106 年中興工程數學與輸送現象

Problem 1

【Solution】

(1)

For incompressible fluid,

$$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+v\cdot\nabla\rho=0$$

By the equation of continuity,

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)=\underbrace{\frac{\partial\rho}{\partial t}+v\cdot\nabla\rho}_{=0}+\rho\nabla\cdot v=0 \text{ ， } \nabla\cdot v=0$$

(2)

$$\nabla\cdot v=\frac{\partial}{\partial x}(\frac{y}{x^2-y^2})+\frac{\partial}{\partial y}(\frac{x}{x^2-y^2})=\frac{-2xy}{(x^2-y^2)^2}+\frac{2xy}{(x^2-y^2)^2}=0$$

<u>The flow is incompressible.</u>

(3)

$$\nabla\times v=\frac{\partial}{\partial x}(\frac{x}{x^2-y^2})-\frac{\partial}{\partial y}(\frac{y}{x^2-y^2})=\frac{(x^2-y^2)-2x^2}{(x^2-y^2)^2}+\frac{(x^2-y^2)+2y^2}{(x^2-y^2)^2}\neq 0$$

<u>The flow is irrotational</u>

Problem 2

【Solution】

令 $P_A(\lambda)=\det(A-\lambda I)=(\lambda-2)(\lambda-5)=0$

$$\lambda=2,5$$

1. $\lambda=2$ 代入 $A\vec{X}=\vec{0}$

$$\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=c_1\begin{bmatrix}1\\ 2\end{bmatrix}, c_1\neq 0$$

2. $\lambda=5$ 代入 $A\vec{X}=\vec{0}$

$$\begin{bmatrix}-1 & -1\\ -2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=c_2\begin{bmatrix}1\\ -1\end{bmatrix}, c_2\neq 0$$

令 $P^{-1}AP=D=\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}$, 其中 $P=\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$, $P^{-1}=\begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{-1}{3}\end{bmatrix}$

$$A=PDP^{-1}$$

$$A^{55}=PD^{55}P^{-1}=\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}2^{55} & 0\\ 0 & 5^{55}\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{-1}{3}\end{bmatrix}$$

$$A^{55}=\frac{1}{3}\begin{bmatrix}2^{55}+2\times 5^{55} & 2^{55}-5^{55}\\ 2^{56}-2\times 5^{55} & 2^{56}+5^{55}\end{bmatrix}$$

Problem 3

【Solution】

(1)

令 $f(x)=\sum_{n=1}^{\infty} b_n \sin(n\pi x)$

$$b_n = 2\int_0^1 (-x)\sin(n\pi x)dx = \frac{2(-1)^n}{n\pi}$$

$$\underline{\underline{f(x)=\sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n\pi}\right)\sin(n\pi x)}}$$

(2)

By Parseval’s Relation,

$$\sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n\pi}\right)^2 = \int_{-1}^{1}(-x)^2\,dx$$

$$\sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\right)=\frac{2}{3}$$

$$\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)=\frac{1}{3}$$

$$\frac{2}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^2}+\ldots\right)=\frac{1}{3}$$

$$\underline{\underline{1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}}}$$

Problem 4

【Solution】

(1)

將 P.D.E.做拉氏轉換，令 $\theta(x,s)=L\left[T(x,t)\right]$

$$\begin{cases}\theta(0,s)=\dfrac{T_0}{s}\\ \theta(\infty,s)=0\end{cases}$$

$$\frac{d^2\theta(x,s)}{dx^2}-\frac{s}{\alpha}\theta(x,s)=0$$

$$\theta(x,s)=C_1(s)e^{\sqrt{\frac{s}{\alpha}}x}+C_2(s)e^{-\sqrt{\frac{s}{\alpha}}x}$$

代入 B.C.

$$C_1(s)=0,\ C_2(s)=\frac{T_0}{s}$$

$$\theta(x,s)=T_0\frac{1}{s}e^{-\sqrt{\frac{s}{\alpha}}x}$$

$$\underline{\underline{T(x,t)=T_0\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)}}$$

(2)

代入題目條件到(1)之結果，

$$0.4T_0=T_0\mathrm{erfc}\left(\frac{8}{2\sqrt{4t}}\right)$$

$$0.4=\mathrm{erfc}\left(\frac{2}{\sqrt{t}}\right)$$

$$0.4=1-\mathrm{erf}\left(\frac{2}{\sqrt{t}}\right)$$

$$\mathrm{erf}\left(\frac{2}{\sqrt{t}}\right)=0.6$$

由查表可知，

$$\frac{2}{\sqrt{t}}=0.6039$$

$$\underline{\underline{t=10.97\ (\mathrm{sec})}}$$

Problem 5

【Solution】

By shell balance of energy in the region of $r_1 \le r \le r_2$

$$q^{"}(2\pi rL)dr\Big|_{r} - q^{"}(2\pi rL)dr\Big|_{r+dr} + q_0(2\pi rL)dr = 0$$

同除 $2\pi Ldr \to 0$

$$\frac{d(rq^{"})}{dr} + q_0 r = 0$$

$$q^{"} = -k\frac{dT}{dr} = -\frac{q_0 r}{2} + \frac{c_1}{r}$$

$$T = \frac{q_0 r^2}{4k} + C_1 \ln r + C_2 \begin{cases} r = r_1, T = T_1 \\ r = r_2, T = T_2 \end{cases}$$

$$T_1 = \frac{q_0 r_1^2}{4k} + C_1 \ln r_1 + C_2$$

$$T_2 = \frac{q_0 r_2^2}{4k} + C_1 \ln r_2 + C_2$$

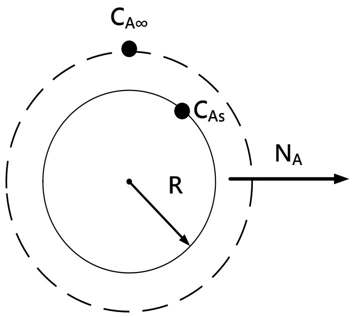
$$C_1 = \frac{T_2 - T_1 - \dfrac{q_0}{4k}(r_2^2 - r_1^2)}{\ln \dfrac{r_2}{r_1}}\text{，}C_2 = T_1 - \frac{q_0 r_1^2}{4k} - C_1 \ln r_1$$

$$T = \frac{q_0 r^2}{4k} + C_1 \ln r + C_2 = \frac{q_0 r^2}{4k} + C_1 \ln r + T_1 - \frac{q_0 r_1^2}{4k} - C_1 \ln r_1$$

$$= \frac{q_0 r_1^2}{4k}[(\frac{r}{r_1})^2 - 1] + C_1 \ln \frac{r}{r_1} + T_1 = \underline{\underline{T_1 + \frac{q_0 r_1^2}{4k}[(\frac{r}{r_1})^2 - 1] + \frac{T_2 - T_1 - \dfrac{q_0}{4k}(r_2^2 - r_1^2)}{\ln \dfrac{r_2}{r_1}} \ln \frac{r}{r_1}}}$$

Problem 6

【Solution】



By shell balance of mass in spherical cooridinate,

$$N_A(4\pi r^2)\Big|_r - N_A(4\pi r^2)\Big|_{r+dr} = 0$$

同除 $4\pi dr \to 0$

$$\frac{d}{dr}(r^2 N_A) = 0$$

By Fick's law,

$$N_A = -D_{AB}\frac{dC_A}{dr} + \cancel{y_A}(N_A + N_B) = -D_{AB}\frac{dC_A}{dr}$$

代回：

$$\frac{d}{dr}(r^2\frac{dC_A}{dr}) = 0 \text{，} C_A = \frac{-c_1}{r} + c_2 \begin{cases} r = R, C_A = C_{As} \\ r = \infty, C_A = C_{A\infty} \end{cases}$$

$$\begin{cases} c_1 = R(C_{A\infty} - C_{As}) \\ c_2 = C_{A\infty} \end{cases}$$

$$C_A = \frac{R}{r}(C_{As} - C_{A\infty}) + C_{A\infty}$$

$@r = R$， $N_{A,diffusion} = N_{A,convection}$

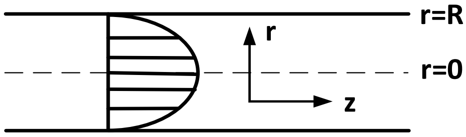
$$-D_{AB}\frac{dC_A}{dr}\bigg|_{r=R} = k(C_{As} - C_{A\infty})$$

$$\frac{R}{R^2}D_{AB}(C_{As} - C_{A\infty}) = k(C_{As} - C_{A\infty}) \text{，} \frac{kR}{D_{AB}} = 1$$

$$\underline{\underline{Sh = \frac{kD}{D_{AB}} = \frac{2kR}{D_{AB}} = 2}}$$

Problem 7

【**Solution**】



By the equation of motion,

$$\cancel{\rho\frac{D\vec{v}}{Dt}} = -\nabla P - [\nabla \cdot \overset{=}{\tau}] + \cancel{\rho\vec{g}}$$

For flow in z-direction,

$$\nabla \cdot \overset{=}{\tau} + \nabla P = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial P}{\partial z} = 0$$

$$\tau_{rz} = \frac{r}{2}\left(\frac{-\partial P}{\partial z}\right) + \frac{c_1}{r} = m\left(\frac{-du_z}{dr}\right)^n$$

$$\left(m = \frac{K}{g_c}\right)$$

$$\left(\frac{-du_z}{dr}\right)^n = \frac{r}{2m}\left(\frac{-\partial P}{\partial z}\right) + \frac{\alpha_1}{r}$$

$$\frac{-du_z}{dr} = \left[\frac{r}{2m}\left(\frac{-\partial P}{\partial z}\right)\right]^{\frac{1}{n}} + \beta_1 r^{\frac{-1}{n}}$$

$$u_z = \frac{-n}{n+1} r\left[\frac{r}{2m}\left(\frac{-\partial P}{\partial z}\right)\right]^{\frac{1}{n}} - \frac{n}{n-1}\beta_1 r^{1-\frac{1}{n}} + \beta_2 \begin{cases} r = 0, \ \dfrac{du_z}{dr} = 0 \\ x = R, \ v_z = 0 \end{cases}$$

$$\begin{cases} \beta_1 = 0 \\ \beta_2 = \dfrac{n}{n+1} R\left[\dfrac{R}{2m}\left(\dfrac{-\partial P}{\partial z}\right)\right]^{\frac{1}{n}} \end{cases}$$

$$\underline{\underline{u_z = \frac{n}{n+1}\left[\frac{1}{2m}\left(\frac{-\partial P}{\partial z}\right)\right]^{\frac{1}{n}} R^{\frac{n+1}{n}}\left[1 - \left(\frac{r}{R}\right)^{\frac{n+1}{n}}\right]}}$$

(本題可參考：Bird, R.; Stewart, W.; Lightfoot, E. *Transport Phenomena*, 2nd ed.; p 245.)

Problem 8

【**Solution**】

D'Alembert paradox describes that for an uniform incompressible, inviscid potential flow, the mathematical results show that there is no drag if the flow is approaching an object. However, in real cases, the drag is really larger even when the Reynolds number of a system is very large.

對於一個均勻不可壓縮、非黏性勢流系統中，當一物體於此系統中穩定相對運動時，其於數學上將不受任何拖曳力。然而，我們知道於高雷諾數時，於流場中的物體依舊會受到可觀的拖曳力(可參考 drag coefficient-Reynolds number 關係圖)。

To solve this problem, Prandtl stated that there is a thin boundary layer even for flow with high velocity, where the viscous force can't be negligible, thus satisfying the non-slip boundary condition and the existence of viscous force.

此種問題直至 Prandtl 提出邊界層理論才有完整的模型描述，亦即即便在雷諾數很大的流場中，依舊會有一區域，其黏滯力效應不可忽略。

(本題可參考：Bird, R.; Stewart, W.; *Lightfoot, E. Transport Phenomena*, 2nd ed.; p 128~130.

Munson, B. *Fundamentals of Fluid Mechanics*, 6th ed.; p 300~302/489~492

Philip J. Pritchard. *Fox and McDonald's Intoduction to Fluid Mechanics*, 8th ed.; p 38~41.)

107 年中興工程數學與輸送現象

Problem 1

【**Solution**】

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x}=\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right)=\frac{x}{\sqrt{x^2+y^2}}=\cos\theta$$

$$\frac{\partial \theta}{\partial x}=\frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=\frac{-y}{x^2+y^2}=\frac{-\sin\theta}{r}$$

$$\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}+\frac{-\sin\theta}{r}\frac{\partial}{\partial \theta}$$

同理可知，

$$\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\nabla=\left(\cos\theta\vec{e}_r-\sin\theta\vec{e}_\theta\right)\left(\cos\theta\frac{\partial}{\partial r}+\frac{-\sin\theta}{r}\frac{\partial}{\partial \theta}\right)+\left(\sin\theta\vec{e}_r+\cos\theta\vec{e}_\theta\right)\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)+\vec{e}_z\frac{\partial}{\partial z}$$

$$\nabla=\left(\sin^2\theta+\cos^2\theta\right)\vec{e}_r\frac{\partial}{\partial r}+\frac{1}{r}\left(\sin^2\theta+\cos^2\theta\right)\vec{e}_\theta\frac{\partial}{\partial \theta}+\vec{e}_z\frac{\partial}{\partial z}$$

$$\underline{\underline{\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\vec{e}_z\frac{\partial}{\partial z}}}$$

Problem 2

【Solution】

Let x_1, x_2, x_3, x_4 be the weight of Felspar, Diatomite, Magnesite, and Talc, respectively.

a.

