

Stochastic Processes

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1 Review of Martingales

- $(X_n)_{n \geq 0}$ is L^2 -bounded martingale $\Rightarrow X_n$ converges in L^2 .
- $(X_n)_{n \geq 0}$ is L^1 -bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n \geq 0}$ is L^p -bounded martingale for $p > 1$, then X_n converges in $L^{p'}$ for $p' \in [1, p)$.
- Statement is false when $p = 1$. Example: $\Omega = [0, 1)$, $\mathcal{F}_n = \sigma\{\frac{i}{2^n}, \frac{i+1}{2^n}\}_{i=0}^{2^n-1}$, $X_n(\omega) := \begin{cases} 2^n & \omega \in [0, \frac{1}{2^n}) \\ 0 & \text{otherwise} \end{cases}$.
- Let $p > 1$ and $(X_n)_{n \geq 0}$ be L^p bounded martingale w.r.t. \mathcal{F}_n . Then $\exists X \in L^p(\Omega, \mathcal{F}_\infty, P)$ s.t. $X_n \rightarrow X$ in L^p and a.s. and $X_n = \mathbb{E}(X|\mathcal{F}_n)$.
- Let $(Z_n)_{n \geq 0}$ be a nonnegative sub-martingale and $Z_n^* = \sup_{0 \leq k \leq n} Z_k$, then $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda} \mathbb{E}(Z_n 1_{\{Z_n^* > \lambda\}}) \leq \frac{1}{\lambda} \mathbb{E}Z_n$. Corollary: $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda^p} \mathbb{E}(Z_n^p 1_{\{Z_n^* > \lambda\}}) \leq \frac{1}{\lambda^p} \mathbb{E}(Z_n^p)$.
- Doob's maximal inequality: Let $p > 1, \exists C = C_p$ s.t. \forall martingale $(X_n)_{n \geq 0}$, we have $\mathbb{E}|X_n^*|^p \leq C_p \mathbb{E}|X_n|^p$ where $|X_n^*| = \sup_{0 \leq k \leq n} |X_k|$.
- If $(X_n)_{n \geq 0}$ is a martingale with $\sup_n \mathbb{E}(|X_n| \log(1 + |X_n|)) < +\infty$, then X_n converges in L^1 .

Proof $\mathbb{E}|X_n^*| = \int_0^{+\infty} \mathbb{P}(X_n^* > \lambda) d\lambda \leq 1 + \int_1^{+\infty} \frac{1}{\lambda} (\int_{|X_n^*| > \lambda} |X_n| d\mathbb{P}) d\lambda = 1 + \int |X_n| 1_{X_n^* > 1} (\int_1^{X_n^*} \frac{1}{\lambda} d\lambda) d\mathbb{P} \Rightarrow \mathbb{E}|X_n^*| \leq 1 + \mathbb{E}(|X_n| \log(X_n^* \vee 1)) \Rightarrow \mathbb{E}(X_n^* \vee 1) \leq 2 + \mathbb{E}(|X_n| \log(X_n^* \vee 1))$. Since $x \log y \leq 10^{10}(2+x) \log(2+x) + \frac{y}{2}$ when x, y are large enough (insight: if $y \gg x^2$ then $x \log y \leq \frac{y}{2}$; else $x \log y \leq 10^{10}(2+x) \log(2+x)$), $\mathbb{E}X_n^* \leq 10^{100}[1 + \mathbb{E}(|X_n| + 2) \log(|X_n| + 2)]$. Then use dominated convergence theorem. \square

- Two probability measures \mathbb{P} and \mathbb{Q} on (Ω, \mathcal{F}) , $\mathbb{Q} \ll \mathbb{P}$ on \mathcal{F}_n for every n and $M_n = \frac{d\mathbb{Q}|_{\mathcal{F}_n}}{d\mathbb{P}|_{\mathcal{F}_n}}$. $(M_n)_{n \geq 0}$ is a \mathbb{P} -martingale w.r.t. $(\mathcal{F}_n)_{n \geq 0}$. $\mathbb{Q} \ll \mathbb{P}$ on \mathcal{F}_∞ if and only if $M_n \rightarrow M$ in L^1 . $\mathbb{Q}(A) = \int_A M d\mathbb{P} + \mathbb{Q}(A \cap \{M = +\infty\})$.

