Student Projects in Differential Equations

http://online.redwoods.edu/instruct/darnold/deproj/index.htm

A Double Spring Pendulum

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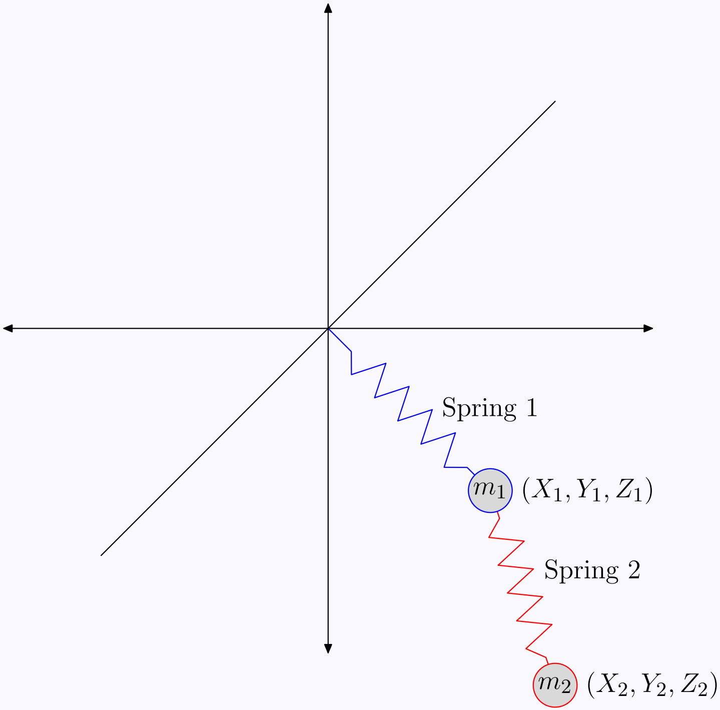
The Model



Getting Started with Vectors

What we know:

- The force on mass 1 is always in the direction of the origin and mass 2
- The force on mass 2 is always in the direction of mass 1
- Gravity always pulls in the -$\hat{\boldsymbol{\kappa}}$ direction

What we need:

- A unit vector pointing from the origin to mass 1

$$r_1 = (X_1\hat{\boldsymbol{\imath}} + Y_1\hat{\boldsymbol{\jmath}} + Z_1\hat{\boldsymbol{\kappa}})$$

$$\|r_1\| = \sqrt{X_1^2 + Y_1^2 + Z_1^2}$$

$$\hat{r_1} = \frac{r_1}{\|r_1\|} = \frac{(X_1\hat{\boldsymbol{\imath}} + Y_1\hat{\boldsymbol{\jmath}} + Z_1\hat{\boldsymbol{\kappa}})}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}}$$

- Unit vectors pointing from mass 2 to mass 1 and mass 1 to mass 2

$$r_2 = ((X_1 - X_2)\hat{\boldsymbol{\imath}} + (Y_1 - Y_2)\hat{\boldsymbol{\jmath}} + (Z_1 - Z_2)\hat{\boldsymbol{\kappa}})$$

$$\|r_2\| = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}$$

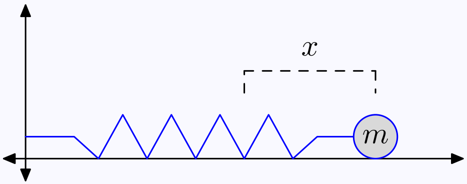
$$\hat{r_2} = \frac{r_2}{\|r_2\|} = \frac{((X_1 - X_2)\hat{\boldsymbol{\imath}} + (Y_1 - Y_2)\hat{\boldsymbol{\jmath}} + (Z_1 - Z_2)\hat{\boldsymbol{\kappa}})}{\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}}$$

$$r_3 = ((X_2 - X_1)\hat{\boldsymbol{\imath}} + (Y_2 - Y_1)\hat{\boldsymbol{\jmath}} + (Z_2 - Z_1)\hat{\boldsymbol{\kappa}})$$

$$\|r_3\| = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

$$\hat{r_3} = \frac{r_3}{\|r_3\|} = \frac{((X_2 - X_1)\hat{\boldsymbol{\imath}} + (Y_2 - Y_1)\hat{\boldsymbol{\jmath}} + (Z_2 - Z_1)\hat{\boldsymbol{\kappa}})}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}}$$

