CSE 301 Assignment

Mathemetical Analysis for Computer Science

Md Toki Tahmid : 1805030

Department of Computer Science and Engineering
Bangladesh University of Engineering and Technology

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Contents

Chapter 1

Chapter 1

1.1 Problem 1

All horses are the same color; we can prove this by induction on the warmups in all the chapters! — The Mgm't number of horses in a given set. Here's how: there's just one horse then it's the same color as itself, so the basis is trivial. For the induction step, assume that there are n horses numbered 1 to n. By the induction hypothesis, horses 1 through n 1 are the same color, and similarly horses 2 through n are the same color. But the middle horses, 2 through n 1, can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and n must be the same color as well, by transitivity. Thus all n horses are the same color; QED." What, if anything, is wrong with this reasoning?
Solve:
The issue with this solution is in the base case. If we have only 2 horses, then there is no way we can make partition of these two horses according to the above described way. So this proof does not work.

1.2 Problem 2

Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg C, if direct moves between A and C are disallowed. (Each move must be to or from the middle peg. As usual, a larger disk must never appear above a smaller one.)
Solve:
The steps of the disks transfers are as follows:
1. $A \rightarrow C(n-1)$
2. $A \rightarrow B(1)$
3. $C \rightarrow A(n-1)$
4. $B \rightarrow C(1)$

5. $A \longrightarrow C(n-1)$

$$\begin{aligned}
T_n &= 3T_{n-1} + 2 \\
&= 3 \cdot (3T_{n-2} + 2) + 2 \\
&= 3^2 T_{n-2} + 3 \cdot 2 + 2 \\
&= 3^2 (3T_{n-3} + 2) + 3 \cdot 2 + 2 \\
&= 3^3 T_{n-3} + 3^2 \cdot 2 + 3 \cdot 2 + 2. \\
&= 3^n \cdot T_0 + 2 \left[3^{n-1} + 3^{n-2} \cdots 3^0\right] + 2
\end{aligned}$$

Finally we get,

$$T_n = 3^n - 1$$

1.3 Problem 5

A Venn diagram with three overlapping circles is often used to illustrate the eight possible subsets are associated with three given sets: Can the sixteen possibilities that arise with

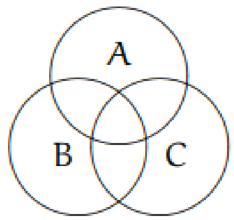


Figure 1.1: *Three intersecting circles*

four given sets be illustrated by four overlapping circles?
Solve:
We know that, 2 circles can intersect each other at most two points. We see from the above figure also that, the maximum number of regions is 14, 16 regions will not be possible just with circles. However, if we have ovals, then 16 regions is possible.

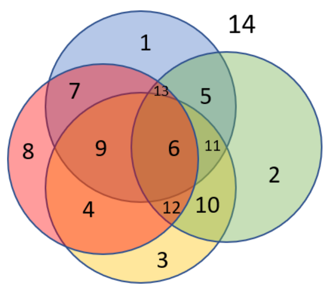


Figure 1.2: *Four intersecting circles*

1.4 Problem 6

Some of the regions defined by n lines in the plane are infinite, while others are bounded. What's the maximum possible number of bounded regions?
Solve:
at first let's notice some small cases:

n	**B**
5	6
0	0
1	0
2	0
3	1
4	3
5	6
6	10

From the table, we see.

$$B_n = B_{n-1} + n - 2$$

our guess is: (for closed form)

$$\begin{aligned} B_n &= S_{n-1} \\ &= \frac{(n-1)(n-2)}{2} \end{aligned}$$

We Proof this by induction: Basic ste: for $n = 3$,

$$B_3 = \frac{(3-1)(3-2)}{2} = 1$$

inductive step:

$$\begin{aligned} B_n &= B_{n-1} + n - 2 \\ &= S_{n-3} + n - 2 \\ &= \frac{(n-3)(n-2)}{2} + \frac{2(n-2)}{2} \\ &= \frac{(n-1)(n-2)}{2}. \end{aligned}$$

(proved)

1.5 Problem 10

Let Qn be the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be clockwise — that is, from A to B, or from B to the other peg, or from the other peg to A. Also let Rn be the minimum number of moves needed to go from B back to A under this restriction. Prove that:

$$Q_n = \begin{cases} 0, & \textbf{if } n = 0; \\ 2R_{n-1} + 1, & \textbf{if } n > 0; \end{cases} \qquad R_n = \begin{cases} 0, & \textbf{if } n = 0 \\ Q_n + Q_{n-1} + 1, & \textbf{if } n > 0 \end{cases}$$

Solve:
First we find the value of Rn for transferring disks from B to A.
(Rn)

1. $B \to A[n-1](R_{n-1})$
2. $B \to C[1](1)$
3. $A \to B[n-1]\,(Q_{n-1})$
4. $C \to A[1](1)$
5. $B \to A[n-1]\,(R_{n-1})$

$$R_n = 2R_{n-1} + Q_{n-1} + 2$$

Now,
(Qn)
1. $A \to C[n-1]\,(R_{n-1})$
2. $A \to B(1)[1]$
3. $C \to B[n-1]\,(R_{n-1})$

$$Q_n = 1 + 2R_{n-1}$$
$$R_n = Q_n + Q_{n-1} + 1$$

1.6 Problem 11

A Double Tower of Hanoi contains 2n disks of n different sizes, two of each size. As usual, we're required to move only one disk at a time, without putting a larger one over a smaller one.
a) How many moves does it take to transfer a double tower from one peg to another, if disks of equal size are indistinguishable from each other?
b) What if we are required to reproduce the original top-to-bottom order of all the equal-size disks in the final arrangement? [Hint: This is difficult. it's really a bonus problem.]
Solve:
a) For the given description, we can approach similar to as the standard tower of hannoi problem as follows:

$$\begin{aligned} T_{2n} =& T_{2(n-1)} + 2 + T_{2(n-1)} \\ \Rightarrow T_{2n} =& 2T_{2(n-1)} + 2 \end{aligned}$$

We guess the solution is:

$$T_{2n} = 2(2^n - 1)$$

We proof this by induction.

