106 年台大工程數學

Problem 1
【Solution】

$$令\ g(x) = f(x) - \pi = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx = \frac{2(-1)^{n+1}}{n}$$

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{2(-1)^{n+1}}{n}\right) \sin(nx)$$

$$f(x) = g(x) + \pi = \pi + \sum_{n=1}^{\infty} \left(\frac{2(-1)^{n+1}}{n}\right) \sin(nx)$$

Problem 2
【Solution】

(a)

$$L\left[t^2 \sin(3t)\right] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{d}{ds}\left(\frac{-6s}{(s^2+9)^2}\right) = -6\frac{(s^2+9)^2 - 4s^2(s^2+9)}{(s^2+9)^4}$$

$$= -6\frac{-3s^2+9}{(s^2+9)^3} = \frac{18s^2 - 54}{(s^2+9)^3}$$

(b)

$$L^{-1}\left[\frac{6s+7}{2s^2+4s+10}\right] = \frac{1}{2}L^{-1}\left[\frac{6s+7}{(s+1)^2+4}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{6(s+1)+1}{(s+1)^2+4}\right] = \frac{1}{2}e^{-t}L^{-1}\left[\frac{6s+1}{s^2+4}\right]$$

$$= \frac{1}{2}e^{-t}\left(6\cos(2t) + \frac{1}{2}\sin(2t)\right), t > 0$$

【Solution】

令 $f(t) = 4t\big(H(t) - H(t-1)\big) + 8H(t-1)$

將原式做拉式轉換，令 $Y(s) = L\big[y(t)\big]$

$$s^2Y(s)\underbrace{-sy(0) - y'(0)}_{0} + 3sY(s)\underbrace{-3y(0)}_{0} + 2Y(s) = \frac{4}{s^2} + e^{-s}\left(\frac{-4}{s^2} + \frac{4}{s}\right)$$

$$Y(s) = \frac{4}{s^2\big(s^2 + 3s + 2\big)} + e^{-s}\left(\frac{-4}{s^2\big(s^2 + 3s + 2\big)} + \frac{4}{s\big(s^2 + 3s + 2\big)}\right)$$

$$Y(s) = \frac{4}{s^2\big(s+1\big)\big(s+2\big)} + 4e^{-s}\left(\frac{-1}{s^2\big(s+1\big)\big(s+2\big)} + \frac{1}{s\big(s+1\big)\big(s+2\big)}\right)$$

$$Y(s) = \left(\frac{2}{s^2} - \frac{3}{s} + \frac{4}{s+1} - \frac{1}{s+2}\right) - e^{-s}\left(\frac{2}{s^2} - \frac{5}{s} + \frac{8}{s+1} - \frac{3}{s+2}\right)$$

$$y(t) = L^{-1}\big[Y(s)\big] = \big(2t - 3 + 4e^{-t} - e^{-2t}\big) - H(t-1)\Big[2\big(t-1\big) - 5 + 8e^{-(t-1)} - 3e^{-2(t-1)}\Big], t > 0$$

Problem 4
【Solution】

令 $y(x) = \sum_{n=0}^{\infty} C_n(r)x^{n+r}, 0 < x < \infty$

可得 $y'(x) = \sum_{n=0}^{\infty}(n+r)C_n(r)x^{n+r-1}, y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)C_n(r)x^{n+r-2}$ 代入原式

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n(r)x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)C_n(r)x^{n+r}$$

$$+\sum_{n=0}^{\infty}(n+r)C_n(r)x^{n+r-1} + \sum_{n=0}^{\infty}C_n(r)x^{n+r+1} + \sum_{n=0}^{\infty}C_n(r)x^{n+r} = 0$$

$$\sum_{n=-1}^{\infty}(n+r)(n+r+1)C_{n+1}(r)x^{n+r} + \sum_{n=0}^{\infty}2(n+r)C_n(r)x^{n+r}$$

$$+\sum_{n=-1}^{\infty}(n+r+1)C_{n+1}(r)x^{n+r} + \sum_{n=1}^{\infty}C_{n-1}(r)x^{n+r} + \sum_{n=0}^{\infty}C_n(r)x^{n+r} = 0$$

$$r^2C_0x^{r-1} + \left((2r+1)C_0 + (r+1)^2C_1\right)x^r$$

$$+\sum_{n=1}^{\infty}\left(C_{n-1} + (2n+2r+1)C_n + (n+r+1)^2C_{n+1}\right)x^{n+r} = 0$$

令 $C_0 \neq 0$，$r = 0,0$ 代入

$$C_0 + C_1 = 0$$

$$C_1 = -C_0$$

$$C_{n-1} + (2n+1)C_n + (n+1)^2 C_{n+1} = 0, \forall n \in \mathbb{N}$$

$$C_{n+1} = -\frac{C_{n-1} + (2n+1)C_n}{(n+1)^2}, \forall n \in \mathbb{N}$$

$$C_2 = \frac{1}{2!}C_0$$

$$C_3 = \frac{-1}{3!}C_0$$

$$C_4 = \frac{1}{4!}C_0$$

$$C_5 = \frac{-1}{5!}C_0$$

$$\phi_1(x) = x^r \sum_{n=0}^{\infty} C_n(r)x^n \bigg|_{r=0} = C_0\left(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \ldots\right) = C_0 e^{-x}$$

取 $\phi_1(x) = e^{-x}$ 為 O.D.E.之一解

By reduction of order,

$$\phi_2 = e^{-x} \int \frac{1}{e^{-2x}xe^{2x}} dx = e^{-x}\ln(x)$$

$$y = c_1 e^{-x} + c_2 e^{-x}\ln(x), x > 0$$

Problem 5

【Solution】

令 $u(x,t) = X(x)T(t)$ 代入邊界條件及 P.D.E.

$$\begin{cases} u(0,t) = X(0)T(t) = 0 \\ u(L,t) = X(L)T(t) = 0 \end{cases} \overset{T(t) \neq 0}{\Rightarrow} X(0) = X(L) = 0$$

$$XT' = cX''T$$

$$\frac{X''}{X} = \frac{T'}{cT} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + c\lambda T = 0 \end{cases}, \text{ 其中} X'' + \lambda X = 0 \text{為特徵方程式}$$

1. 令 $\lambda = 0$ 代入特徵方程式

$$X''(x) = 0$$

$$X(x) = C_1 + C_2 x$$

代入 B.C.

$$C_1 = C_2 = 0$$

2. 令 $\lambda = -k^2, k > 0$ 代入特徵方程式

$$X''(x) - k^2 X(x) = 0$$

$$X(x) = C_1 \cosh(kx) + C_2 \sinh(kx)$$

代入 B.C.

$$C_1 = C_2 = 0$$

3. 令 $\lambda = k^2, k > 0$ 代入特徵方程式

$$X''(x) + k^2 X(x) = 0$$

$$X(x) = C_1 \cos(kx) + C_2 \sin(kx)$$

代入 B.C.

$$C_1 = 0$$

令 $C_2 \neq 0$

$$X(L) = C_2 \sin(kL) = 0$$

$$k = \frac{n\pi}{L}, n = 1, 2, 3, \ldots$$

$$\lambda = k^2 = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \ldots$$

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, 3, \ldots$$

$$T' + \left(\frac{cn^2\pi^2}{L^2}\right)T = 0$$

$$T_n = \alpha_n e^{-\frac{cn^2\pi^2}{L^2}t}$$

$$u_n(x,t) = X_n(x)T_n(t) = C_n e^{-\frac{cn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{cn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

代入 I.C.

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} x, & \text{if } 0 < x < \dfrac{L}{2} \\ L - x, & \text{if } \dfrac{L}{2} < x < L \end{cases}$$

$$C_n = \frac{2}{L}\left(\int_0^{\frac{L}{2}} x \sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right) = \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)e^{-\frac{cn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

Problem 6

【Solution】

(a)

令 $y = x^m$ 代入原式

$$\left(m(m-1) + 3m + 2\right)x^m = 0$$

令 $x^m \neq 0$ 代入上式

$$m(m-1) + 3m + 2 = m^2 + 2m + 2 = 0$$

$$m = -1 \pm i$$

$$y = x^{-1}\left(C_1 \cos\left(\ln(x)\right) + C_2 \sin\left(\ln(x)\right)\right), x > 0$$

(b)

將原式同乘 $\dfrac{r^2}{D}$

$$\frac{d}{dr}\left(r^2 \frac{dC}{dr}\right) - \frac{k}{D}r^2 C = 0$$

$$r^2 \frac{d^2 C}{dr^2} + 2r \frac{dC}{dr} - \frac{k}{D}r^2 C = 0$$

令 $1 - 2\alpha = 2, \alpha = \dfrac{-1}{2}$

令 $C = r^\alpha R = r^{\frac{-1}{2}} R$ 代入上式

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \left(-\frac{k}{D}r^2 R - \left(\frac{1}{2}\right)^2\right) = 0$$

1. 令 $-\dfrac{k}{D} = \lambda^2, \lambda > 0$ 代入上式

$$R(r) = c_1 J_{\frac{1}{2}}(\lambda r) + c_2 Y_{\frac{1}{2}}(\lambda r) = c_1 J_{\frac{1}{2}}\left(\sqrt{-\frac{k}{D}}r\right) + c_2 Y_{\frac{1}{2}}\left(\sqrt{-\frac{k}{D}}r\right)$$

$$C(r) = r^{\frac{-1}{2}}\left(c_3 \frac{1}{\sqrt{r}}\sin\left(\sqrt{-\frac{k}{D}}r\right) + c_4 \frac{1}{\sqrt{r}}\cos\left(\sqrt{-\frac{k}{D}}r\right)\right)$$

2. 令 $-\dfrac{k}{D} = -\lambda^2, \lambda > 0$　代入上式

$$R(r) = c_1 I_{\frac{1}{2}}(\lambda r) + c_2 K_{\frac{1}{2}}(\lambda r) = c_1 I_{\frac{1}{2}}\left(\sqrt{\dfrac{k}{D}}\,r\right) + c_2 K_{\frac{1}{2}}\left(\sqrt{\dfrac{k}{D}}\,r\right)$$

$$C(r) = r^{\frac{-1}{2}}\left(c_3 \dfrac{1}{\sqrt{r}}\sinh\left(\sqrt{\dfrac{k}{D}}\,r\right) + c_4 \dfrac{1}{\sqrt{r}}\cosh\left(\sqrt{\dfrac{k}{D}}\,r\right)\right)$$

3. 令 $k = 0$　代入上式

$$r^2 \dfrac{d^2 C}{dr^2} + 2r \dfrac{dC}{dr} = 0$$

$$C(r) = c_1 + c_2 r^{-1}$$

Problem 7

【Solution】

(a)

令 $P_B(\lambda) = \det(B - \lambda I) = 0$

$$\Rightarrow \quad \lambda(\lambda - 5) = 0$$

$$\Rightarrow \quad \underline{\underline{\lambda = 0,\ 5}} \quad \text{為 eigenvalues}$$

(i) 令 $\lambda = 0$ 代入 $(B - \lambda I)\vec{y} = \vec{0}$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{y} = C_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \forall C_1 \neq 0 \text{ 為 eigen vectors}$$

(ii) 令 $\lambda = 5$ 代入 $(B - \lambda I)\vec{y} = \vec{0}$

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{y} = C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \forall C_2 \neq 0 \text{ 為 eigen vectors}$$

(b)

$$P^{-1}BP = D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}, \quad \boxed{P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}$$

(c)

$$P^{-1}BP = D \quad , \quad B = PDP^{-1}$$

$$B^3 = PD^3P^{-1}$$

$$B^3 = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 5^3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = 25\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

(d)

令 $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \vec{f} = \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$

$$\frac{d\vec{x}}{dt} = B\vec{x} + \vec{f}$$

令 $\vec{x} = P\vec{z}, \ \vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$P\vec{z}' = BP\vec{z} + \vec{f}$$

$$\vec{z}' = P^{-1}BP\vec{z} + P^{-1}\vec{f} = D\vec{z} + P^{-1}\vec{f}$$

$$P^{-1}\vec{f} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} = \begin{bmatrix} \dfrac{1}{5}e^t \\ \dfrac{7}{5}e^t \end{bmatrix}$$

$$\begin{cases} z_1' = \dfrac{1}{5}e^t \\ z_2' = 5z_2 + \dfrac{7}{5}e^t \end{cases} \rightarrow \begin{cases} z_1 = \dfrac{1}{5}e^t + C_1 \\ z_2 = -\dfrac{7}{20}e^t + C_2e^{5t} \end{cases}$$

$$\vec{x} = P\vec{z} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{5}e^t + C_1 \\ -\dfrac{7}{20}e^t + C_2e^{5t} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2}e^t + C_1 + 2C_2e^{5t} \\ -\dfrac{3}{4}e^t - 2C_1 + C_2e^{5t} \end{bmatrix}$$

