & waiting time distribution :



$\{T_n, n = 1, 2, \ldots\}$ interarrival times

$$\begin{aligned} P\{T_1 > t\} &= P\{\text{no events in } [0,t]\} \\ &= P\{N(t) = 0\} \\ &= e^{-\lambda t} \qquad [n=0] \end{aligned}$$

Hence, T_1 has exp. distr. with mean $\frac{1}{\lambda}$.

$$P\{T_2 > t\} = E\left[P\{T_2 > t \mid T_1\}\right]$$

$$\begin{aligned} P\{T_2 > t \mid T_1 = s\} &= P\{0 \text{ events in } (s, s+t] \mid T_1 = s\} \\ &= P\{0 \text{ events in } (s, s+t]\} && [\text{independent inc.}] \\ &= e^{-\lambda t} && [\text{stationary inc.}] \end{aligned}$$

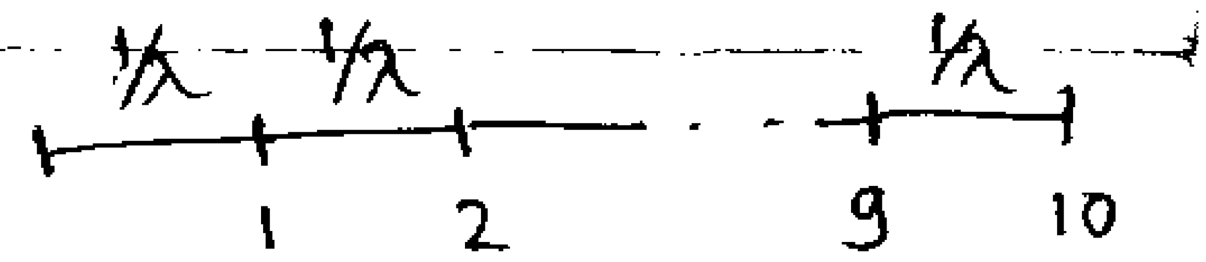
So, T_2 is also an exp. random variable with mean $\frac{1}{\lambda}$ and furthermore, T_2 is independent of T_1.

Similarly, T_n is and exp. random vari.

Ex:

people migrate into a territory at a poisson rate $\lambda = 1$ per day.

(i) $E[S_{10}] = \frac{10}{\lambda} = 10$ days



(ii) $P\{T_{11} > 2\} = e^{-2\lambda} = e^{-2} \approx 0.133$

Chap 8
Queueing Theory

Cost Equations:

Let,

L: avg # of customers in the system.

L_Q: avg. # of customers waiting in the queue.

W: avg. amount of time customers spend in the system

W_Q: " " " " " " " Queue

basic cost identity,

avg. rate at which the system earns

$= \lambda_a \times$ avg amount an entering customer pays.

where, λ_a = avg. arrival rate of enterning customer.

$$= \lim_{t \to \infty} \frac{N(t)}{t}$$

where, $N(t)$ = number of customer arrival by time t.

* if each customer pays \$1 per unit time while in the system,

$$L.1 = \lambda_a . W$$

$$L_Q = \lambda_a W_Q$$

* avg # of customer in the system $= \lambda_a E[S]$

$E[S]$ = avg. amount of time a customer spends in service.

Steady-state Probabilities:

Let,

$x(t)$ - # of customers in the system at time t.

$P_n = \lim_{t\to\infty} P\{x(t) = n\}$

= long run prob. that there will be exactly n customers in the system.

= steady-state prob. of exactly n customers in the system.

a_n = proportion of customers that find n in the system when they arrive.

= proportion of time an entering customer sees n person in the system.

d_n = proportion of time a leaving customer leaves n person in the system.

Proposition:

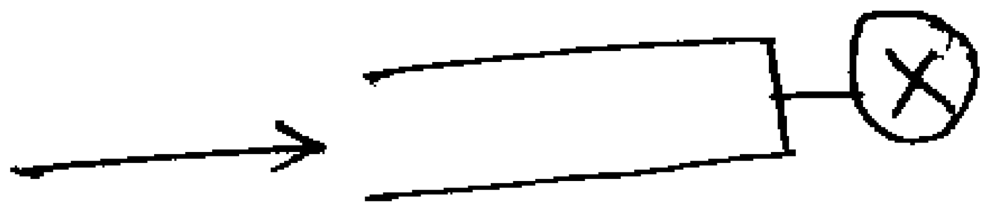
In any system in which customers arrive one at a time and are served one at a time $a_n = d_n$, $n \geq 0$.

Prop.

Poisson arrivals always see time averages, for poisson arrivals, $P_n = a_n$.