Basic Step:

$$\begin{aligned} T_0 &= 0; \\ For, n &= 0, \\ 2(2^0 - 1) = 2(1-1) &= 0 \end{aligned}$$

Thus the base case holds.
Hypothesis:
True for all integer less than $(n-1)$.
Inductive Step:

$$\begin{aligned} T_{2n} &= 2T_{2n-2} + 2 = 2T_{2(n-1)} + 2 \\ &= 2\left[2\left(2^{n-1} - 1\right) + 2\right. \\ &= 2\left[2^n - 2\right] + 2 \\ &= 2^{n+1} - 2 \\ &= 2\left(2^n - 1\right) \end{aligned}$$

(b) In this question, we assume upside means the ordering is reversed, and ok means the order is maintained.

$$\begin{aligned} & A \to C(n-1)[upside]. \\ & A \to B[1](ok]. \\ & C \to B(n-1)[ok] \\ & A \to C[1](ok) \\ & B \to A(n-1)[upside] \\ & B \to C[1](ok) \\ & A \to C(n-1)[ok] \\ Thus, B_n = 4A_{n-1} + 3. \end{aligned}$$

1.7 Problem 13

What's the maximum number of regions definable by n zig-zag lines, each of which consists of two parallel infinite half-lines joined by a straight segment?
Solve:

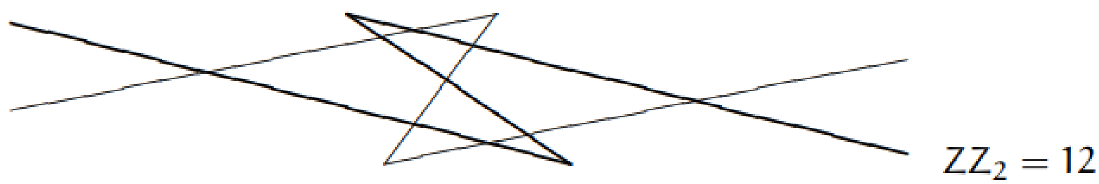


Figure 1.3: *Regions formed by 2 zig-zag lines*

A zig-zag is basinlly 3 lines. we know for each new added line:

$$L_n = L_{n-1} + n$$

Now, in z_{n-1} region there are $3n - 3$ lines in total. For 1 added line:

$$\begin{aligned}\text{new region } &= 3n - 3 + 1 \\ &= 3n - 2\end{aligned}$$

For second added line:

$$\rightarrow (3n - 2 + 1) = 3n - 1 \text{ new region}$$

add last line:

$$3n \text{ new regions.}$$

we need to remove 5 regions. So,

$$Z_n = Z_{n-1} + 9n - 8$$

$$\begin{aligned}
z_0 &= 1 \\
z_n &= z_{n-1} + 9n - 8 + 9[n=0] \\
G(z) &= zG(z) + 9\sum^{n} z^n - 8\sum z^n + 9 \\
&= zG(z) + 9\frac{z}{(1-z)^2} - \frac{8}{1-z} + 9 \\
\Rightarrow G(z)[1-z] &= \frac{9z}{(1-z)^2} - \frac{8}{(1-z)} + 9 \\
\Rightarrow G(z) &= \frac{9z}{(1-z)^3} - \frac{8}{(1-z)^2} + \frac{9}{1-z} \\
&= 9z\sum_k \left(\begin{array}{c} 2+k \\ 2 \end{array} \right) z^k - 8\sum(1+k)z^k + 9\sum z^k \\
\text{Now, } &= \frac{(k+1)k}{2} - 8(k+1) + 9 \\
z^k[G(z)] &= \frac{9k^2}{2} + \frac{9k}{2} - 8k - 8 + 9 \\
&= \frac{9k^2}{2} - \frac{7k}{2} + 1
\end{aligned}$$

Thus,

$$Z_n = \frac{9n^2}{2} - \frac{7n}{2} + 1$$

1.8 Problem 14

How many pieces of cheese can you obtain from a single thick piece by making five straight slices? (The cheese must stay in its original position and each slice must correspond to a plane in 3D.) Find a recurrence relation for Pn, the maximum number of three-dimensional regions that can be defined by n different planes
Solve:
Number of regions that can be formed by n planes can be found if we consider that, each plane newly inserted, will intersect with the previously defined planes in a line. So if in the (n-1)th space, there were (n-1) lines, then the new plane will intersect with each of these (n-1) lines and form a new region. With the intersection of (n-1) lines, new regions can be formed: L_{n-1}.

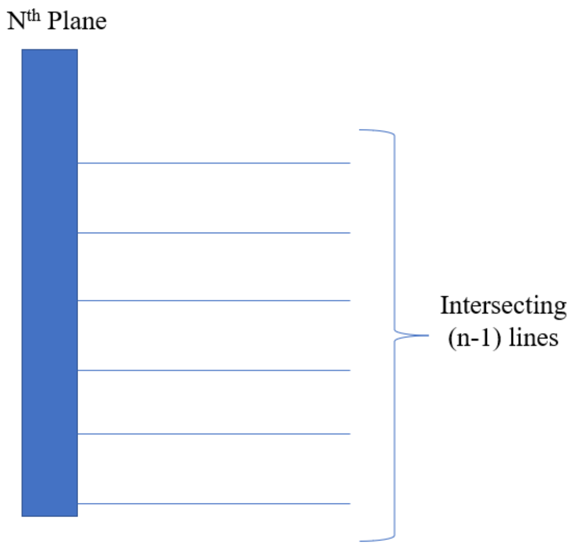


Figure 1.4: *Regions formed by intersecting planes*

Thus, the number of regions with n planes will be:

$$\begin{aligned} P_n =& P_{n-1} + L_{n-1} \\ =& P_{n-1} + 1 + S_n \\ =& P_{n-1} + 1 + \frac{n(n-1)}{2} \\ =& \frac{(6 + 5n + n^3)}{6} \text{ (In closed form)} \end{aligned}$$

1.9 Problem 19

Is it possible to obtain Zn regions with n bent lines when the angle at each zig is 30 **?**
Solve:

Figure 1.5: *Regions formed by intersecting planes*

We see that, if one zig is located inside the bent line, we cannot obtain Zn regions with the bent lines. So, as the lines are 30 degree separated, after adding 5 such bent lines avoiding the intersection between wedges, we cannot add any more bent line, as the angle becomes 180 degree.

