Review for Precalculus of Midterm Exam

I. TOOLKITS

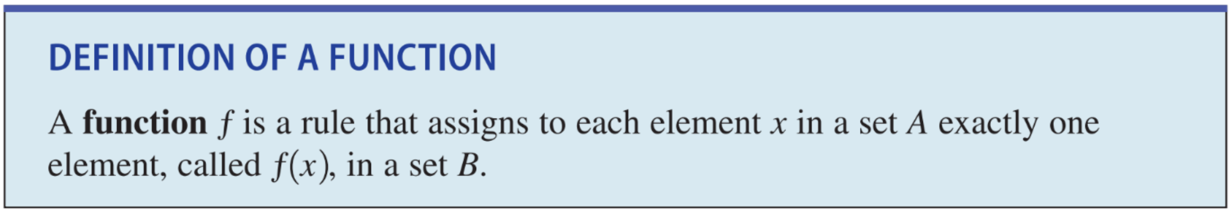
本次期中考试范围包括：

1. Determine the average rates of change for sequences and functions, including linear, quadratic, and other function types.
2. Apply combinations of vertical and horizontal translation and dilation.
3. Know properties of polynomial functions from graphs and description (range, intercepts, increasing/decreasing intervals, relative max/min, even/odd, concavity, point of inflection, maximum number of turning points).
4. Understand factor theorem, remainder theorem and be familiar with long division method.
5. Solve the comprehensive problems of rational functions (x-intercepts, holes, vertical, horizontal or slant asymptotes).
6. Solve the polynomial or rational inequalities using geometrical or algebraic ways.
7. Understand the properties of complex numbers, or use conjugate pairs theorem to solve complex zeros problems.
8. Construct a linear, quadratic or cubic polynomial regression, and answer questions about a data set or contextual.
9. Know the general formula of arithmetic or geometric progression or rewrite the repeated product of binomials using the binomial theorem.

题型包括 25 道选择题（不允许用计算器），三道大题（两道不允许用计算器，一道允许）。

有以下内容要求同学们掌握：

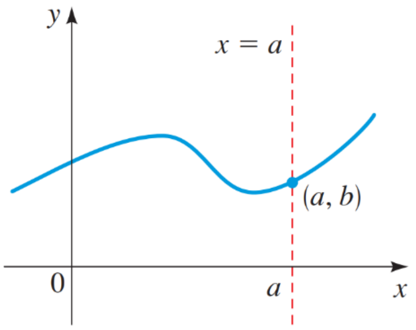
1. Definition of Functions



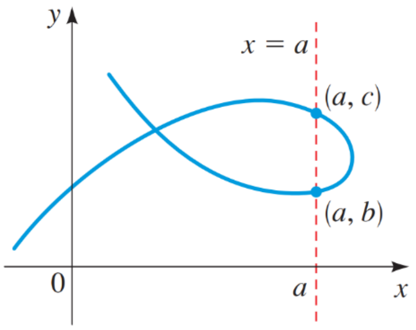
DEFINITION OF A FUNCTION

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

Use the vertical line test to determine which graphs represent functions.

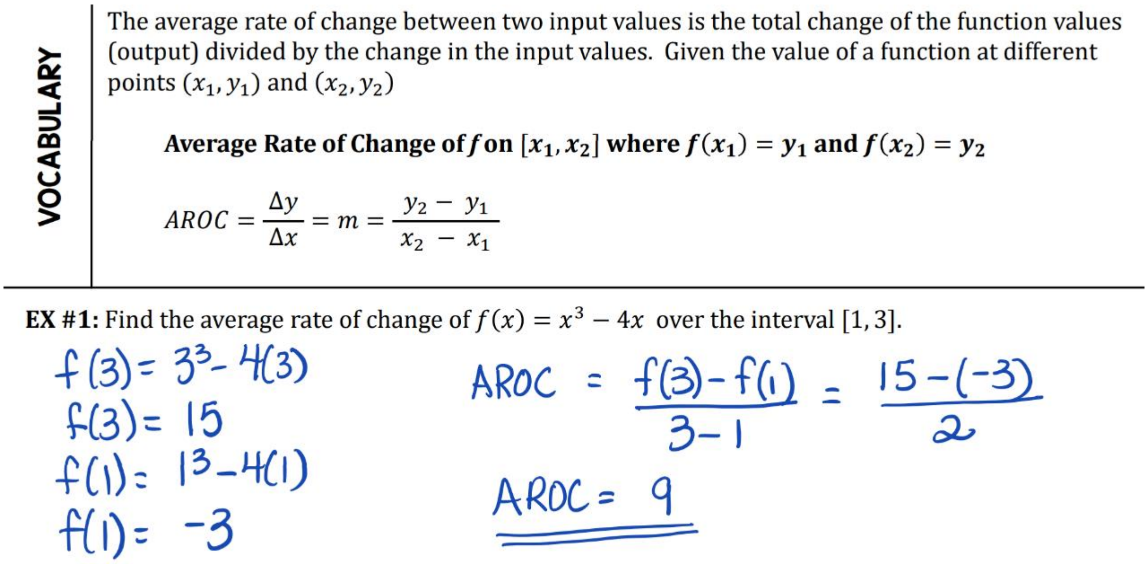


Graph of a function



Not a graph of a function

2. Average Rate of Change (AROC)



VOCABULARY

The average rate of change between two input values is the total change of the function values (output) divided by the change in the input values. Given the value of a function at different points (x_1, y_1) and (x_2, y_2)

Average Rate of Change of f on $[x_1, x_2]$ where $f(x_1) = y_1$ and $f(x_2) = y_2$

$$AROC = \frac{\Delta y}{\Delta x} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

EX #1: Find the average rate of change of $f(x) = x^3 - 4x$ over the interval $[1, 3]$.