Getting the Forces



Notice that $\|r_1\|$ is the length of spring 1, stretched or compressed. The force of spring 1 on mass 1 follows the equation $F = -kx$, where x is the displacement of the string from un-stretched.

$$F_1 = -k_1(\|r_1\| - L_1)\frac{r_1}{\|r_1\|}$$

$$F_1 = k_1\left(\frac{L_1}{\|r_1\|} - 1\right) r_1$$

Follow the same steps to get the other forces

$$F_2 = -k_2(\|r_2\| - L_2)\frac{r_2}{\|r_2\|}$$

$$F_2 = k_2\left(\frac{L_2}{\|r_2\|} - 1\right) r_2$$

$$F_3 = -k_2(\|r_3\| - L_2)\frac{r_3}{\|r_3\|}$$

$$F_3 = k_2\left(\frac{L_2}{\|r_3\|} - 1\right) r_3$$

Putting the Forces Together

The force on mass 1 is

$$F_{m1} = F_1 + F_2 - m_1 g\hat{\boldsymbol{\kappa}}$$

$$F_{m1} = k_1 \left(\frac{L_1}{\|r_1\|} - 1 \right) r_1 + k_2 \left(\frac{L_2}{\|r_2\|} - 1 \right) r_2 - m_1 g\hat{\boldsymbol{\kappa}}$$

The force on mass 2 is

$$F_{m2} = F_3 - m_2 g\hat{\boldsymbol{\kappa}}$$

$$F_{m2} = k_2 \left(\frac{L_2}{\|r_3\|} - 1 \right) r_3 - m_2 g\hat{\boldsymbol{\kappa}}$$

To find the accelerations of the masses, use the formula $F = ma$

$$m_1 a_{m1} = F_{m1}$$
$$a_{m1} = \frac{F_{m1}}{m_1}$$

$$m_2 a_{m2} = F_{m2}$$
$$a_{m2} = \frac{F_{m2}}{m_2}$$

So for the acceleration of mass 1 and mass 2

$$a_{m1} = \frac{k_1}{m_1}\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}} - 1\right)(X_1\hat{\boldsymbol{\imath}} + Y_1\hat{\boldsymbol{\jmath}} + Z_1\hat{\boldsymbol{\kappa}})$$
$$+ \frac{k_2}{m_1}\left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}} - 1\right)\ldots$$
$$* \left((X_1-X_2)\hat{\boldsymbol{\imath}} + (Y_1-Y_2)\hat{\boldsymbol{\jmath}} + (Z_1-Z_2)\hat{\boldsymbol{\kappa}}\right) - g\hat{\boldsymbol{\kappa}}$$

$$a_{m2} = \frac{k_2}{m_2}\left(\frac{L_2}{\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2+(Z_2-Z_1)^2}} - 1\right)\ldots$$
$$* \left((X_2-X_1)\hat{\boldsymbol{\imath}} + (Y_2-Y_1)\hat{\boldsymbol{\jmath}} + (Z_2-Z_1)\hat{\boldsymbol{\kappa}}\right) - g\hat{\boldsymbol{\kappa}}$$

Let me show the $\hat{\boldsymbol{\imath}}$ part of a_{m1}, which is

$$\ddot{X}_1 = \frac{k_1}{m_1} X_1 \left(\frac{L_1}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} - 1 \right) + \frac{k_2}{m_1}(X_1 - X_2) \ldots$$
$$* \left(\frac{L_2}{\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}} - 1 \right)$$

This is not the only way to get the solution. The Euler Lagrange equation can also be used. The equation is

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \dot{q}}\right) = \frac{\partial \ell}{\partial q}$$

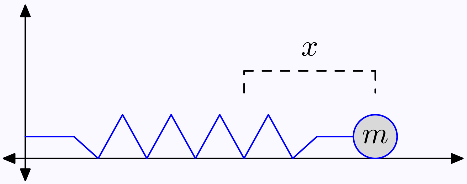
Where $\ell = T - V$, and $T =$ kinetic energy, and $V =$ potential energy.

