Partitioning Edge-Coloured Graphs into Monochromatic Subgraphs

Jess Ryan

University of Glasgow

25th Postgraduate Combinatorial Conference, June 2018

Problem Statement

Question:

Can we partition the vertices of an edge-coloured K_n into a number of monochromatic subgraphs depending only on the number of colours used (and not on n)?

Motivation:

- Calculating generalised Ramsey numbers.

Generalised Ramsey Number:

The generalised Ramsey number $R(H_1, \ldots, H_r)$ is the smallest n for which any r-edge-coloured K_n contains a monochromatic H_i for at least one i.

- Finding vertex covers in intersecting hypergraphs.

Partitioning into paths and cycles

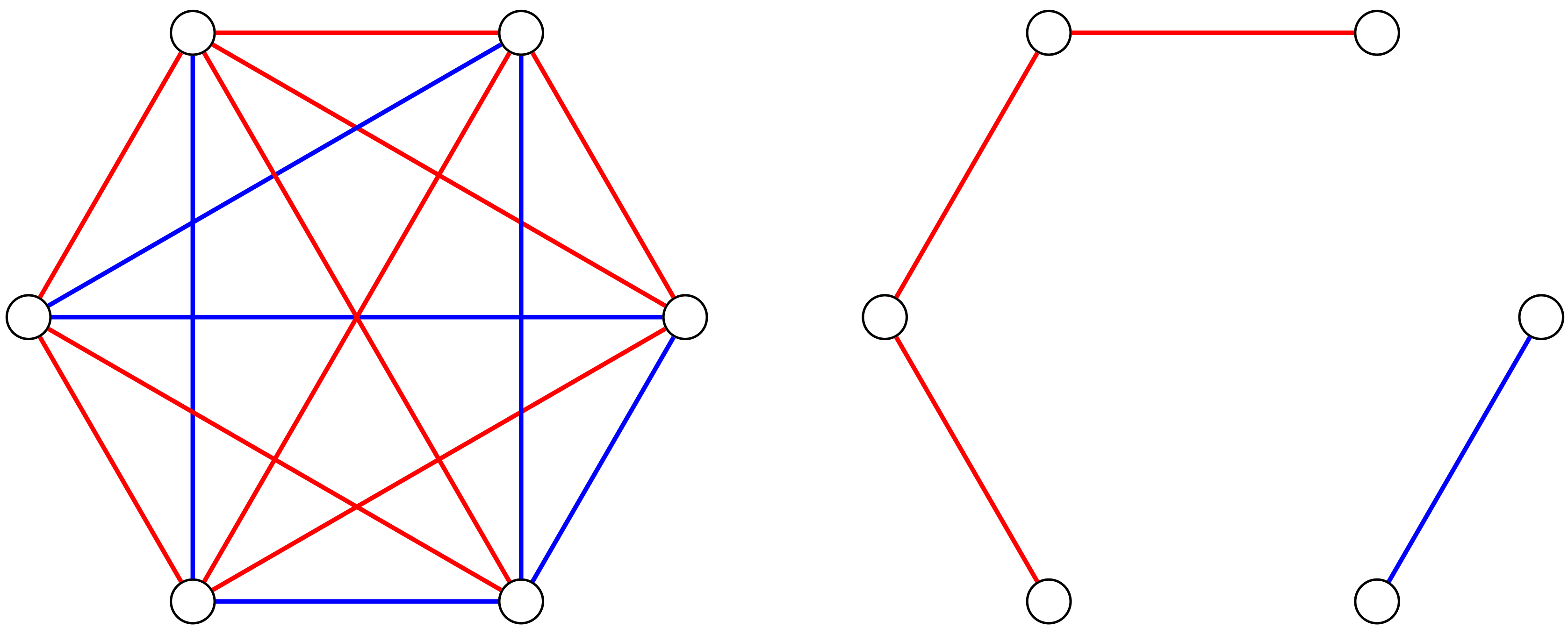
Conjecture (Gyárfás, 1989)

At most r disjoint monochromatic paths are needed to partition the vertices of any r-edge-coloured K_n.