Single Server Exponential Queueing System:

poisson process rate λ



$\therefore$ inter-arrival time ind. exp. random vari with mean $\frac{1}{\lambda}$.

service time " " " " " " $\frac{1}{\mu}$

M/M/1 Queue:

Single server.

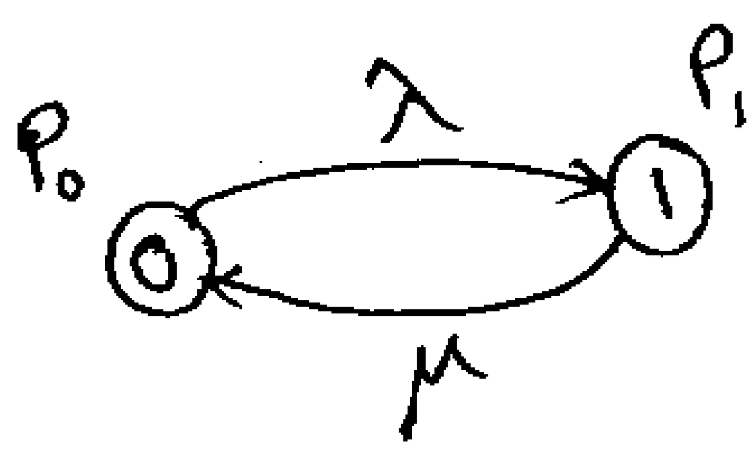
Service distribution is memoryless.

inter-arrival time is memory less

Arrival rate λ

Service rate μ

* For stable system $\lambda \leq \mu$

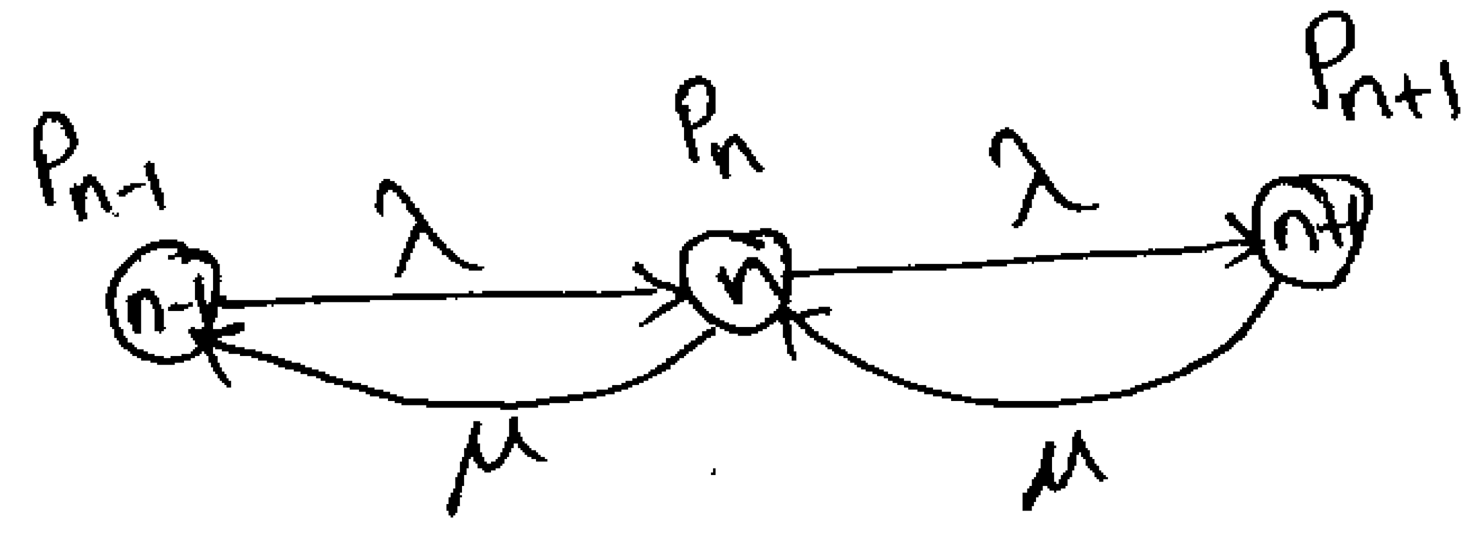


$$\lambda P_0 = \mu P_1$$

$$P_1 = \frac{\lambda}{\mu} P_0$$

balance equ^{n}s:

State	Rate at which process leaves = rate at which it enters
0	$\lambda P_0 = \mu P_1$
$n,\ n \geq 1$	$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$



Now, $P_1 = \frac{\lambda}{\mu} P_0$

$$P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right) \quad , n \geq 1$$

