106 年台大工程數學

Problem 1

【Solution】

令 $g(x)=f(x)-\pi=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{\pi}\right)=\sum_{n=1}^{\infty}b_n\sin(nx)$

$$b_n=\frac{2}{\pi}\int_0^{\pi}x\sin(nx)dx=\frac{2(-1)^{n+1}}{n}$$

$$g(x)=\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)\sin(nx)$$

$$f(x)=g(x)+\pi=\underline{\underline{\pi+\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)\sin(nx)}}$$

Problem 2

【Solution】

(a)

$$L\left[t^2\sin(3t)\right]=(-1)^2\frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$=\frac{d}{ds}\left(\frac{-6s}{(s^2+9)^2}\right)=-6\frac{(s^2+9)^2-4s^2(s^2+9)}{(s^2+9)^4}$$

$$=-6\frac{-3s^2+9}{(s^2+9)^3}=\underline{\underline{\frac{18s^2-54}{(s^2+9)^3}}}$$

(b)

$$L^{-1}\left[\frac{6s+7}{2s^2+4s+10}\right]=\frac{1}{2}L^{-1}\left[\frac{6s+7}{(s+1)^2+4}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{6(s+1)+1}{(s+1)^2+4}\right]=\frac{1}{2}e^{-t}L^{-1}\left[\frac{6s+1}{s^2+4}\right]$$

$$=\underline{\underline{\frac{1}{2}e^{-t}\left(6\cos(2t)+\frac{1}{2}\sin(2t)\right),t>0}}$$

Problem 3

【**Solution**】

令 $f(t)=4t\left(H(t)-H(t-1)\right)+8H(t-1)$

將原式做拉式轉換，令 $Y(s)=L\left[y(t)\right]$

$$s^2Y(s)\underbrace{-sy(0)-y'(0)}_{0}+3sY(s)\underbrace{-3y(0)}_{0}+2Y(s)=\frac{4}{s^2}+e^{-s}\left(\frac{-4}{s^2}+\frac{4}{s}\right)$$

$$Y(s)=\frac{4}{s^2\left(s^2+3s+2\right)}+e^{-s}\left(\frac{-4}{s^2\left(s^2+3s+2\right)}+\frac{4}{s\left(s^2+3s+2\right)}\right)$$

$$Y(s)=\frac{4}{s^2\left(s+1\right)\left(s+2\right)}+4e^{-s}\left(\frac{-1}{s^2\left(s+1\right)\left(s+2\right)}+\frac{1}{s\left(s+1\right)\left(s+2\right)}\right)$$

$$Y(s)=\left(\frac{2}{s^2}-\frac{3}{s}+\frac{4}{s+1}-\frac{1}{s+2}\right)-e^{-s}\left(\frac{2}{s^2}-\frac{5}{s}+\frac{8}{s+1}-\frac{3}{s+2}\right)$$

$$y(t)=L^{-1}\left[Y(s)\right]=\underline{\underline{\left(2t-3+4e^{-t}-e^{-2t}\right)-H(t-1)\left[2\left(t-1\right)-5+8e^{-(t-1)}-3e^{-2(t-1)}\right],t>0}}$$

Problem 4

【**Solution**】

令 $y(x)=\sum_{n=0}^{\infty}C_n(r)x^{n+r},0<x<\infty$

可得 $y'(x)=\sum_{n=0}^{\infty}(n+r)C_n(r)x^{n+r-1},y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n(r)x^{n+r-2}$ 代入原式

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n(r)x^{n+r-1}+\sum_{n=0}^{\infty}2(n+r)C_n(r)x^{n+r}$$

$$+\sum_{n=0}^{\infty}(n+r)C_n(r)x^{n+r-1}+\sum_{n=0}^{\infty}C_n(r)x^{n+r+1}+\sum_{n=0}^{\infty}C_n(r)x^{n+r}=0$$

$$\sum_{n=-1}^{\infty}(n+r)(n+r+1)C_{n+1}(r)x^{n+r}+\sum_{n=0}^{\infty}2(n+r)C_n(r)x^{n+r}$$

$$+\sum_{n=-1}^{\infty}(n+r+1)C_{n+1}(r)x^{n+r}+\sum_{n=1}^{\infty}C_{n-1}(r)x^{n+r}+\sum_{n=0}^{\infty}C_n(r)x^{n+r}=0$$

$$r^2C_0x^{r-1}+\left(\left(2r+1\right)C_0+\left(r+1\right)^2C_1\right)x^r$$

$$+\sum_{n=1}^{\infty}\left(C_{n-1}+\left(2n+2r+1\right)C_n+\left(n+r+1\right)^2C_{n+1}\right)x^{n+r}=0$$

令 $C_0 \neq 0$ ， $r = 0, 0$ 代入

$$C_0 + C_1 = 0$$
$$C_1 = -C_0$$
$$C_{n-1} + (2n+1)C_n + (n+1)^2 C_{n+1} = 0, \forall n \in \mathbb{N}$$
$$C_{n+1} = -\frac{C_{n-1} + (2n+1)C_n}{(n+1)^2}, \forall n \in \mathbb{N}$$
$$C_2 = \frac{1}{2!}C_0$$
$$C_3 = \frac{-1}{3!}C_0$$
$$C_4 = \frac{1}{4!}C_0$$
$$C_5 = \frac{-1}{5!}C_0$$

$$\phi_1(x) = x^r \sum_{n=0}^{\infty} C_n(r)x^n \bigg|_{r=0} = C_0\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \dots\right) = C_0 e^{-x}$$

取 $\phi_1(x) = e^{-x}$ 為 O.D.E.之一解

By reduction of order,

$$\phi_2 = e^{-x}\int \frac{1}{e^{-2x} x e^{2x}} dx = e^{-x}\ln(x)$$

$$\underline{\underline{y = c_1 e^{-x} + c_2 e^{-x}\ln(x), x > 0}}$$

Problem 5

【Solution】

令 $u(x,t)=X(x)T(t)$ 代入邊界條件及 P.D.E.

$$\begin{cases} u(0,t)=X(0)T(t)=0 \\ u(L,t)=X(L)T(t)=0 \end{cases} \overset{T(t)\neq 0}{\Rightarrow} X(0)=X(L)=0$$

$$XT'=cX''T$$

$$\frac{X''}{X}=\frac{T'}{cT}=-\lambda$$

$$\begin{cases} X''+\lambda X=0 \\ T'+c\lambda T=0 \end{cases}\text{，其中}X''+\lambda X=0\text{為特徵方程式}$$

1. 令 $\lambda=0$ 代入特徵方程式

$$X''(x)=0$$

$$X(x)=C_1+C_2x$$

代入 B.C.

$$C_1=C_2=0$$

2. 令 $\lambda=-k^2, k>0$ 代入特徵方程式

$$X''(x)-k^2X(x)=0$$

$$X(x)=C_1\cosh(kx)+C_2\sinh(kx)$$

代入 B.C.

$$C_1=C_2=0$$

3. 令 $\lambda=k^2, k>0$ 代入特徵方程式

$$X''(x)+k^2X(x)=0$$

$$X(x)=C_1\cos(kx)+C_2\sin(kx)$$

代入 B.C.

$$C_1=0$$

令 $C_2 \neq 0$

$$X(L) = C_2 \sin\left(kL\right) = 0$$

$$k = \frac{n\pi}{L}, n = 1, 2, 3, \ldots$$

$$\lambda = k^2 = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \ldots$$

$$X_n\left(x\right) = \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, \ldots$$

$$T' + \left(\frac{cn^2\pi^2}{L^2}\right)T = 0$$

$$T_n = \alpha_n e^{-\frac{cn^2\pi^2}{L^2}t}$$

$$u_n(x,t) = X_n(x)T_n(t) = C_n e^{-\frac{cn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{cn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

代入 I.C.

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} x,\ \text{if}\ \ 0 < x < \dfrac{L}{2} \\ L - x,\ \text{if}\ \ \dfrac{L}{2} < x < L \end{cases}$$

$$C_n = \frac{2}{L}\left(\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}\left(L - x\right)\sin\left(\frac{n\pi x}{L}\right)dx\right) = \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\underline{\underline{u(x,t) = \sum_{n=1}^{\infty}\left(\frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)e^{-\frac{cn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)}}$$

Problem 6

【Solution】

(a)

令 $y=x^m$ 代入原式

$$\left(m\left(m-1\right)+3m+2\right)x^m=0$$

令 $x^m\neq 0$ 代入上式

$$m\left(m-1\right)+3m+2=m^2+2m+2=0$$

$$m=-1\pm i$$

$$y=x^{-1}\left(C_1\cos\left(\ln\left(x\right)\right)+C_2\sin\left(\ln\left(x\right)\right)\right),x>0$$

(b)

將原式同乘 $\dfrac{r^2}{D}$

$$\frac{d}{dr}\left(r^2\frac{dC}{dr}\right)-\frac{k}{D}r^2C=0$$

$$r^2\frac{d^2C}{dr^2}+2r\frac{dC}{dr}-\frac{k}{D}r^2C=0$$

令 $1-2\alpha=2,\alpha=\dfrac{-1}{2}$

令 $C=r^{\alpha}R=r^{\frac{-1}{2}}R$ 代入上式

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left(-\frac{k}{D}r^2R-\left(\frac{1}{2}\right)^2\right)=0$$

1. 令 $-\dfrac{k}{D}=\lambda^2,\lambda>0$ 代入上式

$$R(r)=c_1J_{\frac{1}{2}}\left(\lambda r\right)+c_2Y_{\frac{1}{2}}\left(\lambda r\right)=c_1J_{\frac{1}{2}}\left(\sqrt{-\frac{k}{D}}r\right)+c_2Y_{\frac{1}{2}}\left(\sqrt{-\frac{k}{D}}r\right)$$

$$\underline{\underline{C(r)=r^{\frac{-1}{2}}\left(c_3\frac{1}{\sqrt{r}}\sin\left(\sqrt{-\frac{k}{D}}r\right)+c_4\frac{1}{\sqrt{r}}\cos\left(\sqrt{-\frac{k}{D}}r\right)\right)}}$$

2. 令 $-\dfrac{k}{D}=-\lambda^2,\lambda>0$ 代入上式

$$R(r)=c_1I_{\frac{1}{2}}(\lambda r)+c_2K_{\frac{1}{2}}(\lambda r)=c_1I_{\frac{1}{2}}\left(\sqrt{\frac{k}{D}}r\right)+c_2K_{\frac{1}{2}}\left(\sqrt{\frac{k}{D}}r\right)$$

$$\underline{\underline{C(r)=r^{\frac{-1}{2}}\left(c_3\frac{1}{\sqrt{r}}\sinh\left(\sqrt{\frac{k}{D}}r\right)+c_4\frac{1}{\sqrt{r}}\cosh\left(\sqrt{\frac{k}{D}}r\right)\right)}}$$

3. 令 $k=0$ 代入上式

$$r^2\frac{d^2C}{dr^2}+2r\frac{dC}{dr}=0$$

$$\underline{\underline{C(r)=c_1+c_2r^{-1}}}$$

Problem 7

【Solution】

(a)

令 $P_B(\lambda)=\det(B-\lambda I)=0$

$$\Rightarrow \lambda(\lambda-5)=0$$

$$\Rightarrow \underline{\underline{\lambda=0,\ 5}} \text{ 為 eigenvalues}$$

(i) 令 $\lambda=0$ 代入 $(B-\lambda I)\vec{y}=\vec{0}$

$$\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\vec{y}=C_1\begin{bmatrix}1\\ -2\end{bmatrix},\ \forall C_1\neq 0 \text{ 為 eigen vectors}$$

(ii) 令 $\lambda=5$ 代入 $(B-\lambda I)\vec{y}=\vec{0}$

$$\begin{bmatrix}-1 & 2\\ 2 & -4\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\vec{y}=C_2\begin{bmatrix}2\\ 1\end{bmatrix},\ \forall C_2\neq 0 \text{ 為 eigen vectors}$$

(b)

$$P^{-1}BP=D=\begin{bmatrix}0 & 0\\ 0 & 5\end{bmatrix}$$

$$\boxed{P=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}},\ \boxed{P^{-1}=\frac{1}{\det(P)}\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}=\frac{1}{5}\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}}$$

(c)

$$P^{-1}BP = D \text{，} B = PDP^{-1}$$

$$B^3 = PD^3P^{-1}$$

$$B^3 = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 5^3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \underline{\underline{25\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}}}$$

(d)

