Logic Proofs

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1. *Detective James is solving a case. The four suspects , , and made the following statements:*
   * *If is guilty then was an accomplice*
   * *If is guilty then either was an accomplice or A is innocent*
   * *If is guilty then is guilty and is innocent*
   * *If is guilty then is guilty*

* *Is D guilty based on these statements?*

### Proof By Truth Table

* First, let us convert the above statements to their CNF logical equivalents. To clean up our proof, let us assume that if a constant is true, then it means that suspect is guilty. For this problem, the term “accomplice” will be used interchangeablely with “guilty”.
* Next, we use these sentences to generate the following truth table:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | * (a) | * (b) | * (c) | * (d) |
| * T | * T | * T | * F | * T | * T | * T | * T |
| * T | * T | * F | * F | * T | * F | * T | * T |
| * T | * F | * T | * F | * F | * T | * T | * T |
| * T | * F | * F | * F | * F | * T | * T | * T |
| * F | * T | * T | * F | * T | * T | * T | * T |
| * F | * T | * F | * F | * T | * T | * T | * T |
| * F | * F | * T | * F | * T | * T | * T | * T |
| * F | * F | * F | * F | * T | * T | * T | * T |
| * T | * T | * F | * T | * T | * F | * T | * T |
| * T | * F | * T | * T | * F | * T | * F | * T |
| * T | * F | * F | * T | * F | * T | * T | * T |
| * F | * T | * T | * T | * T | * T | * F | * F |
| * F | * T | * F | * T | * T | * T | * F | * F |
| * F | * F | * T | * T | * T | * T | * F | * F |
| * F | * F | * F | * T | * T | * T | * F | * F |
| * T | * T | * T | * T | * T | * T | * F | * T |

* Looking at this table, we see that for all cases in which is guilty, there are no cases in which all four statements are true. Therefore, we cannot prove that is guilty.  
    
  Following the above sentences, this makes sense. If is guilty, then is guilty and is innocent. However, if is guilty, then was an accomplice (guilty), and if is guilty, then was either an accomplice (guilty) or was innocent. However, we just said is innocent and is guilty! This is a contradiction, and we will prove this more formally later.

### Proving By Resolution

* Let us formalize this proof using resolution:

|  |  |  |
| --- | --- | --- |
| * 1 |  | * Given |
| * 2 |  | * Given |
| * 3 |  | * Given |
| * 4 |  | * Given |
| * 5 |  | * Resolution between 1 and 3 |
| * 6 |  | * Resolution between 2 and 4 |
| * 7 |  | * Resolution between 2 and 5 |

* No further resolutions can be drawn. As a result, we cannot prove is true/guilty, as it cannot be resolved using the given statements.

### Adding Refutation

* Now let us add to the premise:

|  |  |  |
| --- | --- | --- |
| * 1 |  | * Given |
| * 2 |  | * Given |
| * 3 |  | * Given |
| * 4 |  | * Given |
| * 5 |  | * Negated conclusion |
| * 6 |  | * Resolution between 1 and 3 |
| * 7 |  | * Resolution between 2 and 4 |
| * 8 |  | * Resolution between 2 and 5 |

* We still cannot resolve any further, so we cannot prove whether suspect is guilty or not.
* p = Congress refuses to vote for new laws
* q = Strike finished
* s = Strike continues for more than a month
* r = company’s CEO retires

### Proving By Resolution

* 1 implies that if congress refuses to vote for new laws and strike finished, then strike continues for more than a month and company’s CEO retires.

### Adding Refutation

* Thus, if congress refuses to vote for new laws and strike finished, then strike continues for more than a month and company’s CEO retires.

### Proof by Truth Table

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **p** | **q** | **s** | **r** | **(¬q∨¬p)∨(s∧r)** | **q∧p** | **KB** |
| FALSE | FALSE | FALSE | FALSE | TRUE | FALSE | FALSE |
| TRUE | FALSE | FALSE | FALSE | TRUE | FALSE | FALSE |
| FALSE | TRUE | FALSE | FALSE | TRUE | FALSE | FALSE |
| FALSE | FALSE | TRUE | FALSE | TRUE | FALSE | FALSE |
| FALSE | FALSE | FALSE | TRUE | TRUE | FALSE | FALSE |
| TRUE | TRUE | FALSE | FALSE | FALSE | TRUE | FALSE |
| TRUE | FALSE | TRUE | FALSE | TRUE | FALSE | FALSE |
| TRUE | FALSE | FALSE | TRUE | TRUE | FALSE | FALSE |
| FALSE | TRUE | TRUE | FALSE | TRUE | FALSE | FALSE |
| FALSE | TRUE | FALSE | TRUE | TRUE | FALSE | FALSE |
| FALSE | FALSE | TRUE | TRUE | TRUE | FALSE | FALSE |
| FALSE | TRUE | TRUE | TRUE | TRUE | FALSE | FALSE |
| TRUE | FALSE | TRUE | TRUE | TRUE | FALSE | FALSE |
| TRUE | TRUE | FALSE | TRUE | FALSE | TRUE | FALSE |
| TRUE | TRUE | TRUE | FALSE | FALSE | TRUE | FALSE |
| TRUE | TRUE | TRUE | TRUE | TRUE | TRUE | TRUE |

1. *If 2 is a prime number then 2 is the smallest prime number. If 2 is the smallest prime number then 1 is not a prime number. 1 is not a prime number. Are the following propositions correct based on the aforementioned statements?*
   * *2 is the smallest prime number*
   * *2 is a prime number*

