AI for Finance HW5

20210695 Seongje Hong

May 17, 2025

Settings

- Retire in 20 years
- \$550,000 in the retirement account
- Income = \$160,000 $(1.03)^t$
- Saving = \$19,200 $(1.03)^t$
- 4 asset classes – stocks, bonds, commodities, and cash

Part A. Single-period approach

Settings

Data were used from 1991-01-01 to 2012-12-31 (22 years) via yfinance to compute the efficient frontiers.

- Stocks – `^SP500TR` (S&P 500 Total Return Index)
- Bonds – `VBMFX` (Vanguard Total Bond Market Index Fund Investor Shares)
- Commodities – `^SPGSCI` (S&P GSCI)
- Cash – `^IRX` (Treasury Yld Index-13 Wk Tbl)

Note that `VBMFX` is used as a proxy of Barclays Capital Aggregate Bond index.
Also, for the simplicity, assume that returns of each year is independent.
Every optimalization problem is solved by "cvxpy".

You may check my Python work here: `https://github.com/WQGGSEY/AI4Finance_HW5`

Questions

1. Draw the efficient frontiers in the mean-variance plane when she employs the Buy–and–Hold strategy for the whole planning horizon when (a) shorting is not allowed and (b) shorting is allowed.

In this problem, we first fix the initial weight of the portfolio and hold it for the whole investment horizon. In this case, we have following variables.

$$R_p^{bh} = \sum_{i=1}^{4} w_i \prod_{t=1}^{T} (1 + r_{t,i}), \qquad \mu_p = E[R_p^{bh}], \qquad \sigma_p = Var(R_p^{bh})$$

Also, our optimization problem is

$$\min_w Var(R_p^{bh}) \quad \text{s.t.} \quad E[R_p^{bh}] = \mu_p, \quad \mathbf{1}^\top w = 1$$

Before solving the optimization problem to obtain efficient frontiers, consider μ and Σ first. **The assignment explicitly states that all asset-return parameters (μ, Σ) remain *constant* over the 20-year horizon .** Even under this stationarity assumption the *true* (μ, Σ) are unknown; we therefore treat them as random and integrate over their posterior distribution via MCMC. Conditional on each draw, the same μ and Σ are used in *every* simulated year, so the assumption is fully respected.

To deal with it, we must consider the distribution of r_t's and prior distribution of μ and Σ. The most simple way is choosing multivariate normal distribution for μ, $r_t$and inverse-Wishart Distribution for Σ.

$$r_t|\mu, \Sigma \overset{\text{i.i.d}}{\sim} \mathcal{MVN}(\mu, \Sigma), \qquad \mu \sim \mathcal{MVN}(\mu_0, \Lambda_0), \qquad \Sigma \sim \mathcal{IW}(\nu_0, \Lambda_0^{-1})$$

I used $\mu_0 = \mathbf{0}$ (in the sense of mean-reversion). For Λ_0, I choose its diagonal terms as $\Lambda_{0,ii} = 0.609$ satisfying $P(\mu_i > -1.00) = 0.9$ for $i = 1, 2, 3$ (stocks, bonds, commodities). For cash, I give a very small positive value for variance,0.001. Also, historically, it is known that stocks and bonds are negatively correlated, stocks and commodities are positively correlated, and bonds and commodities are also negatively correlated. Since the variance of cash is really small, we can ignore the covariance term of cash. Considering these relations, I decided to use

$$\rho_{stocks,bonds} = -0.3, \qquad \rho_{stocks,commodities} = 0.3, \qquad \rho_{bonds,commodities} = -0.3, \qquad \rho_{*,cash} = 0$$

Finally, we have

$$\Lambda_0 = \begin{bmatrix} 0.609 & -0.183 & 0.183 & 0 \\ -0.183 & 0.609 & -0.183 & 0 \\ 0.183 & -0.183 & 0.609 & 0 \\ 0 & 0 & 0 & 0.001 \end{bmatrix}$$

Also, I choose $\nu_0 = 4 + 2 = 6$ (Low confidence of the prior). Our assumption for the distribution of samples is that $r|\mu, \Sigma \sim \mathcal{MVN}(\mu, \Sigma)$. Therefore, by the conjugacy, we have the posterior distribution,

$$\mu|r, \Sigma \sim \mathcal{MVN}(\mu_T, \Lambda_T), \qquad \Sigma|r, \mu \sim \mathcal{IW}(\nu_T, S_T^{-1})$$

where

$$\mu_T = (\Lambda_0^{-1} + T\Sigma^{-1})^{-1}(\Lambda_0^{-1}\mu_0 + T\Sigma^{-1}\bar{r}), \qquad \Lambda_T = (\Lambda_0^{-1} + T\Sigma^{-1})^{-1}$$

$$\nu_T = \nu_0 + T = 6 + T, \qquad S_T = \Lambda_0 + \sum_{t=1}^{T}(r_t - \mu)(r_t - \mu)^\top$$

From this conditional posterior distribution, 50,000 Gibb's sampling is conducted and means of μ and Σ after 49,000 samplings are computed.

$$\hat{\mu} \approx (0.038, 0.050, 0.113, 0.001)^\top, \qquad \hat{\Sigma} \approx \begin{bmatrix} 0.070 & -0.007 & 0.015 & 0.003 \\ -0.007 & 0.033 & -0.012 & 0.001 \\ 0.015 & -0.012 & 0.098 & -0.001 \\ 0.003 & 0.001 & -0.001 & 0.001 \end{bmatrix}$$

Considering its the shrinkage plot, Gelman-Rubin statistic, and ESS, we can consider that μ and Σ converge to true parameters sufficiently.

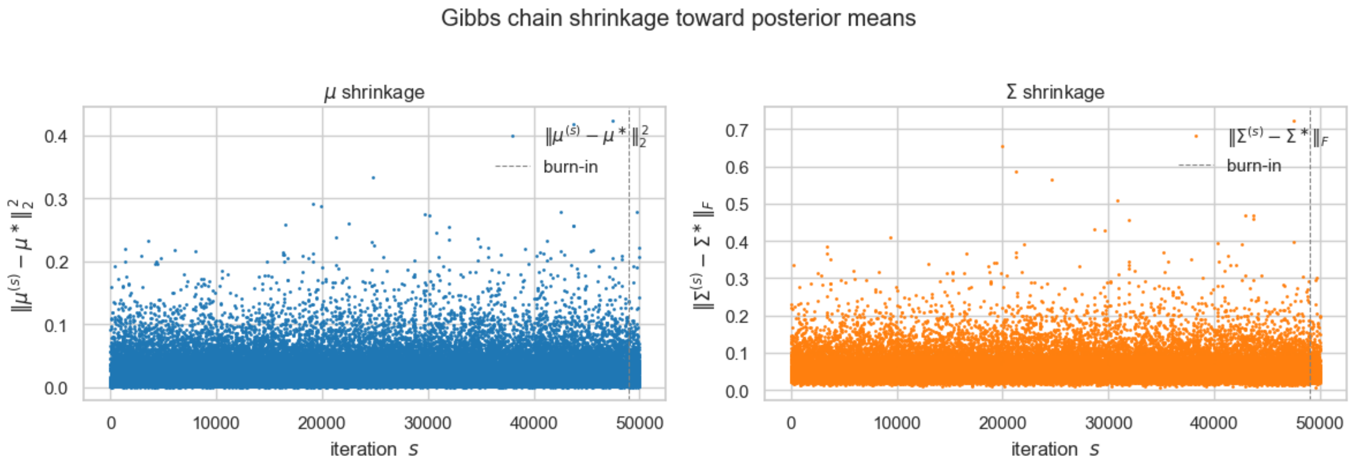


Figure 1: Gibbs' Chain shrinkage toward posterior means after 49,000 samplings

Table 1: Posterior–sampling diagnostics (one split–chain $\hat{R}$, ESS per scalar)