Silica balance: $\underline{\underline{0.6x_1 + 0.6x_2 + 0.2x_3 + 0.6x_4 = 55}}$

Alumina balance: $\underline{\underline{0.2x_1 + 0.1x_2 + 0.1x_3 + 0.1x_4 = 15}}$

Calcium Oxide balance: $\underline{\underline{0.1x_1 + 0.2x_2 + 0.1x_3 + 0.1x_4 = 12}}$

Magnesium Oxide balance: $\underline{\underline{0.1x_1 + 0.1x_2 + 0.6x_3 + 0.2x_4 = 20}}$

b.

$$\underline{\underline{\begin{bmatrix} 0.6 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 55 \\ 15 \\ 12 \\ 20 \end{bmatrix}}}$$

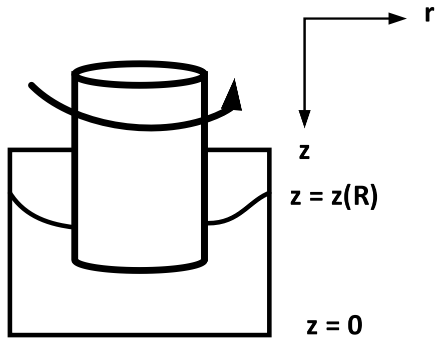
c.

$$\begin{bmatrix} 0.6 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 55 \\ 15 \\ 12 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 & 0.2 & 0.6 \\ 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{bmatrix}^{-1} \begin{bmatrix} 55 \\ 15 \\ 12 \\ 20 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 48 \\ 18 \\ 15.5 \\ 20.5 \end{bmatrix}}}$$

Problem 3

【Solution】



To determine the shape of the liquid phase, we first derive the velocity profile.

Assume：

(1) The flow is steady and fully developed in θ-direction

(2) $v_r = v_z = 0$.

(3) Incompressible Newtonian fluid.

By the equation of continuity,

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \cancel{\frac{\partial}{\partial z}(\rho v_z)} = 0$$

$$\frac{\partial v_\theta}{\partial \theta} = 0 \text{，} v_\theta = f(\cancel{t}, r, \cancel{\theta}, \cancel{z}) = f(r)\text{only}$$

By the Navier Stokes equation in θ direction,

$$\rho(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}} + \cancel{\frac{v_\theta v_r}{r}})$$

$$= \cancel{\frac{-1}{r}\frac{\partial p}{\partial \theta}} + \mu[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}] + \cancel{\rho g_\theta}$$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta))=0 \text{ ， } v_\theta=\frac{c_1}{2}r+\frac{c_2}{r}\begin{cases} r=\kappa R, v_\theta=\Omega_i\kappa R \\ r=R, v_\theta=0 \end{cases}$$

$$c_1=\frac{-2\kappa}{1-\kappa^2}\Omega_i \text{ ， } c_2=\frac{\kappa^2}{1-\kappa^2}\Omega_i R^2$$

$$v_\theta=(\frac{-2\kappa^2}{1-\kappa^2}\Omega_i)\frac{r}{2}+(\frac{\kappa^2}{1-\kappa^2}\Omega_i R^2)\frac{1}{r}=\frac{\kappa^2}{1-\kappa^2}(\frac{R^2-r^2}{r})\Omega_i$$

To determine the shape of the free liquid surface, by Navier-Stokes equation with r and z components,

$$\frac{-\partial p}{\partial r}=\frac{-\rho v_\theta^2}{r}$$

$$\frac{-\partial p}{\partial z}=-\rho g_z$$

Integrate the z-direction,

$$p=\rho g_z z+f(r)$$

代回 r 方向，

$$\frac{\partial p}{\partial r}=f\,'(r)=\frac{\rho v_\theta^2}{r}=\frac{\rho[\frac{\kappa^2}{1-\kappa^2}(\frac{R^2-r^2}{r})\Omega_i]^2}{r}=\rho(\frac{\kappa^2\Omega_i}{1-\kappa^2})^2(\frac{R^4}{r^3}-\frac{2R^2}{r}+r)$$

Integrate with r,

$$f=\rho(\frac{\kappa^2\Omega_i}{1-\kappa^2})^2(-\frac{R^4}{2r^2}-2R^2\ln r+\frac{r^2}{2})+C$$

And the pressure distribution becomes,

$$p(r,z)=\rho g_z z+\rho(\frac{\kappa^2\Omega_i}{1-\kappa^2})^2(-\frac{R^4}{2r^2}-2R^2\ln r+\frac{r^2}{2})+C$$

Because the liquid is open to the atmosphere at the top, and z is used to describe the height of the liquid-gas interface,

$$p(r,z)=p(R,z_R)=p_{atm} \text{ ， } p(r,z)=p(R,z_R)$$

$$\rho g_z z+\rho(\frac{\kappa^2\Omega_i}{1-\kappa^2})^2(-\frac{R^4}{2r^2}-2R^2\ln r+\frac{r^2}{2})+C=\rho g_z z_R+\rho(\frac{\kappa^2\Omega_i}{1-\kappa^2})^2(-\cancel{\frac{R^2}{2}}-2R^2\ln R+\cancel{\frac{R^2}{2}})+C$$

$$\rho g_z(z-z_R)=\frac{\rho}{2}(\frac{\kappa^2\Omega_i R}{1-\kappa^2})^2[\frac{R^2}{r^2}-4\ln\frac{R}{r}-\frac{r^2}{R^2}]$$

$$(z-z_R)=\frac{\rho}{2g_z}(\frac{\kappa^2\Omega_i R}{1-\kappa^2})^2[\frac{R^2}{r^2}-4\ln\frac{R}{r}-\frac{r^2}{R^2}]$$

將 $r/R=\xi$ 代入：

$$\underline{\underline{(z-z_R)=\frac{\rho}{2g_z}(\frac{\kappa^2\Omega_i R}{1-\kappa^2})^2[\xi^{-2}+4\ln\xi-\xi^2]}}$$

(本題改編自：Bird, R.; Stewart, W.; Lightfoot, E. *Transport Phenomena*, 2nd ed.; p 110, Problem 3B.15.)

Problem 4

【**Solution**】

Assume $T = T(r)$ only,

by equation of energy,

$$\rho c_p\left(\frac{\partial T}{\partial t} + \cancel{\bar{v}\cdot\nabla T}\right) = k\nabla^2 T + \cancel{\dot{q}}$$

$$\rho c_p\left(\frac{\partial T}{\partial t}\right) = k\left[\frac{1}{r^2}\frac{\partial}{r}\left(r^2\frac{\partial T}{\partial r}\right)\right]\begin{cases} I.C.: t = 0, T = T_0 \\ B.C.: r = 0, T = \mathit{finite} \\ B.C.: r = R, -k\dfrac{\partial T}{\partial r} = h(T - T_b) \end{cases}$$

Problem 5

【Solution】

By mechanical energy balance of the liquid, from 1 to 2

$$\cancel{g(h_1-h_2)}+\frac{P_1-P_2}{\rho}+\frac{1}{2}(v_1^2-v_2^2)-h_f=0$$

where the loss term, h_f, denotes the energy change

By mass balance between 1 and 2,

$$A_1v_1=A_2v_2\text{ ， }v_2=\frac{A_1}{A_2}v_1$$

代回：

$$h_f=\frac{1}{2}(v_1^2-v_2^2)+\frac{P_1-P_2}{\rho}=\underline{\underline{\frac{1}{2}v_1^2[1-(\frac{A_1}{A_2})^2]+\frac{P_1-P_2}{\rho}}}$$

Problem 6

【Solution】Ans: (A)

Assume, (1) The flow is fully-developed in z-direction

(2) $v_r = v_\theta = 0$, $v_z = f(\cancel{t}, r, \cancel{\theta}, z)$

(3) Incompressible Newtonian fluid

(4) The gravity effect can be neglected

By the equation of continuity,

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)} + \cancel{\frac{1}{r}\frac{\partial \rho v_\theta}{\partial \theta}} + \frac{\partial \rho v_z}{\partial z} = 0 \text{，} v_z = f(r, \cancel{z})$$

By Navier-Stokes equation in z-direction,

$$\mu[(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}))] - \frac{\partial p}{\partial z} = 0$$

$$\mu[(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}))] = \frac{\partial p}{\partial z}$$

$$\frac{\partial v_z}{\partial r} = \frac{\partial v_z}{\partial r} = \frac{1}{2\mu}(\frac{\partial p}{\partial z})r + \frac{c_1}{r}$$

$$v_z = \frac{1}{4\mu}(\frac{\partial p}{\partial z})r^2 + c_1 \ln r + c_2 \begin{cases} r = \dfrac{d}{2}, v_z = 0 \\ r = \dfrac{D}{2}, v_z = 0 \end{cases}$$

$$c_1 = \frac{\dfrac{-1}{4\mu}(\dfrac{\partial p}{\partial z})(\dfrac{D^2}{4} - \dfrac{d^2}{4})}{\ln\dfrac{D}{d}}$$

For maximum velocity, let $\dfrac{\partial v_z}{\partial r} = 0$

$$r = \sqrt{\frac{-2\mu c_1}{(\dfrac{\partial p}{\partial z})}} = \sqrt{\frac{\dfrac{2\mu \times \dfrac{1}{4\mu}(\cancel{\dfrac{\partial p}{\partial z}})(\dfrac{D^2}{4} - \dfrac{d^2}{4})}{\ln(\dfrac{D}{d})}}{(\cancel{\dfrac{\partial p}{\partial z}})}} = \underline{\underline{\sqrt{\frac{(D^2 - d^2)}{8\ln(\dfrac{D}{d})}}}}$$

Problem 7

【Solution】

Firstly, check the Biot number of the system,

$$Bi=\frac{h(\frac{V}{A})}{k}=\frac{h(\frac{\pi}{4}D^2L)}{k(\pi DL+\frac{\pi}{4}D^2\times 2)}=\frac{8500\times(\frac{\pi}{4}\times 0.1^2\times 0.1)}{1.21(\pi\times 0.1\times 0.1+\frac{\pi}{2}\times 0.1^2)}=117>0.1$$

We can't use lumped-capacity analysis here.

To use the chart attached, we need to calculate the following parameter,

$$\frac{T-T_s}{T_0-T_s}=\frac{310-373}{292-373}=0.778$$

對照代表圓柱的線，可得橫軸 $\frac{\alpha t}{x_1^2}\approx 0.1$

$$\frac{\alpha t}{x_1^2}=\frac{kt}{\rho c_p x_1^2}=\frac{1.21t}{2310\times 0.21\times 4184\times(\frac{0.1}{2})^2}=0.1$$

$$\underline{\underline{t=419.35\ (s)}}$$

(本題改編自：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p 257, Example 2.)

Problem 8

【Solution】

(1)

$$B_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi - x}{2}\right)\sin(nx)\,dx$$

$$B_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin(nx)\,dx = \frac{1}{\underline{\underline{n}}}$$

(2)

By Parseval's Relation,

$$\sum_{n=1}^{\infty}(B_n)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\pi - x}{2}\right)^2 dx$$

$$\sum_{n=1}^{\infty}(B_n)^2 = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\pi^2 - 2\pi x + x^2\right)dx$$

$$\sum_{n=1}^{\infty}(B_n)^2 = \frac{1}{4\pi}\left(\frac{8}{3}\pi^3\right) = \underline{\underline{\frac{2}{3}\pi^2}}$$

Problem 9

【**Solution**】

令 $D=\dfrac{d}{dx}$

$$\left(D^2+3D+2.25\right)y=-10e^{-1.5x}$$

1. 求 y_h

$$D^2+3D+2.25=0$$

$$\left(D+1.5\right)^2=0$$

$$D=-1.5,-1.5$$

$$y_h=C_1e^{-1.5x}+C_2xe^{-1.5x}$$

2. 求 y_p，利用 Heaviside 反微分運算

$$y_p=\frac{1}{\left(D+1.5\right)^2}\left(-10e^{-1.5x}\right)$$

$$y_p=-10e^{-1.5x}\frac{1}{\left(D\right)^2}\left(1\right)$$

$$y_p=-10e^{-1.5x}\left(\frac{1}{2}x^2\right)=-5x^2e^{-1.5x}$$

$$\underline{\underline{y=C_1e^{-1.5x}+C_2xe^{-1.5x}-5x^2e^{-1.5x}}}$$

Problem 10

【**Solution**】

將原式化成標準式

$$y''+\frac{6}{x}y'+\left(\frac{6}{x^2}-4\right)y=0$$

令 $p(x)=\frac{6}{x}$

已知一齊性解為 $y_1=x^{-3}\cosh(2x)$

By reduction of order method,

$$y_2=y_1\int\frac{e^{-\int p(x)dx}}{y_1^{\ 2}}dx$$

$$y_2=x^{-3}\cosh(2x)\int\frac{e^{-\int\frac{6}{x}dx}}{x^{-6}\cosh^2(2x)}dx$$

$$y_2=x^{-3}\cosh(2x)\int\operatorname{sech}^2(2x)dx$$

$$y_2=x^{-3}\cosh(2x)\left(\frac{1}{2}\tanh(2x)\right)$$

$$\underline{\underline{y_2=\frac{1}{2}x^{-3}\cosh(2x)\tanh(2x)}}$$

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Problem 1

【Solution】

令 $D=\dfrac{d}{dx}$

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1. 求 y_h

$$D^2+3D+2.25=0$$

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2. 求 y_p，利用 Heaviside 反微分運算

$$y_p=\frac{1}{(D+1.5)^2}\left(-10e^{-1.5x}\right)$$

$$y_p=-10e^{-1.5x}\frac{1}{(D)^2}(1)$$

$$y_p=-10e^{-1.5x}\left(\frac{1}{2}x^2\right)=-5x^2e^{-1.5x}$$

$$y=C_1e^{-1.5x}+C_2xe^{-1.5x}-5x^2e^{-1.5x}$$

$$y'=-1.5C_1e^{-1.5x}+C_2e^{-1.5x}-1.5C_2xe^{-1.5x}-10xe^{-1.5x}+7.5x^2e^{-1.5x}$$

