Chapter-5

5.2: Exponential Distribution

$X \longrightarrow$ A continuous Random Variable

$\lambda \longrightarrow$ parameter

(Exponential distribution)

PDF (Probability density function),

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

CDF (Cumulative Distribution Function),

$$F(x) = \int_{-\infty}^{x} f(y)\, dy = \begin{cases} 1 - e^{-\lambda x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

Proof:

$$F(x) = \int_{-\infty}^{x} f(y)\, dy$$

$$= \int_{-\infty}^{x} \lambda e^{-\lambda \cancel{x}}\, dy$$

$$= \lambda \left[\frac{e^{-\lambda y}}{-\lambda} \right]_{\infty}^{x}$$

$$= 1 - \lambda e^{-\lambda x}$$

$$\boxed{\begin{aligned} P\{X \le x\} &= 1 - e^{-\lambda x} = F(x) \\ P\{X > x\} &= 1 - F(x) = e^{-\lambda x} \end{aligned}}$$

$$\rightarrow \quad E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} \lambda x e^{-\lambda x}\, dx$$

$$= \left[\lambda x \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{(-\lambda)^2} \lambda \right]_{0}^{\infty} = \frac{1}{\lambda}$$

$$\therefore \boxed{E[X] = \frac{1}{\lambda}}$$ // mean of Exponential distb^n.

$$\rightarrow \quad \varphi(t) = E\left[e^{tx}\right] = \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_{0}^{\infty} = \frac{\lambda}{\lambda - t} \quad \therefore t < \lambda$$

$$\boxed{t < \lambda}$$

$$\therefore \boxed{\varphi(t) = \frac{\lambda}{\lambda - t}}$$

$\boxed{\varphi(t) \rightarrow \text{Moment Generating Function}}$

$$E[X^2] = \frac{d^2}{dt^2}[\varphi(t)]\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\right)\Big|_{t=0}$$

$$= \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0}$$

$$= \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

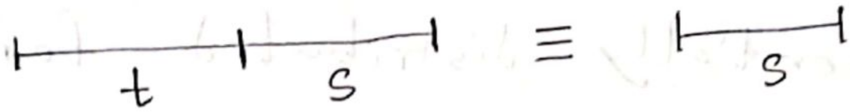
$\therefore$

$E[X] = \frac{1}{\lambda}$	$\varphi(t) = \frac{\lambda}{\lambda - t}$
$E[X^2] = \frac{2}{\lambda^2}$	$P\{X \le x\} = 1 - e^{-\lambda x}$
$Var(X) = \frac{1}{\lambda^2}$	$P\{X > x\} = e^{-\lambda x}$

Properties of Exponential Distribution

A random variable X is said to be without memory or memoryless if

$$P\{X>s+t \mid X>t\} = P\{X>s\}, \quad \boxed{s,t \geq 0}$$



→ Doesn't deteriorate with time.

→ An item that has been used for 't' hours is as good as a new item in regards to the amount of time remaining until the item fails.

→ The item doesn't remember that it has already been in use for time t.

So,

$$P\{X > s+t \mid X > t\} = P\{X > s\}$$

$$\Rightarrow \frac{P\{X>s+t,\ X>t\}}{P\{X>t\}} = P\{X>s\}$$

$$\therefore \boxed{P\{X > s+t\} = P\{X > s\} P\{X > t\}}$$

// $[X > s+t]$ and $[X > t]$

// means $[X > s+t]$

Since,

This eq^n is satisfied if X is exponentially distributed for $e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t}$, it follows that exponentially distributed RVs are memoryless.

$$\boxed{\begin{aligned} P\{X > s+t\} &= P\{X > s\} P\{X > t\} \\ e^{-\lambda(s+t)} &= e^{-\lambda s} \cdot e^{-\lambda t} \end{aligned}}$$

Example: 5.2

X : Amount of time a customer spends in a bank.

λ : Arrival rate, $\lambda = \frac{1}{10}$ customer/sec

(i) $P\{X>15\} = e^{-\lambda x}$

$= e^{-\lambda \times 15} = e^{-\frac{1}{10} \times 15}$

$= e^{-3/2}$

(ii) $P\{X>15 \mid X>10\}$

$= P\{X > \underset{s + t}{10+5} \mid X > \underset{t}{10}\}$

$= P\{X>5\}$

$= e^{-\frac{1}{10} \times 5}$

$= e^{-1/2}$

15 second এর বেশি লাগার probability given 10 second already passed is equal to 5 second এর বেশি লাগার probability for a new customer (Memoryless)

Exponential Distribution is the only memoryless distribution

Proof:

