Chapter 4

Markov Chains

4.1. Intro

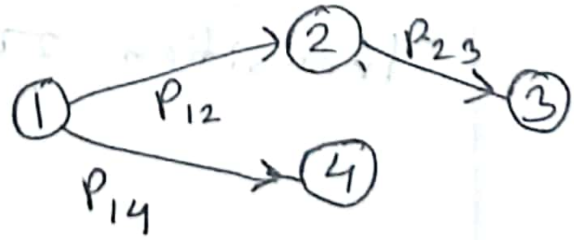
Chain → Previous state এর উপর depend করে সামনের state predict করা।

Stochastic Process :

- A collection of Random Variables.
- time এর উপর define করা হয়।
- $\{X(t), t \in T\}$ → A state of the process at time t.

Markov Chain

Markov chain is an stochastic process $\{X_n, n = 1, 2, 3, \dots\}$ such that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j.



$$P_{ij} = P\{X_{n+1} = j \mid X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0\}$$

The $\underbrace{P_{ij}}_{\text{future}}$ is independent of ~~past states~~ and depends only on the present state.

(i) $P_{ij} \geqslant 0 \quad , \quad i, j \geqslant 0$

(ii) $\sum_j P_{ij} = 1 \quad , \quad i = 0, 1, \dots$

at each step, there must be a transition

Transition Matrix :

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & \cdots P & \cdots \end{bmatrix}$$

Transition 20 # row $\longrightarrow$ # column

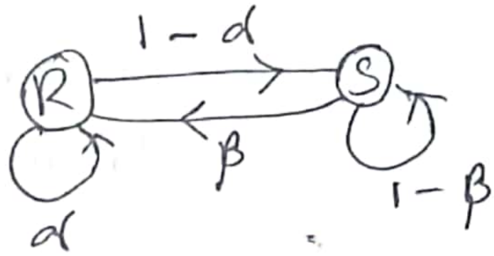
Example: Weather (4-1)

Weather : Rainy (R) , Sunny (S), [States]

$$P = \begin{array}{c|cc} & R & S \\ \hline R & \alpha & \cancel{\beta}\ 1-\alpha \\ S & \beta & 1-\beta \end{array}$$

row → today

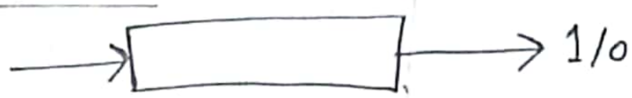
col → tomorrow



α → today rainy → tomorrow rainy

β → " sunny → " rainy

Example: Communication System



$$P = \begin{array}{c|cc} \text{(col) output} & 0 & 1 \\ \hline 0 & p & 1-p \\ 1 & 1-p & p \end{array}$$

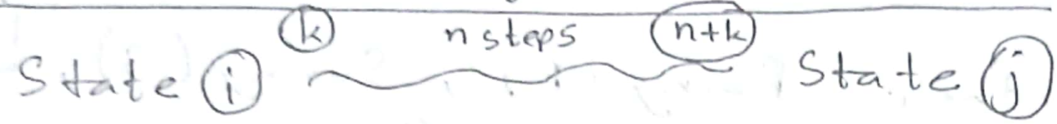
input (row)

Given

$p(1 \to 1) = p$

$p(0 \to 0) = p$

4.2 Chapman-Kolmogorov Equation



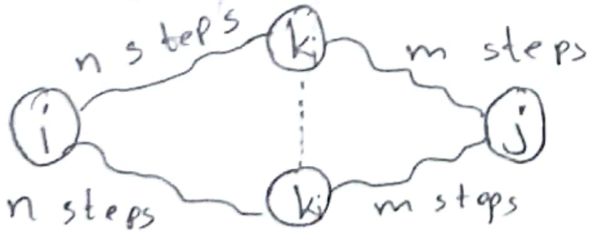
P_{ij} = One step transition probability

P_{ij}^{n} = n step transition probability that a process in state i, will reach state j, after n transitions.

$$\boxed{P_{ij}^{n} = P\{X_{n+k} = j \mid X_k = i\}} \quad , \boxed{n \geq 0 ; i, j \geq 0}$$

$$\boxed{P_{ij}^{(n+m)} = \sum_{k} P_{ik}^{n} P_{kj}^{m}} \quad \text{for } \forall\, n, m \geq 0 \;\; \forall\, i, j$$