令 $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{f} = \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$

$$\frac{d\vec{x}}{dt} = B\vec{x} + \vec{f}$$

令 $\vec{x} = P\vec{z}$, $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$P\vec{z}\,' = BP\vec{z} + \vec{f}$$

$$\vec{z}\,' = P^{-1}BP\vec{z} + P^{-1}\vec{f} = D\vec{z} + P^{-1}\vec{f}$$

$$P^{-1}\vec{f} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} = \begin{bmatrix} \frac{1}{5}e^t \\ \frac{7}{5}e^t \end{bmatrix}$$

$$\begin{cases} z_1' = \frac{1}{5}e^t \\ z_2' = 5z_2 + \frac{7}{5}e^t \end{cases} \rightarrow \begin{cases} z_1 = \frac{1}{5}e^t + C_1 \\ z_2 = -\frac{7}{20}e^t + C_2e^{5t} \end{cases}$$

$$\vec{x} = P\vec{z} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{5}e^t + C_1 \\ -\frac{7}{20}e^t + C_2e^{5t} \end{bmatrix} = \underline{\underline{\begin{bmatrix} -\frac{1}{2}e^t + C_1 + 2C_2e^{5t} \\ -\frac{3}{4}e^t - 2C_1 + C_2e^{5t} \end{bmatrix}}}$$

<u>Problem 8</u>

【Solution】

(a)

$$\vec{\nabla}\bullet\vec{G}=\frac{\partial}{\partial x}(xe^{x^2+y^2}+2xy)+\frac{\partial}{\partial y}(ye^{x^2+y^2}+x^2)=e^{x^2+y^2}(1+2x^2+1+2y^2)+2y$$

$$=e^{x^2+y^2}(2x^2+2y^2+2)+2y$$

(b)

$$\begin{cases}\dfrac{\partial f}{\partial x}=xe^{x^2+y^2}+2xy\rightarrow f=\dfrac{1}{2}e^{x^2+y^2}+x^2y+g(y)\\ \dfrac{\partial f}{\partial y}=ye^{x^2+y^2}+x^2\rightarrow f=\dfrac{1}{2}e^{x^2+y^2}+x^2y+h(x)\end{cases}$$

$$\Rightarrow\quad f(x,y)=\frac{1}{2}e^{x^2+y^2}+x^2y+C$$

(c)

$$\oint_c\vec{G}\bullet d\vec{r}=\oint_c df=\oint_c d(\frac{1}{2}e^{x^2+y^2}+x^2y)=0$$

(d)

若在一平滑且封閉的曲線平面上，$P(x,y)$，$Q(x,y)$之一階偏導數存在且連續，

則：

$$\oint_c(Pdx+Qdy)=\iint_R(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$$

(e)

By Green’s theorem

$$\oint_c(Pdx+Qdy)=\iint_R(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$$
$$=\oint_c\iint_R(2xye^{x^2+y^2}+2x-2xye^{x^2+y^2}-2x)dxdy$$
$$=\underline{\underline{0}}$$

107 年台大工程數學

Problem 1

【Solution】

(a)

$$y dx + 2xy dy - e^{-2y} dy = 0$$

同乘 $\frac{e^{2y}}{y}$ ，

$$e^{2y} dx + 2xe^{2y} dy - \frac{1}{y} dy = 0$$

$$d\left(xe^{2y}\right) - \frac{1}{y} dy = 0$$

$$\underline{\underline{xe^{2y} - \ln|y| = C}}$$

(b)

令 $D = \frac{d}{dx}$

$$\left(D^2 - 2D + 1\right) y = e^x \ln\left(x\right)$$

1. 求 y_h

令 $\left(D^2 - 2D + 1\right) y_h = 0$

$$D^2 - 2D + 1 = \left(D - 1\right)^2 = 0$$

$$D = 1,1$$

$$y_h = C_1 e^x + C_2 xe^x$$

2. 求 y_p ，利用參數變異法

令 $\phi_1 = e^x, \phi_2 = xe^x, r(x) = e^x \ln\left(x\right)$

$$W\left(\phi_1, \phi_2\right) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}$$

$$y_p = -\phi_1\int\frac{\phi_2 r(x)}{W}dx+\phi_2\int\frac{\phi_1 r(x)}{W}dx = -e^x\int x\ln(x)dx + xe^x\int\ln(x)dx$$

$$y_p = -e^x\left(\frac{1}{2}x^2\ln(x)-\frac{1}{4}x^2\right)+xe^x\left(x\ln(x)-x\right)$$

$$y_p = \frac{1}{2}x^2e^x\ln(x)-\frac{3}{4}x^2e^x$$

$$\underline{\underline{y = C_1e^x + C_2xe^x + \frac{1}{2}x^2e^x\ln(x)-\frac{3}{4}x^2e^x}}$$

Problem 2

【Solution】

(a)

令 $\vec{X}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}, D=\frac{d}{dt}$

$$\frac{d^2\vec{X}}{dt^2}+2\frac{d\vec{X}}{dt}-3\vec{X}=\vec{0}$$

$$\left(D^2+2D-3\right)\vec{X}=0, \vec{X}\neq 0$$

$$D^2+2D-3=0$$

$$D=-3,1$$

$$\vec{X}=\begin{bmatrix}C_1e^{-3t}+C_2e^t\\C_3e^{-3t}+C_4e^t\end{bmatrix}$$

$$\frac{d\vec{X}}{dt}=\begin{bmatrix}-3C_1e^{-3t}+C_2e^t\\-3C_3e^{-3t}+C_4e^t\end{bmatrix}$$

代入 I.C.

$$C_1=1, C_2=4, C_3=-1, C_4=1$$

$$\underline{\underline{\vec{X}=\begin{bmatrix}e^{-3t}+4e^t\\-e^{-3t}+e^t\end{bmatrix}}}$$

(b)

$$\text{Slope}\quad m_1=\lim_{t\to\infty}\left(\frac{-e^{-3t}+e^t}{e^{-3t}+4e^t}\right)=\frac{1}{4}$$

$$\text{Intercept}\quad b_1=\lim_{t\to\infty}\left(-e^{-3t}+e^{t}-\frac{1}{4}e^{-3t}-e^{t}\right)=0$$

$\underline{\underline{y_1=\frac{1}{4}x}}$ 為一條斜漸近線

$$\text{Slope}\quad m_2=\lim_{t\to-\infty}\left(\frac{-e^{-3t}+e^{t}}{e^{-3t}+4e^{t}}\right)=-1$$

$$\text{Intercept}\quad b_2=\lim_{t\to-\infty}\left(-e^{-3t}+e^{t}+e^{-3t}+4e^{t}\right)=0$$

$\underline{\underline{y_2=-x}}$ 為另一條斜漸近線

<u>Problem 3</u>

【Solution】

將原式做拉氏轉換，令 $Y(s)=L\left[y(t)\right],G(s)=L\left[g(t)\right]$

$$s^2Y(s)-sy(0)-y'(0)+4sY(s)-4y(0)+4Y(s)=G(s)$$

$$\left(s^2+4s+4\right)Y(s)=G(s)+\left(2s+5\right)$$

$$Y(s)=\frac{G(s)}{s^2+4s+4}+\frac{2s+5}{s^2+4s+4}=\frac{G(s)}{\left(s+2\right)^2}+\frac{2s+5}{\left(s+2\right)^2}$$

$$y(t)=L^{-1}\left[Y(s)\right]=te^{-2t}*g(t)+2e^{-2t}+te^{-2t}$$

$$=\underline{\underline{\int_0^t \tau e^{-2\tau}g(t-\tau)d\tau+2e^{-2t}+te^{-2t},t>0}}$$

Problem 4

【Solution】

令 $w(x,t)=U(x,t)-1$

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}$$

$$\begin{cases} w(0,t)=0,\ t>0 \\ w(1,t)=0,\ t>0 \\ w(x,0)=\cos(\pi x),\ 0<x<1 \end{cases}$$

(a)

令 $w(x,t)=X(x)T(t)$ 代入上式

$$XT'=X''T$$

同乘 $\frac{1}{XT}$ ，

$$\frac{X''}{X}=\frac{T'}{T}=-\lambda$$

$$\begin{cases} X''+\lambda X=0 \\ T'+\lambda T=0 \end{cases}，其中 X''+\lambda X=0 為特徵方程式$$

1. 令 $\lambda=0$ 代入特徵方程式

$$X''(x)=0$$

$$X(x)=C_1+C_2x$$

代入 B.C.

$$C_1=C_2=0$$

2. 令 $\lambda=-k^2, k>0$ 代入特徵方程式

$$X''(x)-k^2X(x)=0$$

$$X(x)=C_1\cosh(kx)+C_2\sinh(kx)$$

代入 B.C.

$$C_1=C_2=0$$

3. 令 $\lambda=k^2, k>0$ 代入特徵方程式

$$X''(x)+k^2X(x)=0$$

$$X(x)=C_1\cos(kx)+C_2\sin(kx)$$

代入 B.C.

$$C_1=0$$

令 $C_2\neq 0$

$$X(1)=C_2\sin(k)=0$$

$$k=n\pi, n=1,2,3,\ldots$$

$$\lambda=k^2=(n\pi)^2, n=1,2,3,\ldots$$

$$X_n(x)=\sin(n\pi x), n=1,2,3,\ldots$$

$$T'+(n^2\pi^2)T=0$$

$$T_n=\alpha_n e^{-n^2\pi^2 t}$$

$$w_n(x,t)=X_n(x)T_n(t)=C_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$w(x,t)=\sum_{n=1}^{\infty}C_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

代入 I.C.

$$w(x,0)=\sum_{n=1}^{\infty}C_n\sin(n\pi x)=\cos(\pi x)$$

$$C_n=2\int_0^1\cos(\pi x)\sin(n\pi x)dx=\frac{2n\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)}, n\neq 1, C_1=0$$

$$w(x,t)=\sum_{n=2}^{\infty}\left(\frac{2n\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)}\right)e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$u(x,t)=1+\sum_{n=2}^{\infty}\left(\frac{2n\left(1+(-1)^{n}\right)}{\pi\left(n^{2}-1\right)}\right)e^{-n^{2}\pi^{2}t}\sin\left(n\pi x\right)$$

(b)

將 $\frac{\partial w}{\partial t}=\frac{\partial^{2}w}{\partial x^{2}}$ 做拉式轉換，令 $W(x,s)=L\left[w(x,t)\right]$

$$\frac{\partial^{2}W(x,s)}{\partial x^{2}}-sW(x,s)=-w(x,0)$$

$$\frac{\partial^{2}W(x,s)}{\partial x^{2}}-sW(x,s)=-\cos\left(\pi x\right)$$

$$W(x,s)=C_{1}(s)\cosh\left(\sqrt{s}x\right)+C_{2}(s)\sinh\left(\sqrt{s}x\right)+\frac{1}{s+\pi^{2}}\cos\left(\pi x\right)$$

$$\begin{cases}W(0,s)=0\\W(1,s)=0\end{cases}$$

$$C_{1}(s)=\frac{-1}{s+\pi^{2}},C_{2}(s)=\frac{1}{s+\pi^{2}}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)}$$

$$W(x,s)=\frac{-1}{s+\pi^{2}}\cosh\left(\sqrt{s}x\right)+\left(\frac{1}{s+\pi^{2}}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)}\right)\sinh\left(\sqrt{s}x\right)+\frac{1}{s+\pi^{2}}\cos\left(\pi x\right)$$

$$w(x,t)=L^{-1}\left[\frac{-1}{s+\pi^{2}}\cosh\left(\sqrt{s}x\right)+\left(\frac{1}{s+\pi^{2}}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)}\right)\sinh\left(\sqrt{s}x\right)+\frac{1}{s+\pi^{2}}\cos\left(\pi x\right)\right]$$

$$U(x,t)=1+L^{-1}\left[\frac{-1}{s+\pi^{2}}\cosh\left(\sqrt{s}x\right)+\left(\frac{1}{s+\pi^{2}}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)}\right)\sinh\left(\sqrt{s}x\right)+\frac{1}{s+\pi^{2}}\cos\left(\pi x\right)\right]$$