Finding the Kinetic Energy

Kinetic energy $= 1/2mv^2$. For the masses, $v =$ velocity of the masses is just the derivative of their position with respect to time.

$$
\begin{aligned}
T =& \frac{1}{2}m_1\dot{X_1}^2 + \frac{1}{2}m_1\dot{Y_1}^2 + \frac{1}{2}m_1\dot{Z_1}^2 \\
&+ \frac{1}{2}m_2\dot{X_2}^2 + \frac{1}{2}m_2\dot{Y_2}^2 + \frac{1}{2}m_2\dot{Z_2}^2 \\
T =& \frac{1}{2}m_1\left(\dot{X_1}^2 + \dot{Y_1}^2 + \dot{Z_1}^2\right) + \frac{1}{2}m_2\left(\dot{X_2}^2 + \dot{Y_2}^2 + \dot{Z_2}^2\right)
\end{aligned}
$$

Finding the Potential Energy



The potential energy of a spring is $1/2kx^2$, where x is the displacement from un-stretched.

$$
\begin{aligned}
V =& \frac{1}{2}k_1\left(\sqrt{X_1^2+Y_1^2+Z_1^2}-L_1\right)^2\\
&+\frac{1}{2}k_2\left(\sqrt{(X_1-X_2)^2+(Y_1-Y-2)^2+(Z_1-Z_2)^2}-L_2\right)^2\\
&+m_1gZ_1+m_2gZ_2
\end{aligned}
$$

Using the Lagrangian

So the Lagrangian is

$$\begin{aligned}
\ell =& T - V \\
\ell =& \frac{1}{2}m_1\left(\dot{X_1}^2 + \dot{Y_1}^2 + \dot{Z_1}^2\right) + \frac{1}{2}m_2\left(\dot{X_2}^2 + \dot{Y_2}^2 + \dot{Z_2}^2\right) \\
&- \frac{1}{2}k_1\left(\sqrt{X_1^2 + Y_1^2 + Z_1^2} - L_1\right)^2 \\
&- \frac{1}{2}k_2\left(\sqrt{(X_1 - X_2)^2 + (Y_1 - Y - 2)^2 + (Z_1 - Z_2)^2} - L_2\right)^2 \\
&- m_1 g Z_1 - m_2 g Z_2
\end{aligned}$$

Remember the Euler Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \ell}{\partial \dot{q}}\right) = \frac{\partial \ell}{\partial q}$$

where q is any variable of differentiation. So,

$$\begin{aligned}\frac{\partial \ell}{\partial \dot{X}_1} =& m_1 \dot{X}_1 \\ \frac{d}{dt}\left(\frac{\partial \ell}{\partial \dot{X}_1}\right) =& m_1 \ddot{X}_1\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial X_1} =& k_1 X_1 \left(\frac{L_1}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} - 1\right) + k_2(X_1 - X_2)... \\ & * \left(\frac{L_2}{\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}} - 1\right)\end{aligned}$$

And doing this for Y_1 and Z_1 will yield

$$m_1\ddot{Y_1} = k_1 Y_1\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}}-1\right)+k_2(Y_1-Y_2)...$$
$$* \left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}-1\right)$$

$$m_1\ddot{Z_1} = k_1 Z_1\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}}-1\right)+k_2(Z_1-Z_2)...$$
$$* \left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}-1\right)$$
$$- m_1 g$$

Remember that $(\partial\ell)/(\partial X_1)$ is in the $\hat{\boldsymbol{\imath}}$ direction. Putting the last three equation from the Euler Lagrange equation together, and dividing by m_1, I get

$$
\begin{aligned}
a_{m1} = &= \frac{k_1}{m_1}\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}}-1\right)(X_1\hat{\boldsymbol{\imath}}+Y_1\hat{\boldsymbol{\jmath}}+Z_1\hat{\boldsymbol{\kappa}}) \\
&+ \frac{k_2}{m_1}\left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}-1\right)\ldots \\
&* ((X_1-X_2)\hat{\boldsymbol{\imath}}+(Y_1-Y_2)\hat{\boldsymbol{\jmath}}+(Z_1-Z_2)\hat{\boldsymbol{\kappa}})-g\hat{\boldsymbol{\kappa}}
\end{aligned}
$$