Now, $P_2 = \frac{\lambda}{\mu} P_1 + \left(P_1 - \frac{\lambda}{\mu} P_0\right) = \left(\frac{\lambda}{\mu}\right)^2 P_0$

$$P_3 = \frac{\lambda}{\mu} P_2 + \left(P_2 - \frac{\lambda}{\mu} P_1\right) = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\vdots$$

$$P_{n+1} = \left(\frac{\lambda}{\mu}\right)^{n+1} P_0$$

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow \frac{P_0}{1 - \frac{\lambda}{\mu}} = 1 \qquad \left[\frac{\lambda}{\mu} < 1\right]$$

$$\Rightarrow P_0 = 1 - \frac{\lambda}{\mu}$$

$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

* avg no. of customers in the sys.

$$L = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \frac{\lambda/\mu}{\left(1 - \frac{\lambda}{\mu}\right)^2}$$

$$= \frac{\lambda}{\mu - \lambda}$$

$$L = \lambda W$$

$$\therefore W = \frac{1}{\mu - \lambda}$$

$$W_Q = W - E[S]$$

$$= \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$

$$= \frac{\lambda}{\mu(\mu - \lambda)}$$

$$L_Q = \lambda W_Q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

Single Server Exponential Queueing System having finite capacity

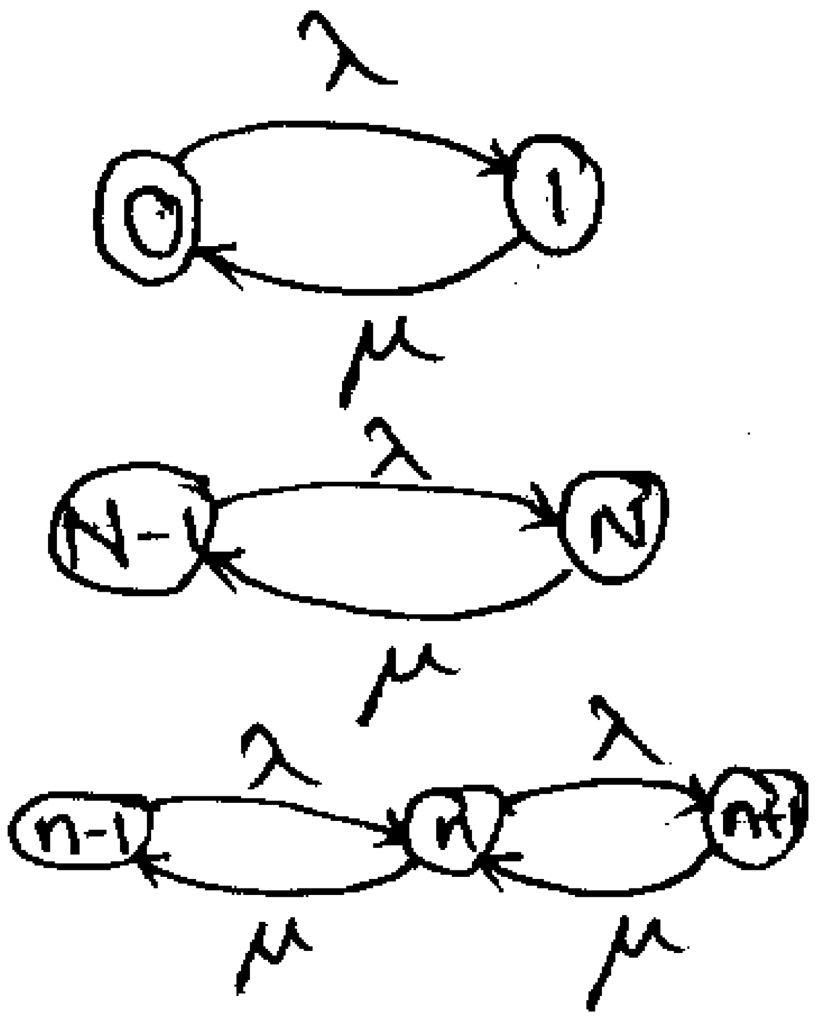
- finite system capacity N

Let,

P_n, $0 \leq n \leq N$: the prob. that there are n customers in the system.

balanced eqn:

State	Rate at which process leaves = rate at which it enters
0	$\lambda P_0 = \mu P_1$
N	$\lambda P_{N-1} = \mu P_N$
$1 \leq n \leq N-1$	$\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu) P_n$



