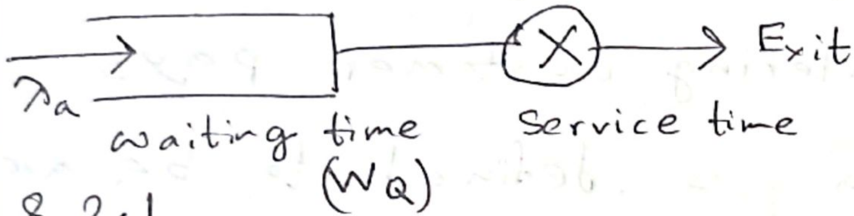
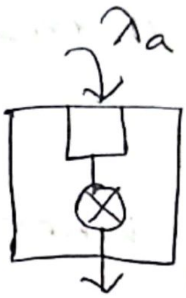
Chapter 8

Queueing Theory

8.1: Intro





8.2.1

Cost Equations:

Let,

L = Average number of customers in the system.

L_Q = Average number of customers waiting in the queue.

W = Average amount of time a customer spends in the ~~queue.~~ system

W_Q = Average amount of time a customer spends in the queue.

Basic cost identity,

average rate at which the system earns $= \lambda_a \times$ average amount an entering customer pays

where, λ_a is defined to be average arrival rate of entering customers.

$$\lambda_a = \lim_{t \to \infty} \frac{N(t)}{t}$$

where, $N(t)$ = total number of customer arrival by time t.

Little's formula

If each customer pays \$1 per unit time while in the system.

$$L \cdot 1 = \lambda_a \cdot W$$

$$\Rightarrow \boxed{L = \lambda_a W}$$

L = System এ কত জন আছে on avg

λ_a = arrival rate

W = System এ একজন কত সময় ধরে থাকে

→ Similarly, if each customer pays \$1 per unit time while in the queue.

$L_Q = \lambda_a W_Q$

[Same, শুধু system এর জায়গায় waiting queue]

→ Average number of customers in service $= \lambda_a E[S]$

where, $E[S]$ = average amount of time a customer spends in service.

8.2.2 Steady State Probabilities

Let,

$X(t) \rightarrow$ Number of customers in the system at time t.

$$P_n = \lim_{t \to \infty} P\{X(t) = n\}, \quad n \geq 0$$

$P_n \rightarrow$ Long run probabilities that there will exactly n customers in the system.

→ ~~Pf~~ P_n is sometimes referred to as the steady-state probability of exactly n customers in the system.

→ P_n → The proportion of time the system contains exactly n customers. [Dual Interpretation]

$P_0 = 0.3$ means, system will be empty for 30% of the time.

$P_1 = 0.2$ means, 20% of the time, there will exacty 1 customer in the system.

→ 2 other sets of limiting probabilities are $\{a_n, n \geq 0\}$ and $\{d_n, n \geq 0\}$.

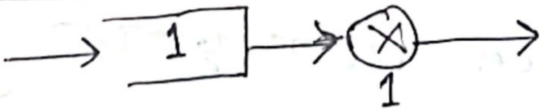
where, a_n = proportion of customers that find n in the system when they arrive.

also → a_n = proportion of the time an entering customer sees n person in the system. [কতজন system এ ঢুকে দেখে আগে থেকে n জন আছে, তার proportion]

d_n = proportion of customers leaving behind n customers in the system when they depart. [কত শতাংশ/জন যাওয়ার সময় n জনকে system এ রেখে বের হয়]

Proposition:

In any system in which customers arrive one at a time and served one at a time. $\boxed{a_n = d_n}$; $n \geq 0$



Proposition

Poisson arrivals always see time averages, for poisson arrivals,

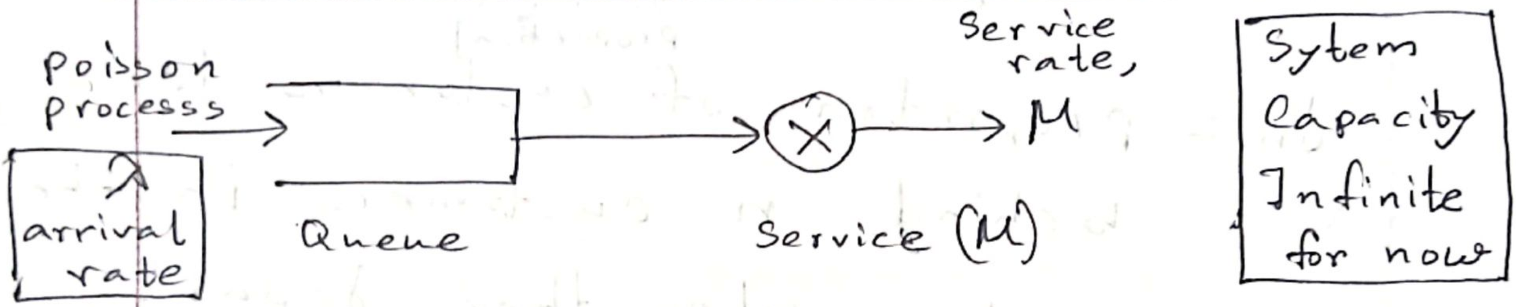
$\boxed{P_n = a_n}$ // read 8.2.2 from book for
// better understanding

~~কত~~ সময়ের কত অংশ জুড়ে system এ exactly n জন থাকে আর কত অংশ জুড়ে কেউ এসে আগে থেকেই n জনকে system এ থাকতে দেখে তা same

8.3 Exponential Models

8.3.1

Single Server Exponential Queueing System



→ Times between successive arrivals or, "Inter-arrival time's" are Independent Exponential Random Variables with mean $\frac{1}{\lambda}$

$\frac{1}{\lambda}$

→ Server free থাকলে customer সরাসরি service নিতে ঢুকবে আর server busy থাকলে queue তে দাঁড়াবে।

→ Service নেয়া শেষ হলে system leave করবে, queue তে কেউ waiting থাকলে সে ঢুকবে।

→ The successive service times are assumed to be Independent exponential Random Variables with mean $\frac{1}{\mu}$

$\frac{1}{\mu}$

M/M/1 Queue [Last page এ যে queue এর কথা বলা হয়েছে]

