Theoretical Machine Learning

Lectured by Zhihua Zhang   LaTeXed by Chengxin Gong

May 5, 2024

Contents

1 Introduction

Outline 1.1 (Main tasks in machine learning) Generation, prediction, decision. Generation: $X_1, \cdots, X_n \sim F$, infer and analyse F, unsupervised learning, e.g. GAN, GPT, $\cdots$. Prediction: data pairs $(X^{(1)}, Y^{(1)}), \cdots, (X^{(n)}, Y^{(n)})$, input variables $X^{(i)} \in \mathbb{R}^d$, $f : \mathcal{X} \to \mathcal{Y}, x \in \mathcal{X}, y \in \mathcal{Y}$, ascribe, supervised learning. Decision: Reinforcement learning, Agent $\leftarrow$ action, state, reward $\rightarrow$ environment.

Outline 1.2 (Methods for solving tasks) Parameterized/Non-parameterized, frequency(MLE)/Bayesian.

Outline 1.3 (Modeling error) Supervised: Fix $X = (X_1, \cdots, X_d)^T \in \mathbb{R}^d$, for regression $Y \in \mathbb{R}$, for classificataion $Y \in \{0,1\}$(also $\{-1,1\}, \{1, \cdots, M\}, \{0,1\}^M$). Random design for X(known as generative models): $Y^{(i)} = g(X^{(i)}, Z^{(i)})$. Fixed design for X(known as discriminative models): $Y^{(i)} = g(x^{(i)}, Z^{(i)})$. Unsupervised: $X = g(Z)$(e.g. factor model: $X = AZ + \varepsilon, Z \in \mathcal{N}(0,1), \varepsilon \sim \mathcal{N}(0, \Sigma)$).

2 Statistical Decision Theory

Definition 2.1 (Basic concepts) Consider a state space Ω, data space $\mathcal{D}$, model $\mathcal{P} = \{p(\theta, x)\}$, action space $\mathscr{A}$. Loss function: $\mathcal{L} : \Omega \times \mathscr{A} \to [-\infty, +\infty]$, measurable, nonnegative. A measurable function $\delta : \mathcal{D} \to \mathscr{A}$ is called a nonrandomized decision rule. Risk function is defined as $\mathcal{R}(\theta, \delta) = \int \mathcal{L}(\theta, \delta(x)) \mathrm{d}P_\theta(x) = \mathbb{E}_\theta \mathcal{L}(\theta, \delta(X))$. Randomized decision: for each $X = x$, $\delta(x)$ is a probability distribution: $[A|X = x] \sim \delta_x$. Risk function for δ: $\mathcal{R}(\theta, \delta) = \mathbb{E}_\theta \mathcal{L}(\theta, A) = \mathbb{E}_\theta \mathbb{E}_a \mathcal{L}(\theta, A|X) = \iint \mathcal{L}(\theta, a) \mathrm{d}\delta_x(a) \mathrm{d}P_\theta(x)$.

Example 2.1 (Parameter estimation) $\theta \in \Omega, \mathscr{A} = \Omega, \mathcal{L}(\theta, a) = \|\theta - a\|_p^p (p \geq 1) \overset{\text{or}}{=} \int \log \frac{P_\theta(x)}{P_a(x)} P_\theta(x) \mathrm{d}m(x)$(KL divergence). $\mathcal{R} = \mathrm{Var}(a) + \mathrm{bias}^2(a)$. Bregmass loss: $\phi : \mathbb{R}^d \to \mathbb{R}$ describe any strictly convex differentiable function. Then $\mathcal{L}_\phi(\theta, a) = \phi(a) - \phi(\theta) - (\phi - a)^T \nabla \phi(a)$.

Example 2.2 (Testing) $\mathscr{A} = \{0, 1\}$ with action "0" associated with accepting $H_0 : \theta \in \Omega_0$ and "1": $H_1 : \theta \in \Omega_1$. δ_x is a Bernolli distribution. $\mathcal{L}(\theta, a) = I\{a = 1, \theta \in \Omega_0\} + I\{a = 0, \theta \in \Omega_1\}$. Risk $\mathcal{R}(\theta, \delta) = \mathbb{P}_\theta(A = 1) 1_{\theta \in \Omega_0} + \mathbb{P}_\theta(A = 0) 1_{\theta \in \Omega_1}$.

Definition 2.2 (Admissibility) A decision rule δ is called inadmissible if a competing rule δ^* such that $\mathcal{R}(\theta, \delta^*) \leq \mathcal{R}(\theta, \delta)$ for all $\theta \in \Omega$ and $\mathcal{R}(\theta, \delta^*) < \mathcal{R}(\theta, \delta)$ for at least one $\theta \in \Omega$. Otherwise, δ is admissible.

Definition 2.3 (Bayes rule) The maximum risk $\bar{\mathcal{R}}(\delta) = \sup_{\theta \in \Omega} \mathcal{R}(\theta, \delta)$ and the Bayes risk $r(\Lambda, \delta) = \int \mathcal{R}(\theta, \delta) \mathrm{d}\Lambda(\theta) = \int \mathcal{L}(\theta, \delta) \mathrm{d}\mathbb{P}(x, \theta)$ ($\Lambda(\theta)$ is a prior). A decision rule that minimizes the Bayes risk is called a Bayes rule, that is, $\hat{\delta} : r(\Lambda, \hat{\delta}) = \inf_\delta r(\Lambda, \delta)$. Minimax rule $\delta^* : \sup_{\theta \in \Omega} \mathcal{R}(\theta, \delta^*) = \inf_\delta \sup_{\theta \in \Omega} \mathcal{R}(\theta, \delta)$.

Theorem 2.1 If risk functions for all decision rules are continuous in θ, if δ is Bayesian for Λ and has finite integrated risk $r(\Lambda, \delta) < \infty$, and if the support of Λ is the whole state space Ω, then δ is admissible.

Property 2.1 $p(\theta|x) = \dfrac{p_\theta(x)\lambda(\theta)}{\int p_\theta(x)\lambda(\theta)\mathrm{d}\theta} := \dfrac{p_\theta(x)\lambda(\theta)}{m(x)}$. Define the posterior risk of δ: $r(\delta|X = x) = \int \mathcal{L}(\theta, \delta(x)) \mathrm{d}\mathbb{P}(\theta|x)$. The Bayes risk $r(\Lambda, \delta)$ satisfies that $r(\Lambda, \delta) = \int r(\delta|x) \mathrm{d}M(x)$. Let $\hat{\delta}(x)$ be the value of δ that minimizes $r(\delta|x)$. Then $\hat{\delta}$ is the Bayes rule.

Example 2.3 (Application to supervised learning: regression) $(X, Y) \in \mathcal{X} \times \mathcal{Y}, f : \mathcal{X} \to \mathcal{Y}, \mathscr{A} = \Omega = \mathcal{Y}, \mathcal{D} = \mathcal{X}, \delta = f, \mathcal{L}(Y, f(X)) = \|Y - f(X)\|_p^p, p \geq 1$, risk $R_f = \iint \mathcal{L}(y, f(x)) \mathrm{d}\mathbb{P}(x, y) = \mathbb{E}[\mathcal{L}(Y, f(X))] = \mathbb{E}[\mathbb{E}\mathcal{L}(Y, f(X))|X]$. When $p = 2$, $r(f|X = x) = \int \mathcal{L}(y, f(x)) \mathrm{d}\mathbb{P}(y|x) = \int |y - f(x)|^2 \mathrm{d}\mathbb{P}(y|x)$. Regression function is $g(x) := \int y \mathrm{d}\mathbb{P}(y|x) \Rightarrow R_f = \mathbb{E}|Y - f(X)|^2 = \mathbb{E}|Y - g(X) + g(X) - f(X)|^2 = \mathbb{E}|Y - g(X)|^2 + \mathbb{E}|g(X) - f(X)|^2 \geq \mathbb{E}|Y - g(X)|^2$.

Example 2.4 (Application to supervised learning: pattern classification) $Y \in \{0, 1\}, p_0 = P(Y = 0), p_1 = \mathbb{P}(Y = 1) = 1 - p_0, \mathbb{E}[\mathcal{L}(Y, f(X))] = \mathbb{P}(Y \neq f(X))$. The Bayesian predictor is given by $f(x) = 1_{\{\mathbb{P}(Y=1|X=x) \geq \frac{\mathcal{L}(1,0) - \mathcal{L}(0,0)}{\mathcal{L}(0,1) - \mathcal{L}(1,1)} \mathbb{P}(Y=0|X=x)\}}$.

Proof $\mathbb{E}[\mathcal{L}(Y, f(X))|X = x] = \begin{cases} \mathbb{E}[\mathcal{L}(Y, 0)|X = x] = \mathcal{L}(0, 0)\mathbb{P}(Y = 0|X = x) + \mathcal{L}(1, 0)\mathbb{P}(Y = 1|X = x) \\ \mathbb{E}[\mathcal{L}(Y, 1)|X = x] = \mathcal{L}(0, 1)\mathbb{P}(Y = 0|X = x) + \mathcal{L}(1, 1)\mathbb{P}(Y = 1|X = x) \end{cases}$, compare the sizes of the two. $\square$

Property 2.2 (Continuation) $\mathbb{P}(Y=1|X=x)=\mathbb{E}(Y|X=x):=g(x)$, $f(x)=1_{\{g(x)\geq\frac{1}{2}\}}$. Then $0\leq\mathbb{P}(\hat{f}(X)\neq Y)-\mathbb{P}(f(X)\neq Y)\leq 2\int_{\mathcal{X}}|\hat{g}(x)-g(x)|\mu(\mathrm{d}x)\leq 2(\int_{\mathcal{X}}|\hat{g}(x)-g(x)|^2\mu(dx))^{\frac{1}{2}}$. In Example 2.4, $f(x)=1_{\{\frac{p(x|y=1)}{p(x|y=0)}\geq\frac{p_0(\mathcal{L}(0,1)-\mathcal{L}(0,0))}{p_1(\mathcal{L}(1,0)-\mathcal{L}(1,1))}\}}$, which takes the same form as the likelihood ratio test (LRT): Likelihood $L(X):=\frac{p(X|Y=1)}{p(X|Y=0)}$ and $f(x)=1_{\{L(x)\geq\eta\}}$.

Definition 2.4 (Confusion table) Ture Positive Rate: $\mathrm{TPR}=\mathbb{P}(\hat{Y}=1|Y=1)$; False Negative Rate: $\mathrm{FNR}=1-\mathrm{TPR}$, type II error; False Positive Rate: $\mathrm{FPR}=\mathbb{P}(\hat{Y}=1|Y=0)$, type I error; True Negative Rate: $\mathrm{TNR}=1-\mathrm{FPR}$. Precision: $\mathbb{P}(Y=1|\hat{Y}=1)=\frac{p_1\mathrm{TPR}}{p_0\mathrm{FPR}+p_1\mathrm{TPR}}$. F_1-score: F_1 is the harmonic mean of precision and recall, which can be written as $F_1=\frac{2\mathrm{TPR}}{1+\mathrm{TPR}+\frac{p_0}{p_1}\mathrm{FPR}}$.

	$Y=0$	$Y=1$
$\hat{Y}=0$	true negative	false negative
$\hat{Y}=1$	false positive	true positive

Theorem 2.2 (N-P lemma) Optimization: maximize TPR subject to $\mathrm{FPR}\leq\alpha,\alpha\in[0,1]$. Randomized rule: Q return 1 with probability $Q(x)$ and 0 with probability $1-Q(x)$. Maximize $\mathbb{E}[Q(x)|Y=1]$ subject to $\mathbb{E}[Q(x)|Y=0]\leq\alpha$. Suppose the likelihood functions $p(x|y)$ are continuous. Then the optimal predictor is a deterministic LRT.

Proof Let η be the threshold for an LRT such that the predictor $Q_\eta(x)=1\{\alpha(x)\geq\eta\}$ has $\mathrm{FPR}=\alpha$. Such an LRT exists because likelihood functions are continuous. Let β denote the TPR of Q_η. Prove that Q_η is optimal for risk minimization problem corresponding to the loss functions $\mathcal{L}(0,1)=\eta\frac{p_1}{p_0},\mathcal{L}(1,0)=1,\mathcal{L}(1,1)=\mathcal{L}(0,0)=0$ since $\frac{p_0(\mathcal{L}(0,1)-\mathcal{L}(0,0))}{p_1(\mathcal{L}(1,0)-\mathcal{L}(1,1))}=\frac{p_0\mathcal{L}(0,1)}{p_1\mathcal{L}(1,0)}=\eta$. Under these loss functions, the risk of Bayes predictor for Q is $\mathcal{R}_Q=p_0\mathrm{FPR}(Q)\mathcal{L}(0,1)+p_1(1-\mathrm{TPR}(Q))\mathcal{L}(1,0)=p_1\eta\mathrm{FPR}(Q)+p_1(1-\mathrm{TPR}(Q))$. Now let Q be any other rule with $\mathrm{FPR}(Q)\leq\alpha$, $\mathcal{R}_{Q_\eta}=p_1\eta\alpha+p_1(1-\beta)\leq p_1\eta\mathrm{FPR}(Q)+p_1(1-\mathrm{TPR}(Q))\leq p_1\eta\alpha+p_1(1-\mathrm{TPR}(Q))\Rightarrow\mathrm{TPR}(Q)\leq\beta$. $\square$

Definition 2.5 (ROC (Receiver operating character) curve) y-axis is TPR and x-axis is FPR.

Proposition 2.1 (1) The points $(0,0)$ and $(1,1)$ are on the ROC curve; (2) The ROC must lie above the main diagnal; (3) The ROC curve is concave.

Proof We only prove (2). Fix $\alpha\in(0,1)$ and consider a randomized rate $\mathrm{TPR}=\mathrm{FPR}=\alpha$, $Q(x)\equiv\alpha$; (3): Consider two rules $(\mathrm{FPR}(\eta_1),\mathrm{TPR}(\eta_1))$ and $(\mathrm{FPR}(\eta_2),\mathrm{TPR}(\eta_2))$. Flip a biased coin and use the first rule with probability t and the second rule with probability $1-t$. Then this yields a randomized rule with $(\mathrm{FPR},\mathrm{TPR})=(t\mathrm{FPR}(\eta_1)+(1-t)\mathrm{FPR}(\eta_2),$ $t\mathrm{TPR}(\eta_1)+(1-t)\mathrm{FPR}(\eta_2))$. Fixing $\mathrm{FPR}\leq t\mathrm{FPR}(\eta_1)+(1-t)\mathrm{FPR}(\eta_2)$, $\mathrm{TPR}\geq t\mathrm{TPR}(\eta_1)+(1-t)\mathrm{TPR}(\eta_2)$. $\square$

3 Markov Decision Process

Definition 3.1 (Basic concepts) Five elements: decision epoches, states, actions, transition probabilities and rewards. (1) Decision epoches: Let T denote the set of decision epoches, discrete: $\{1,2,\cdots,N\}$; continuous: $[0,N]$; $N</=\infty$: finite or infinite. (2) State and action sets: decision epoch $t\in T$, the system occupies a state $S_t\in\mathcal{S}$, the decision maker $a\in\mathcal{A}$. (3) Reward and transition probabilities: t, in state s, choose action a, (i) the decision maker receives a reward $r_t(s,a)$, (ii) the system state at the next decision epoch is determined by the probability distribution $p_t(\cdot|s_t,a)$.