Chapter 2

Chapter 2

2.1 Problem 11

The general rule (2.56) for summation by parts is equivalent to

$$\sum_{0\le k<n}(a_{k+1}-a_k)b_k = a_nb_n - a_0b_0 - \sum_{0\le k<n} a_{k+1}(b_{k+1}-b_k), for, n \ge 0.$$

Prove this formula directly by using the distributive, associative, and commutative laws.
Solve:

$$\begin{aligned}
&\sum_{0\le k<n}(a_{k+1}-a_k)b_k \\
&= \sum_{0\le k<n} a_{k+1}b_k - \sum_{0\le k<n} a_kb_k
\end{aligned}$$

$$\begin{aligned}
Now, &\sum_{0\le k<n} a_kb_k \\
&= \sum_{0\le k<n} a_kb_k + a_nb_n - a_nb_n \\
&= \sum_{0\le k\le n} a_kb_k - a_nb_n \\
&= \sum_{1\le k\le n} a_kb_k - a_nb_n + a_0b_0 \\
&= \sum_{0\le k<n} a_{k+1}b_{k+1} - a_nb_n + a_0b_0
\end{aligned}$$

$$\begin{aligned}
&Thus, \\
\sum_{0\le k<n}(a_{k+1}-a_k)b_k = a_nb_n - a_0b_0 - \sum_{0\le k<n} a_{k+1}(b_{k+1}-b_k)
\end{aligned}$$

2.2 Problem 12

Show that the function $p(k) = k+(1)^k c$ **is a permutation of the set of all integers, whenever c is an integer.**
Solve:

We see that, each even number is shifted forward by **C**
and each odd number is shifted backward by **C**
As $\mathbb{Z}$ is infite, the term p(k) completely maps to $\mathbb{Z}$. Thus P(k) is a contribution of the set of all integers.

2.3 Problem 13

Use the repertoire method to find a closed form for $\mathbf{T}_k = \sum_{k=0}^{n} (-1)^k k^2$
Solve:
Let,

Now, A(n-1)=(-1)$^{n-1}(n-1)^3$
$Thus, A(n) - A(n-1) = (-1)^n(2n^3 - 3n^2 + 3n - 1)$

Similarly,
B(n)-B(n-1)=(-1)$^n(2n^2 - 2n + 1)$

And,
C(n)-C(n-1)=(-1)$^n(2n - 1)$

Let, $\mathrm{T}_n = \alpha A(n) + \beta B(n) + \gamma C(n)$
$= (-1)^n((2n^3 - 3n^2 + 3n - 1)\alpha + (2n^2 - 2n + 1)\beta + (2n - 1)\gamma) + A(n-1) + B(n-1) + C(n-1)$

By equating the coefficients:
$\alpha = 0$
$\beta = \gamma = \frac{1}{2}$

And thus, $\mathrm{T}_n = (-1)^n(n^2 + n)/2$.

2.4 Problem 14

Evaluate $\sum_{k=1}^{n} k2^k$ by rewriting it as the multiple sum $\sum_{1\le j\le k\le n} 2^k$
Solve:

$$
\begin{aligned}
\sum_{k=1}^{n} k2^k &= \sum_{k=1}^{n}\sum_{j=1}^{k} 1\cdot 2^k \\
&= \sum_{k=1}^{n} 2^k \sum_{j=1}^{k} 1 \\
&= \sum_{1\le j\le kn} 2^k \\
&= \sum_{j=1}^{n}\sum_{k=j}^{n} 2^k \\
&= \sum_{j=1}^{n}[\sum_{k=0}^{n} 2^k - \sum_{0\le k<j} 2^k] \\
&= \sum_{j=1}^{n} [2^{n+1} - 2^{j+1}] \\
&= n2^{n+1} - 4(2^n - 1) \\
&= 2^{n+1}(n-1) + 2
\end{aligned}
$$

2.5 Problem 15

Evaluate $Q_n = \sum_{k=1}^{n} k^3$ by the text's Method 5
Solve:

$$
\begin{aligned}
Q_n &= \sum_{k=1}^{n} k^3 \\
&= \sum_{k=1}^{n} k^2\cdot k \\
&= \sum_{k=1}^{n} k^2 \sum_{j=1}^{k} 1 \\
&= \sum_{1\le j\le k\le n} k^2 \\
&= \sum_{j=1}^{n}\sum_{j\le k\le n} k^2
\end{aligned}
\tag{2.1}
$$

Now,

$$
\begin{aligned}
Now, \sum_{j\le k\le n} k^2 = \sum_{1\le k\le n} k^2 - \sum_{1\le k(j-1)} k^2 \\
Finally, \\
Q_n =& \frac{n^2(n+1)(2n+1)}{6} - \frac{1}{3}\sum_n j^3 + \frac{1}{2}\sum_n j^2 + \frac{1}{4}\sum_n j \\
=& \frac{n^2(n+1)(2n+1)}{6} - \frac{Q_n}{3} + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{12} \\
\text{Thus,} \\
Q_n =& \frac{3}{4}[\frac{(n+1)(2n+1)n}{6}(1+\frac{n}{2}) + \frac{n(n+1)}{12}]
\end{aligned}
$$