$\boxed{M/M/1}$

1 → Single Server

M → [Inter arrival time is memoryless] [Exp. Distr^n যেহেতু]

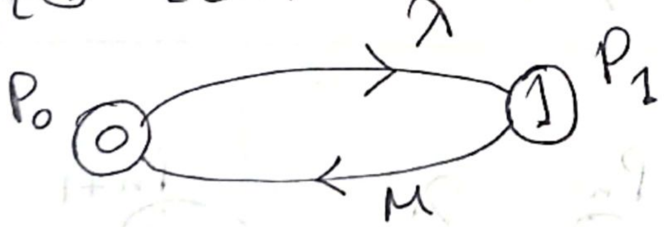
M → [Service Distributions are memoryless]

$\boxed{\text{Arrival Rate} = \lambda}$

$\boxed{\text{Service Rate} = \mu}$

For stable system, $\lambda \leq \mu$

অর্থাৎ, system stable হতে হলে arrival rate এ service rate এর চেয়ে বেশি দিতে হবে।



Rate Quality Principal

$$\boxed{\lambda P_0 = \mu P_1}$$

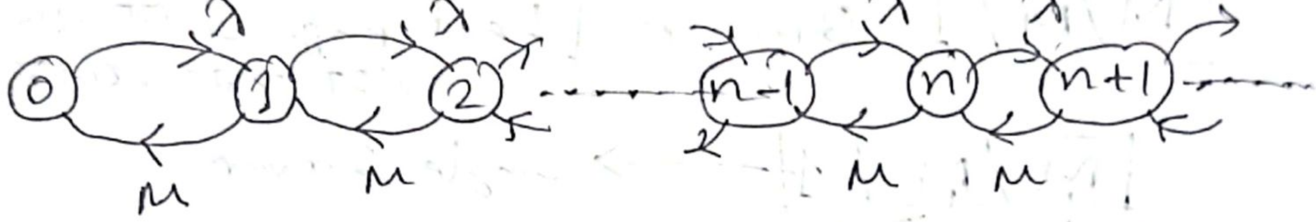
$\boxed{\text{State 0}}$ system এ কেউ না থাকার probability (P_0)

$\boxed{\text{State 1}}$ " " " 1 জন থাকার " (P_1)

যখনই কেউ আসবে state 0 to state 1 এ যাবে।

λP_0 = Rate at which the process leaves 0

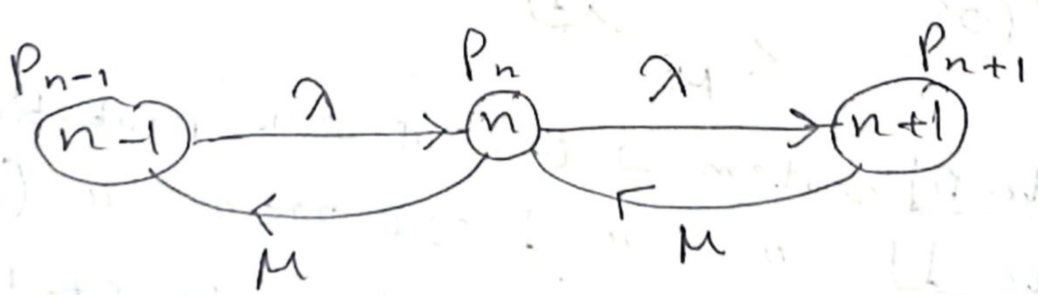
μP_1 = Rate " " " enter 0 from 1 [μ rate এ service দিয়ে ~~চলে~~ ওই 1 জন চলে গেলে, system empty]



যে rate এ কোনো state এ ঢুকবে সে rate এই ঐ state leave করবে।

Balance Equation:

State	Rate at which process leaves = Rate ath which it enters
0	$\lambda P_0 = \mu P_1$
$n, n \geq 1$	$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$



Now, $\lambda P_0 = \mu P_1 \Rightarrow P_1 = \frac{\lambda}{\mu} P_0$

and, $P_n (\lambda + \mu) = \lambda P_{n-1} + \mu P_{n+1}$

$$\Rightarrow \mu P_{n+1} = \lambda P_n + \mu P_n - \lambda P_{n-1}$$

$$\therefore \boxed{P_{n+1} = \frac{\lambda}{\mu} P_n + P_n - \frac{\lambda}{\mu} P_{n-1}}$$

So,

$$P_2 = \frac{\lambda}{\mu} P_1 + P_1 - \frac{\lambda}{\mu} P_0 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 + P_2 - \frac{\lambda}{\mu} P_1 = \frac{\lambda}{\mu} P_2 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\vdots$$

$$P_{n+1} = \left(\frac{\lambda}{\mu}\right)^{n+1} P_0$$

$$\therefore \boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n P_0}$$

Now, let's determine P_0.

We know, $\sum_{n=0}^{\infty} P_n = 1$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow \frac{1}{1-\left(\frac{\lambda}{\mu}\right)} P_0 = 1 \quad \left[\frac{\lambda}{\mu} < 1\right]$$

$\because \lambda < \mu$

$$\therefore \boxed{P_0 = 1 - \frac{\lambda}{\mu}}$$

$$\therefore \boxed{P_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n} \quad \boxed{\lambda < \mu}$$

Note:

If $\lambda \geq \mu$, $\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$ would be infinite and all the P_n would be 0.

By time t, (λt) enters, (μt) leaves.

So, $\lambda t - \mu t = (\lambda - \mu)t$ is the expected number of customers in the system.

As $t \to \infty$, $(\lambda - \mu)t \to \infty$ if $\lambda > \mu$

So, the queue size increases without limit.

So, we shall assume that $\lambda < \mu$

The general condition for limited probabilities to exist in most single server queueing system.