代入 I.C.

$$C_1=1,\ C_2=1.5$$

$$\underline{\underline{y=e^{-1.5x}+1.5xe^{-1.5x}-5x^2e^{-1.5x}}}$$

Problem 2

【Solution】

$$f(t)=m\frac{d^2x}{dt^2}$$

$$20gx=mx''$$

$$x''-4x=0$$

(a)

$$\frac{d^2x}{dt^2}=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}$$

$$v\frac{dv}{dx}-4x=0$$

$$\underline{\underline{v\frac{dv}{dx}=4x}}$$

(b)

$$v\frac{dv}{dx}=4x$$

$$vdv=4xdx$$

$$\frac{1}{2}v^2=2x^2+C_1$$

$$v=\pm\sqrt{4x^2+C_2}\ (\text{取正})$$

代入 I.C.

$$C_2=-36$$

$$\underline{\underline{v=\sqrt{4x^2-36}}}$$

(c)

$$x''-4x=0$$

$$x=C_3e^{2t}+C_4e^{-2t}$$

$$v=\frac{dx}{dt}=2C_3e^{2t}-2C_4e^{-2t}$$

代入 I.C.

$$C_3=C_4=\frac{3}{2}$$

$$x=\frac{3}{2}e^{2t}+\frac{3}{2}e^{-2t}$$

$$\underline{\underline{x=3\cosh(2t)}}$$

(d)

$$v=\frac{dx}{dt}=\frac{d}{dt}\left(3\cosh\left(2t\right)\right)$$

$$\underline{\underline{v=6\sinh\left(2t\right)}}$$

(e)

As the rest of the chain slides off the platform,

$$x=8$$

$$8=3\cosh\left(2t\right)$$

$$\underline{\underline{t=0.85\ (\text{sec})}}$$

(f)

$$v=6\sinh\left(17\right)$$

$$\underline{\underline{v=15.9\ (\text{ft}/\text{sec})}}$$

(g)

$$LWx''=xWg$$

$$x''=\frac{g}{L}x$$

$$v\frac{dv}{dx}=\frac{g}{L}x$$

$$\frac{1}{2}v^2=\frac{1}{2}\frac{g}{L}x^2+C_5$$

$$v=\sqrt{\frac{g}{L}x^2+C_6}$$

代入 I.C.

$$C_6=-\frac{g}{L}{x_0}^2$$

$$v=\sqrt{\frac{g}{L}\left(x^2-{x_0}^2\right)}$$

As the end of the chain leaves the edge of the platform,

$$x=L$$

$$\underline{\underline{v=\sqrt{\frac{g}{L}\left(L^2-{x_0}^2\right)}}}$$

Problem 3

【Solution】

將原式做拉氏轉換，令 $Y(s)=L\left[y(t)\right],\ y(0)=c_1,\ y'(0)=c_2$

$$s^2Y(s)-c_1s-c_2+3sY(s)-3c_1+2Y(s)=e^{-t}$$

$$(s+1)(s+2)Y(s)=c_1s+c_2+3c_1+e^{-t}$$

$$Y(s)=\frac{c_1s+c_2+3c_1}{(s+1)(s+2)}+\frac{e^{-t}}{(s+1)(s+2)}$$

$$Y(s)=\frac{\alpha}{s+1}+\frac{\beta}{s+2}+e^{-t}\left(\frac{1}{s+1}-\frac{1}{s+2}\right)$$

$$\underline{\underline{y(t)=\alpha e^{-t}+\beta e^{-2t}+H(t-1)\left(e^{-(t-1)}-e^{-2(t-1)}\right),t>0}}$$

Problem 4

【**Solution**】

令 $P_A(\lambda)=\det(A-\lambda I)=-(\lambda+1)(\lambda^2-2)=0$

$$\lambda=-1,\sqrt{2},-\sqrt{2}$$

$$\lambda^6=Q(\lambda)(\lambda+1)(\lambda^2-2)+\alpha(\lambda+1)(\lambda-\sqrt{2})+\beta(\lambda+1)+\gamma$$

1. $\lambda=-1$ 代入

$$\gamma=1$$

2. $\lambda=\sqrt{2}$ 代入

$$\beta=7\left(\sqrt{2}-1\right)$$

3. $\lambda=-\sqrt{2}$ 代入

$$\alpha=7$$

By Cayley-Hamilton theorem,

$$(A+I)(A^2-2I)=0$$

$$A^6=Q(A)(A+I)(A^2-2I)+\left(\alpha A^2+\beta A+\gamma I\right)$$

$$A^6=7A^2+7\left(\sqrt{2}-1\right)A+I$$

$$\underline{\underline{A^6=\begin{bmatrix} 8 & -7+7\sqrt{2} & 7\sqrt{2} \\ 7 & 22-7\sqrt{2} & -14+7\sqrt{2} \\ -7+7\sqrt{2} & -7+7\sqrt{2} & 15 \end{bmatrix}}}$$

Problem 5

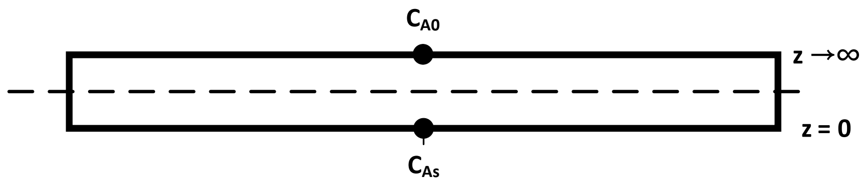
【**Solution**】

$$\iint_R xe^{y^2}dA=\int_0^4\int_0^{\sqrt{y}} xe^{y^2}dxdy$$

$$=\frac{1}{2}\int_0^4 ye^{y^2}dy=\frac{1}{4}\int_0^4 e^{y^2}d\left(y^2\right)=\underline{\underline{\frac{1}{4}\left(e^{16}-1\right)}}$$

Problem 6

【**Solution**】



By Fick's second law of diffusion (no homogeneous reaction),

$$\frac{\partial C_A}{\partial t} = D_{AB}\frac{\partial^2 C_A}{\partial z^2}$$

By combination of variables, let $\eta = \dfrac{z}{\sqrt{4D_{AB}t}}$

$$\frac{\partial C_A}{\partial t} = \frac{\partial C_A}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{-\eta}{2t}\frac{\partial C_A}{\partial \eta}$$

$$\frac{\partial^2 C_A}{\partial z^2} = \frac{\partial}{\partial z}[\frac{\partial C_A}{\partial \eta}\frac{\partial \eta}{\partial z}] = \frac{\partial}{\partial z}[\frac{1}{\sqrt{4D_{AB}t}}\frac{\partial C_A}{\partial \eta}] = \frac{\partial}{\partial \eta}[\frac{1}{\sqrt{4D_{AB}t}}\frac{\partial C_A}{\partial \eta}]\frac{\partial \eta}{\partial z} = \frac{1}{4D_{AB}t}\frac{\partial^2 C_A}{\partial \eta^2}$$

$$\frac{-\eta}{2t}\frac{\partial C_A}{\partial \eta} = D_{AB}\cdot\frac{1}{4D_{AB}t}\frac{\partial^2 C_A}{\partial \eta^2}\text{ , }\frac{\partial^2 C_A}{\partial \eta^2} + 2\eta\frac{\partial C_A}{\partial \eta} = 0$$

The solution is,

$$C_A = c_1\int_0^{\eta} e^{-\eta^2}d\eta + c_2\begin{cases} z = 0 \to \eta = 0, C_A = C_{As} \\ z \to \infty, \eta \to \infty, C_A = C_{A0} \\ t = 0, \eta \to \infty, C_A = C_{A0} \end{cases}$$

B.C.1 代入：

$$C_A = c_2 = C_{As}$$

B.C.2 代入：

$$C_{A0} = c_1\int_0^{\infty} e^{-\eta^2}d\eta + C_{As}\text{ , }c_1 = \frac{C_{A0} - C_{As}}{\frac{\sqrt{\pi}}{2}}$$

$$C_A = \frac{2}{\sqrt{\pi}}(C_{A0} - C_{As})\int_0^{\infty} e^{-\eta^2}d\eta + C_{As}$$

To determine the flux of oxygen,

$$N_A = -D_{AB}\left.\frac{dC_A}{dz}\right|_{z=0} = -D_{AB}\frac{d}{dz}\left(\frac{2}{\sqrt{\pi}}(C_{A0}-C_{As})\int_0^{\infty} e^{-\eta^2}d\eta + C_{As}\right)$$

$$= -D_{AB}(C_{A0}-C_{As})\cdot\frac{2}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{4D_{AB}t}}$$

$$= -1\times10^{-11}(0.5-5)\cdot\frac{2}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{4\times1\times10^{-11}\times10}} = \underline{\underline{2.54\times10^{-6}\ (mol/s/m^2)}}$$

<u>Problem 7</u>

【Solution】

The viscosity is the resistance of a fluid to deformation rate,

$$\underline{\underline{\mu = \frac{\tau}{-\frac{dv}{dx}}[=]kg/(m\cdot s)}}$$

<u>Problem 8</u>

【Solution】

$$\tau = -\mu(\nabla v + (\nabla v)^{\dagger}) + (\frac{2}{3}\mu - \kappa)(\nabla\cdot v)]$$

(本題可參考：Bird, R.; Stewart, W.; Lightfoot, E. *Transport Phenomena*, 2nd ed.;

p 843.)

<u>Problem 9</u>

【Solution】

(a)

For turbulent flow on a plate, the boundary layer thickness can be described by,

$$\frac{\delta}{x}=\frac{0.376}{\mathrm{Re}_x^{1/5}}\ ,\ \frac{\delta}{5}=\frac{0.376}{(2.5\times10^8)^{1/5}}$$

$$\underline{\underline{\delta=0.039\ (m)}}$$

(本題可參考：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p 163~164.)

※若為 Laminar flow，則其邊界層厚度可近似為：

$$\frac{\delta}{x}=\frac{4.64}{\sqrt{\mathrm{Re}_x}}$$ **(von Karman momentum integral)**

(b) As the result of (a), the flow system is **<u>turbulent flow</u>**

(c)

For turbulent flow past a plate, the friction coefficient is,

$$C_{fx}=\frac{0.0576}{\mathrm{Re}_x^{1/5}}=\frac{0.0576}{(\frac{\rho v}{\mu})^{1/5}}x^{-1/5}$$

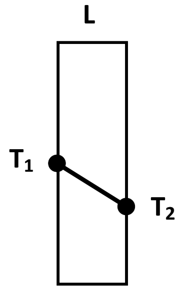
$$F=2\times\frac{0.0576}{(\frac{\rho v}{\mu})^{1/5}}\int_0^W\int_0^L x^{-1/5}dx\times\frac{1}{2}\rho v_\infty^2=\frac{0.0576\times\frac{5}{4}W}{(\frac{\rho v_\infty}{\mu})^{1/5}}L^{4/5}\times\rho v_\infty^2=\frac{0.0576\times\frac{5}{4}WL}{(\frac{\rho v_\infty L}{\mu})^{1/5}}\times\rho v_\infty^2$$

$$=\frac{0.0576\times\frac{5}{4}\times500}{(\frac{998\times50\times5}{9.93\times10^{-4}})^{1/5}}\times998\times50^2=\underline{\underline{2.97\times10^6\ (N)}}$$

Problem 10

【Solution】

For a plane wall in steady state without heat generation, the temperature prfile is linear.