Proof Sufficiency. $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F} = \mathcal{F}_\infty$, thus let $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$, we need to show M_n converges to Z in L^1 . $\forall A \in \mathcal{F}_n, \int_A M_n d\mathbb{P} = \mathbb{Q}(A) = \int_A Z d\mathbb{P} \Rightarrow M_n = \mathbb{E}(Z|\mathcal{F}_n)$. Thus M_n is uniformly integrable, thus converges in L^1 .

Necessity. Suppose $M_n \rightarrow M$ a.s. and in L^1 . We need to show $M_n = \mathbb{E}(M|\mathcal{F}_n)$ and $M = \frac{d\mathbb{Q}}{d\mathbb{P}}$. It suffices to show $\mathbb{Q}(A) = \int_A M d\mathbb{P}$ for all $A \in \cup_n \mathcal{F}_n$. Suppose $A \in \mathcal{F}_N$. Then $\mathbb{Q}(A) = \int_A M_N d\mathbb{P} = \int_A M_{N+k} d\mathbb{P} \rightarrow \int_A M d\mathbb{P}$. By $\pi - \lambda$ theorem we can get the desired result.

Special situation: Suppose $\mathbb{P} \perp \mathbb{Q}$ on $\mathcal{F} (\exists E \text{ s.t. } \mathbb{P}(E) = 1, \mathbb{Q}(E^c) = 1)$ and $\mathbb{P} \ll \mathbb{Q}$ on \mathcal{F}_n . Then $\frac{1}{M_n}$ converges \mathbb{Q} -a.s. Let $\mathbb{R} = \frac{1}{2}(\mathbb{P} + \mathbb{Q})$, $\mathbb{P}, \mathbb{Q} \ll \mathbb{R}$ on \mathcal{F} , $\frac{d\mathbb{P}|_{\mathcal{F}_n}}{d\mathbb{R}|_{\mathcal{F}_n}} = \frac{2}{1+M_n} \rightarrow \frac{2M}{1+M}$ in $L^1(\mathbb{R})$, $\frac{d\mathbb{Q}}{d\mathbb{R}} = \frac{2M_n}{1+M_n} \rightarrow \frac{2}{1+M}$ in $L^1(\mathbb{R})$. Then $\mathbb{Q}(A) = \mathbb{Q}(A \cap E^c) = \int_{A \cap E^c} \frac{2M}{1+M} d\mathbb{R} = \int_A \frac{2M}{1+M} 1_{E^c} d\mathbb{R} \stackrel{\mathbb{P}(E^c)=0}{=} 2\mathbb{R}(A \cap E^c) = 2 \int_A 1_{E^c} d\mathbb{R} \Rightarrow \frac{2M}{1+M} 1_{E^c} = 2 \cdot 1_{E^c} \Rightarrow M = +\infty$ on $E^c \Rightarrow \mathbb{Q}(M = +\infty) = 1$. Similarly $\mathbb{P}(M = 0) = \mathbb{Q}(M = +\infty) = 1$.

General situation: $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2, \mathbb{Q}_1 \ll \mathbb{P}, \mathbb{Q}_2 \perp \mathbb{P}$ on \mathcal{F} . Therefore we can decompose M_n as $M_n = Y_n + Z_n$ where $Y_n \rightarrow Y$ in $L^1(\mathbb{P})$ and $Z_n \rightarrow 0$ \mathbb{P} -a.s. $\mathbb{Q}_1(A) = \int_A Y d\mathbb{P} = \int_A M d\mathbb{P}$. $\mathbb{Q}_2(A) = \mathbb{Q}_2(A \cap \{Z = +\infty\})$. Since $Z = 0$ \mathbb{P} -a.s., $M < +\infty$ \mathbb{P} -a.s. and $\mathbb{Q}_2(M = +\infty) = 1$, we have $\mathbb{Q}_2(A) = \mathbb{Q}(A \cap \{Z = +\infty\}) = \mathbb{Q}_2(A \cap \{M = +\infty\}) = \mathbb{Q}(A \cap \{M = +\infty\})$. To sum up, $\mathbb{Q}(A) = \int_A M d\mathbb{P} + \mathbb{Q}(A \cap \{M = +\infty\})$. \square