Metric	Observed range	Rule of thumb	Interpretation
$\hat{R}$ (bulk)[1]	1.00 – 1.01	≤ 1.01	Chains have mixed; convergence OK
ESS_{bulk}	8.6×10^3 – 4.8×10^4	> 400	Mean/variance estimates are precise
ESS_{tail}	1.6×10^4 – 5.0×10^4	> 100	5 % / 95 % tail quantiles reliable

Now, we will obtain efficient frontiers with $\hat{\mu}$ and $\hat{\Sigma}$. By the definition of each variable,

$$\mu_p = E[R_p^{bh}] = \sum_{i=1}^{4} w_i E\Big[\prod_{t=1}^{T}(1+r_{t,i})\Big] = \sum_{i=1}^{4} w_i(1+\mu_i)^T = w^\top a \quad (\because \text{Independency by the assumption})$$

$$Var(R_p^{bh}) = E[(R_p^{bh})^2] - (E[R_p^{bh}])^2 = w^\top(P - aa^T)w$$

where

$$P_{ij} := (1 + \mu_i + \mu_j + \Sigma_{ij} + \mu_i\mu_j)^T, \qquad a_i := (1+\mu_i)^T, \qquad 1 \leq i,j \leq 4$$

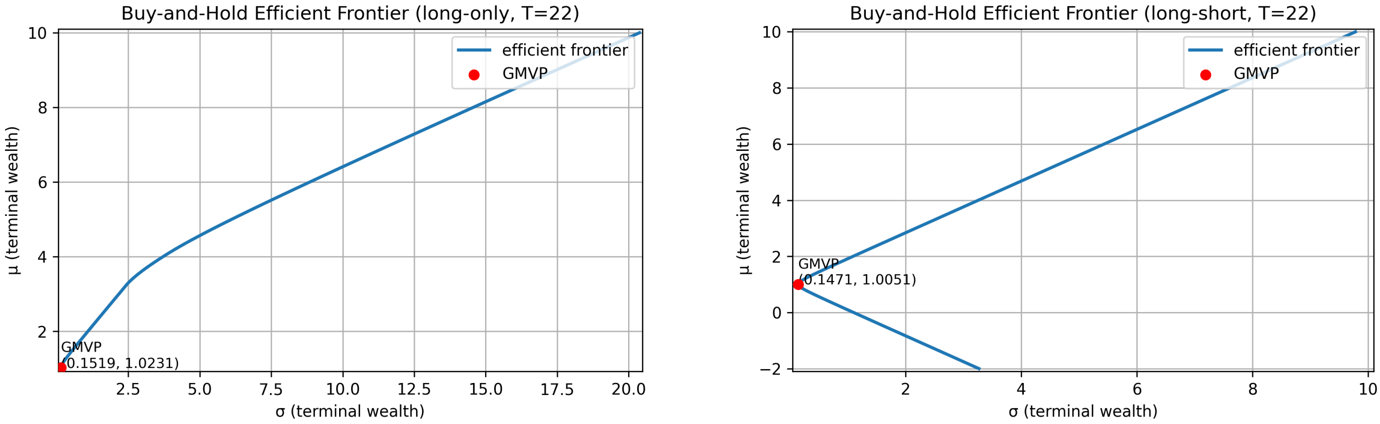
By putting $\mu = \hat{\mu}$ and $\Sigma = \hat{\Sigma}$,

$$K := P - aa^\top = \begin{bmatrix} 17.477 & -1.029 & 8.526 & 0.149 \\ -1.028 & 8.528 & -6.228 & 0.039 \\ 8.526 & -6.228 & 492.194 & -0.235 \\ 0.149 & 0.039 & -0.235 & 0.028 \end{bmatrix}, \qquad eigenvalues(K) = \begin{bmatrix} 492.428 & 17.416 & 8.087 & 0.028 \end{bmatrix}^\top$$

Since K is positive definite, we can get the optimal solution by solving $A\mathbf{x} = C$ where

$$A = \left[\begin{array}{c:c:c} \mathbf{1}^\top & & \\ \hdashline a^\top & & \\ \hdashline K & -\mathbf{1} & -a \end{array}\right] \in \mathbb{R}^{6\times6}, \qquad C = \begin{bmatrix} 1 & \mu_p & 0 & 0 & 0 & 0 \end{bmatrix}^\top.$$

For each feasible μ_p, solve $A\mathbf{x} = C$; the first four components of $\mathbf{x}$ give the efficient-frontier weights. We can obtain efficient frontiers without shorting by adding a constraint $w \geq 0$



(a) No shorting allowed

(b) Shorting allowed

Figure 2: Buy–and–Hold Efficient frontiers of the four-asset portfolio with and without shorting

2. Draw the efficient frontiers in the mean-variance plane when she employs the annual rebalancing strategies when a) shorting is not allowed and b) shorting is allowed.

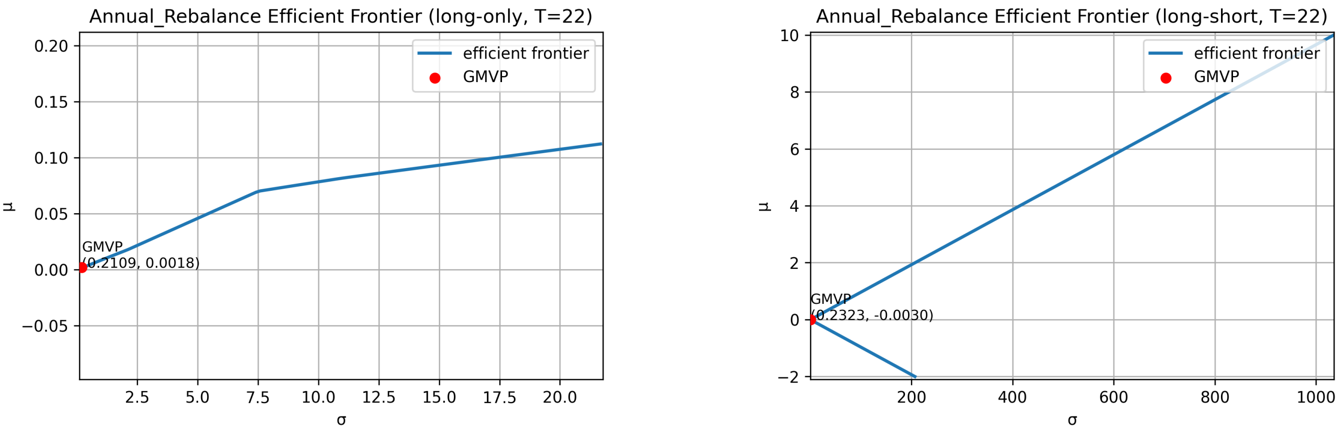
In this problem, we rebalance the portfolio every year. In this case, for annual expected return, μ we have

$$R_p^{reb} = \sum_{t=1}^{T} w^\top \mu, \qquad \mu_p = E[R_p^{reb}], \qquad \sigma_p = Var(R_p^{reb})$$