Average # of customers in the system, L

$$L = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n} \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n}$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{d}{dx} \sum_{n=0}^{\infty} x^{n} \quad // x = \frac{\lambda}{\mu}$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{d}{dx} \left(1 + x + x^2 + \cdots\right)$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{d}{dx} \left(\frac{1}{1-x}\right)$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{1}{(1-x)^2}$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{1}{\left(1-\frac{\lambda}{\mu}\right)^2} \quad // \frac{\lambda}{\mu} = x$$

$$\therefore \boxed{L = \frac{\lambda}{\mu - \lambda}}$$

As, $\lambda_a = \lambda$, from Little's formula,

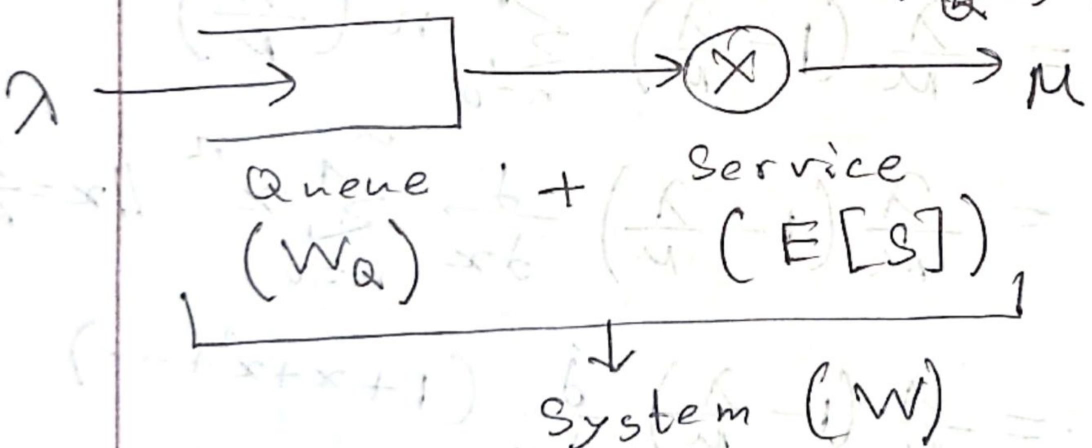
$$L = \lambda W$$

$$\Rightarrow W = \frac{L}{\lambda} = \frac{\frac{\lambda}{\mu - \lambda}}{\lambda} = \frac{1}{\mu - \lambda}$$

$$\therefore \boxed{W = \frac{1}{\mu - \lambda}}$$

$W \rightarrow$ avg time customers spend in the system.

$W_Q \rightarrow$ in the queue.



So,

$$W = W_Q + E[S]$$

$$\Rightarrow W_Q = W - E[S]$$

$$= \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$

$$\boxed{W_Q = \frac{\lambda}{\mu(\mu - \lambda)}}$$

$$L_Q = \lambda W_Q$$ // $L_Q \rightarrow$ Avg #cust. waiting in the queue.

$$= \lambda \frac{\lambda}{\mu(\mu-\lambda)}$$

$$\therefore \boxed{L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)}}$$

Note that, when $\lambda \longrightarrow \mu$, $\frac{\lambda}{\mu} \rightarrow 1$, a slight increase in $\frac{\lambda}{\mu}$ will lead to a large increase in L and W.

8.3.2 M/M/1 Queuing system with finite Capacity

[A Single Server Exponential Queueing System having finite capacity, N]

Previously, no limit on the # of customers in the system at a time. Now, no more than N customers can stay in the system at any time.

[Reality]

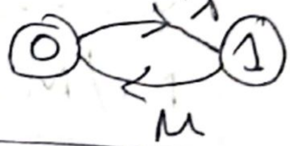
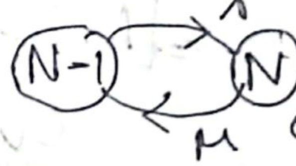
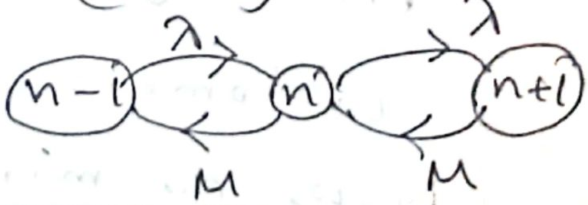
Derivations:

সবই আগের মতো, শুধু $n = [0, \infty]$ এর জায়গায় এখন, $n = [0, N]$; $0 \leq n \leq N$.

And no need for the condition $(\lambda < \mu)$.

Let, P_n, $0 \leq n \leq N \rightarrow$ The probability that there are n customers in the system.

Balanced Equation:

State	Rate at which process leaves = " " " " enters	
0 আগের মত	$\lambda P_0 = \mu P_1$	
N (Exception)	$\lambda P_{N-1} = \mu P_N$	 শেষ
$1 \leq n \leq N-1$ আগের মত	$\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu) P_n$	

So, we get,

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right), \quad 1 \le n \le N-1$$

$$P_N = \frac{\lambda}{\mu} P_{N-1}$$

$$P_2 = \frac{\lambda}{\mu} P_1 + \left(P_1 - \frac{\lambda}{\mu} P_0\right) = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 + \left(P_2 - \frac{\lambda}{\mu} P_1\right) = \frac{\lambda}{\mu} P_2 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\vdots$$

$$P_{N-1} = \frac{\lambda}{\mu} P_{N-2} + \left(P_{N-2} - \frac{\lambda}{\mu} P_{N-3}\right) = \left(\frac{\lambda}{\mu}\right)^{N-1} P_0$$

$$P_N = \frac{\lambda}{\mu} P_{N-1} = \left(\frac{\lambda}{\mu}\right)^N P_0$$

Now, $\sum_{n=0}^{N} P_n = 1$

$$\Rightarrow \sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^N P_0 = 1$$

$$\Rightarrow P_0 \left[\frac{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}{1 - \frac{\lambda}{\mu}}\right] = 1$$

$$\Rightarrow P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}}$$

As, N is not infinite,

$$\boxed{P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}}$$

$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$$

$$\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1-\lambda/\mu}{1-\left(\lambda/\mu\right)^{N+1}}} \quad n = 0,1,2,..,N$$

So, there is no need for the condⁿ $\frac{\lambda}{\mu} < 1$ or $\lambda < \mu$ [We haven't applied the formula, $S_\infty = \frac{a}{1-r}$, when $|r| < 1$.]