Definition 3.2 (Decision rules) Prescribe a procedure for action selection in each state at a specified decision epoch. Four cases: (1) Markovian and Deterministic (MD): $\delta_t:\mathcal{S}\to\mathcal{A}$; (2) M and Randomized (MR): $\delta_t:\mathcal{S}\to\Delta(\mathcal{A})(q_{\delta_t(s)}(a))$; (3) History-dependent and D (HD): $h_t=(s_1,a_1,\cdots,s_{t-1},a_{t-1},s_t)=(h_{t-1},a_{t-1},s_t),\mathcal{H}_1=\mathcal{S},\mathcal{H}_2=\mathcal{S}\times\mathcal{A}\times\mathcal{S},\cdots,\delta_t:\mathcal{H}_t\to\mathcal{A}$; (4) HR: $\delta_t:\mathcal{H}_t\times\Delta(\mathcal{A})$. A policy $\pi=(\delta_1,\delta_2,\cdots,\delta_{N-1})$ is stationary if $\delta_1=\delta_2=\cdots=\delta$ for $t\in T$.

Definition 3.3 Let $\pi=(\delta_1,\cdots,\delta_{N-1})$ in HR and $R_t:=r_t(X_t,Y_t)$ denote the random reward, $R_N:=r_N(X_N)$, $R:=(R_1,\cdots,R_N)$. The expected total reward $U_N^\pi(s):=\mathbb{E}^\pi\{\sum_{t=1}^{N-1}r_t(X_t,Y_t)+r_N(X_N)|X_1=s\}$. Assume $|r_t(s,a)|\leq M<\infty$ for all $(s,a)\in\mathcal{S}\times\mathcal{A}$. Optimal policy: $U_N^{\pi^*}(s)\geq U_N^\pi(s),s\in\mathcal{S}$. ε-optimal policy: $U_N^{\pi_\varepsilon^*}(s)+\varepsilon>U_N^\pi(s),s\in\mathcal{S}$. The value of the MDP: $U_N^*(s)=\sup_{\pi\in\mathcal{D}^{\mathrm{HR}}}U_N^\pi(s),s\in\mathcal{S}$.

Property 3.1 (Finite-Horizon Policy Evaluation) $V_t^\pi(h_t) = \mathbb{E}^\pi\{\sum_{k=t}^{N-1} r_k(X_k, Y_k) + r_N(X_N)|h_t\}, V_N^\pi(h_N) = r_N(s), \pi \in \mathcal{D}^{\mathrm{HD}}$. By the formula of total expectation,

$$V_t^\pi(h_t) = r_t(s_t, \delta_t(h_t)) + \mathbb{E}_{h_t}^\pi V_{t+1}^\pi(h_t, \delta_t(h_t), X_{t+1}) = r_t(s_t, \delta_t(h_t)) + \sum_{j\in\mathcal{S}} V_{t+1}^\pi(h_t, \delta_t(h_t), j)p(j|s_t, \delta_t(h_t)).$$

Consider randomness, i.e. $\pi \in \mathcal{D}^{\mathrm{HR}}$,

$$V_t^\pi(h_t) = \sum_{a\in\mathcal{A}} q_{\delta_t(h_t)}(a)\{r_t(s_t, a) + \sum_{j\in\mathcal{S}} V_{t+1}^\pi(h_t, a, j)p(j|s_t, a)\}.$$

Computational complexity: let $K = |\mathcal{S}|, L = |\mathcal{A}|$, at decision epoch t, $K^{t+1}L^t$ histories, $K^2\sum_{i=0}^{N-1}(KL)^i$ multiplications. If $\pi \in \mathcal{D}^{\mathrm{MD}}$,

$$V_t^\pi(s_t) = r_t(s_t, \delta_t(s_t)) + \sum_{j\in\mathcal{S}} V_{t+1}^\pi(j)p(j|s_t, \delta_t(s_t)),$$

only $(N-1)K^2$ multiplications. On the other hand, given π, this yields a valid and accurate calculation method for $U_N^\pi(s)$.

Theorem 3.1 (The Bellman Equations) Let $V_t^*(h_t) = \sup_{\pi\in\mathcal{D}^{\mathrm{HR}}} V_t^\pi(h_t)$. The optimality equations:

$$V_t(h_t) = \sup_{a\in\mathcal{A}}\{r_t(s_t, a) + \sum_{j\in\mathcal{S}} V_{t+1}(h_t, a, j)p_t(j|s_t, a)\} \text{ for } t = 1, 2, \cdots, N-1 \text{ and } h_t = (h_{t-1}, a_{t-1}, s_t) \in \mathcal{H}_t.$$

For $t = N, V_N(h_N) = r_N(s_N)$. Suppose V_t is a solution and V_N satisfies $V_N(h_N) = r_N(s_N)$. Then $V_t(h_t) = V_t^*(h_t)$ for all $h_t \in \mathcal{H}_t, t = 1, \cdots, N$ and $V_1(s_1) = V_1^*(s_1) = U_N^*(s_1)$ for all $s_1 \in \mathcal{S}$.

Proof We divide the proof into two parts.

Step 1: Prove $V_n(h_n) \geq V_n^*(h_n)$ for all $h_n \in \mathcal{H}_n$. By induction: For $t = N$, $V_N(h_N) = r_N(s_N) = V_N^*(h_N)$ for all h_t, π. Now assume that $V_t(h_t) \geq V_t^*(h_t)$ for all $h_t \in \mathcal{H}_t$ for $t = n+1, \cdots, N$. Let $\pi' = (\delta_1', \cdots, \delta_{N-1}')$ be an arbitrary policy in $\mathcal{D}^{\mathrm{HR}}$. On the one hand, for $t = n$, it is trivial that

$$\begin{aligned}V_n(h_n) &= \sup_{a\in\mathcal{A}}\{r_n(s_t, a_t) + \sum_{j\in\mathcal{S}} p(j|s_n, a)V_{n+1}(h_n, a, j)\} \geq \sup_{a\in\mathcal{A}}\{r_n(s_n, a) + \sum_{j\in\mathcal{S}} p_n(j|s_n, a)V_{n+1}^*(h_n, a, j)\} \\ &\geq \sup_{a\in\mathcal{A}}\{r_n(s_n, a) + \sum_{j\in\mathcal{S}} p_n(j|s_n, a)V_{n+1}^{\pi'}(h_n, a, j)\} \geq V_n^{\pi'}(h_n).\end{aligned}$$

Step 2: Prove that for any $\varepsilon > 0$, there exists a $\pi \in \mathcal{D}^{\mathrm{HD}}$ such that

$$V_n^{\pi'}(h_n) + (N-n)\varepsilon \geq V_n(h_n) \Rightarrow V_n^*(h_n) + (N-n)\varepsilon \geq V_n^{\pi'}(h_n) + (N-n)\varepsilon \geq V_n(h_n) \geq V_n^*(h_n).$$

Construct a policy $\pi' = (\delta_1', \cdots, \delta_{N-1}')$ by choosing $\delta_n'(h_n)$ to satisfy

$$r_n(s_n, \delta_n'(h_n)) + \sum_{j\in\mathcal{S}} p_n(j|s_n, \delta_n'(h_n))V_{n+1}(h_n, \delta_n'(h_n)) + \varepsilon \geq V_n(h_n).$$

By induction: For $t = N$, $V_N^{\pi'}(h_N) = V_N(h_N)$. Assume $V_t^{\pi'}(h_t) + (N-t)\varepsilon \geq V_t(h_t)$ for $t = n+1, \cdots, N$. For $t = n$,

$$V_n^{\pi'}(h_n) = r_n(s_n, \pi_n'(h_n)) + \sum_{j\in\mathcal{S}} p_n(j|s_n, \delta_n^{\pi'}(h_n))V_{n+1}^{\pi'}(h_n, \delta_n^{\pi'}(h_n), j) \geq V_n(h_n) - (N-n)\varepsilon.$$ □

Remark 3.1 The equations yield that $\delta_t^*(h_t) \in \arg\max_{a\in\mathcal{A}}\{r_t(s_t, a) + \sum_{j\in\mathcal{S}} p_t(s_t, a)V_{t+1}^*(h_t, a, j)\}$, which means it is HD, i.e. $U_N^*(s) = \sup_{\pi\in\mathcal{D}^{\mathrm{HR}}} U_N^\pi(s) = \sup_{\pi\in\mathcal{D}^{\mathrm{HD}}} U_N^\pi(s) \overset{?}{=} \sup_{\pi\in\mathcal{D}^{\mathrm{MD}}} U_N^\pi(s)$. We will answer "?" in the following theorem.

Theorem 3.2 Let $V_t^*, t = 1, \cdots, N$ be solutions of Bellman Equations. Then (a) For each $t = 1, \cdots, N, V_t^*(h_t)$ depends on h_t only through s_t; (b) For any $\varepsilon > 0$, there exists an ε-optimal policy which is D and M; (c) Maximum can be achieved, it is optimal, which is MD.

Proof We only prove (a). By induction, $V_N^*(h_N) = V_N^*(h_{N-1}, a_{N-1}, s) = r_N(s)$ for all $h_{N-1} \in \mathcal{H}_{N-1}$. Assume (a) is valid for $t = n+1, \cdots, N$. Then $V_n^*(h_n) = \sup_{a \in \mathcal{A}}\{r_t(s_t, a) + \sum\limits_{j \in \mathcal{S}} p_t(j|s_t, a)V_{t+1}^*(j)\} = V_n^*(s_t)$. ☐

Definition 3.4 (Backward Indcution (Dynamic Programming) Algorithm) 1. Set $t = N$ and $V_N^*(s_N) = r_N(s_N)$ for all $s_N \in \mathcal{S}$; 2. Substitute $t-1$ for t and compute $V_t^*(s_t)$ for each $s_t \in \mathcal{S}$ according to

$$V_t^*(s_t) = \max_{a \in \mathcal{A}}\{r_t(s_t, a) + \sum_{j \in \mathcal{S}} p_t(j|s_t, a)V_{t+1}^*(s_t)\},$$

and set $\mathcal{A}_{s_t} = \arg\max_{a \in \mathcal{A}}\{r_t(s_t, a) + \sum\limits_{j \in \mathcal{S}} p_t(j|s_t, a)V_{t+1}^*(s_t)\}$; 3. If $t = 1$, stop. Otherwise return to Step 2.

Remark 3.2 (1) At time t, specialized $\mathcal{S}_t$ and $\mathcal{A}_s$, special structure for r_t and p_t; (2) $K = |\mathcal{S}|$ and $L = |\mathcal{A}|$, at eact t, only $(N-1)LK^2$ multiplications, ease computation and storage cost (because there are $(L^K)^{N-1}$ DM policies).

Definition 3.5 (Infinite-Horizon MDPs) Assumptions: Stationary reward and transition probabilities, i.e. $r_t(s, a) \equiv r(s, a), p_t(j|s, a) \equiv p(j|s, a)$; Bounded rewards, i.e. $|r(s, a)| \leq M < \infty$ for all $a \in \mathcal{A}$ and $s \in \mathcal{S}$; Discounting coefficient $\lambda, 0 \leq \lambda < 1$; Discrete state space $\mathcal{S}$. The expected total reward of policy $\pi = (\delta_1, \delta_2, \cdots) \in \mathcal{D}^{\mathrm{HR}}$:

$$U^\pi(s) = \lim_{N \to +\infty} \mathbb{E}_s^\pi\{\sum_{t=1}^{N} \lambda^{t-1} r(X_t, Y_t)\} = \mathbb{E}_s^\pi\{\sum_{t=1}^{+\infty} \lambda^{t-1} r(X_t, Y_t)\}.$$

We say that a policy π^* is optimal when $U^{\pi^*}(s) \geq U^\pi(s)$ for each $s \in \mathcal{S}$ and all $\pi \in \mathcal{D}^{\mathrm{HR}}$. Define the value of the MDP $U^*(s) = \sup_{\pi \in \mathcal{D}^{\mathrm{HR}}} U^\pi(s)$. Let $U_\nu^\pi(s)$ denote the expected reward obtained by using π when the horizon ν is random. Then $U_\nu^\pi(s) = \mathbb{E}_s^\pi\{\mathbb{E}_{\nu \sim P} \sum\limits_{t=1}^{\nu} r(X_t, Y_t)\}$.

Theorem 3.3 Suppose ν has a GD(λ), i.e. $\mathbb{P}(\nu = n) = \lambda^{n-1}(1-\lambda)$. Then $U^\pi(s) = U_\nu^\pi(s)$ for all $s \in \mathcal{S}$.

Proof $\mathbb{E}_\nu^\pi(s) = \mathbb{E}_s^\pi\{\sum\limits_{n=1}^{+\infty}\sum\limits_{t=1}^{n} r(X_t, Y_t)(1-\lambda)\lambda^{n-1}\} = \mathbb{E}_s^\pi\{\sum\limits_{t=1}^{+\infty}\sum\limits_{n=t}^{+\infty} r(X_t, Y_t)(1-\lambda)\lambda^{n-1}\} = \mathbb{E}_s^\pi\{\sum\limits_{t=1}^{+\infty} \lambda^{t-1} r(X_t, Y_t)\}$. ☐

Theorem 3.4 Suppose $\pi \in \mathcal{D}^{\mathrm{HR}}$, then for each $s \in \mathcal{S}$, there exists a $\pi' \in \mathcal{D}^{\mathrm{MR}}$ for which $U^{\pi'}(s) = U^\pi(s)$.

Proof Note that

$$U^\pi(s) = \mathbb{E}_s^\pi\{\sum_{t=1}^{+\infty} \lambda^{t-1} r(X_t, Y_t)\} = \sum_{t=1}^{+\infty}\sum_{j \in \mathcal{S}}\sum_{a \in \mathcal{A}} \lambda^{t-1} r(j, a) p^\pi(X_t = j, Y_t = a|X_1 = s).$$

Fixing $s \in \mathcal{S}$, we only need to check

$$p^\pi(X_t = j, Y_t = a|X_1 = s) = p^{\pi'}(X_t = j, Y_t = a|X_1 = s).$$

For each $j \in \mathcal{S}$ and $a \in \mathcal{A}$, define the randomized Markov decision rule δ_t' by

$$q_{\delta_t'(j)}(a) = p^\pi(Y_t = a|X_t = j, X_1 = s).$$

Then

$$p^{\pi'}(Y_t = a|X_t = j) = p^\pi(Y_t = a|X_t = j, X_1 = s).$$

Assume the conclusion holds for $t = 0, 1, \cdots, n-1$. Then

$$\begin{aligned} p^{\pi'}(X_n = j, Y_n = a|X_1 = s) &= p^{\pi'}(Y_n = a|X_n = j, X_1 = s)p^{\pi'}(X_n = j|X_1 = s) \\ &= p^\pi(Y_n = a|X_n = j, X_1 = s)p^{\pi'}(X_n = j|X_1 = s). \end{aligned}$$

Then by induction assumption,

$$\begin{aligned} p^\pi(X_n = j|X_1 = s) &= \sum_{k \in \mathcal{S}}\sum_{a \in \mathcal{A}} p^\pi(X_{n-1} = k, Y_{n-1} = a|X_1 = s)p(j|k, a) \\ &= \sum_{k \in \mathcal{S}}\sum_{a \in \mathcal{A}} p^{\pi'}(X_{n-1} = k, Y_{n-1} = a|X_1 = s)p(j|k, a) = p^{\pi'}(X_n = j|X_1 = s) \end{aligned}$$ ☐

Proposition 3.1 (Vector expression for MDP) Let δ be MD, define $r_\delta(s)$ and $p_\delta(j|s)$ by

$$r_\delta(s) := r(s, \delta(s)), p_\delta(j|s) := p(j|s, \delta(s)).$$

Denote $r_\delta = (r_\delta(1), \cdots, r_\delta(|\mathcal{S}|))^T \in \mathbb{R}^{|\mathcal{S}|}, p_\delta = (p_\delta)_{(s,j)} = p(j|s, \delta(s))$. For MR δ, define

$$r_\delta(s) = \sum_{a\in\mathcal{A}} q_{\delta(s)}(a) r(s,a), p_\delta(j|s) = \sum_{a\in\mathcal{A}} q_{\delta(s)}(a) p(j|s,a).$$

The (s,j)-th component of the t-step transition probability matrix p_π^t satisfies

$$\begin{aligned}
p_\pi^t(j|s) &= [p_{\delta_1} p_{\delta_2} \cdots p_{\delta_t}](j|s) = p^\pi(X_{t+1} = j | X_1 = s)\\
\mathbb{E}_s^\pi g(X_t) &= \sum_{j\in\mathcal{S}} p_\pi^{t-1}(j|s) g(j) = (p_\pi^t g)_s\\
U^\pi &= \sum_{t=1}^{+\infty} \lambda^{t-1} p_\pi^{t-1} r_{\delta_t} = r_{\delta_1} + \lambda p_{\delta_1}(r_{\delta_1} + \lambda p_{\delta_2} r_{\delta_2} + \cdots) = r_{\delta_1} + \lambda p_{\delta_1} U^{\pi_1}.
\end{aligned}$$

When π is stationary, $U = r_\delta + \lambda p_\delta U$.