$P_{ij}^{(n+m)} \longrightarrow$ i থেকে j তে (n+m) steps এ যাওয়ার probability



$$\cancel{P(a|b) = \sum_c P(a|b,c) = \sum_c P(a}$$

$$P(a|b) = \sum_c P(a,c|b) = \sum_c P(a|b,c)\,P(c|b)$$

Proof:

$$P_{ij}^{(n+m)} = P\{X_{n+m} = j \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j \mid X_n = k, X_0 = i\}\, P\{X_n = k \mid X_0 = i\}$$

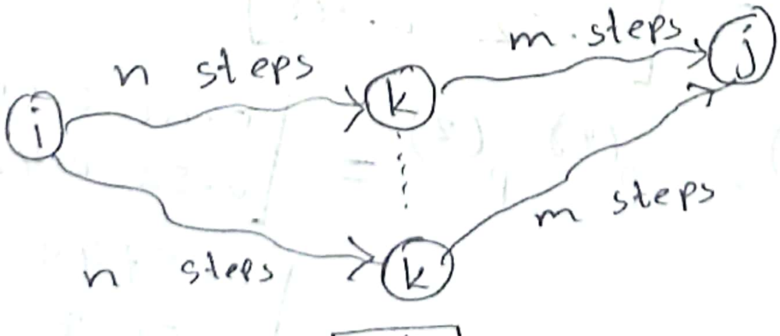
$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j \mid X_n = k\}\, P\{X_n = k \mid X_0 = i\}$$

$$\therefore \boxed{P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{kj}^{m}\, P_{ik}^{n}} \quad \text{(proved)}$$

The n step transition probability may be obtained by multiplying the matrix by itself n times.

// Basically,
// $P^{m+n} = P^m P^n$
// $n = 1,\ m = 1,$
// P^2 হবে

$$\underset{\text{matrix}}{P^{(n+m)}} = P^{(n)} P^{(m)}$$



$\boxed{\forall k}$

Example: 4.8

Consider example of the weather (4.1)

$\alpha = 0.7$, $\beta = 0.4$.

Calculate the probability that it will rain 4 days from today, given it is raining today.

Solve

$$P = \begin{array}{c} \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix} \end{array} = \begin{array}{c} \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$$P^{(2)} = P \cdot P$$

$$= \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$$P^{(4)} = P^{(2)} P^{(2)} = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix} \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{bmatrix}$$

$\therefore \; P_{00}^{(4)} = 0.5749$ (Ans)

R R

Today Rain $\rightarrow$ Sunny, 4 days Later, $P_{01}^{(4)} = 0.4251$

" Sunny $\rightarrow$ Rain 4 " " $P_{1}^{(4)} = 0.5668$

$P_{00}^{(2)} = 0.61$ means, Today Rainy, 2 দিন পরেও Rainy

$P_{11}^{(2)} = 0.48$ " , " Sunny, 2 " " Sunny

4.3 → Classification of States

Markov chain is irreducible, if there is only one class, that is, if all states communicate each other.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Here, all state communicates with each other, only 1 class.

It is possible to go to 2 from 0.

$0 \longrightarrow 1 \longrightarrow 2$

#

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

Here,

$\{0, 1\} \rightarrow$ Recurrent State

state 2 and 3 are not accessible

[0,1 এ একবার গেলে আর বের হতে পারে না, 0,1 এই ঘুরবে]

$\{2\} \rightarrow$ Transient State

all states are accessible from it

[এটার state থেকে long term এ বের হবেই]

$\{3\} \rightarrow$ Absorbing State

(no other state is accessible from it)

[long term এ যেখানে গিয়ে terminate করবে]

So, 3 Classes $\rightarrow \{0, 1\}, \{2\}, \{3\}$

Example [4-4]

$$\begin{array}{c|c} {}_{MT}\diagdown^{TW} & \begin{array}{cccc} RR & RS & SR & SS \end{array} \\ \begin{array}{c} RR \\ RS \\ SR \\ SS \end{array} & \begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

MT $\rightarrow$ Monday and Tuesday

TW $\rightarrow$ Tuesday and Wednesday

MT RR means, Monday an Tuesday

Rain Given.

Given RR in Monday, What is the probability of Raining on Thursday?