Avg # of customers in the system, L

$$L = \sum_{n=0}^{N} n P_n = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} \frac{\lambda}{\mu} \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} \frac{\lambda}{\mu} \frac{\partial}{\partial x} \sum_{n=0}^{N} x^n \quad \left[x = \frac{\lambda}{\mu}\right]$$

$$= \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} \frac{\lambda}{\mu} \frac{\partial}{\partial x}\left(\frac{1-x^{N+1}}{1-x}\right)$$

$$= \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} \frac{\lambda}{\mu} \frac{-(1-x)(N+1)x^N + (1-x^{N+1})}{(1-x)^2}$$

$$= \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}} \cdot \frac{\lambda}{\mu} \cdot \frac{\left(1-\frac{\lambda}{\mu}\right)^{N+1} - \left(1-\frac{\lambda}{\mu}\right)(N+1)\left(\frac{\lambda}{\mu}\right)^N}{\left(1-\frac{\lambda}{\mu}\right)^2}$$

$$// \; x = \frac{\lambda}{\mu}$$

$$\therefore L = \frac{\lambda \left\{ 1 + N\left(\frac{\lambda}{\mu}\right)^{N+1} - (N+1)\left(\frac{\lambda}{\mu}\right)^{N} \right\}}{(\mu - \lambda)\left(1 - \frac{\lambda}{\mu}\right)^{N+1}}$$

Avg time customers spend in the system, W

$$\boxed{W = \frac{L}{\lambda_a}} \longrightarrow \lambda_{actual}$$ [যারা আসলেই ঢুকতে পারবে]

(i) $\lambda_a = \lambda$ [Including those customers, who arrive and find the system full, so doesn't spend any time in the system] No waiting when N full system

(ii) $\lambda_a = \lambda(1 - P_N)$ [Only those Customers, actually entered the system]

$(1 - P_N)$ is the probability that the system can allow new customer to enter.

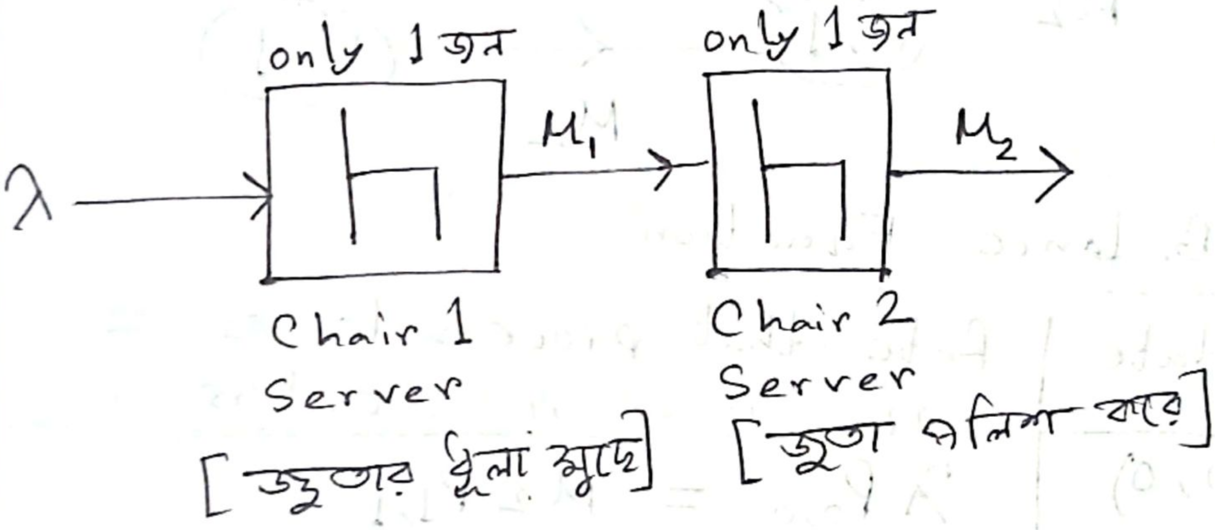
λ is the arrival rate of customers.

λ_a " " actual arrival rate [Actually enters]

$$\therefore W = \frac{L}{\lambda_a} = \frac{L}{\lambda(1-P_N)}$$

8.3.4

A Shoe Shine Shop



C 1 → μ_1 rate এ service নেয়

C 2 → μ_2 " " " "

Possible States and their interpretations

(0, 0) → Both server/chair empty

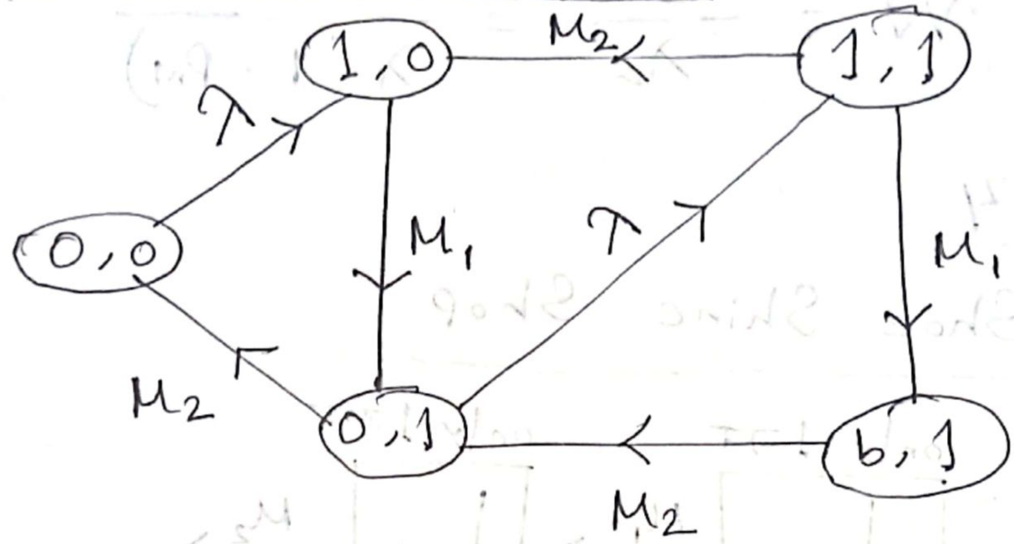
(0, 1) → One customer in chair 2

(1, 0) → " " " " 1

(1, 1) → Both being served

(b, 1) → 1st chair's customer waiting, (C 1 এর service নেয়া শেষ)
2nd " " getting service.