For the steady state heat transfer rate, by Ohm's law,

$$q = \frac{(T_1 - T_2)}{\dfrac{L}{kA}} = \underline{\underline{\frac{kA}{L}(T_1 - T_2)}}$$

Problem 10

【Solution】

(a) By shell balance of energy, assume,

(1) Steady state

(2) Consider only x-direction

$$q_x^{"}(dydz)\Big|_x - q_x^{"}(dydz)\Big|_{x+dx} + q_k dxdydz = 0$$

同除 $dxdydz \to 0$

$$\frac{dq_x^{"}}{dx} = q_0(1 - \frac{x}{L}) \text{ ， } q_x^{"} = q_0(x - \frac{x^2}{2L}) + c_1$$

$\because$ $x = L$, the wall is insulated,

$$q_x^{"}\Big|_{x=L} = \frac{q_0 L}{2} + c_1 = 0 \text{ ， } c_1 = -\frac{q_0 L}{2}$$

$$q_x^{"} = -k\frac{dT}{dx} = q_0(x - \frac{x^2}{2L}) - \frac{q_0 L}{2}$$

$$T = \frac{q_0}{k}(\frac{x^3}{6L} - \frac{x^2}{2}) + \frac{q_0 L}{2k}x + c_2$$

$\because x=0, T=T_0$

$$c_2 = T_0$$

$$\underline{\underline{T=\frac{q_0}{k}\left(\frac{x^3}{6L}-\frac{x^2}{2}\right)+\frac{q_0 L}{2k}x+T_0}}$$

(b) By the discussion of (a), we know that when $x=L$, $\frac{dT}{dx}=0$, which indicated the maximum temperature occurs at $x=L$.

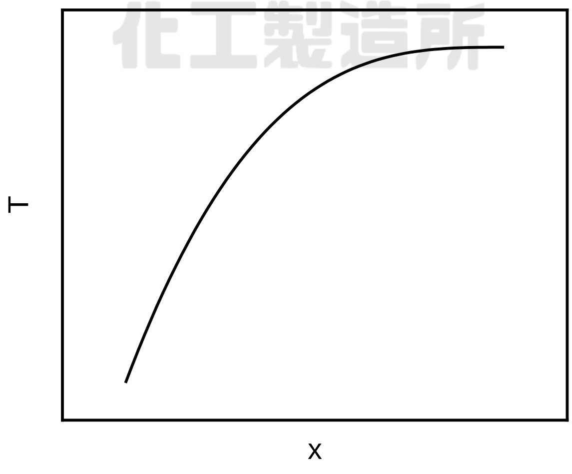
(c)

At $x=L$,

$$T=\frac{q_0}{k}\left(\frac{L^2}{6}-\cancel{\frac{L^2}{2}}\right)+\cancel{\frac{q_0 L^2}{2k}}+T_0=\frac{q_0 L^2}{6k}+T_0=\frac{180\times10^3\times L^2}{6\times0.6}+320=\underline{\underline{320+50000L^2}}$$

(本題改編自：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p 248, Problem 17.24.，原題有給定厚度 L 為 0.06 m，代入可得系統最高溫為 500K。)

※Temperature profile：



Problem 11

【**Solution**】

$$D_{AB}[=]m^2/s$$

$$k_c[=]m/s^2$$

Problem 12

【**Solution**】

If we consider that Knudsen diffusion and molecular diffusion compete with each other by a "resistance in series" approach, the effective diffusivity is,

$$\frac{1}{D_{Ae}}=\frac{1-\alpha y_A}{D_{AB}}+\frac{1}{D_{KA}}$$

where,

$$D_{KA}=\frac{d_{pore}}{3}\sqrt{\frac{8\kappa NT}{\pi M_A}}$$

$$\alpha=1+\frac{N_B}{N_A}$$

The diffusion process is a combination of Kndusen diffusion in pore and normal molecular diffusion.

For special case with $\alpha=0$ or $N_A=-N_B$

$$\frac{1}{D_{Ae}}=\frac{1}{D_{AB}}+\frac{1}{D_{KA}}$$

(本題可參考：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p420~421.。)

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Problem 1

【Solution】

(1)

令 $f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}xdx=\frac{-\pi}{2}$$

$$a_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx=\frac{2}{\pi}\int_{-\pi}^{0}x\cos(nx)dx=\frac{2\left(1-(-1)^n\right)}{n^2\pi}$$

$$b_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx=\frac{2}{\pi}\int_{-\pi}^{0}x\sin(nx)dx=\frac{2(-1)^{n+1}}{n}$$

$$\underline{\underline{f(x)=-\frac{\pi}{2}+\sum_{n=1}^{\infty}\left(\left(\frac{2\left(1-(-1)^n\right)}{n^2\pi}\right)\cos(nx)+\left(\frac{2(-1)^{n+1}}{n}\right)\sin(nx)\right)}}$$

(2)

令 $x=\frac{\pi}{2}$ 代入 $f(x)$

$$0=-\frac{\pi}{2}+\sum_{n=1}^{\infty}\left(\left(\frac{2(-1)^{n+1}}{n}\right)\sin\left(\frac{n\pi}{2}\right)\right)$$

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\left(\left(\frac{(-1)^{n+1}}{n}\right)\sin\left(\frac{n\pi}{2}\right)\right)$$

$$\underline{\underline{\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots}}$$

Problem 2

【Solution】

令 $y(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},\ A=\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix},\ b=\begin{bmatrix} -7\sin(4t)-4\cos(4t) \\ 4\sin(4t)+3\cos(4t) \end{bmatrix}$

$$y'=Ay+b$$

將上式進行拉氏轉換，令 $Y(s)=L\left[y(t)\right],\ B(s)=L\left[b(t)\right]$

$$sY(s)-y(0)=AY(s)+B(s)$$

$$\left(s-AI\right)Y(s)=B(s)+y(0)$$

$$Y(s)=\left(s-AI\right)^{-1}B(s)+\left(s-AI\right)^{-1}y(0)$$

$$Y(s)=\frac{1}{-2s+1}\begin{bmatrix} s+1 & -s-2 \\ -s+2 & s-3 \end{bmatrix}\begin{bmatrix} \dfrac{-4s-28}{s^2+16} \\ \dfrac{3s+16}{s^2+16} \end{bmatrix}+\frac{1}{-2s+1}\begin{bmatrix} s+1 & -s-2 \\ -s+2 & s-3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$Y(s)=\begin{bmatrix} \dfrac{-7s^2-54s-60}{\left(s^2+16\right)\left(-2s+1\right)}-1+\dfrac{1}{-2s+1} \\ \dfrac{7s^2-s+36}{\left(s^2+16\right)\left(-2s+1\right)}+1+\dfrac{1}{-2s+1} \end{bmatrix}$$

$$Y(s)=\begin{bmatrix} \dfrac{10}{13}\dfrac{s}{s^2+16}+\dfrac{89}{13}\dfrac{4}{s^2+16}+\dfrac{29}{13}\dfrac{1}{s-\frac{1}{2}}-1 \\ \dfrac{-153}{65}\dfrac{s}{s^2+16}-\dfrac{11}{65}\dfrac{4}{s^2+16}-\dfrac{107}{65}\dfrac{1}{s-\frac{1}{2}}+1 \end{bmatrix}$$

$$y(t)=\begin{bmatrix} \dfrac{10}{13}\cos(4t)+\dfrac{89}{13}\sin(4t)+\dfrac{29}{13}e^{\frac{1}{2}t}-\delta(t) \\ \dfrac{-153}{65}\cos(4t)-\dfrac{11}{65}\sin(4t)-\dfrac{107}{65}e^{\frac{1}{2}t}+\delta(t) \end{bmatrix},t>0$$

Problem 3

【Solution】

將 P.D.E.做拉氏轉換，令 $U(x,s)=L\left[u(x,t)\right]$

$$U(0,s)=\frac{s}{s^2+9},\ U_x(\infty,s)=0$$

$$s^2U(x,s)-\underbrace{su(x,0)}_{0}-\underbrace{u_t(x,0)}_{0}=25\frac{d^2U(x,s)}{dx^2}+\frac{4}{s}$$

$$\frac{d^2U(x,s)}{dx^2}-\frac{s^2}{25}U(x,s)=\frac{4}{25s}$$

$$U(x,s)=C_1(s)e^{\frac{s}{5}x}+C_2(s)e^{-\frac{s}{5}x}-\frac{4}{s^3}$$

$$U_x(x,s)=\frac{s}{5}C_1(s)e^{\frac{s}{5}x}-\frac{s}{5}C_2(s)e^{-\frac{s}{5}x}$$

代入 B.C.

$$C_1(s)=0$$

$$C_2(s)=\frac{s}{s^2+9}+\frac{4}{s^3}$$

$$U(x,s)=\frac{s}{s^2+9}e^{-\frac{s}{5}x}+\frac{4}{s^3}e^{-\frac{s}{5}x}-\frac{4}{s^3}$$

$$u(x,t)=\cos\left(3\left(t-\frac{1}{5}x\right)\right)H\left(t-\frac{1}{5}x\right)+2\left(t-\frac{1}{5}x\right)^2H\left(t-\frac{1}{5}x\right)-2t^2,t>0$$

Problem 4

【Solution】

(1)

If the flow is incompressible,

$$\nabla \cdot v = 0,$$

and we can find a stream function, Ψ , satisfying,

$$\begin{cases} v_x = \dfrac{\partial \Psi}{\partial y} = \dfrac{-2y}{x^2 - y^2} \\ v_y = -\dfrac{\partial \Psi}{\partial x} = \dfrac{-2x}{x^2 - y^2} \end{cases}$$

Back into the continuity equation, we have

$$\nabla \cdot v = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2 - y^2}\right) + \frac{\partial}{\partial y}\left(\frac{-2x}{x^2 - y^2}\right) = \frac{4xy}{(x^2 - y^2)^2} - \frac{4xy}{(x^2 - y^2)^2} = 0$$

<u>**The flow is incompressible**</u>

(2)

If $\nabla \times v = 0$, we can find a function ϕ such that the relation is satisfied,

$$\frac{\partial v_x}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial v_y}{\partial x}$$

For $\dfrac{\partial v_x}{\partial y}$,

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-2y}{x^2 - y^2}\right) = \frac{-2(x^2 - y^2) - (-2y)(-2y)}{(x^2 - y^2)^2} = \frac{-2x^2 - 2y^2}{(x^2 - y^2)^2}$$

For $\dfrac{\partial v_y}{\partial x}$,

$$\frac{\partial v_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-2x}{x^2 - y^2}\right) = \frac{-2(x^2 - y^2) - (2x)(-2x)}{(x^2 - y^2)^2} = \frac{2x^2 + 2y^2}{(x^2 - y^2)^2}$$

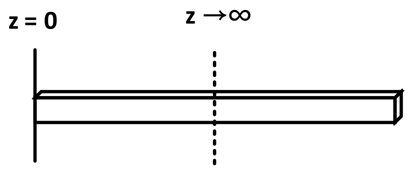
<u>**The flow doesn't have a velocity potential**</u>

(本題可參考：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p 114~118.)

Problem 5

【Solution】

(1)



Consider only the z-direction diffusion, by Fick's law,

$$\frac{\partial C_A}{\partial t} = D_{AB}\frac{\partial^2 C_A}{\partial x^2}$$

By combination of variables, let $\eta = \dfrac{z}{\sqrt{4D_{AB}t}}$

$$\frac{\partial C_A}{\partial t} = \frac{\partial C_A}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{-\eta}{2t}\frac{\partial C_A}{\partial \eta}$$

$$\frac{\partial^2 C_A}{\partial z^2} = \frac{\partial}{\partial z}[\frac{\partial C_A}{\partial \eta}\frac{\partial \eta}{\partial z}] = \frac{\partial}{\partial z}[\frac{1}{\sqrt{4D_{AB}t}}\frac{\partial C_A}{\partial \eta}] = \frac{\partial}{\partial \eta}[\frac{1}{\sqrt{4D_{AB}t}}\frac{\partial C_A}{\partial \eta}]\frac{\partial \eta}{\partial z} = \frac{1}{4D_{AB}t}\frac{\partial^2 C_A}{\partial \eta^2}$$

$$\frac{-\eta}{2t}\frac{\partial C_A}{\partial \eta} = D_{AB}\cdot\frac{1}{4D_{AB}t}\frac{\partial^2 C_A}{\partial \eta^2}\text{，}\frac{\partial^2 C_A}{\partial \eta^2} + 2\eta\frac{\partial C_A}{\partial \eta} = 0$$

The solution is,

$$C_A = c_1\int_0^{\eta} e^{-\eta^2}d\eta + c_2\begin{cases} z = 0 \to \eta = 0, C_A = C_{As} \\ z \to \infty, \eta \to \infty, C_A = 0 \\ t = 0, \eta \to \infty, C_A = 0 \end{cases}$$

B.C.1 代入：

$$C_A = c_2 = C_{As}$$

B.C.2 代入：

$$0 = c_1\int_0^{\infty} e^{-\eta^2}d\eta + C_{As}\text{，}c_1 = \frac{-C_{As}}{\frac{\sqrt{\pi}}{2}}$$

$$C_A = C_{As}[1 - \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\eta^2}d\eta] = C_{As}[1 - erf(\eta)]$$

(2)

For η,

$$\eta = \frac{z}{\sqrt{4D_{AB}t}} = \frac{2\times10^{-3}}{\sqrt{4\times2.5\times10^{-11}\times10\times3600}} = 1.054$$

For $erf(1.054)$,

$$erf(1.054) \approx \frac{erf(1)+erf(1.1)}{2} = 0.86145$$

$$C_A = 0.001[1-0.86145] = 1.3855\times10^{-4} = \underline{\underline{0.01385\%}}$$

