Stochastic Processes

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目录

1 Review of Martingales

- $(X_n)_{n\geq 0}$ is L^2-bounded martingale $\Rightarrow X_n$ converges in L^2.
- $(X_n)_{n\geq 0}$ is L^1-bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n\geq 0}$ is L^p-bounded martingale for $p>1$, then X_n converges in $L^{p'}$ for $p' \in [1,p)$.
- Statement is false when $p=1$. Example: $\Omega = [0,1), \mathscr{F}_n = \sigma\{[\frac{i}{2^n}, \frac{i+1}{2^n})\}_{i=0}^{2^n-1}, X_n(\omega) := \begin{cases} 2^n & \omega \in [0, \frac{1}{2^n}) \\ 0 & \text{otherwise} \end{cases}$.
- Let $p>1$ and $(X_n)_{n\geq 0}$ be L^p bounded martingale w.r.t. $\mathscr{F}_n$. Then $\exists X \in L^p(\Omega, \mathscr{F}_\infty, P)$ s.t. $X_n \to X$ in L^p and a.s. and $X_n = \mathbb{E}(X|\mathscr{F}_n)$.
- Doob's maximal inequality: Let $p>1, \exists C = C_p$ s.t. $\forall$ martingale $(X_n)_{n\geq 0}$, we have $\mathbb{E}|X_n^*|^p \leq C_p \mathbb{E}|X_n|^p$ where $|X_n^*| = \sup_{0\leq k\leq n} \sup |X_k|$.
- Let $(Z_n)_{n\geq 0}$ be a nonnegative sub-martingale and $Z_n^* = \sup_{0\leq k\leq n} Z_k$, then $P(Z_n^* > \lambda) \leq \frac{1}{\lambda}\mathbb{E}(Z_n 1_{\{Z_n^* > \lambda\}}) \leq \frac{1}{\lambda}\mathbb{E}Z_n$. Corollary: $P(Z_n^* > \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p 1_{\{Z_n^*>\lambda\}}) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p)$.
- If $(X_n)_{n\geq 0}$ is a martingale with $\sup_n \mathbb{E}(|X_n|\log(1+|X_n|)) < +\infty$, then X_n converges in L^1.
- Two prob measures P and Q on $(\Omega, \mathscr{F})$, $Q << P$ on $\mathscr{F}_n$ for every n and $M_n = \frac{dQ}{dP}|_{\mathscr{F}_n}$. $(M_n)_{n\geq 0}$ is a P-martingale w.r.t. $(\mathscr{F}_n)_{n\geq 0}$. $Q << P$ on $\mathscr{F}_\infty$ if and only if $M_n \to M$ in L^1. $Q(A) = \int_A M dP + Q(A \cap \{M = +\infty\})$.
- Statement is false if $M_n \not\to M$ in L^1. Example: $\Omega = \{\omega = (\omega_1, \cdots, \omega_n, \cdots) \in \{\pm 1\}^{\mathbb{N}}\}, X_n(\omega) = \omega_n$. X_n's are i.i.d. under P and Q, but $P(X_n = 1) = \frac{1}{2}, P(X_n = -1) = \frac{1}{2}, Q(X_n = 1) = \frac{1}{3}, Q(X_n = -1) = \frac{2}{3}$. $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$. $P(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k = 0) = 1, Q(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n X_k = -\frac{1}{3}) = 1$.
- Monotone class theorem for functions: Suppose $\mathcal{A}$ us a π-system and $\mathcal{H}$ be a class of functions from Ω to $\mathbb{R}$ s.t. (1) $1_A \in \mathcal{H}$ for every $A \in \mathscr{A}$, (2) if $f,g \in \mathcal{H}$ then $af + bg \in \mathcal{H}$, (3) if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ then $f \in \mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$-measurable functions are in $\mathcal{H}$.
- Let $(Y_n)_{n\geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E}Y_k = 1$. Then $M_n = \prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n \equiv 1$. Otherwise $M_n \to 0$ a.s.
- Kakutani's theorem: $M_n = \prod_{k=1}^n Y_k$, $Y_k \geq 0$ are independent, $\mathbb{E}Y_k = 1$, $\lambda_k = \mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k \lambda_k > 0$, then $M_n \to M$ in L^1; (2) If $\prod_k \lambda_k = 0$, then $M_n \to 0$ a.s.

2 Markov Chains

- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain on a discrete space S. P^x : law of $(X_n)_{n\geq 0}$ conditioned on $X_0 = x$. $P(X_{n+1} \in A|\mathscr{F}_n) = P^{X_n}(X_1 \in A) = P(X_1 \in A|X_0 = X_n)$. $\mathbb{E}^x$: expectation under P^x. $P^x(X_1 = y) = p(x,y)$.
- For every $f : S \to \mathbb{R}$ bounded, define $(Pf)(x) = \sum_{y\in S} p(x,y)f(y) = \mathbb{E}^x(f(X_1))$, $(Lf)(x) = \sum_{y\in S} p(x,y)f(y) - f(x)$. $L = P - \text{id}$, the generator.
- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with generator L. Then for every bounded $f : S \to \mathbb{R}$, $M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1}(Lf)(X_k)$ is a martingale.