Final Exam for Math3315/CSE3365 (Fall 2017)

December 11, 2017

I pledge that I do not give or receive help on this exam.

Signature: ____

Name: (please print) ______________________________________

Note: If a problem requires hand-written answers, write on the empty space provided in this exam. Every problem in this exam requires coding. You should submit your code together with any plots your code generate to canvas. There is **no need** to submit log files for this exam. (The grader will run your code and see the outputs.)

Time: 180 minutes
4 problems, 30 points each, total 120 points.
Good Luck, and Thank You for the semester.

P1. (30 points) Plot the Gaussian distribution function

$$G(x,a,b) = \frac{1}{\sqrt{2\pi b^2}} e^{\frac{-(x-a)^2}{2b^2}}$$

on $x \in [-10, 10]$ for the specified (a, b).

You need to plot 4 curves associated the four (a, b) pairs specified in the script `p1_plot.py`, using only one plot command. The structure of your plotting code would look like this

```
for (a, b) in abList:
    construct a vector that stores y = G(x, a, b)   #where x is a numpy array
    plot (x, y) with correct label

add labels, titles, legend
save plot;  show plot
```

The result should be similar to Figure 1. Don't worry about the color in this black and white printing, the color should be fine in electronic format.

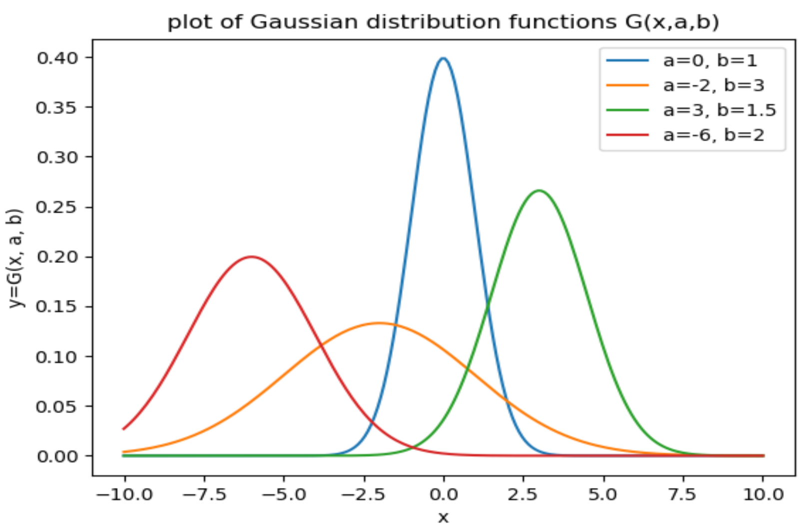


Figure 1: Plot of Gaussian distributions

P2. (30 points) Write a function to compute the following series

$$p1 = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \tag{1}$$

$$p2 = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^3} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots \tag{2}$$

$$p3 = \sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots \tag{3}$$

Stop each summation when the absolute value of the new term to be added to the sum becomes less than or equal to a given tol (i.e., stop sum when $|\text{new term}| \le tol$).

The return values of your function should be two tuples, the first tuple should be $(p1, p2, p3)$, the second tuple should be $(\sqrt{p1 * 8},\ (p2 * 32)^{1/3},\ (p3 * 90)^{1/4})$.

Write down the three values of **the second tuple** that you get from your code here:

(______________________ , ______________________ , ______________________)

You only need to write down 10 decimal digits of each value.
(These three values should all be close to each other if your code is right.)

P3. (30 points) First order ordinary differential equations of form

$$y'(x) = f(x), \qquad y(x_0) = y_0$$

can be solved via numerical integration. This is because (via basic calculus) the solution is

$$y(x) = y_0 + \int_{x_0}^{x} f(t)dt.$$

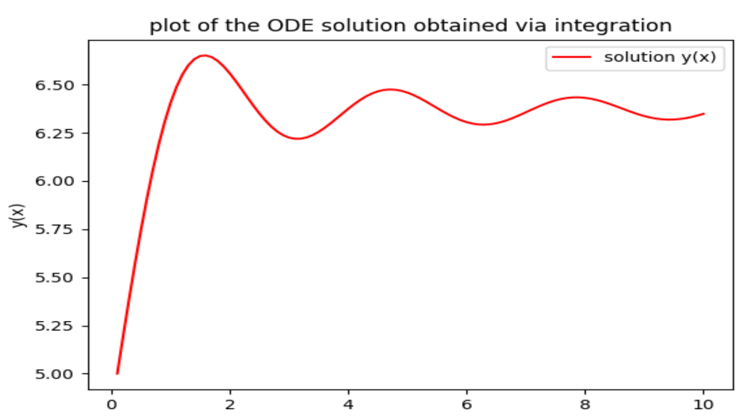
A code for numerical integration to compute $\int_a^b f(t)dt$ is provided in the file `p3_quad_ode.py`. Use this code to solve