108 年台大工程數學

Problem 1

【**Solution**】

$$ydx-2xdy+x^2ydy=0$$

$$\frac{d\left(xy^{-2}\right)}{y^{-3}}+x^2ydy=0$$

同乘 $\frac{1}{x^2y^{-1}}$，

$$\frac{d\left(xy^{-2}\right)}{x^2y^{-4}}+y^2dy=0$$

$$\left(xy^{-2}\right)^{-2}d\left(xy^{-2}\right)+y^2dy=0$$

$$\underline{\underline{-\left(xy^{-2}\right)^{-2}+\frac{1}{3}y^3=C}}$$

Problem 2

【**Solution**】

令 $D=\frac{d}{dx}$

$$\left(D^4-2D^2+1\right)y=e^{-x}$$

1. 求 y_h

令 $\left(D^4-2D^2+1\right)y_h=0$

$$D^4-2D^2+1=\left(D-1\right)^2\left(D+1\right)^2=0$$

$$D=-1,-1,1,1$$

$$y_h=C_1e^{-x}+C_2xe^{-x}+C_3e^{x}+C_4xe^{x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p=\frac{1}{(D-1)^2(D+1)^2}e^{-x}$$

$$y_p=\frac{1}{4}e^{-x}\frac{1}{D^2}(1)=\frac{1}{4}e^{-x}\left(\frac{1}{2}x^2\right)=\frac{1}{8}x^2e^{-x}$$

$$\underline{\underline{y=C_1e^{-x}+C_2xe^{-x}+C_3e^{x}+C_4xe^{x}+\frac{1}{8}x^2e^{-x}}}$$

Problem 3

【Solution】

(a)

令 $x=e^t, t=\ln(x), x>0, D=\frac{d}{dt}$

$$\left(D(D-1)-2\right)y=e^{(m+2)t}$$

1. 求 y_h

令 $\left(D(D-1)-2\right)y_h=0$

$$D(D-1)-2=D^2-D-2=(D-2)(D+1)=0$$

$$D=-1,2$$

$$y_h(t)=C_1e^{-t}+C_2e^{2t}$$

$$y_h(x)=C_1\frac{1}{x}+C_2x^2, x>0$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t)=\frac{1}{(D-2)(D+1)}e^{(m+2)t}$$

$$\because m+2>0 \ \therefore m+3\neq 0$$

Case 1. $m\neq 0$

$$y_p(t)=\frac{1}{m(m+3)}e^{(m+2)t}$$

$$y_p(x)=\frac{1}{m(m+3)}x^{m+2}, x>0$$

$$\underline{\underline{y(x)=C_1\frac{1}{x}+C_2x^2+\frac{1}{m(m+3)}x^{m+2}, x>0}}$$

Case 2. $m = 0$

$$y_p(t) = \frac{1}{3}e^{2t}\frac{1}{D}(1)$$

$$y_p(t) = \frac{1}{3}te^{2t}$$

$$y_p(x) = \frac{1}{3}x^2\ln(x), x > 0$$

$$\underline{\underline{y(x) = C_1\frac{1}{x} + C_2x^2 + \frac{1}{3}x^2\ln(x), x > 0}}$$

(b)

Case 1. $m \neq 0$

$$y(x) = C_1\frac{1}{x} + C_2x^2 + \frac{1}{m(m+3)}x^{m+2}, x > 0$$

代入 I.C.

$$C_1 = 0, C_2 = -\frac{1}{m(m+3)}$$

$$\underline{\underline{y(x) = \frac{1}{m(m+3)}\left(x^{m+2} - x^2\right), x > 0}}$$

Case 2. $m = 0$

$$y(x) = C_1\frac{1}{x} + C_2x^2 + \frac{1}{3}x^2\ln(x), x > 0$$

代入 I.C.

$$C_1 = C_2 = 0$$

$$\underline{\underline{y(x) = \frac{1}{3}x^2\ln(x), x > 0}}$$

<u>Problem 4</u>

【Solution】

將原式做拉氏轉換，令 $Y(s)=L\left[y(t)\right]$

$$-\frac{d}{ds}\left[s^2Y(s)-sy(0)-y'(0)\right]+\frac{d}{ds}\left[sY(s)-y(0)\right]-Y(s)=0$$

$$\left(s-1\right)\frac{dY(s)}{ds}+2Y(s)=0$$

$$\frac{dY(s)}{ds}+\frac{2}{s-1}Y(s)=0$$

$$Y(s)=C\frac{1}{\left(s-1\right)^2}$$

$$y(t)=L^{-1}\left[Y(s)\right]=Cte^t$$

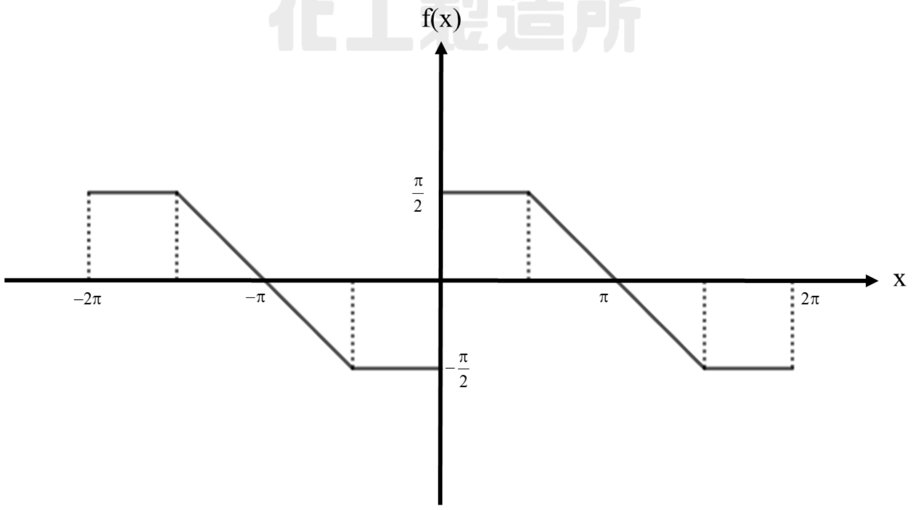
代入 I.C.

$$C=3$$

$$\underline{\underline{y(t)=3te^t,t>0}}$$

<u>Problem 5</u>

【Solution】



Problem 6

【**Solution**】

由圖形可知，此函數應為週期為2π之偶函數，利用 Fourier cosine series 展開

令 $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(nx)$

$$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)dx=\frac{1}{\pi}\int_0^{\pi}(-x+\pi)dx=\frac{\pi}{2}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos(nx)dx=\frac{2}{\pi}\int_0^{\pi}(-x+\pi)\cos(nx)dx=\frac{2(1-\cos(n\pi))}{n^2\pi}$$

$$\underline{\underline{f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\left(\frac{2(1-\cos(n\pi))}{n^2\pi}\right)\cos(nx)}}$$

Problem 7

【**Solution**】

令 $\vec{Y}=\begin{bmatrix}y_1\\y_2\end{bmatrix},A=\begin{bmatrix}3&-4\\1&-1\end{bmatrix}$

$$\frac{d\vec{Y}}{dt}=A\vec{Y}$$

令 $P_A(\lambda)=\det(A-\lambda I)=(\lambda-1)^2=0$

$$\lambda=1,1$$

代入 $(A-\lambda I)\vec{X}_1=\vec{0}$

$$\begin{bmatrix}2&-4\\1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\vec{X}_1=\begin{bmatrix}x_1\\x_2\end{bmatrix}=c_1\begin{bmatrix}2\\1\end{bmatrix},c_1\neq 0$$

取 $\begin{bmatrix}2\\1\end{bmatrix}$ 作為特徵向量

代入 $(A-\lambda I)\vec{X}_2=\vec{X}_1$

$$\begin{bmatrix}2&-4\\1&-2\end{bmatrix}\begin{bmatrix}x_3\\x_4\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}，\vec{X}_2=\begin{bmatrix}x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

取$\begin{bmatrix}1\\0\end{bmatrix}$作為廣義特徵向量

$$\vec{Y}=C_1\vec{X}_1e^t+C_2\left(\vec{X}_2+t\vec{X}_1\right)e^t$$

$$\underline{\underline{\vec{Y}=C_1\begin{bmatrix}2\\1\end{bmatrix}e^t+C_2\left(\begin{bmatrix}1\\0\end{bmatrix}+t\begin{bmatrix}2\\1\end{bmatrix}\right)e^t}}$$

Problem 8

【Solution】

By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}dA=\iiint_V\left(\nabla\cdot\vec{F}\right)dV$$

$$\iint_S \vec{F}\cdot\hat{n}dA=\iiint_V\left(1+1+2z\right)dV=\underline{\underline{96}}$$

Problem 9

【Solution】

令$u(x,t)=w(x,t)+\phi(x)$，其中$\phi(x)=\lim\limits_{t\to\infty}u(x,t)$為穩態解

$$\begin{cases}u(0,t)=w(0,t)+\phi(0)=3\\u(1,t)=w(1,t)+\phi(1)=4\end{cases}$$

$$\phi(x)=x+3$$

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}$$

(a)

令$w(x,t)=X(x)T(t)$代入上式

$$XT'=X''T$$

同乘$\dfrac{1}{XT}$，

$$\frac{X''}{X}=\frac{T'}{T}=-\lambda$$

$$\begin{cases}X''+\lambda X=0\\T'+\lambda T=0\end{cases}\text{，其中}X''+\lambda X=0\text{為特徵方程式}$$

1. 令$\lambda=0$代入特徵方程式

$$X''(x)=0$$

$$X(x)=C_1+C_2x$$

代入 B.C.

$$C_1=C_2=0$$

2. 令 $\lambda=-k^2, k>0$ 代入特徵方程式

$$X''(x)-k^2X(x)=0$$

$$X(x)=C_1\cosh(kx)+C_2\sinh(kx)$$

代入 B.C.

$$C_1=C_2=0$$

3. 令 $\lambda=k^2, k>0$ 代入特徵方程式

$$X''(x)+k^2X(x)=0 \text{，} X(x)=C_1\cos(kx)+C_2\sin(kx)$$

代入 B.C.

$$C_1=0$$

令 $C_2\neq 0$

$$X(1)=C_2\sin\left(k\right)=0$$

$$k=n\pi, n=1,2,3,\ldots$$

$$\lambda=k^2=\left(n\pi\right)^2, n=1,2,3,\ldots$$

$$X_n\left(x\right)=\sin\left(n\pi x\right), n=1,2,3,\ldots$$

$$T'+\left(n^2\pi^2\right)T=0$$

$$T_n=\alpha_n e^{-n^2\pi^2t}$$

$$w_n(x,t)=X_n(x)T_n(t)=C_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$w(x,t)=\sum_{n=1}^{\infty}C_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$u(x,t)=x+3+\sum_{n=1}^{\infty}C_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

代入 I.C.

$$u(x,0)=x+3+\sum_{n=1}^{\infty}C_n\sin(n\pi x)=0$$

$$\sum_{n=1}^{\infty}C_n\sin(n\pi x)=-x-3$$

$$C_n=2\int_0^1(-x-3)\sin(n\pi x)dx=\frac{2\left(4(-1)^n-3\right)}{n\pi}$$

$$u(x,t)=x+3+\sum_{n=1}^{\infty}\left(\frac{2\left(4(-1)^n-3\right)}{n\pi}\right)e^{-n^2\pi^2 t}\sin(n\pi x)$$

109 年台大工程數學

Problem 1

【Solution】

(a)

$$\text{velocity} = \vec{v} = \frac{d\vec{F}}{dt} = \underline{\underline{3\hat{i} - 2t\hat{k}}}$$

(b)

$$\text{speed} = |\vec{v}| = \sqrt{3^2 + (-2t)^2} = \underline{\underline{\sqrt{9+4t^2}}}$$

(c)

$$\text{unit tangent vector} = \vec{T}(t) = \frac{\frac{d\vec{F}}{dt}}{\left|\frac{d\vec{F}}{dt}\right|} = \underline{\underline{\frac{3\hat{i} - 2t\hat{k}}{\sqrt{9+4t^2}}}}$$

(d)

$$\vec{T}'(t) = \frac{-12t}{\left(9+4t^2\right)^{3/2}}\hat{i} + \frac{18}{\left(9+4t^2\right)^{3/2}}\hat{k}$$

$$\text{unit normal vector} = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|} = \underline{\underline{\frac{-2t}{\sqrt{9+4t^2}}\hat{i} + \frac{3}{\sqrt{9+4t^2}}\hat{k}}}$$

(e)

$$\text{acceleration} = \vec{a} = \frac{d\vec{v}}{dt} = \underline{\underline{-2\hat{k}}}$$

(f)

$$\text{tangent component of acceleration} = \vec{a} \cdot \vec{T}(t) = \underline{\underline{\frac{4t}{\sqrt{9+4t^2}}}}$$

(g)

$$\text{normal component of acceleration} = \vec{a} \cdot \vec{N}(t) = \underline{\underline{\frac{-6}{\sqrt{9+4t^2}}}}$$

(h)

$$\kappa = \left|\frac{\vec{T}'(t)}{\vec{F}'(t)}\right| = \underline{\underline{\frac{6}{\left(9+4t^2\right)^{3/2}}}}$$