Theorem 3.5 Define $\mathscr{L}U = \sup_{d\in\mathcal{D}^{\mathrm{MD}}}\{r_d + \lambda p_d U\}$. Suppose there exists a $U \in \mathcal{U}$ for which (a) $U \geq \mathscr{L}U$, then $U \geq U^*$; (b) $U \leq \mathscr{L}U$, then $U \leq U^*$; (c) $U = \mathscr{L}U$, then $U = U^*$.

Proof (a) By the given conditions,

$$\begin{aligned}
U &\geq \sup_{\delta\in\mathcal{D}^{\mathrm{MR}}}\{r_d + \lambda p_d U\} \geq r_{\delta_1} + \lambda p_{\delta_1} U \geq r_{\delta_1} + \lambda p_{\delta_1}(r_{\delta_2} + \lambda p_{\delta_2} U)\\
&\geq r_{\delta_1} + \lambda p_{\delta_1} r_{\delta_2} + \cdots + \lambda^{n-1} p_{\delta_1} p_{\delta_2} \cdots p_{\delta_{n-1}} r_{\delta_n} + \lambda^n p_\pi^n U\\
\Rightarrow U - U^\pi &\geq \lambda^n p_\pi^n U - \sum_{k=n}^{+\infty} \lambda^k p_\pi^k r_{\delta_{k+1}} \geq 0.
\end{aligned}$$

(b) $U \leq \mathscr{L}U \Rightarrow U \leq r_d + \lambda p_d U + \varepsilon 1 \Rightarrow (I - \lambda p_d) U \leq r_d + \varepsilon 1 \Rightarrow U \leq (I - \lambda p_d)^{-1}(r_d + \varepsilon 1) = U^\pi + \varepsilon(1-\lambda)^{-1} 1_{|\mathcal{S}|}$.
(c) Omitted. □

Theorem 3.6 If $0 \leq \lambda < 1$, $\mathscr{L}$ is a contraction mapping on $\mathcal{U}$.

Proof Let u and v in $\mathcal{U}$. For each $s \in \mathcal{S}$, assume $\mathscr{L}v(s) \geq \mathscr{L}u(s)$ and let $a_s^* = \arg\max_{a\in\mathcal{A}}\{r(s,a) + \sum_{j\in\mathcal{S}} \lambda p(j|s,a) v(j)\}$. Then

$$\begin{aligned}
0 &\leq \mathscr{L}v(s) - \mathscr{L}u(s) \leq r(s, a_s^*) + \sum_{j\in\mathcal{S}} \lambda p(j|s, a_j^*) v(j) - r(s, a_j^*) - \sum_{j\in\mathcal{S}} \lambda p(j|s, a_s^*) u(j)\\
&= \lambda \sum_{j\in\mathcal{S}} p(j|s, a_s^*)(v(j) - u(j)) \leq \lambda \sum_{j\in\mathcal{S}} p(j|s, a_s^*) \| |u - v\| = \lambda \|u - v\|.
\end{aligned}$$

□

4 Statistical Learning Theory

Definition 4.1 (Basic concepts) $(X,Y) \sim P \in \mathcal{P}$, definite $(X_1, Y_1), \cdots, (X_n, Y_n)$ i.i.d., $\mathcal{D}_n = \{(X_1, Y_1), \cdots, (X_n, Y_n)\}$, risk $\mathcal{R}_n(f) = \mathbb{E}_{(X,Y)\in\mathcal{D}_n} l(X,Y)$. An algorithm A is a mapping from $\mathcal{D}_n$ to a function $\mathcal{X} \to \mathcal{Y}$. Excess risk: $\mathcal{R}_P(A(\mathcal{D}_n)) - \mathcal{R}_P^*$. Expected error: $\mathbb{E}[\mathcal{R}_P(A(\mathcal{D}_n))]$. An algorithm is called consistent in expectation for P iff $\mathbb{E}[\mathcal{R}_P(A(\mathcal{D}_n))] - \mathcal{R}_P^* \to 0$. PAC (probability approximately correct): for a given $\delta \in (0,1)$ and $\varepsilon > 0$, $\mathbb{P}(\mathcal{R}_P(A(\mathcal{D}_n)) - \mathcal{R}_P^* \leq \varepsilon) \geq 1 - \delta$.

Definition 4.2 (Consistency) $g(x) = \mathbb{E}[Y|X=x], g_n(x, \mathcal{D}_n) = g_n(x), \mathbb{E}\{|g_n(X) - Y|^2 | \mathcal{D}_n\} = \int_{\mathbb{R}^d} |g_n(x) - g(x)|^2 \mu(\mathrm{d}x) + \mathbb{E}|g(X) - Y|^2$. A sequence of regression function estimates $\{g_n\}$ is called (a) weakly consistent for a certain distribution of (X,Y) if $\lim_{n\to+\infty} \mathbb{E}\{\int [g_n(x) - g(x)] \mu(\mathrm{d}x)\} = 0$; (b) strongly consistent for a certain distribution if $\lim_{n\to+\infty} \int [g_n(x) - g(x)]^2 \mu(\mathrm{d}x) = 0$ with probability 1; (c) weakly universally consistent if for all distributions of (X,Y) with $\mathbb{E}[Y^2] < \infty$, $\cdots$; (d) strongly universally consistent $\cdots$.

Definition 4.3 (Penalized model) $g_n = \arg\min_f\{\frac{1}{n}\sum_{i=1}^n |f(X_i) - Y_i|^2 + J_n(f)\}$. Penalized term for f:

$$J_n(f) = \lambda_n \int |f''(t)|^2 \mathrm{d}t \text{ or } J_{n,k}(f) = \lambda_n \int \sum_{t_1,\cdots,t_k\in\{1,\cdots,d\}} \left|\frac{\partial f^k}{\partial x_{t_1} \cdots \partial x_{t_d}}\right|^2 \mathrm{d}t, \cdots$$

Proposition 4.1 (Curse of dimensionality) Let $X, X_1, \cdots, X_n$ i.i.d. $\mathbb{R}^d$ uniformly distributed in $[0,1]^d$.

$$\begin{aligned} d_\infty(d,n) = \mathbb{E}\{\min_{i=1,\cdots,n} \|X - X_i\|_\infty\} &= \int_0^\infty \mathbb{P}\{\min_{i=1,\cdots,n} \|X - X_i\|_\infty > t\}\mathrm{d}t \\ &= \int_0^\infty (1 - \mathbb{P}\{\min_{i=1,\cdots,n} \|X - X_i\|_\infty < t\})\mathrm{d}t. \end{aligned}$$

Since $\mathbb{P}\{\min_i \|X - X_i\|_\infty < t\} \leq n\mathbb{P}(\|X - X_1\|_\infty \leq t) \leq n(2t)^d$, $d_\infty(d,n) \geq \frac{d}{2(d+1)} n^{-\frac{1}{d}}$.

Theorem 4.1 (No-Free lunch theorem) Let $\{a_n\}$ be a sequence of positive numbers converging to 0. For every sequence of regression estimates, there exists a distribution of (X,Y) such that X is uniformly distributed on $[0,1]$, $Y = g(X)$, g is ± 1 valued, and $\limsup\limits_{n\to+\infty} \frac{\mathbb{E}\|g_n - g\|^2}{a_n} \geq 1$.

Proof Let $\{p_j\}$ be a probability distribution and let $A = \{A_j\}$ be a partition of $[0,1]$ such that A_j is an interval of length p_j. Consider regression function indexed by a parameter $c = (c_1, c_2, \cdots)$ with $c_j \in \{\pm 1\}$. Define $g^{(c)} : [0,1] \to \{-1,1\}$ by $g^{(c)}(x) = c_j$ iff $x \in A_j$ and $Y = g^{(c)}(X)$. For $x \in A_j$, define $\bar{g}_n(x) = \frac{1}{p_j}\int_{A_j} g_n(z)\mu(\mathrm{d}z)$ to be the projection of g_n on A. Then

$$\begin{aligned} \int_{A_j} |g_n(x) - g^{(c)}(x)|^2\mu(\mathrm{d}x) &= \int_{A_j} |g_n(x) - \bar{g}_n(x)|^2\mu(\mathrm{d}x) + \int_A |\bar{g}_n(x) - g^{(c)}(x)|^2\mu(\mathrm{d}x) \\ &\geq \int_A |\bar{g}_n(x) - g^{(c)}(x)|^2\mu(\mathrm{d}x). \end{aligned}$$

Set $\hat{c}_{nj} = \begin{cases} 1 & \text{if } \int_{A_j} g_n(z)\mu(\mathrm{d}z) \geq 0 \\ -1 & \text{otherwise} \end{cases}$. For $x \in A_j$, if $\hat{c}_{nj} = 1$ and $c_j = -1$, then $\bar{g}_n(x) \geq 0$ and $g^{(c)}(x) = -1$, implying $|\bar{g}_n(x) - g^{(c)}(x)| \geq 1$; if $\hat{c}_{nj} = -1$ and $c_j = 1$, then $\bar{g}_n(x) < 0$ and $g^{(c)}(x) = 1$, also implying $|\bar{g}_n(x) - g^{(c)}(x)|^2 \geq 1$. Therefore,

$$\begin{aligned} &\int_A |\bar{g}_n(x) - g^{(c)}(x)|^2\mu(\mathrm{d}x) \geq 1_{\{\hat{c}_{nj}\neq c_j\}} \int_{A_j} 1\mu(\mathrm{d}x) \geq 1_{\{\hat{c}_{nj}\neq c_j\}} p_j \geq 1_{\{\hat{c}_{nj}\neq c_j\}} 1_{\{\mu_n(A_j)=0\}} p_j \\ \Rightarrow\ &\mathbb{E}\left\{\int |g_n(x) - g^{(c)}(x)|^2\mu(\mathrm{d}x)\right\} \geq \sum_{j=1}^{+\infty} \mathbb{P}(\hat{c}_{nj} \neq c_j, \mu_n(A_j) = 0)p_j := R_n(c). \end{aligned}$$

Now we randomize c. Let $C_1, C_2, \cdots$ be a sequence of i.i.d. random variables independent of $X_1, X_2, \cdots$ which satisfy $\mathbb{P}(c_1 = 1) = \mathbb{P}(c_1 = -1) = \frac{1}{2}$. Thus

$$\begin{aligned} \mathbb{E}R_n(C) &= \sum_{j=1}^{+\infty} \mathbb{E}\mathbb{P}(\hat{C}_{nj} \neq C_j, \mu_n(A_j) = 0)p_j \overset{\text{total expectation}}{=} \sum_{j=1}^{+\infty} \mathbb{E}\{1_{\{\mu_n(A_j)=0\}}\mathbb{P}(\hat{C}_{nj} \neq C_j | X_1, \cdots, X_n)\}p_j \\ &= \frac{1}{2}\sum_{j=1}^{+\infty} \mathbb{P}(\mu_n(A_j) = 0)p_j = \frac{1}{2}\sum_{j=1}^{+\infty}(1 - p_j)^n p_j. \end{aligned}$$

On the other hand,

$$R_n(c) \leq \sum_{j=1}^{+\infty} \mathbb{P}(\mu_n(A_j) = 0)p_j = \sum_{j=1}^{+\infty}(1 - p_j)^n p_j \Rightarrow \frac{R_n(c)}{\mathbb{E}R_n(C)} \leq 2.$$

By Fatou's lemma,

$$\mathbb{E}\left\{\limsup_{n\to+\infty} \frac{R_n(C)}{\mathbb{E}R_n(C)}\right\} \geq \limsup_{n\to+\infty} \left\{\frac{R_n(C)}{\mathbb{E}R_n(C)}\right\} = 1,$$

which implies that there exists $c \in C$ such that

$$\limsup_{n\to+\infty} \frac{R_n(C)}{\mathbb{E}R_n(C)} \geq 1 \Rightarrow \limsup_{n\to+\infty} \frac{\mathbb{E}\{\int |g_n(x) - g(x)|^2\mu(\mathrm{d}x)\}}{\frac{1}{2}\sum\limits_{j=1}^{+\infty}(1 - p_j)^n p_j} \geq 1.$$

Let $\{a_n\}$ be a sequence of positive numbers converging to 0 with $\frac{1}{2} \geq a_1 \geq a_2 \geq \cdots$, then there exists a probability $\{p_j\}$ such that $\sum\limits_{j=1}^{+\infty}(1 - p_j)^n p_j \geq a_n, \forall n$. $\square$

Definition 4.4 (Minimax lower bounds) (a) The sequence of positive numbers a_n is called the lower minimax rate of convergence for the $\mathcal{P}$ if $\liminf_{n\to+\infty}\inf_{g_n}\sup_{P\in\mathcal{P}}\frac{\mathbb{E}\|g_n-g\|^2}{a_n}=c_1>0$. (b) a_n is called optimal rate of convergence for the class $\mathcal{P}$ if it is a lower minimax rate of convergence and there is an estimate g_n such that $\limsup_{n\to+\infty}\sup_{P\in\mathcal{P}}\frac{\mathbb{E}\|g_n-g\|^2}{a_n}=c_n<\infty$.

Definition 4.5 (Smoothness) Let $q=k+\beta$ for some $k\in\mathbb{N}$ and $0<\beta\le 1$ and let $\rho>0$. A function $f:\mathbb{R}^d\to\mathbb{R}$ is called (q,ρ)-smooth if for every $\alpha=(\alpha_1,\cdots,\alpha_d),\alpha_i\in\mathbb{N},\sum_{i=1}^d\alpha_i=k$, the partial derivative $\frac{\partial^k f}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$ exists and satisfies $\left|\frac{\partial^k f}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}(x)-\frac{\partial^k f}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}(z)\right|\le\rho\|x-z\|^\beta$. Let $\mathscr{F}^{(q,\rho)}$ be the set of all (q,ρ)-smooth functions f. Let $\mathcal{P}^{(q,\rho)}$ be the class of distributions (X,Y) such that (i) X is uniformly distributed on $[0,1]^d$; (ii) $Y=g(X)+N$, where $X\perp\!\!\!\perp N$, and N is standard normal; (iii) $g\in\mathscr{F}^{q,\rho}$.

Lemma 4.1 Let u be an l-dimensional real vector, let C be a zero means random variables takeing values in $\{-1,1\}$ and let N be an l-dimensional standard normal independent of C. Set $Z=Cu+N$. Then the error probability of the Bayesian decision for C based on Z is $\mathcal{R}^*=\min_{g:\mathbb{R}^l\to\mathbb{R}}\mathbb{P}(g(Z)\neq C)=\Phi(-\|u\|)$.