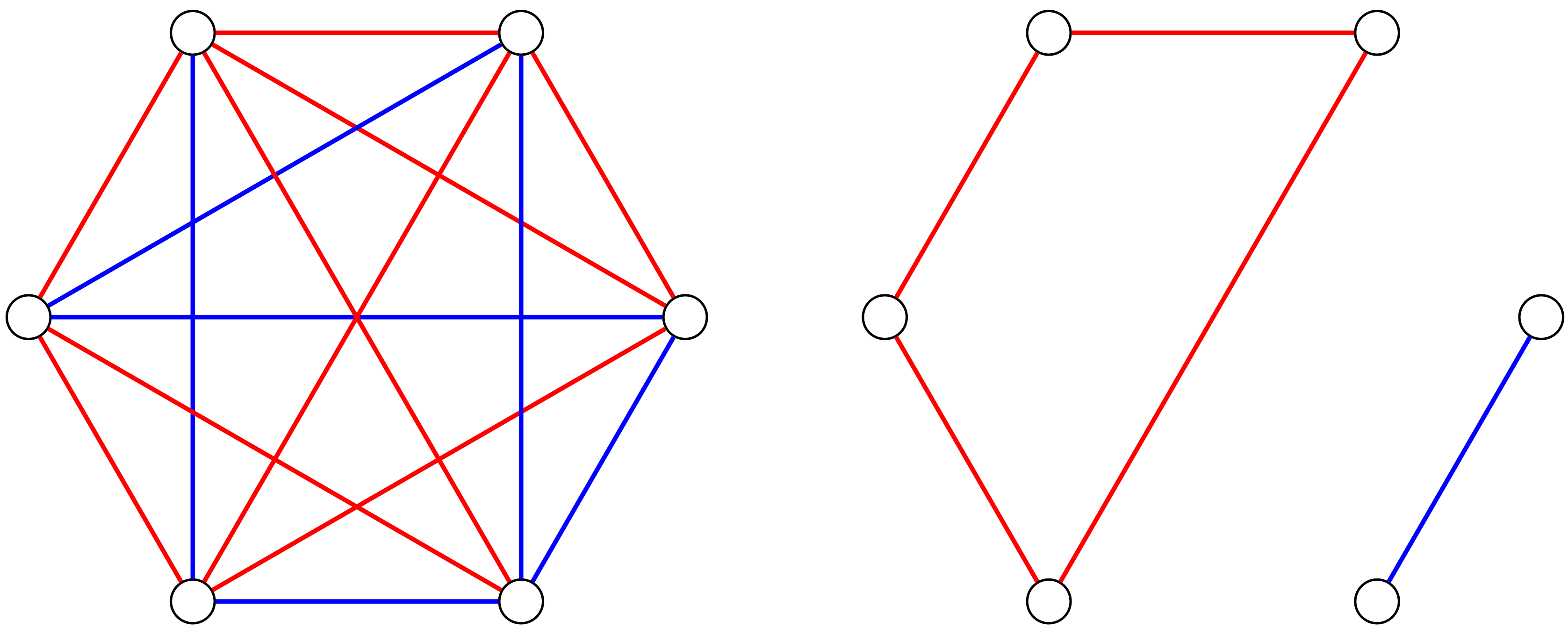
Conjecture (Erdös, Gyárfás and Pyber, 1991)

At most r disjoint monochromatic cycles are needed to partition the vertices of any r-edge-coloured K_n.

Partitioning into paths



Partitioning into cycles



Partitioning into paths

Conjecture (Gyárfás, 1989)

At most r disjoint monochromatic paths are needed to partition the vertices of any r-edge-coloured K_n.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (Gyárfás and Gerencsér, 1967)
  - $\rightarrow R(P_m, P_n) \leq m + n - 3$ for $m, n \geq 2$.