Transition Diagram



Balance Equation

State	Rate that process leaves = " " " enters
(0, 0)	$\lambda P_{0,0} = \mu_2 P_{1,1}$
(0, 1)	$(\lambda + \mu_2) P_{0,1} = \mu_1 P_{1,0} + \mu_2 P_{b,1}$
(1, 0)	$\mu_1 P_{1,0} = \lambda P_{0,0} + \mu_2 P_{1,1}$
(1, 1)	$(\mu_1 + \mu_2) P_{1,1} = \lambda P_{0,1}$
(b, 1)	$\mu_2 P_{b,1} = \mu_1 P_{1,1}$

$$P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1} + P_{b,1} = 1$$

6 টা equation, 5 টা unknown.

L = Avg # of customers in the system

$= P\{1 \text{ জন আছে}\} + P\{2 \text{ জন আছে}\} \times 2$

$= 1 \times (P_{0,1} + P_{1,0}) + 2 \times (P_{1,1} + P_{b,1})$

$$\therefore \boxed{L = (P_{0,1} + P_{1,0}) + 2(P_{1,1} + P_{b,1})}$$

W = Avg time a customer spends in a system

$$\boxed{W = \frac{L}{\lambda_a}}$$

// $\lambda_a = \lambda_{actual}$ = actual entering rate

Proportion of arriving customers actually entering the system $= (P_{0,0} + P_{0,1})$ → 1st chair যেহেতু খালি, এই state এও system এ ঢুকতে পারবে,

So, $\boxed{\lambda_a = \lambda(P_{0,0} + P_{0,1})}$

$$\therefore W = \frac{L}{\lambda_a} = \frac{(P_{0,1} + P_{1,0}) + 2(P_{1,1} + P_{b,1})}{\lambda(P_{0,0} + P_{0,1})}$$

Find π_b [Sir পড়ায় নাই, বই এ আছে]

$\pi_b \rightarrow$ Probabity that a customer will be a blocker. That is, the fraction of entering customers that will have to wait after completing service with server 1 before they can enter server 2/chair 2.

Way 1

An entering customers see 2 states, (0,0) or (0,1)

$$P\{\text{finding } (0,1) \text{ when entering}\} = P\{(0,1) \mid (0,0) \text{ or } (0,1)\}$$

$$= \frac{P_{0,1}}{P_{0,0} + P_{0,1}}$$

System's server 1 will block when a customer enters in (0,1) state and completes service in chair 1, before server 2 has finished

$$\pi_b = \left(\frac{P_{0,1}}{P_{0,0} + P_{0,1}}\right)\left(\frac{\mu_1}{\mu_1 + \mu_2}\right)$$

Way 2

$\lambda_b \rightarrow$ The rate at which customers become blockers.

The proportion of entering customers that are blockers $= \dfrac{\lambda_b}{\lambda_a}$

Blocking occurs when state is $(1,1)$ and a service at 1 is done, 2 is not.

So, $\lambda_b = \mu_1 P_{1,1}$

$$\therefore \pi_b = \frac{\lambda_b}{\lambda_a} = \frac{\mu_1 P_{1,1}}{\lambda (P_{0,0} + P_{0,1})}$$

8.3.5

Queueing system with Bulk Service

Serves 2 or more customers at the same time. Upon completing a service, Server is able to simultaneously serves all customers waiting in the queue. [Bulk service to waiting Customers]

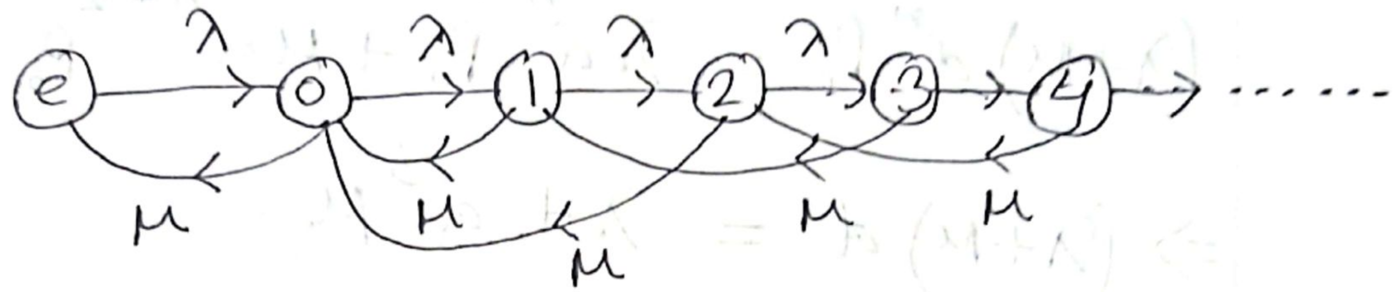
Example : 8.11 [But modified, Only serves 2 customers at a time]

$\mu \rightarrow$ The service rate, no matter how many customers are waiting in queue.

$\lambda \rightarrow$ Arrival rate.

States	Interpretation
e	No one in service, empty
0	Server busy, but no one waiting
n; n>0	n customers waiting in queue

Transition Diagram



Here, only 2 waiting customers get a bulk service, remaining ones should wait.

Balance Equation

State	Rate that process leaves = Rate that process enters
e	$\lambda P_e = \mu P_0$
0	$(\lambda+\mu) P_0 = \lambda P_e + \mu P_1 + \mu P_2$
n, n>0	$(\lambda+\mu) P_n = \lambda P_{n-1} + \mu P_{n+2}$

Now, the equation,

$$(\lambda+\mu) P_n = \lambda P_{n-1} + \mu P_{n+2} \quad ; n = 1, 2, \dots$$

It has a solution of the form,

$$\boxed{P_n = \alpha^n P_0}$$

Putting in the equation,

$$(\lambda+\mu)\,\alpha^{n} P_0 = \lambda \alpha^{n-1} P_0 + \mu \alpha^{n+2} P_0$$

$$\Rightarrow (\lambda+\mu)\,\alpha = \lambda + \alpha^{3}\mu$$

$$\Rightarrow \alpha^{3}\mu - \alpha(\lambda+\mu) + \lambda = 0$$

$$\Rightarrow \alpha^{3}\mu - \alpha^{2}\mu + \alpha^{2}\mu - \mu\alpha - \lambda\alpha + \lambda = 0$$

$$\Rightarrow \mu\alpha^{2}(\alpha-1) + \mu\alpha(\alpha-1) - \lambda(\alpha-1) = 0$$

$$\Rightarrow (\alpha-1)(\mu\alpha^{2} + \mu\alpha - \lambda) = 0$$

$$\therefore \alpha = 1, \text{ or, } \alpha = \frac{-\mu \pm \sqrt{\mu^{2} + 4\mu\lambda}}{2\mu}$$