Proof $\mathbb{P}(C=1)=\mathbb{P}(C=-1)=\frac{1}{2},\mathbb{P}(Z|C=1)=\mathcal{N}(u,I),\mathbb{P}(Z|C=-1)=\mathcal{N}(-u,I)$. By the Bayes formula,

$$\mathbb{P}(C=1|Z=z)=\frac{\mathbb{P}(C=1)\mathbb{P}(Z|C=1)}{\mathbb{P}(C=1)\mathbb{P}(Z|C=1)+\mathbb{P}(C=-1)\mathbb{P}(Z|C=-1)}=\frac{1}{1+\exp\left(\frac{\|Z-u\|^2}{2}-\frac{\|Z+u\|^2}{2}\right)}=\frac{1}{1+\exp(-2Z^Tu)}.$$

Therefore, the optimal Bayes decision is $g^*(Z)=\mathrm{sgn}(Z^Tu)$, and the risk is

$$\begin{aligned}\mathcal{R}^*&=\mathbb{P}(g^*(Z)\neq C)=\mathbb{P}(Z^Tu<0,C=1)+\mathbb{P}(Z^Tu>0,C=-1)\\&=\mathbb{P}(\|u\|^2+u^TN<0,C=1)+\mathbb{P}(-\|u\|^2+u^TN>0,C=-1)\\&=\frac{1}{2}\mathbb{P}(u^TN\le-\|u\|^2)+\frac{1}{2}\mathbb{P}(u^TN>\|u\|^2)=\Phi(-\|u\|).\end{aligned}$$

□

Theorem 4.2 For the class $\mathcal{P}^{(q,\rho)}$, the sequence $a_n=n^{-\frac{2q}{2q+d}}$ is a lower minimax rate of convergence. In particular,

$$\liminf_{n\to\infty}\inf_{g_n}\sup_{P_{(X,Y)}\in\mathcal{P}^{(q,\rho)}}\frac{\mathbb{E}\|g_n-g\|^2}{\rho^{\frac{2d}{2q+d}}n^{-\frac{2q}{2q+d}}}\ge c_1>0.$$

Proof Step 1: Construct an auxiliary function $g^{(c)}(x)$. Set $M_n=\lceil(\rho^2n)^{\frac{1}{2q+d}}\rceil$. Partition $[0,1]^d$ into M_n^d cubes $\{A_{n,j}\}$ of side length $\frac{1}{M_n}$ and with centers $\{a_{n,j}\}$. Choose a function $\bar{f}:\mathbb{R}^d\to\mathbb{R}$ such that the support of $\bar{f}$ is a subset of $[-\frac{1}{2},\frac{1}{2}]^d,\int\bar{f}^2(x)\mathrm{d}x>0$ and $\bar{f}\in\mathscr{F}^{(q,2^{\beta-1})}$. Define $f:\mathbb{R}^d\to\mathbb{R}$ by $f=\rho\bar{f}$. Let $c_n=(c_{n,1},\cdots,c_{n,M_n^d})\in\mathcal{C}_n$ take values in $\{\pm1\}$. Define $g^{(c_n)}(x)=\sum_{j=1}^{M_n^d}c_{n,j}f_{n,j}(x)$ where $f_{n_j}(x)=M_n^{-q}f(M_n(x-a_{n,j}))$.

Step 2: Show that $g^{(c_n)}\in\mathscr{F}^{(q,\rho)}$. Let $\alpha=(\alpha_1,\cdots,\alpha_d),\alpha_i\in\mathbb{N},\sum_{j=1}^d\alpha_j=k$ and $D^\alpha=\frac{\partial^k}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$. If $x,z\in A_{n,j}$,

$$|D^\alpha g^{c_n}(x)-D^\alpha g^{(c_n)}(z)|=|c_{n,k}||D^\alpha f_{n,j}(x)-D^\alpha f_{n,j}(z)|\le\rho\|x-z\|^\beta.$$

If $x\in A_{n,i},z\in A_{n,j}$, choose $\bar{x},\bar{z}$ on the line between x and z such that $\bar{x}$ is on the boundary of $A_{n,i}$ and $\bar{z}$ is on the boundary of $A_{n,j}$. Then

$$\begin{aligned}|D^\alpha g^{(c_n)}(x)-D^\alpha g^{(c_n)}(z)|&\le|c_{n,i}D^\alpha f_{n,i}(x)|+|c_{n,j}D^\alpha f_{n,j}(z)|\\&=|c_{n,i}||D^\alpha f_{n,i}(x)-D^\alpha f_{n,i}(\bar{x})|+|c_{n,j}||D^\alpha f_{n,j}(z)-D^\alpha f_{n,j}(\bar{z})|\\&\le\rho2^{\beta-1}(\|x-\bar{x}\|^\beta+\|z-\bar{z}\|^\beta)=\rho2^\beta\left(\frac{\|x-\bar{x}\|^\beta}{2}+\frac{\|z-\bar{z}\|^\beta}{2}\right)\\&\le\rho2^\beta\left(\frac{\|x-\bar{x}\|}{2}+\frac{\|z-\bar{z}\|}{2}\right)^\beta\le\rho\|x-z\|^\beta.\end{aligned}$$

Step 3: Prove that

$$\liminf_{n\to+\infty}\inf_{g_n}\sup_{Y=g^{(c)}(X)+N,c\in\mathcal{C}_n}\frac{M_n^{2q}}{\rho^2}\mathbb{E}\|g_n-g^{(c)}\|^2>0.$$

$\{f_{n,j}\}$ forms a set of orthogonal basis. Let g_n be an arbitrary estimate, and the projection $\bar{g}_n$ of g_n to $\{g^{(c)} : c \in \mathcal{C}_n\}$ is given by $\bar{g}_n = \sum_{j=1}^{M_n} \tilde{c}_{n,j} f_{n,j}(x)$. Then

$$\begin{aligned}\|g_n - g^{(c)}\|^2 &= \|g_n - \bar{g}_n\|^2 + \|g_n - g^{(c)}\|^2 \geq \|\bar{g}_n - g^{(c)}\|^2 = \sum_{j=1}^{M_n^d} \int_{A_{n,j}} (\tilde{c}_{n,j} f_{n,j}(x) - c_{n,j} f_{n,j}(x))^2 \mathrm{d}x \\ &= \sum_{j=1}^{M_n^d} \int_{A_{n,j}} (\tilde{c}_{n,j} - c_{n,k})^2 f_{n,j}^2(x) \mathrm{d}x = \int f^2(x) \mathrm{d}x \sum_{j=1}^{M_n^d} (\tilde{c}_{n,j} - c_{n,j})^2 \frac{1}{M_n^{2q+d}}.\end{aligned}$$

Define $\bar{c}_{n,j} = \mathrm{sgn}(\tilde{c}_{n,j})$, then

$$|\tilde{c}_{n,j} - c_{n,j}| \geq \frac{|\bar{c}_{n,j} - c_{n,j}|}{2} \Rightarrow \|g_n - g^{(c)}\|^2 \geq \int f^2(x)\mathrm{d}x \frac{1}{4} \frac{1}{M_n^{2q+d}} \sum_{j=1}^{M_n^d} (\bar{c}_{n,j} - c_{n,j})^2 = \frac{\rho^2}{M_n^{2q}} \int \bar{f}^2(x) \mathrm{d}x \frac{1}{M_n^d} \sum_{j=1}^{M_n^d} 1_{\{\bar{c}_{n,j} \neq c_{n,j}\}}.$$

Step 4: Prove that

$$\lim_{n \to +\infty} \inf_{\bar{c}_n} \inf_{c_n} \sup \frac{1}{M_n^d} \sum_{j=1}^{M_n^d} \mathbb{P}(\bar{c}_{n,j} \neq c_{n,j}) > 0.$$

Now we randomize c_n. Let $c_{n,1}, \cdots, c_{n,M_n^d}$ be i.i.d. random variables independent of $(X_1, N_1), \cdots, (X_n, N_n)$, $\mathbb{P}(C_{n,1} = 1) = \mathbb{P}(C_{n,1} = -1) = \frac{1}{2}$. $\bar{c}_{n,j}$ can be interpreted as a decision on $C_{n,j}$ using $\mathcal{D}_n$. Let $\bar{C}_{n,j} = 1$ if $\mathbb{P}(\bar{C}_{n,j} = 1 | \mathcal{D}_n) \geq \frac{1}{2}$. Therefore,

$$\begin{aligned}\inf_{\bar{c}_n} \sup_{c_n} \frac{1}{M_n^d} \sum_{j=1}^{M_n^d} \mathbb{P}(\bar{c}_{n,j} \neq c_{n,,j}) &\geq \inf_{\bar{c}_n} \frac{1}{M_n^d} \sum_{j=1}^{M_n^d} \mathbb{P}(\bar{c}_{n,j} \neq C_{n,j}) \geq \frac{1}{M_n^d} \sum_{j=1}^{M_n^d} \mathbb{P}(\bar{C}_{n,j} \neq C_{n,j}) \\ &= \mathbb{P}(\bar{C}_{n,1} \neq C_{n,1}) = \mathbb{E}\{\mathbb{P}(\bar{C}_{n,1} \neq C_{n,1} | X_1, \cdots, X_n)\}.\end{aligned}$$

Let $X_{i_1}, \cdots, X_{i_t}$ be those $X_i \in A_{n,1}$, $(Y_{i,1}, \cdots, Y_{i_t}) = C_{n,1}(f_{n,1}(X_{i_1}), \cdots, f_{n,1}(X_{i_t})) + (N_{i_1}, \cdots, N_{i_t})$. By lemma 4.1,

$$\begin{aligned}\mathbb{E}\{\mathbb{P}(\bar{C}_{n,1} \neq C_{n,1} | X_1, \cdots, X_n)\} &= \mathbb{E}\Phi\left(-\sqrt{\sum_{r=1}^{t} f_{n,1}^2(X_{i_r})}\right) = \mathbb{E}\Phi\left(-\sqrt{\sum_{i=1}^{n} f_{n,1}^2(X_i)}\right) \\ &\geq \Phi\left(-\sqrt{\mathbb{E}\sum_{i=1}^{n} f_{n,1}^2(X_i)}\right) \geq \Phi\left(-\sqrt{\int f^2(x)\mathrm{d}x}\right) > 0. \qquad \square\end{aligned}$$

5 Uniform Laws of Large Numbers

Definition 5.1 (Background) Set $Z = (X, Y), Z_i = (X_i, Y_i), g_f(x, y) = |f(x) - y|^2$ for $f \in \mathscr{F}_n, G_n = \{g_f : f \in \mathscr{F}_n\}$, consider the limit $\lim_{n \to +\infty} \sup_{g \in \mathscr{G}_n} \left| \frac{1}{n} \sum_{i=1}^{n} g(Z_i) - \mathbb{E}g(Z) \right|$.

Lemma 5.1 (Hoeffding's inequality) For $g : \mathbb{R}^d \to [0, B]$, the following inequalities hold:

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon\right) \leq 2e^{-\frac{2n\varepsilon^2}{B^2}} \Rightarrow \mathbb{P}\left(\sup_{g \in \mathscr{G}_n}\left|\frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon\right) \leq 2|\mathscr{G}_n| e^{-\frac{2n\varepsilon^2}{B^2}}.$$

For finite class $\mathscr{G}$ satisfying $\sum_{n=1}^{+\infty} |\mathscr{G}_n| e^{-\frac{2n\varepsilon^2}{B^2}} < \infty$ for all $\varepsilon > 0$, by Borel-Cantelli lemma,

$$\mathbb{P}\left(\sup_{g \in \mathscr{G}_n}\left|\frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon \text{ i.o.}\right) = 0$$

Definition 5.2 (Covering number) Let $\varepsilon > 0$ and $\mathscr{G}$ be a set of functions $\mathbb{R}^d \to \mathbb{R}$. Every finite collection of functions $g_1, \cdots, g_N : \mathbb{R}^d \to \mathbb{R}$ with the property that for every $g \in \mathscr{G}$ there is a $j = j(g) \in [N]$ such that $\|g - g_j\|_\infty < \varepsilon$ is called an ε-cover of $\mathscr{G}$ w.r.t. $\|\cdot\|_\infty$. Let $\mathscr{N}(\varepsilon, \mathscr{G}, \|\cdot\|_\infty)$ or $\mathscr{N}_\infty(\varepsilon, \mathscr{G})$ be the smallest ε-cover of $\mathscr{G}$ w.r.t. $\|\cdot\|_\infty$.

Theorem 5.1 For $n \in \mathbb{N}$, let $\mathscr{G}_n$ be a set of functions $g : \mathbb{R}^d \to [0, B]$ and let $\varepsilon > 0$. Then

$$\mathbb{P}\left(\sup_{g\in\mathscr{G}_n}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon\right) \le 2\mathscr{N}_\infty\left(\frac{\varepsilon}{3}, \mathscr{G}_n\right)\exp\left(-\frac{2n\varepsilon^2}{9B^2}\right).$$

Proof Let $\mathscr{G}_{n,\frac{\varepsilon}{3}}$ be an $\frac{\varepsilon}{3}$-cover of $\mathscr{G}_n$ w.r.t. $\|\cdot\|_\infty$ of minimal cardinality. Fix $g \in \mathscr{G}_n$, there exists $\bar{g} \in \mathscr{G}_{n,\frac{\varepsilon}{3}}$ such that $\|g - \bar{g}\|_\infty < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z)\right| &\le \left|\frac{1}{n}\sum_{i=1}^n (g(Z_i) - \bar{g}(Z_i))\right| + \left|\frac{1}{n}\sum_{i=1}^n \bar{g}(Z_i) - \mathbb{E}\{\bar{g}(Z)\}\right| + |\mathbb{E}\bar{g}(Z) - \mathbb{E}g(Z)| \\
&\le \frac{2\varepsilon}{3} + \left|\frac{1}{n}\sum_{i=1}^n \bar{g}(Z_i) - \mathbb{E}\{\bar{g}(Z)\}\right|, \\
\Rightarrow \mathbb{P}\left(\sup_{g\in\mathscr{G}_n}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon\right) &\le \mathbb{P}\left(\sup_{g\in\mathscr{G}_{n,\frac{\varepsilon}{3}}}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \frac{\varepsilon}{3}\right)
\end{aligned}$$

Then use Hoeffding's inequality. □

Definition 5.3 Let $\varepsilon > 0$, $\mathscr{G}$ be a set of functions $\mathbb{R}^d \to \mathbb{R}$, $1 \le p < \infty$, and ν be a probability measure on $\mathbb{R}^d$. (a) Every finite collection of functions $g_1, \cdots, g_N : \mathbb{R}^d \to \mathbb{R}$ with the property that for every $g \in \mathscr{G}$ there is a $j = j(g) \in [N]$ such that $\|g - g_j\|_{L_p(\nu)} < \varepsilon$ is called a ε-cover of $\mathscr{G}$. Similarly define $\mathscr{N}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)})$. (b) Let $Z^{1:n} = (Z_1, \cdots, Z_n) \subset \mathbb{R}^d$ and ν_n be the corresponding empirical measure, then $\|f\|_{L_p(\nu_n)} := \left\{\frac{1}{n}\sum_{i=1}^n |f(Z_i)|^p\right\}^{\frac{1}{p}}$ and similarly define $\mathscr{N}_p(\varepsilon, \mathscr{G}, Z^{1:n})$.

Definition 5.4 (Packing number) (a) Every finite collection of functions $g_1, \cdots, g_N \in \mathscr{G}$ with $\|g_j - g_k\|_{L_p(\nu)} \ge \varepsilon$ for all $1 \le j < k \le N$ is called ε-packing of $\mathscr{G}$ with $\|\cdot\|_{L_p(\nu)}$. The largest ε-packing is denoted as $\mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)})$. Similarly define $\mathscr{M}(\varepsilon, \mathscr{G}, Z^{1:n})$.