Problem 6

【Solution】

By shell balance of energy, assume,

(1) Steady state (2) Consider only r-direction

$$q_r''(2\pi rL)\Big|_r - q_r''(2\pi rL)\Big|_{r+dr} + q_k(2\pi rLdr) = 0$$

同除 $2\pi Ldr \to 0$

$$\frac{d(rq_r'')}{dr} + q_k r = 0 \text{ , } \frac{d(rq_r'')}{dr} + \frac{I^2}{k_e} r = 0$$

$$(\,k_e = electrical\ conductivity \propto \frac{1}{resistance}\,)$$

By definition, the current density, $I[=]amp/m^2$ is,

$$I = \frac{k_e(\Delta V)}{L} \text{ , } \frac{d(rq_r'')}{dr} + \frac{k_e(\Delta V)^2}{L^2} r = 0$$

$$q_r'' = -k\frac{dT}{dr} = \frac{-k_e(\Delta V)^2}{2L^2} r + \frac{c_1}{r}$$

$\because\ r = 0,\ T = finite,$

$$c_1 = 0 \text{ , } \frac{dT}{dr} = \frac{k_e(\Delta V)^2}{2kL^2} r$$

$$T = \frac{k_e(\Delta V)^2}{4kL^2} r^2 + c_2$$

$r = R, T = T_R,$

$$c_2 = T_R - \frac{k_e(\Delta V)^2 R^2}{4kL^2}$$

$$T = \frac{k_e(\Delta V)^2}{4kL^2} r^2 + T_R - \frac{k_e(\Delta V)^2 R^2}{4kL^2}$$

At the wire axis, $R = 0$, $T = T_0$

$$T_R - T_0 = \frac{k_e(\Delta V)^2 R^2}{4kL^2}$$

$$20 - 10 = \frac{59.6\times10^6(\Delta V)^2 \times (2\times10^{-3})^2}{4\times390\times(5^2)} \text{ , } \underline{\underline{\Delta V = 40.4\ (V)}}$$

(本題改編自：Bird, R.; Stewart, W.; Lightfoot, E. *Transport Phenomena*, 2nd ed.; p 295, Example 10.2-1.)

Problem 7

【Solution】

(1) For a given complex velocity potential, we have, by definition,

$$w(z) = \phi(x, y) + i\Psi(x, y)$$

$$\frac{dw}{dz} = -v_x + iv_y$$

$\because w(z) = v_\infty(z + \frac{R^2}{z})$,

$$\frac{dw}{dz} = v_\infty(1 - \frac{R^2}{z^2})$$

$\because z = x + iy = r\sin\theta + ir\cos\theta = re^{i\theta}$

$$\frac{dw}{dz} = v_\infty(1 - \frac{R^2}{z^2}) = v_\infty(1 - \frac{R^2}{r^2}e^{-2i\theta}) = v_\infty(1 - \frac{R^2}{r^2}\cos 2\theta) + iv_\infty(\frac{R^2}{r^2}\sin 2\theta)$$

$$\underline{\underline{v_x = -v_\infty(1 - \frac{R^2}{r^2}\cos 2\theta)}} \text{ 、 } \underline{\underline{v_y = v_\infty(\frac{R^2}{r^2}\sin 2\theta)}}$$

(b)

By Bernoulli equation for incompressible, potential flow,

$$(\frac{1}{2}\rho v^2 + p)_{r=R} = \frac{1}{2}\rho v_\infty^2 + p_\infty$$

For $(v^2)_{r=R}$

$$(v^2)_{r=R} = (v_x^2 + v_y^2)_{r=R} = v_\infty^2[(1-\cos 2\theta)^2 + (\sin 2\theta)^2]$$

$$= v_\infty^2[1 - 2\cos 2\theta + \cos^2 2\theta + (\sin 2\theta)^2]$$

$$= v_\infty^2[2 - 2\cos 2\theta] = v_\infty^2[2 - 2(1 - 2\sin^2\theta)]$$

$$= 4v_\infty^2 \sin^2\theta$$

代回：

$$\frac{1}{2}\rho \times (4v_\infty^2 \sin^2\theta) + p = \frac{1}{2}\rho v_\infty^2 + p_\infty$$

$$\underline{\underline{p = \frac{1}{2}\rho v_\infty^2(1 - 4\sin^2\theta) + p_\infty}}$$

(本題改編自：Bird, R.; Stewart, W.; Lightfoot, E. *Transport Phenomena*, 2nd ed.; p 128~129, Example 4.3-1.)

在 106 年中興單操輸送 Problem 8 所提及之 D'Alembert paradox，可以用此範例做說明：

若考慮 tengential force

$$p=\int_0^{2\pi}[\frac{1}{2}\rho v_\infty^2(1-4\sin^2\theta)+p_\infty]R\sin\theta d\theta=0$$

若考慮 normal force

$$p=\int_0^{2\pi}[\frac{1}{2}\rho v_\infty^2(1-4\sin^2\theta)+p_\infty]R\cos\theta d\theta=0$$

也就代表在此狀態下，數學證明出物體所受到的 drag force 為 0，然而卻與實際情況矛盾，就算在高雷諾數下，物體所受到之拖曳力依舊可觀。直至 Prandtl 提出邊界層理論，並且可用於說明 Stagnation point 的產生以及分離點後渦流的生成後，才得以解釋與解決。

(關於 D'Alembert apradox 之描述可參考：Bird, R.; Stewart, W.; *Lightfoot, E. Transport Phenomena*, 2nd ed.; p 128~130.

Munson, B. *Fundamentals of Fluid Mechanics*, 6th ed.; p 300~302(有壓力造成之拖曳力計算方式)

Philip J. Pritchard. *Fox and McDonald's Intoduction to Fluid Mechanics*, 8th ed.; p 38~41.)

110 年中興工程數學與輸送現象

Problem 1

【Solution】

(a)

$\because v_x = v_y = 0$, $v_z = v_z(x,y)$, and the gravity effect can be ignored, by Navier Stoke equation,

$$\rho(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x \frac{\partial v_z}{\partial x}} + \cancel{v_y \frac{\partial v_z}{\partial y}} + \cancel{v_z \frac{\partial v_z}{\partial z}}) = -\frac{\partial P}{\partial z} + \mu(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}) + \cancel{\rho g_z}$$

$$\underline{\underline{\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} = \frac{1}{\mu}(\frac{\partial P}{\partial z}) = K}}$$

(b)

Let $v_z = W(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})$

$$-W(\frac{2}{a^2} + \frac{2}{b^2}) = K$$

$$W = \frac{-K}{(\frac{2}{a^2} + \frac{2}{b^2})} = \frac{-K}{(\frac{2a^2 + 2b^2}{a^2 b^2})} = -\frac{a^2 b^2 K}{2a^2 + 2b^2}$$

$$v_z = W(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}) = \underline{\underline{-\frac{a^2 b^2 K}{2a^2 + 2b^2}(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})}}$$

(c)

For the maxinum velocity, $(x, y) = (0, 0)$ 代入：

$$v_{z,\max} = \underline{\underline{-\frac{a^2 b^2 K}{2a^2 + 2b^2}}}$$

For the average velocity, calculate the volumatric flow rate first, which is,

$$Q=\iint v_z dA=\iint -\frac{a^2b^2K}{2a^2+2b^2}(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})dxdy$$

Let $x=au$， $y=bv$

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}a & 0\\ 0 & b\end{vmatrix}=ab$$

原式：

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\rightarrow u^2+v^2=1$$

$$Q=\iint -\frac{a^2b^2K}{2a^2+2b^2}(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})dxdy=-\frac{a^3b^3K}{2a^2+2b^2}\iint(1-u^2-v^2)dudv$$

$$=-\frac{a^3b^3K}{2a^2+2b^2}\int_0^{2\pi}\int_0^1(1-r^2)rdrd\theta=-\frac{a^3b^3\pi K}{4a^2+4b^2}$$

The average velocity can be calculated as,

$$\underline{\underline{v_z=\frac{Q}{\pi ab}=-\frac{a^2b^2K}{4a^2+4b^2}}}$$

Problem 2

【Solution】

(a)

$$\delta[=]L \qquad x[=]L \qquad U[=]\frac{m}{s}[=]\frac{L}{T}$$

$$\mu[=]\frac{kg}{m\cdot s}[=]\frac{M}{LT} \qquad \rho[=]\frac{kg}{m^3}[=]\frac{M}{L^3} \qquad a[=]\frac{m}{s}[=]\frac{L}{T}$$

Fundamental units： M, L, T

Total variable： $\delta, x, U, \mu, \rho, a$

共可決定$6-3=3$個無因次群，定義為π_1, π_2, π_3

令 repeating unit 為 ρ, U, x

For π_1

$$\pi_1 = \delta\rho^a U^b x^c$$

$$L\cdot(ML^{-3})^a\cdot(LT^{-1})^b\cdot(L)^c = 1$$

$$\begin{cases} L: 1-3a+b+c=0 \\ T: -b=0 \\ M: a=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=-1 \end{cases}$$

$$\underline{\underline{\pi_1 = \frac{\delta}{x}}}$$

For π_2

$$\pi_2 = \mu\rho^d U^e x^f$$

$$(ML^{-1}T^{-1})\cdot(ML^{-3})^d\cdot(LT^{-1})^e\cdot(L)^f = 1$$

$$\begin{cases} L: -1-3d+e+f=0 \\ T: -1-e=0 \\ M: 1+d=0 \end{cases} \Rightarrow \begin{cases} d=-1 \\ e=-1 \\ f=-1 \end{cases}$$

$$\underline{\underline{\pi_2 = \frac{\mu}{\rho U x}}}$$

For π_3

$$\pi_3 = a\rho^g U^h x^i$$

$$(LT^{-1})\cdot(ML^{-3})^g\cdot(LT^{-1})^h\cdot(L)^i = 1$$

$$\begin{cases} L: 1-3g+h+i=0 \\ T: -1-h=0 \\ M: g=0 \end{cases} => \begin{cases} g=0 \\ h=-1 \\ i=0 \end{cases}$$

$$\underline{\underline{\pi_3 = \frac{a}{U}}}$$

(b)

$$\pi_1 = \frac{\delta}{x} = dimensionless\ boundary\ layer\ thickness$$

$$\pi_2 = \frac{\mu}{\rho U x} = \frac{1}{Reynolds\ number} = \frac{viscous\ force}{inertial\ force}$$

$$\pi_3 = \frac{a}{U} = \frac{1}{Mach\ number} = \frac{speed\ of\ sound}{speed\ of\ fluid}$$

(Mach number 定義可參考：Welty, J.; Rorrer, G.; Foster, D. ***Fundamentals of Momentum, Heat, and Mass Transfer*****, 5th ed.; p 9~10，為流體速度與音速之比值。)**

(c)

For Reynolds number,

$$\mathrm{Re} = \frac{\rho U x}{\mu} = \frac{1.165\times 30\times 1}{1.86\times 10^{-5}} = 1.88\times 10^{6}$$

For Mach number,

$$M = \frac{U}{a} = \frac{30}{349} = 0.086$$

<u>Reynolds number is the dominating parameter to determine the boundary layer thickness.</u>

Problem 3

【Solution】

令 $C(r,\theta)=R(r)\phi(\theta)$ 代入 $\dfrac{\partial^2 C(r,\theta)}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial C(r,\theta)}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 C(r,\theta)}{\partial \theta^2}=0$

$$R''\phi+\frac{1}{r}R'\phi+\frac{1}{r^2}R\phi''=0$$

$$\frac{r^2R''+rR'}{R}=-\frac{\phi''}{\phi}=\lambda$$

$$\begin{cases} r^2R''+rR'-\lambda R=0 \\ \phi''+\lambda\phi=0 \end{cases}$$

$$\phi(0)=\phi(\pi)=0$$

1. $\lambda=0$ 代入 $\phi''+\lambda\phi=0$

$$\phi=C_1+C_2\theta$$

代入 B.C.

$$C_1=C_2=0$$

2. $\lambda=-k^2, k>0$ 代入 $\phi''+\lambda\phi=0$

$$\phi=C_1\cosh(k\theta)+C_2\sinh(k\theta)$$

代入 B.C.

$$C_1=C_2=0$$

3. $\lambda=k^2, k>0$ 代入 $\phi''+\lambda\phi=0$

$$\phi=C_1\cos(k\theta)+C_2\sin(k\theta)$$

代入 B.C.

$$C_1=0$$

令 $C_2\neq 0$

$$k=n$$
$$\lambda=n^2$$

$$\phi_n(\theta)=\sin(n\theta)$$

$$r^2R''+rR'-n^2R=0$$
$$R(r)=\alpha r^n+\beta r^{-n}$$

$R(r)$ is finite at $r=0$,

$$\beta=0$$

$$R_n(r)=\alpha_n r^n$$

$$C_n(r,\theta) = R_n(r)\phi_n(\theta) = \alpha_n r^n \sin(n\theta)$$

$$C(r,\theta) = \sum_{n=1}^{\infty} d_n r^n \sin(n\theta)$$

代入 B.C.

