

# Advanced Theory of Statistics

Lectured by Wang Miao

L<sup>A</sup>T<sub>E</sub>Xed by Chengxin Gong

2022 年 11 月 7 日

## 目录

<b>1</b>	<b>Probability Theory</b>	<b>2</b>
1.1	Measure space, measurable function, and integration . . . . .	2
1.2	Integration theory and Radon-Nikodym derivative . . . . .	3
1.3	Densities, moments, inequalities, and generating functions . . . . .	4
1.4	Conditional expectation and independence . . . . .	5
1.5	Convergence modes and relationships . . . . .	6
1.6	Uniform integrability and weak convergence . . . . .	7
1.7	Convergence of transformations and law of large numbers . . . . .	8
1.8	The central limit theorem . . . . .	9
<b>2</b>	<b>Fundamentals of Statistics</b>	<b>9</b>
2.1	Models, data, statistics, and sampling distributions . . . . .	9
2.2	Sufficiency and minimal sufficiency . . . . .	10
2.3	Completeness . . . . .	11
2.4	Statistical decision . . . . .	11
2.5	Statistical inference . . . . .	13
<b>3</b>	<b>Unbiased Estimation</b>	<b>15</b>
3.1	UMVUE: functions of sufficient and complete statistics . . . . .	15
3.2	Characteristic of UMVUE and Fisher information bound . . . . .	17
3.3	U- and V-statistics . . . . .	18
3.4	Construction of unbiased or approximately unbiased estimators and method of moments . . . . .	19
<b>4</b>	<b>Estimation in Parametric Models</b>	<b>20</b>
4.1	Bayesian approach . . . . .	20
4.2	Bayes rule and computation . . . . .	21
4.3	Minimaxity and admissibility . . . . .	22
4.4	Simultaneous estimation and shrinkage estimators . . . . .	23

# 1 Probability Theory

## 1.1 Measure space, measurable function, and integration

**Definition 1:** A collection of subsets of  $\Omega, \mathcal{F}$ , is a  $\sigma$ -field (or  $\sigma$ -algebra) if (i) The empty set  $\emptyset \in \mathcal{F}$ ; (ii) If  $A \in \mathcal{F}$ , then the complement  $A^c \in \mathcal{F}$ ; (iii) If  $A_i \in \mathcal{F}, i = 1, 2, \dots$ , then their union  $\cup A_i \in \mathcal{F}$ .  $(\Omega, \mathcal{F})$  is a measurable space if  $\mathcal{F}$  is a  $\sigma$ -field on  $\Omega$ .

**Example 1:**  $\mathcal{C}$  = a collection of subsets of interest.  $\sigma(\mathcal{C})$  = the smallest  $\sigma$ -field containing  $\mathcal{C}$  (the  $\sigma$ -field generated by  $\mathcal{C}$ ).  $\sigma(\mathcal{C}) = \mathcal{C}$  if  $\mathcal{C}$  itself is a  $\sigma$ -field.  $\sigma(\{A\}) = \{\emptyset, A, A^c, \Omega\}$ .

**Example 2 (Borel  $\sigma$ -field):**  $\mathbb{R}^k$ : the  $k$ -dimensional Euclidean space ( $\mathbb{R}^1 = \mathbb{R}$  is the real line).  $\mathcal{O}$  = all open sets,  $\mathcal{C}$  = all closed sets.  $\mathcal{B}^k = \sigma(\mathcal{O}) = \sigma(\mathcal{C})$ : the Borel  $\sigma$ -field on  $\mathbb{R}^k$ .  $C \in \mathcal{B}^k, \mathcal{B}_C = \{C \cap B : B \in \mathcal{B}^k\}$  is the Borel  $\sigma$ -field on  $C$ .

**Definition 2:** Let  $(\Omega, \mathcal{F})$  be a measurable space. A set function  $\nu$  defined on  $\mathcal{F}$  is a measure if (i)  $0 \leq \nu(A) \leq \infty$  for any  $A \in \mathcal{F}$ ; (ii)  $\nu(\emptyset) = 0$ ; (iii) If  $A_i \in \mathcal{F}, i = 1, 2, \dots$ , and  $A_i$ 's are disjoint, i.e.  $A_i \cap A_j = \emptyset$  for any  $i \neq j$ , then  $\nu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i)$ .  $(\Omega, \mathcal{F}, \nu)$  is a measure if  $\nu$  is a measure on  $\mathcal{F}$  in  $(\Omega, \mathcal{F})$ .

**Convention 1:** For any  $x \in \mathbb{R}$ ,  $\infty + x = \infty$ ,  $x\infty = \infty$  if  $x > 0$ ,  $x\infty = -\infty$  if  $x < 0$ .  $0\infty = 0$ ,  $\infty + \infty = \infty$ ,  $\infty^a = \infty$  for any  $a > 0$ .  $\infty - \infty$  or  $\infty/\infty$  is not defined.

**Example 3 (Important examples of measures):** (a) Let  $x \in \Omega$  be a fixed point and  $\delta_x(A) = \begin{cases} c & x \in A \\ 0 & x \notin A \end{cases}$ . This is called a point mass at  $x$ . (b) Let  $\mathcal{F}$  = all subsets of  $\Omega$  and  $\nu(A)$  = the number of elements in  $A \in \mathcal{F}$  ( $\nu(A) = \infty$  if  $A$  contains infinitely many elements). Then  $\nu$  is a measure on  $\mathcal{F}$  and is called the counting measure. (c) There is a unique measure  $m$  on  $(\mathbb{R}, \mathcal{B})$ , that satisfies  $m([a, b]) = b - a$  for every finite interval  $[a, b]$ ,  $-\infty < a \leq b < \infty$ . This is called the Lebesgue measure.

**Proposition 1 (Properties of measures):** Let  $(\Omega, \mathcal{F}, \nu)$  be a measure space. (1) Monotonicity: If  $A \subset B$ , then  $\nu(A) \leq \nu(B)$ . (2) Subadditivity: For any sequence  $A_1, A_2, \dots$ ,  $\nu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \nu(A_i)$ . (3) Continuity: If  $A_1 \subset A_2 \subset A_3 \subset \dots$  (or  $A_1 \supset A_2 \supset A_3 \supset \dots$  and  $\nu(A_1) < \infty$ ), then  $\nu(\lim_{n \rightarrow \infty} A_n) = \lim_{n \rightarrow \infty} \nu(A_n)$  where  $\lim_{n \rightarrow \infty} A_n = \cup_{i=1}^{\infty} A_i$  (or  $= \cap_{i=1}^{\infty} A_i$ ).

**Definition 3:** Let  $P$  be a probability measure on  $(\mathbb{R}, \mathcal{B})$ . The cumulative distribution function (c.d.f.) of  $P$  is defined to be  $F(x) = P((-\infty, x])$ ,  $x \in \mathbb{R}$ .

**Proposition 2 (Properties of c.d.f.'s):** (i) Let  $F$  be a c.d.f. on  $\mathbb{R}$ . (a)  $F(-\infty) = \lim_{x \rightarrow -\infty} F(x) = 0$ ; (b)  $F(\infty) = \lim_{x \rightarrow \infty} F(x) = 1$ ; (c)  $F$  is nondecreasing, i.e.  $F(x) \leq F(y)$  if  $x \leq y$ ; (d)  $F$  is right continuous, i.e.  $\lim_{y \rightarrow x+0} F(y) = F(x)$ . (ii) Suppose a real-valued function  $F$  on  $\mathbb{R}$  satisfies (a)-(d) in part (i). Then  $F$  is the c.d.f. of a unique probability measure on  $(\mathbb{R}, \mathcal{B})$ .

**Definition 4 (Product space):**  $\mathcal{I} = \{1, \dots, k\}$ ,  $k$  is finite or  $\infty$ .  $\Gamma_i, i \in \mathcal{I}$ , are some sets.  $\prod_{i \in \mathcal{I}} \Gamma_i = \Gamma_1 \times \dots \times \Gamma_k = \{(a_1, \dots, a_k) : a_i \in \Gamma_i, i \in \mathcal{I}\}$ . Let  $(\Omega_i, \mathcal{F}_i), i \in \mathcal{I}$  be measurable spaces.  $\sigma(\prod_{i \in \mathcal{I}} \mathcal{F}_i)$  is called the product  $\sigma$ -field on the product space  $\prod_{i \in \mathcal{I}} \Omega_i$ .  $(\prod_{i \in \mathcal{I}} \Omega_i, \sigma(\prod_{i \in \mathcal{I}} \mathcal{F}_i))$  is denoted by  $\prod_{i \in \mathcal{I}} (\Omega_i, \mathcal{F}_i)$ .

**Definition 5 ( $\sigma$ -finite):** A measure  $\nu$  on  $(\Omega, \mathcal{F})$  is said to be  $\sigma$ -finite iff there exists a sequence  $\{A_1, A_2, \dots\}$  such that  $\cup A_i = \Omega$  and  $\nu(A_i) < \infty$  for all  $i$ . Any finite measure is clearly  $\sigma$ -finite. The Lebesgue measure on  $\mathcal{F}$  is  $\sigma$ -finite.

**Proposition 3 (Product measure theorem):** Let  $(\Omega_i, \mathcal{F}_i, \nu_i), i = 1, \dots, k$ , be measure spaces with  $\sigma$ -finite measures. There exists a unique  $\sigma$ -finite measure on  $\sigma$ -field  $\sigma(\mathcal{F}_1 \times \dots \times \mathcal{F}_k)$ , called the product measure and denoted by  $\nu_1 \times \dots \times \nu_k$ , such that  $\nu_1 \times \dots \times \nu_k(A_1 \times \dots \times A_k) = \nu_1(A_1) \dots \nu_k(A_k)$  for all  $A_i \in \mathcal{F}_i, i = 1, \dots, k$ .

**Definition 6 (Measurable function):** Let  $(\Omega, \mathcal{F})$  and  $(\Lambda, \mathcal{G})$  be measurable spaces. Let  $f$  be a function from  $\Omega$  to  $\Lambda$ .  $f$  is called a measurable function from  $(\Omega, \mathcal{F})$  to  $(\Lambda, \mathcal{G})$  iff  $f^{-1}(\mathcal{G}) \subset \mathcal{F}$ .

**Definition 7 (Integration):** (a) The integral of a nonnegative simple function  $\phi$  w.r.t.  $\nu$  is defined as  $\int \phi d\nu = \sum_{i=1}^k a_i \nu(A_i)$ . (b) Let  $f$  be a nonnegative Borel function and let  $\mathcal{S}_f$  be the collection of all nonnegative simple functions satisfying  $\phi(\omega) \leq f(\omega)$  for any  $\omega \in \Omega$ . The integral of  $f$  w.r.t.  $\nu$  is defined as  $\int f d\nu = \sup\{\int \phi d\nu : \phi \in \mathcal{S}_f\}$  (Hence, for any Borel function  $f \geq 0$ , there exists a sequence of simple functions  $\phi_1, \phi_2, \dots$  such that  $0 \leq \phi_i \leq f$  for all  $i$  and  $\lim_{n \rightarrow \infty} \int \phi_n d\nu = \int f d\nu$ ). (c) Let  $f$  be a Borel function,  $f_+(\omega) = \max\{f(\omega), 0\}$  be the positive part of  $f$ , and  $f_-(\omega) = \max\{-f(\omega), 0\}$  be the negative part of  $f$ . We say that  $\int f d\nu$  exists if and only if at least one of  $\int f_+ d\nu$  and  $\int f_- d\nu$  is finite, in which case  $\int f d\nu = \int f_+ d\nu - \int f_- d\nu$ . (d) When both  $\int f_+ d\nu$  and  $\int f_- d\nu$  are finite, we say that  $f$  is integrable. Let  $A$  be a measurable set and  $I_A$  be its indicator function. The integral of  $f$  over  $A$  is defined as  $\int_A f d\nu = \int I_A f d\nu$ .

**Example 4 (Extended set):** For convenience, we define the integral of a measurable  $f$  from  $(\Omega, \mathcal{F}, \nu)$  to  $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ , where  $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ ,  $\bar{\mathcal{B}} = \sigma(\mathcal{B} \cup \{\infty, -\infty\})$ . Let  $A_+ = \{f = \infty\}$  and  $A_- = \{f = -\infty\}$ . If  $\nu(A_+) = 0$ , we define  $\int f_+ d\nu$  to be  $\int I_{A_+} f_+ d\nu$ ; otherwise  $\int f_+ d\nu = \infty$ .  $\int f_- d\nu$  is similarly defined. If at least one of  $\int f_+ d\nu$  and  $\int f_- d\nu$  is finite, then  $\int f d\nu = \int f_+ d\nu - \int f_- d\nu$  is well defined.

## 1.2 Integration theory and Radon-Nikodym derivative

**Proposition 1:**  $(\Omega, \mathcal{F}, \nu)$  be a measure space and  $f$  and  $g$  be Borel functions. (i) If  $f \leq g$  a.e., then  $\int f d\nu \leq \int g d\nu$ , provided that the integrals exist. (ii) If  $f \geq 0$  a.e. and  $\int f d\nu = 0$ , then  $f = 0$  a.e.