$f(3) = 3^3 - 4(3)$

$f(3) = 15$

$f(1) = 1^3 - 4(1)$

$f(1) = -3$

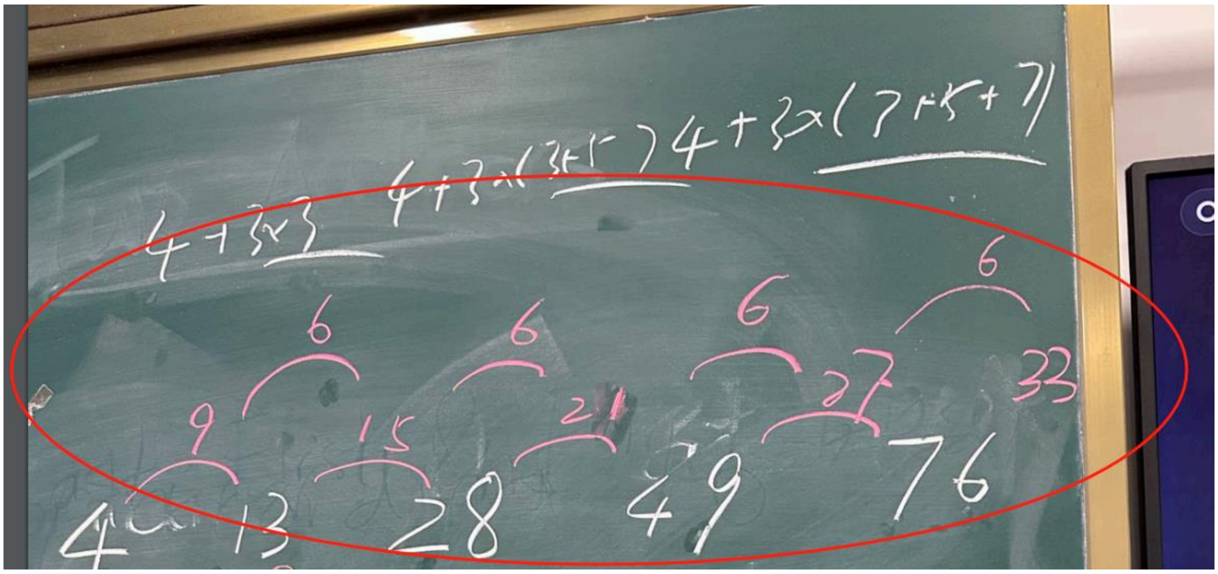
$AROC = \frac{f(3) - f(1)}{3 - 1} = \frac{15 - (-3)}{2}$

$\underline{\underline{AROC = 9}}$

3. ★★

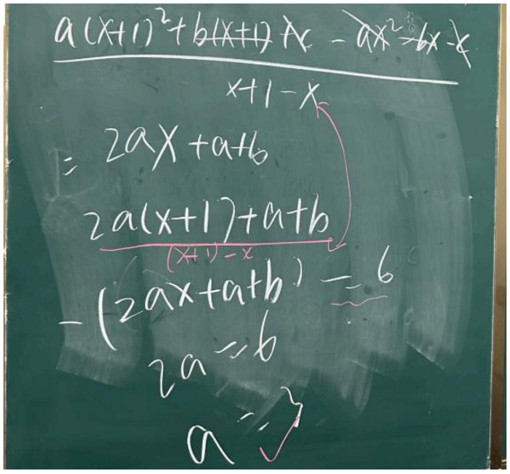
A polynomial function of degree n models data sets or contextual scenarios that demonstrate roughly constant nonzero nth differences. (给一组数据，作差判断 type of function；如果作一次差得到常数，则为 linear model；如果作两次差得到常数，则为 quadratic function；依次类推。)

EX#2 Try to determine the general formula when the first several terms are 4, 13, 28, 49, 76.



Hence, it is a quadratic formula, and then the following methods:

M1 (using AROC)



$$\frac{a(x+1)^2+b(x+1)+c-ax^2-bx-c}{x+1-x}$$

$$=2ax+a+b$$

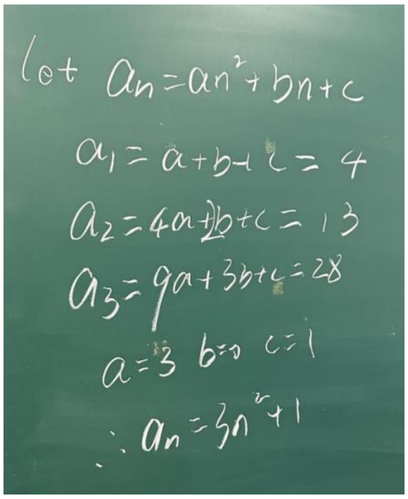
$$\frac{2a(x+1)+a+b}{(x+1)-x}$$

$$-(2ax+a+b)=6$$

$$2a=6$$

$$a=3$$

M2 (using quadratic function formula)