Property 5.1 (Covering number v.s. packing number)

$$\begin{gathered}
\mathscr{M}(2\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}) \le \mathscr{N}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}) \le \mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}), \\
\mathscr{M}(2\varepsilon, \mathscr{G}, Z^{1:n}) \le \mathscr{N}(\varepsilon, \mathscr{G}, Z^{1:n}) \le \mathscr{M}(\varepsilon, \mathscr{G}, Z^{1:n}).
\end{gathered}$$

Theorem 5.2 Let $\mathscr{F}$ be a set of functions $\mathbb{R}^d \to \mathbb{R}$. Assume that $\mathscr{F}$ is a linear vector space of dimension D. Then for arbitrary $R > 0, \varepsilon > 0$, and $z_1, \cdots, z_n \in \mathbb{R}^d$,

$$\mathscr{N}_2\left(\varepsilon, \left\{f \in \mathscr{F} : \frac{1}{n}\sum_{i=1}^n |f(z_i)|^2 \le R^2\right\}, Z^{1:n}\right) \le \left(\frac{4R + \varepsilon}{\varepsilon}\right)^D.$$

Definition 5.5 Let $\mathscr{A}$ be a class of subsets of $\mathbb{R}^d$ and $n \in \mathbb{N}$. For $z_1, \cdots, z_n \in \mathbb{R}^d$, define $s(\mathscr{A}, \{z_1, \cdots, z_n\}) = |\{A \cap \{z_1, \cdots, z_n\} : A \in \mathscr{A}\}|$.

Definition 5.6 Let $\mathscr{G}$ be a subset of $\mathbb{R}^d$ of size n. We say $\mathscr{A}$ shatters $\mathscr{G}$ if $s(\mathscr{A}, \mathscr{G}) = 2^n$. The nth shatter coefficient of $\mathscr{A}$ is $S(\mathscr{A}, n) = \max_{\{z_1,\cdots,z_n\}\subset\mathbb{R}^d} s(\mathscr{A}, \{z_1, \cdots, z_n\})$, the maximum number of different subsets of n points that can be picked out by set from $\mathscr{A}$.

Definition 5.7 (VC dimension) Let $\mathscr{A}$ be a class of subsets of $\mathbb{R}^d$ with $\mathscr{A} \neq \emptyset$. The VC dimension $V_\mathscr{A}$ of $\mathscr{A}$ is defined by $V_\mathscr{A} = \sup\{n \in \mathbb{N}, S(\mathscr{A}, n) = 2^n\}$.

Proposition 5.1 $S(\mathscr{A}, n) \le \sum_{i=0}^{V_\mathscr{A}} \binom{n}{i}$.

Theorem 5.3 Let $\mathscr{G}$ be a set of functions $g : \mathbb{R}^d \to [0, B]$. For any $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n g(Z) - \mathbb{E}[g(Z)]\right| > \varepsilon\right\} \le 8\mathbb{E}\mathscr{N}_1(\frac{\varepsilon}{8}, \mathscr{G}, Z^{1:n})\exp\left(-\frac{n\varepsilon^2}{128B^2}\right).$$

Proof Step 1: Symmetrization. Let $Z'^{1:n}$ be i.i.d. samples from the same distribution and independent of $Z^{1:n}$ and $g^* \in \mathscr{G}$ be a function such that $\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z)\right| > \varepsilon$ if there exists such one. Otherwise, let g^* be an arbitrary function in $\mathscr{G}$. $g^*(z)$ depends on $Z^{1:n}$ and $\mathbb{P}\left\{\left|\mathbb{E}[g^*(Z)|Z^{1:n}] - \frac{1}{n}\sum_{i=1}^n g^*(Z_i')\right| > \frac{\varepsilon}{2}\middle|Z^{1:n}\right\} \le \frac{\mathrm{Var}(g^*(Z)|Z^{1:n})}{n(\frac{\varepsilon}{2})^2} \le \frac{B^2/4}{n\varepsilon^2/4} = \frac{B^2}{n\varepsilon^2} \le \frac{1}{2}$ holds for $n \ge \frac{2B^2}{\varepsilon^2}$. Thus we have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \frac{1}{n}\sum_{i=1}^n g(Z_i')\right| > \frac{\varepsilon}{2}\right\} &\ge \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n g^*(Z_i) - \frac{1}{n}\sum_{i=1}^n g^*(Z_i')\right| > \frac{\varepsilon}{2}\right\} \\
&\ge \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}]\right| > \varepsilon, \left|\frac{1}{n}\sum_{i=1}^n g^*(Z_i') - \mathbb{E}[g^*(Z)|Z^{1:n}]\right| \le \frac{\varepsilon}{2}\right\} \\
&= \mathbb{E}\left\{1_{\left\{\left|\frac{1}{n}\sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}]\right| > \varepsilon\right\}} \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n g^*(Z_i') - \mathbb{E}[g^*(Z)|Z^{1:n}]\right| \le \frac{\varepsilon}{2}\middle|Z^{1:n}\right)\right\} \\
&\ge \frac{1}{2}\mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}]\right| > \varepsilon\right\}
\end{aligned}$$

Therefore, $2\mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \frac{1}{n}\sum_{i=1}^n g(Z_i')\right| > \frac{\varepsilon}{2}\right\} \ge \mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}[g(Z)]\right| > \varepsilon\right\}$.

Step 2: Introduction of additive randomness by random signs. Let $U_1, \cdots, U_n$ be independent and uniformly distributed over $\{-1, 1\}$ and independent $Z^{1:n}$ and $Z'^{1:n}$.

$$\begin{aligned}
\mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n [g(Z_i) - g(Z_i')]\right| > \frac{\varepsilon}{2}\right\} &= \mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n U_i[g(Z_i) - g(Z_i')]\right| > \frac{\varepsilon}{2}\right\} \\
&\le \mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n U_i g(Z_i)\right| > \frac{\varepsilon}{4}\right\} + \mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}U_i g(Z_i')\right| > \frac{\varepsilon}{4}\right\} \\
&= 2\mathbb{P}\left\{\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n U_i g(Z_i)\right| > \frac{\varepsilon}{4}\right\}
\end{aligned}$$

Step 3: Conditioning and introduction of a covering on $Z^{1:n}$. Let $\mathscr{G}_{\frac{\varepsilon}{8}}$ be an L_1 $\frac{\varepsilon}{8}$-cover of $\mathscr{G}$ in $Z^{1:n}$. Fix $g \in \mathscr{G}$, then there exists $\bar{g} \in \mathscr{G}_{\frac{\varepsilon}{8}}$ s.t. $\frac{1}{n}\sum_{i=1}^n |g(Z_i) - \bar{g}(Z_i)| < \frac{\varepsilon}{8}$. $\left|\frac{1}{n}\sum_{i=1}^n U_i g(Z_i)\right| = \left|\frac{1}{n}\sum_{i=1}^n U_i \bar{g}(Z_i) + \frac{1}{n}\sum_{i=1}^n U_i[g(Z_i) - \bar{g}(Z_i)]\right| \le \left|\frac{1}{n}\sum_{i=1}^n U_i \bar{g}(Z_i)\right| + \frac{\varepsilon}{8}$. Thus

$$\mathbb{P}\left\{\exists g \in \mathscr{G} : \left|\frac{1}{n}\sum_{i=1}^n U_i g(Z_i)\right| > \frac{\varepsilon}{4}\right\} \le \mathbb{P}\left\{\exists g \in \mathscr{G}_{\frac{\varepsilon}{8}} : \left|\frac{1}{n}\sum_{i=1}^n U_i \bar{g}(Z_i)\right| > \frac{\varepsilon}{8}\right\} \le |\mathscr{G}_{\frac{\varepsilon}{8}}| \max_{g\in\mathscr{G}_{\frac{\varepsilon}{8}}} \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n U_i g(Z_i)\right| > \frac{\varepsilon}{8}\right\}$$

Step 4: Application of Hoeffding's inequality: $|U_i g(Z_i)| \le B \Rightarrow \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n U_i g(Z_i)\right| > \frac{\varepsilon}{8}\right\} \le 2\exp\left(-\frac{2n(\frac{\varepsilon}{8})^2}{(2B)^2}\right) = 2\exp\left(-\frac{n\varepsilon^2}{128B^2}\right)$. □

Theorem 5.4 Let $\mathscr{G}$ be a class of functions $g : \mathbb{R}^d \to [0, B]$ with $V_{\mathscr{G}^+} \ge 2$ where $\mathscr{G}^+ := \{(z, t) \in \mathbb{R}^d \times \mathbb{R} : t \le g(z), g \in \mathscr{G}\}$. Let $p \ge 1$, ν be a probability measure on $\mathbb{R}^d$ and $0 < \varepsilon < \frac{B}{4}$. Then

$$\mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}) \le 3\left(\frac{2eB^p}{\varepsilon^p}\log\frac{3eB^p}{\varepsilon^p}\right)^{V_{\mathscr{G}^+}}.$$

Proof Step 1: Set $p = 1$. Relate $\mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)})$ to a shatter coefficient of $\mathscr{G}^+$. Set $m = \mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)})$ and let $\bar{\mathscr{G}} = \{g_1, \cdots, g_m\}$ be a ε-packing of $\mathscr{G}$ w.r.t. $\|\cdot\|_{L_p(\nu)}$. Let $Q_1, \cdots, Q_K \in \mathbb{R}^d$ be K independent r.v.'s with common ν. Generate K independent r.v.'s $T_1, \cdots, T_K$ uniformly distributed on $[0, B]$. Denote $R_i = (Q_i, T_i), i = 1, \cdots, K, \mathscr{G}_f = \{(x, t) : t \le f(x)\}$ for $f : \mathbb{R}^d \to [0, B]$. Then

$$S(\mathscr{G}^+, K) = \max_{\{z_1,\cdots,z_K\}\in\mathbb{R}^d\times\mathbb{R}} s(\mathscr{G}^+, \{z_1, \cdots, z_K\}) \ge \mathbb{E}s(\mathscr{G}_+, \{R_1, \cdots, R_K\}) \ge \mathbb{E}s(\{\mathscr{G}_f : f \in \mathscr{G}\}, \{R_1, \cdots, R_K\})$$

$$\begin{aligned}
&\geq \mathbb{E}s(\{\mathscr{G}_f : f \in \mathscr{G}, \mathscr{G}_f \cap R^{1:K} \neq \mathscr{G}_g \cap R^{1:K} \text{ for all } g \in \bar{\mathscr{G}}, g \neq f\}, R^{1:K}) \\
&= \mathbb{E}\left\{\sum_{f\in\bar{\mathscr{G}}} 1_{\{\mathscr{G}_f \cap R^{1:K} \neq \mathscr{G}_g \cap R^{1:K} \text{ for all } g \in \mathscr{G}, g \neq f\}}\right\} = \sum_{f\in\bar{\mathscr{G}}} \mathbb{P}(\mathscr{G}_f \cap R^{1:K} \neq \mathscr{G}_g \cap R^{1:K} \text{ for all } g \in \mathscr{G}, g \neq f) \\
&= \sum_{f\in\bar{\mathscr{G}}} (1 - \mathbb{P}(\exists g \in \bar{\mathscr{G}}, g \neq f, \mathscr{G}_f \cap R^{1:K} = \mathscr{G}_g \cap R^{1:K})) \geq \sum_{f\in\bar{\mathscr{G}}} \left(1 - m \max_{g\in\bar{\mathscr{G}}, g\neq f} \mathbb{P}(\mathscr{G}_f \cap R^{1:K} = \mathscr{G}_g \cap R^{1:K})\right).
\end{aligned}$$

For $f, g \in \bar{\mathscr{G}}, f \neq g$,

$$\mathbb{P}(\mathscr{G}_f \cap R^{1:K} = \mathscr{G}_g \cap R^{1:K}) = \mathbb{P}(\mathscr{G}_f \cap \{R_1\} = \mathscr{G}_g \cap \{R_1\})^K,$$

and

$$\begin{aligned}
&\mathbb{P}(\mathscr{G}_f \cap \{R_1\} = \mathscr{G}_g \cap \{R_1\}) = 1 - \mathbb{P}(\mathscr{G}_f \cap \{R_1\} \neq \mathscr{G}_g \cap \{R_1\}) = 1 - \mathbb{E}[\mathbb{P}(\mathscr{G}_f \cap \{R_1\} \neq \mathscr{G}_g \cap \{R_1\} | Q_1)] \\
&\qquad = 1 - \mathbb{E}[\mathbb{P}(f(Q_1) < T \leq g(Q_1) \text{ or } g(Q_1) < T \leq f(Q_1) | Q_1)] = 1 - \mathbb{E}\left[\frac{|f(Q_1) - g(Q_1)|}{B}\right] \\
&\qquad = 1 - \frac{1}{B}\int |f(x) - g(x)|\nu(\mathrm{d}x) \leq 1 - \frac{\varepsilon}{B} \Rightarrow \mathbb{P}(\mathscr{G}_f \cap \{R_1\} = \mathscr{G}_g \cap \{R_1\})^K \leq \left(1 - \frac{\varepsilon}{B}\right)^K \leq \exp\left(-\frac{\varepsilon K}{B}\right) \\
&\Rightarrow S(\mathscr{G}^+, K) \geq m\left(1 - m\exp\left(-\frac{\varepsilon K}{B}\right)\right).
\end{aligned}$$

Set $K = \left\lfloor \frac{B}{\varepsilon}\log(2m)\right\rfloor$. Then

$$1 - m\exp\left(-\frac{\varepsilon K}{B}\right) \geq 1 - m\exp\left(-\frac{\varepsilon}{B}\left(\frac{B}{\varepsilon}\log(2m) - 1\right)\right) = 1 - \frac{1}{2}\exp\left(\frac{\varepsilon}{B}\right) \geq 1 - \frac{1}{2}\exp\left(\frac{1}{4}\right) \geq \frac{1}{3} \Rightarrow m \leq 3S(\mathscr{G}_+, K).$$

Step 2: Relate $S(\mathscr{G}_+, K)$ to $V_{\mathscr{G}_+}$. Set $K = \lfloor \frac{B}{\varepsilon}\log(2\mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}))\rfloor \leq V_{\mathscr{G}_+} \Rightarrow \mathscr{M}(\varepsilon, \mathscr{G}, \|)\cdot\|_{L_p(\nu)} \leq \frac{\varepsilon}{2}\exp(V_{\mathscr{G}_+}) \leq 3\left(\frac{2eB}{\varepsilon}\log\frac{3eB}{\varepsilon}\right)^{V_{\mathscr{G}_+}}$. In the case $K > V_{\mathscr{G}_+}$, use the following lemma:

Lemma 5.2 Let $\mathscr{A} \in \mathbb{R}^d$ and $V_{\mathscr{A}} < \infty$. Then $\forall n \in \mathbb{N}, S(\mathscr{A}, n) \leq (n+1)^{V_{\mathscr{A}}}$ and $\forall n \geq V_{\mathscr{A}}, S(\mathscr{A}, n) \leq (\frac{en}{V_{\mathscr{A}}})^{V_{\mathscr{A}}}$.

Then $\mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}) \leq 3\left(\frac{eK}{V_{\mathscr{G}_+}}\right)^{V_{\mathscr{G}_+}} \leq 3\left(\frac{eB}{\varepsilon V_{\mathscr{G}_+}}\log(2\mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}))\right)^{V_{\mathscr{G}_+}}$.

Step 3: Setting $a = \frac{eB}{\varepsilon}$ and $b = V_{\mathscr{G}_+}, \mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}) := x \leq 3(\frac{a}{b}\log(2x))^b \Rightarrow x \leq 3(2a\log(3a))^b$.

