High-Dimensional Probability

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Contents

0 Appetizer

- Convex combination: For $z_1, z_2, \cdots, z_m \in \mathbb{R}^n$, the form of $\sum_{i=1}^m \lambda_i z_i$ with $\lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1$. Convex hull of $T \subset \mathbb{R}^n$: $\text{conv}(T) = \{\text{convex combinations of } z_1, \cdots, z_m \in T, m \in \mathbb{N}\}$.

- Caratheodory's theorem: Every point in the convex hull of a set $T \subset \mathbb{R}^n$ can be expressed as a convex combination of at most $n+1$ points from T.

- Approximate Caratheodory's theorem: Consider $T \subset \mathbb{R}^n$, $\text{diam}(T) = \sup\{||s-t||_2, s, t \in T\} < 1$. Then for any $x \in \text{conv}(T)$ and any k, one can find points $x_1, x_2, \cdots, x_k \in T$ such that $||x - \frac{1}{k}\sum_{i=1}^k x_i||_2 \leq \frac{1}{\sqrt{k}}$ (repetition is allowed).

  Proof WLOG assume $||t||_2 \leq 1, \forall t \in T$. Fix $x \in \text{conv}(T), x = \sum_{i=1}^m \lambda_i z_i, z_i \in T$. Define $Z, \mathbb{P}(Z = z_i) = \lambda_i, \mathbb{E}Z = x$. Consider i.i.d. $Z_1, Z_2, \cdots$ of Z, $\frac{1}{n}\sum_{j=1}^n Z_j \to x$ a.s. $n \to +\infty$. $\mathbb{E}||x - \frac{1}{k}\sum_{j=1}^k Z_j||_2^2 = \frac{1}{k^2}\sum_{j=1}^k \mathbb{E}||Z_j - x||_2^2 = \frac{1}{k^2}\sum_{j=1}^k(\mathbb{E}||Z_j||^2 - ||\mathbb{E}Z_j||_2^2) \leq \frac{1}{k} \Rightarrow \exists$ a realization of $Z_1, \cdots, Z_k$ such that $||x - \frac{1}{k}\sum_{j=1}^k Z_j||_2 \leq \frac{1}{\sqrt{k}}$. □

- Corollary (Covering polytopes by balls): P is a polytope in $\mathbb{R}^n$ with N vertices, $\text{diam}(P) \leq 1$. Then P can be covered by at most $N^{\lfloor 1/\epsilon^2 \rfloor}$ Euclidean balls of radii $\epsilon > 0$.

1 Preliminaries on random variables

- Jensen's inequality: convex ϕ, $\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X)$. $\Rightarrow ||X||_{L^p} \leq ||X||_{L^q}$ for $p \leq q$.

- Minkowski inequality: $p \geq 1, ||X+Y||_{L^p} \leq ||X||_{L^p} + ||Y||_{L^p}$.

- Cauchy-Schwarz inequality: $\mathbb{E}|XY| \leq ||X||_{L^2}||Y||_{L^2}$.

- Holder inequality: $p, q \in (1, +\infty), \frac{1}{p} + \frac{1}{q} = 1$ or $p = 1, q = \infty$, $\mathbb{E}||XY|| \leq ||X||_{L^p}||Y||_{L^q}$.

- $X \geq 0$, then $\mathbb{E}X = \int_0^{+\infty} \mathbb{P}(X > t)\text{d}t$.

- Markov inequality: $X \geq 0, t > 0, \mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}$.

- LLN: $X_1, \cdots, X_n, \cdots$ i.i.d., $\mathbb{E}X_i = \mu, \text{Var}(X_i) = \sigma^2, S_N = X_1 + X_2 + \cdots + X_N$. Then: (WLLN) $\mathbb{P}(|\frac{S_N}{N} - \mu| > \epsilon) \to 0, \forall \epsilon > 0$; (SLLN) $\mathbb{P}(\frac{S_N}{N} \to \mu, N \to +\infty) = 1$.

- CLT: $Z_N = \frac{S_N - \mathbb{E}S_N}{\sqrt{\text{Var}(S_N)}} \xrightarrow{d} \mathcal{N}(0,1)$.