Problem 8

【Solution】

(a)

$$\vec{\nabla} \cdot \vec{G} = \frac{\partial}{\partial x}(xe^{x^2+y^2} + 2xy) + \frac{\partial}{\partial y}(ye^{x^2+y^2} + x^2) = e^{x^2+y^2}(1 + 2x^2 + 1 + 2y^2) + 2y$$

$$= e^{x^2+y^2}(2x^2 + 2y^2 + 2) + 2y$$

(b)

$$\begin{cases} \dfrac{\partial f}{\partial x} = xe^{x^2+y^2} + 2xy \rightarrow f = \dfrac{1}{2}e^{x^2+y^2} + x^2y + g(y) \\ \dfrac{\partial f}{\partial y} = ye^{x^2+y^2} + x^2 \rightarrow f = \dfrac{1}{2}e^{x^2+y^2} + x^2y + h(x) \end{cases}$$

$$\Rightarrow \quad f(x,y) = \frac{1}{2}e^{x^2+y^2} + x^2y + C$$

(c)

$$\oint_c \vec{G} \cdot d\vec{r} = \oint_c df = \oint_c d(\frac{1}{2}e^{x^2+y^2} + x^2y) = 0$$

(d)

若在一平滑且封閉的曲線平面上，$P(x,y)$，$Q(x,y)$ 之一階偏導數存在且連續，

則：

$$\oint_c (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

(e)

By Green's theorem

$$\oint_c (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

$$= \oint_c \iint_R (2xye^{x^2+y^2} + 2x - 2xye^{x^2+y^2} - 2x)dxdy$$

$$= \underline{\underline{0}}$$

107 年台大工程數學

【Solution】

(a)

$$ydx + 2xydy - e^{-2y}dy = 0$$

同乘 $\dfrac{e^{2y}}{y}$,

$$e^{2y}dx + 2xe^{2y}dy - \frac{1}{y}dy = 0$$

$$d\left(xe^{2y}\right) - \frac{1}{y}dy = 0$$

$$\underline{\underline{xe^{2y} - \ln|y| = C}}$$

(b)

令 $D = \dfrac{d}{dx}$

$$\left(D^2 - 2D + 1\right)y = e^x \ln(x)$$

1. 求 y_h

令 $\left(D^2 - 2D + 1\right)y_h = 0$

$$D^2 - 2D + 1 = \left(D - 1\right)^2 = 0$$

$$D = 1, 1$$

$$y_h = C_1 e^x + C_2 x e^x$$

2. 求 y_p ,利用參數變異法

令 $\phi_1 = e^x, \phi_2 = xe^x, r(x) = e^x \ln(x)$

$$W\left(\phi_1, \phi_2\right) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}$$

$$y_p = -\phi_1 \int \frac{\phi_2 r(x)}{W} dx + \phi_2 \int \frac{\phi_1 r(x)}{W} dx = -e^x \int x \ln(x) dx + xe^x \int \ln(x) dx$$

$$y_p = -e^x \left(\frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 \right) + xe^x \left(x \ln(x) - x \right)$$

$$y_p = \frac{1}{2} x^2 e^x \ln(x) - \frac{3}{4} x^2 e^x$$

$$y = C_1 e^x + C_2 xe^x + \frac{1}{2} x^2 e^x \ln(x) - \frac{3}{4} x^2 e^x$$

Problem 2

【Solution】

(a)

令 $\vec{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, D = \frac{d}{dt}$

$$\frac{d^2 \vec{X}}{dt^2} + 2 \frac{d\vec{X}}{dt} - 3\vec{X} = \vec{0}$$

$$\left(D^2 + 2D - 3 \right) \vec{X} = 0, \vec{X} \neq 0$$

$$D^2 + 2D - 3 = 0$$

$$D = -3, 1$$

$$\vec{X} = \begin{bmatrix} C_1 e^{-3t} + C_2 e^t \\ C_3 e^{-3t} + C_4 e^t \end{bmatrix}$$

$$\frac{d\vec{X}}{dt} = \begin{bmatrix} -3C_1 e^{-3t} + C_2 e^t \\ -3C_3 e^{-3t} + C_4 e^t \end{bmatrix}$$

代入 I.C.

$$C_1 = 1, C_2 = 4, C_3 = -1, C_4 = 1$$

$$\vec{X} = \begin{bmatrix} e^{-3t} + 4e^t \\ -e^{-3t} + e^t \end{bmatrix}$$

(b)

$$\text{Slope} \quad m_1 = \lim_{t \to \infty} \left(\frac{-e^{-3t} + e^t}{e^{-3t} + 4e^t} \right) = \frac{1}{4}$$

Intercept $b_1 = \lim\limits_{t\to\infty}\left(-e^{-3t} + e^t - \dfrac{1}{4}e^{-3t} - e^t\right) = 0$

$\underline{\underline{y_1 = \dfrac{1}{4}x}}$ 為一條斜漸近線

Slope $m_2 = \lim\limits_{t\to-\infty}\left(\dfrac{-e^{-3t} + e^t}{e^{-3t} + 4e^t}\right) = -1$

Intercept $b_2 = \lim\limits_{t\to-\infty}\left(-e^{-3t} + e^t + e^{-3t} + 4e^t\right) = 0$

$\underline{\underline{y_2 = -x}}$ 為另一條斜漸近線

Problem 3
【Solution】

將原式做拉氏轉換，令 $Y(s) = L[y(t)], G(s) = L[g(t)]$

$$s^2 Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = G(s)$$

$$\left(s^2 + 4s + 4\right)Y(s) = G(s) + (2s + 5)$$

$$Y(s) = \frac{G(s)}{s^2 + 4s + 4} + \frac{2s + 5}{s^2 + 4s + 4} = \frac{G(s)}{(s+2)^2} + \frac{2s + 5}{(s+2)^2}$$

$$y(t) = L^{-1}[Y(s)] = te^{-2t} * g(t) + 2e^{-2t} + te^{-2t}$$

$$\underline{\underline{= \int_0^t \tau e^{-2\tau} g(t - \tau)d\tau + 2e^{-2t} + te^{-2t}, t > 0}}$$

Problem 4

【Solution】

令 $w(x, t) = U(x, t) - 1$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}$$

$$\begin{cases} w(0, t) = 0, \ t > 0 \\ w(1, t) = 0, \ t > 0 \\ w(x, 0) = \cos(\pi x), \ 0 < x < 1 \end{cases}$$

(a)

令 $w(x, t) = X(x)T(t)$ 代入上式

$$XT' = X''T$$

同乘 $\frac{1}{XT}$，

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda T = 0 \end{cases}，其中 X'' + \lambda X = 0 為特徵方程式$$

1. 令 $\lambda = 0$ 代入特徵方程式

$$X''(x) = 0$$

$$X(x) = C_1 + C_2 x$$

代入 B.C.

$$C_1 = C_2 = 0$$

2. 令 $\lambda = -k^2, k > 0$ 代入特徵方程式

$$X''(x) - k^2 X(x) = 0$$

$$X(x) = C_1 \cosh(kx) + C_2 \sinh(kx)$$

代入 B.C.

$$C_1 = C_2 = 0$$

3. 令 $\lambda = k^2, k > 0$ 代入特徵方程式

$$X''(x) + k^2 X(x) = 0$$

$$X(x) = C_1 \cos(kx) + C_2 \sin(kx)$$

代入 B.C.

$$C_1 = 0$$

令 $C_2 \neq 0$

$$X(1) = C_2 \sin(k) = 0$$

$$k = n\pi, n = 1, 2, 3, \ldots$$

$$\lambda = k^2 = (n\pi)^2, n = 1, 2, 3, \ldots$$

$$X_n(x) = \sin(n\pi x), n = 1, 2, 3, \ldots$$

$$T' + (n^2 \pi^2) T = 0$$

$$T_n = \alpha_n e^{-n^2 \pi^2 t}$$

$$w_n(x, t) = X_n(x) T_n(t) = C_n e^{-n^2 \pi^2 t} \sin(n\pi x)$$

$$w(x, t) = \sum_{n=1}^{\infty} C_n e^{-n^2 \pi^2 t} \sin(n\pi x)$$

代入 I.C.

$$w(x, 0) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) = \cos(\pi x)$$

$$C_n = 2\int_0^1 \cos(\pi x) \sin(n\pi x) dx = \frac{2n\left(1 + (-1)^n\right)}{\pi\left(n^2 - 1\right)}, n \neq 1, C_1 = 0$$

$$w(x, t) = \sum_{n=2}^{\infty} \left(\frac{2n\left(1 + (-1)^n\right)}{\pi\left(n^2 - 1\right)} \right) e^{-n^2 \pi^2 t} \sin(n\pi x)$$

$$u(x,t) = 1 + \sum_{n=2}^{\infty} \left(\frac{2n\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)} \right) e^{-n^2\pi^2 t} \sin(n\pi x)$$

(b)

將 $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}$ 做拉式轉換，令 $W(x,s) = L\big[w(x,t)\big]$

$$\frac{\partial^2 W(x,s)}{\partial x^2} - sW(x,s) = -w(x,0)$$

$$\frac{\partial^2 W(x,s)}{\partial x^2} - sW(x,s) = -\cos(\pi x)$$

$$W(x,s) = C_1(s)\cosh\left(\sqrt{s}\,x\right) + C_2(s)\sinh\left(\sqrt{s}\,x\right) + \frac{1}{s+\pi^2}\cos(\pi x)$$

$$\begin{cases} W(0,s) = 0 \\ W(1,s) = 0 \end{cases}$$

$$C_1(s) = \frac{-1}{s+\pi^2}, \; C_2(s) = \frac{1}{s+\pi^2}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)}$$

$$W(x,s) = \frac{-1}{s+\pi^2}\cosh\left(\sqrt{s}\,x\right) + \left(\frac{1}{s+\pi^2}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)} \right)\sinh\left(\sqrt{s}\,x\right) + \frac{1}{s+\pi^2}\cos(\pi x)$$

$$w(x,t) = L^{-1}\left[\frac{-1}{s+\pi^2}\cosh\left(\sqrt{s}\,x\right) + \left(\frac{1}{s+\pi^2}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)} \right)\sinh\left(\sqrt{s}\,x\right) + \frac{1}{s+\pi^2}\cos(\pi x) \right]$$

$$U(x,t) = 1 + L^{-1}\left[\frac{-1}{s+\pi^2}\cosh\left(\sqrt{s}\,x\right) + \left(\frac{1}{s+\pi^2}\frac{1+\cosh\left(\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)} \right)\sinh\left(\sqrt{s}\,x\right) + \frac{1}{s+\pi^2}\cos(\pi x) \right]$$

108 年台大工程數學

【Solution】

$$ydx - 2xdy + x^2ydy = 0$$

$$\frac{d(xy^{-2})}{y^{-3}} + x^2ydy = 0$$

同乘 $\dfrac{1}{x^2y^{-1}}$ ，

$$\frac{d(xy^{-2})}{x^2y^{-4}} + y^2dy = 0$$

$$(xy^{-2})^{-2}d(xy^{-2}) + y^2dy = 0$$

$$-(xy^{-2})^{-2} + \frac{1}{3}y^3 = C$$

Problem 2
【Solution】

令 $D = \dfrac{d}{dx}$

$$(D^4 - 2D^2 + 1)y = e^{-x}$$

1. 求 y_h

令 $(D^4 - 2D^2 + 1)y_h = 0$

$$D^4 - 2D^2 + 1 = (D-1)^2(D+1)^2 = 0$$

$$D = -1, -1, 1, 1$$

$$y_h = C_1e^{-x} + C_2xe^{-x} + C_3e^x + C_4xe^x$$

2. 求 y_p ，利用 Heaviside 反運算值法

$$y_p = \frac{1}{(D-1)^2(D+1)^2} e^{-x}$$

$$y_p = \frac{1}{4} e^{-x} \frac{1}{D^2}(1) = \frac{1}{4} e^{-x}\left(\frac{1}{2}x^2\right) = \frac{1}{8}x^2 e^{-x}$$