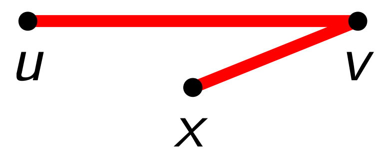
$r = 2$ proof:

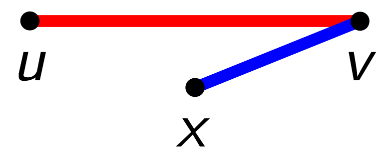
Basic idea:

- Construct a maximal red path followed by a maximal blue path.
- While there are uncovered vertices, extend the path while maintaining the single colour change property.
- Continue until all vertices are covered.

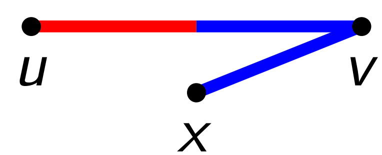
$r = 2$ proof:

Proof:
First suppose that the path is only red. Either vx is red or it is blue.

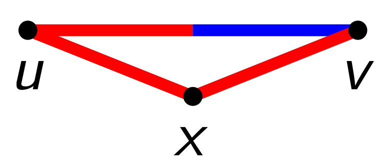


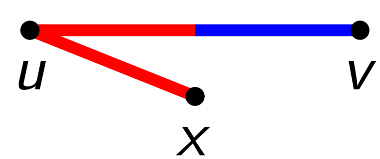


Now suppose that the path has at least one blue edge. First suppose vx is blue.



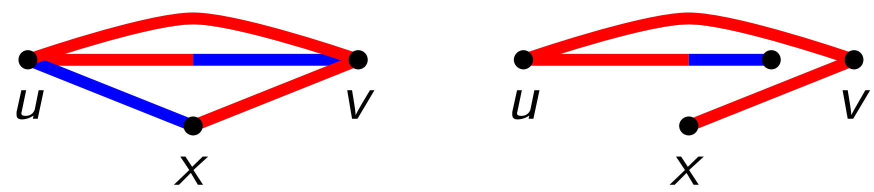
Now suppose vx is red First suppose ux is red.



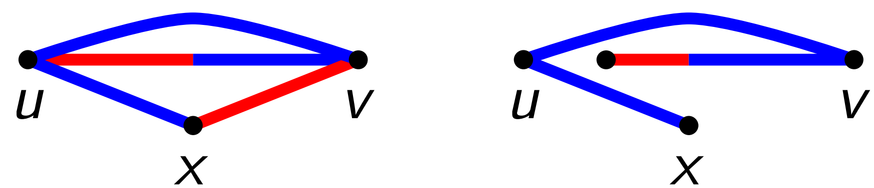


$r = 2$ proof:

Now suppose ux is blue. First suppose uv is red.



Now suppose uv is blue.



Partitioning into paths

Conjecture (Gyárfás, 1989)

At most r disjoint monochromatic paths are needed to partition the vertices of any r-edge-coloured K_n.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (Gyárfás and Gerencsér, 1967)
- $r = 3$ ✓ (Pokrovskiy, 2014)
- $r \geq 4$?

Partitioning into cycles

Conjecture (Erdös, Gyárfás and Pyber, 1991)

At most r disjoint monochromatic cycles are needed to partition the vertices of any r-edge-coloured K_n.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (Bessy and Thomassé, 2010)
- $r \geq 3$ ✗ (Pokrovskiy, 2014)

Conjecture (Pokrovskiy, 2014)

The vertices of any r-edge-coloured K_n can be partitioned into r (not necessarily disjoint) monochromatic cycles.

Conjecture (Pokrovskiy, 2014)

At most r disjoint monochromatic cycles are needed to cover all but c_r of the vertices of any r-edge-coloured K_n.

Partitioning into cycles

Conjecture (Erdös, Gyárfás and Pyber, 1991)

At most r disjoint monochromatic cycles are needed to partition the vertices of any r-edge-coloured K_n.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (Bessy and Thomassé, 2010)
- $r \geq 3$ ✗ (Pokrovskiy, 2014)

Conjecture (Pokrovskiy, 2014)

The vertices of any r-edge-coloured K_n can be partitioned into r (not necessarily disjoint) monochromatic cycles.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (follows from Bessy and Thomassé)
- $r \geq 3$?