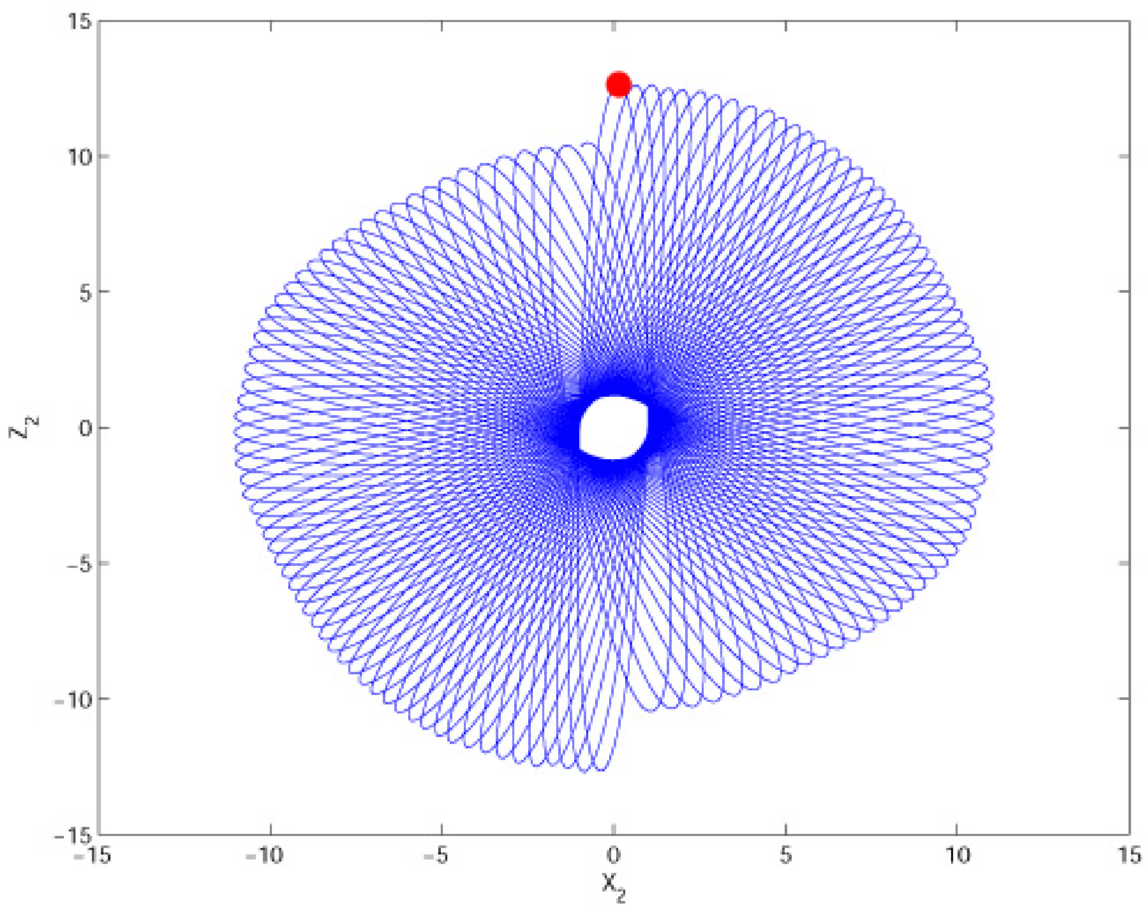
Which is the same as from using vectors to get the acceleration of the first mass.

