where σ_S is the square root of the cross section of the sphere. From (11), the scattering matrix of a sphere measured in the presence of errors is given by

$$\tilde{C}_S = \sigma_S e^{\epsilon_1} \begin{pmatrix} 2\epsilon_{21} & 1 \\ 1 & 2\epsilon_{12} \end{pmatrix}. \tag{34}$$

By taking ratios $\tilde{C}_{11}/\tilde{C}_3$ and $\tilde{C}_{22}/\tilde{C}_3$, the quantities ϵ_{12} and ϵ_{21} can be determined directly without absolute measurements. Also, $\tilde{C}_{12}/\tilde{C}_{21}$ can identify the presence of Faraday rotation [6]. When practical, the antenna feed structure should be adjusted until a null is obtained for the diagonal elements of $\tilde{C}$ so as to insure the transmission of purely circular polarization. If absolute cross section is known, then a determination of ϵ_1 can be made. Imperfections in the calibrating sphere can be neglected only if $|C_{ii}/C_3| \ll \epsilon_{ij}$; $i \neq j$, $i=1, 2$.

To locate a reference axis for the antenna it is necessary to measure ϵ_2. As indicated by (34), no information is obtained about this difference quantity using returns from a sphere. This is to be expected since a sphere is a nondepolarizing body and, as such, scatters independent of rotation. Thus, to specify a reference axis, it is necessary to measure the return from a second scatterer, such as a dipole.

The scattering matrix for a dipole whose axis lies along the $\hat{L}+\hat{R}$ direction is given by

$$C_d = \frac{\sigma_d}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{35}$$

where σ_d is the square root of the cross section of the dipole. Assuming that calibration has been performed on the sphere so as to make ϵ_{12} and ϵ_{21} zero in (34), then the measured scattering matrix return from the dipole becomes

$$\tilde{C}_d = \frac{\sigma_d e^{\epsilon_1}}{2}\begin{pmatrix} e^{\epsilon_2} & 1 \\ 1 & e^{-\epsilon_2} \end{pmatrix}. \tag{36}$$

Then $\tilde{C}_{11}/\tilde{C}_{22}$ in (36) gives ϵ_2, and hence the antenna axes are readily determined for known body orientation.

The calibration also offers a convenient method to evaluate the behavior of the errors as a function of line-of-sight direction.

Another calibration procedure is possible using a single body of revolution at different orientations. For bodies of revolution in the absence of errors, C_{11} is equal to C_{22}. Therefore, the real part of (28) can be written as

$$\mathrm{Re}\ln\frac{\tilde{C}_{11}}{\tilde{C}_{22}} = 2\,\mathrm{Re}\,\epsilon_2 + \mathrm{Re}\,\tilde{C}_3\left(\frac{\epsilon_{21}}{\tilde{C}_{11}} - \frac{\epsilon_{12}}{\tilde{C}_{22}}\right). \tag{37}$$

All the elements of (37) are either ratios of observables or error terms. Since there are five error quantities associated with the three real and two imaginary parts of the error terms, it is necessary to make measurements at five different aspects of the body. Since (37) does not involve the orientation angle explicitly, these error terms (i.e., axial ratios and difference between the power in orthogonal circular channels) can be determined without knowing the orientation of the scatterer. This is analogous to calibration from a sphere. If the orientation of the projection of the body axis on the radar line of sight is known, then from (31) the imaginary part of ϵ_2 which locates the antenna axis can be found.

Application

Of the various considerations which can be derived from the results of this paper, one of particular interest relates the variation in μ_δ to axial errors for rotationally symmetric scatterers.