2.6 Problem 19

Use a summation factor to solve the recurrence:

$$
\begin{aligned}
T_0 &= 5 \\
2T_n &= nT_{n-1} + 3\cdot n!
\end{aligned}
$$

Solve: Comparing with the standard equation of recurrence

$$
\begin{aligned}
a_nT_n = b_nT_{n-1} + C_n \\
We get, \\
a_n = 2 \\
b_n = n \\
C_n = 3n! \\
Now, \\
S_n =& \frac{2.2.2.2...}{n(n-1)...2.1} \\
=& \frac{2^{n-1}}{n!} \\
Now, \\
T_n =& \frac{1}{S_na_n}(S_1b_1T_0 + \sum_{k=1} nS_kC_k) \\
=& \frac{n!}{2^n}(5 + 3\sum_{k=1} n2^{k-1} \\
=& \frac{n!}{2^n}(5 + 3\sum_{r=0} n - 12^r \\
=& n!(2^{1-n} + 3)
\end{aligned}
$$

2.7 Problem 20

Try to evaluate $\sum_{k=0}^{n} kH_k$ by the perturbation method, but deduce the value of $\sum_{k=0}^{n} H_k$ instead

Solve: Let,

$$
\begin{aligned}
S_n =& \sum_{k=0}^{n} kH_k \\
S_{n+1} =& \sum_{k=0}^{n+1} kH_k \\
=& S_n + (n+1)H_{n+1} \\
Again, & \\
S_{n+1} =& \sum_{k=1}^{n+1} kH_k \\
=& \sum_{k=0}^{n} (k+1)H_{K+1} \\
=& \sum_{k=0}^{n} (k+1)(H_k + \frac{1}{1+k}) \\
=& \sum_{k=0}^{n} 1 + \sum_{k=0}^{kH_k} + \sum_{k=0}^{H_k} \\
(n+1)H_{n+1} =& \sum_{k=0}^{n} 1 + \sum_{k=0}^{H_k} \\
\sum_{k=0}^{n} H_k =& (n+1)H_{n+1} - \sum_{k=0}^{n} 1 \\
=& (n+1)H_{n+1} - (n+1)
\end{aligned}
$$

2.8 Problem 21

Evaluate the sums:

$$
\begin{aligned}
\textbf{a)} S_n &= \sum_{k=0}^{n} (-1)^{n-k} \\
\textbf{b)} T_n &= \sum_{k=0}^{n} (-1)^{n-k} k \\
\textbf{c)} U_n &= \sum_{k=0}^{n} (-1)^{n-k} k^2
\end{aligned}
$$

Solve:

a)

$$S_n = \sum_{k=0}^{n} (-1)^{n-k}$$

$$S_{n+1} = \sum_{k=0}^{n+1} (-1)^{n-k}$$

Spliting the first term we get:

$$\begin{aligned} S_{n+1} =& (-1)^{n+1} + \sum_{k=0}^{n} (-1)^{n-k} \\ =& (-1)^{n+1} + S_n \end{aligned}$$

Spliting the first last term we get:

$$\begin{aligned} S_{n+1} =& -\sum_{k=0}^{n} (-1)^{n-k} + 1 \\ =& - S_n + 1 \end{aligned}$$

From the above two equations, we get,

$$S_n = \frac{1}{2}(1 + (-1)^n)$$

b)

$$T_n = \sum_{k=0}^{n} (-1)^{n-k} k$$

$$T_{n+1} = \sum_{k=0}^{n+1} (-1)^{n-k} k$$

Spliting the first term we get:

$$\begin{aligned} S_{n+1} =& (-1)^{n+1} + \sum_{k=0}^{n} (-1)^{n-k} k \\ =& \sum_{k=0}^{n} (-1)^{n-k} k + \sum_{k=0}^{n} (-1)^{n-k} \\ =& T_n + S_n \end{aligned}$$

Spliting the first last term we get:

$$\begin{aligned} T_{n+1} =& -\sum_{k=0}^{n} (-1)^{n-k} + (-1)^{n+1-(n+1)}(n+1) \\ =& (n+1) - T_n \end{aligned}$$

From the above two equations,

$$\begin{aligned} T_n + S_n =& (n+1) - T_n \\ T_n =& \frac{1}{2}(n - (-1)^n). \end{aligned}$$

2.9 Problem 29

Evaluate the sum $\sum_1^n (-1)^k \frac{k}{4k^2-1}$
Solve: *Let,*

$$\begin{aligned}
\sum_1^n (-1)^k \frac{k}{4k^2-1} &= \frac{k}{(2k+1)(2k-1)} \\
&= \frac{1}{4(2k+1)} + \frac{1}{4(2k-1)} \\
\sum_1^n (-1)^k \frac{k}{4k^2-1} &= \frac{1}{4}[\sum (-1)^k \frac{1}{2k+1} + \sum (-1)^k \frac{1}{2k+1}] \\
&= \frac{1}{4}[(-1 + \frac{(-1)^n}{2n+1})]
\end{aligned}$$

Chapter 3

Chapter 4

3.1 Problem 2

Prove that gcd(m, n)· lcm(m, n) = m·n, and use this identity to express lcm(m, n) in terms of lcm(n mod m, m), when n mod m $\neq 0$.
Solve:

$$\begin{aligned}
&Let,\ \mathbf{A} = \gcd(m,n) \Leftrightarrow Ap = \min(mp, n_p) \\
&B = \mathrm{lcm}(m,n) \Leftrightarrow Bp = \max(m_p, n_p) \\
&A \cdot B = \gcd(m,n) \cdot lcm(m,n) \Leftrightarrow A_p + B_P \\
&= \min(m_p, n_p) + \max(m_p, n_p) \\
&\textbf{Therefore, } mp + np \Leftrightarrow mn\textbf{.}
\end{aligned}$$

again

$$k = mp \Leftrightarrow kp = mp + np.$$

We can conclude that,

$$gcd(m,n).lcm(m,n) = m.n$$

Now,

$$lcm(m,n) = \frac{mn}{gcd(m,n)} = \frac{mn}{gcd(nmodm, m)}$$

$$= \frac{mn.lcm(nmodm, m)}{m.nmodm} = \frac{nlcm(nmodm, m)}{nmodm}$$

3.2 Problem 14

Prove or disprove:
a) gcd(km, kn) = k gcd(m, n);
b) lcm(km, kn) = k lcm(m, n).
Solve:

a)

$$\begin{aligned}
&x = \gcd(km, kn) \Leftrightarrow x_p = \Big(\min\Big((km)p, (kn)_p\Big)\Big)\\
&\Leftrightarrow x_p = \min\left(kp + m_p, k_p + n_p\right)\\
&= kp + \min(m_p, n_p) \Leftrightarrow k \cdot \gcd(\min)\\
&\gcd(km, kn) = k.gcd(m, n)
\end{aligned}$$

b)

$$\begin{aligned}
X = lcm(km, kn) \Leftrightarrow x_p &= max((km)p, (kn)p)\\
&= max(kp + mp, kp + np)\\
&= kp + max(mp, np) \Leftrightarrow k \cdot lcm(m, n).
\end{aligned}$$

3.3 Problem 18

Show that if $2^n + 1$ is prime then n is a power of 2.
Solve: let,

$$n = xy$$

where, x is proper power of 2 and ; y is odd. Now,

$$\begin{aligned}
&2^n + 1\\
=&2^{x \cdot y} + 1\\
=&\left(2^x\right)^y + 1\\
=&\left(2^x\right)^y (-1)^y \mid \textbf{ as y is odd}\\
=&\left(2^x - 1\right)\Big[\left(2^x\right)^{n-1} + \cdots + (-1)^{n-1}\Big]
\end{aligned}$$

divisible by $(2^x - 1) \rightarrow$ not prime.

3.4 Problem 24

Express in terms of Vp(n), the sum of the digits in the radix p representation of n, thereby generalizing (4.24)
Solve:
$V_p(n) \rightarrow$ **highest power of p that divides $n =$**
$\longrightarrow$ **sum of the digits in the radix p representation of n**
At first, we can represent a number n as follows:

$n = d_m p^m + d_{m-1} p^{m-1} + ... d_0 p^0$

$Now,$

$\epsilon_p(n!) = \lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor ... + \lfloor n/p^m \rfloor$

$= d_m \left(p^{m-1} + p^{m-2} + \cdots + p^0\right) + d_{m-1} \left(p^{m-2} + p^{m-3} + \cdots + p^0\right) +$

... d_0

$= d_m \dfrac{p^m - 1}{p-1} + d_{m-1} \dfrac{p^{m-1} - 1}{p-1} + \cdots + d_1 \dfrac{p-1}{p-1} + d_0 p^0$

$= \dfrac{1}{p-1} \left[n - v_p\left(n\right)\right]$

This is because:

$d_m P^m + d_{m-1} P_x^{m-1} + \cdots x_n = n$

and, $(d_m + d_{m-1} \cdots + d_0 = v_p(n))$

3.5 Problem 31

A number in decimal notation is divisible by 3 if and only if the sum of its digits is divisible by 3. Prove this well-known rule, and generalize it.
Solve:

$$\textbf{Decimal means the base is 10.} \Rightarrow 10 \equiv 1(\mathrm{mod}3) \Rightarrow 10^n = 1(\mathrm{mod}(3))$$

let the number be X

$$x = x_n 10^{n-1} + x_{n-1}10^{n-2} + \cdots x_0$$

Now,

$$X = (x_n + x_{n-1} + ... + x_0)\,(\mathrm{mod}3) \mid as, 10^{n-1} \equiv (\mathrm{mod}3), 10^{n-2} \equiv 1(\mathrm{mod}3)...$$

so, if

$$3 \mid (x_n + x_{n-1} + ... + x_0) \Rightarrow X \equiv 0(mod3)$$

and thus,

$$3 \mid X$$

3.6 Problem 38

Prove that if $a \perp b$ **and** $a > b$ **then**

$$\gcd\left(a^m - b^m, a^n - b^n\right) = a^{gcd(m,n)} - b^{\gcd(m,n)}, \quad 0m < n.$$

(All variables are integers.) Hint: Use Euclid's algorithm.
Solve:
We need to find the gcd of : $(a^m - b^m, a^n - b^n)$

$$\begin{array}{l} a^m - b^m \, |a^n - b^n| \, a^{n-m} + a^{n-2m}b^m + a^{n-3m} \cdot b^{2m} ... \\ \qquad\qquad \underline{a^n - a^{n-m}b^m} \\ \qquad\qquad a^{n-m}b^m - b^n \\ \qquad\qquad \underline{a^{n-m}b^m - a^{n-2m}b^{2m}} \\ \qquad\qquad\quad a^{n-2m}b^{2m} - b^n \\ \qquad\qquad\qquad\quad \vdots \end{array}$$

Lowest power of a will be $nmodm \Rightarrow a^{nmodm}$ **Thus, power of b: n-nmodm** $= m \cdot \lfloor \frac{n}{m} \rfloor$

$$\begin{aligned} & a^n - b^n \\ & \quad = (a^m - b^m)[a^{n-m} + ...] + (a^{nmodm} b^{m\lfloor \frac{n}{m} \rfloor} - b^n) \\ & Now, \\ & \quad a^{nmodm} b^{m\lfloor \frac{n}{m} \rfloor} - b^n \\ & \quad = b^{m\lfloor \frac{n}{m}} [a^{nmodm} - b^{nmodm}] \end{aligned}$$

Now,

$$\begin{aligned} & \gcd\left(a^n - b^n, a^m - b^m\right) \\ = & \gcd\left(a^m - b^m \% a^n - b^n; a^n - b^n\right) \\ = & \gcd\left(b^{n\lfloor m/n \rfloor}\left(a^{m \bmod n} \cdot b^{m \bmod n}\right), a^n - b^n\right) \end{aligned}$$

lowest power of $a = \gcd(m, n)$ **lowest power of** $b = \gcd(m, n)$ **one part will be zero eventually**

$$\gcd\left(a^n - b^n, a^m - b^m\right) = a^{gcd(m,n)} - b^{gcd(m,n)}$$

3.7 Problem 41

a) Show that if $pmod4 \equiv 3$, there is no integer n such that p divides n^2+1. Hint: Use Fermat's theorem.
Solve:

$P \bmod 4 \equiv 3$
Let there is an integer n such that,

$$p|n^2+1 \leftrightarrow n^2 \equiv -1(\text{mod}p) \leftrightarrow \left(n^2\right)^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}}(\text{mod}p)$$

Now

$$\begin{aligned} &Pmod4 \equiv 3 \\ &\Leftrightarrow 4 \mid P-3 \\ &\Leftrightarrow \frac{P-1}{2} = 2K+1 \end{aligned}$$

$$\frac{P-1}{2}\textbf{is odd}.$$

So we get,

$$n^{p-1} \equiv -1(modP)$$

Which is violation of FLT. This there is no such integer.