Problem 2

【Solution】

$$\left[\sin(t)-\cos(t)\right]^2=\sin^2(t)-2\sin(t)\cos(t)+\cos^2(t)=1-\sin(2t)$$

$$L\left[f(t)\right]=L\left[1-\sin(2t)\right]=\underline{\underline{\frac{1}{s}-\frac{2}{s^2+4}}}$$

Problem 3

【Solution】

將原式做拉氏轉換，令 $Y(s)=L\left[y(t)\right]$

$$\left(s^2Y(s)-sy(0)-y'(0)\right)+\frac{d}{ds}\left(sY(s)-y(0)\right)+Y(s)=\frac{1}{s}$$

$$\frac{dY(s)}{ds}+\left(s+\frac{2}{s}\right)Y(s)=1+\frac{2}{s}+\frac{1}{s^2}$$

$$Y(s)=\frac{1}{s}+\frac{2}{s^2}+C\frac{1}{s^2}e^{\frac{-s^2}{2}}$$

利用初值定理

$$\lim_{s\to\infty}sY(s)=\lim_{t\to 0}y(t)=y(0)=1$$

$$\lim_{s\to\infty}sY(s)=\lim_{s\to\infty}\left(1+\frac{2}{s}+C\frac{1}{s}e^{\frac{-s^2}{2}}\right)=1 \text{ ， } C=0$$

$$Y(s)=\frac{1}{s}+\frac{2}{s^2}$$

$$y(t)=L^{-1}\left[Y(s)\right]=\underline{\underline{1+2t, t>0}}$$

Problem 4

【**Solution**】

令 $A=\begin{bmatrix}-7&0&1&0\\0&1&1&0\\-4&0&2&0\\0&0&0&0\end{bmatrix}$ ，$P^{-1}AP=D$

令 $P_A(\lambda)=\det(A-\lambda I)=\lambda(\lambda-1)(\lambda^2+5\lambda-10)=0$

$$\lambda=0,1,\frac{-5\pm\sqrt{65}}{2}$$

1. $\lambda=0$ 代入 $(A-\lambda I)\vec{X}=\vec{0}$

$$\begin{bmatrix}-7&0&1&0\\0&1&1&0\\-4&0&2&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=c_1\begin{bmatrix}0\\0\\0\\1\end{bmatrix},c_1\neq 0$$

取特徵向量為 $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

2. $\lambda=1$ 代入 $(A-\lambda I)\vec{X}=\vec{0}$

$$\begin{bmatrix}-8&0&1&0\\0&0&1&0\\-4&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=c_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix},c_2\neq 0$$

取特徵向量為$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

3. $\lambda=\dfrac{-5+\sqrt{65}}{2}$ 代入 $(A-\lambda I)\vec{X}=\vec{0}$

$$\begin{bmatrix}\frac{-9-\sqrt{65}}{2} & 0 & 1 & 0\\ 0 & \frac{7-\sqrt{65}}{2} & 1 & 0\\ -4 & 0 & \frac{9-\sqrt{65}}{2} & 0\\ 0 & 0 & 0 & \frac{5-\sqrt{65}}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=c_3\begin{bmatrix}\frac{9-\sqrt{65}}{8}\\ \frac{-2}{7-\sqrt{65}}\\ 1\\ 0\end{bmatrix},c_3\neq 0$$

取特徵向量為$\begin{bmatrix}\frac{9-\sqrt{65}}{8}\\ \frac{-2}{7-\sqrt{65}}\\ 1\\ 0\end{bmatrix}$

4. $\lambda=\dfrac{-5-\sqrt{65}}{2}$ 代入 $(A-\lambda I)\vec{X}=\vec{0}$

$$\begin{bmatrix}\frac{-9+\sqrt{65}}{2} & 0 & 1 & 0\\ 0 & \frac{7+\sqrt{65}}{2} & 1 & 0\\ -4 & 0 & \frac{9+\sqrt{65}}{2} & 0\\ 0 & 0 & 0 & \frac{5+\sqrt{65}}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_3 \begin{bmatrix} \frac{9+\sqrt{65}}{8} \\ \frac{-2}{7+\sqrt{65}} \\ 1 \\ 0 \end{bmatrix}, c_3 \neq 0$$

取特徵向量為$\begin{bmatrix} \frac{9+\sqrt{65}}{8} \\ \frac{-2}{7+\sqrt{65}} \\ 1 \\ 0 \end{bmatrix}$

$$P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-5+\sqrt{65}}{2} & 0 \\ 0 & 0 & 0 & \frac{-5-\sqrt{65}}{2} \end{bmatrix}$$

$$\underline{\underline{P = \begin{bmatrix} 0 & 0 & \frac{9-\sqrt{65}}{8} & \frac{9+\sqrt{65}}{8} \\ 0 & 1 & \frac{-2}{7-\sqrt{65}} & \frac{-2}{7+\sqrt{65}} \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}}$$

Problem 5

【Solution】

令 $D=\dfrac{d}{dx}$

$$\left(D^2-6D+9\right)y=6x+2-12e^{3x}$$

1. 求 y_h

令 $\left(D^2-6D+9\right)y_h=0$

$$D^2-6D+9=\left(D-3\right)^2=0$$

$$D=3,3$$

$$y_h=C_1e^{3x}+C_2xe^{3x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p=\frac{1}{D^2-6D+9}\left(6x+2-12e^{3x}\right)$$

$$y_p=\frac{1}{9}\frac{1}{1-\left(-\dfrac{D^2}{9}+\dfrac{2D}{3}\right)}\left(6x+2\right)-12\frac{1}{\left(D-3\right)^2}e^{3x}$$

$$y_p=\frac{1}{9}\left(1+\left(-\frac{D^2}{9}+\frac{2D}{3}\right)+\left(-\frac{D^2}{9}+\frac{2D}{3}\right)^2+\ldots\right)\left(6x+2\right)-12e^{3x}\frac{1}{\left(D\right)^2}\left(1\right)$$

$$y_p=\frac{1}{9}\left(1+\frac{2D}{3}\right)\left(6x+2\right)-12e^{3x}\left(\frac{1}{2}x^2\right)$$

$$y_p=\frac{2}{3}\left(x+1\right)-6x^2e^{3x}$$

$$\underline{\underline{y=C_1e^{3x}+C_2xe^{3x}+\frac{2}{3}\left(x+1\right)-6x^2e^{3x}}}$$

Problem 6

【**Solution**】

(a)

非線性 P.D.E.無法分離變數

(b)

線性且齊性 P.D.E.可以分離變數

(c)

非線性 P.D.E.無法分離變數

(d)

非齊性 P.D.E.無法分離變數

(e)

線性且齊性 P.D.E.可以分離變數

Problem 7

【**Solution**】

令 $u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)$

$$\nabla^2u(x,y)=\nabla^2u_1(x,y)+\nabla^2u_2(x,y)+\nabla^2u_3(x,y)+\nabla^2u_4(x,y)=0$$

令 $\begin{cases}\nabla^2u_1(x,y)=0,u_1(x,0)=0,u_1(x,L)=g(x),u_1(0,y)=0,u_1(K,y)=0\\ \nabla^2u_2(x,y)=0,u_2(x,0)=0,u_2(x,L)=0,u_2(0,y)=0,u_2(K,y)=g_1(y)\\ \nabla^2u_3(x,y)=0,u_3(x,0)=f(x),u_3(x,L)=0,u_3(0,y)=0,u_3(K,y)=0\\ \nabla^2u_4(x,y)=0,u_4(x,0)=0,u_4(x,L)=0,u_4(0,y)=f_1(y),u_4(K,y)=0\end{cases}$

a. $\nabla^2u_1(x,y)=0,u_1(x,0)=0,u_1(x,L)=g(x),u_1(0,y)=0,u_1(K,y)=0$

令$u_1(x,y)=X(x)Y(y)$代入$\nabla^2u_1(x,y)=0$

$$\begin{cases}u_1(0,y)=X(0)Y(y)=0\Rightarrow X(0)=0\\ u_1(K,y)=X(K)Y(y)=0\Rightarrow X(K)=0\end{cases}$$

$$X''Y+XY''=0$$

同乘$\dfrac{1}{XY}$，

$$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda$$

$$\begin{cases}X''+\lambda X=0\\ Y''-\lambda Y=0\end{cases}$$

$$X''+\lambda X=0,\ X(0)=X(K)=0$$

$$X_n(x)=\sin\left(\frac{n\pi x}{K}\right), n=0,1,2,3,\cdots$$

$$\lambda_n=\left(\frac{n\pi}{K}\right)^2$$

$$Y''-\left(\frac{n\pi}{K}\right)^2 Y=0$$

$$Y_n(y)=\alpha_n\cosh\left(\frac{n\pi y}{K}\right)+\beta_n\sinh\left(\frac{n\pi y}{K}\right)$$

$$u_{1n}(x,y)=X_n(x)Y_n(y)=\left(\alpha_n\cosh\left(\frac{n\pi y}{K}\right)+\beta_n\sinh\left(\frac{n\pi y}{K}\right)\right)\sin\left(\frac{n\pi x}{K}\right)$$

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(\alpha_n\cosh\left(\frac{n\pi y}{K}\right)+\beta_n\sinh\left(\frac{n\pi y}{K}\right)\right)\sin\left(\frac{n\pi x}{K}\right)$$

$$\because u_1(x,0)=0 \;\therefore \alpha_n=0$$

$$\because u_1(x,L)=g(x) \;\therefore \beta_n=\frac{2\int_0^K g(x)\sin\left(\frac{n\pi x}{K}\right)dx}{K\sinh\left(\frac{n\pi L}{K}\right)}$$

$$\underline{\underline{u_1(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\int_0^K g(x)\sin\left(\frac{n\pi x}{K}\right)dx}{K\sinh\left(\frac{n\pi L}{K}\right)}\right)\sinh\left(\frac{n\pi y}{K}\right)\sin\left(\frac{n\pi x}{K}\right)}}$$

b. $\nabla^2 u_2(x,y)=0, u_2(x,0)=0, u_2(x,L)=0, u_2(0,y)=0, u_2(K,y)=g_1(y)$

$$\underline{\underline{u_2(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\int_0^L g_1(y)\sin\left(\frac{n\pi y}{L}\right)dy}{L\sinh\left(\frac{n\pi K}{L}\right)}\right)\sinh\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)}}$$

c. $\nabla^2 u_3(x,y)=0, u_3(x,0)=f(x), u_3(x,L)=0, u_3(0,y)=0, u_3(K,y)=0$

$$\underline{\underline{u_3(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\int_0^K f(x)\sin\left(\frac{n\pi x}{K}\right)dx}{K\sinh\left(\frac{n\pi L}{K}\right)}\right)\sinh\left(\frac{n\pi(L-y)}{K}\right)\sin\left(\frac{n\pi x}{K}\right)}}$$

d. $\nabla^2 u_4(x,y)=0, u_4(x,0)=0, u_4(x,L)=0, u_4(0,y)=f_1(y), u_4(K,y)=0$

$$u_4(x,y)=\sum_{n=1}^{\infty}\left(\frac{2\int_0^L f_1(y)\sin\left(\frac{n\pi y}{L}\right)dy}{L\sinh\left(\frac{n\pi K}{L}\right)}\right)\sinh\left(\frac{n\pi(K-x)}{L}\right)\sin\left(\frac{n\pi y}{L}\right)$$

$$\underline{\underline{u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)}}$$

110 年台大工程數學

Problem 1

【Solution】

(a)

$$dy = ye^{3x}dx$$

同乘 $\frac{1}{y}$，

$$\frac{1}{y}dy = e^{3x}dx$$

$$\ln|y| = \frac{1}{3}e^{3x} + C_1$$

$$\underline{\underline{y = C_2 e^{\frac{1}{3}e^{3x}}}}$$

(b)

令 $D = \frac{d}{dx}$

$$\left(D^2 + 3D + 2.25\right)y = -10e^{-1.5x}$$

1. 求 y_h

令 $\left(D^2 + 3D + 2.25\right)y_h = 0$

$$D^2 + 3D + 2.25 = \left(D + 1.5\right)^2 = 0$$

$$D = -1.5, -1.5 \text{，} y_h = C_1 e^{-1.5x} + C_2 x e^{-1.5x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p = \frac{1}{\left(D+1.5\right)^2}\left(-10e^{-1.5x}\right)$$

$$y_p = -10e^{-1.5x}\frac{1}{D^2}(1)$$

$$y_p = -10e^{-1.5x}\left(\frac{1}{2}x^2\right) = -5x^2e^{-1.5x}$$

$$\underline{\underline{y = C_1 e^{-1.5x} + C_2 x e^{-1.5x} - 5x^2 e^{-1.5x}}}$$

(c)