Partitioning into cycles

Conjecture (Erdös, Gyárfás and Pyber)

At most r disjoint monochromatic cycles are needed to partition the vertices of any r-edge-coloured K_n.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (Bessy and Thomassé)
- $r \geq 3$ ✗ (Pokrovskiy, 2014)

Conjecture (Pokrovskiy, 2014)

At most r disjoint monochromatic cycles are needed to cover all but c_r of the vertices of any r-edge-coloured K_n.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (follows from Bessy and Thomassé)
- $r = 3$ ✓ $c_3 = 60$ (Letzer, 2016)
- $r \geq 4$?

Partitioning into cycles

Conjecture (Erdös, Gyárfás and Pyber, 1991)

At most r disjoint monochromatic cycles are needed to partition the vertices of any r-edge-coloured K_n.

- $r = 1$ ✓ (trivial)
- $r = 2$ ✓ (Bessy and Thomassé, 2010)
- $r \geq 3$ ✗ (Pokrovskiy, 2014)
  - $r = 3$: All but $o(n)$ vertices *can* be covered by 3 disjoint cycles (Gyárfás et al., 2011).
  - $r = 3$: 10 disjoint cycles cover *all* the vertices (Lang et al., 2015).
- General r:
  - At most $O(r^2 \log r)$ cycles needed (Gyárfás et al., 2006).
  - For large enough n, at most $100r \log r$ cycles needed (Gyárfás et al., 2011).

Partitioning complete bipartite graphs

Conjecture (Erdös, Gyárfás and Pyber, 1991)

Can the vertices of any r-edge-coloured $K_{n,n}$ be partitioned into a number of cycles depending only on r? ✓ (Haxell, 1997)

Conjecture (Pokrovskiy, 2014)

At most $2r-1$ disjoint monochromatic paths are needed to partition the vertices of any r-edge-coloured $K_{n,n}$.

- $r=1$ ✓ (trivial)
- $r=2$ ✓ (Pokrovskiy, 2014)
- $r\geq 3$?
  - $r=3$: At most 1695 cycles needed (Haxell, 1997).
  - $r=3$: At most 5 cycles needed to cover $2n-o(n)$ vertices (Lang et al., 2015).
- General r:
  - At most $O(r^2\log r)$ cycles needed (Peng et al., 2002).

$r = 2$ Algorithmic proof

Theorem (Pokrovskiy), 2014

At most 3 disjoint monochromatic paths are needed to partition the vertices of any 2-edge-coloured $K_{n,n}$.

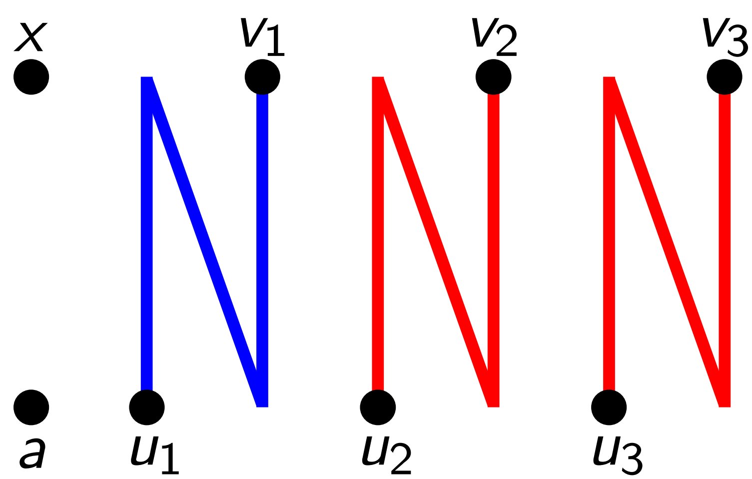
Proof:

Basic idea:

- Construct 3 maximal disjoint monochromatic paths in $K_{n,n}$.
- Consider the induced subgraph $K_{m,m}$ of $K_{n,n}$ formed by the three paths and at least one uncovered vertex.
- We can always find 3 disjoint monochromatic paths covering the vertices on our original paths and at least one additional vertex in $K_{m,m}$.
- Extend new paths so that they are maximal and repeat until all vertices are covered.