So, we get,

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right) \quad , \quad 1 \leq n \leq N-1$$

$$P_N = \frac{\lambda}{\mu} P_{N-1} = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$\therefore P_2 = \frac{\lambda}{\mu} P_1 + \left(P_1 - \frac{\lambda}{\mu} P_0\right) = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 + \left(P_2 - \frac{\lambda}{\mu} P_1\right) = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\vdots$$

$$P_{N-1} = \frac{\lambda}{\mu} P_{N-2} + \left(P_{N-2} - \frac{\lambda}{\mu} P_{N-3}\right) = \left(\frac{\lambda}{\mu}\right)^{N-1} P_0$$

$$P_N = \frac{\lambda}{\mu} P_{N-1} = \left(\frac{\lambda}{\mu}\right)^N P_0$$

Now,

$$\sum_{n=0}^{N} P_n = 1 \Rightarrow \sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^N P_0 = 1 \Rightarrow P_0 \frac{1 - (\lambda/\mu)^{N+1}}{1 - \frac{\lambda}{\mu}} = 1$$

$$\therefore P_0 = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}, \quad n = 0, 1, \dots, N$$

There is no need for condition $\frac{\lambda}{\mu} < 1$

$$L = \sum_{n=0}^{N} n P_n$$

$$= \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}} \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} \cdot \frac{\lambda}{\mu} \, \frac{\left(1-\frac{\lambda}{\mu}\right)^{N+1} - (N+1)\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^N}{\left(1-\frac{\lambda}{\mu}\right)^2}$$

$$= \frac{\lambda\left\{1 + N\left(\frac{\lambda}{\mu}\right)^{N+1} - (N+1)\left(\frac{\lambda}{\mu}\right)^N\right\}}{\left(1-\frac{\lambda}{\mu}\right)^{N+1}(\mu-\lambda)}$$

$$\sum_{n=0}^{N} n x^n = \sum_{n=0}^{N} x \frac{d}{dx}(x^n) = x \cdot \frac{d}{dx}\left(\sum_{n=0}^{N} x^n\right) = x \frac{d}{dx} \cdot \frac{1-x^{n+1}}{1-x} = x \frac{-(1-x)(n+1)x^n + (1-x^{n+1})}{(1-x)^2}$$

$$W = \frac{L}{\lambda_a}$$

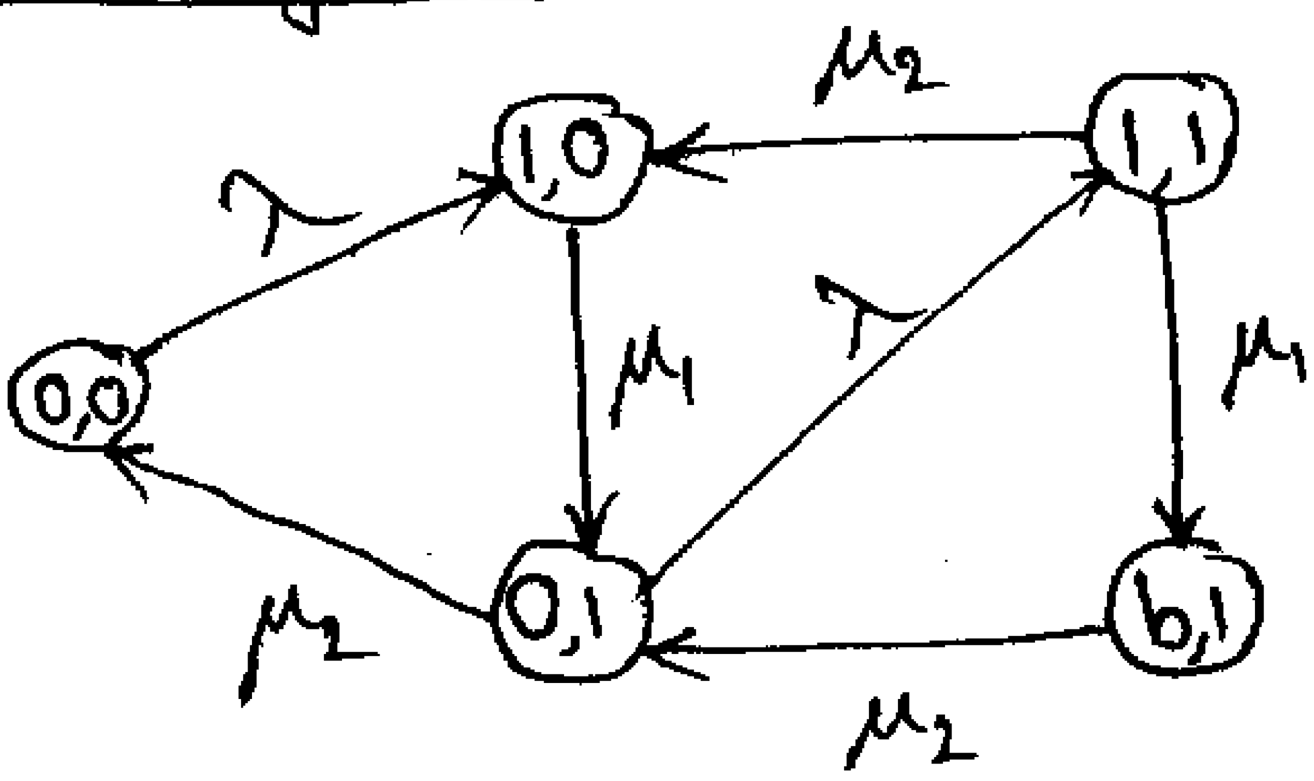
(i) $\lambda = \lambda_a$, if full system & cust. does not wait