Therefore, efficient frontier of this problem is equivalent to single period Markowitz efficient frontiers. When exploring the efficient frontiers in 2013, we assume that the historical *annualized* average return of each asset over the past three years (2010–2013) is the expected return. From Q1, we have $\hat{\mu}$ and $\hat{\Sigma}$. Let

$$A = \left[\begin{array}{c:c:c} \mathbf{1}^\top & & \\ \hdashline \hat{\mu}^\top & & \\ \hdashline \hat{\Sigma} & -\mathbf{1} & -\hat{\mu} \end{array}\right] \in \mathbb{R}^{6\times 6}, \qquad C = \begin{bmatrix} 1 & \mu_p & 0 & 0 & 0 & 0 \end{bmatrix}^\top.$$

For each feasible μ_p, solve $A\mathbf{x} = C$; the first four components of $\mathbf{x}$ give the efficient-frontier weights. We can obtain efficient frontiers without shorting by adding a constraint $w \geq 0$



(a) No shorting allowed

(b) Shorting allowed

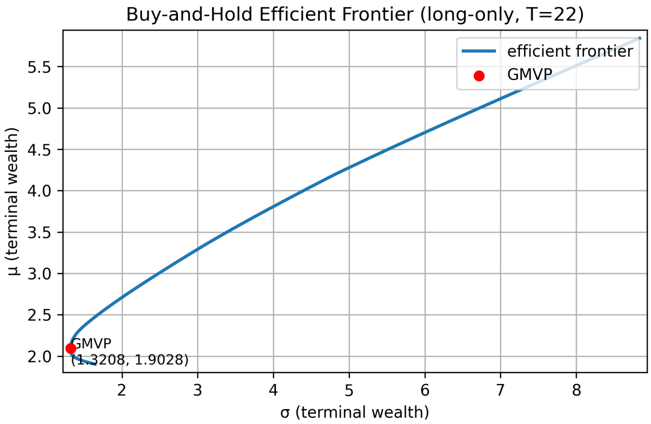
Figure 3: Annual Rebalance Efficient frontiers of the four-asset portfolio with and without shorting

3. For the problems 1 and 2, draw the efficient frontiers when the maximum weight on a single asset is 40% and the maximum value of the short position on a single asset is 25%.

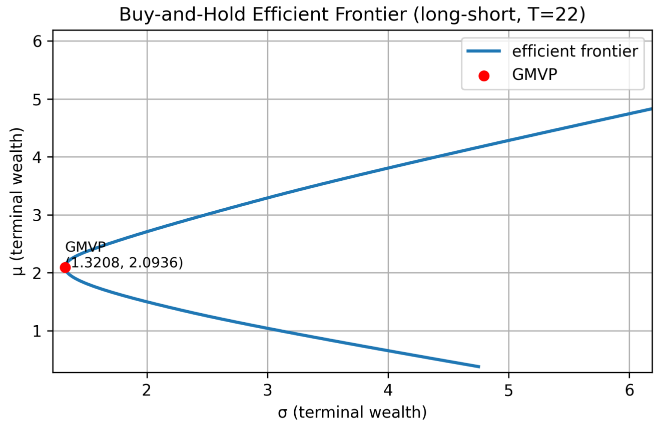
Now, we have a new restriction which is

$$-0.25 \leq w_i \leq 0.4, \qquad 1 \leq i \leq 4$$

Since the restriction is linear, similar to Q1 and Q2, we can construct the Lagrangian solvable with linear equations.

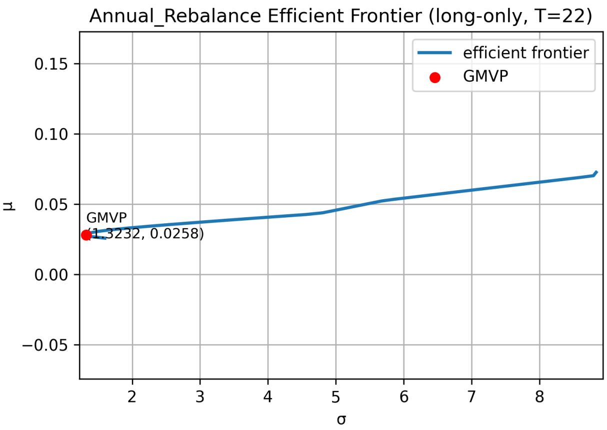


(a) No shorting allowed

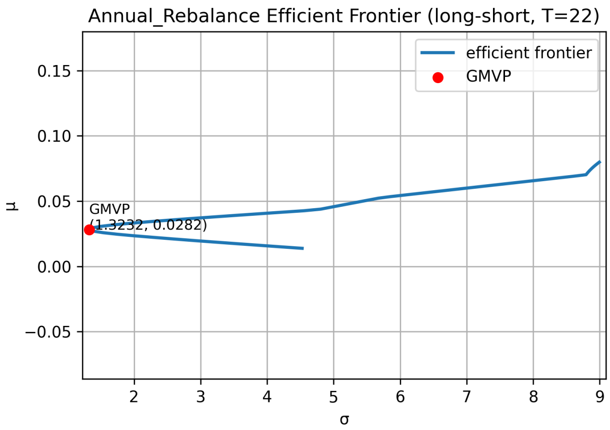


(b) Shorting allowed

Figure 4: Constrained Buy–and–Hold Efficient frontiers of the four-asset portfolio with and without shorting



(a) No shorting allowed



(b) Shorting allowed

Figure 5: Constrained Annual Rebalance Efficient frontiers of the four-asset portfolio with and without shorting

4. Pick various points on the efficient frontiers in problems 1 – 3. What's the probability that the investor will be able to retire with a reasonable level of prosperity for these strategies? Assume that reasonable level is defined as 80% of the average of her last five years of salary. To simplify the analysis, we assume that a 65 year old retiree can spend 5% of her endowment per year. Hence, if she wants to spend \$200,000 per year, she will need to have saved \$4,000,000 by age 65.

Her reasonable level of prospsperity is computed as

$$0.8 \times \frac{1}{5} \sum_{t=1}^{5} (1.03)^{15+t} \times \$160,000 = \$218,102$$

Her reasonable level of prosperity is \$144,191 and she will need to have saved $S := \$218,102/0.05 = \$4,362,037$ by her age 65. Since we assumed that $r_t|\mu, \Sigma \overset{\text{i.i.d}}{\sim} \mathcal{MVN}(\mu, \Sigma)$, we will do MCMC with some efficient frontiers and find the efficient frontier which gives the greatest $P(W_p \geq S)$ where W_p is the random variable representing her total savings in her investment horizon.