令 $x=e^{t},t=\ln(x),x>0,D=\dfrac{d}{dt}$

$$\left(D(D-1)(D-2)-3D(D-1)+6D-6\right)y=te^{4t}$$

1. 求 y_h

令 $\left(D(D-1)(D-2)-3D(D-1)+6D-6\right)y_h=0$

$$D(D-1)(D-2)-3D(D-1)+6D-6=(D-1)(D-2)(D-3)=0$$

$$D=1,2,3$$

$$y_h(t)=C_1e^{t}+C_2e^{2t}+C_3e^{3t}$$

$$y_h(x)=C_1x+C_2x^2+C_3x^3,x>0$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t)=\frac{1}{(D-1)(D-2)(D-3)}\left(te^{4t}\right)$$

$$y_p(t)=e^{4t}\frac{1}{(D+3)(D+2)(D+1)}(t)$$

$$y_p(t)=e^{4t}\frac{1}{D^3+6D^2+11D+6}(t)$$

$$y_p(t)=\frac{1}{6}e^{4t}\frac{1}{1-\left(-\frac{1}{6}D^3-D^2-\frac{11}{6}D\right)}(t)$$

$$y_p(t)=\frac{1}{6}e^{4t}\left(1+\left(-\frac{1}{6}D^3-D^2-\frac{11}{6}D\right)+\left(-\frac{1}{6}D^3-D^2-\frac{11}{6}D\right)^2+\ldots\right)(t)$$

$$y_p(t)=\frac{1}{6}e^{4t}\left(1-\frac{11}{6}D\right)(t)=\frac{1}{6}e^{4t}\left(t-\frac{11}{6}\right)$$

$$y_p(x)=\frac{1}{6}x^4\left(\ln(x)-\frac{11}{6}\right),x>0$$

$$\underline{\underline{y(x)=C_1x+C_2x^2+C_3x^3+\frac{1}{6}x^4\left(\ln(x)-\frac{11}{6}\right),x>0}}$$

Problem 2

【Solution】

(a)

$$L\left[e^{-t}+\sinh(t)-t\right]=\underline{\underline{\frac{1}{s+1}+\frac{1}{s^2-1}-\frac{1}{s^2}}}$$

(b)

將原式做拉氏轉換，令 $Y(s)=L\left[y(t)\right]$

$$s^2Y(s)-sy(0)-y'(0)-Y(s)=\frac{1}{s^2}$$

$$\left(s^2-1\right)Y(s)-s-1=\frac{1}{s^2}$$

$$Y(s)=\frac{1}{s^2\left(s^2-1\right)}+\frac{s}{s^2-1}+\frac{1}{s^2-1}$$

$$Y(s)=-\frac{1}{s^2}+\frac{s}{s^2-1}+\frac{2}{s^2-1}$$

$$y(t)=L^{-1}\left[Y(s)\right]=\underline{\underline{-t+\cosh(t)+2\sinh(t),t>0}}$$

Problem 3

【Solution】

By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}dA=\iiint_V\left(\nabla\cdot\vec{F}\right)dV$$

$$\iint_S \vec{F}\cdot\hat{n}dA=\iiint_V 3dV=3\left(\frac{4}{3}\pi(3)^3\right)=\underline{\underline{108\pi}}$$

Problem 4

【Solution】

$$\left(x^2-x\right)y''+\left(3x-1\right)y'+y=0$$

$$\left[\left(x^2-x\right)y''+\left(2x-1\right)y'\right]+\left(xy'+y\right)=0$$

$$\frac{d}{dx}\left[\left(x^2-x\right)y'\right]+\frac{d}{dx}\left(xy\right)=0$$

$$\left(x^2-x\right)y'+xy=C_1$$

同乘 $\dfrac{1}{x^2-x}$ ，

$$y'+\frac{1}{x-1}y=\frac{C_1}{x\left(x-1\right)}$$

$$\underline{\underline{y=C_1\frac{\ln|x|}{x-1}+C_2\frac{1}{x-1}}}$$

<u>Problem 5</u>

【Solution】

令 $u(x,t)=\sum_{n=0}^{\infty}C_n(x)\cos\left(\frac{n\pi y}{2}\right)$ 代入 P.D.E.

$$\sum_{n=0}^{\infty}\left[C_n''(x)-\left(\frac{n^2\pi^2}{4}\right)C_n(x)\right]\cos\left(\frac{n\pi y}{2}\right)=0$$

$$C_n''(x)-\left(\frac{n^2\pi^2}{4}\right)C_n(x)=0$$

$$C_n(x)=\alpha_n\cosh\left(\frac{n\pi x}{2}\right)+\beta_n\sinh\left(\frac{n\pi x}{2}\right)$$

$$u(x,t)=\sum_{n=0}^{\infty}\left[\alpha_n\cosh\left(\frac{n\pi x}{2}\right)+\beta_n\sinh\left(\frac{n\pi x}{2}\right)\right]\cos\left(\frac{n\pi y}{2}\right)$$

代入 B.C.

$$\sum_{n=0}^{\infty}\alpha_n\cos\left(\frac{n\pi y}{2}\right)=0$$

$$\alpha_n=0$$

代入 B.C.

$$\sum_{n=0}^{\infty}\beta_n\sinh(n\pi)\cos\left(\frac{n\pi y}{2}\right)=2-y$$

$$\beta_n=\frac{1}{\sinh(n\pi)}\int_0^2(2-y)\cos\left(\frac{n\pi y}{2}\right)dy=\frac{4\left[1-(-1)^n\right]}{n^2\pi^2}$$

$$\underline{\underline{u(x,t)=\sum_{n=0}^{\infty}\left(\frac{4\left[1-(-1)^n\right]}{n^2\pi^2}\right)\sinh\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi y}{2}\right)}}$$

Problem 6

【Solution】

令 $u(x,t)=w(x,t)+\phi(x,t)$

$$\begin{cases} u(0,t)=w(0,t)+\phi(0,t)=t \\ u(1,t)=w(1,t)+\phi(1,t)=3 \end{cases}$$

$$\begin{cases} w(0,t)=w(1,t)=0 \\ \phi(0,t)=t,\ \phi(1,t)=3 \end{cases}$$

令 $\phi(x,t)=\alpha(t)x+\beta(t)$ 代入上式

$$\phi(x,t)=(3-t)x+t$$

將 $u(x,t)=w(x,t)+(3-t)x+t$ 代入 P.D.E.

$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}=x-1$$

令 $x-1=\sum_{n=1}^{\infty} d_n \sin(n\pi x)$

$$d_n=\frac{2}{1}\int_0^1 (x-1)\sin(n\pi x)dx=-\frac{2}{n\pi}$$

$$x-1=\sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)\sin(n\pi x)$$

令 $w(x,t)=\sum_{n=1}^{\infty} C_n(t)\sin(n\pi x)$ 代入 $\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}=\sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)\sin(n\pi x)$

$$\sum_{n=1}^{\infty}\left[C_n'(t)+n^2\pi^2 C_n(t)\right]\sin(n\pi x)=\sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)\sin(n\pi x)$$

$$C_n'(t)+n^2\pi^2 C_n(t)=\left(-\frac{2}{n\pi}\right)$$

$$C_n(t)=c_n e^{-n^2\pi^2 t}-\frac{2}{n^3\pi^3}$$

$$u(x,t)=\sum_{n=1}^{\infty}\left(c_n e^{-n^2\pi^2 t}-\frac{2}{n^3\pi^3}\right)\sin(n\pi x)+(3-t)x+t$$

代入 I.C.

$$\sum_{n=1}^{\infty}\left(c_n-\frac{2}{n^3\pi^3}\right)\sin(n\pi x)+3x=0$$

$$c_n=\frac{2}{n^3\pi^3}+\frac{2}{1}\int_0^1(-3x)\sin(n\pi x)dx=\frac{2}{n^3\pi^3}+\frac{6(-1)^n}{n\pi}$$

$$u(x,t)=\sum_{n=1}^{\infty}\left[\left(\frac{2}{n^3\pi^3}+\frac{6(-1)^n}{n\pi}\right)e^{-n^2\pi^2 t}-\frac{2}{n^3\pi^3}\right]\sin(n\pi x)+(3-t)x+t$$

Problem 7

【**Solution**】

令 $f(x)=\int_0^{\infty} B(w)\sin(wx)dw$

$$B(w)=\frac{2}{\pi}\int_0^{\pi}\sin(x)\sin(wx)dx=\frac{2\sin(\pi w)}{\pi(1-w^2)}$$

$$f(x)=\int_0^{\infty}\left(\frac{2\sin(\pi w)}{\pi(1-w^2)}\right)\sin wxdx$$

Problem 8

【**Solution**】

$$F^{-1}\left[\frac{6e^{iw}\cos(4w)}{9+w^2}\right]=F^{-1}\left[\frac{6e^{iw}}{9+w^2}\left(\frac{e^{i4w}+e^{-i4w}}{2}\right)\right]$$

$$=\frac{1}{2}F^{-1}\left[\frac{6e^{iw}}{9+w^2}\left(e^{i4w}+e^{-i4w}\right)\right]=\frac{1}{2}\left(e^{-3|x+5|}+e^{-3|x-3|}\right)$$

111 年台大工程數學

Problem 1

【Solution】

(a)

令 $y = x^m$ 代入原式

$$\left(m\left(m-1\right)+2m+1\right)x^m = 0$$

令 $x^m \neq 0$

$$m\left(m-1\right)+2m+1 = m^2+m+1 = 0$$

$$m = \frac{-1\pm\sqrt{3}i}{2}$$

$$\underline{\underline{y = x^{-\frac{1}{2}}\left(c_1\cos\left(\frac{\sqrt{3}}{2}\ln x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right), x>0}}$$

(b)

令 $D = \dfrac{d}{dx}$

$$\left(D^2-1\right)y = e^x$$

1. 求 y_h

令 $\left(D^2-1\right)y_h = 0$

$$D^2-1=0 \text{ ， } D=-1,1$$

$$y_h = C_1e^{-x}+C_2e^x$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p = \frac{1}{D^2-1}\left(e^x\right) = \frac{1}{\left(D+1\right)\left(D-1\right)}\left(e^x\right)$$

$$y_p = \frac{1}{2}\frac{1}{\left(D-1\right)}\left(e^x\right) = \frac{1}{2}e^x\frac{1}{D}\left(1\right) = \frac{1}{2}xe^x$$

$$\underline{\underline{y = C_1e^{-x}+C_2e^x+\frac{1}{2}xe^x}}$$

Problem 2

【Solution】

$$y(t)+y(t)*1=1$$

將上式做拉氏轉換，令 $Y(s)=L\left[y(t)\right]$

$$Y(s)+\frac{1}{s}Y(s)=\frac{1}{s}$$

$$\left(1+\frac{1}{s}\right)Y(s)=\frac{1}{s}$$

$$Y(s)=\frac{1}{s+1}$$

$$y(t)=L^{-1}\left[Y(s)\right]=\underline{\underline{e^{-t},t>0}}$$

Problem 3

【Solution】

(a)

此為 Bessel equation

$$x^2y''+xy'+\left((5x)^2-1^2\right)y=0$$

$$y=C_1J_1(5x)+C_2Y_1(5x)$$

(b)

y(x) is a finite physical quantity, and

$$\lim_{x\to 0}Y_1(5x)=-\infty$$

$$C_2=0$$

代入 $y(1)=2$

$$y(1)=C_1J_1(5)=2$$

$$C_1=\frac{2}{J_1(5)}$$

$$\underline{\underline{y=\frac{2}{J_1(5)}J_1(5x)}}$$

Problem 4

【Solution】

(a)

$$F\left[4H(t-2)e^{-3t}\cos(t-2)\right]=e^{-i2w}F\left[4H(t)e^{-3(t+2)}\cos t\right]$$

$$=2e^{-6}e^{-i2w}F\left[H(t)e^{-3t}\left(e^{it}+e^{-it}\right)\right]$$

$$=2e^{-i2w-6}\left(\frac{1}{3+i(w-1)}+\frac{1}{3+i(w+1)}\right)$$

(b)

$$F^{-1}\left[\frac{e^{(20-4w)i}}{3-(5-w)i}\right]=F^{-1}\left[\frac{e^{-i4(w-5)}}{3+i(w-5)}\right]=e^{i5t}F^{-1}\left[\frac{e^{-4iw}}{3+iw}\right]$$

$$=e^{i5t}F^{-1}\left[\frac{e^{-4iw}}{3+iw}\right]=e^{i5t}H(t-4)e^{-3(t-4)}$$

<u>Problem 5</u>

【Solution】

令 $u(x,t)=w(x,t)+\phi(x)$，其中 $\phi(x)=\lim_{t\to\infty}u(x,t)$ 為穩態解

$$\begin{cases} u(0,t)=w(0,t)+\phi(0)=T_1 \\ u(9,t)=w(9,t)+\phi(9)=T_2 \end{cases}$$

$$\begin{cases} w(0,t)=w(9,t)=0 \\ \phi(x)=\left(\dfrac{T_2-T_1}{9}\right)x+T_1 \end{cases}$$