(ii) $\lambda_a = \lambda(1-P_N)$, persons actually entered.

A Shoeshine Shop:

State	Interpretation
(0,0)	no cust in system
(0,1)	one cust. in chair 2
(1,0)	" " " " 1
(1,1)	both being served
(b,1)	1st chair waiting

Transition Diagram:



balance equation:

State	Rate that process leaves = rate that it enters
(0,0)	$\lambda P_{00} = \mu_2 P_{01}$
(1,0)	$\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$
(0,1)	$(\mu_2+\lambda) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$
(1,1)	$\lambda P_{01} = (\mu_1+\mu_2) P_{11}$
(b,1)	$\mu_2 P_{b,1} = \mu_1 P_{11}$

$$P_{00} + P_{01} + P_{10} + P_{11} + P_{b1} = 1$$

Proportion of customers entering the system $P_{00} + P_{01}$

$\therefore$ Avg. no. of customers in the system, $L = (P_{01}+P_{10}) + 2(P_{11}+P_{b1})$

Avg. amount of time an entering customer spends in the system,

$$W = \frac{L}{\lambda_a}$$

$$\lambda_a = \lambda\left(P_{00} + P_{01}\right)$$

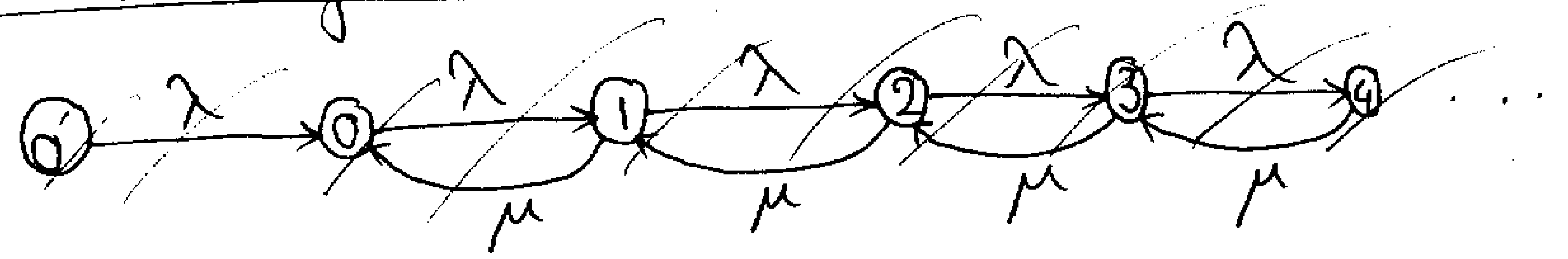
$$W = \frac{P_{01} + P_{10} + 2\left(P_{11} + P_{b1}\right)}{\lambda\left(P_{00} + P_{01}\right)}$$

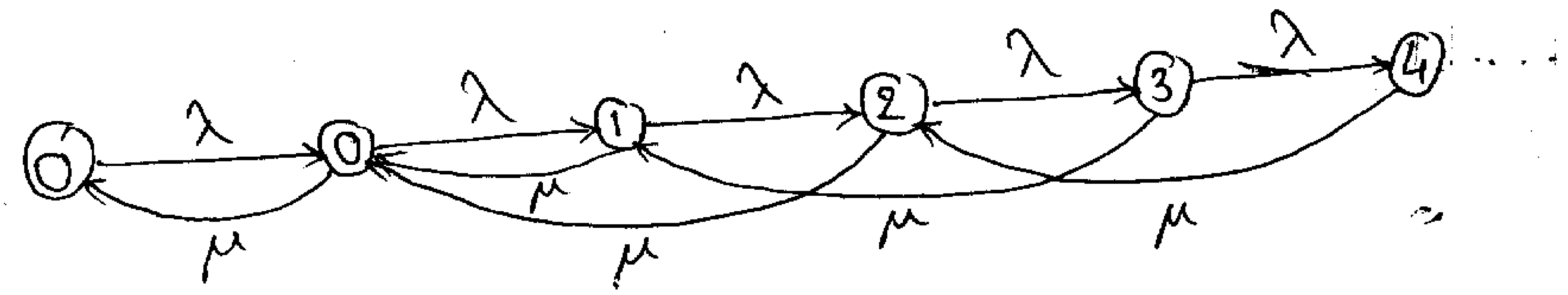
Queueing System with Bulk Service:

* serves 2 customers at the same time.