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 e^x + C_4 x e^x + \frac{1}{8}x^2 e^{-x}$$

Problem 3

【Solution】

(a)

令 $x = e^t, t = \ln(x), x > 0, D = \dfrac{d}{dt}$

$$\left(D(D-1)-2\right)y = e^{(m+2)t}$$

1. 求 y_h

令 $\left(D(D-1)-2\right)y_h = 0$

$$D(D-1)-2 = D^2 - D - 2 = (D-2)(D+1) = 0$$

$$D = -1, 2$$

$$y_h(t) = C_1 e^{-t} + C_2 e^{2t}$$

$$y_h(x) = C_1 \frac{1}{x} + C_2 x^2, x > 0$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t) = \frac{1}{(D-2)(D+1)} e^{(m+2)t}$$

$$\because m + 2 > 0 \quad \therefore m + 3 \neq 0$$

Case 1. $m \neq 0$

$$y_p(t) = \frac{1}{m(m+3)} e^{(m+2)t}$$

$$y_p(x) = \frac{1}{m(m+3)} x^{m+2}, x > 0$$

$$y(x) = C_1 \frac{1}{x} + C_2 x^2 + \frac{1}{m(m+3)} x^{m+2}, x > 0$$

Case 2. $m = 0$

$$y_p(t) = \frac{1}{3}e^{2t}\frac{1}{D}(1)$$

$$y_p(t) = \frac{1}{3}te^{2t}$$

$$y_p(x) = \frac{1}{3}x^2\ln(x), x > 0$$

$$y(x) = C_1\frac{1}{x} + C_2x^2 + \frac{1}{3}x^2\ln(x), x > 0$$

(b)

Case 1. $m \neq 0$

$$y(x) = C_1\frac{1}{x} + C_2x^2 + \frac{1}{m(m+3)}x^{m+2}, x > 0$$

代入 I.C.

$$C_1 = 0, C_2 = -\frac{1}{m(m+3)}$$

$$y(x) = \frac{1}{m(m+3)}\left(x^{m+2} - x^2\right), x > 0$$

Case 2. $m = 0$

$$y(x) = C_1\frac{1}{x} + C_2x^2 + \frac{1}{3}x^2\ln(x), x > 0$$

代入 I.C.

$$C_1 = C_2 = 0$$

$$y(x) = \frac{1}{3}x^2\ln(x), x > 0$$

Problem 4
【Solution】

將原式做拉氏轉換，令 $Y(s) = L[y(t)]$

$$-\frac{d}{ds}\left[s^2 Y(s) - sy(0) - y'(0)\right] + \frac{d}{ds}\left[sY(s) - y(0)\right] - Y(s) = 0$$

$$(s-1)\frac{dY(s)}{ds} + 2Y(s) = 0$$
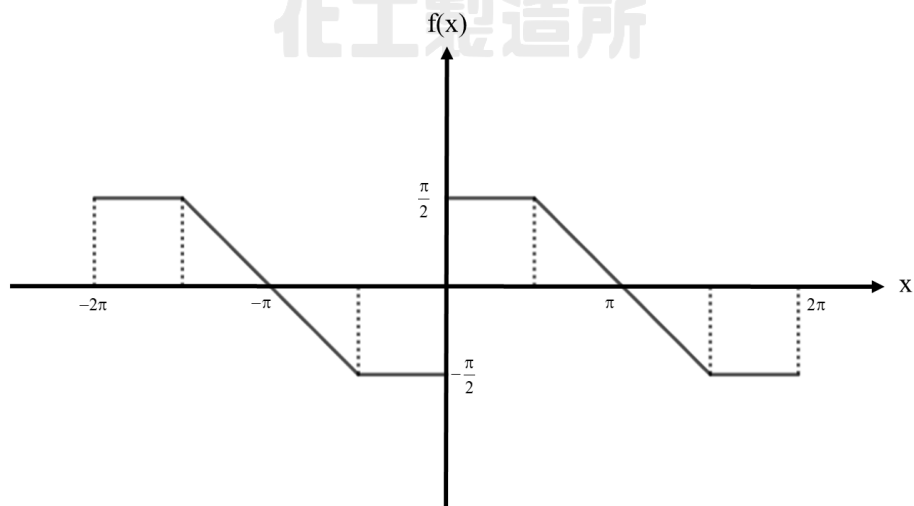
$$\frac{dY(s)}{ds} + \frac{2}{s-1}Y(s) = 0$$

$$Y(s) = C\frac{1}{(s-1)^2}$$

$$y(t) = L^{-1}[Y(s)] = Cte^t$$

代入 I.C.

$$C = 3$$

$$\underline{\underline{y(t) = 3te^t, \, t > 0}}$$

Problem 5
【Solution】

Problem 6

【Solution】

由圖形可知，此函數應為週期為 2π 之偶函數，利用 Fourier cosine series 展開

令 $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(-x+\pi)dx = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi}(-x+\pi)\cos(nx)dx = \frac{2(1-\cos(n\pi))}{n^2\pi}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2(1-\cos(n\pi))}{n^2\pi}\right)\cos(nx)$$

Problem 7

【Solution】

令 $\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

令 $P_A(\lambda) = \det(A-\lambda I) = (\lambda-1)^2 = 0$

$$\lambda = 1,1$$

代入 $(A-\lambda I)\vec{X}_1 = \vec{0}$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{X}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}, c_1 \neq 0$$

取 $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ 作為特徵向量

代入 $(A-\lambda I)\vec{X}_2 = \vec{X}_1$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{X}_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

取 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 作為廣義特徵向量

$$\vec{Y} = C_1 \vec{X}_1 e^t + C_2 \left(\vec{X}_2 + t\vec{X}_1 \right) e^t$$

$$\vec{Y} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) e^t$$

Problem 8
【Solution】

By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \left(\nabla \cdot \vec{F} \right) dV$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V \left(1 + 1 + 2z \right) dV = \underline{96}$$

Problem 9
【Solution】

令 $u(x,t) = w(x,t) + \phi(x)$ ，其中 $\phi(x) = \lim_{t \to \infty} u(x,t)$ 為穩態解

$$\begin{cases} u(0,t) = w(0,t) + \phi(0) = 3 \\ u(1,t) = w(1,t) + \phi(1) = 4 \end{cases}$$

$$\phi(x) = x + 3$$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}$$

(a)

令 $w(x,t) = X(x)T(t)$ 代入上式

$$XT' = X''T$$

同乘 $\dfrac{1}{XT}$ ，

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda T = 0 \end{cases}$$，其中 $X'' + \lambda X = 0$ 為特徵方程式

1. 令 $\lambda = 0$ 代入特徵方程式

$$X''(x) = 0$$

$$X(x) = C_1 + C_2 x$$

代入 B.C.

$$C_1 = C_2 = 0$$

2. 令 $\lambda = -k^2, k > 0$ 代入特徵方程式

$$X''(x) - k^2 X(x) = 0$$

$$X(x) = C_1 \cosh(kx) + C_2 \sinh(kx)$$

代入 B.C.

$$C_1 = C_2 = 0$$

3. 令 $\lambda = k^2, k > 0$ 代入特徵方程式

$$X''(x) + k^2 X(x) = 0 \quad , \quad X(x) = C_1 \cos(kx) + C_2 \sin(kx)$$

代入 B.C.

$$C_1 = 0$$

令 $C_2 \neq 0$

$$X(1) = C_2 \sin(k) = 0$$

$$k = n\pi, n = 1, 2, 3, \ldots$$

$$\lambda = k^2 = (n\pi)^2, n = 1, 2, 3, \ldots$$

$$X_n(x) = \sin(n\pi x), n = 1, 2, 3, \ldots$$

$$T' + (n^2 \pi^2) T = 0$$

$$T_n = \alpha_n e^{-n^2 \pi^2 t}$$

$$w_n(x,t) = X_n(x)T_n(t) = C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$w(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$u(x,t) = x + 3 + \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

代入 I.C.

$$u(x,0) = x + 3 + \sum_{n=1}^{\infty} C_n \sin(n\pi x) = 0$$

$$\sum_{n=1}^{\infty} C_n \sin(n\pi x) = -x - 3$$

$$C_n = 2\int_0^1 (-x-3)\sin(n\pi x)\,dx = \frac{2\left(4(-1)^n - 3\right)}{n\pi}$$

$$u(x,t) = x + 3 + \sum_{n=1}^{\infty} \left(\frac{2\left(4(-1)^n - 3\right)}{n\pi} \right) e^{-n^2\pi^2 t} \sin(n\pi x)$$

109 年台大工程數學

Problem 1
【Solution】

(a)

$$\text{velocity} = \vec{v} = \frac{d\vec{F}}{dt} = \underline{\underline{3\hat{i} - 2t\hat{k}}}$$

(b)

$$\text{speed} = |\vec{v}| = \sqrt{3^2 + (-2t)^2} = \underline{\underline{\sqrt{9 + 4t^2}}}$$

(c)

$$\text{unit tangent vector} = \vec{T}(t) = \frac{\dfrac{d\vec{F}}{dt}}{\left|\dfrac{d\vec{F}}{dt}\right|} = \underline{\underline{\frac{3\hat{i} - 2t\hat{k}}{\sqrt{9 + 4t^2}}}}$$

(d)

$$\vec{T}'(t) = \frac{-12t}{\left(9 + 4t^2\right)^{3/2}}\hat{i} + \frac{18}{\left(9 + 4t^2\right)^{3/2}}\hat{k}$$

$$\text{unit normal vector} = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|} = \underline{\underline{\frac{-2t}{\sqrt{9 + 4t^2}}\hat{i} + \frac{3}{\sqrt{9 + 4t^2}}\hat{k}}}$$

(e)

$$\text{acceleration} = \vec{a} = \frac{d\vec{v}}{dt} = \underline{\underline{-2\hat{k}}}$$

(f)

$$\text{tangent component of acceleration} = \vec{a} \cdot \vec{T}(t) = \underline{\underline{\frac{4t}{\sqrt{9 + 4t^2}}}}$$

(g)

$$\text{normal component of acceleration} = \vec{a} \cdot \vec{N}(t) = \underline{\underline{\frac{-6}{\sqrt{9 + 4t^2}}}}$$

(h)

$$\kappa = \left|\frac{\vec{T}'(t)}{\vec{F}'(t)}\right| = \underline{\underline{\frac{6}{\left(9 + 4t^2\right)^{3/2}}}}$$

Problem 2

【Solution】

$$\left[\sin(t)-\cos(t)\right]^2 = \sin^2(t)-2\sin(t)\cos(t)+\cos^2(t) = 1-\sin(2t)$$

$$L\left[f(t)\right] = L\left[1-\sin(2t)\right] = \frac{1}{s}-\frac{2}{s^2+4}$$

Problem 3

【Solution】

將原式做拉氏轉換，令 $Y(s)=L\left[y(t)\right]$

$$\left(s^2Y(s)-sy(0)-y'(0)\right)+\frac{d}{ds}\left(sY(s)-y(0)\right)+Y(s)=\frac{1}{s}$$

$$\frac{dY(s)}{ds}+\left(s+\frac{2}{s}\right)Y(s)=1+\frac{2}{s}+\frac{1}{s^2}$$

$$Y(s)=\frac{1}{s}+\frac{2}{s^2}+C\frac{1}{s^2}e^{\frac{-s^2}{2}}$$

利用初值定理

$$\lim_{s\to\infty}sY(s)=\lim_{t\to0}y(t)=y(0)=1$$

$$\lim_{s\to\infty}sY(s)=\lim_{s\to\infty}\left(1+\frac{2}{s}+C\frac{1}{s}e^{\frac{-s^2}{2}}\right)=1 \text{，} C=0$$

$$Y(s)=\frac{1}{s}+\frac{2}{s^2}$$

$$y(t)=L^{-1}\left[Y(s)\right]=1+2t, t>0$$

Problem 4

【Solution】

令 $A = \begin{bmatrix} -7 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$，$P^{-1}AP = D$

令 $P_A(\lambda) = \det(A - \lambda I) = \lambda(\lambda - 1)(\lambda^2 + 5\lambda - 10) = 0$

$$\lambda = 0, 1, \frac{-5 \pm \sqrt{65}}{2}$$

1. $\lambda = 0$ 代入 $(A - \lambda I)\vec{X} = \vec{0}$

$$\begin{bmatrix} -7 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, c_1 \neq 0$$