**Theorem 1:** Let  $f_1, f_2, \dots$  be a sequence of Borel functions on  $(\Omega, \mathcal{F}, \nu)$ . (i) Fatou's lemma: If  $f_n \geq 0$ , then  $\int \liminf_n f_n d\nu \leq \liminf_n \int f_n d\nu$ . (ii) Dominated convergence theorem: If  $\lim_{n \rightarrow \infty} f_n = f$  a.e. and  $|f_n| \leq g$  a.e. for integrable  $g$ , then  $\int \lim_{n \rightarrow \infty} f_n d\nu = \lim_{n \rightarrow \infty} \int f_n d\nu$ . (iii) Monotone convergence theorem: If  $0 \leq f_1 \leq f_2 \leq \dots$  and  $\lim_{n \rightarrow \infty} f_n = f$  a.e., then  $\int \lim_{n \rightarrow \infty} f_n d\nu = \lim_{n \rightarrow \infty} \int f_n d\nu$ .

**Example 1 (Interchange of differentiation and integration):** Let  $(\Omega, \mathcal{F}, \nu)$  be a measure space and, for any fixed  $\theta \in \mathbb{R}$ , let  $f(\omega, \theta)$  be a Borel function on  $\Omega$ . Suppose that  $\partial f(\omega, \theta)/\partial \theta$  exists a.e. for  $\theta \in (a, b) \subset \mathbb{R}$  and that  $|\partial f(\omega, \theta)/\partial \theta| \leq g(\omega)$  a.e., where  $g$  is an integrable function on  $\Omega$ . Then for each  $\theta \in (a, b)$ ,  $\partial f(\omega, \theta)/\partial \theta$  is integrable and, by Theorem 1(ii),  $\frac{d}{d\theta} \int f(\omega, \theta) d\nu = \int \frac{\partial f(\omega, \theta)}{\partial \theta} d\nu$ .

**Theorem 2 (Change of variables):** Let  $f$  be measurable from  $(\Omega, \mathcal{F}, \nu)$  to  $(\Lambda, \mathcal{G})$  and  $g$  be Borel on  $(\Lambda, \mathcal{G})$ . Then  $\int_\Omega g \circ f d\nu = \int_\Lambda g d(\nu \circ f^{-1})$ , i.e., if either integral exists, then so does the other, and the two are the same.

**Theorem 3 (Fubini's theorem):** Let  $\nu_i$  be a  $\sigma$ -finite measure on  $(\Omega_i, \mathcal{F}_i), i = 1, 2$ , and  $f$  be a Borel function on  $\prod_{i=1}^2 (\Omega_i, \mathcal{F}_i)$  with  $f \geq 0$  or  $\int |f| d\nu_1 \times \nu_2 < \infty$ . Then  $g(\omega_2) = \int_{\Omega_1} f(\omega_1, \omega_2) d\nu_1$  exists a.e.  $\nu_2$  and defines a Borel function on  $\Omega_2$  whose integral w.r.t.  $\nu_2$  exists, and  $\int_{\Omega \times \Omega} f(\omega_1, \omega_2) d\nu_1 \times \nu_2 = \int_{\Omega_2} [\int_{\Omega_1} f(\omega_1, \omega_2) d\nu_1] d\nu_2$ .

## PROBABILITY THEORY

**Definition 1 (Absolutely continuous):** Let  $\lambda$  and  $\nu$  be two measures on a measurable space  $(\Omega, \mathcal{F}, \nu)$ . We say  $\lambda$  is absolutely continuous w.r.t.  $\nu$  and write  $\lambda \ll \nu$  iff  $\nu(A) = 0$  implies  $\lambda(A) = 0$ .

**Theorem 4 (Radon-Nikodym theorem):** Let  $\nu$  and  $\lambda$  be two measure on  $(\Omega, \mathcal{F})$  and  $\nu$  be  $\sigma$ -finite. If  $\lambda \ll \nu$ , then there exists a nonnegative Borel function  $f$  on  $\Omega$  such that  $\lambda(A) = \int_A f d\nu, A \in \mathcal{F}$ . Furthermore,  $f$  is unique a.e.  $\nu$ , i.e. if  $\lambda(A) = \int_A g d\nu$  for any  $A \in \mathcal{F}$ , then  $f = g$  a.e.  $\nu$ .

**Example 2:** A continuous c.d.f. may not have a p.d.f. w.r.t. Lebesgue measure. A necessary and sufficient condition for a c.d.f.  $F$  having a p.d.f. w.r.t. Lebesgue measure is that  $F$  is absolute continuous in the sense that for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for each finite collection of disjoint bounded open intervals  $(a_i, b_i)$ ,  $\sum(b_i - a_i) < \delta$  implies  $\sum[F(b_i) - F(a_i)] < \epsilon$ .

**Proposition 2 (Calculus with Radon-Nikodym derivatives):** Let  $\nu$  be a  $\sigma$ -finite measure on a measure space  $(\Omega, \mathcal{F})$ . (i) If  $\lambda$  is a measure,  $\lambda \ll \nu$ , and  $f \geq 0$ , then  $\int f d\lambda = \int f \frac{d\lambda}{d\nu} d\nu$ . (ii) If  $\lambda_i, i = 1, 2$ , are measures and  $\lambda_i \ll \nu$ , then  $\lambda_1 + \lambda_2 \ll \nu$  and  $\frac{d(\lambda_1 + \lambda_2)}{d\nu} = \frac{d\lambda_1}{d\nu} + \frac{d\lambda_2}{d\nu}$  a.e.  $\nu$ . (iii) If  $\tau$  is a measure,  $\lambda$  is a  $\sigma$ -finite measure, and  $\tau \ll \lambda \ll \nu$ , then  $\frac{d\tau}{d\nu} = \frac{d\tau}{d\lambda} \frac{d\lambda}{d\nu}$  a.e.  $\nu$ . In particular, if  $\lambda \ll \nu$  and  $\nu \ll \lambda$  (in which case  $\lambda$  and  $\nu$  are equivalent), then  $\frac{d\lambda}{d\nu} = \left(\frac{d\nu}{d\lambda}\right)^{-1}$  a.e.  $\nu$  or  $\lambda$ . (iv) Let  $(\Omega_i, \mathcal{F}_i, \nu_i)$  be a measure space and  $\nu_i$  be  $\sigma$ -finite,  $i = 1, 2$ . Let  $\lambda_i$  be a  $\sigma$ -finite measure on  $(\Omega, \mathcal{F}_i)$  and  $\lambda_i \ll \nu_i, i = 1, 2$ . Then  $\lambda_1 \times \lambda_2 \ll \nu_1 \times \nu_2$  and  $\frac{d(\lambda_1 \times \lambda_2)}{d(\nu_1 \times \nu_2)}(\omega_1, \omega_2) = \frac{d\lambda_1}{d\nu_1}(\omega_1) \frac{d\lambda_2}{d\nu_2}(\omega_2)$  a.e.  $\nu_1 \times \nu_2$ .

### 1.3 Densities, moments, inequalities, and generating functions

**Example 1:** Let  $X$  be a random variable on  $(\Omega, \mathcal{F}, P)$  whose c.d.f.  $F_X$  has a Lebesgue p.d.f.  $f_x$  and  $F_x(c) < 1$ , where  $c$  is a fixed constant. Let  $Y = \min\{X, c\}$ . Note that  $Y^{-1}((-\infty, X]) = \Omega$  if  $x \geq c$  and  $Y^{-1}((-\infty, x]) = X^{-1}((-\infty, x])$  if  $x < c$ . Hence  $Y$  is a random variable and the c.d.f. of

$$Y \text{ is } F_Y(x) = \begin{cases} 1 & x \geq c \\ F_X(x) & x < c \end{cases}. \text{ This c.d.f. is discontinuous at } c, \text{ since } F_x(c) < 1. \text{ Thus, it does}$$

not have a Lebesgue p.d.f. It is not discrete either. Does  $P_Y$ , the probability measure corresponding to  $F_y$ , have a p.d.f. w.r.t. some measure? Consider the point mass probability measure on  $(\mathbb{R}, \mathcal{B})$ :

$$\delta_c(A) = \begin{cases} 1 & c \in A \\ 0 & c \notin A \end{cases}, A \in \mathcal{B}. \text{ Then } P_Y \ll m + \delta_c, \text{ and the p.d.f. of } P_Y \text{ is } f_Y(x) = \frac{dP_Y}{d(m + \delta_c)}(x) =$$

$$\begin{cases} 0 & x > c \\ 1 - F_X(c) & x = c \\ f_X(x) & x < c \end{cases}. \text{ To show this, it suffices to show that } \int_{(-\infty, x]} f_Y(t) d(m + \delta_c) = P_Y((-\infty, x])$$

for any  $x \in \mathcal{B}$ .

**Proposition 1 (Transformation):** Let  $X$  be a random  $k$ -vector with a Lebesgue p.d.f.  $f_X$  and let  $Y = g(X)$ , where  $g$  is a Borel function from  $(\mathbb{R}^k, \mathcal{B}^k)$  to  $(\mathbb{R}^l, \mathcal{B}^l)$ . Let  $A_1, \dots, A_m$  be disjoint sets in  $\mathcal{B}^k$  such that  $\mathcal{B}^k - (A_1 \cup \dots \cup A_m)$  has Lebesgue measure 0 and  $g$  on  $A_j$  is one-to-one with a nonvanishing Jacobian, i.e., the determinant  $\text{Det}(\partial g(x)/\partial x) \neq 0$  on  $A_j, j = 1, \dots, m$ . Then  $Y$  has the following Lebesgue p.d.f.:  $f_Y(x) = \sum_{j=1}^m |\text{Det}(\partial h_j(x)/\partial x)| f_X(h_j(x))$ , where  $h_j$  is the inverse function of  $g$  on  $A_j, j = 1, \dots, m$ .

**Example 2 (F-distribution):** Let  $X_1$  and  $X_2$  be independent random variables having the chi-

## PROBABILITY THEORY

square distributions  $\chi_{n_1}^2$  and  $\chi_{n_2}^2$ , respectively. One can show that the p.d.f. of  $Y = (X_1/n_1)/(X_2/n_2)$  is the p.d.f. of the F-distribution  $F_{n_1, n_2}$ .

**Example 3 (t-distribution):** Let  $U_1$  be a random variable having the standard normal distribution  $N(0, 1)$  and  $U_2$  a random variable having the chi-square distribution  $\chi_n^2$ . One can show that if  $U_1$  and  $U_2$  are independent, then the distribution of  $T = U_1/\sqrt{U_2/n}$  is the t-distribution  $t_n$ .

**Example 4 (Noncentral chi-square distribution):** Let  $X_1, \dots, X_n$  be independent random variables and  $X_i \sim N(\mu_i, \sigma^2)$ . The distribution of  $Y = (X_1^2 + \dots + X_n^2)/\sigma^2$  is called the noncentral chi-square distribution and denoted by  $\chi_n^2(\delta)$ , where  $\delta = (\mu_1^2 + \dots + \mu_n^2)/\sigma^2$  is the noncentrality parameter. If  $Y_1, \dots, Y_k$  are independent random variables and  $Y_i$  has the noncentral independent chi-square distribution  $\chi_{n_i}^2(\delta_i)$ ,  $i = 1, \dots, k$ , then  $Y = Y_1 + \dots + Y_k$  has the noncentral chi-square distribution  $\chi_{n_1 + \dots + n_k}^2(\delta_1 + \dots + \delta_k)$ .

**Definition 1 (Moments):** If  $\mathbb{E}X^k$  is finite, where  $k$  is a positive integer,  $\mathbb{E}X^k$  is called the  $k$ -th moment of  $X$  or  $P_X$ . If  $\mathbb{E}|X|^a < \infty$  for some real number  $a$ ,  $\mathbb{E}|X|^a$  is called the  $a$ -th absolute moment of  $X$  or  $P_X$ . If  $\mu = \mathbb{E}X$ ,  $\mathbb{E}(X - \mu)^k$  is called the  $k$ -th central moment of  $X$  or  $P_X$ .  $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2$  is called the variance of  $X$  or  $P_X$ . For random matrix  $M = (M_{ij})$ ,  $\mathbb{E}M = (\mathbb{E}M_{ij})$ . For random vector  $X$ ,  $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)(X - \mathbb{E}X)^T$  is its covariance matrix, whose  $(i, j)$ -th element,  $i \neq j$ , is called the covariance of  $X_i$  and  $X_j$  and denoted by  $\text{Cov}(X_i, X_j)$ . If  $\text{Cov}(X_i, X_j) = 0$ , then  $X_i$  and  $X_j$  are said to be uncorrelated. Independence implies uncorelation, not converse. If  $X$  is random and  $c$  is fixed, then  $\mathbb{E}(c^T X) = c^T \mathbb{E}(X)$  and  $\text{Var}(c^T X) = c^T \text{Var}(X)c$ .