For this situation, and with μ_{12}^2 and μ_{21}^2 both taken equal to μ^2, we have from (32),

$$\mu_\delta^2 = \frac{1}{4}\frac{|C_{12}|^2}{|C_{11}|^2}\mu^2. \tag{38}$$

The depolarization [3], D, for the case at hand is given by

$$D = \frac{1}{1 + |C_{12}|^2/|C_{11}|^2}. \tag{39}$$

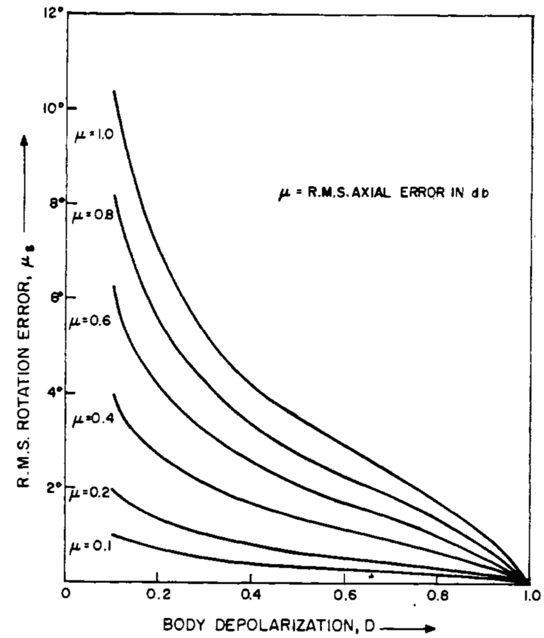

Fig. 1. Rotation error vs. body depolarization and axial errors.

We thus obtain the relationship

$$\mu_\delta = \frac{1}{2}\left(\frac{1-D}{D}\right)^{1/2}\mu. \tag{40}$$

A plot of μ_δ in degrees as a function of the body depolarization D for various axial errors specified in decibels by μ is given in Fig. 1.

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Effects of Magneto-Ionic Propagation on the Polarization Scattering Matrix

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***Abstract*—Magneto-ionic propagation effects are considered at radio frequencies that are sufficiently high so that there is negligible physical separation between the ordinary and extraordinary propagation paths. It is recognized that the sensible magneto-ionic propagation effects are birefringence and Faraday rotation. A matrix representation for magneto-ionic propagation is employed in order to facilitate the determination of the effects of magneto-ionic propagation on the polarization scattering matrix of an arbitrary, monostatic radar target. It is shown that the propagation path must be calibrated with a known, four-way symmetric target when both birefringence**

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and Faraday rotation are present, if it is desired to eliminate the effects of magneto-ionic propagation. However, the determinant of the polarization scattering matrix and the trace of the power scattering matrix are, in general, unaffected by magneto-ionic propagation. If birefringence is absent, then even if the radar target is arbitrary and unknown, the effects of Faraday rotation can be accounted for in a particularly simple manner if the scattering matrix is referred to circular polarization.

Introduction

The polarization scattering matrix is a powerful analytical device for the interpretation of monostatic radar data [1], [2]. The reasoning used to demonstrate the usefulness of the scattering matrix is based largely upon the reciprocity theorem. Many radar systems have propagation paths that traverse the ionosphere. Since the latter introduces nonreciprocal and anisotropic propagation effects, it is pertinent to inquire whether or not the useful properties of the scattering matrix are irrevocably destroyed if the radar wave passes through the ionosphere.

Magneto-ionic theory [3] shows that, when an arbitrary radio wave passes through the ionosphere, it splits into the well-known ordinary and extraordinary rays. The medium has different refractive indexes for the two waves, so that they have a tendency to follow different propagation paths. However, almost all radar systems operate at frequencies well above the resonant frequencies of the ionosphere so that the physical separation between the ordinary and extraordinary propagation paths is negligible. The sensible effects are Faraday rotation (caused by the component of the earth's magnetic field along the propagation path) and birefringence (caused by the component of the earth's magnetic field transverse to the propagation path). Faraday rotation is a nonreciprocal effect that causes linear polarizations to rotate by, say, α radians per unit length of propagation path. Birefringence is a reciprocal but anisotropic effect that introduces a differential phase shift of, say, 2β radians per unit length between two particular, orthogonal, linear polarizations (these polarizations are directed parallel to what are commonly called the optical axes of the medium). In this paper it is shown that:

1) If the propagation path exhibits both Faraday rotation and birefringence, and if the propagation path is uncalibrated, then the magneto-ionic propagation effects cannot be removed from the measured scattering matrix of an arbitrary target. However, the determinant of the polarization scattering matrix and the trace of the power scattering matrix [6] are in general unaffected by magneto-ionic propagation.