Consider Buy–and–Hold Strategy. Recall that the total return for her investment horizon, T years, is $R_p^{bh} = \sum_{i=1}^4 w_i \prod_{j=1}^T (1 + r_{j,i})$. Her saving for each year is $S_t = \$19,200 \times (1.03)^t$. Considering her initial investment $V_0 = \$550,000$, assume that she will invest her savings with the initial portfolio weight w. Then, we have

$$W_p^{bh} = V_0 \sum_{i=1}^{4} w_i \prod_{j=1}^{20} (1 + r_{j,i}) + \sum_{t=1}^{20} S_t \sum_{i=1}^{4} w_i \prod_{j=t}^{20} (1 + r_{j,i})$$

Similarly, for annual rebalance strategy, we have

$$W_p^{reb} = V_0 \prod_{j=1}^{20} (1 + w^\top r_j) + \sum_{t=1}^{20} S_t \prod_{j=t}^{20} (1 + w^\top r_j)$$

Now, we will use MCMC to compute $P(W_p \geq S)$ for each strategy, because we expect that as $n \to \infty$, $\frac{1}{n} \sum_{k=1}^{n} 1(W_p \geq S) \to P(W_p \geq S)$.

Table 2: Efficient Frontiers giving the greatest $P(W_p \geq S)$

Strategy	$\max_w(P(W_p \geq S))$	w_1	w_2	w_3	w_4
Buy–and–Hold Long Only	0.4660	0.0000	0.3886	0.6114	0.0000
Buy–and–Hold Long–Short	0.7051	0.9529	2.9333	0.2315	-3.1177
Annual Rebalance Long Only	0.4538	0.0000	0.0792	0.9208	0.0000
Annual Rebalance Long–Short	0.6177	0.4350	1.6117	0.9638	-2.0106
Constrained Buy–and–Hold Long Only	0.4629	0.2009	0.4000	0.3991	0.0000
Constrained Buy–and–Hold Long–Short	0.4977	0.4000	0.4000	0.4000	-0.2000
Constrained Annual Rebalance Long Only	0.3330	0.1786	0.4000	0.4000	0.0214
Constrained Annual Rebalance Long–Short	0.3861	0.3948	0.4000	0.4000	-0.1948

5. What if the investor increases her annual saving to 15% of her salary? What if she waits to age 69 to retire at 70% of final salary? Here you can assume that the investor can spend 7% of assets per year. Show the impact of these changes.

In this condition, her reasonable level of prosperity is computed as

$$0.7 \times \$160,000 \times (1.03)^{24} = \$227,673$$

From this prosperity, she will need to save $S := \$227,673/0.07 = \$3,252,471$ by her age 69. Similar to problem 4, by doing MCMC for each strategy, we have the following table.

Table 3: Efficient Frontiers giving the greatest $P(W_p \geq S)$ with new conditions

Strategy	$\max_w(P(W_p \geq S))$	w_1	w_2	w_3	w_4
Buy–And–Hold Long Only	0.6849	0.0000	0.5426	0.4574	0.0000
Buy–And–Hold Long–Short	0.8245	0.9526	2.9323	0.2315	-3.1163
Annual Rebalance Long Only	0.6339	0.0000	0.4126	0.5874	0.0000
Annual Rebalance Long–Short	0.7197	0.3275	1.2436	0.7497	-1.3208
Constrained Buy–And–Hold Long Only	0.7066	0.2037	0.4000	0.3963	0.0000
Constrained Buy–And–Hold Long–Short	0.7265	0.4000	0.4000	0.4000	-0.2000
Constrained Annual Rebalance Long Only	0.6011	0.1786	0.4000	0.4000	0.0214
Constrained Annual Rebalance Long–Short	0.6108	0.3408	0.4000	0.4000	-0.1408

We can see that the probability that she can achieve her target wealth S is increased.

6. Suggest alternative risk measures to the one mentioned above (probability of meeting a goal). Calculate these measures as appropriate for your recommendations.

VaR(Value at Risk) and CVaR(Conditional Value at Risk) can be considered as alternative risk measures to $P(W_p \geq S)$. With the Gibb's sampling, we have

$$W_p^{(k)}|w, \mathcal{D} \sim p(W_p|w, \mathcal{D}) = \int p(W_p|r, w)p(r|\mu, \Sigma)drd\mu d\Sigma$$

Based on this probability measure, we can compute VaR and CVaR with confidence level $1-\alpha$ and data $\mathcal{D}$ as

$$\text{VaR}_\alpha(w) := \inf\{x \in \mathbb{R} : P(W_p \leq x|\mathcal{D}) \geq \alpha\} = F^{-1}_{W_p|\mathcal{D}}(\alpha), \qquad \text{CVaR}_\alpha(w) := E[W_p(w)|W_p(w) \leq Var_\alpha(w), \mathcal{D}]$$

In this context, minimizing VaR and CVaR will give the best efficient frontiers for her.

Table 4: The best VaR and CVaR at the 95% confidence level

Strategy	$\text{VaR}_{5\%}$(\$)	$\text{CVaR}_{5\%}$(\$)	w_1	w_2	w_3	w_4
Buy–And–Hold Long Only	-1,243,683	-1,084,044	0.0234	0.0935	0.0084	0.8748
Buy–And–Hold Long–Short	-1,243,683	-1,084,044	0.0234	0.0935	0.0084	0.8748
Annual Rebalance Long Only	-1,200,944	-1,104,266	0.0000	0.0584	0.0548	0.8868
Annual Rebalance Long–Short	-1,191,697	-1,086,609	-0.0161	0.0670	0.0653	0.8837
Constrained Buy–And–Hold Long Only	-1,131,798	-575,189	0.1613	0.4000	0.0421	0.3967
Constrained Buy–And–Hold Long–Short	-1,131,798	-575,189	0.1613	0.4000	0.0421	0.3967
Constrained Annual Rebalance Long Only	-1,042,243	-819,438	0.0827	0.3333	0.1841	0.4000
Constrained Annual Rebalance Long–Short	-1,042,243	-819,438	0.0827	0.3333	0.1841	0.4000

Table 5: Goal–probability vs. tail–risk for the eight candidate strategies

Strategy	$P(W_p \geq S)$	Tail loss (\$)		Rank	
		$\text{VaR}_{5\%}$	$\text{CVaR}_{5\%}$	Goal	Tail
Buy–and–Hold Long–Short	0.8245	-1,243,683	-1,084,044	1	5
Constrained Buy–and–Hold Long–Short	0.7265	-1,131,798	-575,189	2	1
Rebalance Long–Short	0.7197	-1,191,697	-1,086,609	3	7
Constrained Buy–and–Hold Long–Only	0.7066	-1,131,798	-575,189	4	1
Buy–and–Hold Long–Only	0.6849	-1,243,683	-1,084,044	5	5
Rebalance Long–Only	0.6339	-1,200,944	-1,104,266	6	8
Constrained Rebalance Long–Short	0.6108	-1,042,243	-819,438	7	3
Constrained Rebalance Long–Only	0.6011	-1,042,243	-819,438	8	3

7. Clearly, the assumption that the parameters for the asset returns are the same throughout the whole period is flawed. Since this approach uses the in-sample testing, the results are not necessarily representative for the future performance. One possible fix for this issue is to solve the portfolio optimization problem every year using the data available up to that point. Draw the efficient frontiers using this approach. Is this efficient frontier better than the previous ones? If so, what is the reason? If no, can you propose an alternative approach for improve the performance? You have to describe the data you employed carefully as well.