Solve

$$P^{(2)} = \begin{array}{c|c} {}_{MT}\diagdown^{WTh} & \begin{array}{cccc} RR & RS & SR & SS \end{array} \\ \begin{array}{c} RR \\ RS \\ SR \\ SS \end{array} & \begin{bmatrix} 0.49 & 0.21 & 0.12 & 0.18 \\ 0.20 & 0.20 & 0.12 & 0.48 \\ 0.35 & 0.15 & 0.20 & 0.30 \\ 0.1 & 0.1 & 0.16 & 0.64 \end{bmatrix} \end{array}$$

$$P_{00}^{(2)} + P_{02}^{(2)} = 0.49 + 0.12 = 0.61$$ (Ans)

4.4. Limiting Probability

Theorem :

If $\lim_{n\to\infty} P_{ij}^{n}$ exists and is independent of i and $\pi_j = \lim_{n\to\infty} P_{ij}^{n}$, $j \geqslant 0$

then, π_j is the unique non-negative solution of :

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad , j \geqslant 0$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$

π_j is limiting probability of a column

Example: 4.22

$$P = \begin{array}{c} \\ R \\ S \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix} \\ \begin{array}{cc} \pi_0 & \pi_1 \end{array} \end{array}$$

State 0 if it rains tomorrow.

" 1 " " doesn't rain "

Long run proportions π_0 and π_1 will be,

$$\pi_0 = \alpha \pi_0 + \beta \pi_1$$

$$\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$$

$$\pi_0 + \pi_1 = 1$$

So, $\pi_0 \cancel{+} = \alpha \pi_0 + \beta(1-\pi_0)$

$$\Rightarrow \pi_0 - \alpha \pi_0 + \beta \pi_0 = \beta$$

$$\therefore \boxed{\pi_0 = \frac{\beta}{1-\alpha+\beta}}$$ → Probability of Rainy in the long run

$$\boxed{\pi_1 = \frac{1-\alpha}{1-\alpha+\beta}}$$ → " " Sunny " " "

2 properties of Markov Chain

(i) Period $\longrightarrow$ State (i) is said to have period of d if $P_{ii}^{n} = 0$, whenever n is not divisible by d and d is the largest integer with this property

(ii) A state with period 1 is said to be aperiodic.

Periodic chain doesn't have a limiting probability.

Example : 4.25

Markov Chain in Genetics

Hardy-Weinberg Law:

→ 2 types of genes $\{A, a\}$

→ Each individual possesses a particular pair of genes → AA, aa or Aa.

→ $P\{AA\} = P_0$, $P\{Aa\} = r_0$, $P\{aa\} = q_0$

→ During mating, each individuals contribute one type of his/her genes at random.

→ Mating occurs at random.

Let, next generation probability,

$$P\{AA\} = p, \quad P\{Aa\} = r, \quad P\{aa\} = q$$

Now,
A randomly choosen gene will be type A with probability,

$$P\{A\} = P\{A|AA\}P_0 + P\{A|aa\}q_0 + P\{A|Aa\}r_0$$

// $P\{A|Aa\} = \frac{1}{2}$

$$\therefore P\{A\} = P_0 + \frac{r_0}{2}$$

// $P\{A|AA\} = 1$

// $P\{A|aa\} = 0$

Similarly,

$$P\{a\} = q_0 + \frac{r_0}{2}$$

Now, $p = P\{AA\} = \left(p_0 + \frac{r_0}{2}\right)^2$

$$q = P\{aa\} = \left(q_0 + \frac{r_0}{2}\right)^2$$

$$r = P\{Aa\} = {}^2C_1 \left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$$

$$p+q+r = \left(p_0 + \frac{r_0}{2}\right)^2 + 2\left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right) + \left(q_0 + \frac{r_0}{2}\right)^2$$

$$= \left(p_0 + \frac{r_0}{2} + q_0 + \frac{r_0}{2}\right)^2$$

$$= \left(p_0 + q_0 + r_0\right)^2 = 1^2$$

$\therefore p+q+r = 1$

In new generation, $P\{A\} = p + \frac{r}{2}$

$$p + \frac{r}{2} = \left(p_0 + \frac{r_0}{2}\right)^2 + \left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$$

$$= \left(p_0 + \frac{r_0}{2}\right)\left(p_0 + \frac{r_0}{2} + q_0 + \frac{r_0}{2}\right)$$

$$= \left(p_0 + \frac{r_0}{2}\right)\left(p_0 + q_0 + r_0\right)$$

$$= p_0 + \frac{r_0}{2} \quad \text{(Unchanged!)}$$

Under random mating, in all successive generations after the initial one, the percentages of the population having gene pairs AA, aa, Aa will remain same at the values p, q and r.