State	Interpretation
$0'$	No one in service
0	server busy, no one waiting
$n, n\geq 0$	n cust. waiting

Transition Diagram:





balance equ^n:

State	Rate that process leaves = " " " enters
$0'$	$\lambda P_{0'} = \mu P_0$
0	$(\lambda+\mu)P_0 = \lambda P_{0'} + \mu P_1 + \mu P_2$
$n, n\geq 1$	$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+2}$

Now, the equ^n,

$$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+2} \quad , \quad n=1,2,\ldots$$

it has a sol^n of the form, $P_n = \alpha^n P_0$

Putting in equ^n,

$$(\lambda+\mu)\alpha^n P_0 = \lambda \alpha^{n-1} P_0 + \mu \alpha^{n+2} P_0$$

$$\Rightarrow (\lambda+\mu)\alpha = \lambda + \mu\alpha^3$$

$$\Rightarrow \alpha^3\mu - \alpha(\lambda+\mu) + \lambda = 0$$

$$\Rightarrow \mu\alpha^2(\alpha-1) + \mu\alpha(\alpha-1) - \lambda(\alpha-1) = 0$$

$$\Rightarrow (\alpha-1)\left(\mu\alpha^2 + \mu\alpha - \lambda\right) = 0$$

$$\alpha = 1, \quad \alpha = \frac{-\mu \pm \sqrt{\mu^2 + 4\mu\lambda}}{2\mu} = \frac{-1 \pm \sqrt{1 + \frac{4\lambda}{\mu}}}{2}$$

if $\alpha = 1$, $P_0 = P_1 = P_2 = \dots = 0$

$\alpha \ngtr 1$ is not possible,

$$\therefore \alpha = \frac{-1 + \sqrt{1 + \frac{4\lambda}{\mu}}}{2} < 1$$

$$\Rightarrow -1 + \sqrt{1 + \frac{4\lambda}{\mu}} < 2$$

$$\Rightarrow \sqrt{1 + \frac{4\lambda}{\mu}} < 3.$$

$$\Rightarrow 1 + \frac{4\lambda}{\mu} < 9$$

$$\frac{\lambda}{\mu} < 2$$

$\therefore$ max^m service rate 2μ

$$\therefore P_n = \alpha^n P_0$$

$$P_0' = \frac{\mu}{\lambda} P_0$$

To obtain P_0,

$$P_0 + P_0' + \sum_{n=1}^{\infty} P_n = 1$$

$$\Rightarrow P_0\left[1 + \frac{\mu}{\lambda} + \sum_{n=1}^{\infty} \alpha^n\right] = 1$$

$$\Rightarrow P_0\left[\frac{\mu}{\lambda} + \frac{1}{1-\alpha}\right] = 1$$

$$\Rightarrow P_0 = \frac{\lambda(1-\alpha)}{\lambda + \mu(1-\alpha)}$$

$$\therefore P_n = \alpha^n P_0 = \frac{\alpha^n \lambda(1-\alpha)}{\lambda + \mu(1-\alpha)}, \quad n \geq 0$$

$$P_0' = \frac{\mu(1-\alpha)}{\lambda + \mu(1-\alpha)}$$

$\therefore 2\mu > \lambda$ must hold to avoid overload the system

$\therefore$ The rate at which the customers are served alone $= \lambda P_{0'} + \mu P_1$

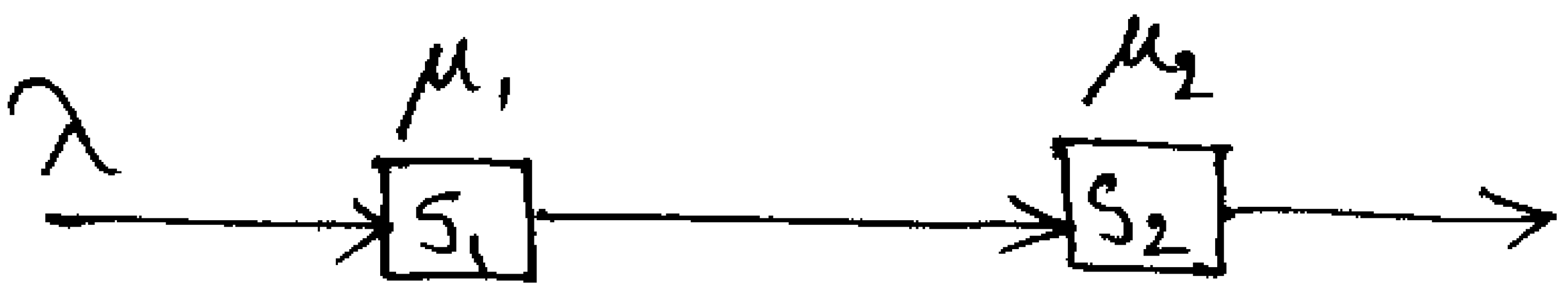
$$\therefore \text{Proportion of cust. served alone} = \frac{\lambda P_{0'} + \mu P_1}{\lambda}$$