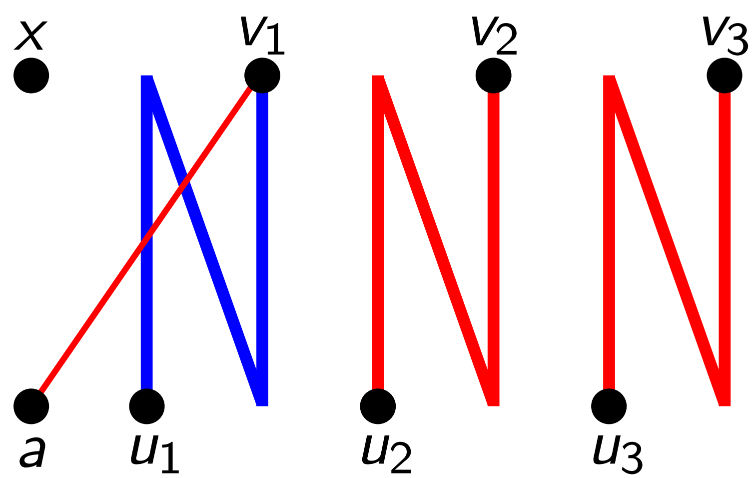
$r = 2$ Algorithmic proof

Denote the three paths by P_1, P_2 and P_3.



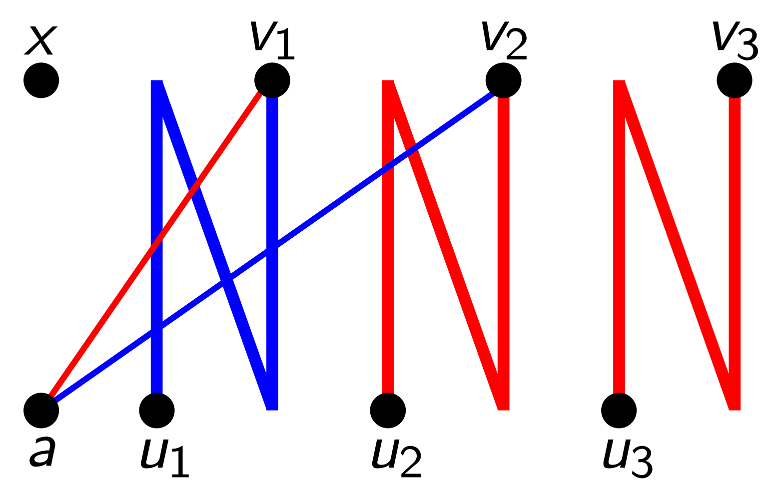
$r = 2$ Algorithmic proof

av_1 must be red (maximality of P_1).



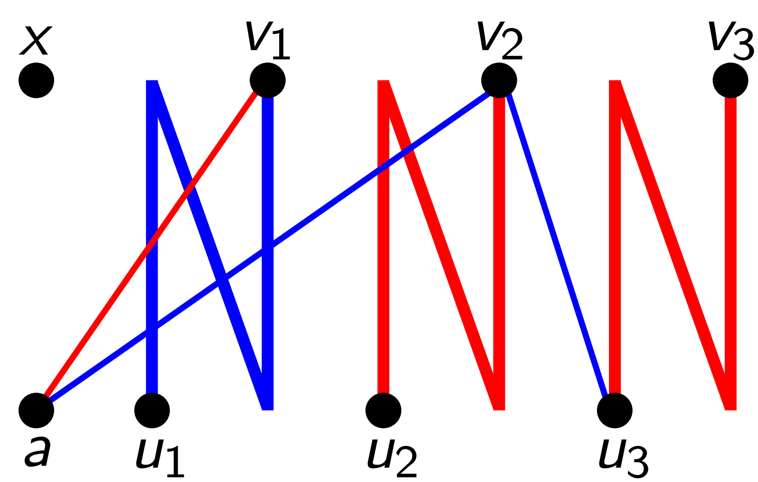
$r = 2$ Algorithmic proof

av_2 must be blue (maximality of P_2).