Getting Results

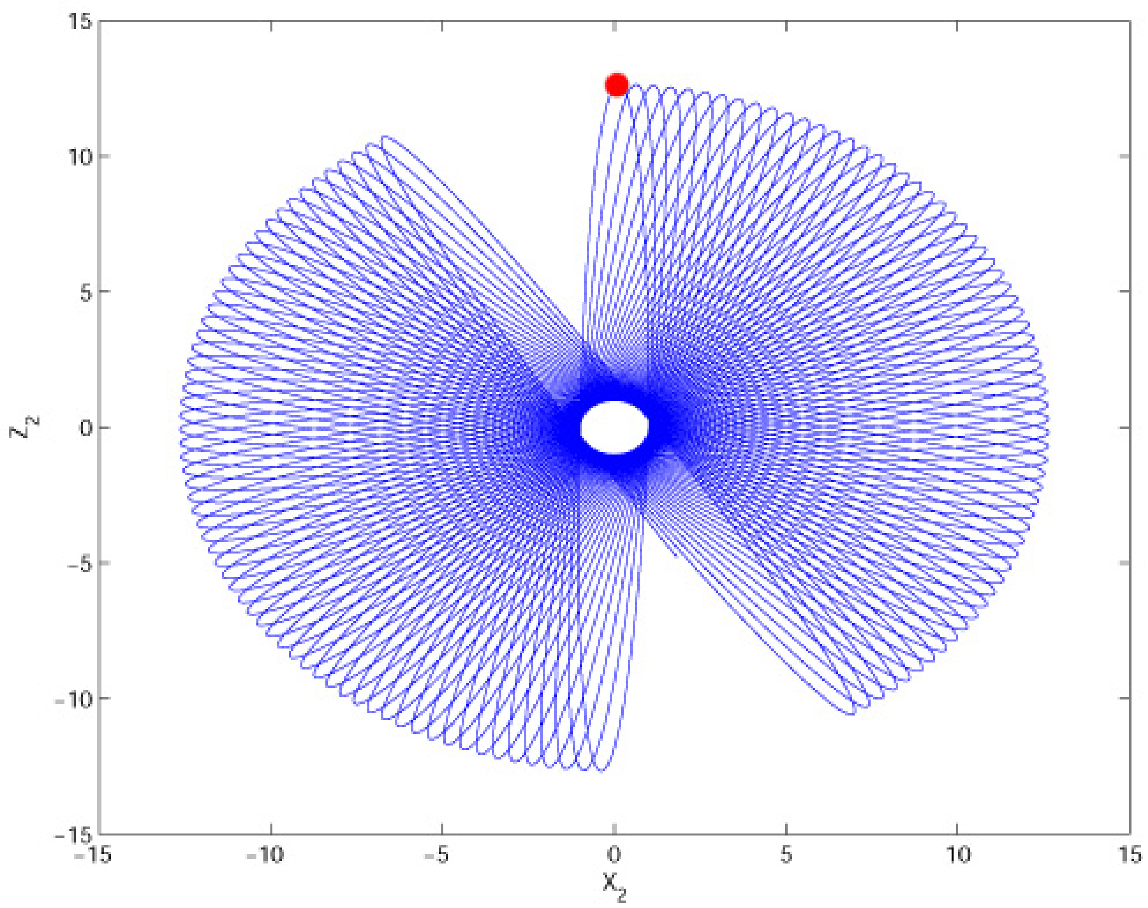
These equations can be solved using MATLAB's ode45 routine. But there really is three different models here

- When spring 1 has constant zero.
- When spring 2 has constant zero.
- When both spring constants are non-zero.

What will happen when spring 1 has constant zero?



15
10
5
0
−5
−10
−15
−15
−10
−5
0
5
10
15
Z_2
X_2


15
10
5
0
−5
−10
−15
Z_2
−15
−10
−5
0
5
10
15
X_2

Getting the Error

We know there was error in this graph because we knew what it should look like. But how do we find the error when we don't know what it should look like?

- There is no friction
- There is no external force
- Total energy has to remain constant

Total energy being constant means that the kinetic energy plus the potential energy has to be constant.

$$E = \left| \frac{(T_F + V_F) - (T_I + V_I)}{T_I + V_I} \right|$$

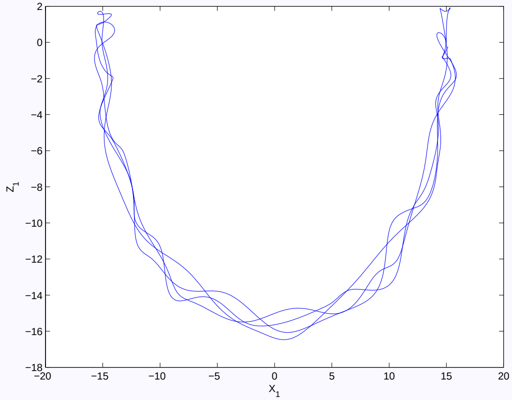
The error of the first graph is

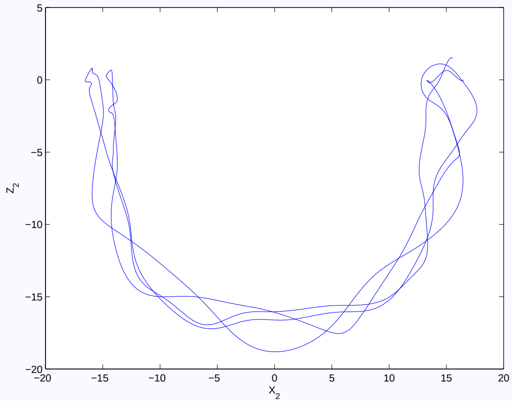
$$\left|\frac{(T_F + V_F) - (T_I + V_I)}{T_I + V_I}\right| = 0.3235$$