$$L_Q = \sum_{n=1}^{\infty} nP_n = \frac{\lambda\alpha}{(1-\alpha)\left[\lambda + \mu(1-\alpha)\right]}$$

$$W_Q = \frac{L_Q}{\lambda}$$

$$W = W_Q + \frac{1}{\mu} \qquad L = W\lambda$$

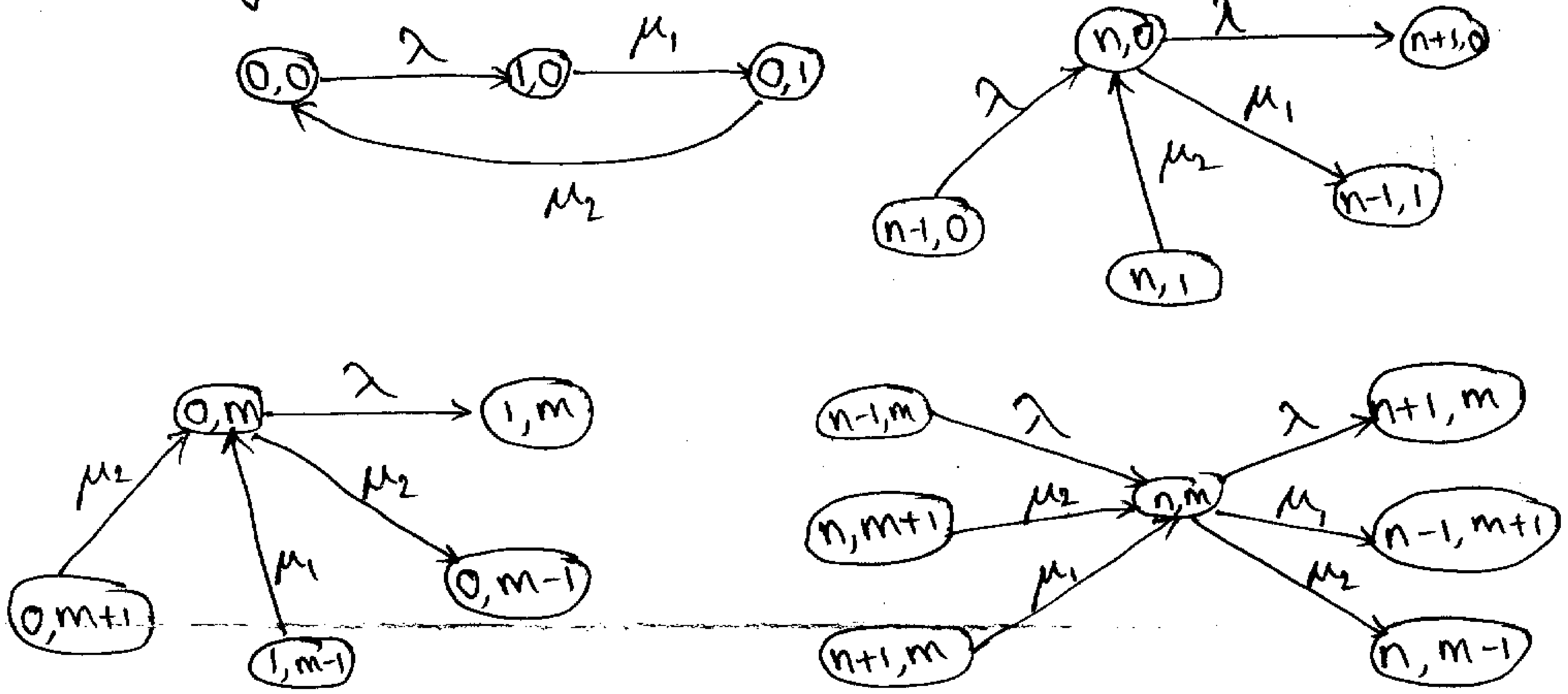
Network of Queues: (Open System)



