Class Test 3 **Date: January 23, 2023**

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY

Department of Computer Science and Engineering

L-3/T-II CSE 301: Mathematical Analysis for Computer Science

1. Evaluate the following sum: (10)

$$\sum_{k=1}^{n} \frac{2k+3}{(2k-1)(2k+1)}$$

You are allowed to include harmonic numbers in the closed form.

Let us look at an incorrect solution to this problem:

$$\sum_{k=1}^{n} \frac{2k+3}{(2k-1)(2k+1)} = \sum_{1\le k\le n} (\frac{2}{2k-1} - \frac{1}{2k+1}) = 2\sum_{1\le k\le n} \frac{1}{2k-1} - \sum_{1\le k\le n} \frac{1}{2k+1}$$

$$= 2\sum_{1\le \frac{j+1}{2}\le n} \frac{1}{2\cdot\frac{j+1}{2}-1} - \sum_{1\le \frac{j-1}{2}\le n} \frac{1}{2\cdot\frac{j-1}{2}+1}$$

$$= 2\sum_{1\le j\le 2n-1} \frac{1}{j} - \sum_{3\le j\le 2n+1} \frac{1}{j}$$

$$= 1+\frac{1}{2} + 2\sum_{1\le j\le 2n-1} \frac{1}{j} - \sum_{1\le j\le 2n+1} \frac{1}{j}$$

$$= \tfrac{3}{2} + 2H_{2n-1} - H_{2n+1}$$

This is a solution written by many of you, which, sadly, is not correct. In the sum manipulation above, we are basically asserting $\sum_{1\le k\le n} \frac{1}{2k-1} = \sum_{1\le j\le 2n-1} \frac{1}{j}$

But if we expand these sums, we have

$$\sum_{1\le k\le n} \frac{1}{2k-1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

and $\sum_{1\le j\le 2n-1} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1}$.

These are not, in fact, equal. Notice that the second sum has additional terms like $\frac{1}{2}, \cdots$. So, where did we go wrong?

The mistake lies in the substitutions $k \leftarrow \frac{j+1}{2}, k \leftarrow \frac{j-1}{2}$. Recall that sum manipulation must satisfy the commutative or permutation law. These substitutions are in violation of this law.

Notice that in $\sum_{1\le k\le n} \frac{1}{2k-1}$, k increments by 1 from 1 to n. But if we make the substitution $k \leftarrow \frac{j+1}{2}$, each increment of 1 in k will be equivalent to an increment of 2 in j. This basically means that j has an added constraint of being an odd integer in addition to ranging from 1 to $2n-1$. Therefore, $\sum_{1\le k\le n} \frac{1}{2k-1} = \sum_{1\le j\le 2n-1} \frac{1}{j}$ will have to be $\sum_{1\le k\le n} \frac{1}{2k-1} = \sum_{1\le j\le 2n-1} \frac{1}{j}[j \text{ odd}]$ to be a correct assertion.