取特徵向量為 $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

2. $\lambda = 1$ 代入 $(A - \lambda I)\vec{X} = \vec{0}$

$$\begin{bmatrix} -8 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, c_2 \neq 0$$

取特徵向量為 $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

3. $\lambda = \dfrac{-5+\sqrt{65}}{2}$ 代入 $(A - \lambda I)\vec{X} = \vec{0}$

$$\begin{bmatrix} \dfrac{-9-\sqrt{65}}{2} & 0 & 1 & 0 \\ 0 & \dfrac{7-\sqrt{65}}{2} & 1 & 0 \\ -4 & 0 & \dfrac{9-\sqrt{65}}{2} & 0 \\ 0 & 0 & 0 & \dfrac{5-\sqrt{65}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_3 \begin{bmatrix} \dfrac{9-\sqrt{65}}{8} \\ \dfrac{-2}{7-\sqrt{65}} \\ 1 \\ 0 \end{bmatrix}, c_3 \neq 0$$

取特徵向量為 $\begin{bmatrix} \dfrac{9-\sqrt{65}}{8} \\ \dfrac{-2}{7-\sqrt{65}} \\ 1 \\ 0 \end{bmatrix}$

4. $\lambda = \dfrac{-5-\sqrt{65}}{2}$ 代入 $(A - \lambda I)\vec{X} = \vec{0}$

$$\begin{bmatrix} \dfrac{-9+\sqrt{65}}{2} & 0 & 1 & 0 \\ 0 & \dfrac{7+\sqrt{65}}{2} & 1 & 0 \\ -4 & 0 & \dfrac{9+\sqrt{65}}{2} & 0 \\ 0 & 0 & 0 & \dfrac{5+\sqrt{65}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_3 \begin{bmatrix} \dfrac{9+\sqrt{65}}{8} \\ \dfrac{-2}{7+\sqrt{65}} \\ 1 \\ 0 \end{bmatrix}, c_3 \neq 0$$

取特徵向量為 $\begin{bmatrix} \dfrac{9+\sqrt{65}}{8} \\ \dfrac{-2}{7+\sqrt{65}} \\ 1 \\ 0 \end{bmatrix}$

$$P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-5+\sqrt{65}}{2} & 0 \\ 0 & 0 & 0 & \dfrac{-5-\sqrt{65}}{2} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & \dfrac{9-\sqrt{65}}{8} & \dfrac{9+\sqrt{65}}{8} \\ 0 & 1 & \dfrac{-2}{7-\sqrt{65}} & \dfrac{-2}{7+\sqrt{65}} \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

<u>Problem 5</u>
【Solution】

令 $D = \dfrac{d}{dx}$

$$\left(D^2 - 6D + 9\right)y = 6x + 2 - 12e^{3x}$$

1. 求 y_h

令 $\left(D^2 - 6D + 9\right)y_h = 0$

$$D^2 - 6D + 9 = \left(D - 3\right)^2 = 0$$

$$D = 3, 3$$

$$y_h = C_1 e^{3x} + C_2 x e^{3x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p = \frac{1}{D^2 - 6D + 9}\left(6x + 2 - 12e^{3x}\right)$$

$$y_p = \frac{1}{9}\frac{1}{1 - \left(-\dfrac{D^2}{9} + \dfrac{2D}{3}\right)}\left(6x + 2\right) - 12\frac{1}{\left(D - 3\right)^2}e^{3x}$$

$$y_p = \frac{1}{9}\left(1 + \left(-\dfrac{D^2}{9} + \dfrac{2D}{3}\right) + \left(-\dfrac{D^2}{9} + \dfrac{2D}{3}\right)^2 + \ldots\right)\left(6x + 2\right) - 12e^{3x}\frac{1}{\left(D\right)^2}(1)$$

$$y_p = \frac{1}{9}\left(1 + \dfrac{2D}{3}\right)\left(6x + 2\right) - 12e^{3x}\left(\dfrac{1}{2}x^2\right)$$

$$y_p = \frac{2}{3}\left(x + 1\right) - 6x^2 e^{3x}$$

$$y = C_1 e^{3x} + C_2 x e^{3x} + \frac{2}{3}\left(x + 1\right) - 6x^2 e^{3x}$$

Problem 6

【Solution】

(a)

非線性 P.D.E.無法分離變數

(b)

線性且齊性 P.D.E.可以分離變數

(c)

非線性 P.D.E.無法分離變數

(d)

非齊性 P.D.E.無法分離變數

(e)

線性且齊性 P.D.E.可以分離變數

Problem 7

【Solution】

令 $u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$

$$\nabla^2 u(x,y) = \nabla^2 u_1(x,y) + \nabla^2 u_2(x,y) + \nabla^2 u_3(x,y) + \nabla^2 u_4(x,y) = 0$$

令 $\begin{cases} \nabla^2 u_1(x,y) = 0, u_1(x,0) = 0, u_1(x,L) = g(x), u_1(0,y) = 0, u_1(K,y) = 0 \\ \nabla^2 u_2(x,y) = 0, u_2(x,0) = 0, u_2(x,L) = 0, u_2(0,y) = 0, u_2(K,y) = g_1(y) \\ \nabla^2 u_3(x,y) = 0, u_3(x,0) = f(x), u_3(x,L) = 0, u_3(0,y) = 0, u_3(K,y) = 0 \\ \nabla^2 u_4(x,y) = 0, u_4(x,0) = 0, u_4(x,L) = 0, u_4(0,y) = f_1(y), u_4(K,y) = 0 \end{cases}$

a. $\nabla^2 u_1(x,y) = 0, u_1(x,0) = 0, u_1(x,L) = g(x), u_1(0,y) = 0, u_1(K,y) = 0$

令 $u_1(x,y) = X(x)Y(y)$ 代入 $\nabla^2 u_1(x,y) = 0$

$$\begin{cases} u_1(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0 \\ u_1(K,y) = X(K)Y(y) = 0 \Rightarrow X(K) = 0 \end{cases}$$

$$X''Y + XY'' = 0$$

同乘 $\dfrac{1}{XY}$，

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$$

$$X'' + \lambda X = 0, \, X(0) = X(K) = 0$$

$$X_n(x) = \sin\left(\frac{n\pi x}{K}\right), \, n = 0, 1, 2, 3, \cdots$$

$$\lambda_n = \left(\frac{n\pi}{K}\right)^2$$

$$Y'' - \left(\frac{n\pi}{K}\right)^2 Y = 0$$

$$Y_n(y) = \alpha_n \cosh\left(\frac{n\pi y}{K}\right) + \beta_n \sinh\left(\frac{n\pi y}{K}\right)$$

$$u_{1n}(x, y) = X_n(x) Y_n(y) = \left(\alpha_n \cosh\left(\frac{n\pi y}{K}\right) + \beta_n \sinh\left(\frac{n\pi y}{K}\right)\right) \sin\left(\frac{n\pi x}{K}\right)$$

$$u_1(x, y) = \sum_{n=1}^{\infty} \left(\alpha_n \cosh\left(\frac{n\pi y}{K}\right) + \beta_n \sinh\left(\frac{n\pi y}{K}\right)\right) \sin\left(\frac{n\pi x}{K}\right)$$

$$\because u_1(x, 0) = 0 \quad \therefore \alpha_n = 0$$

$$\because u_1(x, L) = g(x) \quad \therefore \beta_n = \frac{2\int_0^K g(x) \sin\left(\frac{n\pi x}{K}\right) dx}{K \sinh\left(\frac{n\pi L}{K}\right)}$$

$$u_1(x, y) = \sum_{n=1}^{\infty} \left(\frac{2\int_0^K g(x) \sin\left(\frac{n\pi x}{K}\right) dx}{K \sinh\left(\frac{n\pi L}{K}\right)}\right) \sinh\left(\frac{n\pi y}{K}\right) \sin\left(\frac{n\pi x}{K}\right)$$

b. $\nabla^2 u_2(x, y) = 0, u_2(x, 0) = 0, u_2(x, L) = 0, u_2(0, y) = 0, u_2(K, y) = g_1(y)$

$$u_2(x, y) = \sum_{n=1}^{\infty} \left(\frac{2\int_0^L g_1(y) \sin\left(\frac{n\pi y}{L}\right) dy}{L \sinh\left(\frac{n\pi K}{L}\right)}\right) \sinh\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right)$$

c. $\nabla^2 u_3(x, y) = 0, u_3(x, 0) = f(x), u_3(x, L) = 0, u_3(0, y) = 0, u_3(K, y) = 0$

$$u_3(x, y) = \sum_{n=1}^{\infty} \left(\frac{2\int_0^K f(x) \sin\left(\frac{n\pi x}{K}\right) dx}{K \sinh\left(\frac{n\pi L}{K}\right)}\right) \sinh\left(\frac{n\pi (L - y)}{K}\right) \sin\left(\frac{n\pi x}{K}\right)$$

d. $\nabla^2 u_4(x,y) = 0, u_4(x,0) = 0, u_4(x,L) = 0, u_4(0,y) = f_1(y), u_4(K,y) = 0$

$$u_4(x,y) = \sum_{n=1}^{\infty} \left(\frac{2\int_0^L f_1(y)\sin\left(\dfrac{n\pi y}{L}\right)dy}{L\sinh\left(\dfrac{n\pi K}{L}\right)} \right) \sinh\left(\frac{n\pi(K-x)}{L}\right)\sin\left(\frac{n\pi y}{L}\right)$$

$$\underline{\underline{u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)}}$$

110 年台大工程數學

Problem 1

【Solution】

(a)

$$dy = ye^{3x}dx$$

同乘 $\dfrac{1}{y}$，

$$\frac{1}{y}dy = e^{3x}dx$$

$$\ln|y| = \frac{1}{3}e^{3x} + C_1$$

$$\underline{\underline{y = C_2 e^{\frac{1}{3}e^{3x}}}}$$

(b)

令 $D = \dfrac{d}{dx}$

$$\left(D^2 + 3D + 2.25\right)y = -10e^{-1.5x}$$

1. 求 y_h

令 $\left(D^2 + 3D + 2.25\right)y_h = 0$

$$D^2 + 3D + 2.25 = \left(D + 1.5\right)^2 = 0$$

$$D = -1.5, -1.5 \text{，} \quad y_h = C_1 e^{-1.5x} + C_2 x e^{-1.5x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p = \frac{1}{\left(D + 1.5\right)^2}\left(-10e^{-1.5x}\right)$$

$$y_p = -10e^{-1.5x}\frac{1}{D^2}(1)$$

$$y_p = -10e^{-1.5x}\left(\frac{1}{2}x^2\right) = -5x^2 e^{-1.5x}$$

$$\underline{\underline{y = C_1 e^{-1.5x} + C_2 x e^{-1.5x} - 5x^2 e^{-1.5x}}}$$

(c)

令 $x = e^t, t = \ln(x), x > 0, D = \dfrac{d}{dt}$

$$\left(D(D-1)(D-2) - 3D(D-1) + 6D - 6\right)y = te^{4t}$$

1. 求 y_h

令 $\left(D(D-1)(D-2) - 3D(D-1) + 6D - 6\right)y_h = 0$

$$D(D-1)(D-2) - 3D(D-1) + 6D - 6 = (D-1)(D-2)(D-3) = 0$$

$$D = 1, 2, 3$$

$$y_h(t) = C_1 e^t + C_2 e^{2t} + C_3 e^{3t}$$

$$y_h(x) = C_1 x + C_2 x^2 + C_3 x^3, x > 0$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t) = \frac{1}{(D-1)(D-2)(D-3)}\left(te^{4t}\right)$$

$$y_p(t) = e^{4t}\frac{1}{(D+3)(D+2)(D+1)}(t)$$

$$y_p(t) = e^{4t}\frac{1}{D^3 + 6D^2 + 11D + 6}(t)$$

$$y_p(t) = \frac{1}{6}e^{4t}\frac{1}{1 - \left(-\frac{1}{6}D^3 - D^2 - \frac{11}{6}D\right)}(t)$$

$$y_p(t) = \frac{1}{6}e^{4t}\left(1 + \left(-\frac{1}{6}D^3 - D^2 - \frac{11}{6}D\right) + \left(-\frac{1}{6}D^3 - D^2 - \frac{11}{6}D\right)^2 + \ldots\right)(t)$$

$$y_p(t) = \frac{1}{6}e^{4t}\left(1 - \frac{11}{6}D\right)(t) = \frac{1}{6}e^{4t}\left(t - \frac{11}{6}\right)$$

$$y_p(x) = \frac{1}{6}x^4\left(\ln(x) - \frac{11}{6}\right), x > 0$$

$$\underline{\underline{y(x) = C_1 x + C_2 x^2 + C_3 x^3 + \frac{1}{6}x^4\left(\ln(x) - \frac{11}{6}\right), x > 0}}$$