3.8 Problem 46

Prove that if $n^j \equiv 1$ and $n^k \equiv 1$ (mod m), then $n^{gcd(j,k)} \equiv 1$.
Solve:
a) $j'j - k'k = gcd(J,k)$ **— basic diaphantine eqn. Now,**

$$\begin{aligned} &n^j \equiv 1 \bmod m \\ &\textbf{S. } n^{j'j} \equiv 1 \bmod m \\ &\rightarrow n^{k'k} \cdot n^{\text{gcd}(j,k)} \equiv 1 \bmod m \\ &\Leftrightarrow n^{k'k} \cdot n^{\text{ged}(j,k)} - 1 = mk_1 ...(1) \end{aligned}$$

again,

$$\begin{aligned} n^{k'k} &\equiv 1 \textbf{ modm} \\ \Leftrightarrow n^{k'k} - 1 &= mk_2 ...(2) \end{aligned}$$

from 1 and 2:

$$n^{k'k}\left(1 - n^{\textbf{gcd }(j,k))}\right) = m(k1-k2)$$

⇒ Now,

$$\begin{aligned} &m|n^{k'k} - 1 \Rightarrow \textbf{ so, } mn^{k'k.} \\ &m|1 - n^{gcd(j,k)} \\ &\Rightarrow n^{gcd}(j,k) \equiv 1(modm). \end{aligned}$$

Chapter 4

Chapter 4 Extra

4.1 Problem 1

What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?
Solve:

Dividing $n^3 + 100$ by $n + 10$, we get, $n^3 + 100 = (n + 10)(n^2 - 10n + 100) - 900$
Now, if $n + 10$ divides, $n^3 + 100$, it should also divide 900.
$n + 10 \backslash 900$, **for max n, when n=890.**

4.2 Problem 2

Show that the fraction $\frac{12n+1}{30n+2}$ is irreducible for all positive integers n
Solve:

If the fraction is irreducible, then then the numerator and denominator must be coprime. That is : $gcd(12n+1, 30n+2)$ should be 1. Now, we evaluate gcd(12n+1,30n+2)

$$\begin{aligned} gcd(12n+1, 30n+2) &= gcd(30n+2 mod 12n+1, 12n+1) \\ &= gcd(6n, 12n+1) \\ &= gcd(12n+1 mod 6n, 6n) \\ &= gcd(1, 6n) \\ &= 1 \end{aligned}$$

As they are coprime, the fraction is irreducible.

4.3 Problem 3

Call a number prime looking if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?
Solve:

We use the principle of inclusion-exclusion here. Numbers, which are not divisible by 2,3,or 5 is represented as $n(S2 \cup s3 \cup s5)$**. Now,**

$$\begin{aligned}
&n(S2 \cup s3 \cup s5)\\
&= n(s2) + n(s3) + n(s5) - n(s2 \cap s3) - n(s2 \cap s5) - n(s3 \cap s5) + n(s2 \cap s3 \cap s5)\\
&= 500 + 667 + 800 - 834 - 900 - 934 + 967\\
&= 266
\end{aligned}$$

Now, number of primes under 1000 , excluding 2,3,5 are 165. Moreover, 1 is not also prime-looking. Thus the number of prime-looking numbers less than 1000 are : 266 -165-1=100

4.4 Problem 4

Let m and n be positive integers such that lcm(m, n) + gcd(m, n) = m + n. Prove that one of the two numbers is divisible by the other.
Solve:

Let, gcd(m,n)=a and lcm(m,n)=b. Now, given that, a+b=m+n. We know from the property of gcd and lcm that, $lcm(m, n) \times gcd(m, n) = m \times n$ **Thus,**

$$\begin{aligned}
ab &= mn\\
a &= mn/b
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{mn}{b} + b = m + n\\
&mn + b^2 = b(m + n)\\
&(b - m)(b - n) = 0\\
&b = m, n
\end{aligned}$$

Similarly, a=m,n. Now, a=gcd(m,n)=n. So, m must be divisible by n. Thus, one of the number is divisible by the other one.

4.5 Problem 5

Show that for any positive integers a and b, the number (36a + b)(a + 36b) cannot be a power of 2.
Solve:
First, let $(36a + b)(a + 36b) = 2^m$ **As the multiple of two numbers is a power of two, each individually must be a power of 2.**
Thus,

$$(36a + b) = 2^p$$

$$(36b + a) = 2^q$$

We see that, both a and b must be even. Thus we divide by 2 in both the equations, and get:

$$36a' + b' = 2^{2-1}$$
$$36b' + a' = 2^{q-1}$$

$$\vdots$$

And this goes on, ultimately we will not get any solution. Thus (36a + b)(a + 36b) is not a power of 2.