- Statement is false if $M_n \not\rightarrow M$ in L^1 . Example: $\Omega = \{\omega = (\omega_1, \dots, \omega_n, \dots) \in \{\pm 1\}^{\mathbb{N}}\}$, $X_n(\omega) = \omega_n$. X_n 's are i.i.d. under \mathbb{P} and \mathbb{Q} , but $\mathbb{P}(X_n = 1) = \frac{1}{2}, \mathbb{P}(X_n = -1) = \frac{1}{2}, \mathbb{Q}(X_n = 1) = \frac{1}{3}, \mathbb{Q}(X_n = -1) = \frac{2}{3}$. $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. $\mathbb{P}(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k = 0) = 1, \mathbb{Q}(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k = -\frac{1}{3}) = 1$.
- Monotone class theorem for functions: Suppose \mathcal{A} as a π -system and \mathcal{H} be a class of functions from Ω to \mathbb{R} s.t. (1) $1_A \in \mathcal{H}$ for every $A \in \mathcal{A}$, (2) if $f, g \in \mathcal{H}$ then $af + bg \in \mathcal{H}$, (3) if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ then $f \in \mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$ -measurable functions are in \mathcal{H} .
- Let $(Y_n)_{n \geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E}Y_k = 1$. Then $M_n = \prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n \equiv 1$. Otherwise $M_n \rightarrow 0$ a.s.

Proof Note that $\frac{1}{n} \log M_n = \frac{1}{n} \sum_{k=1}^n \log Y_k \rightarrow \mathbb{E} \log Y$ a.s. If $\mathbb{E} \log Y = 0$ then by Jensen's inequality we have $Y_n \equiv 1$ which means M_n converges in L^1 . If $\mathbb{E} \log Y < 0$ then $M_n \rightarrow 0$ a.s. \square

- Kakutani's theorem: $M_n = \prod_{k=1}^n Y_k$, $Y_k \geq 0$ are independent, $\mathbb{E}Y_k = 1$, $\lambda_k = \mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k \lambda_k > 0$, then $M_n \rightarrow M$ in L^1 ; (2) If $\prod_k \lambda_k = 0$, then $M_n \rightarrow 0$ a.s.

Proof Let $Z_n = \prod_{k=1}^n \frac{\sqrt{Y_k}}{\lambda_k}$. Then Z_n is a martingale and has an a.s. limit Z , and $M_n = (\prod_{k=1}^n \lambda_k)^2 Z_n^2$. If $\prod_k \lambda_k > 0$, then Z_n is L^2 bounded and then convergence in L^2 , which implies $M_n \rightarrow M$ in L^1 . If $\prod_k \lambda_k = 0$, it is obvious that $M_n \rightarrow 0$ a.s. \square

- Martingale LLN: Let $(M_n)_{n \geq 0}$ be a martingale s.t. $\sum_{k=1}^{+\infty} \frac{\mathbb{E}(M_k - M_{k-1})^2}{k^2} < +\infty$. Then $\frac{M_n}{n} \rightarrow 0$ a.s.

Proof Let $Y_n = \sum_{k=1}^n \frac{X_k}{k}$. Then $(Y_n)_{n \geq 0}$ is an L^2 bounded martingale, thus $Y_n \rightarrow Y$ a.s. Then use Kronecker's lemma. \square

- Martingale CLT: Let $(M_n)_{n \geq 0}$ be a martingale with $M_0 = 0$ and $\sigma_n^2 = \sum_{k=1}^n \mathbb{E}X_k^2 = \mathbb{E}\langle M \rangle_n$. Assume that $\frac{1}{\sigma_n^2} \max_{1 \leq k \leq n} (\mathbb{E}X_k^2) \rightarrow 0$, $\frac{1}{\sigma_n^2} \sum_{k=1}^n \mathbb{E}(X_k^2 1_{\{|X_k| > \epsilon \sigma_n\}} | \mathcal{F}_{k-1}) \xrightarrow{P} 0$ for all $\epsilon > 0$, $\frac{1}{\sigma_n^2} \langle M \rangle_n \xrightarrow{P} 1$. Then $\frac{M_n}{\sigma_n} \Rightarrow \mathcal{N}(0, 1)$.