Step 4: Let $1 < p < \infty$. Then for any $g_j, g_k \in \mathscr{G}$,

$$\|g_j - g_k\|^p_{L_p(\nu)} \leq B^{p-1}\|g_j - g_k\|_{L_1(\nu)} \Rightarrow \mathscr{M}(\varepsilon, \mathscr{G}, \|\cdot\|_{L_p(\nu)}) \leq \mathscr{M}\left(\frac{\varepsilon^p}{B^{p-1}}, \mathscr{G}, \|\cdot\|_{L_p(\nu)}\right). \qquad \square$$

Theorem 5.5 (ULLN) Let $\mathscr{G}$ be a class of functions $g : \mathbb{R}^d \to \mathbb{R}$ and $G : \mathbb{R}^d \to \mathbb{R}, G(x) = \sup_{g\in\mathscr{G}} |g(x)|$ be an envelope of $\mathscr{G}$. Assume $\mathbb{E}G(Z) < \infty$ and $V_{\mathscr{G}+} < \infty$. Then

$$\sup_{g\in\mathscr{G}}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z)\right| \to 0 \text{ a.s. as } n \to +\infty$$

Proof For $L > 0$, set $\mathscr{G}_L := \{g \cdot 1_{\{G\leq L\}} : g \in \mathscr{G}\}$. For $g \in \mathscr{G}$,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z)\right| &\leq \left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \frac{1}{n}\sum_{i=1}^n g(Z_i)1_{\{G(Z_i)\leq L\}}\right| \\
&\quad + \left|\frac{1}{n}\sum_{i=1}^n g(Z_i)1_{\{G(Z_i)\leq L\}} - \mathbb{E}\{g(Z)1_{G(Z)\leq L}\}\right| + |\mathbb{E}\{g(Z)1_{G(Z)\leq L}\} - \mathbb{E}\{g(Z)\}| \\
&= \left|\frac{1}{n}\sum_{i=1}^n g(Z_i)1_{\{G(Z_i)>L\}}\right| + \mathbb{E}|g(Z)|1_{\{G(Z)>L\}} + \left|\frac{1}{n}\sum_{i=1}^n g(Z_i)1_{\{G(Z_i)\leq L\}} - \mathbb{E}\{g(Z)1_{G(Z)\leq L}\}\right|
\end{aligned}$$

Since $\mathbb{P}(\sup_{g\in\mathscr{G}_L}\left|\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z)\right| > \varepsilon) \leq 8\mathbb{E}\left\{\mathscr{M}_1(\frac{\varepsilon}{8}, \mathscr{G}_L, Z^{1:n})\exp\left(-\frac{n\varepsilon^2}{128(2L)^2}\right)\right\}$, use the B-C lemma. $\square$

6 Least Square Estimates: Consistency and Convergence Rate

Definition 6.1 (Notation) $\mathbb{E}\{(m(X)-Y)^2\} = \inf_f \mathbb{E}\{(f(X)-Y)^2\} \Rightarrow m(X) = \mathbb{E}[Y|X]$. Define

$$m_n = \arg\min_{f\in\mathscr{F}_n} \frac{1}{n}\sum_{i=1}^n |f(X_i)-Y_i|^2, m^* = \arg\min_{f\in\mathscr{F}_n} \mathbb{E}\{(f(X)-Y)^2\}.$$

Theorem 6.1 Let $\mathscr{F}_n$ be a class of functions $f:\mathbb{R}^d\to\mathbb{R}$ depending on th data $\mathcal{D}_n = \{(X_1,Y_1),\cdots,(X_n,Y_n)\}$. Then

$$\int |m_n(x)-m(x)|^2\nu(\mathrm{d}x) \le 2\sup_{f\in\mathscr{F}_n}\left|\frac{1}{n}\sum_{i=1}^n |f(X_i)-Y_i|^2 - \mathbb{E}\{(f(X)-Y)^2\}\right| + \inf_{f\in\mathscr{F}_n}\int |f(x)-m(x)|^2\nu(\mathrm{d}x).$$

Proof We do the following decomposition:

$$\begin{aligned}\int |m_n(x)-m(x)|^2\nu(\mathrm{d}x) &= \mathbb{E}[|m_n(X)-Y|^2|\mathcal{D}_n] - \mathbb{E}[|m(X)-Y|^2] \\ &= \left\{\mathbb{E}[|m_n(X)-Y|^2|\mathcal{D}_n] - \inf_{f\in\mathscr{F}_n}\mathbb{E}|f(X)-Y|^2\right\} + \left\{\inf_{f\in\mathscr{F}_n}\mathbb{E}|f(X)-Y|^2 - \mathbb{E}|m(X)-Y|^2\right\} \\ &:= I_1 + I_2.\end{aligned}$$

$$I_1 \le 2\sup_{f\in\mathscr{F}_n}\left|\frac{1}{n}\sum_{i=1}^n |f(X_i)-Y_i|^2 - \mathbb{E}|f(X)-Y|^2\right|.\ I_2 = \inf_{f\in\mathscr{F}_n}\int (f(x)-m(x))^2\nu(\mathrm{d}x).$$ □

Proposition 6.1 (Method of Sieves) Let $\psi_1,\psi_2,\cdots,\mathbb{R}^d\to\mathbb{R}$ be bounded functions such that $|\psi_j(x)|\le 1$. Assume the set of functions $\cup_{k=1}^{+\infty}\{\sum_{j=1}^k a_j\psi_j(x): a_1,\cdots,a_k\in\mathbb{R}\}$ is dense in $L_2(\mu)$ for any probability measure μ on $\mathbb{R}^d$. Define the regression function estimate m_n as a function minimizing the empirical L_2 risk $\frac{1}{n}\sum_{i=1}^n (f(X_i)-Y_i)^2$ over the function form $f(x) = \sum_{j=1}^{k_n} a_j\psi_j(x)$ with $\sum_{j=1}^{k_n}|a_j|\le\beta_n$. If $\mathbb{E}(Y^2)<\infty$ and k_n and β_n satisfy $k_n\to\infty,\beta_n\to\infty,\frac{k_n\beta_n^4\log\beta_n}{n}\to 0$ and $\frac{\beta_n^4}{n^{1-\delta}}\to 0$ for some $\delta>0$, then $\int(m_n(x)-m(x))^2\mu(\mathrm{d}x)\to 0$ with probability 1.

Proposition 6.2 Consider $\mathscr{F}_n = \left\{\sum_{j=1}^{k_n} a_j\psi_j(x): \sum_{j=1}^{k_n}|a_j|\le\beta_n\right\}$ and $\widetilde{\mathscr{F}}_n = \left\{\sum_{j=1}^{k_n} a_j\psi_j(x): a_j\in\mathbb{R}\right\}$. Step 1: derive $\widetilde{m}_n$ by using $\widetilde{\mathscr{F}}_n$. Step 2: Trancation of $\widetilde{m}_n$, $m_n(x) = T_{\beta_n}\widetilde{m}_n(x)$ where $T_L u = \begin{cases} u, & \text{if } |u|\le L \\ L\mathrm{sgn}(u), & \text{otherwise}\end{cases}$. (a) If $\mathbb{E}(Y^2)<\infty$ and k_n and β_n satisfy $k_n\to\infty,\beta_n\to\infty,\frac{k_n\beta_n^4\log\beta_n}{n}\to 0$, then $\mathbb{E}\left\{\int(m_n(x)-m(x))^2\mu(\mathrm{d}x)\right\}\to 0$. (b) If adding the extra condition $\frac{\beta_n^4}{n^{1-\delta}}\to 0$ for some $\delta>0$, then $\int(m_n(x)-m(x))^2\mu(\mathrm{d}x)\to 0$ a.s.

Proposition 6.3 Let $\widetilde{F}_n = \widetilde{F}_n(\mathcal{D}_n)$ be a class of functions $f:\mathbb{R}^d\to\mathbb{R}$. If $|Y|\le\beta_n$ a.s., then

$$\int(m_n(x)-m(x))^2\mu(\mathrm{d}x) \le 2\sup_{f\in T_{\beta_n}\widetilde{F}_n}\left|\frac{1}{n}\sum_{i=1}^n|f(X_i)-Y_i|^2 - \mathbb{E}|f(X)-Y|^2\right| + \inf_{f\in\widetilde{F}_n,\|f\|_\infty\le\beta_n}\int|f(x)-m(x)|^2\mu(\mathrm{d}x)$$

Theorem 6.2 Let $\widetilde{\mathscr{F}}_n = \widetilde{\mathscr{F}}_n(\mathcal{D}_n)$ be a class of functions $f:\mathbb{R}^d\to\mathbb{R}$ and $Y_L = T_L Y, Y_{i,L} = T_L Y_i$. (a) If

$$\lim_{n\to+\infty}\beta_n = \infty,\ \lim_{n\to+\infty}\inf_{f\in\widetilde{F}_n,\|f\|_\infty\le\beta_n}\int|f(x)-m(x)|^2\mu(\mathrm{d}x) = 0 \text{ a.s.},$$

$$\lim_{n\to+\infty}\sup_{f\in T_{\beta_n}\widetilde{F}_n}\left|\frac{1}{n}\sum_{i=1}^n|f(X_i)-Y_{i,L}|^2 - \mathbb{E}(f(X)-Y_L)^2\right| = 0 \text{ a.s. for all } L>0,$$

then $\lim_{n\to+\infty}\int|m_n(x)-m(x)|^2\mu(\mathrm{d}x) = 0$ a.s. (b) If $\beta_n\to+\infty, \mathbb{E}\{\sim\}\to 0, \mathbb{E}\{\sim\}\to 0$, then $\mathbb{E}\{\sim\}\to 0$.

Definition 6.2 (Piecewise polynomial partition estimate) $\mathcal{P}_n = \{A_{n,1},A_{n,2},\cdots\}$ be a partition of $\mathbb{R}^d$,

$$\hat{m}_n(x) := \frac{\sum_{i=1}^n Y_i I_{\{X_i\in A_n(x)\}}}{\sum_{i=1}^n I_{\{X_i\in A_n(x)\}}}$$

where $A_n(x)$ denotes the cell $A_{n,j} \in \mathcal{P}_n$ which contains x.

Theorem 6.3 Let $\mathscr{F}$ be a class of function $f : \mathbb{R}^d \to \mathbb{R}$ bounded in abolute value by B. Let $\varepsilon > 0$. Then

$$\mathbb{P}\{\exists f \in \mathscr{F} \text{ s.t.} \|f\|_2 - 2\|f\|_n > \varepsilon\} \leq \mathbb{E}\mathscr{N}_2\left(\frac{\sqrt{2}}{24}\varepsilon, \mathscr{F}, X^{1:2n}\right) \exp\left(-\frac{n\varepsilon^2}{288B^2}\right)$$

where $\|f\|_n^2 = \frac{1}{n}\sum_{i=1}^{n}|f(X_i)|^2$.

Proof Step 1: Replace $L_2(\mu)$ norm by the empirical norm. Let $\widetilde{X}^{1:n} = (X_{n+1}, \cdots, X_{2n})$ be a ghost sample of i.i.d. r.v.'s as X and independent of $X^{1:n}$. Define $\|f\|_{n'}^2 = \frac{1}{n}\sum_{i=n+1}^{2n}|f(X_i)|^2$. Let f^* be a function $f \in \mathscr{F}$ such that $\|f\|_2 - 2\|f\|_n > \varepsilon$ if there exists any such function, and let f^* be an arbitrary function in $\mathscr{F}$ if such a function does not exist. Then

$$\begin{aligned}
\mathbb{P}\{2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2 | X^{1:n}\} &\geq \mathbb{P}\{4\|f^*\|_{n'}^2 + \frac{\varepsilon^2}{4} > \|f^*\|_2^2 | X^{1:n}\} = 1 - \mathbb{P}\{4\|f^*\|_{n'}^2 + \frac{\varepsilon^2}{4} \leq \|f^*\|_2^2 | X^{1:n}\} \\
&= 1 - \mathbb{P}\{3\|f^*\|_2^2 + \frac{\varepsilon^2}{4} \leq 4(\|f^*\|_2^2 - \|f^*\|_{n'}^2) | X^{1:n}\} \geq 1 - \frac{16\mathrm{Var}\left(\frac{1}{n}\sum_{i=n+1}^{2n}|f^*(X_i)|^2 \middle| X^{1:n}\right)}{(3\|f^*\|_2^2 + \frac{\varepsilon^2}{4})^2} \\
&\geq 1 - \frac{\frac{16}{n}B^2\|f^*\|_2^2}{(3\|f^*\|_2^2 + \frac{\varepsilon^2}{4})^2} \geq 1 - \frac{\frac{16}{3}\frac{B^2}{n}}{3\|f^*\|_2^2 + \frac{\varepsilon^2}{4}} \geq 1 - \frac{64}{3\varepsilon^2}\frac{B^2}{n} \geq \frac{2}{3} \text{ for } n \geq \frac{64B^2}{\varepsilon^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}\{\exists f \in \mathscr{F} : \|f\|_{n'} - \|f\|_n > \frac{\varepsilon}{4}\} &\geq \mathbb{P}\{2\|f^*\|_{n'} - 2\|f^*\|_n > \frac{\varepsilon}{2}\} \geq \mathbb{P}\{2\|f^*\|_{n'} + \frac{\varepsilon}{2} - 2\|f^*\|_n > \varepsilon, 2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2\} \\
&\geq \mathbb{P}\{\|f^*\|_2 - 2\|f^*\|_n > \varepsilon, 2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2\} \\
&= \mathbb{E}\{1_{\{\|f^*\|_2 - 2\|f^*\|_n > \varepsilon\}}\mathbb{P}\{2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2 | X^{1:n}\}\} \\
&\geq \frac{2}{3}\mathbb{P}\{\|f^*\|_2 - 2\|f^*\|_n > \varepsilon\} = \frac{2}{3}\mathbb{P}\{\exists f \in \mathscr{F} : \|f\|_2 - 2\|f\|_n > \varepsilon\}.
\end{aligned}$$

This proves $\mathbb{P}\{\exists f \in \mathscr{F} : \|f\|_2 - 2\|f\|_n > \varepsilon\} \leq \frac{3}{2}\mathbb{P}\{\exists f \in \mathscr{F} : \|f\|_{n'} - \|f\|_n > \frac{\varepsilon}{4}\}$.