let $a_n=an^2+bn+c$

$a_1=a+b+c=4$

$a_2=4a+2b+c=13$

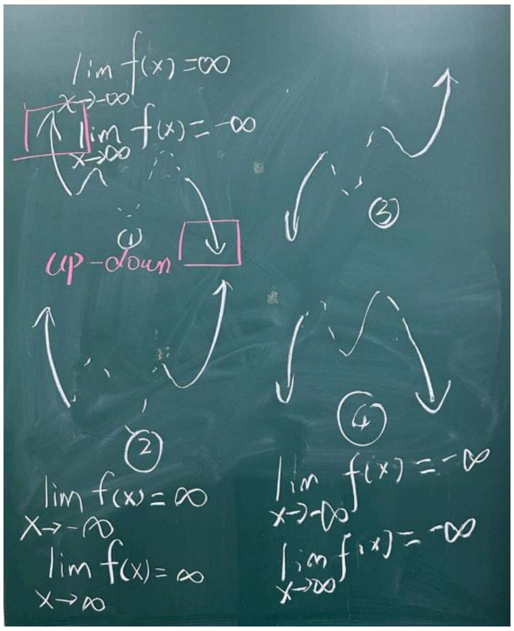
$a_3=9a+3b+c=28$

$a=3$ $b=0$ $c=1$

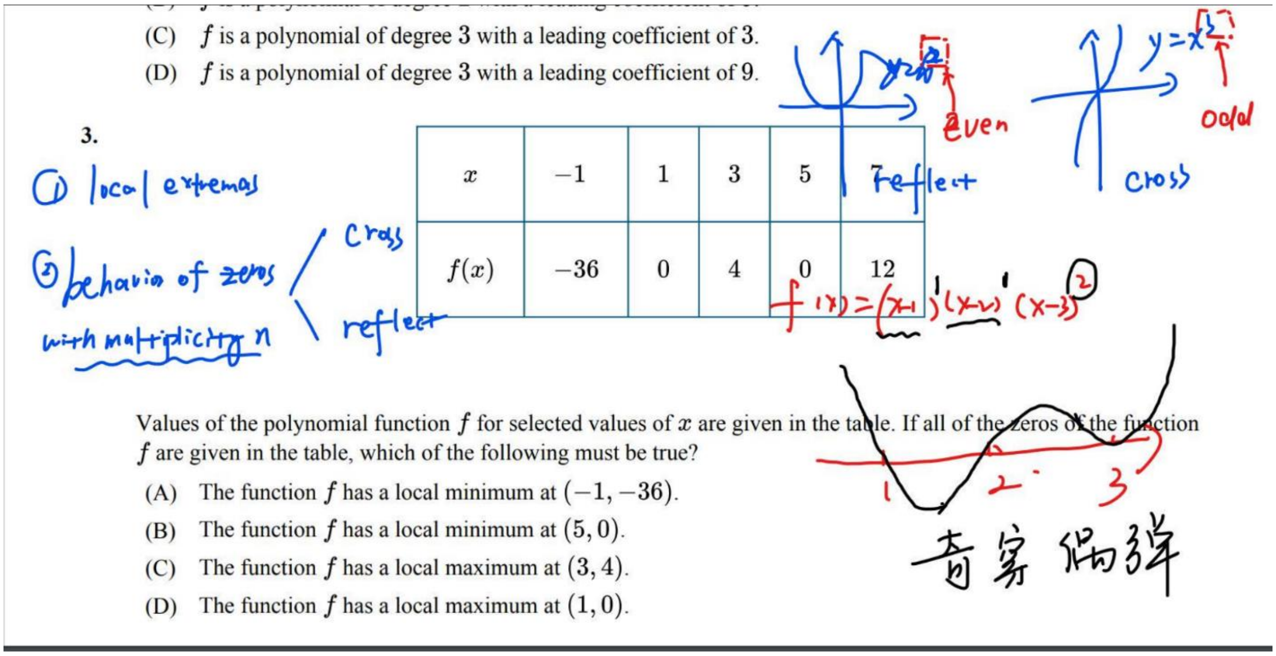
$\therefore a_n=3n^2+1$

4. End behavior and the graph near zeros. ★★

1) End Behavior



2) Zero Behavior



(C) f is a polynomial of degree 3 with a leading coefficient of 3.

(D) f is a polynomial of degree 3 with a leading coefficient of 9.

3.

x	-1	1	3	5	7
$f(x)$	-36	0	4	0	12

Values of the polynomial function f for selected values of x are given in the table. If all of the zeros of the function f are given in the table, which of the following must be true?

(A) The function f has a local minimum at $(-1, -36)$.

(B) The function f has a local minimum at $(5, 0)$.

(C) The function f has a local maximum at $(3, 4)$.

(D) The function f has a local maximum at $(1, 0)$.

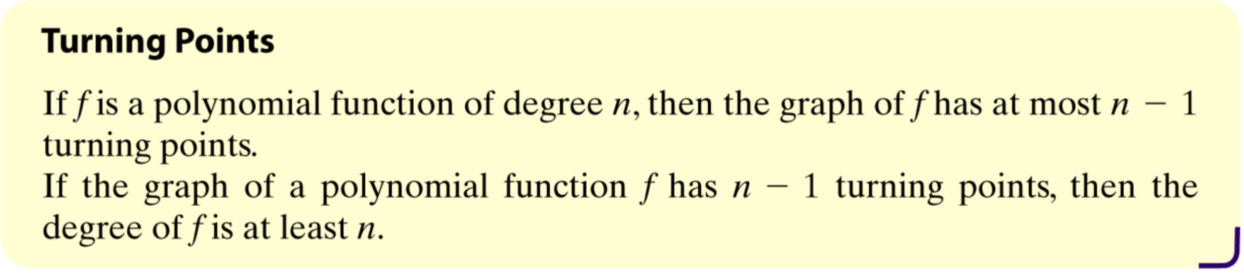
① local extremas

② behavior of zeros with multiplicity n — cross / reflect

$f(x)=(x-1)^1(x-2)^1(x-3)^{②}$

奇穿偶弹

5.



Turning Points

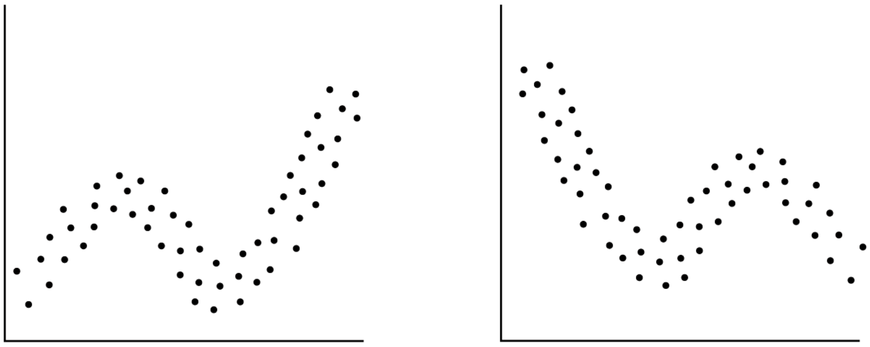
If f is a polynomial function of degree n, then the graph of f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, then the degree of f is at least n.

6. ★★

Construct a linear, quadratic, cubic, quartic, polynomial of degree n.

For example, build cubic models from data.



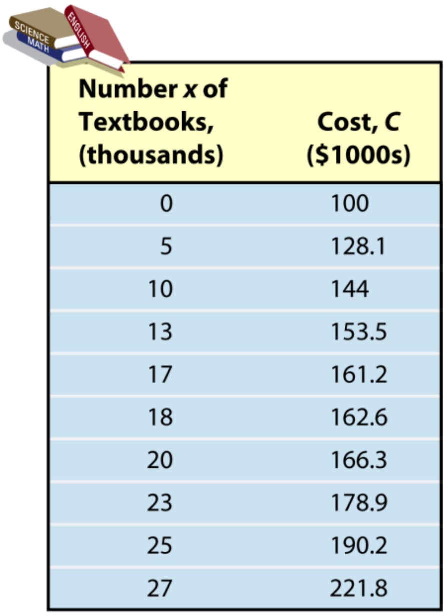
$y = ax^3 + bx^2 + cx + d,\ a > 0$

(a)

$y = ax^3 + bx^2 + cx + d,\ a < 0$

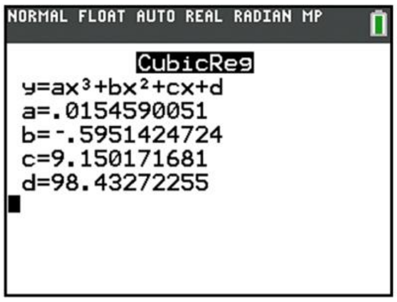
(b)

Given that the data is as followed, can you find the functions by using your graphing calculator?