2) If there is no birefringence, then the effect of Faraday rotation can always be removed from the measured scattering matrix of an arbitrary target.[1] Four times the one-way Faraday rotation angle is given by the phase difference between the off-diagonal terms of the scattering matrix referred to circular polarization. This is an interesting conclusion because the Faraday rotation effect is usually much larger than the birefringence effect.

3) The propagation path can be calibrated, even in the presence of both Faraday rotation and birefringence, by measuring the scattering matrix of a sphere, or other scatterer exhibiting four-way symmetry. The calibration of the propagation path would allow the removal of the magneto-ionic propagation effects from the measured scattering matrix of an arbitrary target.

Basic Theory

The effects of Faraday rotation and birefringence only commute over infinitesimal portions of the propagation path. If $\boldsymbol{a}$ is a complex vector representing the polarization state (elliptical, in general) of a radio wave propagating in the z direction, then the differential equation for $\boldsymbol{a}$ can be written as

$$d\boldsymbol{a}/dz = A\boldsymbol{a} \tag{1}$$

[1] To our knowledge, this possibility was first recognized by our colleague, N. N. Bojarski. The earliest reference that we can find in the open literature is a paper by Brysk [4].

where A is the matrix containing the effects of Faraday rotation and birefringence. In a right-handed, rectangular coordinate system in which z is in the direction of propagation and the x and y axes are the optical axes of the medium, express $\boldsymbol{a}$ as a column vector

$$\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}. \tag{2}$$

From the definitions of α and β given in the Introduction, A has the form

$$A = \begin{pmatrix} j2\beta & \alpha \\ -\alpha & 0 \end{pmatrix}. \tag{3}$$

The polarization vector $\boldsymbol{a}$ is a function of distance along the propagation path and can be written as

$$\boldsymbol{a} = M(z)\boldsymbol{a}_0 \tag{4}$$

where $\boldsymbol{a}_0$ is an initial polarization vector and $M(z)$ is a unitary matrix characteristic of the medium. It follows that [5]

$$M(z) = \exp zA = PD(z)P^{-1} \tag{5}$$

where

$$D(z) = \begin{pmatrix} \exp \lambda_+ z & 0 \\ 0 & \exp \lambda_- z \end{pmatrix}, \tag{6}$$

and where $\lambda_\pm$ are the eigenvalues of A and the columns of the matrix P are the eigenvectors $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ of A. The eigenvectors of A are representations of those polarizations, termed the eigenpolarizations, which can propagate unchanged through the medium. Writing the axial ratio of the polarization ellipse [6] as

$$r = \frac{\text{major axis}}{\text{minor axis}} = \coth\frac{\psi}{2} = \frac{\beta + \sqrt{\alpha^2 + \beta^2}}{\alpha} \tag{7}$$

there results

$$\lambda_\pm = \pm j\alpha\left(\tanh\frac{\psi}{2}\right)^{\mp 1}; \quad \boldsymbol{a}_+ = \begin{pmatrix} r \\ j \end{pmatrix}; \quad \boldsymbol{a}_- = \begin{pmatrix} j \\ r \end{pmatrix}. \tag{8}$$

The major axes of the polarization ellipses for $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ are along the x and y coordinates, respectively. Also, $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ have left- and right-handed directions of circulation along their respective ellipses. It is interesting to note the behavior of the axial ratio in the special limiting cases when either birefringence or Faraday rotation dominates. When birefringence dominates, $\alpha \ll \beta$ and

$$r = \frac{2\beta}{\alpha}\left[1 + O\left(\frac{\alpha^4}{\beta^4}\right)\right]. \tag{9}$$

In the limiting case of infinite axial ratio, the eigenpolarizations are linear. When Faraday rotation dominates,

$$r = 1 + \frac{\beta}{\alpha} + O\left(\frac{\beta^2}{\alpha^2}\right) \tag{10}$$

so that, in the limiting case of no birefringence, the eigenpolarizations are circular.