The sequential Bayesian frontier is obtained by re-estimating (μ_t, Σ_t) every January ($t = 2013, \dots, 2025$) with an expanding-window Normal–Inverse-Wishart posterior and then solving the appropriate mean–variance programme for each target one-year return μ_{target}. All four strategy cases are treated:

Buy-and-Hold, long-only	Buy-and-Hold, short allowed
Annual Rebalance, long-only	Annual Rebalance, short allowed

Key steps (all algebraic, so no simulation is required):

1. *Posterior update.* With n returns observed the NIW mean is
$$\mu_n = \Lambda_n\big(\Lambda_0^{-1}\mu_0 + n\Sigma_n^{-1}\bar{r}\big), \qquad \Sigma_n \approx S_{\text{emp}}(n),$$
where $\Lambda_n^{-1} = \Lambda_0^{-1} + n\Sigma_n^{-1}$ and S_{emp} is the sample covariance. The running sums $\sum r_t$ and $\sum r_t r_t^\top$ make the update $\mathcal{O}(1)$ per year.

2. *Optimal weights.*
$$\begin{aligned} &\min_w \; w^\top \Sigma_n w, \quad \text{s.t. } \mu_n^\top w \ge \mu_{\text{target}}, \; \mathbf{1}^\top w = 1 \quad \text{(annual rebalance)},\\ &\min_w \; w^\top K_n w, \quad \text{s.t. } a_n^\top w = \mu_{\text{target}}, \; \mathbf{1}^\top w = 1 \quad \text{(buy-and-hold)}, \end{aligned}$$
with $a_n = (1+\mu_n)^{20}$ and $K_n = P_n - a_n a_n^\top$, $P_n = (1 + \mu_n \mathbf{1}^\top + \mathbf{1}\mu_n^\top + \Sigma_n + \mu_n\mu_n^\top)^{20}$. When shorting is allowed both problems have a closed-form Lagrange solution; for long-only the negative coordinates are clipped to 0 and renormalised. Buy-and-hold stores the $t = 2013$ weight for all subsequent years, whereas annual rebalance recomputes every January.

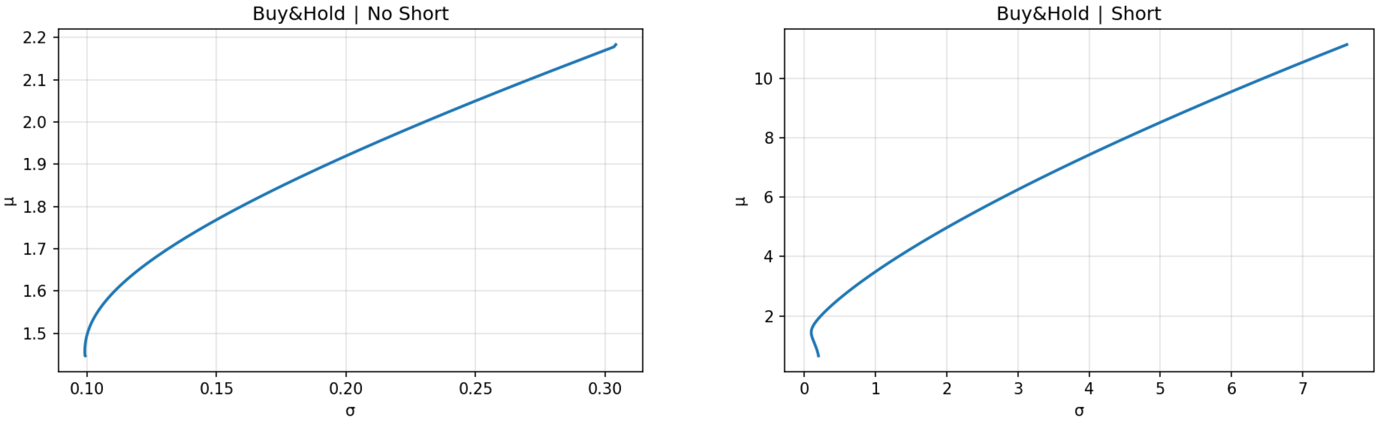
3. *Exact terminal-wealth moments.* Fixing the weight sequence $w_{2013:2025}$ gives $1 + w_t^\top r_t \sim \mathcal{LN}(m_t, s_t^2)$ with $m_t = w_t^\top \mu_*$ and $s_t^2 = w_t^\top \Sigma_* w_t$. Hence the product $W_T = \prod_{t=1}^{13}(1 + w_t^\top r_t)$ is log-normal with
$$\mu_{\log} = \sum_{t=1}^{13}\big[\ln(1+m_t) - \tfrac{1}{2}s_t^2/(1+m_t)^2\big], \quad \sigma_{\log}^2 = \sum_{t=1}^{13}\ln\big(1 + s_t^2/(1+m_t)^2\big).$$
Therefore $\mathbb{E}[W_T] = \exp(\mu_{\log} + 0.5\sigma_{\log}^2)$ and $\sigma_T = \sqrt{(e^{\sigma_{\log}^2} - 1)\, e^{2\mu_{\log} + \sigma_{\log}^2}}$, giving a noise-free $(\sigma_T, \mathbb{E}[W_T])$ pair for every μ_{target}.

4. *Frontier construction.* Repeating steps 1–3 on a lot of $\mu_p \in [-2\%, 10\%]$ grid points yields the four curves plotted in Figure 6 and Figure 7 Runtime is below 4 seconds

Sequentially re-estimating (μ_t, Σ_t) each January eliminates look-ahead bias, because every weight decision uses only information available at that date. As fresh returns arrive, the posterior mean incorporates regime shifts and drift in asset-class Sharpe ratios, so the portfolio automatically tilts toward assets whose expected risk-adjusted performance is rising. The expanding window also shrinks parameter uncertainty over time, reducing estimation error and narrowing the frontier's volatility for later years. Consequently, the resulting frontier better reflects the performance an investor could have achieved in real time, rather than an idealized "full-sample" outcome.

Why Monte–Carlo was omitted.

