Continuum Mechanics of a Wave on a String

A Final Project for AMath 271

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Abstract

1 Introduction

When we study classical mechanics, we typically consider the simplest case of mechanics of point masses. But when we suppose there is a system with a large number of these objects and the mass can be distributed over a region with continuous change, we refer to such mechanics as Continuum Mechanics [3].

Systems such as these share a lot of the features of typical discrete systems in mechanics, but the continuous change may be arbitrary. As such, analyzing them traditionally can be a difficult prospect without knowing the initial conditions of every particle in the system. For example, when studying the motion of fluids, perhaps for studying ocean currents or the aerodynamics of various automotive assemblies, we are forced to study the system as a whole so as to not worry about the minutia of the vast number of individual particles which can influence the change.

However, the running throughline of fluid dynamics, seismic analysis, and analysis of forces on materials is that a system comprises a continuously infinite number of parts and the specification of its configuration requires an infinite number of coordinates [3].

When dealing with continuum mechanics, one is usually led to solutions using partial differential equations due to the continuous change across a space with respect to time. As a bit of background here, a differential equations are, as the name suggests, equations in which some terms are derivatives of others, and so require unique differentiation techniques based on the overall structure and order of the derivatives involved. These differential equations are split into either ordinary differential equations (ODEs) or partial differential equations (PDEs), each with their own methodology for solving.

Specifically, ordinary differential equations contain only a single variable and linear dependent function and associated independent variable, making it elegant to solve a variety of first and second order equations with ease, so long as the integrals exist. Partial differential equations, however, consider multivariable and multilinear dependent functions, with two to many independent variables. As such, depending on the complexity of the problem, the solutions are often much more difficult, and sometimes require multiple equations to actually grasp the physical meaning of the result. Nevertheless, we still note that linear differential equations of any kind are typically easier to solve or analyze than their nonlinear counterparts.

When studying a system of a wave on a string, we consider the wave equation, which is linear and is a linearized partial differential equation describing vibratory motion [4]. The problem concerning the motion of a vibrating string was one of the first problems of contunuum mechanics, which produced a partial differential equation as ascertained earlier. The PDE may be derived simply through the formula for kinetic energy and potential energy of a vibrating string.

Consequently, in this project, we will first derive the wave equation of a string using the respective kinetic and potential energy formulae for a string. We will then perform dimensional analysis to verify the proper units for elements of the equation. Last, we will consider the effects of numerical solutions with specific initial conditions and compare those against those available in textbooks to determine what each initial condition does to mechanics of the overall system.

2 Equations

As set up by [4]: Suppose we have a string that is vibrating in $\mathbb{R}^n$ with its ends tied down at two points, with an origin at 0 and a vector $L\vec{e} \in \mathbb{R}^n$ having length L, with uniform mass density m. This will give us a total mass of mL. If we now describe the motion of the string as $u(t,x)$, then for some time t, the kinetic energy is described by:

$$T(t) = \frac{1}{2}m \int_0^L |u_t(t,x)|^2 \, dx \tag{1}$$

and integrating (1) with respect to time along the interval from a time t_0 to a time t_1, we have:

$$P_0(u) = \frac{1}{2}m \int_{t_0}^{t_1} \int_0^L |u_t(t,x)|^2 \, dx \, dt \tag{2}$$

Continuing on for potential energy at a time t, we use Hooke's law which states that the potential energy across the string is a function of how much the string is stretched, but this may be expanded to small parts giving us that the force exerted by a small piece of string is also proportional by the stretch [2]. This gives us a potential energy function:

$$U(t) = \int_0^L f(u_x(t,x))\,dx \tag{3}$$

as before, we further integrate this with respect to time and get:

$$P_1(u) = \int_{t_0}^{t_1} \int_0^L f(u_x(t,x))\,dx\,dt \tag{4}$$

From here, we make use of Hamilton's principle of stationary action [1] for the fixed endpoints of the string and by using (2) and (4) get that:

$$\frac{d}{ds}(P_0 - P_1)(u+sv)|_{s=0} = 0 \tag{5}$$

We then can use (5) and integration by parts to determine:

$$\begin{aligned} \frac{d}{ds}(P_0)(u+sv)|_{s=0} &= \int_{t_0}^{t_1} \int_0^L m u_t v_t\,dx\,dt \\ &= -\int_{t_0}^{t_1} \int_0^L m u_{tt} v\,dx\,dt \end{aligned} \tag{6}$$

and

$$\begin{aligned} \frac{d}{ds}(P_1)(u+sv)|_{s=0} &= \int_{t_0}^{t_1} \int_0^L f'(u_x(t,x)) \cdot v_x(t,x)\,dx\,dt \\ &= -\int_{t_0}^{t_1} \int_0^L \frac{\partial f'(u_x(t,x))}{\partial x} \cdot v(t,x)\,dx\,dt \end{aligned} \tag{7}$$

With (7) we can see that

$$\frac{\partial f'(u_x(t,x))}{\partial x} = f''(u_x)u_{xx}$$

so we have:

$$\frac{d}{ds}(P_1)(u+sv)|_{s=0} = -\int_{t_0}^{t_1}\int_0^L f''(u_x)(u_x x)\cdot v(t,x)\,dx\,dt \tag{8}$$

We can then combine (5) with the associated (6) and (8) and by linearity, we have:

$$mu_{tt} - f''(u_x)u_x x = 0 \tag{9}$$

And in this phase space, the function f is linear with coefficient based on mass, so overall we use (9) and get:

$$\frac{1}{v^2}u_{tt} - u_{xx} = 0$$

We then rewrite this in its more well-known form:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \tag{10}$$

The equation (10) is also written in Taylor's Classical Mechanics [3] as:

$$c^2\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial t^2} \tag{11}$$

With

$$c = \sqrt{\frac{T}{\mu}}$$

With tension T and mass density μ.

Continuing on, we know that the functions forming the solution of the wave equations are sinusoidal, and while we may need to perform Fourier analysis for more arbitrary waves, a simple wave on a string formed by a single disturbance will have a solution:

$$\begin{aligned} \psi(x,t) &= A\sin[k(x+ct)] \\ \psi(x,t) &= A\sin[kx+\omega t] \end{aligned} \tag{12}$$

The A is the amplitude of the wave, the k is the wave number, with wavelength $\lambda = \frac{2\pi}{k}$, and ω is the angular frequency, with period $\tau = \frac{2\pi}{\omega}$

3 Analysis

The first thing to analyze is a comparison of (10) and (11), are the units on c the same as the units on v? We can test this by performing analysis on c^2, with knowing the dimensions of v as:

$$[v^2] = [v][v] = \frac{L^2}{T^2}$$

We see:

$$
\begin{aligned}
[T] &= \frac{ML}{T^2} \\
[\mu] &= \frac{M}{L} \\
[c^2] &= \left[\frac{T}{\mu}\right] \\
&= \frac{[T]}{[\mu]} \\
&= \frac{ML^2}{MT^2} \\
&= \frac{L^2}{T^2}
\end{aligned}
$$

This gives us a correlation between the velocity-based wave equation and tension-mass-density-based wave equation.

Continuing our analysis on the dimensions of the wave equation, we should confirm the units are correct on either side. Recall (12). There, we see that

$$[\psi] = [A\sin(kx + \omega t)]$$

but the sine term is dimensionless and the amplitude has a distance dimension, we then have

$$[\psi] = [A] = L$$

this now gives us

$$[\psi_{xx}] = \frac{L}{L^2} = L^{-1}$$

and

$$[\psi_{tt}] = \frac{L}{T^2}$$

so if we consider

$$[c^2\psi_{xx}] = [c^2][\psi_{xx}] = \left(\frac{L^2}{T^2}\right)(L^{-1}) = \frac{L}{T^2}$$

since both sides of the wave equation have the same dimension, this evidence supports that the derivation of the wave equation is correct.

