Logic Proofs

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February 14, 2016

1. *Detective James is solving a case. The four suspects A, B, C and D made the following statements:*

   - *If A is guilty then B was an accomplice*
   - *If B is guilty then either C was an accomplice or A is innocent*
   - *If D is guilty then A is guilty and C is innocent*
   - *If D is guilty then A is guilty*

   Is D guilty based on these statements?

   Proof By Truth Table

   First, let us convert the above statements to their CNF logical equivalents. To clean up our proof, let us assume that if a constant X is true, then it means that suspect X is guilty. For this problem, the term "accomplice" will be used interchangeablely with "guilty".

   (a) $\neg A \vee B$
   (b) $\neg B \vee (C \vee \neg A)$
   (c) $(\neg D \vee A) \wedge (\neg D \vee \neg C)$
   (d) $\neg D \vee A$

   Next, we use these sentences to generate the following truth table:

A	B	C	D	(a)	(b)	(c)	(d)
T	T	T	F	T	T	T	T
T	T	F	F	T	F	T	T
T	F	T	F	F	T	T	T
T	F	F	F	F	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	F	T
T	F	F	T	F	T	T	T
F	T	T	T	T	T	F	F
F	T	F	T	T	T	F	F
F	F	T	T	T	T	F	F
F	F	F	T	T	T	F	F
T	T	T	T	T	T	F	T

   Looking at this table, we see that for all cases in which D is guilty, there are no cases in which all four statements are true. Therefore, we cannot prove that D is guilty.

Following the above sentences, this makes sense. If D is guilty, then A is guilty and C is innocent. However, if A is guilty, then B was an accomplice (guilty), and if B is guilty, then C was either an accomplice (guilty) or A was innocent. However, we just said C is innocent and A is guilty! This is a contradiction, and we will prove this more formally later.

Proving By Resolution

Let us formalize this proof using resolution:

1	$\neg A \vee B$	Given
2	$\neg B \vee C \vee \neg A$	Given
3	$\neg D \vee A$	Given
4	$\neg D \vee \neg C$	Given
5	$B \vee \neg D$	Resolution between 1 and 3
6	$\neg A \vee \neg B \vee \neg D$	Resolution between 2 and 4
7	$\neg A \vee C \vee \neg D$	Resolution between 2 and 5

No further resolutions can be drawn. As a result, we cannot prove D is true/guilty, as it cannot be resolved using the given statements.

Adding Refutation

Now let us add $\neg D$ to the premise:

1	$\neg A \vee B$	Given
2	$\neg B \vee C \vee \neg A$	Given
3	$\neg D \vee A$	Given
4	$\neg D \vee \neg C$	Given
5	$\neg D$	Negated conclusion
6	$B \vee \neg D$	Resolution between 1 and 3
7	$\neg A \vee \neg B \vee \neg D$	Resolution between 2 and 4
8	$\neg A \vee C \vee \neg D$	Resolution between 2 and 5

We still cannot resolve any further, so we cannot prove whether suspect D is guilty or not.

2. *If the Congress refuses to vote for new laws then strike would not be finished. Except for the case when it continues for more than a month and a company's CEO retires. The Congress refuses to operate and strike finishes. Therefore strike was going on for more than a month.*

Proof by Truth Table

Let us first convert these sentences into logical constants:

- Let A denote Congress refusing to vote for new laws
- Let B denote the strike finishing
- Let C denote the strike continuing for more than a month
- Let D denote a company's CEO retiring

We can now incorporate these into CNF logical statements as so:

(a) $\neg A \vee \neg B$

(b) $(\neg C \vee \neg D) \vee B$

which generates the following truth table:

A	B	C	D	$\neg A \vee \neg B$	$(\neg C \vee \neg D) \vee \neg B$
T	T	T	T	F	T
T	T	T	F	F	T
T	F	T	T	T	F
T	F	T	F	T	T
F	T	T	T	T	T
F	T	T	F	T	T
F	F	T	T	T	F
F	F	T	F	T	T
T	T	F	T	F	T
T	T	F	F	F	T
T	F	F	T	T	T
T	F	F	F	T	T
F	T	F	T	T	T
F	T	F	F	T	T
F	F	F	T	T	T
F	F	F	F	T	T

Now, remember the statement we are trying to prove: Congress refused to operate, and the strike finished after lasting more than a month. We check our knowledge base by looking at each instance in which A, B, and C are true, then checking to see if statements (a) and (b) are satisfied at these points. Examining the table, we

Proof by Resolution

We formalize this proof through resolution:

1	$\neg A \vee B$	Given
2	$\neg B \vee C$	Given
3	C	Given
4	$\neg A \vee C$	Resolution of 1 and 2

3. *If 2 is a prime number then 2 is the smallest prime number. If 2 is the smallest prime number then 1 is not a prime number. 1 is not a prime number. Are the following propositions correct based on the aforementioned statements?*

   - *2 is the smallest prime number*
   - *2 is a prime number*

Proof by Truth Table

Let us convert these statements into logical constants. Let A denote 2 being a prime number, B denote 2 being the *smallest* prime number, and let C denote 1 as not a prime number. Given these conversion, we are left with the following sentences:

(a) $\neg A \vee B$

(b) $\neg B \vee C$

(c) C

From this we create the following truth table:

A	B	C	$\neg A \vee B$	$\neg B \vee C$	C = True
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	F

Looking at this table, we can see that there are only two case in which all three of our statements are true:

- When A and B are both false and C is true
- When A, B, and C are all true

Therefore, there is only two solutions in which the above propositions are true. In other worlds, these propositions cannot be proven.

Proof by Resolution

We formalize this proof through resolution:

1	$\neg A \vee B$	Given
2	$\neg B \vee C$	Given
3	C	Given
4	$\neg A \vee C$	Resolution of 1 and 2

From this, we cannot resolve neither A nor B. Therefore, both cannot be proven.

Adding Refutation

To further our proof, let us assume B is false. We can add $\neg B$ to the above table, however, we cannot resolve any further statements using it. Therefore, we are left with the same knowledge base, and we still cannot prove neither A nor B.