$$y'(x) = \frac{\sin(2x)}{x}, \qquad y(0.1) = 5. \tag{4}$$

Your code need to address the following two questions: (I suggest you write two functions)

(a) Find the solution $y(x)$ evaluated at $x = 0.2,\ 0.5,\ 0.7,\ 1.1,\ 5,\ 10,\ 19,\ 20$. (I.e., put these x values in a list, loop over this list and printout their related $y(x)$'s in your code.) Using the output of your code, write four values below: (use 5 decimal digits)
$y(0.2) =$ ______________________, $y(1.1) =$ ______________________,
$y(5) =$ ______________________, $y(19) =$ ______________________.
Hint: Since the solution is
$$y(x) = 5 + \int_{0.1}^{x} \frac{\sin(2t)}{t} dt,$$
where the integration can be done by calling the provided function, finding $y(x)$ for a specified x shouldn't be hard to do.

(b) Make a plot of the solution $y(x)$ for $x \in [0.1, 10]$. Your plot should look smooth (try to generate 100 evenly spaced x on $[0.1, 10]$, compute the 100 data points $(x, y(x))$, and plot them by linking them together). Your plot should look like the following Figure. Save your plot to a file named `quad_ode.png`.



P4. (30 points) In the file `p4_ode.py` the forward Euler method is implemented. You are asked to implemented the following 3rd-order RK method

$$y(x+h) = y(x) + \frac{1}{9}(2K_1 + 3K_2 + 4K_3),$$

where

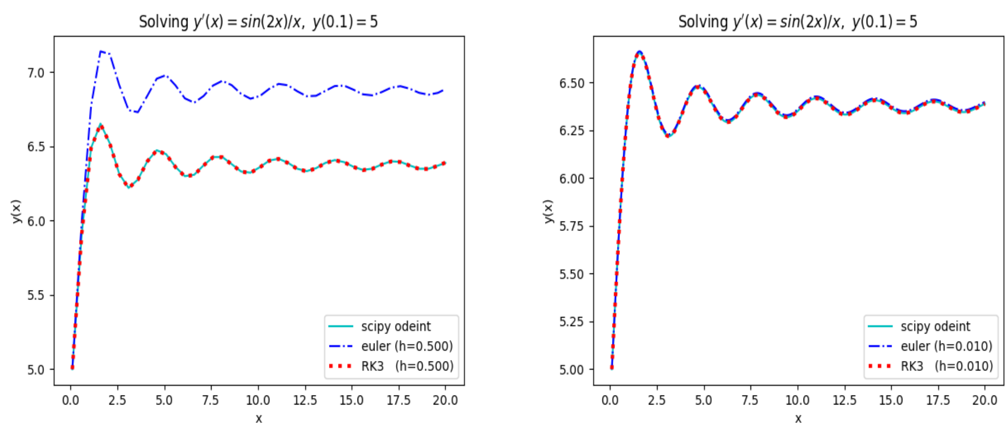
$$\begin{cases} K_1 = & hf(y,\ x) \\ K_2 = & hf(y + K_1/2, \quad x + h/2) \\ K_3 = & hf(y + 3K_2/4, \quad x + 3h/4). \end{cases}$$

After implementing RK3, add code to solve the following two ODEs and plot their solutions (follow instructions in `p4_ode.py`)

(a) The same ODE (4) as in Problem 3, but here you solve for $x \in [0.1, 20]$:

$$y'(x) = \frac{\sin(2x)}{x}, \quad y(0.1) = 5, \qquad x \in [0.1, 20];$$

Solve using $h = 0.5,\ 0.05,\ 0.01$ and plot a figure for each h. Two of your plots should look similar to the following plots.



(b)

$$y'(x) = \cos(xy), \qquad y(-3) = 8, \qquad x \in [-3, 5].$$

Solve using $h = 0.2,\ 0.05,\ 0.01$ and plot a figure for each h. Two of your plots should look similar to the following plots.