4.6 Problem 6

Find all positive integers n for which $n! + 5$ is a perfect cube.
Solve:

To solve this problem, we need to find a upper boundary first above which the search space will not produce any positive result.
As we are talking about cubes, lets consider the prime number 7. From FLT, we get:

$$n^6 \equiv 0, 1 (mod 7) \text{[Depending on if n and 7 are coprime or not]}$$
$$n^3 \equiv 0, +1, -1 (mod 7)$$

Thus, any perfect cube mod7 will produce 0, +1 or -1.
Now, in m!+5, lets consider, $m \geq 7$. Thus, $m! \equiv 0 mod 7$
and, $m! + 5 \equiv 5 mod 7$
which does not satisfy the previous condition for perfect cube.
Therefore, we limit our search below m=6. By inspection, we see only for n=5, n!+5 is a perfect cube.

4.7 Problem 8

Let p be a prime. Show that there are infinitely many positive integers such that p divides

$$2^n - n$$

.
Solve:

Let P be prime, thus p is odd, for$p > 2$.

$$(p-1)^2 \equiv 1 (mod p)$$
$$(p-1)^{2k} \equiv 1 (mod p)$$

Again, from FLT:

$$2^{p-1} \equiv 1 (mod p)$$
$$\Rightarrow 2^{(p-1)m} \equiv 1 (mod p)$$

Let, $m = (p-1)^{2k-1}$

$$\Rightarrow 2^{(p-1)^{2k}} \equiv 1 (mod p)$$

Let, $n = (p-1)^{2k}$

Thus,

$$2^n \equiv n (mod p)$$

Thus there are infinitely many such positive integers.

4.8 Problem 9

Prove that a $a^p \equiv a(modp)$**, where p is any prime.**
Solve:

We assume that p **denotes a prime.**
Now, the $p-1$ **numbers** $n \bmod p, 2n \bmod p, \ldots, (p-1)n \bmod p$ **are the numbers 1,2 ,** $\ldots, p-1$ **in some order, that is the residue list.**
Therefore if we multiply the original terms together we get

$$\begin{aligned} & n \cdot (2n)...((p-1)n) \\ & \qquad \equiv (n \bmod p) \cdot (2n \bmod p) \cdot \ldots \cdot ((p-1)n \bmod p) \\ & \qquad \equiv (p-1)! \end{aligned}$$

where the congruence is modulo p**. This means that**

$$(p-1)!n^{p-1} \equiv (p-1)! \pmod{p}$$

and we can cancel the $(p-1)$ **! since it's not divisible by** p**.**
Finally we multiply both sides by n, and get:

$$n^p \equiv n \pmod{p}, \quad \textbf{integer } n$$

4.9 Problem 11

Evaluate gcd(n! + 1,(n + 1)! + 1).
Solve:

$$\begin{aligned}
gcd(n!+1,(n+1)!+1) & \\
&= gcd((n+1)!+1\% n!+1, n!+1) \\
n!+1 \mid (n+1)!+1 \quad & \mid \quad (n+1) \\
& \underline{(n+1)!+1+n} \\
& -n \\
\textbf{Thus} & \\
gcd(n!+1,(n+1)!+1) &= gcd(-n, n!+1) \\
&= 1
\end{aligned}$$

4.10 Problem 14

Let n ¿ 1 be an odd integer. Prove that n does not divide $3^n + 1$.
Solve:
Let, $n \mid 3^n + 1$
Let P be the least prime factor of n. Thus,

$$\begin{aligned}
3^n &\equiv -1 (mod p) \\
3^{2n} &\equiv 1 (mod p) \\
\textbf{Again from FLT, we have:} & \\
3^{P-1} &\equiv 1 (mod p) \\
\textbf{And thus,} & \\
3^{gcd(2n,P-1)} &\equiv 1 (mod P). \\
\textbf{As 2n is even, and P-1 is also even} & \\
\textbf{gcd(2n,P-1)=2} & \\
3^2 &\equiv 1 (mod P) \\
P &\mid 8 \\
\textbf{Which is a contradiction} &
\end{aligned}$$

Thus, n does not divide $3^n + 1$.

Chapter 5

Chapter 7

5.1 Problem 7

Solve the recurrence

$$\begin{aligned} g_0 =& 1 \\ g_n =& g_{n-1} + 2g_{n-2} + ... + ng_0 \end{aligned}$$

Solve:

$$\begin{aligned} g_n =& g_{n-1} + 2g_{n-2} + ... + ng_0 + [n=0] \\ \sum g_n =& \sum g_{n-1} + 2\sum g_{n-2} + ... + \sum ng_0 + \sum [n=0] \\ \sum g_n Z^n =& \sum g_{n-1}Z^n + 2\sum g_{n-2}Z^n + ... + \sum ng_0 Z^n + \sum [n=0]Z^n \\ G(z) =& zG(z) + 2z^2G(z) + ... + nz^nG(z) + 1 \\ G(z) =& \frac{1-2z+z^2}{1-3z+z^2} \\ =& 1 + \frac{z}{z^2-3z+1} \end{aligned}$$

We get the roots of $z^2 - 3z + 1 = 0$, a=$\frac{3+\sqrt{5}}{2}$, and b=$\frac{3-\sqrt{5}}{2}$.
Now, with partial fraction,

$$
\begin{aligned}
\frac{z}{z^2-3z+1} = \frac{A}{z-a} + \frac{B}{z-b} \\
A =& \frac{3+\sqrt{5}}{2\sqrt{5}} \\
A =& -\frac{3-\sqrt{5}}{2\sqrt{5}} \\
\frac{A}{z-a} + \frac{B}{z-b} =& \frac{A}{-a(1-\frac{z}{a}} + \frac{B}{-b(1-\frac{z}{b}} \\
=& \frac{-A}{a}\sum_i 1^i (z/a)^i + \frac{-B}{b}\sum_i 1^i (z/b)^i \\
G_i =& \frac{-A}{a}(1/a)^i - \frac{B}{b}(1/b)^i \\
=& \frac{-1}{\sqrt{5}}(\frac{2}{3+\sqrt{5}})^i + \frac{1}{\sqrt{5}}(\frac{2}{3-\sqrt{5}})^i \\
[Z^i]G(Z) =& \frac{-1}{\sqrt{5}}(\frac{2}{3+\sqrt{5}})^i + \frac{1}{\sqrt{5}}(\frac{2}{3-\sqrt{5}})^i
\end{aligned}
$$