Step 2: Introduction of additional randomness. Let $U_1, \cdots, U_n$ be independent and uniformly distributed on $\{-1, 1\}$ and independent of $X_1, \cdots, X_{2n}$. Set $Z_i = \begin{cases} X_{i+n} & \text{if } U_i = 1 \\ X_i & \text{if } U_i = -1 \end{cases}$ and $Z_{i+n} = \begin{cases} X_i & \text{if } U_i = 1 \\ X_{i+n} & \text{if } U_i = -1 \end{cases}$. Then

$$\begin{aligned}
\mathbb{P}\left\{\exists f \in \mathscr{F} : \|f\|_{n'} - \|f\|_n > \frac{\varepsilon}{4}\right\} &= \mathbb{P}\left\{\exists f \in \mathscr{F} : \left(\frac{1}{n}\sum_{i=n+1}^{2n}|f(X_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n}\sum_{i=1}^{n}|f(X_i)|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{4}\right\} \\
&= \mathbb{P}\left\{\exists f \in \mathscr{F} : \left(\frac{1}{n}\sum_{i=n+1}^{2n}|f(Z_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n}\sum_{i=1}^{n}|f(Z_i)|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{4}\right\}
\end{aligned}$$

Step 3: Conditioning and introduction of a covery. Let $\mathscr{G} = \{g_j : j = 1, \cdots, \mathscr{N}_2(\frac{\sqrt{2}}{24}\varepsilon, \mathscr{F}, X^{1:2n})\}$ be a $\frac{\sqrt{2}}{24}\varepsilon$-cover of $\mathscr{F}$ w.r.t. $\|\cdot\|_{2n}$ of minimal size. $\|f\|_{2n}^2 = \frac{1}{2n}\sum_{i=1}^{2n}|f(X_i)|^2$. Fix $f \in \mathscr{F}$, $\|f - g\|_{2n} \leq \frac{\sqrt{2}}{24}\varepsilon$. Then

$$\begin{aligned}
&\left\{\frac{1}{n}\sum_{i=n+1}^{2n}|f(Z_i)|^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{i=1}^{n}|f(Z_i)|^2\right\}^{\frac{1}{2}} \\
&= \left\{\frac{1}{n}\sum_{i=n+1}^{2n}|f(Z_i)|^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right\}^{\frac{1}{2}} + \left\{\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right\}^{\frac{1}{2}} + \left\{\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{i=1}^{n}|f(Z_i)|^2\right\}^{\frac{1}{2}} \\
&\leq \left\{\frac{1}{n}\sum_{i=n+1}^{2n}|f(Z_i) - g(Z_i)|^2\right\}^{\frac{1}{2}} + \left\{\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right\}^{\frac{1}{2}} + \left\{\frac{1}{n}\sum_{i=1}^{n}|g(Z_i) - f(Z_i)|^2\right\}^{\frac{1}{2}} \\
&\leq 2\sqrt{2}\|f - g\|_{2n} + \left\{\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{6} + \left\{\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right\}^{\frac{1}{2}} - \left\{\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right\}^{\frac{1}{2}}
\end{aligned}$$

In this way,

$$\mathbb{P}\left\{\exists f \in \mathscr{F} : \left(\frac{1}{n}\sum_{i=n+1}^{2n}|f(Z_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n}\sum_{i=1}^{n}|f(Z_i)|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{4}\middle| X^{1:2n}\right\}$$
$$\leq \mathbb{P}\left\{\exists g \in \mathscr{G} : \left(\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{12}\middle| X^{1:2n}\right\}$$
$$\leq |\mathscr{G}| \max_{g\in\mathscr{G}} \mathbb{P}\left\{\left(\frac{1}{n}\sum_{i=n+1}^{2n}\middle|g(Z_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{12}\middle| X^{1:2n}\right\}$$

Step 4: Application of Hoeffding's inequality.

$$\left(\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right)^{\frac{1}{2}} \leq \left|\frac{\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2 - \frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2}{\left(\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2\right)^{\frac{1}{2}} + \left(\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right)^{\frac{1}{2}}}\right|$$
$$\leq \frac{\left|\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2 - \frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2\right|}{\left(\frac{1}{n}\sum_{i=1}^{2n}|g(Z_i)|^2\right)^{\frac{1}{2}}} = \frac{\left|\frac{1}{n}\sum_{i=1}^{n}U_i|g(X_i)|^2 - \frac{1}{n}\sum_{i=1}^{n}U_i|g(X_{i+n})|^2\right|}{\left(\frac{1}{n}\sum_{i=1}^{2n}|g(Z_i)|^2\right)^{\frac{1}{2}}}$$

Then

$$\mathbb{P}\{(\frac{1}{n}\sum_{i=n+1}^{2n}|g(Z_i)|^2)^{\frac{1}{2}} - (\frac{1}{n}\sum_{i=1}^{n}|g(Z_i)|^2)^{\frac{1}{2}} > \frac{\varepsilon}{12}|X^{1:2n}\} \leq 2\exp\left(-\frac{2n^2\frac{\varepsilon^2}{144}(\frac{1}{n}\sum_{i=1}^{2n}|g(X_i)|^2)}{\sum_{i=1}^{n}4(|g(X_i)|^2 - |g(X_{i+n})|^2)^2}\right)$$
$$\leq 2\exp\left(-\frac{2n^2\frac{\varepsilon^2}{144}(\frac{1}{n}\sum_{i=1}^{2n}|g(X_i)|^2)}{\sum_{i=1}^{n}4B^2(|g(X_i)|^2 + |g(X_{i+n})|^2)}\right)$$
$$= \exp\left(-\frac{n\varepsilon^2}{288B^2}\right). \qquad \square$$

Theorem 6.4 Assume $\sigma^2 = \sup_{x\in\mathbb{R}^d} \mathrm{Var}(Y|X=x) < \infty$. Let $k_n = k_n(x_1, \cdots, x_n)$ be the vector space dimension of $\mathscr{F}_n$. Then

$$\mathbb{E}\{\|\widetilde{m}_n - m\|_n^2 | X^{1:n}\} \leq \frac{\sigma^2 k_n}{n} + \min_{f\in\mathscr{F}_n}\|f - m\|_n^2.$$

Proof Denote $\mathbb{E}^*\{\cdot\} = \mathbb{E}\{\cdot|X^{1:n}\}$. Then

$$\mathbb{E}^*\left\{\|\widetilde{m}_n - m\|_n^2\right\} = \mathbb{E}^*\left\{\frac{1}{n}\sum_{i=1}^{n}|\widetilde{m}_n(X_i) - m(X_i)|^2\right\}$$
$$= \mathbb{E}^*\left\{\frac{1}{n}\sum_{i=1}^{n}|\widetilde{m}_n(X_i) - \mathbb{E}^*(\widetilde{m}_n(X_i)) + \mathbb{E}^*(\widetilde{m}_n(X_i)) - m(X_i)|^2\right\}$$
$$= \mathbb{E}^*\left\{\frac{1}{n}\sum_{i=1}^{n}|\widetilde{m}_n(X_i) - \mathbb{E}^*(\widetilde{m}_n(X_i))|^2\right\} + \mathbb{E}^*\left\{|\mathbb{E}^*(\widetilde{m}_n(X_i)) - m(X_i)|^2\right\}$$
$$= \mathbb{E}^*\left\{\|\widetilde{m}_n - \mathbb{E}^*(\widetilde{m}_n)\|_n^2\right\} + \|\mathbb{E}^*(\widetilde{m}_n) - m\|_n^2.$$

Write that $\widetilde{m}_n = \sum_{j=1}^{k_n} a_j f_{j,n}$ where $f_{1,n}, \cdots, f_{k_n,n}$ is a basis of $\mathscr{F}_n$, and $a = (a_j)_{j=1,\cdots,k_n}$ satisfies that $\frac{1}{n}B^TBa = \frac{1}{n}B^TY$, $B = (f_{j,n}(X_i))_{1\leq i\leq n, 1\leq j\leq k_n}$ and $Y = (Y_1, \cdots, Y_n)^T$. Then

$$\mathbb{E}^*\{\widetilde{m}_n\} = \sum_{j=1}^{k_n}\mathbb{E}^*\{a_j\}f_{j,n} \text{ and } \frac{1}{n}B^TB\mathbb{E}^*a = \frac{1}{n}B^T\mathbb{E}^*Y = \frac{1}{n}B^T(m(X_1), \cdots, m(X_n))^T$$

$$\Rightarrow \|\mathbb{E}^*(\widetilde{m}_n) - m\|_n^2 = \min_{f\in\mathscr{F}_n} \|f - m\|_n^2.$$

Choose a complete orthogonormal system $f_1, \cdots, f_k$ in $\mathscr{F}_n$ w.r.t. the empirical scalar proudct $\langle\cdot,\cdot\rangle_n$ where $\langle f, g\rangle_n = \frac{1}{n}\sum_{i=1}^{n} f(X_i)g(X_i), k \le k_n$. We remind our readers that such a system depends on $X_1, \cdots, X_n$. Then, on $\{X_1, \cdots, X_n\}$, $\text{span}\{f_1, \cdots, f_k\} \subset \mathscr{F}_n$, $\widetilde{m}_n(x) = f(x)^T \frac{1}{n} B^T Y$ where $B = (f_j(X_i))_{1\le j\le n, 1\le j\le k}, B^T B = I$. Therefore,

$$\begin{aligned}
\mathbb{E}^*\{|\widetilde{m}_n(x) - \mathbb{E}^*(\widetilde{m}_n(x))|^2\} &= \mathbb{E}^*\{|f(x)^T \frac{1}{n} B^T Y - f(x)^T \frac{1}{n} B^T (m(X_1), \cdots, m(X_n))^T|^2\} \\
&= f(x)^T \frac{1}{n} B^T (\mathbb{E}^*\{(Y_i - m(X_i))(Y_j - m(X_j))^T\}) \frac{1}{n} B f(x)
\end{aligned}$$

$$\Rightarrow \mathbb{E}^*\{\|\widetilde{m}_n - \mathbb{E}^*(\widetilde{m}_n)\|_n^2\} \le \frac{1}{n^2} f^T B^T \sigma^2 I B f = \frac{\sigma^2}{n}\sum_{j=1}^{k} \|f_j\|_n^2 = \frac{\sigma^2}{n} k \le \frac{\sigma^2}{n} k_n. \qquad \square$$

Theorem 6.5 Assume $\sigma^2 = \sup_{x\in\mathbb{R}^d} \text{Var}(Y|X = x) < \infty$ and $\|m\|_\infty = \sup_{x\in\mathbb{R}^d} |m(x)| \le L \in \mathbb{R}_+, m_n(\cdot) = T_L \widetilde{m}_n(\cdot)$. Then

$$\mathbb{E}\int |m_n(x) - m(x)|^2 \mu(\mathrm{d}x) \le C \cdot \max\{\sigma^2, L^2\} \frac{\log(n) + 1}{n} k_n + 8 \inf_{f\in\mathscr{F}_n} \int |f(x) - m(x)|^2 \mu(\mathrm{d}x).$$

Proof First we note that

$$\begin{aligned}
\int |m_n(x) - m(x)|^2 \mu(\mathrm{d}x) &= (\|m_n - m\|_2 - 2\|m_n - m\|_n + 2\|m_n - m\|_n)^2 \\
&\le (\max\{\|m_n - m\|_2 - 2\|m_n - m\|_m, 0\} + 2\|m_n - m\|_n)^2 \\
&\le 2(\max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\})^2 + 8\|m_n - m\|_n^2.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
\mathbb{E}\{8\|m_n - m\|_n^2\} &\le 8\mathbb{E}\{\mathbb{E}\{\|\widetilde{m}_n - m\|_n^2 | X_1, \cdots, X_n\}\} \\
&\le 8\sigma^2 \frac{k_n}{n} + 8\mathbb{E}\{\min_{f\in\mathscr{F}_n} \|f - m\|_n^2\} \\
&\le 8\sigma^2 \frac{k_n}{n} + 8 \inf_{f\in\mathscr{F}_n} \mathbb{E}\|f - m\|_n^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbb{P}\left(2\max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\} > u\right) &\le \mathbb{P}\left(\exists f \in T_L\mathscr{F}_n : \|f - m\|_2 - 2\|f - m\|_n > \sqrt{\frac{u}{2}}\right) \\
&\le 3\mathbb{E}\mathscr{N}_2\left(\frac{\sqrt{u}}{24}, \mathscr{F}_n, X^{1:2n}\right) \exp\left(-\frac{nu}{576(2L)^2}\right) \\
&\le 9(12en)^{2(k_n+1)} \exp\left(-\frac{nu}{2304L^2}\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathbb{E}(2\max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\}) &\le u + \int_u^\infty \mathbb{P}(2\max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\} > t)\mathrm{d}t \\
\left(\text{take } u \ge \frac{576L^2}{n}\right) &\le CL^2 \frac{\log(n) + 1}{n} k_n.
\end{aligned}$$

Combine these two bounds together. $\square$

Property 6.1 (Nonlinear LSE) $|Y| \le L \le \beta_n$ a.s., $m_n(\cdot) = T_{\beta_n}\widetilde{m}_n(\cdot), \widetilde{m}_n(\cdot) = \arg\min_{f\in\mathscr{F}_n} \frac{1}{n}\sum_{i=1}^{n} |f(X_i) - Y_i|^2$. We do the following decomposition:

$$\begin{aligned}
\int |m_n(x) - m(x)|^2 \mu(\mathrm{d}x) = &\left\{\mathbb{E}\{|m_n(X) - Y|^2 | \mathcal{D}_n\} - \mathbb{E}|m(X) - Y|^2 - \frac{2}{n}\sum_{i=1}^{n} \left[|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2\right]\right\} \\
&+ \frac{2}{n}\sum_{i=1}^{n} [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2].
\end{aligned}$$

On the one hand,

$$\mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2]\right\} \le \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} |\widetilde{m}_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2\right\}$$

$$\begin{aligned}
&\le \mathbb{E}\left\{\inf_{f\in\mathscr{F}_n}\frac{1}{n}\sum_{i=1}^n\left[|f(X_i)-Y_i|^2-|m(X_i)-Y_i|^2\right]\right\}\\
&\le \inf_{f\in\mathscr{F}_n}\mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^n\left[|f(X_i)-Y_i|^2-|m(X_i)-Y_i|^2\right]\right\}\\
&= \inf_{f\in\mathscr{F}_n}\left\{\mathbb{E}|f(X)-Y|^2-\mathbb{E}|m(X)-Y|^2\right\}\\
&= \inf_{f\in\mathscr{F}n}\int|f(x)-m(x)|^2\mu(\mathrm{d}x)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\left\{\mathbb{E}[|m_n(X)-Y|^2|\mathcal{D}_n]-\mathbb{E}|m(X)-Y|^2-\frac{2}{n}\sum_{i=1}^n\left[|m_n(X_i)-Y_i|^2-|m(X_i)-Y_i|^2\right]>\varepsilon\right\}\\
&=\mathbb{P}\left\{\mathbb{E}[|m_n(X)-Y|^2|\mathcal{D}_n]-\mathbb{E}|m(X)-Y|^2-\frac{1}{n}\sum_{i=1}^n\left[|m_n(X_i)-Y_i|^2-|m(X_i)-Y_i|^2\right]>\frac{\varepsilon}{2}+\frac{1}{2}\left[\mathbb{E}[|m_n(X)-Y|^2|\mathcal{D}_n]-\mathbb{E}|m(X)-Y|^2\right]\right\}\\
&\le\mathbb{P}\left\{\exists f\in T_{\beta_n}\mathscr{F}_n:\mathbb{E}|f(X)-Y|^2-\mathbb{E}|m(X)-Y|^2-\frac{1}{n}\sum_{i=1}^n\left[|f(X_i)-Y_i|^2-|m(X_i)-Y_i|^2\right]>\frac{\varepsilon}{2}+\frac{1}{2}\left[\mathbb{E}|f(X)-Y|^2-\mathbb{E}|m(X)-Y|^2\right]\right\}.
\end{aligned}$$

Set $Z=(X,Y),Z_i=(X_i,Y_i),g(Z)=|f(X)-Y|^2-|m(X)-Y|^2$. We can rewrite the above equation as

$$\mathbb{P}\left\{\mathbb{E}g(Z)-\frac{1}{n}\sum_{i=1}^n g(Z_i)>\frac{\varepsilon}{2}+\frac{1}{2}\mathbb{E}g(Z)\right\}.$$

Since $|g(Z)|=|(f(X)+m(X)-2Y)(f(X)-m(X))|\le 4\beta_n|f(X)-m(X)|,\sigma^2:=\mathrm{Var}(g(Z))\le\mathbb{E}g(Z)^2\le 16\beta_n^2\mathbb{E}|f(X)-m(X)|^2=16\beta_n^2(\mathbb{E}|f(X)-Y|^2-\mathbb{E}|m(X)-Y|^2)$, the above equation is upper-bounded by

$$\mathbb{P}\left\{\mathbb{E}g(Z)-\frac{1}{n}\sum_{i=1}^n g(Z_i)>\frac{\varepsilon}{2}+\frac{1}{2}\frac{\mathrm{Var}(g(Z))}{16\beta_n^2}\right\}\overset{\text{Berstein's inequality}}{\le}\exp\left(-\frac{n[\frac{\varepsilon}{2}+\frac{\sigma^2}{32\beta_n^2}]^2}{2\sigma^2+2\frac{8\beta_n^2}{3}[\frac{\varepsilon}{2}+\frac{\sigma^2}{32\beta_n^2}]}\right)\le\exp\left(-\frac{1}{128+\frac{32}{3}}\frac{n\varepsilon}{\beta_n^2}\right).$$