Scattering Matrix of Arbitrary Targets

If $\boldsymbol{b}$ and $\boldsymbol{a}$ are the polarizations of colocated transmitting and receiving antennas, then the voltage V_{ab} developed at the terminals of the receiving antenna is given by [7]

$$V_{ab} = \boldsymbol{a}\cdot S\boldsymbol{b} \tag{11}$$

where S is the scattering matrix of an arbitrary target. For reciprocal targets, $V_{ab} = V_{ba}$. Consequently, S is a symmetric matrix. Suppose that the polarization base vectors are changed to $\boldsymbol{a}'$ and $\boldsymbol{b}'$ such that $\boldsymbol{a} = Q\boldsymbol{a}'$ and $\boldsymbol{b} = Q\boldsymbol{b}'$, where Q is unitary. Then, Q can represent either a rotation of coordinates or a change of polarization ellipticity, or both. The bilinear form in (11) must be preserved, so that S must undergo a congruent transformation of the form [6]

$$\tilde{S} = Q^T S Q = \tilde{S}^T \tag{12}$$

where the superscript T denotes the transpose.

In order to develop the transformation for the effects of a magneto-ionic propagation path on the scattering matrix, it is convenient first of all to restrict consideration to those linear polarizations that are directed parallel to the optical axes of the medium. The vectors $\boldsymbol{b}$ and $\boldsymbol{a}$ then represent the polarizations transmitted and received by an antenna that is separated from the target by a magneto-ionic medium. By analogy with (11), the voltage measured by this antenna is

$$V_{ab} = \boldsymbol{a}\cdot T\boldsymbol{b} \tag{13}$$

where T is the scattering matrix (referred to linear polarizations parallel to the optical axes) which is measured in the presence of a magneto-ionic propagation path. The effects of the medium are given by (4). It is convenient to specify that the coordinate z in (4) is the distance between the antenna and the target. Thus, the functional dependence of M upon z need not be retained explicitly. The polarizations $\boldsymbol{a}'$ and $\boldsymbol{b}'$ at the target are given in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ by

$$\boldsymbol{b}' = M\boldsymbol{b} \quad \text{and} \quad \boldsymbol{a} = M^*\boldsymbol{a}'. \tag{14}$$

The asterisk denotes the complex conjugate and is necessary [6] to account for the reversal of the direction of propagation which takes place when scattering occurs. Since M is unitary, the bilinear form of (13) will be preserved if S undergoes a transformation of the form

$$T = MSM. \tag{15}$$

Since M is unitary,

$$\det T = \det M \det S \det M = \det S, \tag{16}$$

and

$$P_1 = \sum_{m,n} |T_{mn}|^2 = \sum_{m,n} |S_{mn}|^2, \tag{17}$$

where T_{mn} denotes the element in row m and column n of the matrix T. Thus, the total power P_1 (the trace of the power scattering matrix [6]) returned from the target for two orthogonal incident polarizations is unaffected by either birefringence or Faraday rotation. Since both P_1 and the determinant are unchanged by magneto-ionic effects, the eigenvalues of the target [2] are also unaffected, which shows that it is always possible to measure at least two out of the five independent parameters that characterize an arbitrary, monostatic radar target. Performing a congruent transformation, as in (12), on T results in

$$T = Q^TQM'(Q^TQ)^{-1}\tilde{S}M', \tag{18}$$

where the prime indicates a similarity transformation:

$$M' = Q^{-1}MQ. \tag{19}$$

In the special case of a real rotation $Q^T = Q^{-1}$ and (18) reduces to (15) with M' substituted for M so that the operational form of the effects of magneto-ionic propagation is preserved for any set of orthogonal linear polarizations. If Faraday rotation is absent, M is diagonal if the linear polarizations are directed along the optical axes of the medium, in which case it follows that

$$T = DSD = \begin{pmatrix} S_{11}\exp j2\beta z & S_{12} \\ S_{12} & \exp -j2\beta z \end{pmatrix}. \tag{20}$$