$$C(R,\theta) = \sum_{n=1}^{\infty} d_n R^n \sin(n\theta) = C_0$$

$$d_n R^n = \frac{2}{\pi}\int_0^{\pi} C_0 \sin(n\theta) d\theta = \frac{2C_0}{n\pi}\left(1-(-1)^n\right)$$

$$d_n = \frac{2C_0}{n\pi} R^{-n}\left(1-(-1)^n\right)$$

$$\underline{\underline{C(r,\theta) = \sum_{n=1}^{\infty}\left(\frac{2C_0}{n\pi}\left(1-(-1)^n\right)\right)\left(\frac{r}{R}\right)^n \sin(n\theta)}}$$

Problem 4

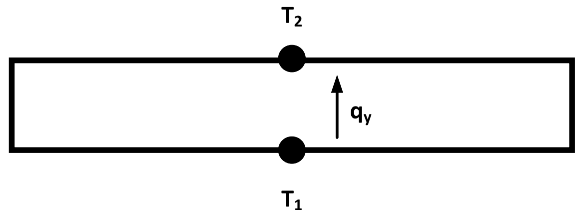
【Solution】

(a)

Fourier’s law of conduction is,

$$\underline{\underline{q=-k\nabla T}}$$

(b)



Assume only y-direction conduction,

$$q_y=-k\frac{dT}{dy}$$

For constant cross-sectional area and steady state,

$$q_k\int_0^L dy=-k\int_{T_1}^{T_2}dT$$

$$q_k=\frac{k(T_1-T_2)}{L}=\frac{Q}{A}$$

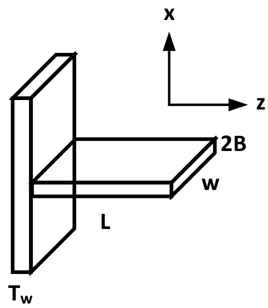
$$\frac{3}{930}=\frac{k(26-24)}{0.64}\text{，}\underline{\underline{k=1.032\times10^{-3}\ (W/cm\cdot K)}}$$

(本題改編自：Bird, R.; Stewart, W.; Lightfoot, E. *Transport Phenomena*, 2nd ed.; p 270, Example 9.1-1.)

Problem 5

【Solution】

(a)



By energy balance of the fin area,

$$2BWq_z\big|_z - 2BWq_z\big|_{z+dz} - h(2Wdz)(T-T_a) = 0$$

同除 $2BWdz \to 0$

$$\frac{-dq_z}{dz} - \frac{h}{B}(T-T_a) = 0$$

$\because q_z = -k\dfrac{dT}{dz}$

$$\frac{d^2T}{dz^2} - \frac{h}{kB}(T-T_a) = 0$$

Let $\theta = \dfrac{T-T_a}{T_w-T_a}$ ， $\zeta = \dfrac{z}{L}$ ， $N^2 = \dfrac{hL^2}{kB}$

$$\frac{d^2\theta}{d\zeta^2} - N^2\theta = 0 \begin{cases} z=0, T=T_w \to \zeta = 0, \theta = 1 \\ z=L, \dfrac{dT}{dz} = 0 \to \zeta = 1, \dfrac{d\theta}{d\zeta} = 0 \end{cases}$$

$$\theta = c_1\sinh(N\zeta) + c_2\cosh(N\zeta) \begin{cases} c_2 = 1 \\ c_1 = -\tanh(N) \end{cases}$$

$$\theta = -\tanh(N)\sinh(N\zeta) + \cosh(N\zeta) = \cosh(N\zeta) - \frac{\sinh(N)}{\cosh(N)}\sinh(N\zeta)$$

$$= \frac{\cosh(N\zeta)\cosh(N) - \sinh(N\zeta)\sinh(N)}{\cosh(N)} = \frac{\cosh(N-N\zeta)}{\cosh(N)} = \frac{\cosh[N(1-\zeta)]}{\cosh(N)}$$

(Note, $\cosh(a)\cosh(b) - \sinh(a)\sinh(b) = \cosh(a-b) = \cosh(b-a)$)

$$\theta=\frac{\cosh[N(1-\zeta)]}{\cosh(N)}\text{ ， }\underline{\underline{\frac{T-T_a}{T_w-T_a}=\frac{\cosh[\sqrt{\frac{hL^2}{kB}}(1-\frac{z}{L})]}{\cosh\sqrt{\frac{hL^2}{kB}}}}}$$

(b)

For the effectiveness of the fin, by definition,

$$\eta=\frac{actual\ rate\ of\ heat\ loss\ from\ the\ fin}{rate\ of\ heat\ loss\ from\ an\ isothermal\ fin\ at\ T_w}$$

$$\eta=\frac{2\times\int_0^W\int_0^L h(T-T_a)dzdy}{2\times\int_0^W\int_0^L h(T_w-T_a)dzdy}=\frac{\int_0^W\int_0^1\theta d\zeta dy}{\int_0^W\int_0^1 d\zeta dy}=\frac{\int_0^1\frac{\cosh[N(1-\zeta)]}{\cosh(N)}d\zeta}{\int_0^1 d\zeta}$$

$$=\frac{1}{\cosh(N)}\{\frac{-1}{N}\sinh[N(1-\zeta)]\}\Big|_0^1=\underline{\underline{\frac{\tanh(N)}{N}}}$$

(※此題題目所需求出之 effectiveness 是根據 Bird 原文書中之定義給出，與 3W 之 efficiency 定義相同，皆是以實際/最大熱傳速率作為定義，並沒有寫錯。)

(c)

By the result of (a),

$$\frac{T-T_a}{T_w-T_a}=\frac{\cosh[\sqrt{\frac{hL^2}{kB}}(1-\frac{z}{L})]}{\cosh\sqrt{\frac{hL^2}{kB}}}$$

$$\frac{T_1-T_a}{T_w-T_a}=\frac{\cosh[\sqrt{\frac{hL^2}{kB}}(1-\frac{L}{L})]}{\cosh\sqrt{\frac{hL^2}{kB}}}=\frac{\cosh(0)}{\cosh\sqrt{\frac{hL^2}{kB}}}$$

$$\frac{500-T_a}{350-T_a}=\frac{\cosh(0)}{\cosh\sqrt{\frac{120\times0.2^2}{60\times\frac{0.08}{12}}}}\text{ ， }\underline{\underline{T_a=510\ (^\circ\mathrm{F})}}$$

(本題組改編自：Bird, R.; Stewart, W.; Lightfoot, E. *Transport Phenomena*, 2nd ed.; p 307~309 之示範與 p 309~310, Example 10.7-1.
3W 中 efficiency 之定義可參考：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p 237.)

111 年中興工程數學與輸送現象

<u>Problem 1</u>

【Solution】

$\because D_{tank} >> D_{jet}$, by pseudo-steady state mass balance,

$$-\rho A_{jet} V = \frac{d(\rho A_{tank} h)}{dt}$$

$$-\cancel{\rho}(\cancel{\frac{\pi}{4}} D_{jet}^2)V = \cancel{\rho}(\cancel{\frac{\pi}{4}} D_{tank}^2)\frac{dh}{dt}$$

$$\int_{h_0}^{h} \frac{1}{\sqrt{h}} dh = -\int_0^t \frac{\sqrt{2g} D_{jet}^2}{D_{tank}^2}$$

$$2(\sqrt{h_0} - \sqrt{h}) = \frac{\sqrt{2g} D_{jet}^2}{D_{tank}^2} t$$

$$\sqrt{h} = \sqrt{h_0} - \frac{\sqrt{2g} D_{jet}^2}{2D_{tank}^2} t = \sqrt{1.2} - \frac{\sqrt{2\times 9.8}\times(1.3\times 10^{-2})^2}{2\times 0.9^2} t = \underline{\underline{1.09 - 4.62\times 10^{-4} t}}$$

(本題改編自：Nakayama Y. *Introduction to Fluid Mechanics*, 2nd ed; p 66~68.)

Problem 2

【Solution】

(a)

Assume：

(1) Fully-developed in r-direction

(2) $v_r = v_\theta = 0$

(3) Gravity effect is negligible

(4) Constant ρ and μ

(5) Steady state

By the equation of continuity,

$$\cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)} + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)} + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$\frac{\partial v_z}{\partial z} = 0 \text{，} v_z = f(\cancel{t}, r, \cancel{\theta}, \cancel{z})$$

By shell balance of momentum,

$$(\rho v_z v_z r dr d\theta)\big|_z - (\rho v_z v_z r dr d\theta)\big|_{z+dz} + (\tau_{rz} r d\theta dz)\big|_r - (\tau_{rz} r d\theta dz)\big|_{r+dr} + (p r dr d\theta)\big|_z - (p r dr d\theta)\big|_{z+dz} = 0$$

同除 $dr d\theta dz \to 0$

$$\cancel{-\frac{\partial (r \rho v_z v_z)}{\partial z}} - \frac{\partial (r\tau_{rz})}{\partial r} - \frac{\partial (rp)}{\partial z} = 0$$

$$-\frac{\partial (r\tau_{rz})}{\partial r} = \frac{\partial p}{\partial z} r \text{，} \tau_{rz} = -\frac{r}{2}(\frac{\partial p}{\partial z}) + \frac{c_1}{r}$$

$\because r = 0, \dfrac{dv_z}{dr} = 0$

$$c_1 = 0$$

$$\tau_{rz} = -\mu \frac{dv_z}{dr} = -\frac{r}{2}(\frac{\partial p}{\partial z}) \text{，} v_z = \frac{r^2}{4\mu}(\frac{\partial p}{\partial z}) + c_2$$

$\because r = R, v_z = 0$

$$c_2 = -\frac{R^2}{4\mu}(\frac{\partial p}{\partial z})$$

$$v_z = \frac{R^2}{4\mu}(\frac{\partial p}{\partial z})[(\frac{r}{R})^2 - 1]$$

For the average velocity,

$$V = \frac{\int_0^{2\pi}\int_0^R v_z r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta} = \frac{R^2}{8\mu}\left(\frac{-\partial p}{\partial z}\right)$$

(b)

$$\tau_w [=] N/m^2 = \frac{kg \cdot m}{s^2 m^2} = MT^{-2}L^{-1}$$

$$V[=]\frac{m}{s}[=]\frac{L}{T} \qquad \mu[=]\frac{kg}{m \cdot s}[=]\frac{M}{LT} \qquad \rho[=]\frac{kg}{m^3}[=]\frac{M}{L^3}$$

$$\varepsilon[=]L \qquad D[=]L$$

Fundamental units： M, L, T

Total variable： $\tau_w, V, \mu, \rho, \varepsilon, D$

共可決定 $6-3=3$ 個無因次群，定義為 π_1, π_2, π_3

令 repeating unit 為 ρ, D, V

For π_1

$$\pi_1 = \varepsilon \rho^a D^b V^c$$

$$L \cdot (ML^{-3})^a \cdot (L)^b \cdot (LT^{-1})^c = 1$$

$$\begin{cases} L: 1-3a+b+c=0 \\ T: -c=0 \\ M: a=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=-1 \\ c=0 \end{cases}$$

$$\pi_1 = \frac{\varepsilon}{D}$$

For π_2

$$\pi_2 = \mu\rho^d D^e V^f$$

$$(ML^{-1}T^{-1})\cdot(ML^{-3})^d\cdot(L)^e\cdot(LT^{-1})^f = 1$$

$$\begin{cases} L: -1-3d+e+f=0 \\ T: -1-f=0 \\ M: 1+d=0 \end{cases} \Rightarrow \begin{cases} d=-1 \\ e=-1 \\ f=-1 \end{cases}$$

$$\underline{\underline{\pi_2 = \frac{\mu}{\rho VD}}}$$

For π_3

$$\pi_3 = \tau_w\rho^g D^h V^i$$

$$(MT^{-2}L^{-1})\cdot(ML^{-3})^g\cdot(L)^h\cdot(LT^{-1})^i = 1$$

$$\begin{cases} L: -1-3g+h+i=0 \\ T: -2-i=0 \\ M: 1+g=0 \end{cases} \Rightarrow \begin{cases} g=-1 \\ h=0 \\ i=-2 \end{cases}$$

$$\underline{\underline{\pi_3 = \frac{\tau_w}{\rho V^2}}}$$

(c)

$$\pi_1 = \frac{\varepsilon}{D} = relative\ roughness$$

$$\pi_2 = \frac{\mu}{\rho VD} = \frac{1}{Reynolds\ number} = \frac{viscous\ force}{inertial\ force}$$

$$\pi_3 = \frac{\tau_w}{\rho V^2} = C_D = drag\ coefficient = \frac{drag\ force}{inertial\ force}$$

(正確之 $C_D = \dfrac{\tau_w}{\frac{1}{2}\rho V^2}$ ，差一個常數，但物理意義相同)

((b)(c)小題改編自：Yunus A. Cengel.; John M. Cimbala. *Fluid Mechanics: Fundamentals and Applications*, 3rd ed.; p 316~318, Example 7-9.)