- $X_{N,i}, 1 \leq i \leq N$ independent $\text{Ber}(p_{N,i}), \max_{i \leq N} p_{N,i} \to 0, S_N = \sum_{i=1}^N X_{N,i}, \mathbb{E}S_N \to \lambda < +\infty$. Then $S_N \xrightarrow{d}$ Poisson(λ).

2 Concentration of sums of independent random variables

- Question: N times, $\mathbb{P}(\text{head} \geq \frac{3}{4}N) = ?$ Let S_N be the number of heads, $\mathbb{E}S_N = \frac{N}{2}, \text{Var}(S_N) = \frac{N}{4}$. (1) Chebyshev's inequality: $\mathbb{P}(S_N \geq \frac{3}{4}N) \leq \mathbb{P}(|S_N - \frac{N}{2}| \geq \frac{N}{4}) \leq \frac{4}{N}$; (2) $Z_N = \frac{S_N - \frac{N}{2}}{\sqrt{N/4}}$, expect: $\mathbb{P}(Z_N \geq \sqrt{N/4}) \approx \mathbb{P}(g \geq \sqrt{N/4}) \leq \frac{1}{\sqrt{2\pi}}e^{-N/8}$ where $g \sim \mathcal{N}(0,1)$.

- For all $t > 0$, $(\frac{1}{t} - \frac{1}{t^3})\frac{1}{\sqrt{2\pi}}e^{-t^2/2} \leq \mathbb{P}(g \sim \mathcal{N}(0,1) \geq t) \leq \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

- Berry-Esseen bound: $|\mathbb{P}(Z_N \geq t) - \mathbb{P}(g \geq t)| \leq \frac{\rho}{\sqrt{N}}$ where $\rho = \mathbb{E}|X_1 - \mu|^3/\sigma^3$. And in general, no improvement since $\mathbb{P}(S_N = \frac{N}{2}) \sim \frac{1}{\sqrt{N}} \Rightarrow \mathbb{P}(Z_N = 0) \sim \frac{1}{\sqrt{N}}$ but $\mathbb{P}(g = 0) = 0$.

- Hoeffding's inequality: $X_1, \cdots, X_N$ i.i.d. symmetric Bernoulli ($\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$), $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$. Then $\forall t \geq 0, \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) \leq e^{-t^2/2||a||_2^2}, \mathbb{P}(|\sum_{i=1}^N a_i X_i \geq t|) \leq 2e^{-t^2/2||a||_2^2}$.

  Proof WLOG, $||a||_2^2 = 1$. For $\lambda > 0, \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) = \mathbb{P}(e^{\lambda \sum a_i X_i} \geq e^{\lambda t}) \leq e^{-\lambda t}\mathbb{E}e^{\lambda \sum_{i=1}^N a_i X_i} = e^{-\lambda t}\prod_{i=1}^N \mathbb{E}e^{\lambda a_i X_i} = e^{-\lambda t}\prod_{i=1}^N \cosh(\lambda a_i) \leq e^{-\lambda t}\prod_{i=1}^N e^{\lambda^2 a_i^2/2} = e^{-\lambda t + \frac{\lambda^2}{2}} \Rightarrow \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) \leq \inf_{\lambda \geq 0} e^{-\lambda t + \frac{\lambda^2}{2}} = e^{-\frac{t^2}{2}} (\lambda = t)$. □

- Bounded r.v.s: $X_1, \cdots, X_N$ independent, $X_i \in [m_i, M_i]$. Then $\forall t \geq 0, \mathbb{P}(\sum_{i=1}^N (X_i - \mathbb{E}X_i) \geq t) \leq e^{-\frac{2t^2}{\sum_{i=1}^N (M_i - m_i)^2}}$.
- Chernoff's inequality: $X_i \sim \mathrm{Ber}(p_i)$ independent, $S_N = \sum_{i=1}^N X_i, \mu = \mathbb{E}S_N \Rightarrow \forall t > \mu, \mathbb{P}(S_N \geq t) \leq e^{-\mu}(\frac{e\mu}{t})^t$.