(n,m) – n cust. at server1
m " " " 2

$\lambda < \mu_1, \mu_2$, exp. poisson process

State Diagram:



balance equ^n

State	Rate process leaves = Rate it enters
0,0	$\lambda P_{00} = \mu_2 P_{01}$
n,0	$(\lambda + \mu_1) P_{n0} = \lambda P_{n-1,0} + \mu_2 P_{n,1}$
0,m	$(\lambda + \mu_2) P_{0,m} = \mu_1 P_{1,m-1} + \mu_2 P_{0,m+1}$
n,m	$(\lambda + \mu_1 + \mu_2) P_{n,m} = \lambda P_{n-1,m} + \mu_1 P_{n+1,m} + \mu_2 P_{n,m+1}$

For M/M/1, system,

$$P\{n \text{ customers at server } 1\} = \left(\frac{\lambda}{\mu_1}\right)^n \left(1 - \frac{\lambda}{\mu_1}\right)$$

$$P\{m \text{ " " " } 2\} = \left(\frac{\lambda}{\mu_2}\right)^m \left(1 - \frac{\lambda}{\mu_2}\right)$$

if the no. of customers at servers 1 and 2 were ind random variables,

$$P_{n,m} = \left(\frac{\lambda}{\mu_1}\right)^n \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^m \left(1-\frac{\lambda}{\mu_2}\right)$$

To verify $P_{n,m}$, we put it to $\lambda P_{00} = \mu_2 P_{01}$

$$\lambda \cdot \left(\frac{\lambda}{\mu_1}\right)^0 \left(1-\frac{\lambda}{\mu_1}\right)\left(1-\frac{\lambda}{\mu_2}\right) = \mu_2 \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)\left(1-\frac{\lambda}{\mu_2}\right)$$, satisfied.

$\therefore$ Avg. no. of cust. in the system,

$$\begin{aligned}
L &= \sum_{n,m} (n+m) P_{n,m} \\
&= \sum_{n,m} (n+m) \left(\frac{\lambda}{\mu_1}\right)^n \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^m \left(1-\frac{\lambda}{\mu_2}\right) \\
&= \sum_n n \left(\frac{\lambda}{\mu_1}\right)^n \left(1-\frac{\lambda}{\mu_1}\right) + \sum_m m \left(\frac{\lambda}{\mu_2}\right)^m \left(1-\frac{\lambda}{\mu_2}\right) \\
&= \frac{\lambda/\mu_1}{\left(1-\frac{\lambda}{\mu_1}\right)^2}\left(1-\frac{\lambda}{\mu_1}\right) + \frac{\lambda/\mu_2}{\left(1-\frac{\lambda}{\mu_2}\right)^2}\left(1-\frac{\lambda}{\mu_2}\right) \\
&= \frac{\lambda}{\mu_1-\lambda} + \frac{\lambda}{\mu_2-\lambda}
\end{aligned}$$

Avg. time a cust. spends in the system,

$$W = \frac{L}{\lambda} = \frac{1}{\mu_1-\lambda} + \frac{1}{\mu_2-\lambda}$$

k-servers:

- cust arrive from outside to server i at rate r_i,
- After being served at S_i, cust has a prob. P_{ij} of joining Q of S_j, $j=1,\dots,k$

$$\therefore \sum_{j=1}^{k} P_{ij} \le 1 \text{, since he may go out after service.}$$

$$\therefore 1-\sum_{j=1}^{k} P_{ij} \longrightarrow \text{prob. that cust departs after being served by server } i.$$

Let,

λ_j = total arrival rate of customer to server j.

$$\lambda_j = r_j + \sum_{j=1}^{k} \lambda_i P_{ij} \quad , i=1,\dots,k$$

$$\therefore P\{n \text{ cust. at } j\} = \left(\frac{\lambda_j}{\mu_j}\right)^n \left(1-\frac{\lambda_j}{\mu_j}\right) \quad , n\geq 1$$

where, μ_j = exponential service rate at S_j. $\frac{\lambda_j}{\mu_j}<1$

$$\therefore P(n_1, n_2, \dots, n_k) = P\{n_j \text{ at server } j, j=1,\dots k\}$$
$$= \prod_{j=1}^{k} \left(\frac{\lambda_j}{\mu_j}\right)^{n_j} \left(1-\frac{\lambda_j}{\mu_j}\right)$$

$\therefore$ Avg. no. of cust in the system,

$$L = \sum_{j=1}^{k} \text{avg. \# at server } j = \sum_{j=1}^{k} \frac{\lambda_j}{\mu_j-\lambda_j}$$

Avg time a cust. spends in system, $W = \dfrac{L}{\lambda_a} = \dfrac{\sum_{j=1}^{k} \frac{\lambda_j}{\mu_j-\lambda_j}}{\sum_{j=1}^{k} r_j}$