Let X is memoryless and $\bar{F}(x) = P\{X > x\}$

$$\bar{F}(s+t) = P\{X > s+t\} = P\{X > s\} P\{X > t\}$$

$$\bar{F}(s+t) = \bar{F}(s)\,\bar{F}(t)$$

$\bar{F}(x)$ satisfies the functional eq^n,

$$g(s+t) = g(s)\,g(t)$$

The only solution of this equation is

$$g(x) = e^{-\lambda x}$$

$$\therefore g(s+t) = e^{-\lambda(s+t)} = e^{-\lambda s} \cdot e^{-\lambda t} = g(s)\,g(t)$$

Since, a distribution function is always right continuous, we must have,

$$\bar{F}(x) = e^{-\lambda x}$$

$$\therefore F(x) = P\{X \le x\} = 1 - e^{-\lambda x}$$

So, X is exponentially distributed

[Proved]

Example:

Lifetime of a bulb is exponentially distributed with mean 10 hours.

$P\{\text{lifetime} > t+5 \mid \text{lifetime} > t\}$

$$= \frac{1 - F(t+5)}{1 - F(t)}$$ // if lifetime was not exponential.

But now,

$P\{\text{lifetime} > t+5 \mid \text{lifetime} > t\}$

$= P\{\text{lifetime} > 5\}$

$= 1 - F(5)$

$= 1 - (1 - e^{-\lambda \times 5})$

$= 1 - (1 - e^{-(\frac{1}{10} \times 5)})$ | $\lambda = \frac{1}{10}$, $\lambda = \frac{1}{\text{mean}}$

$= e^{\frac{1}{2}}$

Note: $P\{X \leq s\} = F(s)$ and $P\{X > s\} = \bar{F}(s) = 1 - F(s)$

Further Properties of Exp. Distr.

1 টি Bulb average এ 5 hours জ্বলে,
আরেকটি " " " 10 " ".
প্রথম Bulb টি | দ্বিতীয় Bulb এর চেয়ে
কম জ্বলার সম্ভাবনা কত?

X_1 and X_2 are independent Random Variables with means $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$

$$P\{X_1 < X_2\} = \int_0^\infty P\{X_1 < X_2, X_1 = x\}\, dx$$

$$= \int_0^\infty P\{X_1 < X_2 \mid X_1 = x\}\, \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_0^\infty P\{x < X_2\}\, \lambda_1 e^{-\lambda_1 x}\, dx$$

$$= \int_0^\infty e^{-\lambda_2 x}\, \lambda_1 e^{-\lambda_1 x}\, dx$$

$$P(X > x) = 1 - F(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

$$= \lambda_1 \int_0^\infty e^{-(\lambda_1 + \lambda_2)x}\, dx$$

$$= \lambda_1 \left[\frac{e^{-(\lambda_1+\lambda_2)x}}{-(\lambda_1+\lambda_2)} \right]_0^{\infty}$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2} \qquad \boxed{P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1+\lambda_2}}$$

So, $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{1}{10}$

$$\therefore P\{X_1 < X_2\} = \frac{\frac{1}{5}}{\frac{1}{5}+\frac{1}{10}} = \frac{2}{3}$$

Example

$\lambda_1 = \frac{1}{1000}$ $\quad \lambda_2 = \frac{1}{500}$

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$= \frac{\frac{1}{1000}}{\frac{1}{500}+\frac{1}{1000}}$$

$$= \frac{1}{3}$$

5.3 Counting Process

A stochastic process is said to be
$\hookrightarrow \{N(t);\ t \geq 0\}$
a counting process if $N(t)$ represents the total number of events that have occured up to time t.

Examples:

\# persons entered store

\# of birth \# of goals.

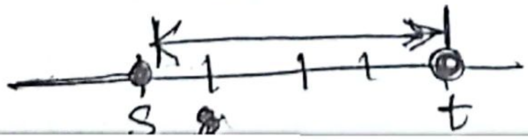
Properties

(i) $N(t) \geq 0$

(ii) $N(t)$ is integer

(iii) if $s < t$, $N(s) \leq N(t)$

(iv) for $s<t$, $N(t) - N(s) =$ \# of events during interval $(s, t]$



① Independent Increment:

A counting process is said to possess independent increments if the number of events occur in disjoint time intervals are independent.

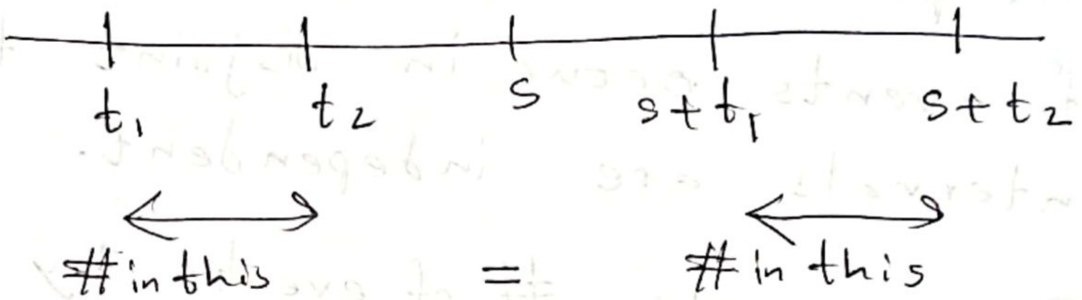
For example, # of events by time 10 ($N(10)$) must be independent of # of events during $[10, 15]$; $(N(15) - N(10))$.