Theorem 6.6 Let $n\in\mathbb{N}$ and $1\le L<\infty$. Assume $|Y|\le L$ a.s. Let estimate m_n be defined by minimization of the empirical l_2 risk over a set of functions $\mathscr{F}_n$ and truncation at L. Then one has

$$\mathbb{E}\int|m_n(x)-m(x)|^2\mu(\mathrm{d}x)\le\frac{c_1}{n}+\frac{(c_2+c_3\log n)V_{\mathscr{F}_n^+}}{n}+2\inf_{f\in\mathscr{F}_n}\int|f(x)-m(x)|^2\mu(\mathrm{d}x)$$

Proof **Lemma** 6.1 Assume $|Y|\le B$ a.s. and $B\ge 1$. Let $\mathscr{F}$ be a set of functions $f:\mathbb{R}^d\to\mathbb{R}$ and let $|f(x)|\le B$. Then for any $n\ge 1$,

$$\begin{aligned}
&\mathbb{P}\left\{\exists f\in\mathscr{F}:\mathbb{E}|f(X)-Y|^2-\mathbb{E}|m(X)-Y|^2-\frac{1}{n}\sum_{i=1}^n\left[|f(X_i)-Y_i|^2-|m(X_i)-Y_i|^2\right]\right.\\
&\qquad\qquad\left.\ge\varepsilon(\alpha+\beta+\mathbb{E}|f(X)-Y|^2-\mathbb{E}|m(X)-Y|^2)\right\}\\
&\le 14\sup_{X^{1:n}}\mathscr{N}_1\left(\frac{\beta\varepsilon}{20B},\mathscr{F},X^{1:n}\right)\exp\left(-\frac{\varepsilon^2(1-\varepsilon)\alpha n}{214(1+\varepsilon)B^2}\right).
\end{aligned}$$

Now let's return to the original theorem.

$$\begin{aligned}
\int|m_n(x)-m(x)|^2\mu(\mathrm{d}x)&=\left\{\mathbb{E}[|m_n(X)-Y|^2|\mathcal{D}_n]-\mathbb{E}[|m(X)-Y|^2]-2\left(\frac{1}{n}\sum_{i=1}^n|m_n(X_i)-Y_i|^2-\frac{1}{n}\sum_{i=1}^n|m(X_i)-Y_i|^2\right)\right\}\\
&+2\left\{\frac{1}{n}\sum_{i=1}^n|m_n(X_i)-Y_i|^2-\frac{1}{n}\sum_{i=1}^n|m(X_i)-Y_i|^2\right\}\\
&:=T_{1,n}+T_{2,n}.
\end{aligned}$$

Since

$$\mathbb{E}(T_{2,n})\le 2\inf_{f\in\mathscr{F}_n}\int|f(x)-m(x)|^2\mu(\mathrm{d}x),\mathbb{E}(T_{1,n})=\int_0^\infty\mathbb{P}(T_{1,n}>t)\mathrm{d}t\le\varepsilon+\int_\varepsilon^\infty\mathbb{P}(T_{1,n}>t)\mathrm{d}t$$

and

$$\begin{aligned}
\mathbb{P}(T_{1,n} > t) &= \mathbb{P}\Bigg\{\mathbb{E}[|m_n(X) - Y|^2|\mathcal{D}_n] - \mathbb{E}[|m(X) - Y|^2] - \frac{1}{n}\sum_{i=1}^n [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \\
&\qquad \geq \frac{1}{2}\left(\frac{t}{2} + \frac{t}{2} + \mathbb{E}[|m_n(X) - Y|^2|\mathcal{D}_n] - \mathbb{E}[|m(X) - Y|^2]\right)\Bigg\} \\
&\leq \mathbb{P}\Bigg\{\exists f \in T_L\mathscr{F}_n : \mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2 - \frac{1}{n}\sum_{i=1}^n [|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \\
&\qquad \geq \frac{1}{2}\left(\frac{t}{2} + \frac{t}{2} + \mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2\right)\Bigg\} \\
&\leq 14 \sup_{X^{1:n}} \mathscr{N}_1\left(\frac{1}{80Ln}, T_L\mathscr{F}_n, X^{1:n}\right)\exp\left(-\frac{nt}{24\cdot 214L^4}\right) \leq 3(480eL^2n)^{2V_{(T_L\mathscr{F}_n)^+}}\exp\left(-\frac{nt}{24\cdot 214L^4}\right).
\end{aligned}$$

Plug this bound into the integral in the previous expectation bound,

$$\mathbb{E}(T_{1,n}) \leq \varepsilon + \frac{24\cdot 214L^4}{n}42(480eL^2n)^{2V_{\mathscr{F}_n^+}}\exp\left(-\frac{n\varepsilon}{24\cdot 214L^4}\right). \qquad \square$$

Lemma 6.2 Let $V_1, \cdots, V_n$ i.i.d. r.v.'s, $0 \leq V_i \leq \beta, 0 < \alpha < 1$ and $\nu > 0$. Then

$$\mathbb{P}\left\{\frac{|\frac{1}{n}\sum_{i=1}^n V_i - \mathbb{E}V_1|}{\nu + \frac{1}{n}\sum_{i=1}^n V_i + \mathbb{E}V_1} > \alpha\right\} \leq \mathbb{P}\left\{\frac{|\frac{1}{n}\sum_{i=1}^n V_i - \mathbb{E}V_1|}{\nu + \mathbb{E}V_1} > \alpha\right\} < \frac{\beta}{4\alpha^2\nu n}.$$

Proof The first inequality is trivial. For the second, note that

$$\mathbb{P}\left\{|\sum_{i=1}^n (V_i - \mathbb{E}V_1)| > \alpha n(\nu + \mathbb{E}V_1)\right\} \leq \frac{\mathbb{E}|\sum_{i=1}^n (V_i - \mathbb{E}V_1)|^2}{[\alpha n(\nu + \mathbb{E}V_1)]^2} = \frac{\mathrm{Var}(V_1)}{n\alpha^2(\nu + \mathbb{E}V_1)^2} \leq \frac{\mathbb{E}V_1(\beta - \mathbb{E}V_1)}{n\alpha^2(\nu + \mathbb{E}V_1)^2}$$

where the last inequality holds since

$$\mathrm{Var}(V_1) = \mathbb{E}\{(V_1 - \mathbb{E}V_1)(V_1 - \mathbb{E}V_1) = \mathbb{E}V_1(V_1 - \mathbb{E}V_1) \leq \mathbb{E}V_1(\beta - \mathbb{E}V_1)\}.$$

In addition,

$$\frac{\mathbb{E}V_1(\beta - \mathbb{E}V_1)}{n\alpha^2(\nu + \mathbb{E}V_1)^2} \leq \max_{x\in[0,\beta]} \frac{x(\beta - x)}{n\alpha^2(\nu + x)^2} = \frac{\beta^2}{4\alpha^2\nu n(\beta + \nu)} < \frac{\beta}{4\alpha^2\nu n}. \qquad \square$$

Theorem 6.7 Let $B \geq 1$ and G be a set of functions $g : \mathbb{R}^d \to [0, B]$. Let $Z_1, \cdots, Z_n$ be i.i.d. $\mathbb{R}^d$-valued r.v.'s. Assume $\alpha > 0, 0 < \varepsilon < 1$ and $n \geq 1$. Then

$$\mathbb{P}\left\{\sup_{g\in\mathscr{G}} \frac{\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z)}{\alpha + \frac{1}{n}\sum_{i=1}^n g(Z_i) + \mathbb{E}g(Z)} > \varepsilon\right\} \leq 4\mathbb{E}\mathscr{N}_1\left(\frac{2\varepsilon}{5}, G, Z^{1:n}\right)\exp\left(-\frac{3\varepsilon^2\alpha n}{40B}\right).$$

Proof Step 1: Replace the expectation with empirical mean. Ghost sample $Z'_{1:n} = (Z'_1, \cdots, Z'_n)$ i.i.d. Let g^* be a function $g \in \mathscr{G}$ such that

$$\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) > \varepsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^n g(Z_i) + \mathbb{E}g(Z)\right)$$

if there exists any such function. Otherwise, let g^* be an arbitrary function in G. g^* depenods on $Z^{1:n}$. Since

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) > \varepsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^n g(Z_i) + \mathbb{E}g(Z)\right) \text{ and } \frac{1}{n}\sum_{i=1}^n g(Z'_i) - \mathbb{E}g(Z) \leq \frac{\varepsilon}{4}\left(\alpha + \frac{1}{n}\sum_{i=1}^n g(Z'_i) + \mathbb{E}g(Z)\right) \\
\Rightarrow &\frac{1}{n}\sum_{i=1}^n g(Z_i) - \frac{1}{n}\sum_{i=1}^n g(Z'_i) > \frac{3}{4}\varepsilon\alpha + \frac{\varepsilon}{n}\sum_{i=1}^n g(Z_i) - \frac{\varepsilon}{n}\sum_{i=1}^n g(Z'_i) + \frac{3\varepsilon}{4}\mathbb{E}g(Z) \\
\Leftrightarrow &\left(1 - \frac{5}{8}\varepsilon\right)\left(\frac{1}{n}\sum_{i=1}^n g(Z_i) - \frac{1}{n}\sum_{i=1}^n g(Z'_i)\right) > \frac{3}{8}\varepsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^n g(Z_i) + \frac{1}{n}\sum_{i=1}^n g(Z'_i)\right) + \frac{3\varepsilon}{4}\mathbb{E}g(Z)
\end{aligned}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \frac{1}{n}\sum_{i=1}^{n} g(Z_i') > \frac{3\varepsilon}{8}\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i) + \frac{1}{n}\sum_{i=1}^{n} g(Z_i')\right),$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \frac{1}{n}\sum_{i=1}^{n} g(Z_i') > \frac{3}{8}\varepsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i) + \frac{1}{n}\sum_{i=1}^{n} g(Z_i')\right)\right\} \\
&\geq \mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{n} g^*(Z_i) - \frac{1}{n}\sum_{i=1}^{n} g^*(Z_i') > \frac{3\varepsilon}{8}\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g^*(Z_i) + \frac{1}{n}\sum_{i=1}^{n} g^*(Z_i')\right)\right\} \\
&\geq \mathbb{P}\Bigg\{\frac{1}{n}\sum_{i=1}^{n} g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] > \varepsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g^*(Z_i) + \mathbb{E}[g^*(Z)|Z^{1:n}]\right) \text{ and} \\
&\qquad \frac{1}{n}\sum_{i=1}^{n} g^*(Z_i') - \mathbb{E}[g^*(Z)|Z^{1:n}] \leq \frac{\varepsilon}{4}\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g^*(Z_i') + \mathbb{E}[g^*(Z)|Z^{1:n}]\right)\Bigg\} \\
&= \mathbb{E}1_{\left\{\frac{1}{n}\sum_{i=1}^{n} g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] > \varepsilon(\cdots)\right\}} \underbrace{\mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{n} g^*(Z_i') - \mathbb{E}[g^*(Z)|Z^{1:n}] \leq \frac{\varepsilon}{4}(\cdots)\right\}}_{>1-\frac{B}{4(\frac{\varepsilon}{4})^2\alpha n}=1-\frac{4B}{\varepsilon^2\alpha n}\geq\frac{1}{2} \text{ for } n>\frac{8B}{\varepsilon^2\alpha} \text{ by lemma 6.2}} \\
&\geq \frac{1}{2}\mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{n} g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] > \varepsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g^*(Z_i) + \mathbb{E}[g^*(Z)|Z^{1:n}]\right)\right\} \\
&= \frac{1}{2}\mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \mathbb{E}[g(Z)] > \varepsilon\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i) + \mathbb{E}[g(Z)]\right)\right\}
\end{aligned}$$

Step 2: Symmetrization. Let $U_1, \cdots, U_n \overset{\text{i.i.d.}}{\sim} U\{-1, 1\}$ independent of $Z'_{1:n}, Z^{1:n}$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \frac{1}{n}\sum_{i=1}^{n} g(Z_i') > \frac{3}{8}\varepsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i) + \frac{1}{n}\sum_{i=1}^{n} g(Z_i')\right)\right\} \\
&= \mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} U_i[g(Z_i) - \frac{1}{n}\sum_{i=1}^{n} g(Z_i')] > \frac{3}{8}\varepsilon\left(2\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i) + \frac{1}{n}\sum_{i=1}^{n} g(Z_i')\right)\right\} \\
&\leq \mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) > \frac{3\varepsilon}{8}\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i)\right)\right\} + \mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i') < -\frac{3\varepsilon}{8}\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i')\right)\right\} \\
&= 2\mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) > \frac{3\varepsilon}{8}\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i)\right)\right\}.
\end{aligned}$$

Step 3: Conditioning and introduction of a covering. Let $\delta > 0$ and G_δ be an L_1 δ-cover of $\mathscr{G}$ on $Z^{1:n}$. For $g \in \mathscr{G}$, there exists a $\bar{g} \in G_\delta$ such that $\frac{1}{n}\sum\limits_{i=1}^{n} |g(Z_i) - \bar{g}(Z_i)| < \delta$. Therefore,

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) &= \frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) - \frac{1}{n}\sum_{i=1}^{n} U_i \bar{g}(Z_i) + \frac{1}{n}\sum_{i=1}^{n} U_i \bar{g}(Z_i) \\
&\leq \frac{1}{n}\sum_{i=1}^{n} U_i \bar{g}(Z_i) + \frac{1}{n}\sum_{i=1}^{n} |g(Z_i) - \bar{g}(Z_i)| \leq \frac{1}{n}\sum_{i=1}^{n} U_i \bar{g}(Z_i) + \delta.
\end{aligned}$$

On the other hand, $\displaystyle\frac{1}{n}\sum_{i=1}^{n} g(Z_i) \geq \frac{1}{n}\sum_{i=1}^{n} \bar{g}(Z_i) - \frac{1}{n}\sum_{i=1}^{n} |g(Z_i) - \bar{g}(Z_i)| \geq \frac{1}{n}\sum_{i=1}^{n} \bar{g}(Z_i) - \delta$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{\exists g \in \mathscr{G} : \frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) > \frac{3\varepsilon}{8}\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i)\right)\right\} \\
&\leq \mathbb{P}\left\{\exists g \in \mathscr{G}_\delta : \frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) + \delta > \frac{3\varepsilon}{8}\left(\alpha + \frac{1}{n}\sum_{i=1}^{n} g(Z_i) - \delta\right)\right\} \\
&\leq |\mathscr{G}_\delta| \max_{g \in \mathscr{G}_\delta} \mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) > \frac{3\varepsilon\alpha}{8} - \frac{3\varepsilon\delta}{8} - \delta + \frac{3\varepsilon}{8}\frac{1}{n}\sum_{i=1}^{n} g(Z_i)\right\}
\end{aligned}$$

$$\left(\text{Take } \delta = \frac{\varepsilon\alpha}{5}\right) \leq |\mathscr{G}_\delta| \max_{g \in \mathscr{G}_\delta} \mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{n} U_i g(Z_i) > \frac{\varepsilon\alpha}{10} + \frac{3\varepsilon}{8}\frac{1}{n}\sum_{i=1}^{n} g(Z_i)\right\}$$

Step 4: Application of Hoeffding's inequality. $\square$