The form of (20) emphasizes the reciprocal nature of birefringence because T is symmetric. If the target is unknown, there is then no way of recognizing whether or not birefringence is present. Thus, the presence of birefringence prevents the removal of magneto-ionic propagation effects from measurements made on an arbitrary target. Now consider the special case when birefringence is absent and Faraday rotation is the only magneto-ionic effect present. The transformation that diagonalizes M is

$$P = 2^{-1/2}\begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix} \tag{21}$$

so that

$$\tilde{T} = D^*\tilde{S}D = \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{12}\exp -j2\alpha z \\ \tilde{S}_{12}\exp j2\alpha z & \tilde{S}_{22} \end{pmatrix}. \tag{22}$$

Since $\tilde{T}_{12} \neq \tilde{T}_{21}$, (22) emphasizes the nonreciprocal nature of Faraday rotation. The Faraday rotation angle is given by

$$\alpha z = \frac{1}{4}[\arg \tilde{T}_{21} - \arg T_{12}] \tag{23}$$

when the matrix $\tilde{T}$ is referred to circular polarization. It is interesting to compare Faraday rotation with rotation of the radar target about the radar line of sight. Let Q represent a real rotation of the target through an angle θ so that (18) becomes, for circu ar polarization,

$$\tilde{T} = \begin{pmatrix} \tilde{S}_{11}\exp j2\theta & \tilde{S}_{12} \\ \tilde{S}_{12} & \tilde{S}_{22}\exp -j2\theta \end{pmatrix}. \tag{24}$$

Since Faraday rotation and angular rotation operate on different diagonals of the scattering matrix, the two effects are separable and Faraday rotation can always be removed from a measured scattering matrix, provided that birefringence is absent.

Scattering Matrix of Four-Way Symmetric Targets

Backscatter from a four-way symmetric body (symmetric about two perpendicular axes and about the axis at 45° to each) cannot exhibit depolarization because of symmetry. Since the radar cross section is the same for all incident polarizations, then the scattering matrix referred to linear polarization is a scalar S_0 multiplied by the identity matrix. Spheres and square plates at broadside incidence are examples of such targets. Let T be the scattering matrix of a four-way symmetric target as measured with linear polarization in the presence of both Faraday rotation and birefringence. It follows that

$$T = S_0M^2 = S_0PD^2P^{-1}; \qquad S_0D^2 = P^{-1}TP, \tag{25}$$

which shows that the eigenvectors of the measured matrix T are the eigenpolarizations of the medium. The eigenvalues of T are the square of the eigenvalues of M. Hence, even though Faraday rotation and birefringence are not separable effects, they can be resolved by measuring the scattering from a four-way symmetric target. For instance, if T is referred to circular polarization, the characteristics of the propagation medium are given by

$$\theta = \frac{1}{4}[\arg T_{11} - \arg T_{22}] + [2n+1]\frac{\pi}{4}, \; n = 0, \pm 1, \pm 2, \cdots \tag{26}$$

$$\tan(2z\sqrt{\alpha^2+\beta^2}) = j\,\frac{\sqrt{[T_{12}-T_{21}] + 4T_{11}T_{22}}}{T_{12}+T_{21}}, \tag{27}$$

and

$$\frac{\alpha}{\beta} = \sinh\psi = j\,\frac{T_{12}-T_{21}}{\sqrt{T_{12}\,T_{22}}}, \tag{28}$$

where arg stands for argument and θ is the angle between the optical axes of the medium and some arbitrary reference axes in the antenna. Notice that (26) through (28) are of considerable practical interest because they only depend upon differences of phases and ratios of moduli, rather than absolute phases and moduli. Since the the phases of T_{11} and T_{22} can be measured *modulo* 2π, then θ, α, and β can be determined *modulo* $\pi/2$.

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