將 $u(x,t)=w(x,t)+\phi(x)$ 代入 P.D.E.

$$\frac{\partial w}{\partial t}=\alpha^2\frac{\partial^2 w}{\partial x^2}$$

令 $w(x,t)=\sum_{n=1}^{\infty}C_n(t)\sin\left(\frac{n\pi x}{9}\right)$ 代入上式

$$\sum_{n=1}^{\infty}\left(C_n'(t)+\alpha^2C_n(t)\right)\sin\left(\frac{n\pi x}{9}\right)=0$$

$$C_n'(t)+\alpha^2C_n(t)=0$$

$$C_n(t)=c_ne^{-\alpha^2t}$$

$$w(x,t)=\sum_{n=1}^{\infty}c_ne^{-\alpha^2t}\sin\left(\frac{n\pi x}{9}\right)$$

$$\underline{\underline{u(x,t)=\left(\frac{T_2-T_1}{9}\right)x+T_1+\sum_{n=1}^{\infty}c_ne^{-\alpha^2t}\sin\left(\frac{n\pi x}{9}\right)}}$$

代入 I.C.

$$u(x,0)=\left(\frac{T_2-T_1}{9}\right)x+T_1+\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{9}\right)=x^2$$

$$\sum_{n=1}^{\infty}c_n\sin\left(\frac{n\pi x}{9}\right)=x^2-\left(\frac{T_2-T_1}{9}\right)x-T_1$$

$$c_n=\frac{2}{9}\int_0^9\left(x^2-\left(\frac{T_2-T_1}{9}\right)x-T_1\right)\sin\left(\frac{n\pi x}{9}\right)dx$$

$$\underline{\underline{c_n=\frac{2\left((T_2-81)(-1)^n+T_1\right)}{n\pi}+\frac{324}{n^3\pi^3}\left((-1)^n-1\right)}}$$

<u>Problem 6</u>

【Solution】

令 $\underline{\underline{u(x,y,z)=u_1(x,y,z)+u_2(x,y,z)}}$

$$\begin{cases} u_1(x,y,z):\begin{cases} u_1(0,y,z)=u_1(1,y,z)=u_1(x,y,0)=u_1(x,y,2)=0 \\ u_1(x,0,z)=0 \\ u_1(x,4,z)=g(z) \end{cases} \\ u_2(x,y,z):\begin{cases} u_2(0,y,z)=u_2(1,y,z)=u_2(x,y,0)=u_2(x,y,2)=0 \\ u_2(x,0,z)=f(x) \\ u_2(x,4,z)=0 \end{cases} \end{cases}$$

令 $u_1(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}(y)\sin\left(\frac{m\pi z}{2}\right)\sin(n\pi x)$ 代入 P.D.E.

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(c_{nm}''(y)-\left(\left(\frac{m\pi}{2}\right)^2+(n\pi)^2\right)c_{nm}(y)\right)\sin\left(\frac{m\pi z}{2}\right)\sin(n\pi x)=0$$

$$c_{nm}''(y)-\left(\left(\frac{m\pi}{2}\right)^2+(n\pi)^2\right)c_{nm}(y)=0$$

令 $\underline{\underline{\lambda=\sqrt{\left(\frac{m\pi}{2}\right)^2+(n\pi)^2}}}$

$$c_{nm}(y)=\alpha_{nm}\cosh(\lambda y)+\beta_{nm}\sinh(\lambda y)$$

代入 $u_1(x,0,z)=0$

$$\alpha_{nm}=0$$

$$u_1(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{nm}\sinh(\lambda y)\sin\left(\frac{m\pi z}{2}\right)\sin(n\pi x)$$

代入 $u_1(x,4,z)=g(z)$

$$u_1(x,4,z)=g(z)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\beta_{nm}\sinh(4\lambda)\sin\left(\frac{m\pi z}{2}\right)\right)\sin(n\pi x)$$

$$\sum_{m=1}^{\infty}\beta_{nm}\sinh(4\lambda)\sin\left(\frac{m\pi z}{2}\right)=2\int_0^1 g(z)\sin(n\pi x)dx$$

$$\underline{\underline{\beta_{nm}=\frac{2}{\sinh(4\lambda)}\int_0^2\left(\int_0^1 g(z)\sin(n\pi x)dx\right)\sin\left(\frac{m\pi z}{2}\right)dz}}$$

$$\underline{\underline{u_1(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{nm}\sinh(\lambda y)\sin\left(\frac{m\pi z}{2}\right)\sin(n\pi x)}}$$

$$\underline{\underline{u_2(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\gamma_{nm}\sinh\big(\lambda(4-y)\big)\sin\left(\frac{m\pi z}{2}\right)\sin(n\pi x)}}$$

代入 $u_2(x,0,z)=f(x)$

$$u_2(x,0,z)=f(x)=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\gamma_{nm}\sinh(4\lambda)\sin\left(\frac{m\pi z}{2}\right)\right)\sin(n\pi x)$$

$$\sum_{m=1}^{\infty}\gamma_{nm}\sinh(4\lambda)\sin\left(\frac{m\pi z}{2}\right)=2\int_0^1 f(x)\sin(n\pi x)dx$$

$$\underline{\underline{\gamma_{nm}=\frac{2}{\sinh(4\lambda)}\int_0^2\left(\int_0^1 f(x)\sin(n\pi x)dx\right)\sin\left(\frac{m\pi z}{2}\right)dz}}$$

112 年台大工程數學

Problem 1

【Solution】

令 $y_2 = u(x)y_1(x)$

$$y_2' = uy_1' + u'y_1$$

$$y_2'' = uy_1'' + 2u'y_1' + u''y_1$$

代入原式

$$u\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{0\text{ (homogeneous)}} + u''y_1 + u'\left(2y_1' + py_1\right) = 0$$

$$u''y_1 + u'\left(2y_1' + py_1\right) = 0$$

$$\frac{du'}{u'} + 2\frac{y_1'}{y_1}dx + pdx = 0$$

$$\int\frac{du'}{u'} + 2\int\frac{y_1'}{y_1}dx + \int pdx = 0$$

$$\ln\left|u'y_1^2\right| = -\int pdx + C_1$$

取 $\ln\left|u'y_1^2\right| = -\int pdx$

$$u'y_1^2 = \pm e^{-\int pdx}$$

取 $u'y_1^2 = e^{-\int pdx}$

$$u = \int\frac{e^{-\int pdx}}{y_1^2}dx + C_2$$

取 $u = \int\frac{e^{-\int pdx}}{y_1^2}dx$，$y_2 = uy_1 = \underline{\underline{y_1\int\frac{e^{-\int pdx}}{y_1^2}dx}}$

Problem 2

【**Solution**】

$$y=-(y')^2+xy'$$

令 $p(x)=y'$ 代入上式

$$y=-p^2+xp$$

$$\frac{dy}{dx}=-2p\frac{dp}{dx}+p+x\frac{dp}{dx}$$

$$\cancel{p}=(-2p+x)\frac{dp}{dx}+\cancel{p}$$

$$(-2p+x)\frac{dp}{dx}=0$$

同乘 $\dfrac{1}{-2p+x}$

$$\frac{dp}{dx}=0$$

$$p=C$$

代入 $y=-p^2+xp$

$$\underline{\underline{y=-C^2+Cx}}$$

其中 $-2p+x=0$ 為奇解

代入 $y=-p^2+xp$

$$y=-\frac{x^2}{4}+\frac{x^2}{2}$$

$$\underline{\underline{y=\frac{x^2}{4}\ \text{為奇解}}}$$

<u>Problem 3</u>

【Solution】

(a)

$$\vec{\nabla} f(x,y,z)=2x\vec{i}+2y\vec{j}+4z\hat{k}$$

$$\vec{\nabla} f\left(1,1,1\right)=2\vec{i}+2\vec{j}+4\hat{k}$$

$$D_{\hat{e}_c} f\left(1,1,1\right)=\vec{\nabla} f\left(1,1,1\right)\cdot\frac{\vec{c}}{\left|\vec{c}\right|}$$

$$D_{\hat{e}_c} f\left(1,1,1\right)=\left(2\hat{i}+2\hat{j}+4\hat{k}\right)\cdot\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}=\underline{\underline{\frac{8}{\sqrt{3}}}}$$

(b)

此題應有誤。

該點不在曲面上，無法求解

Problem 4

【Solution】

令 $P_A(\lambda)=\det(A-\lambda I)=-(\lambda+3)(\lambda^2+4\lambda-27)=0$

$$\lambda=-3,-2\pm\sqrt{31}$$

1. $\lambda=-3$ 代入 $(A+3I)\vec{X}=\vec{0}$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\vec{X}_1=c_1\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix},c_1\neq 0$$

2. $\lambda=-2+\sqrt{31}$ 代入 $(A+3I)\vec{X}=\vec{0}$

$$\begin{bmatrix}-\sqrt{31} & 2 & -3\\ 2 & 3-\sqrt{31} & -6\\ -1 & -2 & -4-\sqrt{31}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\vec{X}_2=c_2\begin{bmatrix}7+\sqrt{31}\\ 14+2\sqrt{31}\\ -1-\sqrt{31}\end{bmatrix},c_2\neq 0$$

3. $\lambda=-2-\sqrt{31}$ 代入 $(A+3I)\vec{X}=\vec{0}$

$$\begin{bmatrix}\sqrt{31} & 2 & -3\\ 2 & 3+\sqrt{31} & -6\\ -1 & -2 & -4+\sqrt{31}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\vec{X}_3=c_3\begin{bmatrix}7-\sqrt{31}\\ 14-2\sqrt{31}\\ -1+\sqrt{31}\end{bmatrix},c_3\neq 0$$

$$\underline{\underline{D = P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2+\sqrt{31} & 0 \\ 0 & 0 & -2-\sqrt{31} \end{bmatrix}}}$$

<u>Problem 5</u>

【Solution】

將原式做拉氏轉換，令 $Y(s) = L[y(t)]$

$$2\left[s^2Y(s) - sy(0) - y'(0)\right] - \frac{d}{ds}\left[sY(s) - y(0)\right] - Y(s) = \frac{6}{s}$$

$$\left(2s^2 - 2\right)Y(s) - s\frac{dY(s)}{ds} = \frac{6}{s}$$

$$\frac{dY(s)}{ds} + \left(-2s + \frac{2}{s}\right)Y(s) = \frac{-6}{s^2}$$

同乘 $s^2e^{-s^2}$ ，

$$\frac{d}{ds}\left(s^2e^{-s^2}Y(s)\right) = -6e^{-s^2}$$

$$s^2e^{-s^2}Y(s) = -6\int_0^s e^{-\tau^2}d\tau + C_1$$

$$Y(s) = C_1\frac{e^{s^2}}{s^2} - 6\frac{e^{s^2}}{s^2}\int_0^s e^{-\tau^2}d\tau$$

$$Y(s) = \frac{e^{s^2}}{s^2}\left(C_1 - 6\int_0^s e^{-\tau^2}d\tau\right)$$

利用初值定理

$$\lim_{s\to\infty} sY(s) = \lim_{t\to 0} y(t) = y(0) = 0$$

$$\lim_{s\to\infty} \underbrace{\frac{e^{s^2}}{s}}_{\to\infty} \underbrace{\left(C_1 - 6\int_0^s e^{-\tau^2}d\tau\right)}_{\text{必須逼近0才可能收斂}} = 0$$

$$C_1 - 6\underbrace{\lim_{s\to\infty}\left(\int_0^s e^{-\tau^2}d\tau\right)}_{\frac{\sqrt{\pi}}{2}} = 0$$

$$C_1 = 3\sqrt{\pi}$$

$$Y(s) = \frac{e^{s^2}}{s^2}\left(3\sqrt{\pi} - 6\int_0^s e^{-\tau^2}d\tau\right)$$

$$Y(s)=3\sqrt{\pi}\frac{e^{s^2}}{s^2}\left(1-\frac{2}{\sqrt{\pi}}\int_0^s e^{-\tau^2}d\tau\right)$$

$$Y(s)=\frac{3\sqrt{\pi}e^{s^2}}{s^2}\operatorname{erfc}(s)$$

$$y(t)=L^{-1}\left[\frac{3\sqrt{\pi}e^{s^2}}{s^2}\operatorname{erfc}(s)\right]=3\sqrt{\pi}L^{-1}\left[\frac{1}{s}\frac{e^{s^2}}{s}\operatorname{erfc}(s)\right]$$

$$y(t)=3\sqrt{\pi}\int_0^t L^{-1}\left[\frac{e^{s^2}}{s}\operatorname{erfc}(s)\right]\Bigg|_{t\to\tau}d\tau=3\sqrt{\pi}\int_0^t \operatorname{erf}\left(\frac{\tau}{2}\right)d\tau$$

$$y(t)=3\sqrt{\pi}\left[\tau\operatorname{erf}\left(\frac{\tau}{2}\right)\Bigg|_0^t-\frac{1}{\sqrt{\pi}}\int_0^t \tau e^{-\frac{\tau^2}{4}}d\tau\right]$$

$$y(t)=3\sqrt{\pi}\left[t\operatorname{erf}\left(\frac{t}{2}\right)-\frac{1}{\sqrt{\pi}}\left(-2e^{-\frac{\tau^2}{4}}\right)\Bigg|_0^t\right]$$

$$y(t)=3\sqrt{\pi}\left[t\operatorname{erf}\left(\frac{t}{2}\right)+\frac{2}{\sqrt{\pi}}\left(e^{-\frac{t^2}{4}}-1\right)\right]$$

$$\underline{\underline{y(t)=3\sqrt{\pi}\left[t\operatorname{erf}\left(\frac{t}{2}\right)+\frac{2}{\sqrt{\pi}}\left(e^{-\frac{t^2}{4}}-1\right)\right],t>0}}$$