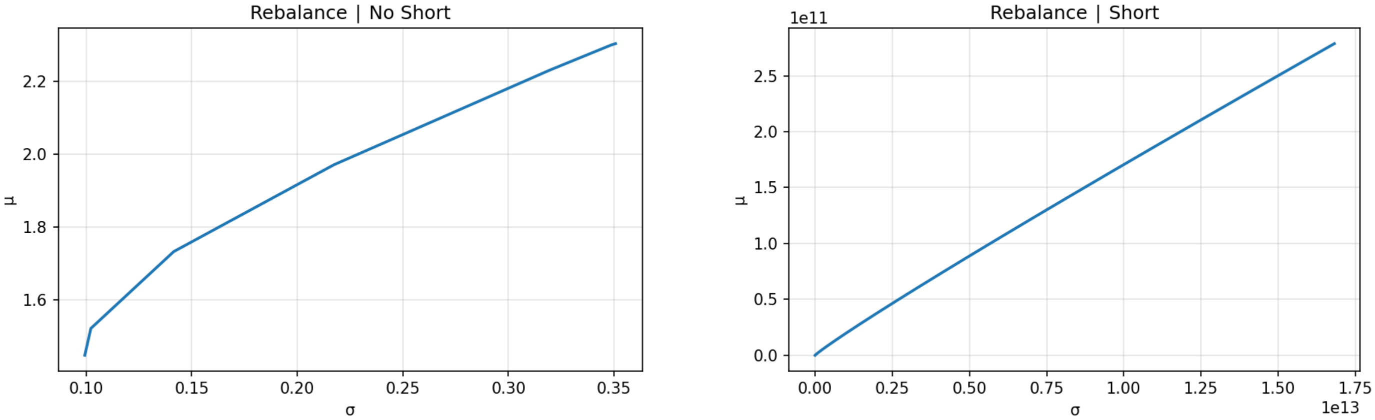
A single frontier point under the original recipe needs $N_{\text{path}} \times T \times S_{\text{draw}}$ quadratic solves. With eleven grid points, $N_{\text{path}} = 5{,}000$, $T = 13$, $S_{\text{draw}} = 100$, the total is $6{,}500{,}000$ QP calls, requiring a lot of time to draw efficient frontiers.



(a) No shorting allowed

(b) Shorting allowed

Figure 6: Portfolio optimization every year with Buy–and–Hold Strategy



(a) No shorting allowed

(b) Shorting allowed

Figure 7: Portfolio optimization every year with Annual Rebalance

8. Provide a carefully prepared summary report with recommendations (2-3 pages) to the investor. Put your analysis, detailed charts, and other background items in the appendix. You will be graded on the conciseness, readability, and thoughtfulness of this report.

I analyzed a range of portfolio strategies for a 20-year retirement investment horizon under various assumptions. The investor's current retirement account is $550,000, with an annual contribution of $19,200 (growing 3% per year) out of a $160,000 salary. Four asset classes were considered – stocks, bonds, commodities, and cash – and I evaluated strategies including a buy-and-hold approach (fixed initial weights) versus annual rebalancing (yearly optimizing weights), each examined both with and without short-selling (leverage). Additionally, I analyzed the impact of imposing a 40% cap on any single asset (and at most 25% short per asset) to limit concentration risk. All analyses assume asset returns follow a stationary distribution estimated from 22 years of historical data, acknowledging that this in-sample view may be optimistic. My goal was to identify the efficient risk–return tradeoffs and recommend a strategy that maximizes the probability of a successful retirement outcome while controlling downside risk.

Risk–Return Tradeoffs and Efficient Frontiers: In mean-variance terms, each strategy produces an efficient frontier of optimal portfolios. As shown in Figure 2 (Appendix), the buy-and-hold strategy yields a classic risk–return tradeoff curve. With short-selling disallowed, the frontier is bounded – expected annualized portfolio returns peak around the high single digits. Allowing short positions (leverage) extends the frontier toward substantially higher returns (and volatility): for example, the maximal point on the long–short frontier entails leveraging the portfolio (borrowing cash at the risk-free rate) to invest more in higher-return assets. Figure 3 (Appendix) illustrates the annual rebalancing strategy's frontier, which under these stable-return assumptions is very similar in shape to buy-and-hold. Notably, rebalancing does not significantly improve the efficient frontier over buy-and-hold when asset return characteristics are constant; both offer comparable sets of attainable risk–return combinations. The primary differences come from leverage and concentration. For instance, without leverage, the highest-return efficient portfolio allocates heavily to commodities (the asset with the greatest historical mean return) – in the long-only annual rebalanced case this was over 90% in commodities (Table 2, Appendix). In contrast, with leverage permitted, the optimal portfolios include borrowing to amplify exposure to assets with favorable risk-reward profiles (e.g. a buy-and-hold long–short solution invested roughly 293% in bonds and 23% in commodities, financed by a -316% position in cash, i.e. borrowing $\tilde{3}$x the portfolio). These extreme allocations achieve higher expected returns but at the cost of much higher risk. Imposing the 40% per-asset cap dramatically changes the frontier (Figures 4 and 5 in Appendix): the efficient set of constrained portfolios is narrower, reflecting how diversification and position limits reduce both risk and return potential. With the constraint, no single asset can dominate the allocation – the optimal constrained portfolios tend to spread ∼40% each into the top-performing asset classes (bonds and commodities in our analysis) and, if leverage is allowed, borrow only modestly (up to 20% of the portfolio) to invest the remainder across assets. Consequently, the gap between the no-short and short-allowed frontiers shrinks under the constraints, since extreme leverage or concentration is no longer possible.

Probability of Meeting Retirement Goal: Beyond abstract risk–return, I assessed each strategy's effectiveness in achieving the investor's retirement goal – defined as accumulating sufficient wealth to fund 70% of final salary spending (this corresponds to a target nest egg S). Using Monte Carlo simulations (incorporating uncertainty in expected returns and covariances via Bayesian posterior sampling), I estimated the probability $P(W_p \geq S)$ that the portfolio's ending wealth meets or exceeds the required target. The results are summarized in Table 2 (Appendix) for the base scenario (retire in 20 years at 65 with current savings and contributions as given). Unconstrained leverage boosts success probability significantly: the buy-and-hold long–short strategy achieved about a 70.5% chance of reaching the goal, compared to only $\tilde{4}$6.6% for its long-only counterpart. The annual rebalancing strategy showed a similar pattern – allowing shorting raised the success probability from $\tilde{4}$5.4% to $\tilde{6}$1.8%. Interestingly, the buy-and-hold approach slightly outperformed annual rebalancing in these goal-based simulations. For example, with no leverage both strategies were around 45–47% successful, but with leverage the buy-and-hold's 70.5% success rate exceeded the rebalancing strategy's 61.8% (Table 2). This suggests that, under static conditions, simply holding an optimal portfolio may be as effective as frequent rebalancing, and in our case buy-and-hold even yielded a higher chance of hitting the target. The intuition is that rebalancing locks in a target mix (here heavily weighted to commodities, the highest-return asset, at about 92% weight), whereas a buy-and-hold portfolio can benefit from compounding if the high-return asset performs well (its weight growing over time). In our historical-based scenario, commodities earned high returns, so the non-rebalanced portfolio enjoyed slightly better growth. However, the trade-off is that rebalancing would provide more stability if relative performance shifts; in this analysis with fixed parameters, that advantage did not materialize.

When constraints are applied, the probability of success naturally drops for the previously aggressive strategies. With a 40% cap per asset, the investor can no longer put, say, nearly all her funds in commodities or lever up 3-to-1, which reduces the upside. Table 2 shows that the unconstrained buy-and-hold (long–short) portfolio's $\tilde{7}$0.5% success probability falls to ∼49.8% under the constrained long–short rule. Similarly, the rebalancing long–short strategy drops from ∼61.8% to ∼38.6% when the 40% cap is imposed. The constrained long-only strategies hover in the mid-40% range of success, similar to the unconstrained long-only case. In effect, constraints level the playing field between strategies by preventing extremely high-return but high-risk allocations. For instance, the optimal unconstrained rebalanced portfolio (long-only) was 92% in commodities, whereas the constrained version was forced into a more balanced mix (roughly 40% commodities, 40% bonds, 18% stocks, 2% cash) – as a result, its success probability dropped from 45.4% to 33.3% (Table 2). While this sacrifice in goal attainment is significant, the benefit is a more stable and diversified portfolio.