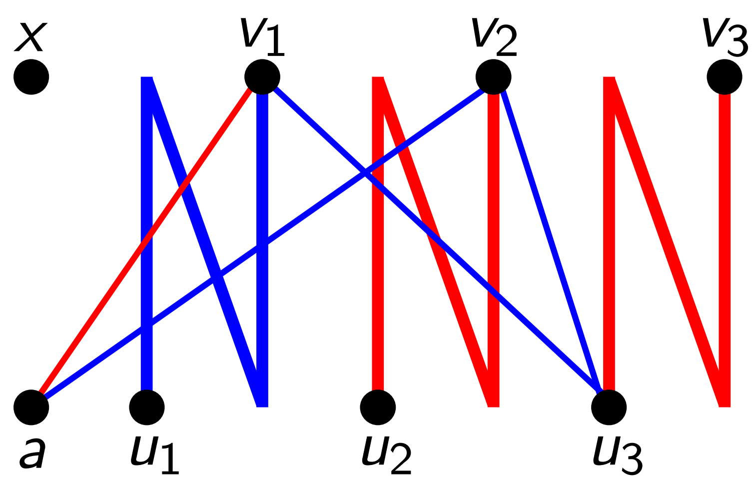
$r = 2$ Algorithmic proof

$v_2 u_3$ must be blue (maximality of P_2 and P_3).



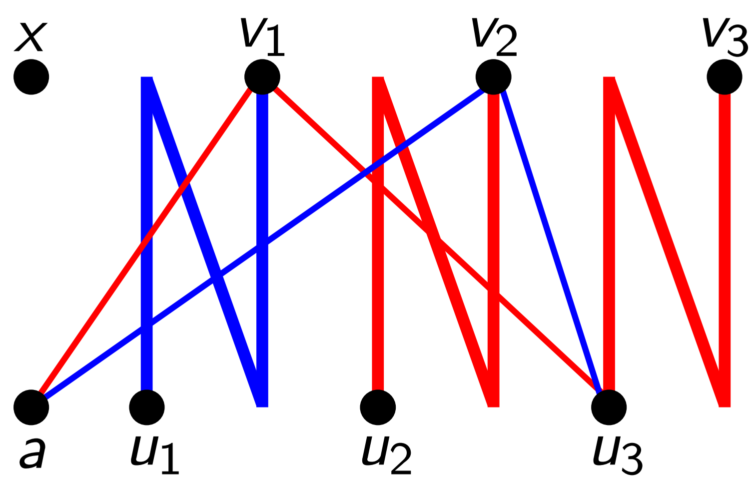
$r = 2$ Algorithmic proof

If $v_1 u_3$ is blue then $P_1 u_3 v_2 a$ is a blue path and we are done.



$r = 2$ Algorithmic proof

So suppose that $v_1 u_3$ is red. Then $P_3 v_1 a$ is a red path, so we are done.



Related open problems

Conjecture (Pokrovskiy, 2014)

The vertices of any r-edge-coloured K_n can be partitioned into r (not necessarily disjoint) monochromatic cycles.

- $r \geq 3$

Conjecture (Pokrovskiy, 2014)

At most r disjoint monochromatic cycles are needed to cover all but c_r of the vertices of any r-edge-coloured K_n.

- $r \geq 4$

- Calculating Ramsey numbers using partitioning results.
- Using partitioning results on $K_{n,n}$ to solve partitioning problems in K_n.

Variants considered so far

- Other kinds of host graph:
  - Bounded min/max degree
  - Bounded independence number
  - Hypergraphs
  - Multipartite graphs
  - Random graphs
  - Infinite graphs
- Not necessarily disjoint subgraphs.
- Partitioning into subgraphs with *distinct* colours.
- Partitioning into other kinds of subgraphs.
  - Powers of cycles
  - Matchings
  - Trees
  - Connected pieces
- Partitioning into more than one kind of subgraph.

More open problems...

Vertex covers by monochromatic pieces - a survey of results and problems (Gyárfás, 2016).