Problem 2

【Solution】

(a)

$$L\left[e^{-t} + \sinh(t) - t\right] = \underline{\underline{\frac{1}{s+1} + \frac{1}{s^2-1} - \frac{1}{s^2}}}$$

(b)

將原式做拉氏轉換，令 $Y(s) = L\left[y(t)\right]$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s^2}$$

$$\left(s^2 - 1\right)Y(s) - s - 1 = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2\left(s^2-1\right)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}$$

$$Y(s) = -\frac{1}{s^2} + \frac{s}{s^2-1} + \frac{2}{s^2-1}$$

$$y(t) = L^{-1}\left[Y(s)\right] = \underline{\underline{-t + \cosh(t) + 2\sinh(t), t > 0}}$$

Problem 3

【Solution】

By Gauss divergence theorem,

$$\iint\limits_{S} \vec{F} \cdot \hat{n} dA = \iiint\limits_{V} \left(\nabla \cdot \vec{F}\right) dV$$

$$\iint\limits_{S} \vec{F} \cdot \hat{n} dA = \iiint\limits_{V} 3 dV = 3\left(\frac{4}{3}\pi(3)^3\right) = \underline{\underline{108\pi}}$$

Problem 4
【Solution】

$$\left(x^2 - x\right) y'' + \left(3x - 1\right) y' + y = 0$$

$$\left[\left(x^2 - x\right) y'' + \left(2x - 1\right) y'\right] + \left(xy' + y\right) = 0$$

$$\frac{d}{dx}\left[\left(x^2 - x\right) y'\right] + \frac{d}{dx}\left(xy\right) = 0$$

$$\left(x^2 - x\right) y' + xy = C_1$$

同乘 $\dfrac{1}{x^2 - x}$ ，

$$y' + \frac{1}{x - 1} y = \frac{C_1}{x\left(x - 1\right)}$$

$$y = C_1 \frac{\ln|x|}{x - 1} + C_2 \frac{1}{x - 1}$$
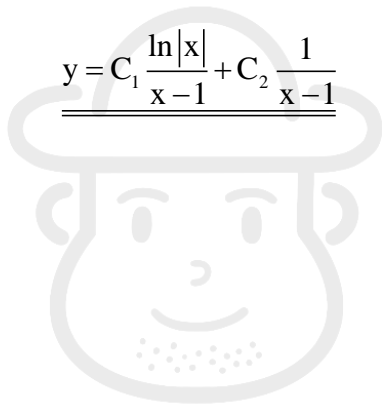
Problem 5

【Solution】

令 $u(x, t) = \sum_{n=0}^{\infty} C_n(x) \cos\left(\frac{n\pi y}{2}\right)$ 代入 P.D.E.

$$\sum_{n=0}^{\infty}\left[C_n''(x) - \left(\frac{n^2\pi^2}{4}\right)C_n(x) \right]\cos\left(\frac{n\pi y}{2}\right) = 0$$

$$C_n''(x) - \left(\frac{n^2\pi^2}{4}\right)C_n(x) = 0$$

$$C_n(x) = \alpha_n \cosh\left(\frac{n\pi x}{2}\right) + \beta_n \sinh\left(\frac{n\pi x}{2}\right)$$

$$u(x, t) = \sum_{n=0}^{\infty}\left[\alpha_n \cosh\left(\frac{n\pi x}{2}\right) + \beta_n \sinh\left(\frac{n\pi x}{2}\right) \right]\cos\left(\frac{n\pi y}{2}\right)$$

代入 B.C.

$$\sum_{n=0}^{\infty} \alpha_n \cos\left(\frac{n\pi y}{2}\right) = 0$$

$$\alpha_n = 0$$

代入 B.C.

$$\sum_{n=0}^{\infty} \beta_n \sinh(n\pi)\cos\left(\frac{n\pi y}{2}\right) = 2 - y$$

$$\beta_n = \frac{1}{\sinh(n\pi)}\int_0^2 (2-y)\cos\left(\frac{n\pi y}{2}\right)dy = \frac{4\left[1-(-1)^n\right]}{n^2\pi^2}$$

$$u(x, t) = \sum_{n=0}^{\infty}\left(\frac{4\left[1-(-1)^n\right]}{n^2\pi^2}\right)\sinh\left(\frac{n\pi x}{2}\right)\cos\left(\frac{n\pi y}{2}\right)$$

Problem 6

【Solution】

令 $u(x,t) = w(x,t) + \phi(x,t)$

$$\begin{cases} u(0,t) = w(0,t) + \phi(0,t) = t \\ u(1,t) = w(1,t) + \phi(1,t) = 3 \end{cases}$$

$$\begin{cases} w(0,t) = w(1,t) = 0 \\ \phi(0,t) = t, \ \phi(1,t) = 3 \end{cases}$$

令 $\phi(x,t) = \alpha(t)x + \beta(t)$ 代入上式

$$\phi(x,t) = (3-t)x + t$$

將 $u(x,t) = w(x,t) + (3-t)x + t$ 代入 P.D.E.

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = x - 1$$

令 $x - 1 = \sum_{n=1}^{\infty} d_n \sin(n\pi x)$

$$d_n = \frac{2}{1} \int_0^1 (x-1)\sin(n\pi x)dx = -\frac{2}{n\pi}$$

$$x - 1 = \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi} \right) \sin(n\pi x)$$

令 $w(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(n\pi x)$ 代入 $\dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} = \sum_{n=1}^{\infty} \left(-\dfrac{2}{n\pi} \right)\sin(n\pi x)$

$$\sum_{n=1}^{\infty} \left[C_n'(t) + n^2\pi^2 C_n(t) \right]\sin(n\pi x) = \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi} \right)\sin(n\pi x)$$

$$C_n'(t) + n^2\pi^2 C_n(t) = \left(-\frac{2}{n\pi} \right)$$

$$C_n(t) = c_n e^{-n^2\pi^2 t} - \frac{2}{n^3\pi^3}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(c_n e^{-n^2\pi^2 t} - \frac{2}{n^3\pi^3} \right)\sin(n\pi x) + (3-t)x + t$$

代入 I.C.

$$\sum_{n=1}^{\infty} \left(c_n - \frac{2}{n^3\pi^3} \right)\sin(n\pi x) + 3x = 0$$

$$c_n = \frac{2}{n^3\pi^3} + \frac{2}{1}\int_0^1 (-3x)\sin(n\pi x)dx = \frac{2}{n^3\pi^3} + \frac{6(-1)^n}{n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{n^3\pi^3} + \frac{6(-1)^n}{n\pi}\right)e^{-n^2\pi^2 t} - \frac{2}{n^3\pi^3}\right]\sin(n\pi x) + (3-t)x + t$$

Problem 7

【Solution】

令 $f(x) = \int_0^\infty B(w) \sin(wx) dw$

$$B(w) = \frac{2}{\pi} \int_0^\pi \sin(x) \sin(wx) dx = \frac{2\sin(\pi w)}{\pi(1-w^2)}$$

$$f(x) = \int_0^\infty \left(\frac{2\sin(\pi w)}{\pi(1-w^2)} \right) \sin wx dx$$

Problem 8

【Solution】

$$F^{-1}\left[\frac{6e^{iw}\cos(4w)}{9+w^2} \right] = F^{-1}\left[\frac{6e^{iw}}{9+w^2} \left(\frac{e^{i4w} + e^{-i4w}}{2} \right) \right]$$

$$= \frac{1}{2} F^{-1}\left[\frac{6e^{iw}}{9+w^2} \left(e^{i4w} + e^{-i4w} \right) \right] = \frac{1}{2} \left(e^{-3|x+5|} + e^{-3|x-3|} \right)$$

111 年台大工程數學

Problem 1

【Solution】

(a)

令 $y = x^m$ 代入原式

$$\left(m(m-1) + 2m + 1 \right) x^m = 0$$

令 $x^m \neq 0$

$$m(m-1) + 2m + 1 = m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{3}i}{2}$$

$$y = x^{-\frac{1}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}}{2} \ln x \right) + c_2 \sin\left(\frac{\sqrt{3}}{2} \ln x \right) \right), \ x > 0$$

(b)

令 $D = \dfrac{d}{dx}$

$$\left(D^2 - 1 \right) y = e^x$$

1. 求 y_h

令 $\left(D^2 - 1 \right) y_h = 0$

$$D^2 - 1 = 0 \ , \ D = -1, 1$$

$$y_h = C_1 e^{-x} + C_2 e^x$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p = \frac{1}{D^2 - 1} \left(e^x \right) = \frac{1}{(D+1)(D-1)} \left(e^x \right)$$

$$y_p = \frac{1}{2} \frac{1}{(D-1)} \left(e^x \right) = \frac{1}{2} e^x \frac{1}{D} (1) = \frac{1}{2} x e^x$$

$$y = C_1 e^{-x} + C_2 e^x + \frac{1}{2} x e^x$$

Problem 2
【Solution】

$$y(t) + y(t) * 1 = 1$$

將上式做拉氏轉換，令 $Y(s) = L\big[y(t)\big]$

$$Y(s) + \frac{1}{s} Y(s) = \frac{1}{s}$$

$$\left(1 + \frac{1}{s}\right) Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s+1}$$

$$y(t) = L^{-1}\big[Y(s)\big] = \underline{\underline{e^{-t}, t > 0}}$$

Problem 3
【Solution】

(a)

此為 Bessel equation

$$x^2 y'' + xy' + \left((5x)^2 - 1^2\right) y = 0$$

$$y = C_1 J_1(5x) + C_2 Y_1(5x)$$

(b)

y(x) is a finite physical quantity, and

$$\lim_{x \to 0} Y_1(5x) = -\infty$$

$$C_2 = 0$$

代入 $y(1) = 2$

$$y(1) = C_1 J_1(5) = 2$$

$$C_1 = \frac{2}{J_1(5)}$$

$$\underline{\underline{y = \frac{2}{J_1(5)} J_1(5x)}}$$

Problem 4

【Solution】

(a)

$$F\left[4H(t-2)e^{-3t}\cos(t-2)\right] = e^{-i2w}F\left[4H(t)e^{-3(t+2)}\cos t\right]$$

$$= 2e^{-6}e^{-i2w}F\left[H(t)e^{-3t}\left(e^{it}+e^{-it}\right)\right]$$

$$= \underline{\underline{2e^{-i2w-6}\left(\frac{1}{3+i(w-1)}+\frac{1}{3+i(w+1)}\right)}}$$

(b)

$$F^{-1}\left[\frac{e^{(20-4w)i}}{3-(5-w)i}\right] = F^{-1}\left[\frac{e^{-i4(w-5)}}{3+i(w-5)}\right] = e^{i5t}F^{-1}\left[\frac{e^{-4iw}}{3+iw}\right]$$

$$= e^{i5t}F^{-1}\left[\frac{e^{-4iw}}{3+iw}\right] = \underline{\underline{e^{i5t}H(t-4)e^{-3(t-4)}}}$$

Problem 5

【Solution】

令 $u(x,t) = w(x,t) + \phi(x)$ ，其中 $\phi(x) = \lim\limits_{t \to \infty} u(x,t)$ 為穩態解

$$\begin{cases} u(0,t) = w(0,t) + \phi(0) = T_1 \\ u(9,t) = w(9,t) + \phi(9) = T_2 \end{cases}$$

$$\begin{cases} w(0,t) = w(9,t) = 0 \\ \phi(x) = \left(\dfrac{T_2 - T_1}{9}\right)x + T_1 \end{cases}$$

將 $u(x,t) = w(x,t) + \phi(x)$ 代入 P.D.E.