Problem 3

【Solution】

$$y''+2y'+2y=f(t)$$

$$f(t)=\begin{cases}10\sin(2t),\ 0<t<\pi\\ 0,\ t>\pi\end{cases}=10\sin(2t)\left(H(t)-H(t-\pi)\right)$$

將上式做拉氏轉換，令 $Y(s)=L\left[y(t)\right]$

$$s^2Y(s)-sy(0)-y'(0)+2sY(s)-2y(0)+2Y(s)=\frac{20}{s^2+4}-e^{-\pi s}\frac{20}{s^2+4}$$

$$Y(s)=\frac{20}{\left(s^2+4\right)\left(s^2+2s+2\right)}-e^{-\pi s}\frac{20}{\left(s^2+4\right)\left(s^2+2s+2\right)}+\frac{s-3}{s^2+2s+2}$$

$$Y(s)=\left(-\frac{2s}{s^2+4}-\frac{2}{s^2+4}+\frac{2(s+1)}{(s+1)^2+1}+\frac{4}{(s+1)^2+1}\right)\left(1-e^{-\pi s}\right)+\frac{s+1}{(s+1)^2+1}-\frac{4}{(s+1)^2+1}$$

$$\underline{\underline{y(t)=-2\cos(2t)-\sin(2t)+3e^{-t}\cos(t)}}$$

$$\underline{\underline{+\left(2\cos(2(t-\pi))+\sin(2(t-\pi))-2e^{-(t-\pi)}\left(\cos(t-\pi)+2\sin(t-\pi)\right)\right)H(t-\pi),t>0}}$$

Problem 4

【**Solution**】

$$\text{Fourier's law of conduction：}\ \underline{\underline{q=-k\nabla T}}$$

$$\text{Thermal diffusivity：}\ \alpha=\frac{k}{\rho C_p}=\frac{\textit{heat transfer rate}}{\textit{heat storage rate}}$$

Problem 5

【**Solution**】

By dimensional analysis,

$$\Gamma[=](\frac{\delta^3\rho^2 g}{\mu})=m^3\times\frac{kg^2}{m^6}\times\frac{m}{s^2}\times\frac{m\cdot s}{kg}[=]\frac{kg}{m\cdot s}$$

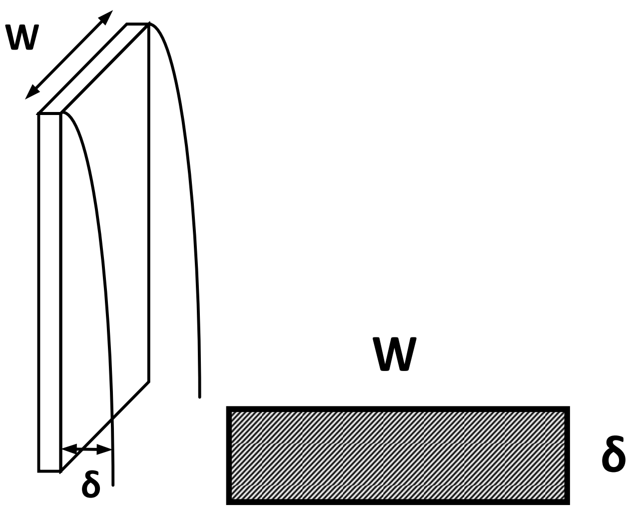
將數據代入：

$$\Gamma=(\frac{\delta^3\rho^2 g}{3\mu})=(\frac{\delta^3\rho g}{3\upsilon})=[\frac{(2\times10^{-3})^3\times(1\times10^3)\times9.8}{3\times2\times10^{-4}}]=\underline{\underline{0.131\ (kg/m\cdot s)}}$$

To check whether the falling oil is laminar or not,

$$\mathrm{Re}=\frac{<\upsilon>(\frac{4A}{P})}{\upsilon}=\frac{<\upsilon>(\frac{4\delta W}{W})}{\upsilon}=\frac{4<\upsilon>\delta}{\upsilon}$$

※Falling film 非傳統圓管，其直徑項需用等效水力直徑(hydraulic diameter)做計算：



For $<v>$,

$$\rho < v > A = Q \text{ ， } \rho < v > = \frac{Q}{A} = \frac{\Gamma}{\delta}$$

$$< v > = \frac{\Gamma}{\delta\rho} = \frac{0.131}{2\times10^{-3}\times1\times10^{3}} = 0.065\ (m/s)$$

$$\mathrm{Re} = \frac{4 < v > \delta}{\upsilon} = \frac{4\times0.065\times2\times10^{-3}}{2\times10^{-4}} = 2.6$$

<u>The falling oil is a laminar flow.</u>

Problem 6

【Solution】

For velocity distribution, assume,

(1) $v_r = v_z = 0$

(2) $v_\theta = v_\theta(\cancel{t}, r, \theta, \cancel{z})$

By the equation of continuity,

$$\cancel{\frac{1}{r}(\frac{\partial}{\partial r} r v_r)} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \cancel{\frac{\partial v_z}{\partial z}} = 0 \text{ ，} v_\theta \neq v_\theta(\theta)$$

(也可在假設時就假設與θ無關，因為對稱性)

By Navier Stokes equation,

$$\rho(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r \frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r v_\theta}{r}} + \cancel{v_z \frac{\partial v_\theta}{\partial z}})$$

$$= \cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}] + \cancel{\rho g_\theta}$$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)) = 0$$

$$v_\theta = C_1 r + \frac{C_2}{r} \begin{cases} r = mR, v_\theta = 0 \\ r = R, v_\theta = \Omega_0 R \end{cases}$$

$$C_1 = \frac{-\Omega_0}{m^2 - 1} \text{ ，} C_2 = \frac{m^2 \Omega_0 R^2}{m^2 - 1}$$

$$\underline{\underline{v_\theta = -\frac{\Omega_0 r}{m^2 - 1} + \frac{m^2 \Omega_0 R^2}{(m^2 - 1)}\frac{1}{r}}}$$

For temperature distribution, consider $T = T(r)$ only.

By the energy equation,

$$\rho C_p(\cancel{\frac{\partial T}{\partial t}} + \cancel{v_r \frac{\partial T}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}} + \cancel{v_z \frac{\partial T}{\partial z}}) = k[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \cancel{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}]$$

$$+2\mu\{\cancel{(\frac{\partial v_r}{\partial r})^2} + [\cancel{\frac{1}{r}(\frac{\partial v_\theta}{\partial \theta} + v_r)}]^2 + \cancel{(\frac{\partial v_z}{\partial z})^2}\} + \mu\{\cancel{(\frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta})^2} + \cancel{(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z})^2} + [\cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}} + r\frac{\partial}{\partial r}(\frac{v_\theta}{r})]^2\}$$

$$k[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})]+\mu[r\frac{\partial}{\partial r}(\frac{v_\theta}{r})]^2=0$$

令 $v_\theta = Ar+\frac{B}{r}$ 代入：

$$\mu[r\frac{\partial}{\partial r}(\frac{v_\theta}{r})]^2=\mu[-\frac{2B}{r^2}]^2=\frac{4\mu B^2}{r^4}$$

代回：

$$k[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})]+\frac{4\mu B^2}{r^4}=0$$

$$T=\frac{-\mu B^2}{kr^2}+\alpha_1\ln r+\alpha_2\begin{cases} r=mR, T=T_k \\ r=R, T=T_1 \end{cases}$$

$$\alpha_1=\frac{\frac{-\mu B^2}{kR^2}[1-\frac{1}{m^2}]+T_k-T_1}{\ln m}\text{，}\alpha_2=T_1+\frac{\mu B^2}{kR^2}-\alpha_1\ln R$$

$$T=\frac{-\mu B^2}{kr^2}+\alpha_1\ln r+T_1+\frac{\mu B^2}{kR^2}-\alpha_1\ln R$$

$$T=\frac{\mu B^2}{kR^2}[1-(\frac{R}{r})^2]+\alpha_1\ln(\frac{r}{R})+T_1=\underline{\underline{\frac{\mu B^2}{kR^2}[1-(\frac{R}{r})^2]+\{\frac{\frac{-\mu B^2}{kR^2}[1-\frac{1}{m^2}]+T_k-T_1}{\ln m}\}\ln(\frac{r}{R})+T_1}}$$

$$\text{where } B=\frac{m^2\Omega_0R^2}{(m^2-1)}$$

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Problem 1

【Solution】

Ans：(B)

令 $u=\frac{y}{x}$

$$dy = xdu + udx$$

將原式同乘 $\frac{1}{x^2}$ 後做變數變換，

$$\left(1+u^2\right)dx + (1-u)(xdu+udx) = 0$$

$$(u+1)dx = x(u-1)du$$

$$\frac{1}{x}dx = \frac{u-1}{u+1}du$$

$$\int \frac{1}{x}dx = \int \frac{u-1}{u+1}du = \int\left(1-\frac{2}{u+1}\right)du$$

$$\ln|x| = u - \ln(u+1)^2 + C_1$$

$$x = C_2\frac{e^u}{(u+1)^2}$$

$$x = C_2\frac{e^{\frac{y}{x}}}{\left(\frac{y}{x}+1\right)^2}$$

$$(x+y)^2 = C_2 x e^{\frac{y}{x}}$$

<u>Problem 2</u>

【Solution】

<u>Ans：(B)</u>

先將原式化成標準式

$$y''+\frac{3x-1}{x^2-x}y'+\frac{1}{x^2-x}y=0$$

令 $p(x)=\dfrac{3x-1}{x^2-x}$

By reduction of order method,

$$y_2=y_1\int\frac{e^{-\int p(x)dx}}{{y_1}^2}dx=\frac{1}{1-x}\int\frac{(1-x)^2}{x(x-1)^2}dx=\frac{1}{1-x}\ln x, x>0$$

<u>Problem 3</u>

【Solution】

<u>Ans：(B)</u>

將原式做拉氏轉換，令 $Y(s)=L\left[y(t)\right]$

$$s^2Y(s)-sy(0)-y'(0)-6sY(s)+6y(0)+9Y(s)=\frac{2}{(s-3)^3}$$

$$\left(s^2-6s+9\right)Y(s)=\frac{2}{(s-3)^3}+2s+5$$

$$Y(s)=\frac{2}{(s-3)^5}+\frac{2s+5}{(s-3)^2}$$

$$y(t)=e^{3t}\left(\frac{1}{12}t^4+11t+2\right), t>0$$

Problem 4

【**Solution**】

Ans：(A)

$$f(t)=3t^2-e^{-t}-f(t)*e^t v$$

將上式做拉氏轉換，令 $F(s)=L[f(t)]$

$$F(s)=\frac{6}{s^3}-\frac{1}{s+1}-F(s)\frac{1}{s-1}$$

$$F(s)=\frac{6(s-1)}{s^4}-\frac{s-1}{s(s+1)}$$

$$F(s)=\frac{6}{s^3}-\frac{6}{s^4}+\frac{1}{s}-\frac{2}{s+1}$$

$$f(t)=3t^2-t^3+1-2e^{-t},t>0$$

Problem 5

【**Solution**】

Ans：(D)

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n=\frac{2}{L}\left(\int_0^{\frac{L}{2}}\left(\frac{2kx}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx+\int_{\frac{L}{2}}^{L}\left(\frac{2k(L-x)}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$b_n=\frac{4k}{L^2}\left(\int_0^{\frac{L}{2}}x\sin\left(\frac{n\pi x}{L}\right)dx+\int_{\frac{L}{2}}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$b_n=\frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Problem 6

【Solution】

利用特徵函數展開法，令 $u(x,t)=\sum_{n=1}^{\infty}C_n(t)\sin\left(\frac{n\pi x}{L}\right)$

代入 $\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$

$$\sum_{n=1}^{\infty}\left(C_n''(t)+\left(\frac{cn\pi}{L}\right)^2 C_n(t)\right)\sin\left(\frac{n\pi x}{L}\right)=0$$

$$C_n''(t)+\left(\frac{cn\pi}{L}\right)^2 C_n(t)=0$$

$$C_n(t)=\alpha_n\cos\left(\frac{cn\pi}{L}t\right)+\beta_n\sin\left(\frac{cn\pi}{L}t\right)$$

$$u(x,t)=\sum_{n=1}^{\infty}\left(\alpha_n\cos\left(\frac{cn\pi}{L}t\right)+\beta_n\sin\left(\frac{cn\pi}{L}t\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$u_t(x,t)=\sum_{n=1}^{\infty}\left(-\frac{cn\pi}{L}\alpha_n\sin\left(\frac{cn\pi}{L}t\right)+\frac{cn\pi}{L}\beta_n\cos\left(\frac{cn\pi}{L}t\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

代入 I.C.

$$\beta_n=0$$

$$u(x,0)=\sum_{n=1}^{\infty}\alpha_n\sin\left(\frac{n\pi x}{L}\right)=f(x)$$

$$\alpha_n=\frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\underline{\underline{u(x,t)=\sum_{n=1}^{\infty}\left(\frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi x}{L}\right)}}$$

Problem 7

【Solution】

(a)

In the specification of the pumps, the pressure at the pump inlet must exceed the vapor pressure of the working fluid by a certain value, which is the *net positive suction head, (NPSH):*

$$\text{NPSH} = \frac{1}{g}\left(\frac{P_a - P_v}{\rho} - h_{fs}\right) + Z_a$$

P_a = absolute pressure at surface of the reservoir.

P_v = vapor pressure

h_{fs} = friction in suction line

(本題可參考：McCabe, W.; Smith, J.; Harriott, P. *Unit Operations of Chemical Engineering*, 7th ed.; p 204.)