**Definition 2 (Moment generating and characteristic functions):** Let  $X$  be a random  $k$ -vector. (i) The moment generating function (m.g.f.) of  $X$  or  $P_X$  is defined as  $\psi_X(t) = \mathbb{E}e^{t^T X}$ ,  $t \in \mathbb{R}^k$ . (ii) The characteristic function (ch.f.) of  $X$  or  $P_X$  is defined as  $\phi_X(t) = \mathbb{E}e^{it^T X} = \mathbb{E}[\cos(t^T X)] + i\mathbb{E}[\sin(t^T X)]$ ,  $t \in \mathbb{R}^k$ .

**Proposition 2 (Properties of m.g.f. and ch.f.):** If the m.g.f. is finite in a neighborhood of  $0 \in \mathbb{R}^k$ , then (i) moments of  $X$  of any order are finite; (ii)  $\phi_X(t)$  can be obtained by replacing  $t$  in  $\psi_X(t)$  by  $it$ . If  $Y = A^T X + c$ , where  $A$  is a  $k \times m$  matrix and  $c \in \mathbb{R}^m$ , then  $\psi_Y(u) = e^{c^T u} \psi_X(Au)$  and  $\phi_Y(u) = e^{ic^T u} \phi_X(Au)$ ,  $u \in \mathbb{R}^m$ . For independent  $X_1, \dots, X_k$ ,  $\psi_{\sum_i X_i}(t) = \prod_i \psi_{X_i}(t)$  and  $\phi_{\sum_i X_i}(t) = \prod_i \phi_{X_i}(t)$ ,  $t \in \mathbb{R}^k$ . For  $X = (X_1, \dots, X_k)$  with m.g.f.  $\psi_X$  finite in a neighborhood of 0,  $\frac{\partial \psi_X(t)}{\partial t}|_{t=0} = \mathbb{E}X$ ,  $\frac{\partial^2 \psi_X(t)}{\partial t \partial t^T}|_{t=0} = \mathbb{E}(XX^T)$ . If  $\mathbb{E}|X_1^{r_1} \dots X_k^{r_k}| < \infty$  for nonnegative integers  $r_1, \dots, r_k$ , then  $\frac{\partial \phi_X(t)}{\partial t}|_{t=0} = i\mathbb{E}X$ ,  $\frac{\partial^2 \phi_X(t)}{\partial t \partial t^T}|_{t=0} = -\mathbb{E}(XX^T)$ .

**Theorem 1 (Uniqueness):** Let  $X$  and  $Y$  be random  $k$ -vectors. (i) If  $\phi_X(t) = \phi_Y(t)$  for all  $t \in \mathbb{R}^k$ , then  $P_X = P_Y$ ; (2) If  $\psi_X(t) = \psi_Y(t) < \infty$  for all  $t$  in a neighborhood of 0, then  $P_X = P_Y$ .

### 1.4 Conditional expectation and independence

**Definition 1:** Let  $X$  be an integrable random variable on  $(\Omega, \mathcal{F}, P)$ . (i) The conditional expectation of  $X$  given  $\mathcal{A}$  (a sub- $\sigma$ -field of  $\mathcal{F}$ ), denoted by  $\mathbb{E}(X|\mathcal{A})$ , is the a.s. unique random variable satisfying the following two conditions: (a)  $\mathbb{E}(X|\mathcal{A})$  is measurable from  $(\Omega, \mathcal{A})$  to  $(\mathbb{R}, \mathcal{B})$ ; (b)  $\int_A \mathbb{E}(X|\mathcal{A}) dP = \int_A X dP$  for any  $A \in \mathcal{A}$ . (ii) The conditional probability of  $B \in \mathcal{F}$  given  $\mathcal{A}$  is defined to be  $P(B|\mathcal{A}) = \mathbb{E}(I_B|\mathcal{A})$ . (iii) Let  $Y$  be measurable from  $(\Omega, \mathcal{F}, P)$  to  $(\Lambda, \mathcal{G})$ . The conditional expectation of  $X$  given  $Y$  is defined to be  $\mathbb{E}(X|Y) = \mathbb{E}[X|\sigma(Y)]$ .

**Theorem 1:** Let  $Y$  be measurable from  $(\Omega, \mathcal{F})$  to  $(\Lambda, \mathcal{G})$  and  $Z$  a function from  $(\Omega, \mathcal{F})$  to  $\mathbb{R}^k$ . Then  $Z$  is measurable from  $(\Omega, \sigma(Y))$  to  $(\mathbb{R}^k, \mathcal{B}^k)$  iff there is a measurable function  $h$  from  $(\Lambda, \mathcal{G})$  such that  $Z = h \circ Y$ .

**Example 1:** Let  $X$  be an integrable random variable on  $(\Omega, \mathcal{F}, P)$ ,  $A_1, A_2, \dots$  be disjoint events on  $(\Omega, \mathcal{F}, P)$  such that  $\cup A_i = \Omega$  and  $P(A_i) > 0$  for all  $i$ , and let  $a_1, a_2, \dots$  be distinct real numbers. Define  $Y = a_1 I_{A_1} + a_2 I_{A_2} + \dots$ . We can show that  $\mathbb{E}(X|Y) = \sum_{i=1}^{\infty} \frac{\int_{A_i} X dP}{P(A_i)} I_{A_i}$ .

**Proposition 1:** Let  $X$  be a random  $n$ -vector and  $Y$  a random  $m$ -vector. Suppose that  $(X, Y)$  has a joint p.d.f.  $f(x, y)$  w.r.t.  $\nu \times \lambda$ , where  $\nu$  and  $\lambda$  are  $\sigma$ -finite measures on  $(\mathbb{R}^n, \mathcal{B}^n)$  and  $(\mathbb{R}^m, \mathcal{B}^m)$ , respectively. Let  $g(x, y)$  be a Borel function on  $\mathbb{R}^{n+m}$  for which  $\mathbb{E}|g(X, Y)| < \infty$ . Then  $\mathbb{E}[g(X, Y)|Y] = \frac{\int g(x, Y)f(x, Y)d\nu(x)}{\int f(x, Y)d\nu(x)}$  a.s.

**Definition 2 (Conditional p.d.f.):** Let  $(X, Y)$  be a random vector with a joint p.d.f.  $f(x, y)$  w.r.t.  $\nu \times \lambda$ . The conditional p.d.f. of  $X$  given  $Y = y$  is defined to be  $f_{X|Y}(x|y)/f_Y(y)$  where  $f_Y(y) = \int f(x, y)d\nu(x)$  is the marginal p.d.f. of  $Y$  w.r.t.  $\lambda$ .

**Proposition 2:** Let  $X, Y, X_1, X_2, \dots$  be integrable random variables on  $(\Omega, \mathcal{F}, P)$  and  $\mathcal{A}$  be a sub- $\sigma$ -field of  $\mathcal{F}$ . (i) If  $X = c$  a.s.,  $c \in \mathbb{R}$ , then  $\mathbb{E}(X|\mathcal{A}) = c$  a.s. (ii) If  $X \leq Y$  a.s., then  $\mathbb{E}(X|\mathcal{A}) \leq \mathbb{E}(Y|\mathcal{A})$  a.s. (iii) If  $a, b \in \mathbb{R}$ , then  $\mathbb{E}(aX + bY|\mathcal{A}) = a\mathbb{E}(X|\mathcal{A}) + b\mathbb{E}(Y|\mathcal{A})$  a.s. (iv)  $\mathbb{E}[\mathbb{E}(X|\mathcal{A})] = \mathbb{E}X$ . (v)  $\mathbb{E}[\mathbb{E}(X|\mathcal{A})|\mathcal{A}_0] = \mathbb{E}(X|\mathcal{A}_0) = \mathbb{E}[\mathbb{E}(X|\mathcal{A}_0)|\mathcal{A}]$  a.s., where  $\mathcal{A}_0$  is a sub- $\sigma$ -field of  $\mathcal{A}$ . (vi) If  $\sigma(Y) \subset \mathcal{A}$  and  $\mathbb{E}|XY| < \infty$ , then  $\mathbb{E}(XY|\mathcal{A}) = Y\mathbb{E}(X|\mathcal{A})$  a.s. (vii) If  $X$  and  $Y$  are independent and  $\mathbb{E}|g(X, Y)| < \infty$  for a Borel function  $g$ , then  $\mathbb{E}[g(X, Y)|Y = y] = \mathbb{E}[g(X, y)]$  a.s.  $P_Y$ . (viii) If  $\mathbb{E}X^2 < \infty$ , then  $[\mathbb{E}(X|\mathcal{A})]^2 \leq \mathbb{E}(X^2|\mathcal{A})$  a.s. (ix) Fatou's lemma: If  $X_n \geq 0$  for any  $n$ , then  $\mathbb{E}(\liminf_n X_n|\mathcal{A}) \leq \liminf_n \mathbb{E}(X_n|\mathcal{A})$  a.s. (x) Dominated convergence theorem: If  $|X_n| \leq Y$  for any  $n$  and  $X_n \rightarrow_{\text{a.s.}} X$ , then  $\mathbb{E}(X_n|\mathcal{A}) \rightarrow_{\text{a.s.}} \mathbb{E}(X|\mathcal{A})$ .

**Definition 3 (Independence):** Let  $(\Omega, \mathcal{F}, P)$  be a probability space. (i) Let  $\mathcal{C}$  be a collection of subsets in  $\mathcal{F}$ . Events in  $\mathcal{C}$  are said to be independent iff for any positive integer  $n$  and distinct events  $A_1, \dots, A_n \in \mathcal{C}$ ,  $P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2) \dots P(A_n)$ . (ii) Collections  $\mathcal{C}_i \subset \mathcal{F}, i \in \mathcal{I}$  are said to be independent iff events in any collection of the form  $\{A_i \in \mathcal{C}_i : i \in \mathcal{J}\}$  are independent. (iii) Random elements  $X_i, i \in \mathcal{I}$ , are said to be independent iff  $\sigma(X_i), i \in \mathcal{I}$  are independent.

**Theorem 2:** Let  $\mathcal{C}_i, i \in \mathcal{I}$  be independent collections of events. If each  $\mathcal{C}_i$  is a  $\pi$ -system, then  $\sigma(\mathcal{C}_i), i \in \mathcal{I}$  are independent.

**Proposition 2:** Let  $X$  be a random variable with  $\mathbb{E}|X| < \infty$  and let  $Y_i$  be random  $k_i$  vectors,  $i = 1, 2$ . Suppose that  $(X, Y_1)$  and  $Y_2$  are independent. Then  $\mathbb{E}[X|(Y_1, Y_2)] = \mathbb{E}(X|Y_1)$  a.s.

**Definition 4 (Conditional independence):** Let  $X, Y, Z$  be random vectors. We say that given  $Z$ ,  $X$  and  $Y$  are conditionally independent iff  $P(A|X, Z) = P(A|Z)$  a.s. for any  $A \in \sigma(Y)$ .

## 1.5 Convergence modes and relationships

**Definition 1 (Convergence modes):** Let  $X, X_1, X_2, \dots$  be a random  $k$ -vectors defined on a probability space. (i) We say that the sequence  $\{X_n\}$  converges to  $X$  almost surely and write  $X_n \rightarrow_{\text{a.s.}} X$  iff  $\lim_{n \rightarrow \infty} X_n = X$  a.s. (ii) We say that  $\{X_n\}$  converges to  $X$  in probability and write  $X_n \rightarrow_p X$  iff for every fixed  $\epsilon > 0$ ,  $\lim_{n \rightarrow \infty} P(\|X_n - X\| > \epsilon) = 0$ . (iii) We say that  $\{X_n\}$  converges to  $X$  in  $L_r$  (or in  $r$ th moment) with a fixed  $r > 0$  and write  $X_n \rightarrow_{L_r} X$  iff  $\lim_{n \rightarrow \infty} \mathbb{E}\|X_n - X\|_r^r = 0$ . (iv)



## PROBABILITY THEORY

Let  $F, F_n, n = 1, 2, \dots$  be c.d.f.'s on  $\mathbb{R}^k$  and  $P, P_n, n = 1, 2, \dots$  be their corresponding probability measures. We say that  $\{F_n\}$  converges to  $F$  weakly (or  $\{P_n\}$  converges to  $P$  weakly) and write  $F_n \rightarrow_w F$  (or  $P_n \rightarrow_w P$ ) iff, for each continuity point  $x$  of  $F$ ,  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ . We say that  $\{X_n\}$  converges to  $X$  in distribution (or in law) and write  $X_n \rightarrow_d X$  iff  $F_{X_n} \rightarrow_w F_X$ .

**Proposition 1:** If  $F_n \rightarrow_w F$  and  $F$  is continuous on  $\mathbb{R}^k$ , then  $\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^k} |F_n(x) - F(x)| = 0$ .

**Theorem 1:** For random  $k$ -vectors  $X, X_1, X_2, \dots$  on a probability space,  $X_n \rightarrow_{a.s.} X$  iff for every  $\epsilon > 0$ ,  $\lim_{n \rightarrow \infty} P(\cup_{m=n}^{\infty} \{\|X_m - X\| > \epsilon\}) = 0$ .