But, $\alpha \neq 1$ because

if $\alpha = 1$,

$P_n = P_0$

so, $\sum P_i = 1$

$P_0 + P_1 + \cdots + P_n = 1$

$n P_0 = 1$

$P_0 = \frac{1}{n}$

if $n \to \infty$, $P_0 = 0$

$\therefore P_1 = P_2 = P_3 = \cdots = 0$

$$\alpha = \frac{-1 \pm \sqrt{1 + \frac{4\lambda}{\mu}}}{2}$$

and $\alpha \nless 0$

$$\text{So, } \alpha = \frac{-1 + \sqrt{1 + \frac{4\lambda}{\mu}}}{2}$$

$$\alpha < 1 \Rightarrow \frac{-1+\sqrt{1+\frac{4\lambda}{\mu}}}{2} < 1$$

$$\Rightarrow -1+\sqrt{1+\frac{4\lambda}{\mu}} < 2$$

$$\Rightarrow \sqrt{1+\frac{4\lambda}{\mu}} < 3$$

$$\Rightarrow 1+\frac{4\lambda}{\mu} < 9$$

$$\Rightarrow \frac{\lambda}{\mu} < 2.$$

$$\therefore \boxed{\lambda < 2\mu} \quad \text{or,} \quad \mu > \frac{\lambda}{2}$$

Now, $\lambda < 2\mu$ msist hold to avoid overloading the system.

$$P_n = \alpha^n P_0$$

$$P_e = \frac{\mu}{\lambda} P_0$$

Now, to obtain P_0,

$$P_e + P_0 + \sum_{n=1}^{\infty} P_n = 1$$

$$\Rightarrow \frac{\mu}{\lambda} P_0 + P_0 + \sum_{n=1}^{\infty} \delta^n P_0 = 1$$

$$\Rightarrow P_0 \left[\frac{\mu}{\lambda} + 1 + \sum_{n=1}^{\infty} \delta^n \right] = 1$$

$$\Rightarrow P_0 \left[\frac{\mu}{\lambda} + 1 + (\delta + \delta^2 + \delta^3 + \cdots) \right] = 1$$

$$\Rightarrow P_0 \left[\frac{\mu}{\lambda} + \frac{1}{1-\delta} \right] = 1 \qquad // \ \delta < 1$$

$$\Rightarrow \boxed{P_0 = \frac{\lambda(1-\delta)}{\lambda + \mu(1-\delta)}}$$

$$P_n = \delta^n P_0 = \delta^n \frac{\lambda(1-\delta)}{\lambda + \mu(1-\delta)}, \qquad n \geq 0$$

$$P_e = \frac{\mu}{\lambda} P_0 = \frac{\mu(1-\delta)}{\lambda + \mu(1-\delta)}$$

→ The rate at which the customers are served alone $= \lambda P_e + \mu P_1$

1 জন system empty থাকা অবস্থায় আসছে, একা service পাবে

1 জন queue তে wait করছিলো, server কাজ শেষ করে তাকে bulk service দিলো

→ $\therefore$ Proportion of customers served alone $= \dfrac{\lambda P_e + \mu P_1}{\lambda}$

→ Average number of customers waiting in the queue, $L_Q = \sum_{n=1}^{\infty} n P_n$

$$= \sum_{n=1}^{\infty} n \frac{d^n \lambda (1-d)}{\lambda + \mu(1-d)}$$

$$= \frac{\lambda(1-d)}{\lambda + \mu(1-d)} \sum_{n=1}^{\infty} n d^n$$

$$= \frac{\lambda(1-d)}{\lambda + \mu(1-d)} d \sum_{n=1}^{\infty} n d^{n-1}$$

$$= \frac{\lambda(1-\alpha)}{\lambda+\mu(1-\alpha)} \alpha \frac{d}{d\alpha} \sum_{n=1}^{\infty} \alpha^{n}$$

$$= \frac{\lambda(1-\alpha)}{\lambda+\mu(1-\alpha)} \alpha \frac{d}{d\alpha} \left(\frac{1}{1-\alpha}\right)$$

$$= \frac{\lambda(1-\alpha)}{\lambda+\mu(1-\alpha)} \alpha \frac{1}{(1-\alpha)^2}$$

$$= \frac{\lambda \alpha}{(1-\alpha)\left[\lambda+\mu(1-\alpha)\right]}$$

$$\therefore \boxed{L_Q = \frac{\lambda \alpha}{(1-\alpha)\left[\lambda+\mu(1-\alpha)\right]}}$$

→ Avg time a customer waits in the queue, $\boxed{W_Q = \frac{L_Q}{\lambda}}$

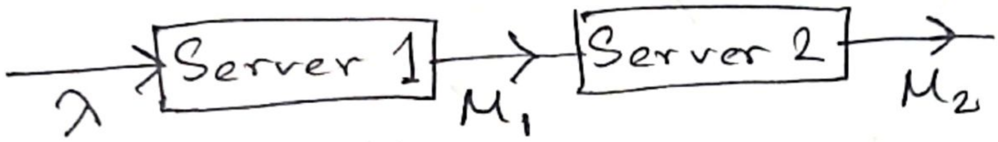
$$\boxed{L = W\lambda}$$

→ $\boxed{\underbrace{W}_{\text{System time}} = \underbrace{W_Q}_{\text{Queue time}} + \underbrace{\frac{1}{\mu}}_{\text{service time}}}$

8.4 Network of Queues

8.4.1

Open Systems



→ Infinite waiting space at both servers.

→ Each server serves one customer at a time.

→ λ, μ_1, μ_2 all are ~~ex~~poisson process,
exponential time.