(b)

If the suction pressure is only slightly greater or even lower than the vapor pressure of the working fluid, some liquid may flash to vapor inside the pump, which is called cavitation, and will greatly reduce the pump capacity and cause severe corrosion.

(本題可參考：McCabe, W.; Smith, J.; Harriott, P. *Unit Operations of Chemical Engineering*, 7th ed.; p 204.)

(c)-(i)

By the volumetric flow rate, 500 c.c., the velocity in the circulation pipe is,

$$Q = \frac{\pi}{4} D^2 v$$

$$v = \frac{Q}{\frac{\pi}{4}D^2} = \frac{500\times 10^{-6}}{\frac{\pi}{4}\times(25\times 10^{-3})^2} = 1.02\ (m/s)$$

The Reynolds number can thus be calculated as,

$$\mathrm{Re} = \frac{\rho v D}{\mu} = \frac{0.8\times 1000\times 1.02\times 25\times 10^{-3}}{0.5\times 10^{-3}} = \underline{\underline{40743}}$$

(c)-(ii)

The Fanning's friction factor is calculated as,

$$f = \frac{0.079}{\mathrm{Re}^{0.25}} = \frac{0.079}{(40743)^{0.25}} = \underline{\underline{5.56\times 10^{-3}}}$$

(c)-(iii)

If the total length of the pipe considered is L,

$$h_{fs} = \frac{4f}{2D}(Lv^2) = \frac{4\times 5.56\times 10^{-3}}{2\times 25\times 10^{-3}}(1.02^2 L) = 0.461\ (m^2/s^2)$$

To convert the unit to m/m, use the specific gravity,

$$h_{fs,(m/m)} = \frac{0.461}{g}L\ \left(\frac{m^2}{s^2}\times\frac{s^2}{m}\right) = \frac{0.461}{9.8}L = 0.047L\ (m)$$

$$\underline{\underline{\frac{h_{fs(m/m)}}{L} = 0.047\ (m/m)}}$$

(The head loss due to the friction in the pipe “per unit length”)

(c)-(iv)

Let the total length of the pipe is l_p, by the definition and requirement of the NPSH,

$$NPSH = \frac{1}{g}(\frac{P_a - P_v}{\rho} - h_{fs}) + Z_a$$

$$3 = \frac{P_a - P_v}{\rho g} - h_{fs(m/m)} + Z_a$$

($h_{fs(m/m)}$ 在上一小題已經換過單位了)

$$3 = \frac{1.013\times10^5 - 1\times10^3}{0.8\times10^3\times9.8} - 0.047 l_p + \Delta z$$

$$\underline{\underline{\Delta z = 0.0417 l_p - 9.79\ (m)}}$$

<u>Problem 8</u>

【Solution】

(a)

Consider a control volume in Cartesian coordinate, by mass balance,

$$\rho v_x \Delta y \Delta z|_x - \rho v_x \Delta y \Delta z|_{x+\Delta x} + \rho v_y \Delta x \Delta z|_y - \rho v_y \Delta x \Delta z|_{y+\Delta y} + \rho v_z \Delta x \Delta y|_z - \rho v_z \Delta x \Delta y|_{z+\Delta z} = \frac{\partial \rho(\Delta x \Delta y \Delta z)}{\partial t}$$

同除 $\Delta x \Delta y \Delta z \to 0$,

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial \rho}{\partial t}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0}$$

(b)

Consider a control volume in cylindrical coordinate, by mass balance,

$$\rho v_r r\Delta\theta\Delta z\big|_r - \rho v_r r\Delta\theta\Delta z\big|_{r+\Delta r} + \rho v_\theta \Delta r\Delta z\big|_\theta - \rho v_\theta \Delta r\Delta z\big|_{\theta+\Delta\theta} + \rho v_z r\Delta\theta\Delta r\big|_z - \rho v_z r\Delta\theta\Delta r\big|_{z+\Delta z}$$
$$= \frac{\partial\rho(r\Delta r\Delta\theta\Delta z)}{\partial t}$$

同除 $\Delta r\Delta\theta\Delta z \to 0$,

$$-\frac{\partial(\rho r v_r)}{\partial r} - \frac{\partial(\rho v_\theta)}{\partial\theta} - \frac{\partial(r\rho v_z)}{\partial z} = \frac{\partial r\rho}{\partial t}$$

同除 r

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z} = 0}$$

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Problem 1

【Solution】

令 $D=\dfrac{d}{dx}$ 代入 O.D.E

$$\left(D^2+1\right)y=\sec x$$

1. 求 y_h

令 $D^2+1=0$

$$D^2=\pm i$$

$$y_h=C_1\cos x+C_2\sin x$$

2. 求 y_p

令 $\phi_1=\cos x,\ \phi_2=\sin x,\ r(x)=\sec x$

$$W\left(\phi_1,\phi_2\right)=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$$

由參數變異法可得

$$y_p=-\phi_1\int\frac{\phi_2 r(x)}{W\left(\phi_1,\phi_2\right)}dx+\phi_2\int\frac{\phi_1 r(x)}{W\left(\phi_1,\phi_2\right)}dx$$

$$y_p=-\cos x\int\frac{\sin x}{\cos x}dx+\sin x\int dx$$

$$y_p=\cos x\cdot\ln\left|\cos x\right|+x\sin x$$

故選(D)

Problem 2

【**Solution**】

$$\left(e^{x+y}+ye^{y}\right)dx+\left(xe^{y}-1\right)dy=0$$

$$\left(e^{x}+y\right)dx+\left(x-e^{-y}\right)dy=0$$

$$e^{x}dx+\left(ydx+xdy\right)-e^{-y}dy=0$$

$$e^{x}dx+d\left(xy\right)-e^{-y}dy=0$$

$$e^{x}+xy+e^{-y}=C$$

故選(A)

Problem 3

【**Solution**】

將 O.D.E.化成 standard form

$$y''-\frac{1}{x-1}y'+\frac{1}{x^{2}-x}y=0$$

By reduction of order,

$$y_2=y_1\int\frac{e^{-\int\frac{-1}{x-1}dx}}{{y_1}^{2}}dx$$

$$y_2=x\int\frac{x-1}{x^{2}}dx$$

$$y_2=x\left(\ln|x|-\frac{1}{x}\right)$$

$$y_2=x\ln|x|-1$$

故選(A)

Problem 4

【Solution】

令 $Y(s)=L\left[y(t)\right]$，將 O.D.E.進行拉氏轉換

$$s^2Y(s)-sy(0)-y'(0)-Y(s)=\frac{1}{s^2}$$

$$\left(s^2-1\right)Y(s)-s-1=\frac{1}{s^2}$$

$$Y(s)=\frac{1}{s^2\left(s^2-1\right)}+\frac{s}{s^2-1}+\frac{1}{s^2-1}$$

$$Y(s)=\frac{-1}{s^2}+\frac{2}{s^2-1}+\frac{s}{s^2-1}$$

$$y(t)=L^{-1}\left[Y(s)\right]=-t+2\sinh t+\cosh t$$

$$y(t)=-t+\sinh t+e^t, t>0$$

故選(A)

Problem 5

【Solution】

$f(x)$ is an even function.

By Fourier cosine series, let $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{2}\right)$

$$\begin{cases} a_0=\frac{1}{2}\int_0^1 k\,dx=\frac{k}{2} \\ a_n=\frac{2}{2}\int_0^1 k\cos\left(\frac{n\pi x}{2}\right)dx=\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right) \end{cases}$$

$$f(x)=\frac{k}{2}+\sum_{n=1}^{\infty}\left(\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{2}\right)$$

Let $x=2$,

$$\begin{cases} f(2)=0 \text{ by definition} \\ f(2)=\frac{k}{2}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right) \text{ by F.C.S} \end{cases}$$

$$0=\frac{k}{2}+\left(-\frac{2k}{\pi}+\frac{2k}{3\pi}-\frac{2k}{5\pi}+\frac{2k}{7\pi}\ldots\right)$$

$$\frac{k}{2}=\frac{2k}{\pi}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\ldots\right)$$

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\ldots=\frac{\pi}{4}$$

故選(C)

Problem 6

【Solution】

此為 Fourier Cosine Integral，令 $f(x)=\int_0^{\infty}A(w)\cos wx dw$

$$A(w)=\frac{2}{\pi}\int_0^{\infty}f(x)\cos wx dx=\frac{2}{\pi}\int_0^{\infty}e^{-kx}\cos wx dx$$

$$A(w)=\frac{2}{\pi}\left[\frac{e^{-kx}}{(-k)^2+w^2}(-k\cos wx+w\sin wx)\right]_0^{\infty}$$

$$A(w)=\frac{2}{\pi}\left[0-\frac{1}{k^2+w^2}(-k)\right]=\frac{2}{\pi}\frac{k}{k^2+w^2}$$

$$f(x)=\int_0^{\infty}\left(\frac{2}{\pi}\frac{k}{k^2+w^2}\right)\cos wx dw$$

$$e^{-kx}=\frac{2k}{\pi}\int_0^{\infty}\frac{\cos wx}{k^2+w^2}dw$$

$$\int_0^{\infty}\frac{\cos wx}{k^2+w^2}dw=\frac{\pi}{2k}e^{-kx}$$

故選(D)

Problem 7

【Solution】

(1) By the definition of velocity potential,

$$v_x = \frac{\partial \varphi}{\partial x} = 6x - 4$$

$$v_y = \frac{\partial \varphi}{\partial y} = -6y + 6$$

(2) For irrotational flow, the criterion is,

$$\nabla \times v = 0$$

$$\nabla \times v = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0 - 0 = 0$$

Therefore, the velocity field is irrotational in the region where φ applies.

(3) By the definition of stream function,

$$v_x = \frac{\partial \Psi}{\partial y} \text{，} \Psi = \int v_x dy = 6xy - 4y + f(x)$$

$$v_y = -\frac{\partial \Psi}{\partial x} \text{，} \Psi = -\int v_y dx = 6xy - 4x + g(y)$$

Combine,

$$\underline{\underline{\Psi = 6xy - 4x - 4y + C}}$$

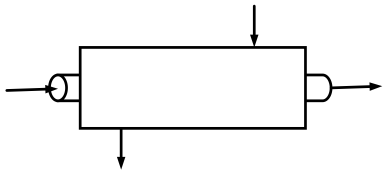
(本題可參考：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p 114~118.)

Problem 8

【**Solution**】

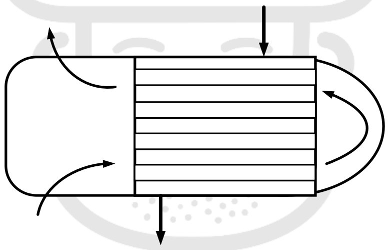
(1) double-pipe exchanger

The double-pipe heat exchanger is the simplest type of heat exchanger, consisting of two pipes where the fluid in the inner tube can flow in the same direction as (parallel flow) or in the opposite direction to (countercurrent flow) the fluid in the outer tube.



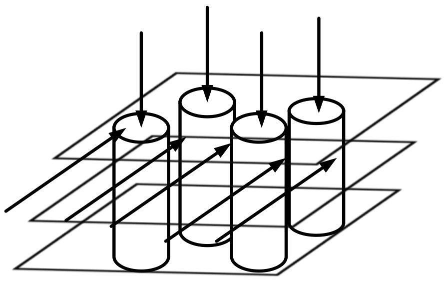
(2) shell-and-tube exchanger

The shell-and-tube exchanger is a multi-pass exchanger where one or both fluids flow through the exchanger more than once.



(3) compact exchanger

A compact exchanger is characterized by its high surface-to-volume ratio, having more compact configurations than that afforded by the shell-and-tube arrangement.



(本題可參考：Welty, J.; Rorrer, G.; Foster, D. ***Fundamentals of Momentum, Heat, and Mass Transfer*****, 5th ed.; p 336~338.)**

Problem 9

【Solution】

(1)

The average velocity is calculated as,

$$v_{avg} = \frac{2\pi c\int_0^R vrdr}{\pi R^2} = \frac{\frac{2}{3}\pi c R^2}{\pi R^2} = \frac{2}{3}c$$

The maximum velocity occurs at $r = 0$,

$$v_{\max} = c$$

The ratio of average velocity and maximum velocity is therefore,

$$\underline{\underline{\frac{v_{avg}}{v_{\max}} = \frac{2}{3}}}$$

(2)

Assume the density of the fluid is ρ, the mass flow rate is,

$$w = \rho Q = \rho(\pi R^2)v_{avg} = \underline{\underline{\frac{2}{3}c\pi R^2\rho}}$$

<u>Problem 10</u>

【Solution】

By the Stokes-Einstein equation,

$$D_{AB} = \frac{kT}{6\pi r \mu_B}$$

$$5.94 \times 10^{-7} \times 10^{-4} = \frac{1.38 \times 10^{-23} \times 293}{6\pi \times 998 \times 10^{-6} \times 1000 r}$$

$$r = 3.62 \times 10^{-9} (m) = \underline{\underline{3.62\ (nm)}}$$

(本題改編自：Welty, J.; Rorrer, G.; Foster, D. *Fundamentals of Momentum, Heat, and Mass Transfer*, 5th ed.; p 431, Problem 24.14.)