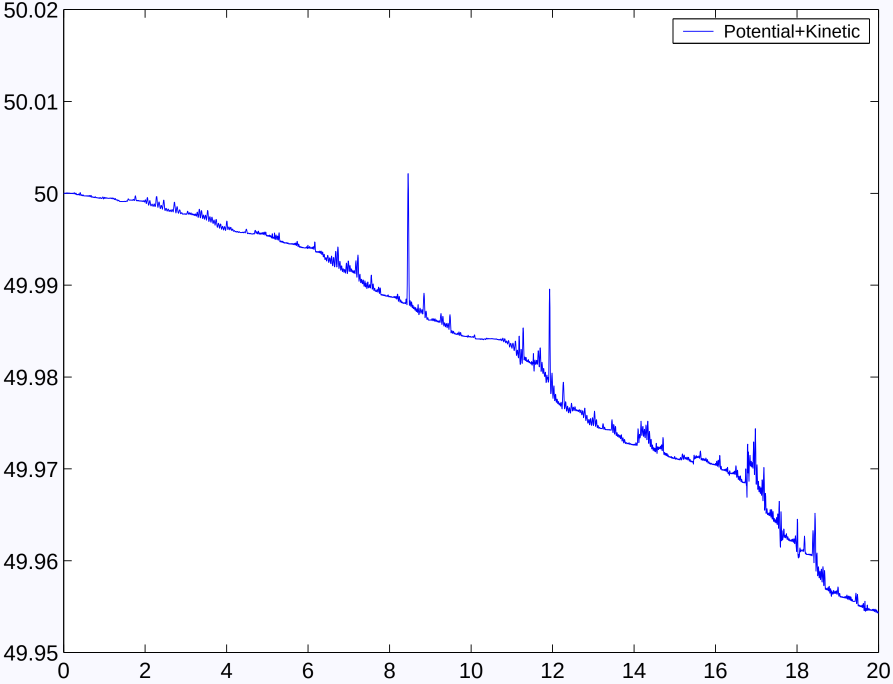
The error of the second graph is

$$\left|\frac{(T_F + V_F) - (T_I + V_I)}{T_I + V_I}\right| = 0.0057$$

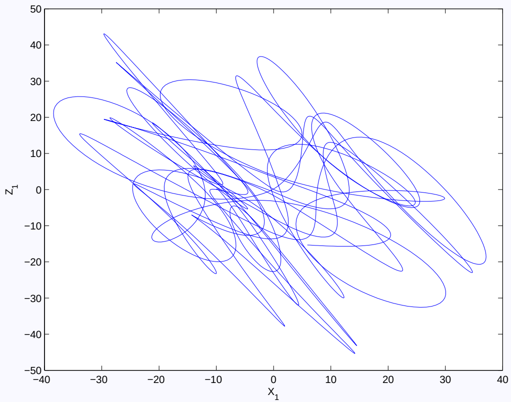
What causes error?