**Theorem 2 (Borel-Cantelli lemma):** Let  $A_n$  be a sequence of events in a probability space and  $\limsup_n A_n = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$ . (i) If  $\sum_{n=1}^{\infty} P(A_n) < \infty$ , then  $P(\liminf_n A_n) = 0$ . (ii) If  $A_1, A_2, \dots$  are pairwise independent and  $\sum_{n=1}^{\infty} P(A_n) = \infty$ , then  $P(\limsup_n A_n) = 1$ .

**Definition 2:** Let  $X_1, X_2, \dots$  be random vectors and  $Y_1, Y_2, \dots$  be random variables defined on a common probability space. (i)  $X_n = O(Y_n)$  a.s. iff  $P(\|X_n\| = O(|Y_n|)) = 1$ . (ii)  $X_n = o(Y_n)$  a.s. iff  $X_n/Y_n \rightarrow_{a.s.} 0$ . (iii)  $X_n = O_p(Y_n)$  iff, for any  $\epsilon > 0$ , there is a constant  $C_\epsilon > 0$  such that  $\sup_n P(\|X_n\| \geq C_\epsilon |Y_n|) < \epsilon$ . (iv)  $X_n = o_p(Y_n)$  iff  $X_n/Y_n \rightarrow_p 0$ .

**Theorem 3:** (i) If  $X_n \rightarrow_{a.s.} X$ , then  $X_n \rightarrow_p X$ . (The converse is not true). (ii) If  $X_n \rightarrow_{L_r} X$  for an  $r > 0$ , then  $X_n \rightarrow_p X$ . (The converse is not true). (iii) If  $X_n \rightarrow_p X$ , then  $X_n \rightarrow_d X$ . (The converse is not true). (iv) (Skorohod's theorem). If  $X_n \rightarrow_d X$ , then there are random vectors  $Y, Y_1, Y_2, \dots$  defined on a common probability space such that  $P_Y = P_X, P_{Y_n} = P_{X_n}, n = 1, 2, \dots$  and  $Y_n \rightarrow_{a.s.} Y$ . (v) If, for every  $\epsilon > 0$ ,  $\sum_{n=1}^{\infty} P(\|X_n - X\| \geq \epsilon) < \infty$ , then  $X_n \rightarrow_{a.s.} X$ . (vi) If  $X_n \rightarrow_p X$ , then there is a subsequence such that  $X_{n_j} \rightarrow_{a.s.} X$  as  $j \rightarrow \infty$ . (vii) If  $X_n \rightarrow_d X$  and  $P(X = c) = 1$ , where  $c \in \mathbb{R}^k$  is a constant vector, then  $X_n \rightarrow_p c$ . (viii) Suppose that  $X_n \rightarrow_d X$ . Then for any  $r > 0$ ,  $\lim_{n \rightarrow \infty} \mathbb{E}\|X_n\|_r^r = \mathbb{E}\|X\|_r^r < \infty$  if  $\{\|X_n\|_r^r\}$  is uniformly integrable in the sense that  $\lim_{t \rightarrow \infty} \sup_n \mathbb{E}(\|X_n\|_r^r I_{\{\|X_n\|_r^r > t\}}) = 0$ .

**Proposition 2 (Sufficient conditions for uniform integrability):**  $\sup_n \mathbb{E}\|X_n\|_r^{r+\delta} < \infty$  for a  $\delta > 0$ .

**Proposition 3 (Properties of the quotient random variables):** (i) Suppose  $X, X_1, X_2, \dots$  are positive random variables. Then  $X_n \rightarrow_{a.s.} X$  iff for every  $\epsilon > 0$ ,  $\lim_{n \rightarrow \infty} P(\sup_{k \geq n} \frac{X_k}{X} > 1 + \epsilon) = 0$ , and  $\lim_{n \rightarrow \infty} P(\sup_{k \geq n} \frac{X}{X_k} > 1 + \epsilon) = 0$ . (ii) Suppose  $X, X_1, X_2, \dots$  are positive random variables. If  $\sum_{n=1}^{\infty} P(X_n/X > 1 + \epsilon) < \infty$  and  $\sum_{n=1}^{\infty} P(X/X_n > 1 + \epsilon) < \infty$ , then  $X_n \rightarrow_{a.s.} X$ .

### 1.6 Uniform integrability and weak convergence

**Definition 1 (Tightness):** A sequence  $\{P_n\}$  of probability measure on  $(\mathbb{R}^k, \mathcal{B}^k)$  is tight if for every  $\epsilon > 0$ , there is a compact set  $C \subset \mathbb{R}^k$  such that  $\inf_n P_n(C) > 1 - \epsilon$ . If  $\{X_n\}$  is a sequence of random  $k$ -vectors, then the tightness of  $\{P_{X_n}\}$  is the same as the boundedness of  $\{\|X_n\|\}$  in probability.

**Proposition 1:** Let  $\{P_n\}$  be a sequence of probability measures on  $(\mathbb{R}^k, \mathcal{B}^k)$ . (i) Tightness of  $\{P_n\}$  is a necessary and sufficient condition that for every subsequence  $\{P_n\}$  there exists a further subsequence  $\{P_{n_j}\} \subset \{P_n\}$  and a probability measure  $P$  on  $(\mathbb{R}^k, \mathcal{B}^k)$  such that  $P_{n_j} \rightarrow_w P$  as  $j \rightarrow \infty$ . (ii) If  $\{P_n\}$  is tight and if each subsequence that converges weakly at all converges to the same probability measure  $P$ , then  $P_n \rightarrow_w P$ .

**Theorem 1 (Useful sufficient and necessary conditions for convergence in distribution):** Let  $X, X_1, X_2, \dots$  be random  $k$ -vectors. (i)  $X_n \rightarrow_d X$  is equivalent to any one of the following conditions:

(a)  $\mathbb{E}[h(X_n)] \rightarrow \mathbb{E}[h(X)]$  for every bounded continuous function  $h$ ; (b)  $\limsup_n P_{X_n}(C) \leq P_X(C)$  for any closed set  $C \subset \mathbb{R}^k$ ; (c)  $\liminf_n P_{X_n}(O) \geq P_X(O)$  for any open set  $O \subset \mathbb{R}^k$ . (ii) Lévy-Cramér continuity theorem. Let  $\phi_X, \phi_{X_1}, \phi_{X_2}, \dots$  be the ch.f.'s of  $X, X_1, X_2, \dots$ , respectively.  $X_n \rightarrow_d X$  iff  $\lim_{n \rightarrow \infty} \phi_{X_n}(t) = \phi_X(t)$  for all  $t \in \mathbb{R}^k$ . (iii) Cramér-Wold device.  $X_n \rightarrow_d X$  iff  $c^T X_n \rightarrow_d c^T X$  for every  $c \in \mathbb{R}^k$ .

**Example 1:** Let  $X_1, \dots, X_n$  be independent random variables having a common c.d.f. and  $T_n = X_1 + \dots + X_n, n = 1, 2, \dots$ . Suppose that  $\mathbb{E}|X_1| < \infty$ . It follows from a result in calculus that the ch.f. of  $X_1$  satisfies  $\phi_{X_1}(t) = \phi_{X_1}(0) + \sqrt{-1}\mu t + o(|t|)$  as  $|t| \rightarrow 0$ , where  $\mu = \mathbb{E}X_1$ . Then, the ch.f. of  $T_n/n$  is  $\phi_{T_n/n}(t) = [\phi_{X_1}(\frac{t}{n})]^n = [1 + \frac{\sqrt{-1}\mu t}{n} + o(\frac{t}{n})]^n \rightarrow e^{\sqrt{-1}\mu t}$  for any  $t \in \mathbb{R}$  as  $n \rightarrow \infty$ .  $e^{\sqrt{-1}\mu t}$  is the ch.f. of the point mass probability measure at  $\mu$ . Thus  $T_n/n \rightarrow_d \mu$  and  $T_n/n \rightarrow_p \mu$ .

**Proposition 2 (Scheffé's theorem):** Let  $\{f_n\}$  be a sequence of p.d.f.'s on  $\mathbb{R}^k$  w.r.t.  $\nu$ . Suppose that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  a.e. and  $f(x)$  is a p.d.f. w.r.t.  $\nu$ . Then  $\lim_{n \rightarrow \infty} \int |f_n(x) - f(x)| d\nu = 0$ .

## 1.7 Convergence of transformations and law of large numbers

**Theorem 1 (Continuous mapping theorem):** Let  $X, X_1, X_2, \dots$  be random  $k$ -vectors defined on a probability space and  $g$  be a measure function from  $(\mathbb{R}^k, \mathcal{B}^k)$  to  $(\mathbb{R}^l, \mathcal{B}^l)$ . Suppose that  $g$  is continuous a.s.  $P_X$ . Then (i)  $X_n \rightarrow_{a.s.} X$  implies  $g(X_n) \rightarrow_{a.s.} g(X)$ ; (ii)  $X_n \rightarrow_p X$  implies  $g(X_n) \rightarrow_p g(X)$ ; (iii)  $X_n \rightarrow_d X$  implies  $g(X_n) \rightarrow_d g(X)$ .

**Theorem 2 (Slutsky's theorem):** Let  $X, X_1, X_2, \dots, Y_1, Y_2, \dots$  be random variables on a probability space. Suppose that  $X_n \rightarrow_d X$  and  $Y_n \rightarrow_p c$ , where  $c$  is a constant. Then (i)  $X_n + Y_n \rightarrow_d X + c$ ; (ii)  $Y_n X_n \rightarrow_d cX$ ; (iii)  $X_n/Y_n \rightarrow_d X/c$  if  $c \neq 0$ .

**Theorem 3:** Let  $X_1, X_2, \dots$  and  $Y = (Y_1, \dots, Y_k)$  be random  $k$ -vectors satisfying  $a_n(X_n - c) \rightarrow_d Y$ , where  $c \in \mathbb{R}^k$  and  $\{a_n\}$  is a sequence of positive numbers with  $\lim_{n \rightarrow \infty} a_n = \infty$ . Let  $g$  be a function from  $\mathbb{R}^k \rightarrow \mathbb{R}$ . (i) If  $g$  is differentiable at  $c$ , then  $a_n[g(X_n) - g(c)] \rightarrow_d [\nabla g(c)^T]Y$ , where  $\nabla g(x)$  denotes the  $k$ -vector of partial derivatives of  $g$  at  $x$ . (ii) Suppose that  $g$  has continuous partial derivatives of order  $m > 1$  in a neighborhood of  $c$ , with all the partial derivatives of order  $j, 1 \leq j \leq m-1$ , vanishing at  $c$ , but with the  $m$ th-order partial derivatives not all vanishing at  $c$ . Then  $a_n^m[g(X_n) - g(c)] \rightarrow_d \frac{1}{m!} \sum_{i_1=1}^k \dots \sum_{i_m=1}^k \frac{\partial^m g}{\partial x_{i_1} \dots \partial x_{i_m}}|_{x=c} Y_{i_1} \dots Y_{i_m}$ .

**Theorem 4 (The  $\delta$ -method):** If  $Y$  has the  $\mathcal{N}_k(0, \Sigma)$  distribution, then  $a_n[g(X_n) - g(c)] \rightarrow_d \mathcal{N}(0, [\nabla g(c)]^T \Sigma \nabla g(c))$ .

**Theorem 5:** Let  $X_1, X_2, \dots$  be i.i.d. random variables. (i) The WLLN. A necessary and sufficient condition for the existence of a sequence of real numbers  $\{a_n\}$  for which  $\frac{1}{n} \sum_{i=1}^n X_i - a_n \rightarrow_p 0$  is that  $nP(|X_1| > n) \rightarrow 0$ , in which case we may take  $a_n = \mathbb{E}(X_1 1_{\{|X_1| \leq n\}})$ . (ii) The SLLN. A necessary and sufficient condition for the existence of a constant  $c$  for which  $\frac{1}{n} \sum_{i=1}^n X_i \rightarrow_{a.s.} c$  is that  $\mathbb{E}|X_1| < \infty$ , in which case  $c = \mathbb{E}X_1$  and  $\frac{1}{n} \sum_{i=1}^n c_i(X_i - \mathbb{E}X_1) \rightarrow_{a.s.} 0$  for any bounded sequence of real numbers  $\{c_i\}$ .

**Theorem 6:** Let  $X_1, X_2, \dots$  be independent random variables with finite expectations. (i) The SLLN. If there is a constant  $p \in [1, 2]$  such that  $\sum_{i=1}^{\infty} \frac{\mathbb{E}|X_i|^p}{i^p} < \infty$ , then  $\frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}X_i) \rightarrow_{a.s.} 0$ . (ii) The WLLN. If there is a constant  $p \in [1, 2]$  such that  $\lim_{n \rightarrow \infty} \frac{1}{n^p} \sum_{i=1}^n \mathbb{E}|X_i|^p = 0$ , then  $\frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}X_i) \rightarrow_p 0$ .