5.2 Problem 8

What is $[z^n](\frac{(ln(1-z))^2}{(1-z)^{m+1}})$
Solve:

$$
\begin{aligned}
p =& (1-z)^{-x-1} \\
\frac{d^2p}{dx^2} =& (\frac{(ln(1-z))^2}{(1-z)^{x+1}})
\end{aligned}
$$

Now, if G(z)=$\frac{1}{(1-z)^{x+1}}$

$$
\begin{aligned}
G(z) =& \sum_x^{x+k} z^k \\
[Z^k]G(Z) =& \binom{x+k}{x} \\
=& \frac{(x+k)!}{x!k!} \\
=& \frac{(x+k)(x+k-1)...(x+1)x!}{x!kI} \\
=& \frac{1}{k!}[(x+k)(x+k-1)...(x+1)]
\end{aligned}
$$

Let's find $\frac{d}{dx}(x+1)(x+2)...(x+k)$
Let,

$$\begin{aligned} f(x) =&(x+1)(x+2)...(x+k) \\ lnf(x) =&ln(x+1)+...+ln(x+k) \\ \frac{f'(x)}{f(x)} =&\frac{1}{x+1}+\frac{1}{x+2}+...+\frac{1}{x+k} \\ f'(x) =&(x+1)(x+2)...(x+k)[H_{x+k}-H_x] \\ Z^k[G'(z)] =&\frac{(x+1)(x+2)...(x+k)}{k!}[H_{x+k}-H_x] \\ =&\binom{x+k}{k}[H_{x+k}-H_x] \end{aligned}$$

Similarly,

$$\begin{aligned} f''(x) =&-[\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+...+\frac{1}{(x+k)^2}] \\ =&[H_x^2-H_{x+k}^2] \\ Z^k[G''(z)] =&\binom{x+k}{k}[H_x^2-H_{x+k}^2]+[H_{x+k}-H_x]^2 \end{aligned}$$

5.3 Problem 21

A robber holds up a bank and demands \$500 in tens and twenties. He also demands to know the number of ways in which the cashier can give him the money. Find a generating function G(z) for which this number is $[z^{500}]G(z)$, and a more compact generating function $\hat{G}$(z) for which this number is $[z^{50}]\hat{G}(z)$. Determine the required number of ways by (a) using partial fractions; (b) using a method like (7.39).
Solve:
With tens and twenties, the number of ways amounts can be made is found after finding their generating function. Let C(x) denote the generating function.

$$\begin{aligned} C(x) =&(1+x^{10}+x^{20}+x^{30}+...)(1+x^{20}+x^{40}+x^{60}+...) \\ =&\frac{1}{1-x^{10}}\cdot\frac{1}{1-x^{20}} \\ Let, & \\ \hat{C}(x) =&\frac{1}{(1-x)(1-x^2)} \\ =&\frac{1}{(1+x)(1-x)^2} \end{aligned}$$

Now,

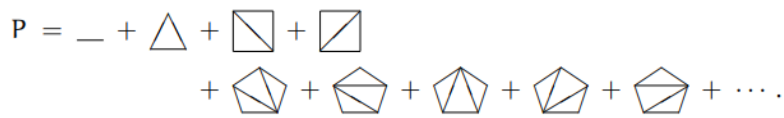
$$\frac{1}{(1+x)(1-x)^2} =\frac{1}{2}\frac{1}{(1-z)^2}+\frac{1}{4(z-1)}+\frac{1}{4(z+1)}$$

We get, the coefficients of $Z^n$from each of the above term as follows:

$$[Z^n]G\hat{(z)} = \frac{(n+1)}{2}+\frac{1}{4}+\frac{(-1)^n}{4}.$$

5.4 Problem 22

(The first term represents a degenerate polygon with only two vertices; every other term shows a polygon that has been divided into triangles. For example, a pentagon can be triangulated in five ways.) Define a multiplication operation $A \triangle B$ on triangulated polygons A and B so that the equation:
P = $—P \triangle P$ is valid. Then replace each triangle with 'z'; what does this tell you about the number of ways to decompose an n-gon into triangles?
Solve:



Each polygon has a base (the line segment at the bottom). If A and B are triangulated polygons, let A be the result of pasting the base of A to the upper left diagonal of $\triangle$, and pasting the base of B to the upper right diagonal.

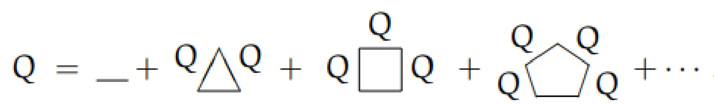
Replacing each triangle by z gives a power series in which the coefficient of Z^n is the number of triangulations with n triangles, namely the number of ways to decompose an (n + 2)-gon into triangles. Since P = $1 + zP^2$, this is the generating function for Catalan numbers $C_0 + C_1 z + C_2 z^2 +$; the number of ways to triangulate an n-gon is:

$$C_{(n2)} = \frac{1}{n-1}\binom{2n-4}{n-2} \tag{5.1}$$

5.5 Problem 50

(Continuing exercise 22, consider the sum of all ways to decompose polygons into polygons: Find a symbolic equation for Q and use it to find a generating function for the number of ways to draw non-intersecting diagonals inside a convex n-gon. (Give a closed form for the generating function as a function of z; you need not find a closed form for the coefficients.)
Solve:

Similar to as the previous problem, we can represent this particular problem in visual way like this:



We see from the above figure that, the generating function is simply:

$$\begin{aligned} Q =& 1 + zQ^2 + z^2Q^3 + ... \\ =& 1 + \frac{zQ^2}{1 - zQ} \end{aligned}$$

This is essentially the generating function for the n-gon problem.
$Q - ZQ^2 = 1 - ZQ + ZQ^2$
Solving the equation, we get the value of Q as root:

$$Q = \frac{1 + Z - \sqrt{1 - 6Z + Z^2}}{4Z}.$$