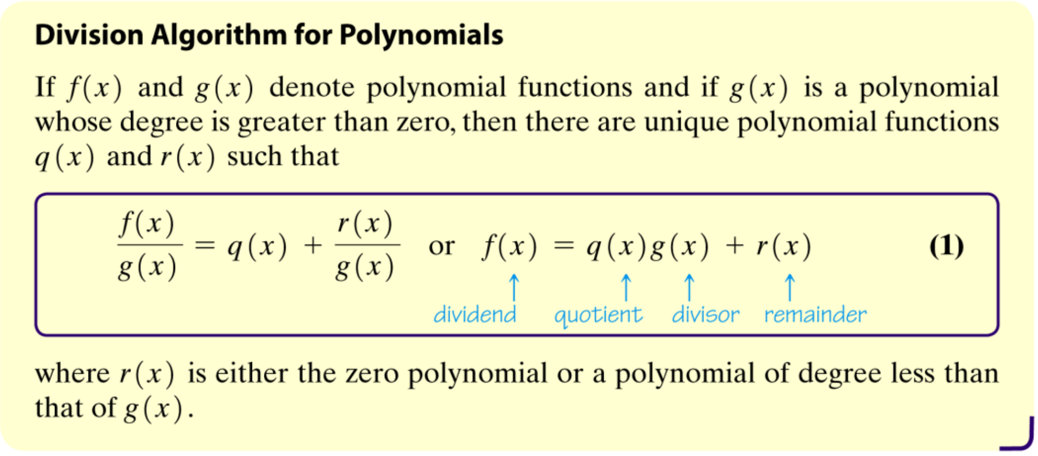
Number *x* of Textbooks, (thousands)	Cost, *C* ($1000s)
0	100
5	128.1
10	144
13	153.5
17	161.2
18	162.6
20	166.3
23	178.9
25	190.2
27	221.8

The answer is as followed.



```
NORMAL FLOAT AUTO REAL RADIAN MP
          CubicReg
 y=ax³+bx²+cx+d
 a=.0154590051
 b=-.5951424724
 c=9.150171681
 d=98.43272255
```

7. Remainder Theorem, Factor Theorem and Long Division Method. ★★



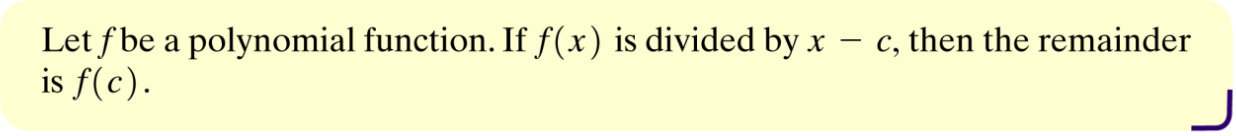
Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\substack{\uparrow\\ \text{dividend}}}{f(x)} = \underset{\substack{\uparrow\\ \text{quotient}}}{q(x)}\,\underset{\substack{\uparrow\\ \text{divisor}}}{g(x)} + \underset{\substack{\uparrow\\ \text{remainder}}}{r(x)} \qquad \textbf{(1)}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

1) Remainder theorem



Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

For example, Let P(x) be a polynomial such that when P(x) is divided by x-19, the remainder is 99, and when P(x) is divided by x-99, the remainder is 19. What is the remainder when P(x) is divided by (x-

19)(x-99).

Key: -x+118.

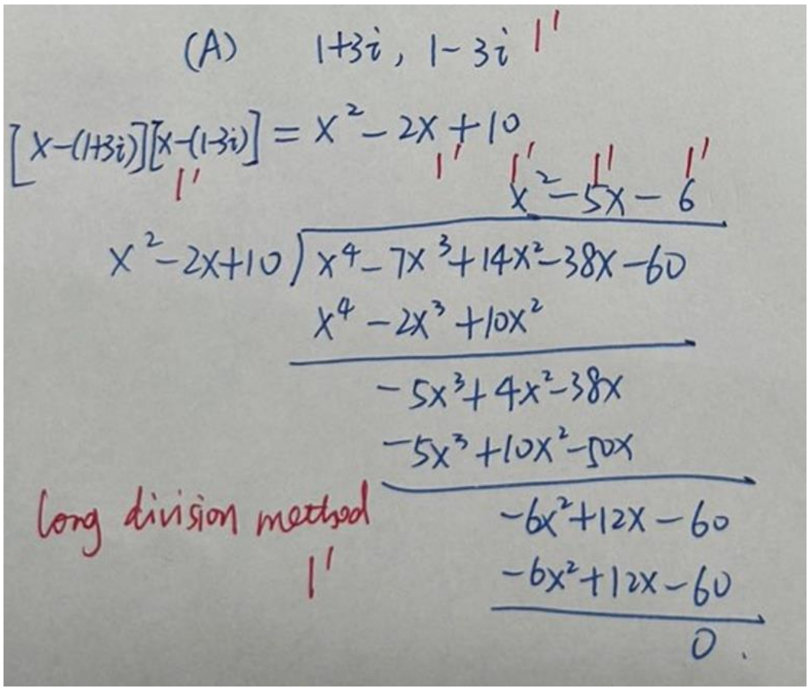
2) Factor theorem

Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

For example, Test2(B 卷)

Use the given zero $1 + 3i$ to find the remaining zeros of the polynomial function $f(x) = x^4 - 7x^3 + 14x^2 - 38x - 60$. [9 points]



(A) $1+3i$, $1-3i$ 1'

$[x-(1+3i)][x-(1-3i)] = x^2-2x+10$ 1' 1'

$x^2 - 5x - 6$ 1' 1' 1'

$$x^2-2x+10 \overline{)\, x^4-7x^3+14x^2-38x-60}$$

$x^4-2x^3+10x^2$

$-5x^3+4x^2-38x$

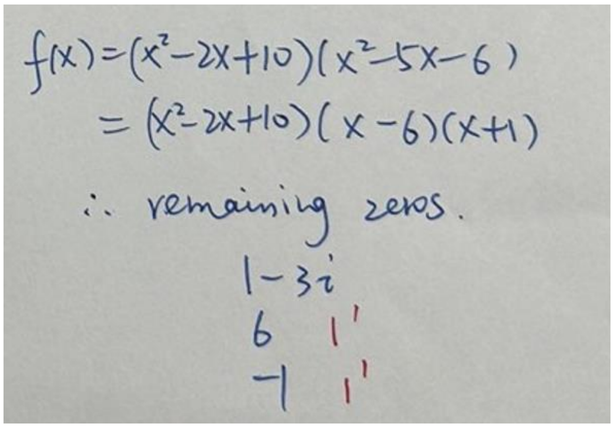
$-5x^3+10x^2-50x$

$-6x^2+12x-60$

$-6x^2+12x-60$

$0.$

long division method 1'



$f(x)=(x^2-2x+10)(x^2-5x-6)$

$=(x^2-2x+10)(x-6)(x+1)$

∴ remaining zeros.