→ $\lambda < \mu_1, \mu_2$

States

$(0,0) \rightarrow$ 0 cust. at S1, 0 cust at S2

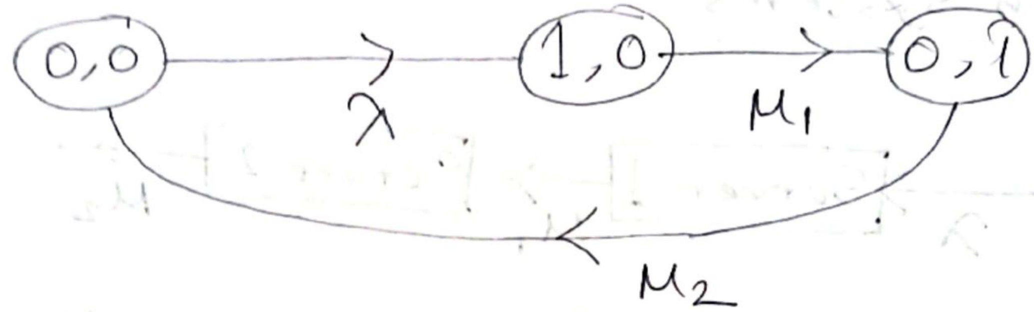
$(n,0) \rightarrow$ n " " S1, 0 " " S2

$(0,m) \rightarrow$ 0 " " S1, m " " S2

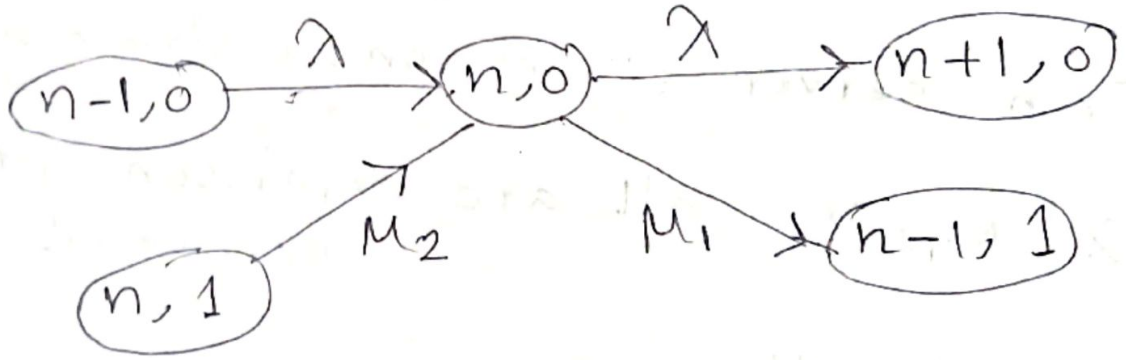
$(n,m) \rightarrow$ n " " S1, m " " S2

Transition Diagram

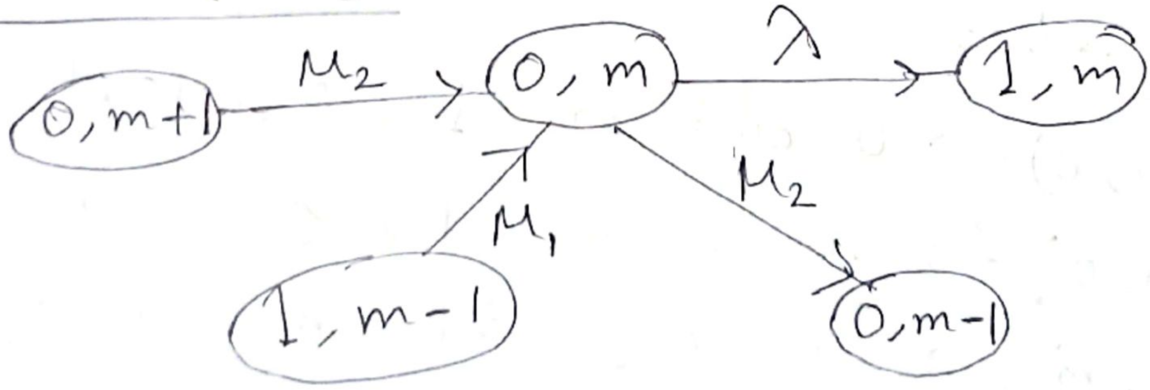
For state (0, 0)



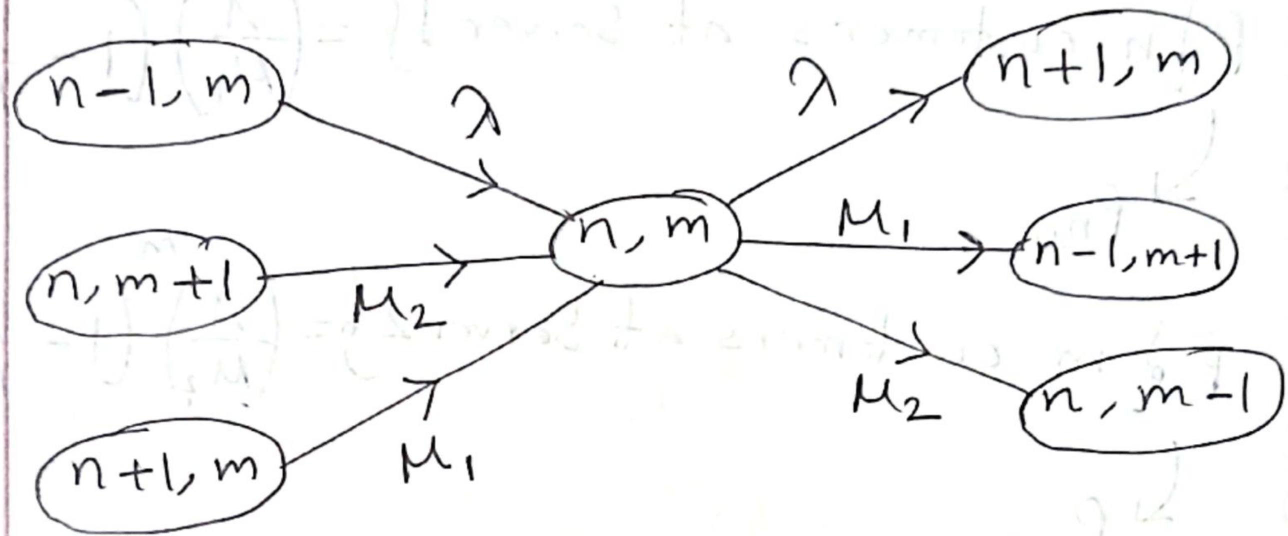
For state (n, 0)