② Stationary Increment

কোনো interval এ যত জন আসবে তা interval এর length এর উপর নির্ভর করে।

A counting process is said to possess stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the interval.

i.e. $N(t_2+s) - N(t_1+s)$
$$= N(t_2) - N(t_1)$$



5.3

Poisson Process:

A counting process $\{N(t)\,;\, t \geq 0\}$ is said to be a poission process having rate λ, $(\lambda > 0)$ if

(i) $N(0) = 0$

(ii) The process has independent increments

(iii) The process has stationary increments.

i.e. The # of events in an interval of length t is poisson distributed with mean (λt), $\lambda > 0$

$$P\{\underbrace{N(t+s) - N(s)}_{=n}\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

$$\therefore E[N(t)] = \lambda t \qquad \forall s, t \geq 0;\ n = 0, 1, 2, \ldots$$

(s) থেকে $(t+s)$ সময়ে n টা count হওয়ার probability আর 0 থেকে t সময়ে n টা count হওয়ার probability same যেহেতু interval সমান $(t+s-s = t)$

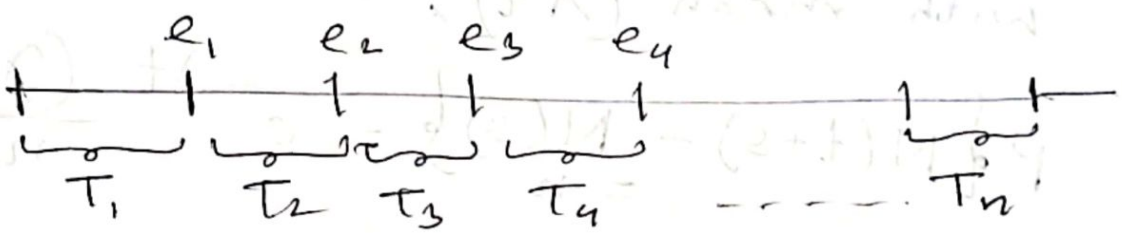
<u>Note:</u>

# of events in length t is poisson distributed but the time is exponentially distributed. (Memoryless)

λ is rate of the process

Distribution of Inter-Arrival time

Inter-arrival time : 2 টি event এর মধ্যবর্তী সময়।



$\{T_n, n = 1, 2, 3, \ldots\}$ interarrival time.

$$P\{T_1 > t\} = P\{\text{No events in } (0,t]\}$$

$$= P\{N(t) = 0\} = \frac{e^{-\lambda t}(\lambda t)^0}{0!}$$

$$= e^{-\lambda t} \quad [n=0]$$

Hence, T_1 has exponential distribution with parameter λ, mean $\frac{1}{\lambda}$.

Note

$P\{T_1 > t\}$ means first event ঘটতে t এর বেশি time লাগবে, so no event in $(0,t]$"

$P\{T_2 > t \mid T_1 = s\}$

$= P\{\text{No events in } (s, s+t] \mid T_1 = s\}$

$= P\{\text{No events in } (s, s+t]\}$

[independent increment, $T_1 = s$ তাহ]

$= P\{0 \text{ events in } (0, t]\}$

[stationary increment]

$= e^{-\lambda t}$

$\therefore T_2$ is exponentially distributed with mean $\frac{1}{\lambda}$, and T_2 is independent of T_1.

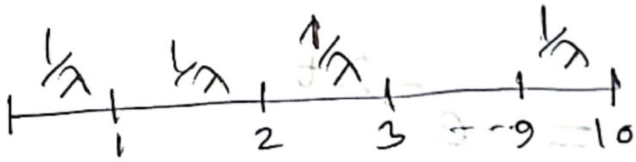
So, T_n is also exp. distr. with parameter λ, mean $\frac{1}{\lambda}$.

$S_n \rightarrow$ the time of nth event.

$$S_n = \sum_{i=1}^{n} T_i \quad , n \geq 1$$

Example

People migrating into a territory at a poisson rate $\lambda = 1$ per day.



(i) $E[S_{10}]$

$$= \frac{10}{\lambda} = 10 \text{ days}$$

(ii) $P\{T_{11} > 2\} = e^{-2\lambda}$

$$= e^{-2}$$

$$\simeq 0.133.$$