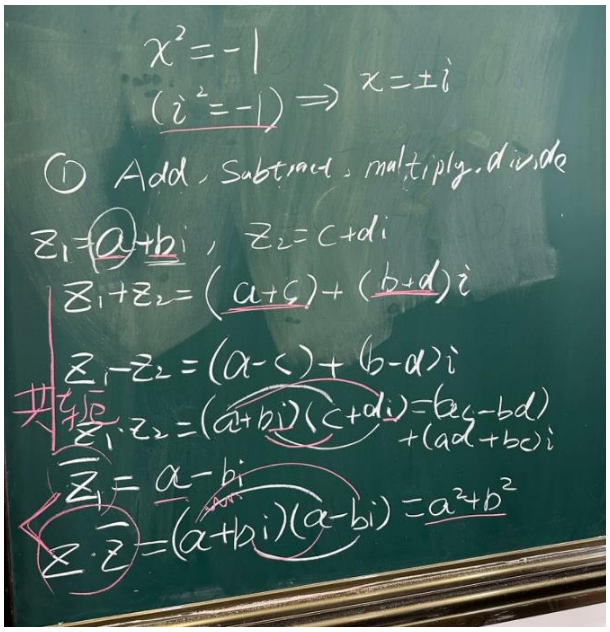
$1-3i$

6 1'

-1 1'

8. Complex Numbers ★★

1) Arithmetic operations on complex numbers



$x^2=-1$

$(i^2=-1) \Rightarrow x=\pm i$

① Add, Subtract, multiply, divide

$z_1=a+bi$, $z_2=c+di$

$z_1+z_2=(a+c)+(b+d)i$

$z_1-z_2=(a-c)+(b-d)i$

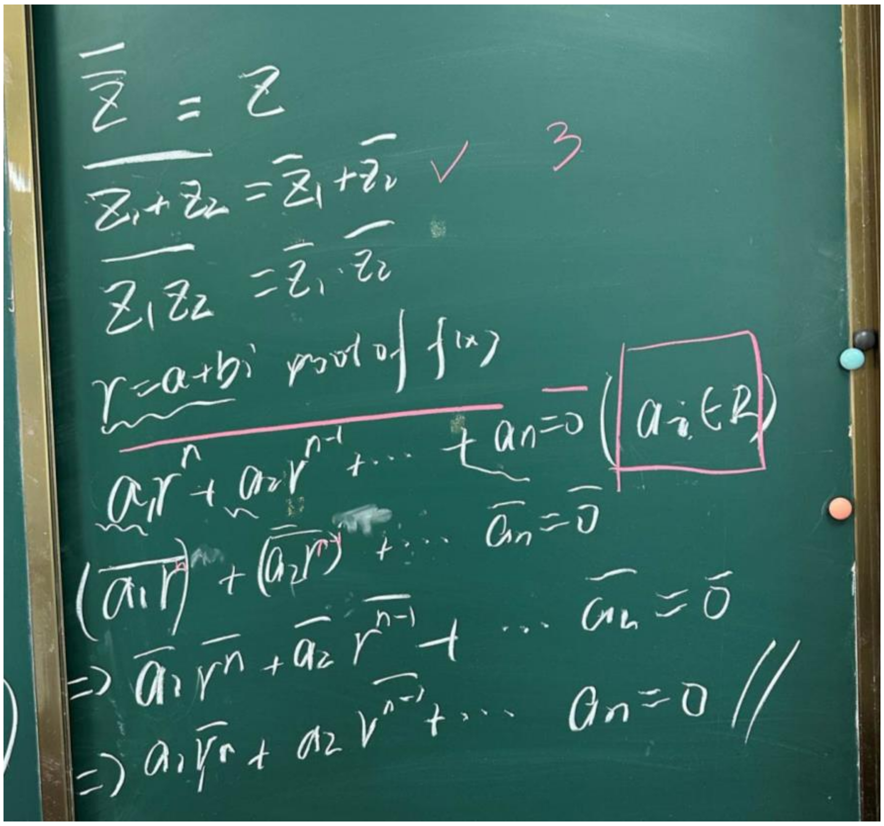
共轭

$z_1\cdot z_2=(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

$\bar{z}_1=a-bi$

$z\cdot\bar{z}=(a+bi)(a-bi)=a^2+b^2$

2) Properties of complex numbers



$\bar{\bar{z}}=z$

$\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$ ✓ 3

$\overline{z_1 z_2}=\bar{z}_1\cdot\bar{z}_2$

$r=a+bi$ root of $f(x)$

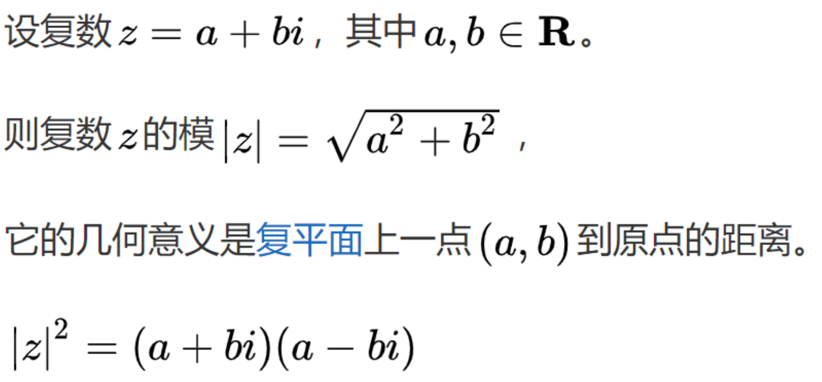
$\overline{a_1r^n+a_2r^{n-1}+\dots+a_n}=\bar{0}$ $\left(\boxed{a_i\in R}\right)$