2 Markov Chains

- Let $(X_n)_{n \geq 0}$ be a homogeneous Markov chain on a discrete space S . \mathbb{P}^x : law of $(X_n)_{n \geq 0}$ conditioned on $X_0 = x$. $\mathbb{P}(X_{n+1} \in A | \mathcal{F}_n) = \mathbb{P}^{X_n}(X_1 \in A) = \mathbb{P}(X_1 \in A | X_0 = X_n)$. \mathbb{E}^x : expectation under \mathbb{P}^x . $\mathbb{P}^x(X_1 = y) = p(x, y)$.
- For every $f : S \rightarrow \mathbb{R}$ bounded, define $(\mathcal{P}f)(x) = \sum_{y \in S} p(x, y)f(y) = \mathbb{E}^x(f(X_1))$, $(\mathcal{L}f)(x) = \sum_{y \in S} p(x, y)f(y) - f(x)$. $\mathcal{L} = \mathcal{P} - \text{id}$, the generator.
- Let $(X_n)_{n \geq 0}$ be a homogeneous Markov chain with generator \mathcal{L} . Then for every bounded $f : S \rightarrow \mathbb{R}$, $M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (\mathcal{L}f)(X_k)$ is a martingale. Conversely, let $(X_n)_{n \geq 0}$ be a process and \mathcal{L} be an operator on $\mathcal{B}(S)$ s.t. M_n^f is a martingale for every f , then $(X_n)_{n \geq 0}$ is a Markov chain with generator \mathcal{L} .
- Given operator \mathcal{L} on $\mathcal{B}(S)$, we say $f : S \rightarrow \mathbb{R}$ is (1) harmonic for \mathcal{L} if $\mathcal{L}f = 0$; (2) sub-harmonic for \mathcal{L} if $\mathcal{L}f \geq 0$; (3) super-harmonic for \mathcal{L} if $\mathcal{L}f \leq 0$.
- Let f be the generator of a Markov chain $(X_n)_{n \geq 0}$. Then f is (sub-/super-)harmonic $\Leftrightarrow f(X_n)_{n \geq 0}$ is a (sub-/super-) martingale.
- f is (sub-/super-)harmonic on $D \subset S$ if $\mathcal{L}f \geq / \leq / = 0$ on D . Let $\tau = \inf\{k \geq 0 : X_k \in D^c\}$, then $(f(X_{n \wedge \tau}))_{n \geq 0}$ is a (sub-/super-)martingale.
- Maximum principle: Let $(X_n)_{n \geq 0}$ be a Markov chain and $D \subset S$ s.t. the stopping time $\tau = \inf\{k \geq 0, X_k \in D^c\}$ is a.s. finite. If f is bounded and sub-harmonic on D , then $\sup_{x \in D} f(x) \leq \sup_{x \in D^c} f(x)$.

Proof f is sub-harmonic implies $(f(X_{n \wedge \tau}))$ is a sub-martingale, hence for $x \in D$ we have $f(x) \leq \mathbb{E}^x f(X_{n \wedge \tau}) \rightarrow \mathbb{E}^x(f(X_\tau)) \leq \sup_{x \in D^c} f(x)$. \square

- $A \subset S, \tau_A = \sup\{k \geq 0 : X_k \in A\}$. (1) $u(x) = \mathbb{P}^x(\tau_A < +\infty) \Rightarrow \begin{cases} \mathcal{L}u = 0 & \text{on } A^c \\ u = 1 & \text{on } A \end{cases}$. (2) $u(x) = \mathbb{P}(\tau_A < \tau_B) \Rightarrow$

$$\begin{cases} \mathcal{L}u = 0 & \text{on } (A \cup B)^c \\ u = 1 & \text{on } A \\ u = 0 & \text{on } B. \end{cases} \quad (3) \quad u(x) = \mathbb{E}^x[\tau_A] \Rightarrow \begin{cases} \mathcal{L}u = -1 & \text{on } A^c \\ u = 0 & \text{on } A \end{cases}.$$

- Any nonnegative solution v to $\begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = 1 & \text{on } A \end{cases}$ satisfies $v \geq u$. Furthermore, if $u \equiv 1$, then \exists 1 bounded solution

$$\text{to } \begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = f & \text{on } A \end{cases} \quad \text{with } v(x) = \mathbb{E}^x(f(X_{\tau_A})).$$

Proof Let $v(x)$ be a non-negative solution, then $v(X_{n \wedge \tau_A})_{n \geq 0}$ is a martingale. $v(x) = \mathbb{E}^x v(X_{n \wedge \tau_A}) = \mathbb{E}^x v(X_{n \wedge \tau_A}) 1_{\tau_A < \infty} + \mathbb{E}^x v(X_{n \wedge \tau_A}) 1_{\tau_A = \infty} \geq \mathbb{E} v(X_{n \wedge \tau_A}) 1_{\tau_A < \infty}$. Let $n \rightarrow \infty$ and by Fatou's lemma, we have $v(x) \geq \mathbb{E}^x v(X_{\tau_A}) 1_{\tau_A < \infty} = \mathbb{P}^x(\tau_A < \infty) = u(x)$. If $u(x) \equiv 1$ and $v(x)$ is bounded, then by bounded convergence theorem, $v(x) = \mathbb{E}^x v(X_{n \wedge \tau_A}) \rightarrow \mathbb{E}^x v(X_{\tau_A}) = \mathbb{E}^x f(X_{\tau_A})$. \square