<補充> 誤差函數與補誤差函數之拉氏轉換

(1) $L\left[\operatorname{erf}(t)\right]=\frac{1}{s}e^{\left(\frac{s}{2}\right)^2}\operatorname{erfc}\left(\frac{s}{2}\right)$

(2) $L\left[\operatorname{erf}\left(\frac{t}{2}\right)\right]=\frac{1}{s}e^{s^2}\operatorname{erfc}(s)$

Problem 6

【**Solution**】

令 $u(r,z)=R(r)\Phi(z)$ 代入P.D.E

$$R''\Phi+\frac{1}{r}R'\Phi+\Phi''=0$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-\frac{\Phi''}{\Phi}=\lambda$$

$$\begin{cases}\Phi''+\lambda\Phi=0\\ r^2R''+rR'-\lambda r^2R=0\end{cases}$$

$$\begin{cases}u(r,1)=R(r)\Phi(1)=0\\ u_z(r,0)=R(r)\Phi'(0)=0\end{cases}$$

$$\begin{cases}\Phi(1)=0\\ \Phi'(0)=0\end{cases}$$

1. 令 $\lambda=0$ 代入 $\Phi''+\lambda\Phi=0$

$$\Phi(z)=C_1+C_2z$$

代入B.C.

$$C_1=C_2=0$$

2. 令 $\lambda=-k^2,k>0$ 代入 $\Phi''+\lambda\Phi=0$

$$\Phi(z)=C_1\cosh(kz)+C_2\sinh(kz)$$

代入B.C.

$$C_1=C_2=0$$

3. 令 $\lambda=k^2,k>0$ 代入 $\Phi''+\lambda\Phi=0$

$$\Phi(z)=C_1\cos(kz)+C_2\sin(kz)$$

$$\Phi'(z)=-kC_1\sin(kz)+kC_2\cos(kz)$$

代入B.C.

$$C_2=0$$

令 $C_1 \neq 0$

$$\Phi(1) = C_1 \cos(k) = 0$$

$$k = \frac{(2n-1)\pi}{2}, n = 1, 2, 3, \ldots$$

$$\lambda = \left(\frac{(2n-1)\pi}{2}\right)^2, n = 1, 2, 3, \ldots$$

$$\Phi_n(z) = \cos\left(\frac{(2n-1)\pi}{2} z\right), n = 1, 2, 3, \ldots$$

$$r^2 R_n''(r) + r R_n'(r) - \left(\frac{(2n-1)^2 \pi^2}{4}\right) r^2 R_n(r) = 0$$

$$R_n(r) = \alpha_n I_0\left(\frac{(2n-1)\pi}{2} r\right) + \underbrace{\beta_n K_0\left(\frac{(2n-1)\pi}{2} r\right)}_{0\ (R_n(r)\text{ is finite})}$$

$$u_n(r,z) = R_n(r)\Phi_n(z) = \alpha_n I_0\left(\frac{(2n-1)\pi}{2} r\right)\cos\left(\frac{(2n-1)\pi}{2} z\right)$$

$$u(r,z) = \sum_{n=1}^{\infty} \alpha_n I_0\left(\frac{(2n-1)\pi}{2} r\right)\cos\left(\frac{(2n-1)\pi}{2} z\right)$$

代入 B.C.

$$u(1,z) = \sum_{n=1}^{\infty} \alpha I_0\left(\frac{(2n-1)\pi}{2}\right)\cos\left(\frac{(2n-1)\pi z}{2}\right) = kz$$

$$\therefore \alpha_n I_0\left(\frac{(2n-1)\pi}{2}\right) = 2\int_0^1 (kz)\cos\left(\frac{(2n-1)\pi}{2}\right) zdz$$

$$\alpha_n = \frac{-4k\left(2 + \pi(2n-1)(-1)^n\right)}{(2n-1)^2 \pi^2 I_0\left(\frac{(2n-1)\pi}{2}\right)}$$

$$u(r,z)=\sum_{n=1}^{\infty}\left(\frac{-4k\left(2+\pi(2n-1)(-1)^{n}\right)}{(2n-1)^{2}\pi^{2}I_0\left(\frac{(2n-1)\pi}{2}\right)}\right)I_0\left(\frac{(2n-1)\pi}{2}r\right)\cos\left(\frac{(2n-1)\pi}{2}z\right)$$

113 年台大工程數學

Problem 1

【Solution】

$$2y^3dx-4dx+3xy^2dy=0$$

$$y^2(2ydx+3xdy)-4dx=0$$

$$y^2\frac{d(x^2y^3)}{xy^2}-4dx=0$$

$$d(x^2y^3)-4xdx=0$$

$$\underline{\underline{x^2y^3-2x^2=C}}$$

代入 I.C.

$$C=-9$$

$$\underline{\underline{x^2y^3-2x^2=-9}}$$

Problem 2

【Solution】

令 $D=\frac{d}{dx}$

$$\left(D^4+2D^2+1\right)y=-\sin\left(x\right)+\cos\left(2x\right)$$

1. 求 y_h

令 $\left(D^4+2D^2+1\right)y_h=0$

$$D^4+2D^2+1=0$$

$$\left(D^2+1\right)^2=0$$

$$D=i,i,-i,-i$$

$$y_h=C_1\cos(x)+C_2\sin(x)+C_3x\cos(x)+C_4x\sin(x)$$

2. 求 y_p，利用 Heaviside 反運算子法

$$y_p = \frac{1}{(D^4+2D^2+1)}\left(-\sin(x)\right) + \frac{1}{(D^4+2D^2+1)}\left(\cos(2x)\right)$$

$$y_p = \frac{-1}{\left(D^2+1\right)^2}\left(\sin(x)\right) + \frac{1}{(D^2)^2-7}\left(\cos(2x)\right)$$

$$\boxed{\left(※\ \frac{1}{(D^2+a^2)}\sin(ax+b) = \frac{-x\cos(ax+b)}{2a}\ ,\ \frac{1}{(D^2+a^2)}\cos(ax+b) = \frac{x\sin(ax+b)}{2a}\right)}$$

$$y_p = \frac{1}{8}x^2\sin\left(x\right) + \frac{1}{9}\left(\cos(2x)\right)$$

$$\underline{\underline{y = C_1\cos(x) + C_2\sin(x) + C_3x\cos(x) + C_4x\sin(x) + \frac{1}{8}x^2\sin\left(x\right) + \frac{1}{9}\left(\cos(2x)\right)}}$$

<u>Problem 3</u>

【Solution】

(a)

$$L\left[t^2\cos\left(at\right)\right] = \left(-1\right)^2\frac{d^2}{ds^2}\left(L\left[\cos\left(at\right)\right]\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{d}{ds}\left[\frac{a^2-s^2}{\left(s^2+a^2\right)^2}\right] = \underline{\underline{\frac{2s^3-6a^2s}{\left(s^2+a^2\right)^3}}}$$

(b)

$$L^{-1}\left[\frac{s}{\left(s+2\right)^2\left(s^2+2s+10\right)}\right] = L^{-1}\left[\frac{\frac{3}{50}}{s+2} - \frac{\frac{1}{5}}{(s+2)^2} - \frac{\frac{3}{50}s - \frac{1}{5}}{s^2+2s+10}\right]$$

$$\underline{\underline{= \frac{3}{50}e^{-2t} - \frac{1}{5}te^{-2t} - \frac{3}{50}e^{-t}\cos\left(3t\right) + \frac{13}{150}e^{-t}\sin\left(3t\right), t>0}}$$

Problem 4

【Solution】

令 $x=e^{t},t=\ln(x),x>0,D=\dfrac{d}{dt}$

$$\left(D\left(D-1\right)-D+1\right)y=4te^{t}$$

1. 求 y_h

$$D(D-1)-D+1=\left(D-1\right)^{2}=0$$

$$D=1,1$$

$$y_h(t)=C_1e^{t}+C_2te^{t}$$

$$y_h(x)=C_1x+C_2x\ln\left(x\right),x>0$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t)=\frac{1}{\left(D-1\right)^{2}}\left(4te^{t}\right)=4e^{t}\frac{1}{D^{2}}\left(t\right)=\frac{2}{3}t^{3}e^{t}$$

$$y_p(x)=\frac{2}{3}x\ln^{3}\left(x\right),x>0$$

$$y=C_1x+C_2x\ln(x)+\frac{2}{3}x\ln^{3}(x),x>0$$

Problem 5

【**Solution**】

令 $P_A(\lambda)=\det(A-\lambda I)=-\left(\lambda-3\right)\left(\lambda^2+2\lambda+2\right)=0$

$$\lambda=3,-1+i,-1-i$$

1. $\lambda=3$ 代入 $(A-\lambda I)\vec{y}=\vec{0}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=c_1\begin{bmatrix} \frac{4}{9} \\ 1 \\ 0 \end{bmatrix},c_1\neq 0$$

2. $\lambda=-1+i$ 代入 $(A-\lambda I)\vec{y}=\vec{0}$

$$\begin{bmatrix} 4-i & 0 & 1 \\ 9 & -i & 2 \\ -9 & 4 & -i \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=c_2\begin{bmatrix} \frac{-4-i}{17} \\ \frac{-9+2i}{17} \\ 1 \end{bmatrix},c_2\neq 0$$

3. $\lambda=-1-i$ 代入 $(A-\lambda I)\vec{y}=\vec{0}$

$$\begin{bmatrix} 4+i & 0 & 1 \\ 9 & i & 2 \\ -9 & 4 & i \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_3 \begin{bmatrix} \dfrac{-4+i}{17} \\ \dfrac{-9-2i}{17} \\ 1 \end{bmatrix}, c_3 \neq 0$$

$$\underline{\underline{\vec{X}(t) = C_1 e^{3t} \begin{bmatrix} \frac{4}{9} \\ 1 \\ 0 \end{bmatrix} + C_2 \left[e^{-t} \left(\begin{bmatrix} \frac{-4}{17} \\ \frac{-9}{17} \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} \frac{-1}{17} \\ \frac{2}{17} \\ 0 \end{bmatrix} \sin(t) \right) \right] + C_3 \left[e^{-t} \left(\begin{bmatrix} \frac{-4}{17} \\ \frac{-9}{17} \\ 1 \end{bmatrix} \sin(t) + \begin{bmatrix} \frac{-1}{17} \\ \frac{2}{17} \\ 0 \end{bmatrix} \cos(t) \right) \right]}}$$

Problem 6

【Solution】

(a)

$$\arg(z) = \theta = \underline{\underline{\pi - \tan^{-1}\left(\frac{2}{3}\right)}}$$

(b)

$$|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\underline{\underline{z = \sqrt{13} e^{\left(\pi - \tan^{-1}\left(\frac{2}{3}\right)\right)i}}}$$

Problem 7

【Solution】

令$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} 2x^2 dx = \frac{8}{3}\pi^2$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} 2x^2 \cos(nx) dx = \frac{8}{n^2}$$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} 2x^2 \sin(nx) dx = -\frac{8\pi}{n}$$

$$\underline{\underline{f(x) = \frac{8}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{8}{n^2}\cos(nx) - \frac{8\pi}{n}\sin(nx)\right)}}$$

Problem 8

【**Solution**】

$$\because \text{adiabatic surface at } x=0 \quad \therefore u_x(0,t)=0$$

令 $u(x,t)=X(x)T(t)$

$$\begin{cases} u_x(0,t)=X'(0)T(t)=0 \\ u(L,t)=X(L)T(t)=0 \end{cases}$$

$$X'(0)=X(L)=0$$

代入 P.D.E.