$\overline{(a_1r^n)}+\overline{(a_2r^{n-1})}+\dots\overline{a_n}=\bar{0}$

$\Rightarrow \bar{a}_1\bar{r}^n+\bar{a}_2\bar{r}^{n-1}+\dots\bar{a}_n=\bar{0}$

$\Rightarrow a_1\bar{r}^n+a_2\bar{r}^{n-1}+\dots a_n=0$ //

3) Modulus of Complex Numbers



设复数 $z=a+bi$，其中 $a,b\in\mathbf{R}$。

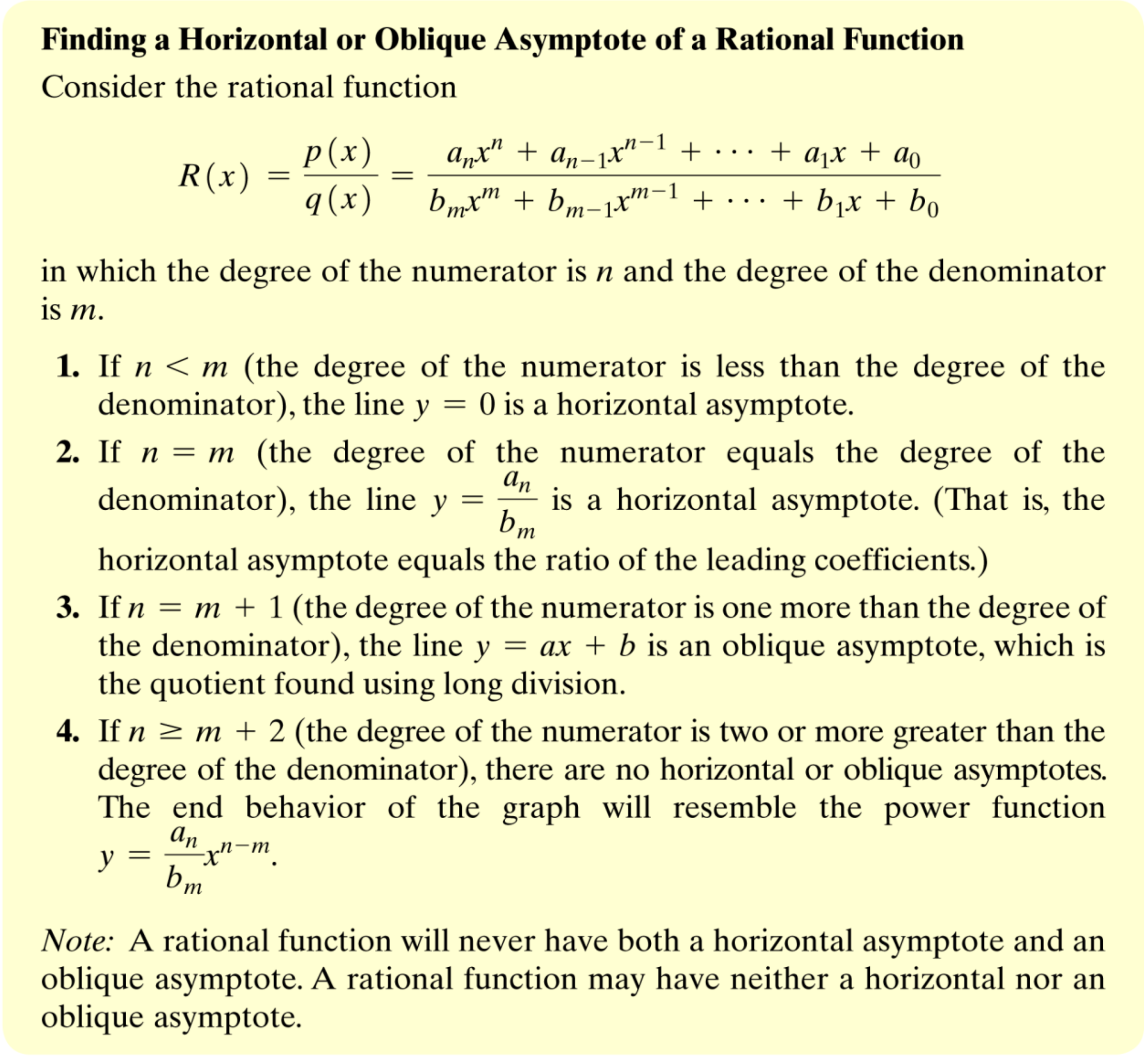
则复数 z 的模 $|z|=\sqrt{a^2+b^2}$，

它的几何意义是复平面上一点 (a,b) 到原点的距离。

$$|z|^2=(a+bi)(a-bi)$$

Hence $|z|^2=z\bar{z}\neq z^2$

9. Intercepts, Holes, Vertical asymptotes, Horizontal or Slant asymptotes of Rational Functions. ★★★



Finding a Horizontal or Oblique Asymptote of a Rational Function

Consider the rational function

$$R(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1 x+a_0}{b_m x^m+b_{m-1}x^{m-1}+\cdots+b_1 x+b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n<m$ (the degree of the numerator is less than the degree of the denominator), the line $y=0$ is a horizontal asymptote.
2. If $n=m$ (the degree of the numerator equals the degree of the denominator), the line $y=\frac{a_n}{b_m}$ is a horizontal asymptote. (That is, the horizontal asymptote equals the ratio of the leading coefficients.)
3. If $n=m+1$ (the degree of the numerator is one more than the degree of the denominator), the line $y=ax+b$ is an oblique asymptote, which is the quotient found using long division.
4. If $n\geq m+2$ (the degree of the numerator is two or more greater than the degree of the denominator), there are no horizontal or oblique asymptotes. The end behavior of the graph will resemble the power function $y=\frac{a_n}{b_m}x^{n-m}$.

Note: A rational function will never have both a horizontal asymptote and an oblique asymptote. A rational function may have neither a horizontal nor an oblique asymptote.