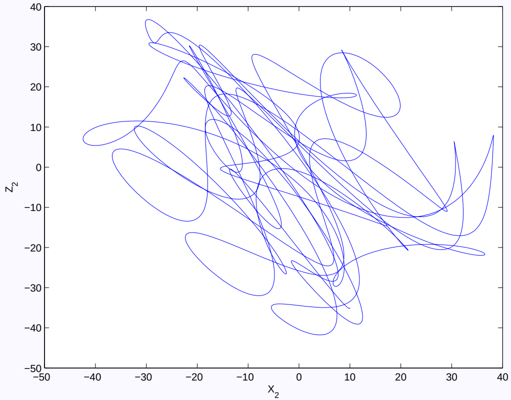


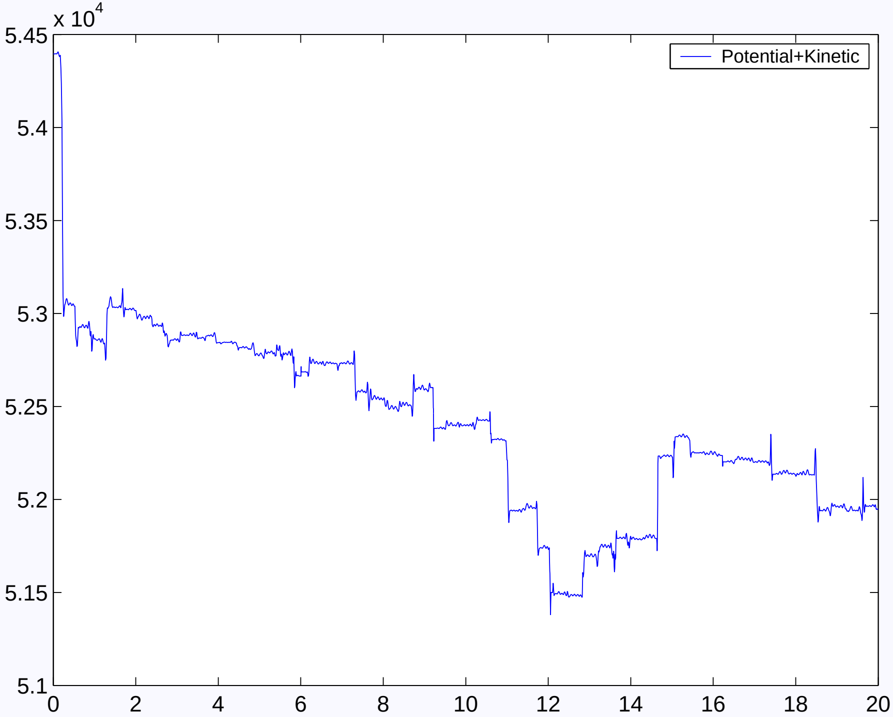




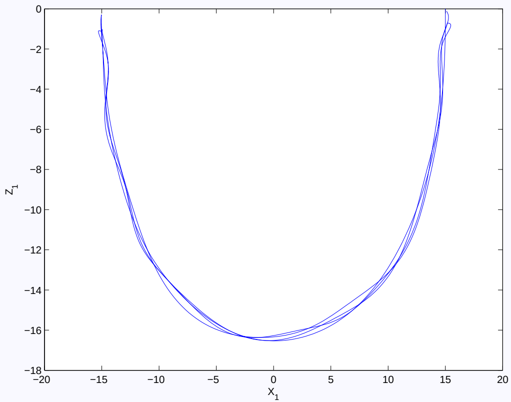
The Potential + Kinetic energy changes only 0.05, or .1 percent

