Class Test 3 **Date: January 23, 2023**

BANGLADESH UNIVERSITY OF ENGINEERING AND TECHNOLOGY

Department of Computer Science and Engineering

L-3/T-II CSE 301: Mathematical Analysis for Computer Science

Time: 30 minutes **Marks: 20**

Student Name: ______________________________ Student No: ____________

1. You have already learned how to formulate the recurrence relation for the shortest number of moves for 'Tower of Hanoi' problem. Now you will have to derive the recurrence relation with an added constraint: direct moves between the leftmost peg (i.e., the source) and the rightmost peg (i.e., the destination peg) are disallowed. You must explain how you have formulated the recursion. Don't forget to write the base case! (10)

   Solution: Let the minimum number of moves needed for moving n disks satisfying the added constraint be denoted by T_n.

   Sufficiency:

   1. We first move the top $(n-1)$ disks from the leftmost peg to the rightmost peg satisfying the added constraint. This will require T_{n-1} moves, by the definition of T_n.
   2. We then move the last disk from the leftmost peg to the middle peg. This will take 1 move.
   3. We again move the $(n-1)$ disks situated at the rightmost peg to the leftmost peg. By symmetry, this will take T_{n-1} moves.
   4. We then move the last disk on the middle peg to the destination peg. This will again take 1 move.
   5. We finally move the $(n-1)$ disks on the leftmost peg to the rightmost peg with T_{n-1} moves.

   Hence, $T_n \leq 3T_{n-1} + 2$. This establishes an upper bound on T_n.

   Necessity:

   1. The disk at the bottom on the leftmost peg will have to be moved to the rightmost peg, which will necessitate moving it to the middle peg first, given the added constraint (1 move).
   2. Moving the bottom disk from the leftmost peg to the middle peg will, in turn, require all the $(n-1)$ disks above to be moved to the rightmost peg as a prerequisite, which will incur T_{n-1} moves.
   3. Again, moving the largest peg on the middle peg to the rightmost peg will require the $(n-1)$ disks on the rightmost peg to be moved to the leftmost peg, which will again cost T_{n-1} moves.
   4. Finally, moving the largest disk to the rightmost peg will require 1 more move, and the $(n-1)$ disks currently on the leftmost peg to the rightmost T_{n-1} moves.

   Hence, $T_n \geq 3T_{n-1} + 2$ and therefore equality will hold on the lower bound:

$$T_n = \begin{cases} 0, \text{ if } n = 0 \text{ (Since there is no disk to move)} \\ 3T_{n-1} + 2, \text{ otherwise} \end{cases}$$

2. You are given that following pair of recurrence relations:

$$A_n = \begin{cases} 2, \text{ if } n = 1 \\ 2A_{n-1} + 2, \text{ otherwise} \end{cases}$$

$$B_n = \begin{cases} 3, \text{ if } n = 1 \\ 2A_{n-1} + B_{n-1} + 4, \text{ otherwise} \end{cases}$$

Here n is a natural number. Prove that $A_n = 2^{n+1} - 2$ and $B_n = 2^{n+2} - 5$. (10)

Solution: We prove using induction on n.

Base case: For $n = 1$, $A_1 = 2^{1+1} - 2 = 2$ and $B_1 = 2^{1+2} - 5 = 3$. Hence, the base case holds.

Induction hypothesis: Let $A_m = 2^{m+1} - 2$ and $B_m = 2^{m+2} - 5, \forall 1 \leq m \leq n - 1$.

Induction step: Now we prove the assertions true for $m = n$.

$$A_n = 2A_{n-1} + 2 = 2(2^{(n-1)+1} - 2) + 2 = 2^{n+1} - 2 \tag{1}$$

$$B_n = 2A_{n-1} + B_{n-1} + 4 = 2(2^{(n-1)+1} - 2) + 2^{(n-1)+2} - 5 + 4 = 2^{n+2} - 5 \tag{2}$$

[Proved]