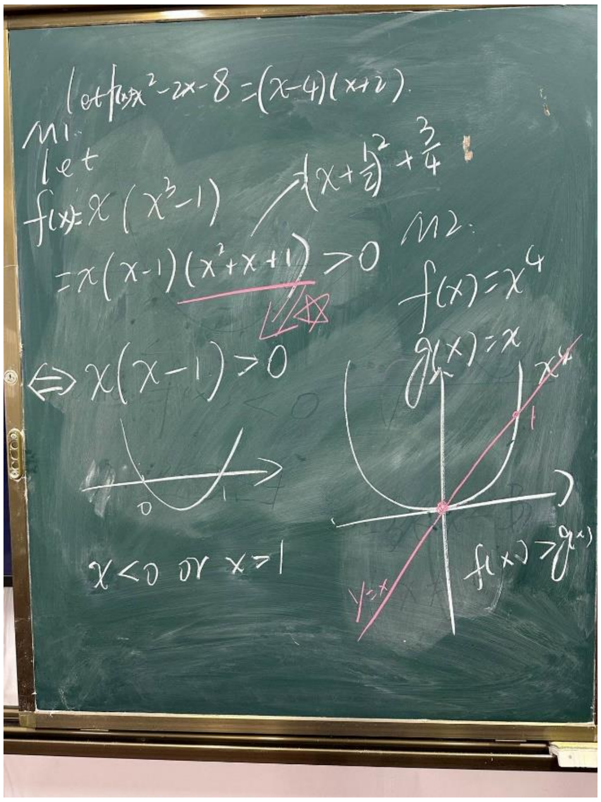
10. Solve polynomial and rational inequalities ★★★

1) polynomial inequalities

For example,

How to Solve a Polynomial Inequality Algebraically

Solve the inequality $x^4 > x$ algebraically, and graph the solution set.



let $f(x)=x^2-2x-8=(x-4)(x+2)$.

M1:

let

$f(x)=x(x^3-1)$

$=x(x-1)(x^2+x+1)>0$

$(x+\frac{1}{2})^2+\frac{3}{4}$

$\Leftrightarrow x(x-1)>0$

$x<0$ or $x>1$

M2:

$f(x)=x^4$

$g(x)=x$

$f(x)>g(x)$

2) Rational Inequalities

$$\frac{f(x)}{g(x)}>0 \Leftrightarrow f(x)g(x)>0$$

$$\frac{f(x)}{g(x)}\geq 0 \Leftrightarrow \begin{cases} f(x)g(x)\geq 0 \\ g(x)\neq 0 \end{cases}$$

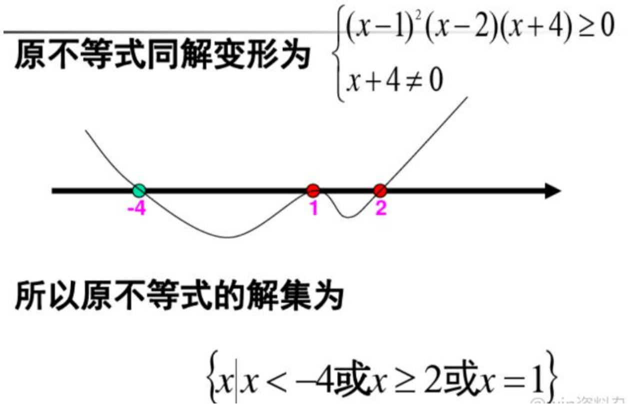
$$\frac{f(x)}{g(x)}<0 \Leftrightarrow f(x)g(x)<0$$

$$\frac{f(x)}{g(x)}\leq 0 \Leftrightarrow \begin{cases} f(x)g(x)\leq 0 \\ g(x)\neq 0 \end{cases}$$

EX1#

Solve $\frac{(x-1)^2(x-2)}{(x+4)}\geq 0$.

Key:



原不等式同解变形为 $\begin{cases}(x-1)^2(x-2)(x+4)\geq 0\\ x+4\neq 0\end{cases}$

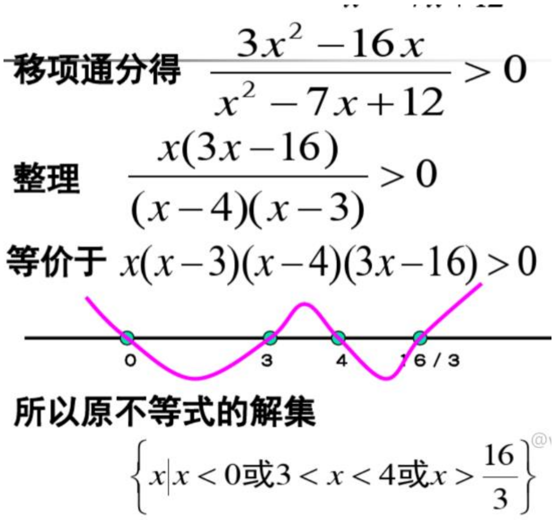
所以原不等式的解集为

$$\{x|x<-4或x\geq 2或x=1\}$$

EX2#

Solve $\dfrac{x^2-2x-24}{x^2-7x+12}>-2$

Key:



移项通分得 $\dfrac{3x^2-16x}{x^2-7x+12}>0$

整理 $\dfrac{x(3x-16)}{(x-4)(x-3)}>0$

等价于 $x(x-3)(x-4)(3x-16)>0$

所以原不等式的解集

$$\left\{x|x<0或3<x<4或x>\frac{16}{3}\right\}$$

11. Transformations★★★

分别研究af(bx+c)+d对横坐标和纵坐标影响；a, d只对纵坐标产生影响，b, c只对横坐标产生影响。

1）函数解析式描述，以从f(x)到f(2x+$\frac{\pi}{2}$)为例，可以表述为f(x)→f(2x)→f(2(x+ $\frac{\pi}{4}$)); 也可以表达为f(x) →f(x+ $\frac{\pi}{2}$)→f(2x+ $\frac{\pi}{2}$);

2) 点描述，假设（$\frac{\pi}{2}$，1）在f(x)上，则f(x)→f(2x)→f(2(x+ $\frac{\pi}{4}$))对应的点坐标变化是（$\frac{\pi}{2}$，1） →（$\frac{\pi}{4}$，1） →（0，1）；或者f(x) →f(x+ $\frac{\pi}{2}$)→f(2x+ $\frac{\pi}{2}$)对应的点坐标变化是（$\frac{\pi}{2}$，1） →（0，1） →（0，1）；

3）变量替换；因为纵坐标始终不变，只有横坐标改变；假设（$\frac{\pi}{2}$，1）在f(x)上，设2x+$\frac{\pi}{2}$= $\frac{\pi}{2}$，则x=0, 则新函数坐标为（0，1）；

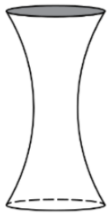
4）图形描述+语言描述，从f(x)到f(2x+$\frac{\pi}{2}$)，先水平压缩一倍（a horizontal dilation by a factor of 2), 再向左移动$\frac{\pi}{4}$（a horizontal translation of $\frac{\pi}{4}$ to the left）或者先向左移动$\frac{\pi}{2}$（a horizontal translation of $\frac{\pi}{2}$ to the left）再水平压缩一倍（a horizontal dilation by a factor of 2)

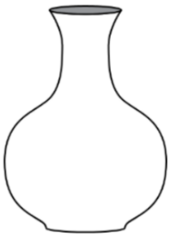
12. Function modeling，重点是实操！请看数学组分享的实操视频★★

II. SAMPLE PROBLMES

1.