$$\frac{\partial w}{\partial t} = \alpha^2 \frac{\partial^2 w}{\partial x^2}$$

令 $w(x,t) = \sum\limits_{n=1}^{\infty} C_n(t) \sin\left(\dfrac{n\pi x}{9}\right)$ 代入上式

$$\sum_{n=1}^{\infty} \left(C_n{}'(t) + \alpha^2 C_n(t)\right) \sin\left(\frac{n\pi x}{9}\right) = 0$$

$$C_n{}'(t) + \alpha^2 C_n(t) = 0$$

$$C_n(t) = c_n e^{-\alpha^2 t}$$

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 t} \sin\left(\frac{n\pi x}{9}\right)$$

$$\underline{\underline{u(x,t) = \left(\frac{T_2 - T_1}{9}\right)x + T_1 + \sum_{n=1}^{\infty} c_n e^{-\alpha^2 t} \sin\left(\frac{n\pi x}{9}\right)}}$$

代入 I.C.

$$u(x,0) = \left(\frac{T_2 - T_1}{9}\right)x + T_1 + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{9}\right) = x^2$$

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{9}\right) = x^2 - \left(\frac{T_2 - T_1}{9}\right)x - T_1$$

$$c_n = \frac{2}{9} \int_0^9 \left(x^2 - \left(\frac{T_2 - T_1}{9}\right)x - T_1\right) \sin\left(\frac{n\pi x}{9}\right) dx$$

$$\underline{\underline{c_n = \frac{2\left((T_2 - 81)(-1)^n + T_1\right)}{n\pi} + \frac{324}{n^3 \pi^3}\left((-1)^n - 1\right)}}$$

Problem 6
【Solution】

令 $\underline{u(x,y,z) = u_1(x,y,z) + u_2(x,y,z)}$

$$\begin{cases} u_1(x,y,z): \begin{cases} u_1(0,y,z) = u_1(1,y,z) = u_1(x,y,0) = u_1(x,y,2) = 0 \\ u_1(x,0,z) = 0 \\ u_1(x,4,z) = g(z) \end{cases} \\ u_2(x,y,z): \begin{cases} u_2(0,y,z) = u_2(1,y,z) = u_2(x,y,0) = u_2(x,y,2) = 0 \\ u_2(x,0,z) = f(x) \\ u_2(x,4,z) = 0 \end{cases} \end{cases}$$

令 $u_1(x,y,z) = \sum\limits_{n=1}^{\infty}\sum\limits_{m=1}^{\infty} c_{nm}(y)\sin\left(\dfrac{m\pi z}{2}\right)\sin(n\pi x)$ 代入 P.D.E.

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(c_{nm}''(y) - \left(\left(\frac{m\pi}{2}\right)^2 + (n\pi)^2\right)c_{nm}(y)\right)\sin\left(\frac{m\pi z}{2}\right)\sin(n\pi x) = 0$$

$$c_{nm}''(y) - \left(\left(\frac{m\pi}{2}\right)^2 + (n\pi)^2\right)c_{nm}(y) = 0$$

令 $\underline{\underline{\lambda = \sqrt{\left(\dfrac{m\pi}{2}\right)^2 + (n\pi)^2}}}$

$$c_{nm}(y) = \alpha_{nm}\cosh(\lambda y) + \beta_{nm}\sinh(\lambda y)$$

代入 $u_1(x,0,z) = 0$

$$\alpha_{nm} = 0$$

$$u_1(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta_{nm}\sinh(\lambda y)\sin\left(\frac{m\pi z}{2}\right)\sin(n\pi x)$$

代入 $u_1(x,4,z) = g(z)$

$$u_1(x,4,z) = g(z) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\beta_{nm}\sinh(4\lambda)\sin\left(\frac{m\pi z}{2}\right)\right)\sin(n\pi x)$$

$$\sum_{m=1}^{\infty}\beta_{nm}\sinh(4\lambda)\sin\left(\frac{m\pi z}{2}\right) = 2\int_0^1 g(z)\sin(n\pi x)\,dx$$

$$\beta_{nm} = \frac{2}{\sinh(4\lambda)} \int_0^2 \left(\int_0^1 g(z)\sin(n\pi x)dx \right) \sin\left(\frac{m\pi z}{2} \right) dz$$

$$u_1(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \beta_{nm} \sinh(\lambda y) \sin\left(\frac{m\pi z}{2} \right) \sin(n\pi x)$$

$$u_2(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \gamma_{nm} \sinh(\lambda(4-y)) \sin\left(\frac{m\pi z}{2} \right) \sin(n\pi x)$$

代入 $u_2(x, 0, z) = f(x)$

$$u_2(x, 0, z) = f(x) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \gamma_{nm} \sinh(4\lambda) \sin\left(\frac{m\pi z}{2} \right) \right) \sin(n\pi x)$$

$$\sum_{m=1}^{\infty} \gamma_{nm} \sinh(4\lambda) \sin\left(\frac{m\pi z}{2} \right) = 2\int_0^1 f(x) \sin(n\pi x)dx$$

$$\gamma_{nm} = \frac{2}{\sinh(4\lambda)} \int_0^2 \left(\int_0^1 f(x)\sin(n\pi x)dx \right) \sin\left(\frac{m\pi z}{2} \right) dz$$

112 年台大工程數學

Problem 1

【Solution】

令 $y_2 = u(x)y_1(x)$

$$y_2' = uy_1' + u'y_1$$

$$y_2'' = uy_1'' + 2u'y_1' + u''y_1$$

代入原式

$$u\underbrace{\left(y_1'' + py_1' + qy_1\right)}_{0 \text{ (homogeneous)}} + u''y_1 + u'\left(2y_1' + py_1\right) = 0$$

$$u''y_1 + u'\left(2y_1' + py_1\right) = 0$$

$$\frac{du'}{u'} + 2\frac{y_1'}{y_1}dx + pdx = 0$$

$$\int\frac{du'}{u'} + 2\int\frac{y_1'}{y_1}dx + \int pdx = 0$$

$$\ln\left|u'y_1^2\right| = -\int pdx + C_1$$

取 $\ln\left|u'y_1^2\right| = -\int pdx$

$$u'y_1^2 = \pm e^{-\int pdx}$$

取 $u'y_1^2 = e^{-\int pdx}$

$$u = \int\frac{e^{-\int pdx}}{y_1^2}dx + C_2$$

取 $u = \int\frac{e^{-\int pdx}}{y_1^2}dx$ ， $y_2 = uy_1 = \underline{\underline{y_1\int\frac{e^{-\int pdx}}{y_1^2}dx}}$

Problem 2

【Solution】

$$y = -\left(y'\right)^2 + xy'$$

令 $p(x) = y'$ 代入上式

$$y = -p^2 + xp$$

$$\frac{dy}{dx} = -2p\frac{dp}{dx} + p + x\frac{dp}{dx}$$

$$\cancel{p} = \left(-2p + x\right)\frac{dp}{dx} + \cancel{p}$$

$$\left(-2p + x\right)\frac{dp}{dx} = 0$$

同乘 $\dfrac{1}{-2p + x}$

$$\frac{dp}{dx} = 0$$
$$p = C$$

代入 $y = -p^2 + xp$

$$\underline{\underline{y = -C^2 + Cx}}$$

其中 $-2p + x = 0$ 為奇解

代入 $y = -p^2 + xp$

$$y = -\frac{x^2}{4} + \frac{x^2}{2}$$

$$\underline{\underline{y = \frac{x^2}{4}}} \ \text{為奇解}$$

Problem 3
【Solution】

(a)

$$\vec{\nabla}f(x, y, z) = 2x\vec{i} + 2y\vec{j} + 4z\hat{k}$$

$$\vec{\nabla}f(1,1,1) = 2\vec{i} + 2\vec{j} + 4\hat{k}$$

$$D_{\hat{e}_c} f(1,1,1) = \vec{\nabla}f(1,1,1) \cdot \frac{\vec{c}}{|\vec{c}|}$$

$$D_{\hat{e}_c} f(1,1,1) = \left(2\hat{i} + 2\hat{j} + 4\hat{k}\right) \cdot \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} = \underline{\underline{\frac{8}{\sqrt{3}}}}$$

(b)
此題應有誤。
該點不在曲面上，無法求解

Problem 4

【Solution】

令 $P_A(\lambda) = \det(A - \lambda I) = -(\lambda + 3)(\lambda^2 + 4\lambda - 27) = 0$

$$\lambda = -3, -2 \pm \sqrt{31}$$

1. $\lambda = -3$ 代入 $(A + 3I)\vec{X} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{X}_1 = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, c_1 \neq 0$$

2. $\lambda = -2 + \sqrt{31}$ 代入 $(A + 3I)\vec{X} = \vec{0}$

$$\begin{bmatrix} -\sqrt{31} & 2 & -3 \\ 2 & 3 - \sqrt{31} & -6 \\ -1 & -2 & -4 - \sqrt{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{X}_2 = c_2 \begin{bmatrix} 7 + \sqrt{31} \\ 14 + 2\sqrt{31} \\ -1 - \sqrt{31} \end{bmatrix}, c_2 \neq 0$$

3. $\lambda = -2 - \sqrt{31}$ 代入 $(A + 3I)\vec{X} = \vec{0}$

$$\begin{bmatrix} \sqrt{31} & 2 & -3 \\ 2 & 3 + \sqrt{31} & -6 \\ -1 & -2 & -4 + \sqrt{31} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{X}_3 = c_3 \begin{bmatrix} 7 - \sqrt{31} \\ 14 - 2\sqrt{31} \\ -1 + \sqrt{31} \end{bmatrix}, c_3 \neq 0$$

$$D = P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2+\sqrt{31} & 0 \\ 0 & 0 & -2-\sqrt{31} \end{bmatrix}$$

Problem 5
【Solution】

將原式做拉氏轉換，令 $Y(s) = L[y(t)]$

$$2\left[s^2Y(s) - sy(0) - y'(0)\right] - \frac{d}{ds}\left[sY(s) - y(0)\right] - Y(s) = \frac{6}{s}$$

$$\left(2s^2 - 2\right)Y(s) - s\frac{dY(s)}{ds} = \frac{6}{s}$$

$$\frac{dY(s)}{ds} + \left(-2s + \frac{2}{s}\right)Y(s) = \frac{-6}{s^2}$$

同乘 $s^2e^{-s^2}$，

$$\frac{d}{ds}\left(s^2e^{-s^2}Y(s)\right) = -6e^{-s^2}$$

$$s^2e^{-s^2}Y(s) = -6\int_0^s e^{-\tau^2}d\tau + C_1$$

$$Y(s) = C_1\frac{e^{s^2}}{s^2} - 6\frac{e^{s^2}}{s^2}\int_0^s e^{-\tau^2}d\tau$$

$$Y(s) = \frac{e^{s^2}}{s^2}\left(C_1 - 6\int_0^s e^{-\tau^2}d\tau\right)$$

利用初值定理

$$\lim_{s\to\infty} sY(s) = \lim_{t\to 0} y(t) = y(0) = 0$$

$$\lim_{s\to\infty} \underbrace{\frac{e^{s^2}}{s}}_{\to\infty} \underbrace{\left(C_1 - 6\int_0^s e^{-\tau^2}d\tau\right)}_{\text{必須逼近0才可能收斂}} = 0$$

$$C_1 - 6\underbrace{\lim_{s\to\infty}\left(\int_0^s e^{-\tau^2}d\tau\right)}_{\frac{\sqrt{\pi}}{2}} = 0$$

$$C_1 = 3\sqrt{\pi}$$

$$Y(s) = \frac{e^{s^2}}{s^2}\left(3\sqrt{\pi} - 6\int_0^s e^{-\tau^2}d\tau\right)$$

$$Y(s) = 3\sqrt{\pi}\frac{e^{s^2}}{s^2}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^s e^{-\tau^2}d\tau\right)$$

$$Y(s) = \frac{3\sqrt{\pi}e^{s^2}}{s^2}\,\text{erfc}(s)$$

$$y(t) = L^{-1}\left[\frac{3\sqrt{\pi}e^{s^2}}{s^2}\,\text{erfc}(s)\right] = 3\sqrt{\pi}L^{-1}\left[\frac{1}{s}\frac{e^{s^2}}{s}\,\text{erfc}(s)\right]$$

$$y(t) = 3\sqrt{\pi}\int_0^t L^{-1}\left[\frac{e^{s^2}}{s}\,\text{erfc}(s)\right]\Bigg|_{t\to\tau}d\tau = 3\sqrt{\pi}\int_0^t \text{erf}\left(\frac{\tau}{2}\right)d\tau$$

$$y(t) = 3\sqrt{\pi}\left[\tau\,\text{erf}\left(\frac{\tau}{2}\right)\Bigg|_0^t - \frac{1}{\sqrt{\pi}}\int_0^t \tau e^{-\frac{\tau^2}{4}}d\tau\right]$$

$$y(t) = 3\sqrt{\pi}\left[t\,\text{erf}\left(\frac{t}{2}\right) - \frac{1}{\sqrt{\pi}}\left(-2e^{-\frac{\tau^2}{4}}\right)_0^t\right]$$

$$y(t) = 3\sqrt{\pi}\left[t\,\text{erf}\left(\frac{t}{2}\right) + \frac{2}{\sqrt{\pi}}\left(e^{-\frac{t^2}{4}} - 1\right)\right]$$

$$y(t) = 3\sqrt{\pi}\left[t\,\text{erf}\left(\frac{t}{2}\right) + \frac{2}{\sqrt{\pi}}\left(e^{-\frac{t^2}{4}} - 1\right)\right], t > 0$$