I also examined how changes in the investor's saving behavior and retirement timing would affect outcomes. Increasing annual savings to 15% of salary (from the current ∼12%) and postponing retirement by 4 years (to age 69) dramatically improves the probabilities for all strategies (Table 3, Appendix). Under these revised assumptions, the target wealth S is higher, but the additional contributions and longer compounding period raise the chances of success substantially. For example, the buy-and-hold long–only strategy's success probability jumps from $\tilde{4}$6.6% to $\tilde{6}$8.5%, and the long–short leveraged strategy climbs from 70.5% to 82.5%. Even the most conservative constrained rebalancing portfolio improves to around 60% likelihood of reaching the goal (up from $\tilde{3}$3%). This highlights that investor-controlled factors (saving more and working longer) can markedly enhance retirement security, often with less risk than purely chasing higher portfolio returns. I would advise the investor to consider these levers in conjunction with the portfolio strategy.

Risk of Shortfall and Tail Risk Measures: Achieving the highest probability of success is not the only concern – the severity of failure in adverse scenarios must also be weighed. I analyzed two tail risk metrics: Value-at-Risk and Conditional VaR at the 95% confidence level, which estimate the threshold and average of the worst 5% of outcomes, respectively. The findings (Tables 4 and 5 in Appendix) reveal a classic risk–return trade-off across the strategies. The very strategy that maximized goal probability – the unconstrained leveraged buy-and-hold – fared poorly in tail risk: its 5% VaR indicates a shortfall of roughly $1.24m̃illion (meaning in the worst 5% cases, the portfolio ended about $1.24M below the target) and a CVaR (average shortfall in that 5%) of $1.08M. In contrast, a constrained buy-and-hold portfolio had far smaller tail losses. Notably, the constrained buy-and-hold (long–short or even long-only) achieved a VaR of about -$1.13M and an average shortfall of only around -$0.58M, roughly half the magnitude of the unconstrained strategy's worst-case loss. In fact, the constrained buy-and-hold strategies provided the best downside protection of all options (ranked #1 in tail risk, see Table 5) while still maintaining respectable success probabilities (50–72%, ranked near the top). On the other hand, the annual rebalancing strategies did not show better tail performance than buy-and-hold in most cases. For instance, the long-only rebalancing portfolio actually had the highest CVaR (worst average loss $1.10M) among the set. The absolute best tail-risk profile was delivered by the constrained portfolios (both buy-and-hold and rebalanced) with their diversified 40/40/40/-20 allocation; these exhibited the smallest 5% losses ($1.04M VaR and $0.82M CVaR for the constrained rebalanced, and even less CVaR for constrained buy-and-hold as noted). However, those same constrained strategies had among the lowest probabilities of reaching the goal when unconstrained alternatives exist. This underscores an important point: aggressive portfolios enhance the chance of meeting the retirement target, but at the cost of potentially severe shortfalls if things go wrong. Conversely, conservative constrained portfolios protect against disastrous outcomes but raise the risk of falling short of the goal in ordinary circumstances.

Recommendation: In light of these findings, I recommend a balanced, constrained buy-and-hold strategy for this investor's retirement portfolio. This approach entails setting an initial allocation with broad diversification (for example, roughly 40% in stocks, 40% in bonds, 20% in commodities, under the given constraints) and then holding these positions for the long term rather than frequent trading. Such a portfolio can be modestly adjusted over time if needed (especially if market forecasts change), but the core idea is to avoid extreme bets. This strategy offers a strong probability of achieving the retirement goal (on the order of 70% or more, especially if the investor can modestly increase savings or extend the working period, per Table 3) while dramatically limiting downside risk. The constrained allocation prevents over-concentration in any one asset and avoids the excessive leverage that, while boosting expected returns, could lead to catastrophic losses. If the investor is comfortable with a controlled amount of leverage, a mild long–short implementation (e.g. borrowing up to ∼20% of the portfolio, as in the optimal constrained long–short solution) can further enhance returns – this variant was identified as an excellent compromise, ranking 2nd in goal attainment and 1st in tail risk (Table 5, Appendix). Otherwise, the long-only constrained portfolio is a very solid choice for a more conservative stance, still yielding one of the highest success probabilities among all low-risk strategies. Frequent annual rebalancing is not strictly necessary in the assumed steady market environment; a buy-and-hold approach should suffice and even slightly outperformed in our analysis. That said, I would monitor the portfolio over time and consider tactical rebalancing if significant market shifts occur or if the investor's circumstances change – flexibility can further improve outcomes if used judiciously. Overall, my recommendation is to pursue the constrained diversified portfolio strategy, complemented by prudent financial planning (increased contributions and/or a later retirement if feasible). This plan provides a high confidence of a comfortable retirement while safeguarding against unacceptable losses, aligning with the investor's goal of a secure and stable retirement fund.

Part B. Multi-Period Approach

Questions

1. Formulate the retirement planning problem into a standard MSP. Use the following assumptions.

- The time step is 10 years. Thus, the problem becomes a two stage problem with recourse.
- Within each time step, she employs a buy-and-hold strategy.
- Define the decision variables, parameters, objective functions, and constraints carefully.

Estimated parameters (from Part A).

$$\hat{\mu} = \begin{bmatrix} 0.038 \\ 0.050 \\ 0.113 \\ 0.001 \end{bmatrix}, \qquad \hat{\Sigma} = \begin{bmatrix} 0.070 & -0.007 & 0.015 & 0.003 \\ -0.007 & 0.033 & -0.012 & 0.001 \\ 0.015 & -0.012 & 0.098 & -0.001 \\ 0.003 & 0.001 & -0.001 & 0.001 \end{bmatrix}.$$

Scenario construction. Eight scenarios $s \in \{1, \ldots, 8\}$ are obtained by assigning each risky asset (stocks, bonds, commodities) an "up" or "down" state. For asset $i \le 3$ in scenario s the mean return is $\mu_i^{(s)} = \hat{\mu}_i \pm 0.02$ ($+0.02$ if up, -0.02 if down); cash ($i = 4$) keeps $\mu_4^{(s)} = \hat{\mu}_4$. Prior probabilities are uniform: $p_s = \frac{1}{8}$.

Time structure. Two ten-year stages ($t = 0 \to 10$ and $t = 10 \to 20$); buy-and-hold inside each stage; rebalancing only at $t = 10$.

Variables.

$$x_{0,i}: \quad \text{initial amount in asset } i \ (i = 1, \ldots, 4),$$
$$x_{1,i}^{(s)}: \quad \text{amount in asset } i \text{ at } t = 10 \text{ if scenario } s \text{ is observed.}$$

Collect $\mathbf{x}_0 \in \mathbb{R}^4$ and $\mathbf{x}_1^{(s)} \in \mathbb{R}^4$; no sign or upper bounds (shorting and leverage permitted).