Water is poured into an empty vase at a constant rate. A graph (not shown) models the depth of the water in the vase over time. The graph can be described as follows: the graph is always increasing; the first portion of the graph is clearly concave up; and the next portion of the graph has a fairly steady and steep increase. Which of the following vases is appropriate for the context described by the graph?

Ⓐ 

Ⓒ 

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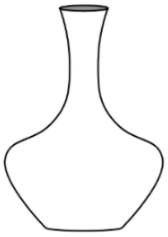
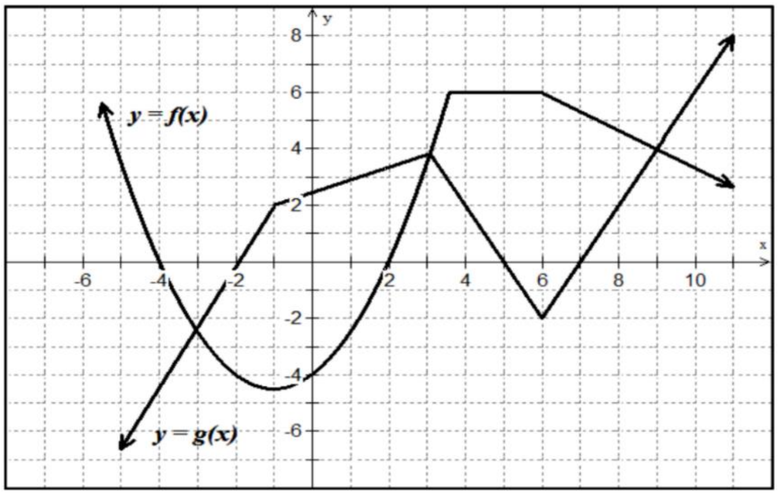
Ⓓ 

2.

Use the graphs of functions $f(x)$ and $g(x)$, shown at right, to answer the following.



A.

The value(s) of x such that

$f(x) = 0$

$g(x) = 0$

B.

The value(s) of x such that $f(x) = g(x)$

C.

The value(s) of x such that $f(x) < 0$

3.

Suppose $(1, 3)$ is a point on the graph of $y = f(x)$.
(a) What point is on the graph of $y = f(x + 3) - 5$?
(b) What point is on the graph of $y = -2f(x - 2) + 1$?
(c) What point is on the graph of $y = f(2x + 3)$?

4.

The function f has domain $[-2, 2]$ and range $[1, 5]$. The function g is given by $g(x) = -2f(x+3) + 4$. What are the domain and range of g ?

(A) domain: $[-5, -1]$, range: $[-6, 2]$

(B) domain: $[-5, -1]$, range: $[-2, 6]$

(C) domain: $[1, 5]$, range: $[-2, 6]$

(D) domain: $[1, 5]$, range: $[-6, 2]$

5.

The function $f(x) = ax^3 + 7x^2 + bx - 8$, where a and b are constants, is such that $2x + 1$ is a factor. Th remainder when $f(x)$ is divided by $x + 1$ is 7.

a Find the value of a and the value of b.

b Factorise $f(x)$ completely.

6 .

The function f is given by $f(x) = (x+3)^4$. When f is rewritten in the form $f(x) = x^4 + ax^3 + bx^2 + cx + d$, which of the following values is greatest?

(A) a

(B) b

(C) c

(D) d

7.

The graph of which of the following functions in the xy-plane has at least one x-intercept, at least one hole, at least one vertical asymptote, and a horizontal asymptote?

(A) $f(x) = \frac{x^2-16}{x^2-x-6}$

(B) $f(x) = \frac{x^2-16}{x^2-x-30}$

(C) $f(x) = \frac{x^2-4}{x^2-x-30}$

(D) $f(x) = \frac{x^2-4}{x^2-x-6}$

8.

Year	2011	2012	2013	2014	2015	2016	2017
US Federal Education Spending(billions of dollars)	112.8	109.3	105.1	104.5	99	99.3	97.7

The table gives amounts of United States federal education spending, in billions of dollars, for selected years. A linear regression is used to construct a function model S that models the spending, in billions of dollars, over the given years. If $t = 1$ corresponds to 2011, $t = 2$ corresponds to 2012, and this pattern continues, which of the following defines function S ?

(A) $S(t) = -2.55t + 114.157$

(B) $S(t) = -2.55t + 5239.657$

(C) $S(t) = -8.099t + 113.820$

(D) $S(t) = -0.369t + 42.308$

9.

Cars and trucks lose value the more they are driven. Is it possible to predict the price of a Ford F-150 SuperCrew 4 x 4 if we know how many miles are on the odometer? A random sample of 16 used Ford F-150 SuperCrew 4 x 4s was selected from among those listed at autotrader.com. The number of miles driven and the price (in dollars) were recorded for each truck as shown in the table below.

Miles driven	70,583	129,484	29,932	29,953	24,495	75,678	8359	4447
Price (in dollars)	21,994	9500	29,875	41,995	41,995	28,986	31,891	37,991
Miles driven	34,077	58,023	44,447	68,474	144,162	140,776	29,397	131,385
Price (in dollars)	34,995	29,988	22,896	33,961	16,883	20,897	27,495	13,997

a) Create a scatter plot on your calculator. Describe any patterns that you see.

b) Use your calculator to determine a linear regression model. Record your model here.

c) Graph your model over the scatter plot. How well does the data fit the model?

d) Interpret the slope and *y*-intercept in the context of this problem.

10. Bonus(选做)

Let z be a complex number satisfying

$$12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31.$$

What is the value of $z + \frac{6}{z}$?

(A) -2

(B) -1

(C) $\frac{1}{2}$

(D) 1

(E) 4