<補充> 誤差函數與補誤差函數之拉氏轉換

(1) $L\left[\text{erf}(t)\right] = \frac{1}{s}e^{\left(\frac{s}{2}\right)^2}\text{erfc}\left(\frac{s}{2}\right)$

(2) $L\left[\text{erf}\left(\frac{t}{2}\right)\right] = \frac{1}{s}e^{s^2}\text{erfc}(s)$

Problem 6
【Solution】

令 $u(r,z) = R(r)\Phi(z)$ 代入P.D.E

$$R''\Phi + \frac{1}{r}R'\Phi + \Phi'' = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{\Phi''}{\Phi} = \lambda$$

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ r^2R'' + rR' - \lambda r^2 R = 0 \end{cases}$$

$$\begin{cases} u(r,1) = R(r)\Phi(1) = 0 \\ u_z(r,0) = R(r)\Phi'(0) = 0 \end{cases}$$

$$\begin{cases} \Phi(1) = 0 \\ \Phi'(0) = 0 \end{cases}$$

1. 令 $\lambda = 0$ 代入 $\Phi'' + \lambda\Phi = 0$

$$\Phi(z) = C_1 + C_2 z$$

代入B.C.

$$C_1 = C_2 = 0$$

2. 令 $\lambda = -k^2, k > 0$ 代入 $\Phi'' + \lambda\Phi = 0$

$$\Phi(z) = C_1\cosh(kz) + C_2\sinh(kz)$$

代入B.C.

$$C_1 = C_2 = 0$$

3. 令 $\lambda = k^2, k > 0$ 代入 $\Phi'' + \lambda\Phi = 0$

$$\Phi(z) = C_1\cos(kz) + C_2\sin(kz)$$

$$\Phi'(z) = -kC_1\sin(kz) + kC_2\cos(kz)$$

代入B.C.

$$C_2 = 0$$

令 $C_1 \neq 0$

$$\Phi(1) = C_1 \cos(k) = 0$$

$$k = \frac{(2n-1)\pi}{2}, n = 1, 2, 3, \ldots$$

$$\lambda = \left(\frac{(2n-1)\pi}{2}\right)^2, n = 1, 2, 3, \ldots$$

$$\Phi_n(z) = \cos\left(\frac{(2n-1)\pi}{2}z\right), n = 1, 2, 3, \ldots$$

$$r^2 R_n''(r) + r R_n'(r) - \left(\frac{(2n-1)^2 \pi^2}{4}\right) r^2 R_n(r) = 0$$

$$R_n(r) = \alpha_n I_0\left(\frac{(2n-1)\pi}{2}r\right) + \underbrace{\beta_n K_0\left(\frac{(2n-1)\pi}{2}r\right)}_{0 \; (R_n(r) \text{ is finite})}$$

$$u_n(r,z) = R_n(r)\Phi_n(z) = \alpha_n I_0\left(\frac{(2n-1)\pi}{2}r\right)\cos\left(\frac{(2n-1)\pi}{2}z\right)$$

$$u(r,z) = \sum_{n=1}^{\infty} \alpha_n I_0\left(\frac{(2n-1)\pi}{2}r\right)\cos\left(\frac{(2n-1)\pi}{2}z\right)$$

代入 B.C.

$$u(1,z) = \sum_{n=1}^{\infty} \alpha I_0\left(\frac{(2n-1)\pi}{2}\right)\cos\left(\frac{(2n-1)\pi z}{2}\right) = kz$$

$$\therefore \alpha_n I_0\left(\frac{(2n-1)\pi}{2}\right) = 2\int_0^1 (kz)\cos\left(\frac{(2n-1)\pi}{2}\right)z\,dz$$

$$\alpha_n = \frac{-4k\left(2 + \pi(2n-1)(-1)^n\right)}{(2n-1)^2 \pi^2 I_0\left(\frac{(2n-1)\pi}{2}\right)}$$

$$u(r, z) = \sum_{n=1}^{\infty} \left(\frac{-4k \left(2 + \pi (2n-1)(-1)^n \right)}{(2n-1)^2 \pi^2 I_0 \left(\frac{(2n-1)\pi}{2} \right)} \right) I_0 \left(\frac{(2n-1)\pi}{2} r \right) \cos \left(\frac{(2n-1)\pi}{2} z \right)$$

113 年台大工程數學

Problem 1
【Solution】

$$2y^3 dx - 4dx + 3xy^2 dy = 0$$

$$y^2(2ydx + 3xdy) - 4dx = 0$$

$$y^2 \frac{d(x^2y^3)}{xy^2} - 4dx = 0$$

$$d(x^2y^3) - 4xdx = 0$$

$$\underline{\underline{x^2y^3 - 2x^2 = C}}$$

代入 I.C.

$$C = -9$$

$$\underline{\underline{x^2y^3 - 2x^2 = -9}}$$

Problem 2
【Solution】

令 $D = \dfrac{d}{dx}$

$$\left(D^4 + 2D^2 + 1\right)y = -\sin(x) + \cos(2x)$$

1. 求 y_h

令 $\left(D^4 + 2D^2 + 1\right)y_h = 0$

$$D^4 + 2D^2 + 1 = 0$$

$$\left(D^2 + 1\right)^2 = 0$$

$$D = i, i, -i, -i$$

$$y_h = C_1 \cos(x) + C_2 \sin(x) + C_3 x \cos(x) + C_4 x \sin(x)$$

2. 求 y_p，利用 Heaviside 反運算子法

$$y_p = \frac{1}{(D^4 + 2D^2 + 1)}\left(-\sin(x)\right) + \frac{1}{(D^4 + 2D^2 + 1)}\left(\cos(2x)\right)$$

$$y_p = \frac{-1}{\left(D^2 + 1\right)^2}\left(\sin(x)\right) + \frac{1}{(D^2)^2 - 7}\left(\cos(2x)\right)$$

$$\left(※ \frac{1}{(D^2 + a^2)}\sin(ax+b) = \frac{-x\cos(ax+b)}{2a} \ , \ \frac{1}{(D^2 + a^2)}\cos(ax+b) = \frac{x\sin(ax+b)}{2a}\right)$$

$$y_p = \frac{1}{8}x^2\sin\left(x\right) + \frac{1}{9}\left(\cos(2x)\right)$$

$$y = C_1\cos(x) + C_2\sin(x) + C_3 x\cos(x) + C_4 x\sin(x) + \frac{1}{8}x^2\sin\left(x\right) + \frac{1}{9}\left(\cos(2x)\right)$$

Problem 3
【Solution】

(a)

$$L\left[t^2\cos\left(at\right)\right] = (-1)^2\frac{d^2}{ds^2}\left(L\left[\cos\left(at\right)\right]\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2 + a^2}\right) = \frac{d}{ds}\left[\frac{a^2 - s^2}{\left(s^2 + a^2\right)^2}\right] = \frac{2s^3 - 6a^2 s}{\left(s^2 + a^2\right)^3}$$

(b)

$$L^{-1}\left[\frac{s}{(s+2)^2\left(s^2 + 2s + 10\right)}\right] = L^{-1}\left[\frac{\dfrac{3}{50}}{s+2} - \frac{\dfrac{1}{5}}{(s+2)^2} - \frac{\dfrac{3}{50}s - \dfrac{1}{5}}{s^2 + 2s + 10}\right]$$

$$= \frac{3}{50}e^{-2t} - \frac{1}{5}te^{-2t} - \frac{3}{50}e^{-t}\cos\left(3t\right) + \frac{13}{150}e^{-t}\sin\left(3t\right), t > 0$$

Problem 4

【Solution】

令 $x = e^t, t = \ln(x), x > 0, D = \dfrac{d}{dt}$

$$\left(D(D-1) - D + 1\right)y = 4te^t$$

1. 求 y_h

$$D(D-1) - D + 1 = (D-1)^2 = 0$$

$$D = 1, 1$$

$$y_h(t) = C_1 e^t + C_2 t e^t$$

$$y_h(x) = C_1 x + C_2 x \ln(x), x > 0$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t) = \frac{1}{(D-1)^2}\left(4te^t\right) = 4e^t \frac{1}{D^2}(t) = \frac{2}{3} t^3 e^t$$

$$y_p(x) = \frac{2}{3} x \ln^3(x), x > 0$$

$$y = C_1 x + C_2 x \ln(x) + \frac{2}{3} x \ln^3(x), x > 0$$

Problem 5
【Solution】

令 $P_A(\lambda) = \det(A - \lambda I) = -(\lambda - 3)(\lambda^2 + 2\lambda + 2) = 0$

$$\lambda = 3, -1+i, -1-i$$

1. $\lambda = 3$ 代入 $(A - \lambda I)\vec{y} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_1 \begin{bmatrix} \dfrac{4}{9} \\ 1 \\ 0 \end{bmatrix}, c_1 \neq 0$$

2. $\lambda = -1+i$ 代入 $(A - \lambda I)\vec{y} = \vec{0}$

$$\begin{bmatrix} 4-i & 0 & 1 \\ 9 & -i & 2 \\ -9 & 4 & -i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_2 \begin{bmatrix} \dfrac{-4-i}{17} \\ \dfrac{-9+2i}{17} \\ 1 \end{bmatrix}, c_2 \neq 0$$

3. $\lambda = -1-i$ 代入 $(A - \lambda I)\vec{y} = \vec{0}$

$$\begin{bmatrix} 4+i & 0 & 1 \\ 9 & i & 2 \\ -9 & 4 & i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_3 \begin{bmatrix} \dfrac{-4+i}{17} \\ \dfrac{-9-2i}{17} \\ 1 \end{bmatrix}, c_3 \neq 0$$

$$\vec{X}(t) = C_1 e^{3t} \begin{bmatrix} \dfrac{4}{9} \\ 1 \\ 0 \end{bmatrix} + C_2 e^{-t} \left(\begin{bmatrix} \dfrac{-4}{17} \\ \dfrac{-9}{17} \\ 1 \end{bmatrix} \cos(t) - \begin{bmatrix} \dfrac{-1}{17} \\ \dfrac{2}{17} \\ 0 \end{bmatrix} \sin(t) \right) + C_3 e^{-t} \left(\begin{bmatrix} \dfrac{-4}{17} \\ \dfrac{-9}{17} \\ 1 \end{bmatrix} \sin(t) + \begin{bmatrix} \dfrac{-1}{17} \\ \dfrac{2}{17} \\ 0 \end{bmatrix} \cos(t) \right)$$

Problem 6
【Solution】
(a)

$$\arg(z) = \theta = \pi - \tan^{-1}\left(\frac{2}{3}\right)$$

(b)

$$|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$z = \sqrt{13} e^{\left(\pi - \tan^{-1}\left(\frac{2}{3}\right)\right)i}$$

Problem 7
【Solution】

令 $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} 2x^2 dx = \frac{8}{3}\pi^2$$

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} 2x^2 \cos(nx) dx = \frac{8}{n^2}$$

$$b_n = \frac{2}{2\pi} \int_0^{2\pi} 2x^2 \sin(nx) dx = -\frac{8\pi}{n}$$

$$f(x) = \frac{8}{3}\pi^2 + \sum_{n=1}^{\infty} \left(\frac{8}{n^2} \cos(nx) - \frac{8\pi}{n} \sin(nx) \right)$$