### Proof by Truth Table

* Let us convert these statements into logical constants. Let denote 2 being a prime number, denote 2 being the *smallest* prime number, and let denote 1 as not a prime number. Given these conversion, we are left with the following sentences:
* From this we create the following truth table:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  | * = True |
| * T | * T | * T | * T | * T | * T |
| * T | * T | * F | * T | * F | * F |
| * T | * F | * T | * F | * T | * T |
| * T | * F | * F | * F | * T | * F |
| * F | * T | * T | * T | * T | * T |
| * F | * T | * F | * T | * F | * F |
| * F | * F | * T | * T | * T | * T |
| * F | * F | * F | * T | * T | * F |

* Looking at this table, we can see that there are only two case in which all three of our statements are true:
  + When and are both false and is true
  + When , , and are all true
* Therefore, there is only two solutions in which the above propositions are true. In other worlds, these propositions cannot be proven.

### Proof by Resolution

* We formalize this proof through resolution:

|  |  |  |
| --- | --- | --- |
| * 1 |  | * Given |
| * 2 |  | * Given |
| * 3 |  | * Given |
| * 4 |  | * Resolution of 1 and 2 |

* From this, we cannot resolve neither nor . Therefore, both cannot be proven.

### Adding Refutation

* To further our proof, let us assume is false. We can add to the above table, however, we cannot resolve any further statements using it. Therefore, we are left with the same knowledge base, and we still cannot prove neither nor .
* 4)
* p = tomorrow would be a snow storm
* q = Patrick would wear a heavy coat
* s = zipper is mended

### Proof by Resolution

* 4 implies that when tomorrow is snow and zipper is not mended, Patrick would not wear a heavy coat. Thus, Patrick would not wear heavy coat.

### Adding Refutation

* Thus, Patrick would not wear heavy coat.

### Proof by Truth Table

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **p** | **q** | **s** | **¬p∨¬s∨q** | **¬q∨s** | **p∧¬s** | **KB** |
| FALSE | FALSE | FALSE | TRUE | TRUE | FALSE | FALSE |
| TRUE | FALSE | FALSE | TRUE | TRUE | TRUE | TRUE |
| FALSE | TRUE | FALSE | TRUE | FALSE | FALSE | FALSE |
| FALSE | FALSE | TRUE | TRUE | TRUE | FALSE | FALSE |
| FALSE | FALSE | FALSE | TRUE | TRUE | FALSE | FALSE |
| TRUE | TRUE | FALSE | TRUE | FALSE | TRUE | FALSE |
| TRUE | FALSE | TRUE | FALSE | TRUE | FALSE | FALSE |
| TRUE | FALSE | FALSE | TRUE | TRUE | TRUE | TRUE |
| FALSE | TRUE | TRUE | TRUE | TRUE | FALSE | FALSE |
| FALSE | TRUE | FALSE | TRUE | FALSE | FALSE | FALSE |
| FALSE | FALSE | TRUE | TRUE | TRUE | FALSE | FALSE |
| FALSE | TRUE | TRUE | TRUE | TRUE | FALSE | FALSE |
| TRUE | FALSE | TRUE | FALSE | TRUE | FALSE | FALSE |
| TRUE | TRUE | FALSE | TRUE | FALSE | TRUE | FALSE |
| TRUE | TRUE | TRUE | TRUE | TRUE | FALSE | FALSE |
| TRUE | TRUE | TRUE | TRUE | TRUE | FALSE | FALSE |

* 6)
* p = Sally and Bob are of the same age
* q = Sally is order than Bob
* s = Nancy and Bob are not of the same age
* r = Bob is older than Walter

### Proof by Resolution

* Thus, either Nancy and Bob are not of the same age or Bob is older than Walter

### Adding Refutation

* Thus, either Nancy and Bob are not of the same age or Bob is older than Walter.

### Proof by Truth Table

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **p** | **q** | **s** | **r** | **p∨q** | **¬p∨s** | **¬q∨r** | **KB** |
| FALSE | FALSE | FALSE | FALSE | FALSE | TRUE | TRUE | FALSE |
| TRUE | FALSE | FALSE | FALSE | TRUE | FALSE | TRUE | FALSE |
| FALSE | TRUE | FALSE | FALSE | TRUE | TRUE | FALSE | FALSE |
| FALSE | FALSE | TRUE | FALSE | FALSE | TRUE | TRUE | FALSE |
| FALSE | FALSE | FALSE | TRUE | FALSE | TRUE | TRUE | FALSE |
| TRUE | TRUE | FALSE | FALSE | TRUE | FALSE | FALSE | FALSE |
| TRUE | FALSE | TRUE | FALSE | TRUE | TRUE | TRUE | TRUE |
| TRUE | FALSE | FALSE | TRUE | TRUE | FALSE | TRUE | FALSE |
| FALSE | TRUE | TRUE | FALSE | TRUE | TRUE | FALSE | FALSE |
| FALSE | TRUE | FALSE | TRUE | TRUE | TRUE | TRUE | TRUE |
| FALSE | FALSE | TRUE | TRUE | FALSE | TRUE | TRUE | FALSE |
| FALSE | TRUE | TRUE | TRUE | TRUE | TRUE | TRUE | TRUE |
| TRUE | FALSE | TRUE | TRUE | TRUE | TRUE | TRUE | TRUE |
| TRUE | TRUE | FALSE | TRUE | TRUE | FALSE | TRUE | FALSE |
| TRUE | TRUE | TRUE | FALSE | TRUE | TRUE | FALSE | FALSE |
| TRUE | TRUE | TRUE | TRUE | TRUE | TRUE | TRUE | TRUE |