1. The Introduction is a bit long, I would suggest making a new title or a sub–title for the introduction of ODE and PDE, or maybe reduce the length of introduction for DEs since the main topic here (I suppose) is continuum mechanics instead of DEs.
2. Draw a diagram for the physical set up for your problem would be helpful.
3. Explain the $\vec{e}$ when you define the vector for the length of the string.
4. Better state that you are integrating the Lagrangian w.r.t time to get the action, you integrate the kinetic and potential energy w.r.t. time separately would makes the readers confused.
5. How did the x term in (9) get eliminated to the next line and how did you eliminate f''(u_x)?
6. Explain what is ϕ when you introduce equation 10 and how it related to u that you originally have.
7. For the dimensional analysis, it will be better to add a subscript to the tension because it may confuse the Symbol of time.
8. How do you get equation 11 from equation 10, since c^2 and v^2 does not equal to each other, so it's confusing.
9. You may add a note that \mu is the mass density for the string.
10. More computation could be done with different constants or initial conditions and explain how would different initial conditions lead to different motions: one–harmonic motion, etc.
11. Could leave some ideas or thoughts about the Three–Dimensional Wave Equation or even the n–Dimensional Wave Equation at the end.