Returns.

$$R_{0,i}^{(s)} = \left(1 + \mu_i^{(s)}\right)^{10} - 1, \qquad R_{1,i}^{(s)} = \left(1 + \mu_i^{(s)}\right)^{10} - 1.$$

Wealth recursion.

$$V_{10}^{(s)} = \sum_{i=1}^{4} x_{0,i}\,(1 + R_{0,i}^{(s)}), \qquad V_{20}^{(s)} = \sum_{i=1}^{4} x_{1,i}^{(s)}\,(1 + R_{1,i}^{(s)}).$$

Stage-1 budget. $\sum_{i=1}^{4} x_{0,i} = V_0.$

Stage-2 (recourse) budgets. $\sum_{i=1}^{4} x_{1,i}^{(s)} = V_{10}^{(s)}, \quad s = 1, \ldots, 8.$

Objective. Maximize expected terminal wealth:

$$\max_{\mathbf{x}_0, \{\mathbf{x}_1^{(s)}\}} \ \mathbb{E}[V_{20}] = \sum_{s=1}^{8} p_s\, V_{20}^{(s)}.$$

Extensive-form LP. Define coefficients $c_{s,i} = p_s\,(1 + R_{1,i}^{(s)})$ and stack all variables

$$\mathbf{z} = (x_{0,1}, \ldots, x_{0,4}, x_{1,1}^{(1)}, \ldots, x_{1,4}^{(8)})^\top \in \mathbb{R}^{36}$$

Let $\mathbf{E}_s \in \mathbb{R}^{1\times 32}$ pick the four components of scenario s.

$$\begin{aligned} \min_{\mathbf{z}} \ & \mathbf{c}^\top \mathbf{z}, \quad \mathbf{c} = \left(0, 0, 0, 0, -c_{1,1}, \ldots, -c_{8,4}\right)^\top, \\ \text{s.t. } & \left[\mathbf{1}_4^\top \,|\, \mathbf{0}_{1\times 32}\right]\mathbf{z} = V_0, \\ & \left[-\operatorname{diag}(1 + R_0^{(s)}) \,|\, \mathbf{E}_s\right]\mathbf{z} = 0, \quad s = 1, \ldots, 8. \end{aligned}$$

Because all $c_{s,i} \le 0$, the program is bounded even with free-sign variables, and its optimal solution $(\mathbf{x}_0^*, \mathbf{x}_1^{(s)*})$ describes the retirement plan that maximizes expected terminal wealth under the stated assumptions.

2. Solve this MSP problem with choice of your optimization software (python, matlab, excel, etc.). Interpret the outcomes so that a freshman at KAIST can understand.

I solved the above problem with the bound $w \in [-10, 10]^4$ and got the following result.

Optimal initial allocation x_0^* (per $1 wealth)

Asset	Stocks	Bonds	Commodities	Cash
Weight	-9.0	10.0	10.0	-10.0

$$\boxed{\mathbb{E}[V_{20}] \;=\; 56.52 \quad \text{(per \$1 initial wealth)}}$$

When the variable bounds are widened the objective value can diverge because the linear programme becomes unbounded above. The expected–wealth objective is

$$\sum_{s=1}^{8} p_s \sum_{i=1}^{4} (1 + R_{1,i}^{(s)})\, x_{1,i}^{(s)} = \mathbf{c}^\top \mathbf{z}, \qquad c_{s,i} = p_s\big(1 + R_{1,i}^{(s)}\big) \;>\; 0$$

for every asset i whose ten–year expected growth rate exceeds zero. Suppose cash (asset 4) earns only $1{+}R_{1,4} = 1.001$ while bonds or commodities earn strictly more in every scenario. Because the stage–two budget constraint $\sum_i x_{1,i}^{(s)} = \sum_i x_{0,i}(1{+}R_{0,i}^{(s)})$ is homogeneous of degree one in the first–stage weights, we may multiply *all* first–stage positions by a common scalar $\lambda > 0$ and obtain feasible second–stage positions that are also scaled by λ. Choosing λ arbitrarily large increases the inner products $\mathbf{c}^\top \mathbf{z}$ proportionally, so the objective tends to $+\infty$. In practical terms the model borrows an unlimited amount of cash (negative $x_{0,4}$) and invests the proceeds in the asset with the highest mean return (positive $x_{0,j}$), repeating this leverage loop indefinitely because nothing—neither borrowing cost nor risk penalty—halts the process. Imposing the box bound $w \in [-10, 10]^4$ simply caps λ at ten; removing or relaxing that cap lets λ grow without limit and the optimiser pushes the allocation to the boundary of the feasible region, sending the expected terminal wealth to infinity. Mathematically the linear objective with strictly positive coefficients together with sign–free variables and homogeneous constraints produces an unbounded optimisation problem unless an explicit leverage, variance, or short–selling constraint is enforced.

Part C. Financial planning for the beloved ones

In this part, I will use the approach of **Part A** for my parents. They expect to retire after 13 years. (They are 70 years old at this time.) After the retirement, they will have monthly cash inflow ₩ 3 millions from the national pension service and their residential real estate. Also, they are also expected to receive ₩ 100 millions in retirement funds from the Yellow Umbrella Mutual Aid Association. They expect their first annual expenses to be ₩ 60 millions after retirement. Suppose that they live to be 100 years old and the inflation rate is 2%. Then, total present value of their asset at their retirement will be ₩ G millions s.t.

$$G = 3 \times 12 \times 30 + 100 - \sum_{t=1}^{30} \frac{60}{1.02^t} = -163.787$$

Note that the cash inflow from NPS will grow with the inflation rate. Also, residential property rents generally increase with inflation.

From these results, we can see that they need to save to cover their loss of ₩ 163 millions. This situation is equivalent to the situation in Problem A-4 where $V_0 = 0$, S = ₩ 163.787 millions, $T = 10$. They currently have debt and plan to pay it off first and save ₩ 3 millions per month starting when they turn 60 years old. Therefore, $S_t = ₩\, 36(1.03)^t$ millions. Now, apply the current KRW/USD currency rate (about 1,400 KRW/USD). Then, we have $S = ₩\, 163.787$ millions $= \$116,991$, $S_t = \$25,714(1.03)^t$. Doing the same exactly same procedure of A-4, we have the following table.

Table 6: Efficient Frontiers giving the greatest $P(W_p \geq S)$

Strategy	$\max_w\big(P(W_p \geq S)\big)$	w_1	w_2	w_3	w_4
Buy–and–Hold Long Only	1.0000	0.0000	0.0000	0.0001	0.9999
Buy–and–Hold Long–Short	1.0000	-0.0199	-0.0388	-0.0020	1.0608
Annual Rebalance Long Only	1.0000	0.0000	0.0000	0.0071	0.9929
Annual Rebalance Long–Short	1.0000	-0.0852	-0.1699	-0.0725	1.3276
Constrained Buy–and–Hold Long Only	1.0000	0.3465	0.2535	0.0000	0.4000
Constrained Buy–and–Hold Long–Short	1.0000	0.2260	0.3768	-0.0028	0.4000
Constrained Annual Rebalance Long Only	1.0000	0.1282	0.3489	0.1229	0.4000
Constrained Annual Rebalance Long–Short	1.0000	0.1282	0.3489	0.1229	0.4000

From the result, we can see that Buy–and–Hold Long Only Strategy will give the lowest uncertainty since it contains .9999% of cash. So, I recommended that they just save the cash for their retirement.