## 1.8 The central limit theorem

**Theorem 1 (Lindeberg's CLT):** Let  $\{X_{nj}, j = 1, \dots, k_n\}$  be independent random variables with  $k_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $0 < \sigma_n^2 = \text{var}(\sum_{j=1}^{k_n} X_{nj}) < \infty, n = 1, 2, \dots$ . If  $\frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} \mathbb{E}[(X_{nj} - \mathbb{E}X_{nj})^2 I_{\{|X_{nj} - \mathbb{E}X_{nj}| > \epsilon \sigma_n\}}] \rightarrow 0$  for any  $\epsilon > 0$ , then  $\frac{1}{\sigma_n} \sum_{j=1}^{k_n} (X_{nj} - \mathbb{E}X_{nj}) \rightarrow_d \mathcal{N}(0, 1)$ .

**Theorem 2 (Multivariate CLT):** For i.i.d. random  $k$ -vectors  $X_1, \dots, X_n$  with a finite  $\Sigma = \text{var}(X_1)$ ,  $\frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \mathbb{E}X_1) \rightarrow_d \mathcal{N}_k(0, \Sigma)$ .

**Theorem 3 (Berry-Esséen bound):** For i.i.d.  $\{X_n\}$  and  $W_n = \sqrt{n}(\bar{X} - \mu)/\sigma$ ,  $\sup_t |F_{W_n}(t) - \phi(t)| \leq \frac{33}{4} \frac{\mathbb{E}|X_1 - \mu|^3}{\sigma^3 \sqrt{n}}, n = 1, 2, \dots$ . Thus, the convergence speed of  $F_{W_n}$  to  $\phi$  is of the order  $n^{-1/2}$ .

## 2 Fundamentals of Statistics

### 2.1 Models, data, statistics, and sampling distributions

**Definition 1:** A set of probability measures  $P_\theta$  on  $(\Omega, \mathcal{F})$  indexed by a parameter  $\theta \in \Theta$  is said to be a parametric family or follow a parametric model iff  $\Theta \subset \mathbb{R}^d$  for some fixed positive integer  $d$  and each  $P_\theta$  is a known probability measure when  $\theta$  is known. The set  $\Theta$  is called the parameter space and  $d$  is called its dimension.  $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$  is identifiable iff  $\theta_1 \neq \theta_2$  and  $\theta_i \in \Theta$  imply  $P_{\theta_1} \neq P_{\theta_2}$ , which may be achieved through reparameterization.

**Definition 2 (Dominated family):** A family of populations  $\mathcal{P}$  is dominated by  $\nu$  (a  $\sigma$ -finite measure) if  $P \ll \nu$  for all  $P \in \mathcal{P}$ , in which case  $\mathcal{P}$  can be identified by the family of densities  $\{\frac{dP}{d\nu} : P \in \mathcal{P}\}$  or  $\{\frac{dP_\theta}{d\nu} : \theta \in \Theta\}$ .

**Definition 3 (Exponential families):** A parametric family  $\{P_\theta : \theta \in \Theta\}$  dominated by a  $\sigma$ -finite measure  $\nu$  on  $(\Omega, \mathcal{F})$  is called an exponential family iff  $\frac{dP_\theta}{d\nu}(\omega) = \exp\{[\eta(\theta)]^T T(\omega) - \xi(\theta)\} h(\omega), \omega \in \Omega$  where  $\xi(\theta) = \log\{\int_\Omega \exp\{[\eta(\theta)]^T T(\omega)\} h(\omega) d\nu(\omega)\}$ . In an exponential family, consider the parameter  $\eta = \eta(\theta)$  and  $f_\eta(\omega) = \exp\{\eta^T T(\omega) - \zeta(\eta)\} h(\omega), \omega \in \Omega$ . This is called the canonical form for the family, and  $\Xi = \{\eta : \zeta(\eta) \text{ is defined}\}$  is called the natural parameter space. An exponential family in canonical form is a natural exponential family. If there is an open set contained in the natural parameter space of an exponential family, then the family is said to be of full rank.

**Theorem 1:** Let  $\mathcal{P}$  be a natural exponential family. (i) Let  $T = (Y, U)$  and  $\eta = (\theta, \phi)$ ,  $Y$  and  $\theta$  have the same dimension. Then,  $Y$  has the p.d.f.  $f_\eta(y) = \exp\{\theta^T y - \zeta(\eta)\}$ . In particular,  $T$  has a p.d.f. in a natural exponential family. Furthermore, the conditional distribution of  $Y$  given  $U = u$  has the p.d.f.  $f_{\theta, u}(y) = \exp\{\theta^T y - \zeta_u(\theta)\}$  w.r.t. a  $\sigma$ -finite measure depending on  $\phi$ . Furthermore, the conditional distribution of  $Y$  given  $U = u$  has the p.d.f.  $f_{\theta, u}(y) = \exp(\theta^T y - \zeta_u(\theta))$  w.r.t. a  $\sigma$ -finite measure depending on  $u$ . (ii) If  $\eta_0$  is an interior point of the natural parameter space, then the m.g.f. of  $P_{\eta_0} \circ T^{-1}$  is finite in a neighborhood of 0 and is given by  $\psi_{\eta_0}(t) = \exp\{\zeta(\eta_0 + t) - \zeta(\eta_0)\}$ .

**Definition 4 (Location-scale families):** Let  $P$  be a known probability measure on  $(\mathbb{R}^k, \mathcal{B}^k)$ ,  $\mathcal{V} \subset \mathbb{R}^k$ , and  $\mathcal{M}_k$  be a collection of  $k \times k$  symmetric positive definite matrices. The family  $\{P_{(\mu, \Sigma)} : \mu \in \mathcal{V}, \Sigma \in \mathcal{M}_k\}$  is called a location-scale family (on  $\mathbb{R}^k$ ), where  $P_{(\mu, \Sigma)}(B) = P(\Sigma^{-1/2}(B - \mu)), B \in \mathcal{B}^k$ . The parameters  $\mu$  and  $\Sigma^{1/2}$  are called the location and scale parameters, respectively.

**Definition 5 (Statistics and their sampling distributions):** Our data set is a realization of a sample

(random vector)  $X$  from an unknown population  $P$ . Statistic  $T(X)$ : A measurable function  $T$  of  $X$ ;  $T(X)$  is a known value whenever  $X$  is known. A nontrivial statistic  $T(X)$  is usually simpler than  $X$ . Finding the form of the distribution of  $T$  is one of the major problems in statistical inference and decision theory.

**Example 1:** Let  $X_1, \dots, X_n$  be i.i.d. random variables having a common distribution  $P$ . The sample mean and sample variance  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$  are two commonly used statistics.

**Example 2 (Order statistics):** Let  $X = (X_1, \dots, X_n)$  with i.i.d. random components. Let  $X_{(i)}$  be the  $i$ th smallest value of  $X_1, \dots, X_n$ . The statistics  $X_{(1)}, \dots, X_{(n)}$  are called the order statistics.

## 2.2 Sufficiency and minimal sufficiency

**Definition 1 (Sufficiency):** Let  $X$  be a sample from an unknown population  $P \in \mathcal{P}$ , where  $\mathcal{P}$  is a family of populations. A statistic  $T(X)$  is said to be sufficient for  $P \in \mathcal{P}$  iff conditional distribution of  $X$  given  $T$  is known.

**Theorem 1 (The factorization theorem):** Suppose that  $X$  is a sample from  $P \in \mathcal{P}$  and  $\mathcal{P}$  is a family of probability measures on  $(\mathbb{R}^n, \mathcal{B}^n)$  dominated by a  $\sigma$ -finite measure  $\nu$ . Then  $T(X)$  is sufficient for  $P \in \mathcal{P}$  iff there are nonnegative Borel functions  $h$  and  $g_p$  on the range of  $T$  such that  $\frac{dP}{d\nu}(x) = g_p(T(x))h(x)$ .

**Theorem 2:** If a family  $\mathcal{P}$  is dominated by a  $\sigma$ -finite measure, then  $\mathcal{P}$  is dominated by a probability measure  $Q = \sum_{i=1}^{\infty} c_i P_i$ , where  $c_i$ 's are nonnegative constants with  $\sum_{i=1}^{\infty} c_i = 1$  and  $P_i \in \mathcal{P}$ .

**Convention 1:** If a statement holds except for outcomes in an event  $A$  satisfying  $P(A) = 0$  for all  $P \in \mathcal{P}$ , then we say that the statement holds a.s.  $\mathcal{P}$ .

**Definition 2 (Minimal sufficiency):** Let  $T$  be a sufficient statistic for  $P \in \mathcal{P}$ .  $T$  is called a minimal sufficient statistic iff, for any other statistic  $S$  sufficient for  $P \in \mathcal{P}$ , there is a measurable function  $\psi$  such that  $T = \psi(S)$  a.s.  $\mathcal{P}$ .

**Theorem 3 (Existence and uniqueness):** Minimal sufficient statistics exist when  $\mathcal{P}$  contains distributions on  $\mathbb{R}^k$  dominated by a  $\sigma$ -finite measure. If both  $T$  and  $S$  are minimal sufficient statistics, then by definition there is one-to-one measurable function  $\psi$  such that  $T = \psi(S)$  a.s.  $\mathcal{P}$ .

**Theorem 4:** Let  $\mathcal{P}$  be a family of distributions on  $\mathbb{R}^k$ . (i) Suppose that  $\mathcal{P}_0 \subset \mathcal{P}$  and a.s.  $\mathcal{P}_0$  implies a.s.  $\mathcal{P}$ . If  $T$  is sufficient for  $P \in \mathcal{P}$  and minimal sufficient for  $P \in \mathcal{P}_0$ , then  $T$  is minimal sufficient for  $P \in \mathcal{P}$ . (ii) Suppose that  $\mathcal{P}$  contains p.d.f.'s  $f_0, f_1, f_2, \dots$  w.r.t. a  $\sigma$ -finite  $\nu$ . Let  $f_{\infty}(x) = \sum_{i=0}^{\infty} c_i f_i(x)$ , where  $c_i > 0$  for all  $i$  and  $\sum_{i=0}^{\infty} c_i = 1$ , and let  $T_i(x) = f_i(x)/f_{\infty}(x)$  when  $f_{\infty}(x) > 0, i = 0, 1, 2, \dots$ . Then  $T(x) = (T_0, T_1, T_2, \dots)$  is minimal sufficient for  $P \in \mathcal{P}$ . Furthermore, if  $\{x : f_i(x) > 0\} \subset \{x : f_0(x) > 0\}$  for all  $i$ , then we may replace  $f_{\infty}(x)$  for  $f_0(x)$ , in which case  $T(x) = (T_1, T_2, \dots)$  is minimal sufficient for  $P \in \mathcal{P}$ . (iii) Suppose that  $\mathcal{P}$  contains p.d.f.'s  $f_p$  w.r.t. a  $\sigma$ -finite measure and that there exists a sufficient statistic  $T(x)$  such that, for any possible values  $x$  and  $y$  of  $X$ ,  $f_p(x) = f_p(y)\phi(x, y)$  for all  $P$  implies  $T(x) = T(y)$ , where  $\phi$  is a measurable function. Then  $T(x)$  is minimal sufficient for  $P \in \mathcal{P}$ .

### 2.3 Completeness

**Definition 1 (Ancillary statistics):** A statistic  $V(x)$  is ancillary iff its distribution does not depend on any unknown quantity. A statistic  $V(X)$  is first-order ancillary iff  $\mathbb{E}[V(X)]$  does not depend on any unknown quantity.

**Remark 1:** If  $V(x)$  is a non-trivial ancillary statistic, then  $\sigma(V)$  does not contain any information about the unknown population  $P$ . If  $T(x)$  is a statistic and  $V(T(x))$  is a non-trivial ancillary statistic, it indicates that the reduced data set by  $T$  contains a non-trivial part that does not contain any information about  $\theta$  and, hence, a further simplification of  $T$  may still be needed.

**Definition 2 (Completeness):** A statistic  $T(x)$  is complete (or boundedly complete) for  $P \in \mathcal{P}$  iff, for any Borel  $f$  (or bounded Borel  $f$ ),  $\mathbb{E}[f(T)] = 0$  for all  $P \in \mathcal{P}$  implies  $f = 0$  a.s.  $\mathcal{P}$ .

**Remark 2:** If  $T$  is complete (or boundedly complete) and  $S = \psi(T)$  for a measurable  $\psi$ , then  $S$  is complete (or boundedly complete). A complete and sufficient statistic should be minimal sufficient. But a minimal sufficient statistic may be not complete.

**Proposition 1:** If  $P$  is in an exponential family of full rank with p.d.f.'s given by  $f_\eta(x) = \exp\{\eta^T T(x) - \zeta(\eta)\}h(x)$ , then  $T(x)$  is complete and sufficient for  $\eta \in \Xi$ .

**Example 1:** Suppose that  $X_1, \dots, X_n$  are i.i.d. random variables having the  $\mathcal{N}(\mu, \sigma^2)$  distribution,  $\mu \in \mathbb{R}$ ,  $\sigma > 0$ . The joint p.d.f. of  $X_1, \dots, X_n$  is  $(2\pi)^{-n/2} \exp\{\eta_1 T_1 + \eta_2 T_2 - n\zeta(\eta)\}$ , where  $T_1 = \sum_{i=1}^n X_i$ ,  $T_2 = -\sum_{i=1}^n X_i^2$  and  $\eta = (\eta_1, \eta_2) = (\frac{\mu}{\sigma^2}, \frac{1}{2\sigma^2})$ . Hence, the family of distributions for  $X = (X_1, \dots, X_n)$  is a natural exponential family of full rank ( $\Xi = \mathbb{R} \times (0, \infty)$ ). Thus  $T(X) = (T_1, T_2)$  is complete and sufficient for  $\eta$ .