- Doob's h -transform: Let h be nonnegative, harmonic with $h(x_0) = 1$ for some $x_0 \in S$. Then $(h(X_n))_{n \geq 0}$ is a martingale with $\mathbb{E}^{\mathbb{P}^{x_0}}(h(X_n)) = 1$. Then \exists ! measure \mathbb{Q}^h on \mathcal{F}_∞ s.t. $\frac{d\mathbb{Q}^h}{d\mathbb{P}^{x_0}}|_{\mathcal{F}_n} = h(X_n), \forall n \geq 0$. $\mathbb{Q}^h(X_0 = x_0) = 1$, $(X_n)_{n \geq 0}$ never visits the set $D = \{x : h(x) = 0\}$. Under \mathbb{Q}^h , $(X_n)_{n \geq 0}$ is again a Markov chain on $S \setminus D$ with transition probability $q(x, y) = \frac{p(x, y)h(y)}{h(x)}$ (or equivalently, $(\mathcal{L}^h f)(x) = \frac{1}{h(x)}(\mathcal{L}(hf))(x)$).

Proof The first two props are trivial. $\mathbb{Q}(X_{n+1} = y | \mathcal{F}_n) = \frac{\mathbb{Q}(X_{n+1}=y, X_n=x_n, \dots, X_0=x_0)}{\mathbb{Q}(X_n=x_n, \dots, X_0=x_0)} = \frac{\int_{\{X_{n+1}=y, X_n=x_n, \dots, X_0=x_0\}} h(X_{n+1}) d\mathbb{P}^{x_0}}{\int_{\{X_n=x_n, \dots, X_0=x_0\}} h(X_n) d\mathbb{P}^{x_0}} = \frac{h(y) \mathbb{P}^{x_0}(X_{n+1}=y, X_n=x_n, \dots, X_0=x_0)}{h(x_n) \mathbb{P}^{x_0}(X_n=x_n, \dots, X_0=x_0)} = \frac{h(y)p(x_n, y)}{h(x_n)}$. Next we show $M_n^f := f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (\mathcal{L}^h f)(X_k)$ is a \mathbb{Q} -martingale for any bounded f . Let $Z_n = \mathbb{E}^{\mathbb{Q}} f(X_{n+1}) | \mathcal{F}_n$. $\forall A \in \mathcal{F}_n$, $\int_A Z_n h(X_n) d\mathbb{P}^{x_0} = \int_A Z_n d\mathbb{Q} = \int_A f(X_{n+1}) d\mathbb{Q} = \int_A f(X_{n+1}) h(X_{n+1}) d\mathbb{P}^{x_0} = \mathbb{E}^{\mathbb{P}^{x_0}} [\mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1})1_A | \mathcal{F}_n)] = \mathbb{E}^{\mathbb{P}^{x_0}} [1_A \mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1}) | \mathcal{F}_n)] = \int_A \mathcal{P}(hf)(X_n) d\mathbb{P}^{x_0}$. Thus $Z_n = \frac{\mathcal{P}(hf)(X_n)}{h(X_n)}$ only depends on X_n , i.e. $(X_n)_{n \geq 0}$ is a MC on \mathbb{Q} with generator \mathcal{L}^h . \square

- An irreducible Markov chain $(X_n)_{n \geq 0}$ (1) is transient if $\exists x$ and $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty | X_0 = x) < 1$; (2) is recurrent if \exists a finite set $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty) = 1$ for all $x \in S$. (3) is positive recurrent if \exists a finite set $A \subset S$ s.t. $\mathbb{E}(\tau_A) < \infty$ for all $x \in S$.
- Foster-Lyapunov criterion: An irreducible MC on a countable state space S (1) is transient iff $\exists v : S \rightarrow \mathbb{R}^+$ and $A \subset S$ non-empty s.t. $\mathcal{L}v \leq 0$ on A^c and $v(x) < \inf_{y \in A} v(y)$ for some $x \in A^c$; (2) is recurrent iff $\exists v : S \rightarrow \mathbb{R}^+$ s.t. $\mathcal{L}v \leq 0$ on A^c where A is a finite set and $\{x : v(x) \leq N\}$ is finite for every N ; (3) is positive recurrent iff $\exists v : S \rightarrow \mathbb{R}^+$, $A \subset S$ finite, $\exists \epsilon > 0$ s.t. $\mathcal{L}v \leq -\epsilon$ on A^c and $\sum_{y \in S} p(x, y)V(y) < +\infty$ for all $x \in A$.