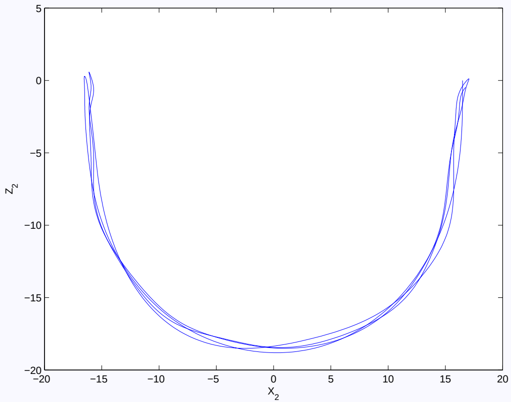

50
40
30
20
10
0
−10
−20
−30
−40
−50
Z1
−40
−30
−20
−10
0
10
20
30
40
X1

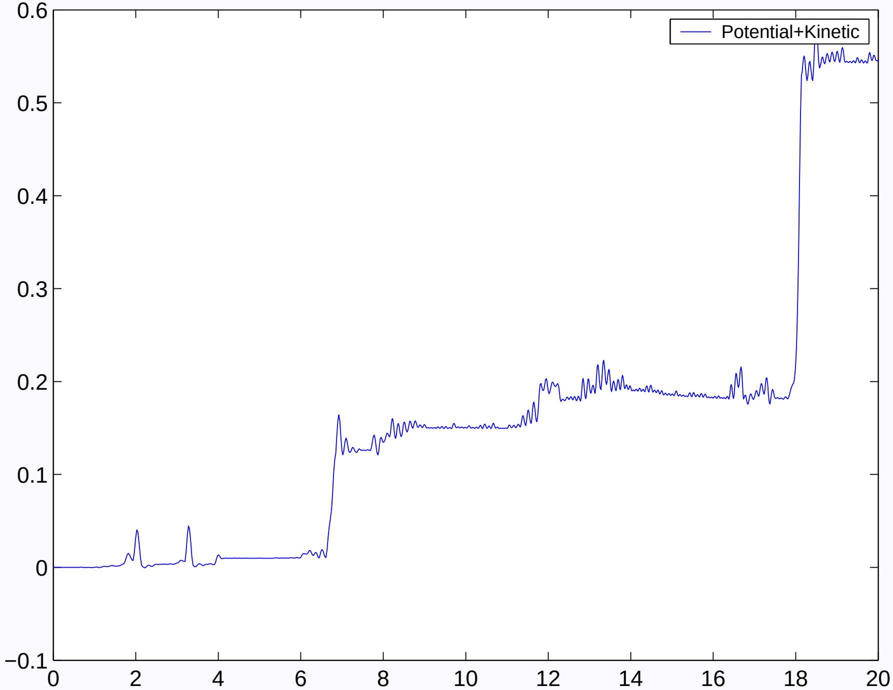

40
30
20
10
0
−10
−20
−30
−40
−50
Z2
−50
−40
−30
−20
−10
0
10
20
30
40
X2



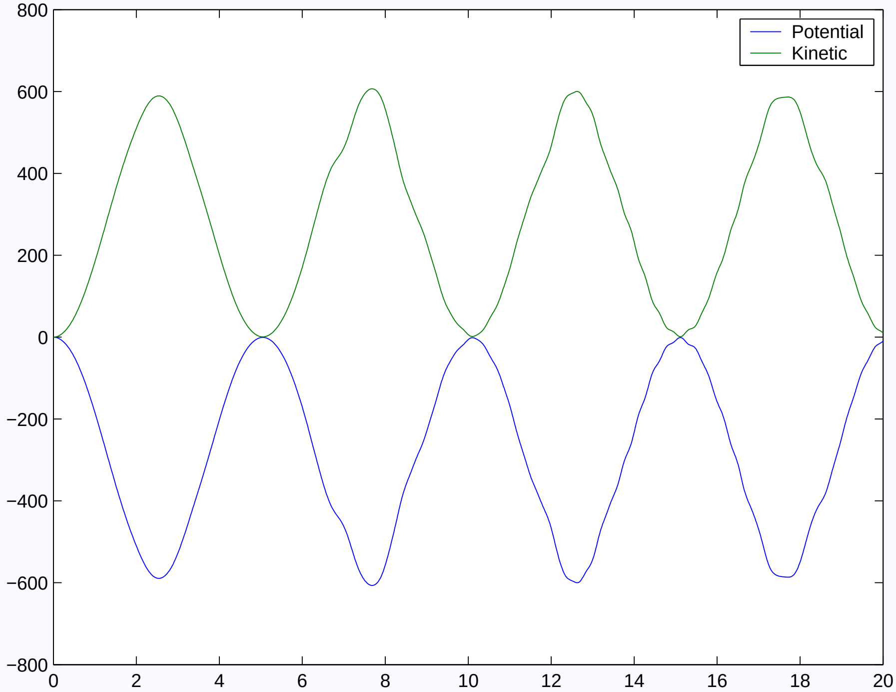
The Potential + Kinetic energy changes 2500, or 4.5 percent.


0
−2
−4
−6
−8
−10
−12
−14
−16
−18
Z_1
−20
−15
−10
−5
0
5
10
15
20
X_1


5
0
−5
−10
−15
−20
Z_2
−20
−15
−10
−5
0
5
10
15
20
X_2



The Potential + Kinetic energy changes 0.55, or ∞ percent.


Potential
Kinetic
800
600
400
200
0
−200
−400
−600
−800
0
2
4
6
8
10
12
14
16
18
20

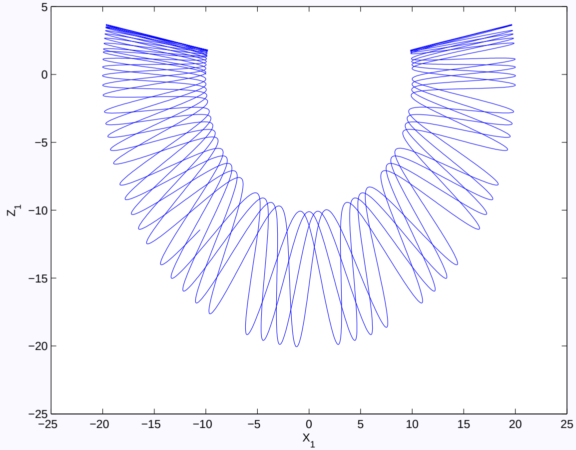


Figure 1: Periodic Motion

Types of Motion

As seen from before, with just the two masses and one spring, the motion is circular. The motion of that system will always be circular. When spring 2 has constant zero, the motion will be periodic.

With both spring constants non-zero, the motion can be quite chaotic, so I made an animating program to view it easier. The amount that it is chaotic is dependent on how much the masses affect each other. If mass 1 is large, and mass 2 is small, the motion of mass 1 will be more periodic.