**Example 2:**  $T(x) = (X_{(1)}, \dots, X_{(n)})$  of i.i.d. random variables  $X_1, \dots, X_n$  is sufficient for  $P \in \mathcal{P}$ , where  $\mathcal{P}$  is the family of distributions on  $\mathbb{R}$  having Lebesgue p.d.f.'s. We can show that  $T(x)$  is also complete for  $P \in \mathcal{P}$ .

**Theorem 1 (Basu's theorem):** Let  $V$  and  $T$  be two statistics of  $X$  from a population  $P \in \mathcal{P}$ . If  $V$  is ancillary and  $T$  is boundedly complete and sufficient for  $P \in \mathcal{P}$ , then  $V$  and  $T$  are independent w.r.t. any  $P \in \mathcal{P}$ .

**Example 3:**  $X_1, \dots, X_n$  is a random sample from uniform( $\theta, \theta + 1$ ),  $\theta \in \mathbb{R}$ , and  $T = (X_{(1)}, X_{(n)})$  is the minimal sufficient statistic for  $\theta$ . We can show that  $T$  is not complete.

**Theorem 2:** Suppose that  $S$  is a minimal sufficient statistic and  $T$  is a complete and sufficient statistic. Then  $T$  must be minimal sufficient and  $S$  must be complete.

### 2.4 Statistical decision

**Convention 1 (Basic elements):**  $X$ : a sample from a population  $P \in \mathcal{P}$ . Decision: an action we take after observing  $X$ .  $\mathcal{A}$ : the set of allowable actions.  $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$ : the action space.  $\mathcal{X}$ : the range of  $X$ . Decision rule: a measurable function  $T$  from  $(\mathcal{X}, \mathcal{F}_{\mathcal{X}})$  to  $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$ . If  $X = x$  is observed, then we take the action  $T(x) \in \mathcal{A}$ .

**Definition 1 (Loss function):**  $L(P, a)$ : a function from  $\mathcal{P} \times \mathcal{A}$  to  $[0, \infty)$ .  $L(P, a)$  is Borel for each  $P$ . If  $X = x$  is observed and our decision rule is  $T$ , then our loss is  $L(P, T(x))$ .

**Definition 2 (Risk):** The averaged loss  $R_T(P) := \mathbb{E}[L(P, T(X))] = \int_{\mathcal{X}} L(P, T(x)) dP_X(x)$ .

**Definition 3 (Comparisons):** For decision rules  $T_1$  and  $T_2$ ,  $T_1$  is as good as  $T_2$  iff  $R_{T_1}(P) \leq R_{T_2}(P)$  for any  $P \in \mathcal{P}$  and is better than  $T_2$  if, in addition,  $R_{T_1}(P) < R_{T_2}(P)$  for some  $P$ .  $T_1$  and  $T_2$  are equivalent iff  $R_{T_1}(P) = R_{T_2}(P)$  for all  $P \in \mathcal{P}$ . Optimal rule: If  $T^*$  is as good as any other rule in  $\mathcal{E}$ , a class of allowable decision rules, then  $T^*$  is  $\mathcal{E}$ -optimal.

**Definition 4 (Randomized decision rules):** A function  $\delta$  on  $\mathcal{X} \times \mathcal{F}_{\mathcal{A}}$ ; for every  $A \in \mathcal{F}_{\mathcal{A}}$ ,  $\delta(\cdot, A)$  is a Borel function and, for every  $x \in \mathcal{X}$ ,  $\delta(x, \cdot)$  is a probability measure on  $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$ . If  $X = x$  is observed, we have a distribution of actions:  $\delta(x, \cdot)$ . A nonrandomized rule  $T$  is a special randomized decision rule with  $\delta(x, \{a\}) = I_{\{a\}}(T(x))$ ,  $a \in \mathcal{A}$ ,  $x \in \mathcal{X}$ . The loss function for a randomized rule  $\delta$  is defined as  $L(P, \delta, x) = \int_{\mathcal{A}} L(P, a) d\delta(x, a)$ , which reduces to the same loss function when  $\delta$  is nonrandomized. The risk of a randomized  $\delta$  is then  $R_{\delta}(P) = \mathbb{E}[L(P, \delta, X)] = \int_{\mathcal{X}} \int_{\mathcal{A}} L(P, a) d\delta(x, a) dP_X(x)$ .

**Example 1:**  $X = (X_1, \dots, X_n)$  is a vector of i.i.d. measurements for a parameter  $\theta \in \mathbb{R}$ . We want to estimate  $\theta$ . Action space:  $(\mathcal{A}, \mathcal{F}_{\mathcal{A}}) = (\mathbb{R}, \mathcal{B})$ . A common loss function in this problem is the squared error loss  $L(P, a) = (\theta - a)^2$ ,  $a \in \mathcal{A}$ . Let  $T(X) = \bar{X}$ , the sample mean. The loss for  $\bar{X}$  is  $(\bar{X} - \theta)^2$ . If the population has mean  $\mu$  and variance  $\sigma^2 < \infty$ , then  $R_{\bar{X}}(P) = (\mu - \theta)^2 + \frac{\sigma^2}{n}$ . This problem is a special case of a general problem called estimation. In an estimation problem, a decision rule  $T$  is called an estimator.

**Example 2:** Let  $\mathcal{P}$  be a family of distributions,  $\mathcal{P}_0 \subset \mathcal{P}$ ,  $\mathcal{P}_1 = \{P \in \mathcal{P} : P \notin \mathcal{P}_0\}$ . A hypothesis testing problem can be formulated as that of deciding which of the following two statements is true:  $H_0 : P \in \mathcal{P}_0$  versus  $H_1 : P \in \mathcal{P}_1$ .  $H_0$  is called the null hypothesis and  $H_1$  is the alternative hypothesis. The action space for this problem contains only two elements, i.e.,  $\mathcal{A} = \{0, 1\}$ , where 0 is accepting  $H_0$  and 1 is rejecting  $H_0$ . This problem is a special case of a general problem called hypothesis testing. A decision rule is called a test, which must have the form  $I_C(X)$ , where  $C \in \mathcal{F}_{\mathcal{X}}$  is called the rejection or critical region.

**Definition 5 (0-1 loss):**  $L(P, a) = 0$  if a correct decision is made and 1 if an incorrect decision is made, which leads to the risk  $R_T(P) = \begin{cases} P(T(X) = 1) = P(X \in C) & P \in \mathcal{P}_0 \\ P(T(X) = 0) = P(X \notin C) & P \in \mathcal{P}_1 \end{cases}$ .

**Definition 6 (Admissibility):** Let  $\mathcal{E}$  be a class of decision rules. A decision rule  $T \in \mathcal{E}$  is called  $\mathcal{E}$ -admissible iff there does not exist any  $S \in \mathcal{E}$  that is better than  $T$  (in terms of the risk).

**Remark 1:** An admissible decision rule is not necessarily good. For example, in an estimation problem a silly estimator  $T(X) \equiv a$  constant may be admissible.

**Proposition 1:** Let  $T(X)$  be a sufficient statistic for  $P \in \mathcal{P}$  and let  $\delta_0$  be a decision rule. Then  $\delta_1(t, A) = \mathbb{E}[\delta_0(X, A) | T = t]$ , which is a randomized decision rule depending only on  $T$ , is equivalent to  $\delta_0$  if  $R_{\delta_0}(P) < \infty$  for any  $P \in \mathcal{P}$ .

**Theorem 1:** Suppose that  $\mathcal{A}$  is a convex subset of  $\mathbb{R}^k$  and that for any  $P \in \mathcal{P}$ ,  $L(P, a)$  is a convex function of  $a$ . (i) Let  $\delta$  be a randomized rule satisfying  $\int_{\mathcal{A}} \|a\| d\delta(x, a) < \infty$  for any  $x \in \mathcal{X}$  and let  $T_1(x) = \int_{\mathcal{A}} a d\delta(x, a)$ . Then  $L(P, T_1(x)) \leq L(P, \delta, x)$  (or  $L(P, T_1(x)) < L(P, \delta, x)$ ) if  $L$  is strictly convex in  $a$  for any  $x \in \mathcal{X}$  and  $P \in \mathcal{P}$ . (ii) Rao-Blackwell theorem. Let  $T$  be a sufficient statistic for  $P \in \mathcal{P}$ ,  $T_0 \in \mathbb{R}^k$  be a nonrandomized rule satisfying  $\mathbb{E}\|T_0\| < \infty$ , and  $T_1 = \mathbb{E}[T_0(X) | T]$ . Then  $R_{T_1}(P) \leq R_{T_0}(P)$  for any  $P \in \mathcal{P}$ . If  $L$  is strictly convex in  $a$  and  $T_0$  is not a function of  $T$ ,

then  $T_0$  is inadmissible.

**Definition 7 (Unbiasedness):** In an estimation problem, the bias of an estimator  $T(X)$  of a parameter  $\theta$  of the unknown population is defined to be  $b_T(P) = \mathbb{E}[T(X)] - \theta$ . An estimator  $T(X)$  is unbiased for  $\theta$  iff  $b_T(P) = 0$  for any  $P \in \mathcal{P}$ .

**Approach 1:** Define a class  $\mathcal{E}$  of decision rules that have some desirable properties and then try to find the best rule in  $\mathcal{E}$ .

**Approach 2:** Consider some characteristic  $R_T$  of  $R_T(P)$ , for a given decision rule  $T$ , and then minimize  $R_T$  over  $T \in \mathcal{E}$ . Methods include the Bayes rule and the minimax rule.

## 2.5 Statistical inference

**Definition 1 (Three components in statistical inference):** Point estimators, hypothesis tests, confidence sets.

**Definition 2 (Point estimators):** Let  $T(X)$  be an estimator of  $\theta \in \mathbb{R}$ . Bias:  $b_T(P) = \mathbb{E}[T(X)] - \theta$ . Mean squared error (mse):  $\text{mse}_T(P) = \mathbb{E}[T(X) - \theta]^2 = [b_T(P)]^2 + \text{Var}(T(X))$ . Bias and mse are two common criteria for the performance of point estimators, i.e., instead of considering risk functions, we use bias and mse to evaluate point estimators.

**Definition 3 (Hypothesis tests):** To test the hypotheses  $H_0 : P \in \mathcal{P}_0$  versus  $H_1 : P \in \mathcal{P}_1$ , there are two types of errors we may commit: rejecting  $H_0$  when  $H_0$  is true (called the type I error) and accepting  $H_0$  when  $H_0$  is wrong (called the type II error). A test  $T$ : a statistic from  $\mathcal{X}$  to  $\{0, 1\}$ .

**Theorem 1 (Probabilities of making two types of errors):** Type I error rate:  $\alpha_T(P) = P(T(X) = 1), P \in \mathcal{P}_0$ . Type II error rate:  $1 - \alpha_T(P) = P(T(X) = 0), P \in \mathcal{P}_1$ .  $\alpha_T(P)$  is also called the power function of  $T$ . Power function is  $\alpha_T(\theta)$  if  $P$  is in a parametric family indexed by  $\theta$ .

**Example 1:** Let  $X_1, \dots, X_n$  be i.i.d. from the  $\mathcal{N}(\mu, \sigma^2)$  distribution with an unknown  $\mu \in \mathbb{R}$  and a known  $\sigma^2$ . Consider the hypotheses  $H_0 : \mu \leq \mu_0$  versus  $H_1 : \mu > \mu_0$ , where  $\mu_0$  is a fixed constant. Since the sample mean  $\bar{X}$  is sufficient for  $\mu \in \mathbb{R}$ , it is reasonable to consider the following class of tests:  $T_c(X) = I_{(c, \infty)}(\bar{X})$ . By the property of the normal distributions,  $\alpha_{T_c}(\mu) = P(T_c(X) = 1) = 1 - \phi(\frac{\sqrt{n}(c-\mu)}{\sigma})$ . Since  $\phi(t)$  is an increasing function of  $t$ ,  $\sup_{P \in \mathcal{P}_0} \alpha_{T_c}(\mu) = 1 - \phi(\frac{\sqrt{n}(c-\mu_0)}{\sigma})$ . In fact, it is also true for  $\sup_{P \in \mathcal{P}_1} [1 - \alpha_{T_c}(\mu)] = \phi(\frac{\sqrt{n}(c-\mu_0)}{\sigma})$ . If we would like to use an  $\alpha$  as the level of significance, then the most effective way is to choose a  $c_\alpha$  such that  $\alpha = \sup_{P \in \mathcal{P}_0} \alpha_{T_{c_\alpha}}(\mu)$ , in which case  $c_\alpha$  must satisfy  $1 - \phi(\frac{\sqrt{n}(c_\alpha-\mu_0)}{\sigma}) = \alpha$ , i.e.,  $c_\alpha = \sigma z_{1-\alpha}/\sqrt{n} + \mu_0$ , where  $z_a = \Phi^{-1}(a)$ . It can be shown that for any test  $T(X)$  satisfying  $\sup_{P \in \mathcal{P}_0} \alpha_T(P) \leq \alpha$ ,  $1 - \alpha_T(\mu) \geq 1 - \alpha_{T_{c_\alpha}}(\mu), \mu > \mu_0$ .