Proof (1) $v(X_{n \wedge \tau_A})_{n \geq 0}$ is a super-martingale, hence $v(x) \geq \mathbb{E}v(X_{n \wedge \tau_A}) \geq \mathbb{E}v(X_{n \wedge \tau_A})1_{\tau_A < \infty}$. Let $n \rightarrow \infty$ we know $v(x) \geq \mathbb{E}v(X_{\tau_A}1_{\tau_A < \infty}) \geq (\inf_{y \in A} v(y))\mathbb{P}^x(\tau_A < \infty) \Rightarrow \mathbb{P}^x(\tau_A < \infty) < \frac{v(x)}{\inf_{y \in A} v(y)} < 1$. (2) On $\{\tau_A = \infty\}$, $\limsup_{n \rightarrow \infty} v(X_{n \wedge \tau_A}) = +\infty$ a.s. Since $(v(X_{n \wedge \tau_A}))_{n \geq 0}$ is a nonnegative super-martingale, hence converges a.s., therefore $\lim_{n \rightarrow \infty} v(X_{n \wedge \tau_A}) = +\infty$ a.s. Note that $v(x) \geq \mathbb{E}v(X_{n \wedge \tau_A})1_{\tau_A = \infty}$. Since LHS is a finite number, we have $\mathbb{P}^x(\tau_A = \infty) = 0$. (3) $\mathbb{E}v(X_{n \wedge \tau_A}) | \mathcal{F}_{n-1} \leq v(X_{(n-1) \wedge \tau_A}) - \epsilon 1_{\tau_A \geq n}$. Taking expectation on the both sides, $\mathbb{E}v(X_{n \wedge \tau_A}) \leq \mathbb{E}v(X_{(n-1) \wedge \tau_A}) - \epsilon \mathbb{E}1_{\tau_A \geq n} \leq \dots \leq v(x) - \epsilon \sum_{k=1}^n \mathbb{P}^x(\tau_A \geq k) \Rightarrow \mathbb{E}^x \tau_A = \sum_{k=1}^{\infty} \mathbb{P}^x(\tau_A \geq k) \leq \frac{v(x)}{\epsilon} < \infty$. \square

- e.g. $h(x) = \frac{\mathbb{P}^x(\tau_A < \tau_B)}{\mathbb{P}^{x_0}(\tau_A < \tau_B)}$ is harmonic on $(A \cup B)^c$ with $h(x_0) = 1$ ($x_0 \in (A \cup B)^c$). Then $\forall x, y \in (A \cup B)^c, q(x, y) = \frac{h(y)p(x, y)}{h(x)} = \frac{\mathbb{P}^y(\tau_A < \tau_B)p(x, y)}{\mathbb{P}^x(\tau_A < \tau_B)} = \frac{\mathbb{P}^x(X_1=y, \tau_A < \tau_B)}{\mathbb{P}^x(\tau_A < \tau_B)} = \mathbb{P}^x(X_1 = y | \tau_A < \tau_B)$.
- e.g. \mathbb{P} is simple symmetric random walk on \mathbb{Z} starting from $X_0 = 0$. Question: what is the law of $(X_n)_{n \geq 0}$ conditioned on $X_n \geq 0$ for all n ? Let $\tau_k = \inf\{n \geq 0, X_n = k\}$. On $\{\tau_N < \tau_{-1}\}$, $\frac{h(y)}{h(x)} = \frac{\mathbb{P}^y(\tau_N < \tau_{-1})}{\mathbb{P}^x(\tau_N < \tau_{-1})} = \frac{y+1}{x+1}$. Thus $q_N(x, y) = \frac{1}{2} \frac{y+1}{x+1}, |x - y| = 1, x \in \{0, \dots, N-1\} \Rightarrow q(x, y) = \frac{1}{2} \frac{y+1}{x+1}, x \geq 0, |x - y| = 1$.

3 Ergodic Theorem

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