[Hardy-Weinberg Law]

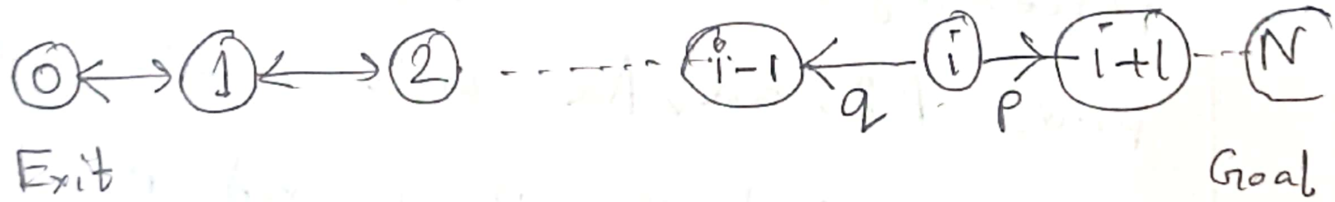
4.5 Some Applications

The Gambler's Ruin Problem

$$\left.\begin{aligned} P(\text{winning one unit}) &= p \\ P(\text{losing one unit}) &= q \end{aligned}\right\} \text{At each game play}$$

$\boxed{p+q=1}$

What is the probability that, starting with i units, the gambler's fortune will reach N before reaching 0?



→ Random model, ফলনের যায় আসতে ফিরতে যায়।

→ 0 হয়ে গেলে আর খেলতে পারবে না।

→ i নিয়ে শুরু করবে।

→ N এ যাওয়ার probability বের করতে হবে

Let, P_i = Probability that starting from i, the Gambler will eventually reach N. $\boxed{\text{Base Case: } P_0 = 0,\ P_N = 1}$

$$P_i = pP_{i+1} + qP_{i-1}$$

$$\Rightarrow (p+q)P_i = pP_{i+1} + qP_{i-1} \quad // \; p+q=1$$

$$\Rightarrow P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1})$$

So, $$P_2 - P_1 = \frac{q}{p}(P_1 - P_0)$$

$$\therefore P_2 - P_1 = \frac{q}{p}(P_1)$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$

$$P_4 - P_3 = \left(\frac{q}{p}\right)^3 P_1$$

$$\vdots$$

$$P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$P_i - P_1 = P_1\left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right]$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} P_1 & , \text{if } \frac{q}{p} \neq 1 \\ & \text{or } q \neq p \text{ or } q \neq \frac{1}{2} \\ i P_1 & ; \text{if } \frac{q}{p} = 1 \end{cases}$$

Now,

$P_N = 1$

$$P_N = \begin{cases} P_1 \dfrac{1-\left(\frac{q}{p}\right)^N}{1-\frac{q}{p}} & , p \neq \frac{1}{2} \\ N P_1 & p = \frac{1}{2} \end{cases}$$

So, $P_1 \left(\dfrac{1-\left(\frac{q}{p}\right)^N}{1-\frac{q}{p}} \right) = 1 \quad // p \neq \frac{1}{2}, P_N = 1$

$$\therefore P_1 = \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} \quad ; p \neq \frac{1}{2}$$

And, $N P_1 = 1$

$$\therefore P_1 = \frac{1}{N} \quad ; p = \frac{1}{2}$$

$$P_1 = \begin{cases} \dfrac{1 - q/p}{1 - \left(q/p\right)^N} & ; p \neq \frac{1}{2} \\ \dfrac{1}{N} & ; p = \frac{1}{2} \end{cases}$$

And, So,

$$P_i = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \frac{q}{p}} \times \dfrac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N} & ; p \neq \frac{1}{2} \\ i \times \dfrac{1}{N} & ; p = \frac{1}{2} \end{cases}$$

$$= \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N} & ; p \neq \frac{1}{2} \\ i/N & ; p = \frac{1}{2} \end{cases}$$

As, $N \to \infty$

$$P_i = \begin{cases} 1 - \left(\frac{q}{p}\right)^i & ; p > \frac{1}{2} \\ 0 & ; p \leq \frac{1}{2} \end{cases}$$

So, if $p \leq \frac{1}{2}$, gambler will lose everything in the long run.