Problem 8
【Solution】

∵ adiabatic surface at $x = 0$ ∴ $u_x(0, t) = 0$

令 $u(x, t) = X(x)T(t)$

$$\begin{cases} u_x(0, t) = X'(0)T(t) = 0 \\ u(L, t) = X(L)T(t) = 0 \end{cases}$$

$$X'(0) = X(L) = 0$$

代入 P.D.E.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

同乘 $\frac{1}{c^2 u}$,

$$\frac{X''}{X} = \frac{T'}{c^2 T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + c^2 \lambda T = 0 \end{cases}$$

1. 令 $\lambda = 0$ 代入 $X'' + \lambda X = 0$

$$X(x) = C_1 + C_2 x$$

代入 B.C.

$$C_1 = C_2 = 0$$

2. 令 $\lambda = -k^2, k > 0$ 代入 $X'' + \lambda X = 0$

$$X(x) = C_1 \cosh(kx) + C_2 \sinh(kx)$$

代入 B.C.

$$C_1 = C_2 = 0$$

3. 令 $\lambda = k^2, k > 0$ 代入 $X'' + \lambda X = 0$

$$X(x) = C_1 \cos(kx) + C_2 \sin(kx)$$

代入 B.C.

$$C_2 = 0$$

令 $C_1 \neq 0$

$$k = \frac{(2n-1)\pi}{2\ell}, n = 1, 2, 3, \ldots$$

$$\lambda = \left(\frac{(2n-1)\pi}{2L} \right)^2$$

$$X_n(x) = \cos\left(\frac{(2n-1)\pi x}{2\ell} \right), n = 1, 2, 3, \ldots$$

$$T' + \left(\frac{(2n-1)\pi c}{2\ell} \right)^2 T = 0$$

$$T_n(t) = c_n e^{-\left(\frac{(2n-1)\pi c}{2L} \right)^2 t}$$

$$u_n(x,t) = X_n(x)T_n(t) = c_n e^{-\left(\frac{(2n-1)\pi c}{2L} \right)^2 t} \cos\left(\frac{(2n-1)\pi x}{2\ell} \right)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{(2n-1)\pi c}{2L} \right)^2 t} \cos\left(\frac{(2n-1)\pi x}{2\ell} \right)$$

代入 I.C.

$$c_n = 0$$

$$\underline{\underline{u(x,t) = 0}}$$

<補充>

此題一端絕熱且另一端恆為 0 且初始溫度是 0，P.D.E.為齊性，溫度始終不會變化，可知此題的溫度始終保持為 0。

114 年台大工程數學

Problem 1
【Solution】
(a)

令 $x = e^t$, $t = \ln(x)$, $x > 0$ 代入 O.D.E.

$$\left(D(D-1)(D-2) + D(D-1) - 2D + 2\right)y = 3te^{3t}$$

1. 求 y_h

$$D(D-1)(D-2) + D(D-1) - 2D + 2 = 0$$

$$(D-2)(D-1)(D+1) = 0$$

$$D = -1, 1, 2$$

$$y_h(x) = C_1 x^{-1} + C_2 x + C_3 x^2$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(t) = \frac{1}{(D-2)(D-1)(D+1)}\left(3te^{3t}\right)$$

$$y_p(t) = 3e^{3t}\frac{1}{(D+1)(D+2)(D+4)}(t)$$

$$y_p(t) = \frac{3}{8}e^{3t}\frac{1}{1-\left(-\dfrac{D^3}{8} - \dfrac{7D^2}{8} - \dfrac{7}{4}D\right)}(t)$$

$$y_p(t) = \frac{3}{8}te^{3t} - \frac{21}{32}e^{3t}$$

$$y_p(x) = \frac{3}{8}x^3\ln(x) - \frac{21}{32}x^3, \; x > 0$$

$$y = C_1 x^{-1} + C_2 x + C_3 x^2 + \frac{3}{8}x^3\ln(x) - \frac{21}{32}x^3, \; x > 0$$

(b)

令 $D = \dfrac{d}{dx}$ 代入 O.D.E.

$$\left(D^2 - 5D + 6\right) y = 9\cos(3x)$$

1. 求 y_h

$$D^2 - 5D + 6 = \left(D - 2\right)\left(D - 3\right) = 0$$

$$D = 2, 3$$

$$y_h(x) = C_1 e^{2x} + C_2 e^{3x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(x) = \frac{1}{D^2 - 5D + 6}\left(9\cos(3x)\right)$$

$$y_p(x) = \frac{-9}{5}\frac{1}{D+1}\cos(3x)$$

$$y_p(x) = \frac{9}{50}\left(-3\sin(3x) - \cos(3x)\right)$$

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{9}{50}\left(-3\sin(3x) - \cos(3x)\right)$$

(c)

令 $D = \dfrac{d}{dx}$ 代入 O.D.E.

$$\left(4D^2 - 12D + 9\right)y = 2e^{1.5x}$$

1. 求 y_h

$$4D^2 - 12D + 9 = \left(2D - 3\right)^2 = 0$$

$$D = 1.5, 1.5$$

$$y_h(x) = C_1 e^{1.5x} + C_2 x e^{1.5x}$$

2. 求 y_p，利用 Heaviside 反運算值法

$$y_p(x) = \frac{1}{\left(2D - 3\right)^2}\left(2e^{1.5x}\right)$$

$$y_p(x) = 2e^{1.5x} \frac{1}{4D^2}(1) = 0.25x^2 e^{1.5x}$$

$$y = C_1 e^{1.5x} + C_2 x e^{1.5x} + 0.25x^2 e^{1.5x}$$

代入 I.C.

$$C_1 = 1, C_2 = -0.5$$

$$\underline{\underline{y = e^{1.5x} - 0.5C_2 x e^{1.5x} + 0.25x^2 e^{1.5x}}}$$

Problem 2

【Solution】

(a)

$$L\left[e^{-t}\left(8\cosh\left(2t\right)-3\sinh\left(4t\right)\right)\right]$$

$$=L\left[4\left(e^{t}+e^{-3t}\right)-\frac{3}{2}\left(e^{3t}-e^{-5t}\right)\right]$$

$$=\frac{4}{s-1}+\frac{4}{s+3}-\frac{\frac{3}{2}}{s-3}+\frac{\frac{3}{2}}{s+5}$$

(b)

$$L^{-1}\left[\frac{2s-5}{s^{2}-6s+25}\right]=L^{-1}\left[\frac{2s-5}{\left(s-3\right)^{2}+16}\right]$$

$$=e^{3t}L^{-1}\left[\frac{2s+1}{s^{2}+16}\right]=2e^{3t}\cos\left(4t\right)+\frac{1}{4}e^{3t}\sin\left(4t\right),t>0$$

(c)

令 $Y\left(s\right)=L\left[y\left(t\right)\right]$，將原式進行拉氏轉換

$$s^{2}Y\left(s\right)-sy\left(0\right)-y'\left(0\right)-2sY\left(s\right)+2y\left(0\right)-3Y\left(s\right)=\frac{e^{-s}}{s}$$

$$\left(s^{2}-2s-3\right)Y\left(s\right)=\frac{e^{-s}}{s}-1$$

$$Y\left(s\right)=\frac{e^{-s}}{s\left(s-3\right)\left(s+1\right)}-\frac{1}{\left(s-3\right)\left(s+1\right)}$$

$$Y\left(s\right)=e^{-s}\left(\frac{-\frac{1}{3}}{s}+\frac{\frac{1}{12}}{s-3}+\frac{\frac{1}{4}}{s+1}\right)-\frac{\frac{1}{4}}{s-3}+\frac{\frac{1}{4}}{s+1}$$

$$y\left(t\right)=\left(\frac{-1}{3}+\frac{1}{12}e^{3\left(t-1\right)}+\frac{1}{4}e^{-\left(t-1\right)}\right)u\left(t-1\right)-\frac{1}{4}e^{3t}+\frac{1}{4}e^{-t},t>0$$

(d)

令 $U(x,s) = L[u(x,t)]$，將原式進行拉氏轉換

$$L\left[\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right] = 0$$

$$\frac{d^2 U(x,s)}{dx^2} - sU(x,s) = -25$$

$$U(x,s) = C_1(s)e^{-\sqrt{s}x} + C_2(s)e^{\sqrt{s}x} + \frac{25}{s}$$

代入 I.C.

$$C_1(s) = \frac{-25}{s}, C_2(s) = 0$$

$$U(x,s) = \frac{-25}{s}e^{-\sqrt{s}x} + \frac{25}{s}$$

$$\underline{\underline{u(x,t) = -25\,erfc\left(\frac{x}{2\sqrt{t}}\right) + 25}}$$

Problem 3

【Solution】

令 $y(x) = \sum_{n=0}^{\infty} C_n(r) x^{n+r}$ 代入 O.D.E.

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n(r) x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) C_n(r) x^{n+r-1} - \sum_{n=0}^{\infty} C_n(r) x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) C_n(r) x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) C_n(r) x^{n+r-1} - \sum_{n=0}^{\infty} C_n(r) x^{n+r} = 0$$

$$\sum_{n=-1}^{\infty} 3(n+r+1)(n+r) C_{n+1}(r) x^{n+r} + \sum_{n=-1}^{\infty} (n+r+1) C_{n+1}(r) x^{n+r} - \sum_{n=0}^{\infty} C_n(r) x^{n+r} = 0$$

$$\left(3r(r-1)+r\right) C_0 x^{r-1} - \sum_{n=0}^{\infty} \left(\left(3(n+r+1)+(n+r+1)\right) C_{n+1}(r) - C_n(r) \right) x^{n+r} = 0$$

$$3r(r-1) + r = 0$$

$$r = 0, \frac{2}{3}$$

$$C_{n+1}(r) = \frac{1}{(n+r+1)(3n+3r+1)} C_n(r)$$

1. $r = 0$

$$\begin{cases} C_1 = C_0 \\ C_2 = \dfrac{1}{8} C_1 = \dfrac{1}{8} C_0 \\ C_3 = \dfrac{1}{21} C_2 = \dfrac{1}{168} C_0 \\ C_4 = \dfrac{1}{65} C_3 = \dfrac{1}{10920} C_0 \end{cases}$$

2. $r = \dfrac{2}{3}$

$$\begin{cases} C_1 = \dfrac{1}{5} C_0 \\ C_2 = \dfrac{1}{16} C_1 = \dfrac{1}{80} C_0 \\ C_3 = \dfrac{1}{33} C_2 = \dfrac{1}{2640} C_0 \\ C_4 = \dfrac{1}{56} C_3 = \dfrac{1}{147840} C_0 \end{cases}$$

$$y = c_1 \left(1 + x + \frac{1}{8} x^2 + \frac{1}{168} x^3 + \frac{1}{10920} x^4 \dots \right) + c_2 x^{\frac{2}{3}} \left(1 + \frac{1}{5} x + \frac{1}{80} x^2 + \frac{1}{2640} x^3 + \frac{1}{147840} x^4 \dots \right)$$

Problem 4
【Solution】

$$\int_{-\pi}^{\pi} 1 \cdot \cos(mx)\,dx = \frac{1}{m}\sin(mx)\bigg|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} 1 \cdot \sin(mx)\,dx = \frac{-1}{m}\cos(mx)\bigg|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(mx)\,dx = \frac{1}{m}\int_{-\pi}^{\pi} \sin(mx)\,d(\sin(mx)) = \frac{1}{2m}\sin^2(mx)\bigg|_{-\pi}^{\pi} = 0$$

因為三者彼此內積為 0，所以此為 orthogonal set。

$$\sqrt{\int_{-\pi}^{\pi} \cos^2(mx)\,dx} = \sqrt{\pi}$$

$$\sqrt{\int_{-\pi}^{\pi} \sin^2(mx)\,dx} = \sqrt{\pi}$$

$$\sqrt{\int_{-\pi}^{\pi} 1\,dx} = \sqrt{2\pi}$$

其 orthonormal set 為 $\underline{\underline{\left\{ \dfrac{1}{\sqrt{2\pi}}, \dfrac{\cos(mx)}{\sqrt{\pi}}, \dfrac{\sin(mx)}{\sqrt{\pi}} \right\}, m = 1, 2, 3}}$

Problem 5
【Solution】

由 Fourier Sine Series，令 $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^3 \sin(nx)\,dx$$

$$b_n = \frac{2\left(6 - \pi^2 n^2\right)(-1)^n}{n^3}$$

$$\underline{\underline{f(x) = \sum_{n=1}^{\infty}\left(\frac{2\left(6 - \pi^2 n^2\right)(-1)^n}{n^3} \right)\sin(nx)}}$$