For (0, m)



For (n, m)



Balance Equation

State	Rate Process leaves = " " enters
$0,0$	$\lambda P_{0,0} = \mu_2 P_{0,1}$
$n,0$	$(\lambda+\mu_1) P_{n,0} = \lambda P_{n-1,0} + \mu_2 P_{n,1}$
$0,m$	$(\lambda+\mu_2) P_{0,m} = \mu_2 P_{0,m+1} + \mu_1 P_{1,m-1}$
n,m	$(\lambda+\mu_1+\mu_2) P_{n,m} = \lambda P_{n-1,m} + \mu_1 P_{n+1,m} + \mu_2 P_{n,m+1}$

For M/M/1 system,

$$P\{n \text{ customers at Server } 1\} = \left(\frac{\lambda}{\mu_1}\right)^n \left(1 - \frac{\lambda}{\mu_1}\right)$$

$\searrow P_{n,0}$

$$P\{m \text{ customers at Server } 2\} = \left(\frac{\lambda}{\mu_2}\right)^m \left(1 - \frac{\lambda}{\mu_2}\right)$$

$\searrow P_{0,m}$

$P_{n,m} = P_{0,m} \times P_{n,0}$ [if the no of customers at server 1 and 2 were independent Random Variables]

$$\therefore P_{n,m} = \left(\frac{\lambda}{\mu_1}\right)^n \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^m \left(1 - \frac{\lambda}{\mu_2}\right)$$

To verify $P_{n,m}$, we put it to

$$\lambda P_{0,0} = \mu_2 P_{0,1}$$

$$\lambda \cdot \left(\frac{\lambda}{\mu_1}\right)^0 \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^0 \left(1 - \frac{\lambda}{\mu_2}\right) \quad [L.H.S]$$

$$= \lambda \left(1 - \frac{\lambda}{\mu_1}\right) \left(1 - \frac{\lambda}{\mu_2}\right)$$

$$\mu_2 \left(\frac{\lambda}{\mu_1}\right)^0 \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right) \left(1-\frac{\lambda}{\mu_2}\right) \quad [\text{R.H.S}]$$

$$= \lambda \left(1-\frac{\lambda}{\mu_1}\right)\left(1-\frac{\lambda}{\mu_2}\right)$$

$\therefore$ It's satisfied.

→ Avg # of customers in the system,

$$L = \sum_{n,m} (n+m) P_{n,m}$$

$$= \sum_{n,m} (n+m) \left(\frac{\lambda}{\mu_1}\right)^n \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^m \left(1-\frac{\lambda}{\mu_2}\right)$$

$$= \sum_n n \left(\frac{\lambda}{\mu_1}\right)^n \left(1-\frac{\lambda}{\mu_1}\right) + \sum_n m \left(\frac{\lambda}{\mu_2}\right)^m \left(1-\frac{\lambda}{\mu_2}\right)$$

$$= \frac{\lambda/\mu_1}{\left(1-\frac{\lambda}{\mu_1}\right)^2}\left(1-\frac{\lambda}{\mu_1}\right) + \frac{\lambda/\mu_2}{\left(1-\lambda/\mu_2\right)^2}\left(1-\frac{\lambda}{\mu_2}\right)$$

$$\boxed{L = \frac{\lambda}{\mu_1-\lambda} + \frac{\lambda}{\mu_2-\lambda}}$$

$$\boxed{L = L_1 + L_2}$$

→ Avg time a customer spends in the system, $W = \frac{L}{\lambda}$

$$\boxed{W = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}}$$

$$\boxed{W = W_1 + W_2}$$

8.4.2

Closed Systems

A system of k servers

- Customers arrive from outside to server i at rate r_i.
- After being served at S_i, customer has a probability of P_{ij} of joining the queue of S_j.

$j = 1, 2, \ldots, k$.

$\therefore \sum_{j=1}^{k} P_{ij} \leq 1$ [Not exactly 1, as he can go out of the system after service]

$\therefore$ Probability that customer departs the system after being served by

$$S_i, = 1 - \sum_{j=1}^{k} P_{ij}$$

Let, λ_j = total arrival rate of customer to server j

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij} \quad ; i = 1, 2, \dots, k$$

r_j: বাইরে থেকে j তে আসার Rate

λ_i: S_i ~~থেকে~~ arrival rate

P_{ij}: i থেকে j তে যাওয়ার Prob

$\therefore P\{n \text{ customer at } S_j\}$

$$= \left(\frac{\lambda_j}{\mu_j}\right)^n \left(1 - \frac{\lambda_j}{\mu_j}\right) \quad ; n \geq 1$$

where; μ_j = exponential service rate

at S_j, $\frac{\lambda_j}{\mu_j} < 1$ for stability

If we let $P_m(n_1, n_2, \ldots, n_k)$ denote the limiting probabilities,

$$P_m(n_1, n_2, n_3, \ldots, n_k) = P\{n_j \text{ customer at } S_j\}$$

$$j = 1, 2, \ldots k$$

$$= \prod_{j=1}^{k} \left(\frac{\lambda_j}{\mu_j}\right)^{n_j} \left(1 - \frac{\lambda_j}{\mu_j}\right)$$

→ Avg number of customer in the system, $L = \sum_{j=1}^{k}$ avg # of customer at S_j

$$= \sum_{j=1}^{k} \underbrace{\frac{\lambda_j}{\mu_j - \lambda_j}}_{\text{for each } j,\ L_j}$$

[যোগ করে দিলাম, যেহেতু Independent]

→ Avg time a customer spends in the system $= \dfrac{L}{\lambda_a}$

$$\therefore \boxed{W = \frac{L}{\lambda_a}}$$

$$W = \frac{L}{\lambda_a} \longrightarrow$$ বাইরে থেকে আসার ra

$$W = \frac{\sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}}{\sum_{j=1}^{k} r_j}$$

Term Final		
Set 1 →	Chapter	1, 2
" 2 →	"	3
" 3 →	"	4, 5
" 4 →	"	8

A mandatory question may be there, most probably from chapter 8.