4 Computations

While in the equation section, we are considering the case one sinusoidal wave moving from left to right on a sting without boundary and intial conditions. In real life, the length of a string is finite, and two ends of thestring should be fixed with $u(0, t) = 0$ and $u(l, t) = 0$. In this case, the analytical solution is

$$u(x, t) = \sin(kx)A\cos(wt - \sigma)$$

as ascribed by Taylor [3]

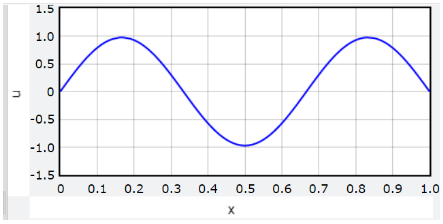
We then solve the wave equation numerically using finite difference method using python. A good choice for solution is to consider a harmonic on a string because it is referenced in Taylor's Classical Mechanics textbook chapter 16.3 equation 16.26 [3], so we can compare to an expected result.

As can be seen in figure 1, this is a numerical solution the third harmonic of the string with amplitude $A = 1$, length $L = 1$, $\sigma = 0$, and $k = \frac{3\pi}{L}$, which describes the analytical solution, as seen in Taylor's Figure 16.6 [3].

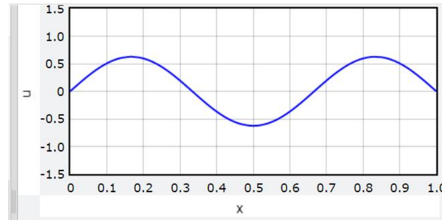
5 Conclusions

To summarize, we derived the one dimensional wave equation and verify it though unit analysis. Then we solved one dimension wave equation on a string numerically in a specific harmonic case. The numerical solution matches the analytical solution. This result tells we could also verify the solved numerical solution of a partial differential equations through comparing it to expected analytical solutions.

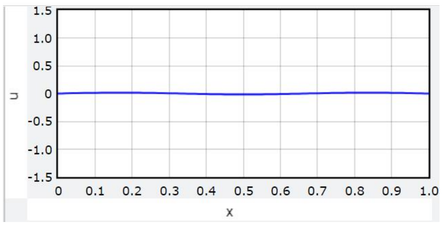
As it stands, the skills of being able to derive a partial differential equation and determine speciifc solutions are very effective in building a lot of the required skills for solving many continuum mechanics problems. So for such a simple problem as determining the vibrations on a finite or infinite string, we may go further and start analyzing the equilibria of heat over composite material in a space, analyze the tremors of of the earth after an earthquake, or even determine the aerodynamics of a new car or plane. Further, while these styles of problem have been and continue to be solved to this day, this problem reveals the first step into applied numerical analysis in the real world. A beautiful art that is shrouded to students by the complexity of complex systems.



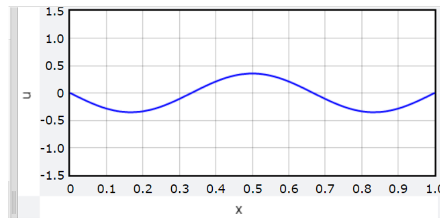
(a) t=0



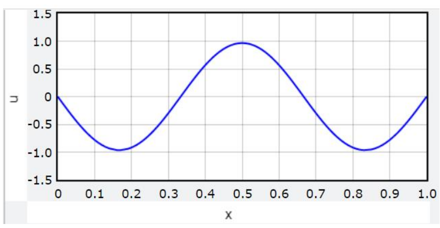
(b) t=2.00



(c) t=2.60



(d) t=3.00



(e) t=4.20

Figure 1: This figure describes the graph of the calculated numerical solutions for the position u of a wave with amplitude $A = 1$, length $L = 1$, $\sigma = 0$, and $k = \frac{3\pi}{L}$ at specific periods of time displaying half a vibration of the string.

References

[1] Hamilton, W.R."On a General Method in Dynamics.", *Philosophical Transactions of the Royal Society Part II*, London, 1834.

[2] Hooke, R. *De Potentia Restitutiva, or of Spring. Explaining the Power of Springing Bodies*, London, 1678.

[3] Taylor, J. *Classical Mechanics*, University Science Books, 2005.

[4] Taylor, M. *Partial Differential Equations*, Springer, 2011.