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$$

同乘 $\dfrac{1}{c^2u}$ ，

$$\frac{X''}{X}=\frac{T'}{c^2T}=-\lambda$$

$$\begin{cases} X''+\lambda X=0 \\ T'+c^2\lambda T=0 \end{cases}$$

1. 令 $\lambda=0$ 代入 $X''+\lambda X=0$

$$X(x)=C_1+C_2x$$

代入 B.C.

$$C_1=C_2=0$$

2. 令 $\lambda=-k^2, k>0$ 代入 $X''+\lambda X=0$

$$X(x)=C_1\cosh(kx)+C_2\sinh(kx)$$

代入 B.C.

$$C_1=C_2=0$$

3. 令 $\lambda=k^2, k>0$ 代入 $X''+\lambda X=0$

$$X(x)=C_1\cos(kx)+C_2\sin(kx)$$

代入 B.C.

$$C_2 = 0$$

令 $C_1 \neq 0$

$$k = \frac{(2n-1)\pi}{2\ell}, n = 1, 2, 3, \ldots$$

$$\lambda = \left(\frac{(2n-1)\pi}{2L}\right)^2$$

$$X_n(x) = \cos\left(\frac{(2n-1)\pi x}{2\ell}\right), n = 1, 2, 3, \ldots$$

$$T' + \left(\frac{(2n-1)\pi c}{2\ell}\right)^2 T = 0$$

$$T_n(t) = c_n e^{-\left(\frac{(2n-1)\pi c}{2L}\right)^2 t}$$

$$u_n(x,t) = X_n(x)T_n(t) = c_n e^{-\left(\frac{(2n-1)\pi c}{2L}\right)^2 t} \cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{(2n-1)\pi c}{2L}\right)^2 t} \cos\left(\frac{(2n-1)\pi x}{2\ell}\right)$$

代入 I.C.

$$c_n = 0$$

$$\underline{\underline{u(x,t) = 0}}$$

<補充>

此題一端絕熱且另一端恆為 0 且初始溫度是 0，P.D.E.為齊性，溫度始終不會變化，可知此題的溫度始終保持為 0。

114 年台大工程數學

Problem 1

【Solution】

(a)

令 $x=e^{t},\ t=\ln(x),\ x>0$ 代入 O.D.E.

$$\left(D(D-1)(D-2)+D(D-1)-2D+2\right)y=3te^{3t}$$

1. 求 y_h

$$D(D-1)(D-2)+D(D-1)-2D+2=0$$

$$(D-2)(D-1)(D+1)=0$$

$$D=-1,1,2$$

$$y_h(x)=C_1x^{-1}+C_2x+C_3x^2$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t)=\frac{1}{(D-2)(D-1)(D+1)}\left(3te^{3t}\right)$$

$$y_p(t)=3e^{3t}\frac{1}{(D+1)(D+2)(D+4)}(t)$$

$$y_p(t)=\frac{3}{8}e^{3t}\frac{1}{1-\left(-\frac{D^3}{8}-\frac{7D^2}{8}-\frac{7}{4}D\right)}(t)$$

$$y_p(t)=\frac{3}{8}te^{3t}-\frac{21}{32}e^{3t}$$

$$y_p(x)=\frac{3}{8}x^3\ln(x)-\frac{21}{32}x^3,\ x>0$$

$$\underline{\underline{y=C_1x^{-1}+C_2x+C_3x^2+\frac{3}{8}x^3\ln(x)-\frac{21}{32}x^3,\ x>0}}$$

(b)

令 $D = \dfrac{d}{dx}$ 代入 O.D.E.

$$\left(D^2 - 5D + 6\right) y = 9\cos\left(3x\right)$$

1. 求 y_h

$$D^2 - 5D + 6 = \left(D-2\right)\left(D-3\right) = 0$$

$$D = 2, 3$$

$$y_h(x) = C_1 e^{2x} + C_2 e^{3x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(x) = \frac{1}{D^2 - 5D + 6}\left(9\cos\left(3x\right)\right)$$

$$y_p(x) = \frac{-9}{5}\frac{1}{D+1}\cos\left(3x\right)$$

$$y_p(x) = \frac{9}{50}\left(-3\sin\left(3x\right) - \cos\left(3x\right)\right)$$

$$\underline{\underline{y = C_1 e^{2x} + C_2 e^{3x} + \frac{9}{50}\left(-3\sin\left(3x\right) - \cos\left(3x\right)\right)}}$$

(c)

令 $D=\dfrac{d}{dx}$ 代入 O.D.E.

$$\left(4D^2-12D+9\right)y=2e^{1.5x}$$

1. 求 y_h

$$4D^2-12D+9=\left(2D-3\right)^2=0$$

$$\mathrm{D}=1.5,1.5$$

$$y_h(x)=C_1e^{1.5x}+C_2xe^{1.5x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(x)=\frac{1}{\left(2D-3\right)^2}\left(2e^{1.5x}\right)$$

$$y_p(x)=2e^{1.5x}\frac{1}{4D^2}(1)=0.25x^2e^{1.5x}$$

$$y=C_1e^{1.5x}+C_2xe^{1.5x}+0.25x^2e^{1.5x}$$

代入 I.C.

$$C_1=1, C_2=-0.5$$

$$\underline{\underline{y=e^{1.5x}-0.5C_2xe^{1.5x}+0.25x^2e^{1.5x}}}$$

Problem 2

【Solution】

(a)

$$L\left[e^{-t}\left(8\cosh(2t)-3\sinh(4t)\right)\right]$$

$$=L\left[4\left(e^{t}+e^{-3t}\right)-\frac{3}{2}\left(e^{3t}-e^{-5t}\right)\right]$$

$$\underline{\underline{=\frac{4}{s-1}+\frac{4}{s+3}-\frac{\frac{3}{2}}{s-3}+\frac{\frac{3}{2}}{s+5}}}$$

(b)

$$L^{-1}\left[\frac{2s-5}{s^2-6s+25}\right]=L^{-1}\left[\frac{2s-5}{(s-3)^2+16}\right]$$

$$=e^{3t}L^{-1}\left[\frac{2s+1}{s^2+16}\right]=\underline{\underline{2e^{3t}\cos(4t)+\frac{1}{4}e^{3t}\sin(4t),t>0}}$$

(c)

令$Y(s)=L\left[y(t)\right]$，將原式進行拉氏轉換

$$s^2Y(s)-sy(0)-y'(0)-2sY(s)+2y(0)-3Y(s)=\frac{e^{-s}}{s}$$

$$\left(s^2-2s-3\right)Y(s)=\frac{e^{-s}}{s}-1$$

$$Y(s)=\frac{e^{-s}}{s(s-3)(s+1)}-\frac{1}{(s-3)(s+1)}$$

$$Y(s)=e^{-s}\left(\frac{-\frac{1}{3}}{s}+\frac{\frac{1}{12}}{s-3}+\frac{\frac{1}{4}}{s+1}\right)-\frac{\frac{1}{4}}{s-3}+\frac{\frac{1}{4}}{s+1}$$

$$\underline{\underline{y(t)=\left(\frac{-1}{3}+\frac{1}{12}e^{3(t-1)}+\frac{1}{4}e^{-(t-1)}\right)u(t-1)-\frac{1}{4}e^{3t}+\frac{1}{4}e^{-t},t>0}}$$

(d)

令$U(x,s)=L\left[u(x,t)\right]$，將原式進行拉氏轉換

$$L\left[\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right]=0$$

$$\frac{d^2U(x,s)}{dx^2}-sU(x,s)=-25$$

$$U(x,s)=C_1(s)e^{-\sqrt{s}x}+C_2(s)e^{\sqrt{s}x}+\frac{25}{s}$$

代入 I.C.

$$C_1(s)=\frac{-25}{s},C_2(s)=0$$

$$U(x,s)=\frac{-25}{s}e^{-\sqrt{s}x}+\frac{25}{s}$$

$$\underline{\underline{u(x,t)=-25erfc\left(\frac{x}{2\sqrt{t}}\right)+25}}$$

Problem 3

【**Solution**】

令 $y(x)=\sum_{n=0}^{\infty}C_n(r)x^{n+r}$ 代入 O.D.E.

$$3x\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n(r)x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)C_n(r)x^{n+r-1}-\sum_{n=0}^{\infty}C_n(r)x^{n+r}=0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)C_n(r)x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)C_n(r)x^{n+r-1}-\sum_{n=0}^{\infty}C_n(r)x^{n+r}=0$$

$$\sum_{n=-1}^{\infty}3(n+r+1)(n+r)C_{n+1}(r)x^{n+r}+\sum_{n=-1}^{\infty}(n+r+1)C_{n+1}(r)x^{n+r}-\sum_{n=0}^{\infty}C_n(r)x^{n+r}=0$$

$$\big(3r(r-1)+r\big)C_0x^{r-1}-\sum_{n=0}^{\infty}\Big(\big(3(n+r+1)+(n+r+1)\big)C_{n+1}(r)-C_n(r)\Big)x^{n+r}=0$$

$$3r(r-1)+r=0$$

$$r=0,\frac{2}{3}$$

$$C_{n+1}(r)=\frac{1}{(n+r+1)(3n+3r+1)}C_n(r)$$

1. $r=0$

$$\begin{cases}C_1=C_0\\ C_2=\frac{1}{8}C_1=\frac{1}{8}C_0\\ C_3=\frac{1}{21}C_2=\frac{1}{168}C_0\\ C_4=\frac{1}{65}C_3=\frac{1}{10920}C_0\end{cases}$$

2. $r=\frac{2}{3}$

$$\begin{cases}C_1=\frac{1}{5}C_0\\ C_2=\frac{1}{16}C_1=\frac{1}{80}C_0\\ C_3=\frac{1}{33}C_2=\frac{1}{2640}C_0\\ C_4=\frac{1}{56}C_3=\frac{1}{147840}C_0\end{cases}$$

$$\underline{\underline{y=c_1\left(1+x+\frac{1}{8}x^2+\frac{1}{168}x^3+\frac{1}{10920}x^4\ldots\right)+c_2x^{\frac{2}{3}}\left(1+\frac{1}{5}x+\frac{1}{80}x^2+\frac{1}{2640}x^3+\frac{1}{147840}x^4\ldots\right)}}$$

Problem 4

【**Solution**】

$$\int_{-\pi}^{\pi} 1\cdot\cos(mx)\,dx=\frac{1}{m}\sin(mx)\Big|_{-\pi}^{\pi}=0$$

$$\int_{-\pi}^{\pi} 1\cdot\sin(mx)\,dx=\frac{-1}{m}\cos(mx)\Big|_{-\pi}^{\pi}=0$$

$$\int_{-\pi}^{\pi}\cos(mx)\sin(mx)\,dx=\frac{1}{m}\int_{-\pi}^{\pi}\sin(mx)\,d\left(\sin(mx)\right)=\frac{1}{2m}\sin^2(mx)\Big|_{-\pi}^{\pi}=0$$

因為三者彼此內積為 0，所以此為 orthogonal set。

$$\sqrt{\int_{-\pi}^{\pi}\cos^2(mx)\,dx}=\sqrt{\pi}$$

$$\sqrt{\int_{-\pi}^{\pi}\sin^2(mx)\,dx}=\sqrt{\pi}$$

$$\sqrt{\int_{-\pi}^{\pi}1dx}=\sqrt{2\pi}$$

其 orthonormal set 為 $\underline{\underline{\left\{\frac{1}{\sqrt{2\pi}},\frac{\cos(mx)}{\sqrt{\pi}},\frac{\sin(mx)}{\sqrt{\pi}}\right\},m=1,2,3}}$

Problem 5

【**Solution**】

由 Fourier Sine Series, 令 $f(x)=\sum_{n=1}^{\infty}b_n\sin(nx)$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin(nx)\,dx=\frac{2}{\pi}\int_0^{\pi}x^3\sin(nx)\,dx$$

$$b_n=\frac{2\left(6-\pi^2n^2\right)(-1)^n}{n^3}$$

$$\underline{\underline{f(x)=\sum_{n=1}^{\infty}\left(\frac{2\left(6-\pi^2n^2\right)(-1)^n}{n^3}\right)\sin(nx)}}$$