**Definition 4 (Significance tests):** A common approach of finding an “optimal” test is to assign a small bound  $\alpha$  to the type I error rate  $\alpha_T(P), P \in \mathcal{P}_0$ , and then to attempt to minimize the type II error rate  $1 - \alpha_T(P), P \in \mathcal{P}_1$ , subject to  $\sup_{P \in \mathcal{P}_0} \alpha_T(P) \leq \alpha$ . The bound  $\alpha$  is called the level of significance. The left-hand side is called the size of the test  $T$ . The level of significance should be positive, otherwise no test satisfies.

**Definition 5 (p-value):** It is good practice to determine not only whether  $H_0$  is rejected for a given  $\alpha$  and a chosen test  $T_\alpha$ , but also the smallest possible level of significance at which  $H_0$  would be rejected for the computed  $T_\alpha(x)$ , i.e.,  $\hat{\alpha} = \inf\{\alpha \in (0, 1) : T_\alpha(x) = 1\}$ . Such an  $\hat{\alpha}$ , which depends on  $x$  and the chosen test and is a statistic, is called the  $p$ -value for the test  $T_\alpha$ .

**Example 2:** Let us calculate the  $p$ -value for  $T_{c_\alpha}$  in Example 1. Note that  $\alpha = 1 - \phi(\frac{\sqrt{n}(c_\alpha - \mu_0)}{\sigma}) > 1 - \Phi(\frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma})$  if and only if  $\bar{X} > c_\alpha$  (or  $T_{c_\alpha}(x) = 1$ ). Hence,  $1 - \phi(\frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}) = \inf\{\alpha \in (0, 1) : T_{c_\alpha}(x) = 1\} = \hat{\alpha}(X)$  is the  $p$ -value for  $T_{c_\alpha}$ . It turns out that  $T_{c_\alpha}(x) = I_{(0, \alpha)}(\hat{\alpha}(X))$ .

**Definition 6 (Confidence sets)**  $\theta$ : a  $k$ -vector of unknown parameters related to the unknown  $P \in \mathcal{P}$ . If a Borel set  $C(X)$  (in the range of  $\theta$ ) depending only on the sample  $X$  such that  $\inf_{P \in \mathcal{P}} P(\theta \in C(X)) \geq 1 - \alpha$ , where  $\alpha$  is a fixed constant in  $(0, 1)$ , then  $C(X)$  is called a confidence set for  $\theta$  with level of significance  $1 - \alpha$ . The left-hand side is called the confidence coefficient of  $C(X)$ , which is the highest possible level of significance for  $C(X)$ . A confidence set is a random element that covers the unknown  $\theta$  with certain probability.

**Example 3:** Let  $X_1, \dots, X_n$  be i.i.d. from the  $\mathcal{N}(\mu, \sigma^2)$  distribution with both  $\mu \in \mathbb{R}$  and  $\sigma^2 > 0$  unknown. Let  $\theta = (\mu, \sigma^2)$  and  $\alpha \in (0, 1)$  be given. Let  $\bar{X}$  be the sample mean and  $S^2$  be the sample variance. Since  $(\bar{X}, S^2)$  is sufficient, we focus on  $C(X)$  that is a function of  $(\bar{X}, S^2)$ . Since  $\sqrt{n}(\bar{X} - \mu)/\sigma$  has the  $\mathcal{N}(0, 1)$  distribution,  $P(-\tilde{c}_\alpha \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \tilde{c}_\alpha) = \sqrt{1 - \alpha}$ , where  $\tilde{c}_\alpha = \Phi^{-1}(\frac{1 + \sqrt{1 - \alpha}}{2})$ . Since the  $\chi^2$  distribution  $\chi_{n-1}^2$  is a known distribution, we can always find two constants  $c_{1\alpha}$  and  $c_{2\alpha}$  such that  $P(c_{1\alpha} \leq \frac{(n-1)S^2}{\sigma^2} \leq c_{2\alpha}) = \sqrt{1 - \alpha}$ . Then  $P(-\tilde{c}_\alpha \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \tilde{c}_\alpha, c_{1\alpha} \leq \frac{(n-1)S^2}{\sigma^2} \leq c_{2\alpha}) = 1 - \alpha$ . The LHS defines a set in the range of  $\theta = (\mu, \sigma^2)$  bounded by two straight lines,  $\sigma^2 = (n-1)S^2/c_{i\alpha}, i = 1, 2$ , and a curve  $\sigma^2 = n(\bar{X} - \mu)^2/\tilde{c}_\alpha^2$ . This set is a confidence set for  $\theta$  with confidence coefficient  $1 - \alpha$ .

**Definition 7 (Randomized tests):** Since the action space contains only two points, 0 and 1, for a hypothesis testing problem, any randomized test  $\delta(X, A)$  is equivalent to a statistic  $T(X) \in [0, 1]$  with  $T(x) = \delta(x, \{1\})$  and  $1 - T(X) = \delta(x, \{0\})$ . A nonrandomized test is obviously a special case where  $T(x)$  does not take any value in  $(0, 1)$ . For any randomized test  $T(X)$ , we define the type I error probability to be  $\alpha_T(P) = \mathbb{E}[T(X)], P \in \mathcal{P}_0$ , and the type II error probability to be  $1 - \alpha_T(P) = \mathbb{E}[1 - T(X)], P \in \mathcal{P}_1$ . For a class of randomized tests, we would like to minimize  $1 - \alpha_T(P)$  subject to  $\sup_{P \in \mathcal{P}_0} \alpha_T(P) = \alpha$ .

**Definition 8 (Consistency of point estimators):** Let  $X = (X_1, \dots, X_n)$  be a sample from  $P \in \mathcal{P}$ ,  $T_n(X)$  be an estimator of  $\theta$  for every  $n$ , and  $\{a_n\}$  be a sequence of positive constants,  $a_n \rightarrow \infty$ . (i)  $T_n(x)$  is consistent for  $\theta$  iff  $T_n(x) \rightarrow_p \theta$  w.r.t. any  $P$ . (ii)  $T_n(x)$  is  $a_n$ -consistent for  $\theta$  iff  $a_n[T_n(X) - \theta] = O_p(1)$  w.r.t. any  $P$ . (iii)  $T_n(x)$  is strongly consistent for  $\theta$  iff  $T_n(x) \rightarrow_{a.s.} \theta$  w.r.t. any  $P$ . (iv)  $T_n(X)$  is  $L_r$ -consistent for  $\theta$  iff  $T_n(x) \rightarrow_{L_r} \theta$  w.r.t. for any  $P$  for some fixed  $r > 0$ ; if  $r = 2$ ,  $L_2$ -consistency is called consistency in mse.

**Remark 1 (Consistency is an essential requirement):** Like the admissibility, consistency is an essential requirement: any inconsistent estimators should not be used, but there are many consistent estimators and some may not be good. Thus, consistency should be used together with other criteria.

**Remark 2 (Approximate and asymptotic bias):** Unbiasedness is a criterion for point estimator. In some cases, however, there is no unbiased estimator. Furthermore, having a “slight” bias in some cases may not be a bad idea.

**Definition 9:** (i) Let  $\xi, \xi_1, \xi_2, \dots$  be random variables and  $\{a_n\}$  be a sequence of positive numbers satisfying  $a_n \rightarrow \infty$  or  $a_n \rightarrow a > 0$ . If  $a_n \xi_n \rightarrow_d \xi$  and  $\mathbb{E}|\xi| < \infty$ , then  $\mathbb{E}\xi/a_n$  is called an asymptotic expectation of  $\xi_n$ . (ii) For a point estimator  $T_n$  of  $\theta$ , an asymptotic expectation of  $T_n - \theta$ , if it exists,



is called an asymptotic bias of  $T_n$  and denoted by  $\tilde{b}_{T_n}(P)$ . If  $\lim_{n \rightarrow \infty} \tilde{b}_{T_n}(P) = 0$  for any  $P$ , then  $T_n$  is asymptotically unbiased.

**Proposition 1 (Asymptotic expectation is essentially unique):** For a sequence of random variables  $\{\xi_n\}$ , suppose both  $\mathbb{E}\xi/a_n$  and  $\mathbb{E}\eta/b_n$  are asymptotic expectations of  $\xi_n$ . Then, one of the following three must hold: (a)  $\mathbb{E}\xi = \mathbb{E}\eta = 0$ ; (b)  $\mathbb{E}\xi \neq 0, \mathbb{E}\eta = 0$ , and  $b_n/a_n \rightarrow 0$ ; (c)  $\mathbb{E}\xi \neq 0, \mathbb{E}\eta \neq 0$ , and  $(\mathbb{E}\xi/a_n)/(\mathbb{E}\eta/b_n) \rightarrow 1$ .

**Example 4 (Functions of sample means):** We consider the case where  $X_1, \dots, X_n$  are i.i.d. random  $k$ -vectors with finite  $\Sigma = \text{Var}(X_1)$ ,  $T_n = g(\bar{X})$ , where  $g$  is a function on  $\mathbb{R}^k$  that is second-order differentiable at  $\mu = \mathbb{E}X_1$ . Consider  $T_n$  as an estimator of  $\theta = g(\mu)$ . By Taylor's expansion,  $T_n - \theta = [\nabla g(\mu)]^T(\bar{X} - \mu) + 2^{-1}(\bar{X} - \mu)^T \nabla^2 g(\mu)(\bar{X} - \mu) + o_p(n^{-1})$ . By the CLT,  $2^{-1}n(\bar{X} - \mu) \nabla^2 g(\mu)(\bar{X} - \mu) \rightarrow_d 2^{-1}Z_\Sigma^T \nabla^2 g(\mu) Z_\Sigma$ , where  $Z_\Sigma = \mathcal{N}_k(0, \Sigma)$ . Thus,  $\frac{\mathbb{E}[Z_\Sigma^T \nabla^2 g(\mu) Z_\Sigma]}{2n} = \frac{\text{tr}(\nabla^2 g(\mu) \Sigma)}{2n}$  is the  $n^{-1}$  order asymptotic bias of  $T_n = g(\bar{X})$ .

**Definition 10 (Asymptotic variance and amse):** Let  $T_n$  be an estimator of  $\theta$  for every  $n$  and  $\{a_n\}$  be a sequence of positive numbers satisfying  $a_n \rightarrow \infty$  or  $a_n \rightarrow a > 0$ . Assume that  $a_n(T_n - \theta) \rightarrow_d Y$  with  $0 < \mathbb{E}Y^2 < \infty$ . (i) The asymptotic mean squared error of  $T_n$ , denoted by  $\text{amse}_{T_n}(P)$ , is defined as the asymptotic expectation of  $(T_n - \theta)^2$ ,  $\text{amse}_{T_n}(P) = \mathbb{E}Y^2/a_n^2$ . The asymptotic variance of  $T_n$  is defined as  $\sigma_{T_n}^2(P) = \text{Var}(Y)/a_n^2$ . (ii) Let  $T'_n$  be another estimator of  $\theta$ . The asymptotic relative efficiency of  $T'_n$  w.r.t.  $T_n$  is defined as  $e_{T'_n, T_n} = \text{amse}_{T_n}(P)/\text{amse}_{T'_n}(P)$ . (iii)  $T_n$  is said to be asymptotically more efficient than  $T'_n$  iff  $\limsup_n e_{T'_n, T_n}(P) \leq 1$  for any  $P$  and  $< 1$  for some  $P$ .

**Proposition 2:** Let  $T_n$  be an estimator of  $\theta$  for every  $n$  and  $\{a_n\}$  be a sequence of positive numbers satisfying  $a_n \rightarrow \infty$  or  $a_n \rightarrow a > 0$ . If  $a_n(T_n - \theta) \rightarrow_d Y$  with  $0 < \mathbb{E}Y^2 < \infty$ , then (i)  $\mathbb{E}Y^2 \leq \liminf_n \mathbb{E}[a_n^2(T_n - \theta)^2]$  and (ii)  $\mathbb{E}Y^2 = \lim_{n \rightarrow \infty} \mathbb{E}[a_n^2(T_n - \theta)^2]$  if and only if  $\{a_n^2(T_n - \theta)^2\}$  is uniformly integrable.

**Example 5:** Let  $X_1, \dots, X_n$  be i.i.d. from the Poisson distribution  $P(\theta)$  with an unknown  $\theta > 0$ . Consider the estimation of  $\theta = P(X_i = 0) = e^{-\theta}$ . Let  $T_{1n} = F_n(0)$ , where  $F_n$  is the empirical c.d.f. Then  $T_{1n}$  is unbiased and has  $\text{mse}_{T_{1n}}(\theta) = e^{-\theta}(1 - e^{-\theta})/n$ . Also,  $\sqrt{n}(T_{1n} - \theta) \rightarrow_d \mathcal{N}(0, e^{-\theta}(1 - e^{-\theta}))$  by the CLT. Thus, in the case  $\text{amse}_{T_{1n}}(\theta) = \text{mse}_{T_{1n}}(\theta)$ . Consider  $T_{2n} = e^{-\bar{X}}$ . Note that  $\mathbb{E}T_{2n} = e^{n\theta(e^{-1/n} - 1)}$ , hence  $nb_{T_{2n}}(\theta) \rightarrow \theta e^{-\theta}/2$ . Using the CLT, we can show that  $\sqrt{n}(T_{2n} - \theta) \rightarrow_d \mathcal{N}(0, e^{-2\theta}\theta)$ . Then  $\text{amse}_{T_{2n}}(\theta) = e^{-2\theta}\theta/n$ . Thus, the asymptotic relative efficiency of  $T_{1n}$  w.r.t.  $T_{2n}$  is  $e_{T_{1n}, T_{2n}} = \theta/(e^\theta - 1) < 1$ . This shows that  $T_{2n}$  is asymptotically more efficient than  $T_{1n}$ .

### 3 Unbiased Estimation

#### 3.1 UMVUE: functions of sufficient and complete statistics

**Definition 1 (Estimable):** If there exists an unbiased estimator of  $\vartheta$ , then  $\vartheta$  is called an estimable parameter.

**Definition 2 (UMVUE):** An unbiased estimator  $T(X)$  of  $\theta$  is called uniformly minimum variance unbiased estimator (UMVUE) iff  $\text{Var}(T(X)) \leq \text{Var}(U(X))$  for any  $P \in \mathcal{P}$  and any other unbiased estimator  $U(X)$  of  $\theta$ .

**Theorem 1 (Lehmann-Scheffé theorem):** Suppose that there exists a sufficient and complete

statistic  $T(X)$  for  $P \in \mathcal{P}$ . If  $\theta$  is estimable, i.e., there is a unique unbiased estimator of  $\theta$ , then there is a unique UMVUE of  $\theta$  that is of the form  $h(T)$  with a Borel function  $h$ .

**The first method (Directly solving for  $h$ ):** Need the distribution of  $T$ . Try some function  $h$  to see if  $\mathbb{E}[h(T)]$  is related to  $\theta$ . If  $\mathbb{E}[h(T)] = \theta$  for all  $P$ , what should  $h$  be?

**Example 1:** Let  $X_1, \dots, X_n$  be i.i.d. from the uniform distribution on  $(0, \theta)$ ,  $\theta > 0$ . Consider  $\vartheta = \theta$ . Since the sufficient and complete statistic  $X_{(n)}$  has the Lebesgue p.d.f.  $n\theta^{-n}x^{n-1}1_{(0,\theta)}(x)$ ,  $\mathbb{E}X_{(n)} = n\theta^{-n} \int_0^\theta x^n dx = \frac{n}{n+1}\theta$ . An unbiased estimator of  $\theta$  is  $(n+1)X_{(n)}/n$ , which is the UMVUE. Consider now  $\vartheta = g(\theta)$ , where  $g$  is a differentiable function on  $(0, \theta)$ . An unbiased estimator  $h(X_{(n)})$  of  $\vartheta$  must satisfy  $\theta^n g(\theta) = n \int_0^\theta h(x)x^{n-1}dx$  for all  $\theta > 0$ . Hence, the UMVUE of  $\vartheta$  is  $h(X_{(n)}) = g(X_{(n)}) + n^{-1}X_{(n)}g'(X_{(n)})$ .

**The second method (When a sufficient and complete statistic is available):** Find an unbiased estimator of  $\theta$ , say  $U(X)$ . Conditioning on a sufficient and complete statistic  $T(X)$ :  $\mathbb{E}[U(X)|T]$  is the UMVUE of  $\theta$ . We need to derive an explicit form of  $\mathbb{E}[U(X)|T]$ .

**Example 2:** Let  $X_1, \dots, X_n$  be i.i.d. from the exponential distribution  $\text{Exp}(0, \theta)$ .  $F_\theta(x) = (1 - e^{-x/\theta})1_{(0,\theta)}(x)$ . Consider the estimation of  $\vartheta = 1 - F_\theta(t)$ .  $\bar{X}$  is sufficient and complete for  $\theta > 0$ .  $1_{(t,\infty)}(X_1)$  is unbiased for  $\vartheta$ ,  $\mathbb{E}[1_{(t,\infty)}(X_1)] = P(X_1 > t) = \vartheta$ . Hence  $T(X) = \mathbb{E}[1_{(t,\infty)}(X_1)|\bar{X}] = P(X_1 > t|\bar{X})$  is the UMVUE of  $\vartheta$ . By Basu's theorem,  $X_1/\bar{X}$  and  $\bar{X}$  are independent. Thus,  $P(X_1 > t|\bar{X} = \bar{x}) = P(X_1/\bar{X} > t/\bar{x}|\bar{X} = \bar{x}) = P(X_1/\bar{X} > t/\bar{x})$ . To compute this unconditional probability, we need the distribution of  $X_1/\sum_{i=1}^n X_i = X_1/(X_1 + \sum_{i=2}^n X_i)$ . Using the transformation technique and the fact that  $\sum_{i=2}^n X_i$  is independent of  $X_1$  and has a gamma distribution, we obtain that  $X_1/\sum_{i=1}^n X_i$  has the Lebesgue p.d.f.  $(n-1)(1-x)^{n-2}1_{(0,1)}(x)$ . Hence  $P(X_1 > t|\bar{X} = \bar{x}) = (n-1) \int_{t/(n\bar{x})}^1 (1-x)^{n-2}dx = (1 - \frac{t}{n\bar{x}})^{n-1}$  and the UMVUE of  $\vartheta$  is  $T(X) = (1 - \frac{t}{n\bar{X}})^{n-1}$ .

**Example 3:** Let  $X_1, \dots, X_n$  be i.i.d. from an unknown population  $P$  in a nonparametric family  $\mathcal{P}$ . In many cases the vector of order statistics,  $T = (X_{(1)}, \dots, X_{(n)})$ , is sufficient and complete for  $P \in \mathcal{P}$ . Note that an estimator  $\phi(X_1, \dots, X_n)$  is a function of  $T$  iff the function  $\phi$  is symmetric in its  $n$  arguments. Hence, if  $T$  is sufficient and complete, then a symmetric unbiased estimator of any estimable  $\vartheta$  is the UMVUE. Specific examples:  $\bar{X}$  is the UMVUE of  $\vartheta = \mathbb{E}X_1$ ,  $S^2$  is the UMVUE of  $\text{Var}(X_1)$ ,  $n^{-1} \sum_{i=1}^n X_i^2 - S^2$  is the UMVUE of  $(\mathbb{E}X_1)^2$ ,  $F_n(t)$  is the UMVUE of  $P(X_1 \leq t)$  for any fixed  $t$ . The previous conclusions are not true if  $T$  is not sufficient and complete for  $P \in \mathcal{P}$ .

**Remark 1 (Nonexistence of any UMVUE):** If  $n > 2$  and  $\mathcal{P}$  contains all symmetric distributions having Lebesgue p.d.f.'s and finite means, then there is no UMVUE for  $\mu = \mathbb{E}X_1$ .

**Example 4 (Survey samples from a finite population):** Let  $\mathcal{P} = \{1, \dots, N\}$  be a finite population of interest. For each  $i \in \mathcal{P}$ , let  $y_i$  be a value of interest associated with unit  $i$ . Let  $s = \{i_1, \dots, i_n\}$  be a subset of distinct elements of  $\mathcal{P}$ , which is a sample selected with selection probability  $p(s)$ , where  $p$  is known. The value  $y_i$  is observed if and only if  $i \in s$ . If  $p(s)$  is constant, the sampling plan is called the simple random sampling without replacement. Consider the estimation of  $Y = \sum_{i=1}^N y_i$ , the population total as the parameter of interest. Let  $X = (X_i, i \in s)$  be the vector such that  $P(X_1 = y_{i_1}, \dots, X_n = y_{i_n}) = p(s)/n!$ . Let  $\mathcal{Y}$  be the range of  $y_i$ ,  $\theta = (y_1, \dots, y_N)$  and  $\Theta = \prod_{i=1}^N \mathcal{Y}$ . Under simple random sampling without replacement, the population under consideration is a parametric family indexed by  $\theta \in \Theta$ .

**Theorem 2 (Watson-Royall theorem):** (i) If  $p(s) > 0$  for all  $s$ , then the vector of order statistics  $X_{(1)} \leq \dots \leq X_{(n)}$  is complete for  $\theta \in \Theta$ . (ii) Under simple random sampling without replacement, the vector of order statistics is sufficient for  $\theta \in \Theta$ . (iii) Under simple random sampling without replacement, for any estimable function of  $\theta$ , its unique UMVUE is the unbiased estimator  $g(X_1, \dots, X_n)$ , where  $g$  is symmetric in its  $n$  arguments.

### 3.2 Characteristic of UMVUE and Fisher information bound

**Remark 1:** When a complete and sufficient statistic is not available, it is usually very difficult to derive a UMVUE. In some cases, the following result can be applied, if we have enough knowledge about unbiased estimators of 0.

**Theorem 1:** Let  $\mathcal{U}$  be the set of all unbiased estimators of 0 with finite variances and  $T$  be an unbiased estimator of  $\theta$  with  $\mathbb{E}(T^2) < \infty$ . (i) A necessary and sufficient condition for  $T(X)$  to be a UMVUE of  $\theta$  is that  $\mathbb{E}[T(X)U(X)] = 0$  for any  $U \in \mathcal{U}$  and any  $P \in \mathcal{P}$ . (ii) Suppose that  $T = h(\tilde{T})$ , where  $\tilde{T}$  is a sufficient statistic for  $P \in \mathcal{P}$  and  $h$  is a Borel function. Let  $\mathcal{U}_{\tilde{T}}$  be the subset of  $\mathcal{U}$  consisting of Borel functions of  $\tilde{T}$ . Then a necessary and sufficient condition for  $T$  to be a UMVUE of  $\theta$  is that  $\mathbb{E}[T(X)U(X)] = 0$  for any  $U \in \mathcal{U}_{\tilde{T}}$  and any  $P \in \mathcal{P}$ . The theorem can be used to find a UMVUE, check whether a particular estimator is a UMVUE and show the nonexistence of any UMVUE.

**Theorem 2:** (i) If  $T_j$  is a UMVUE of  $\theta_j, j = 1, \dots, k$ , then  $\sum_{j=1}^k c_j T_j$  is a UMVUE of  $\theta = \sum_{j=1}^k c_j \theta_j$  for any constants  $c_1, \dots, c_k$ . (ii) If  $T_1$  and  $T_2$  are two UMVUE's of  $\theta$ , then  $T_1 = T_2$  a.s.  $P$  for any  $P \in \mathcal{P}$ .

**Example 1:** Let  $X_1, \dots, X_n$  be i.i.d. from the uniform distribution on the interval  $(0, \theta)$ . We have shown that  $(1+n^{-1})X_{(n)}$  is the UMVUE for  $\theta$  when the parameter space is  $\Theta = (0, \infty)$ . Suppose now that  $\Theta = [1, \infty)$ . Then  $X_{(n)}$  is not complete, although it is still sufficient for  $\theta$ . We now illustrate how to use Theorem 1 to find a UMVUE of  $\theta$ . Let  $U(X_{(n)})$  be an unbiased estimator of 0. Since  $X_{(n)}$  has the Lebesgue p.d.f  $n\theta^{-n}x^{n-1}1_{(0,\theta)}(x)$ ,  $0 = \int_0^1 U(x)x^{n-1}dx + \int_1^\theta U(x)x^{n-1}dx$  for all  $\theta \geq 1$ . This implies that  $U(x) = 0$  a.e. Lebesgue measure on  $[1, \infty)$  and  $\int_0^1 U(x)x^{n-1}dx = 0$ . Consider  $T = h(X_{(n)})$ . To have  $\mathbb{E}(TU) = 0$ , we must have  $\int_0^1 h(x)U(x)x^{n-1}dx = 0$ . Thus, we may consider the

following function:  $h(x) = \begin{cases} c & 0 \leq x \leq 1 \\ bx & x > 1 \end{cases}$ , where  $c$  and  $b$  are some constants. Since  $\mathbb{E}[h(X_{(n)})] = \theta$ ,

we obtain that  $\theta = cP(X_{(n)} \leq 1) + b\mathbb{E}[X_{(n)}1_{(1,\infty)}(X_{(n)})] = c\theta^{-n} + \frac{bn}{n+1}(\theta - \theta^{-n})$ . Thus,  $c = 1$  and

$b = (n+1)/n$ . The UMVUE of  $\theta$  is then  $h(X_{(n)}) = \begin{cases} 1 & 0 \leq X_{(n)} \leq 1 \\ (1+n^{-1})X_{(n)} & X_{(n)} > 1 \end{cases}$ .

**Theorem 3 (Cramér-Rao lower bound):** Let  $X = (X_1, \dots, X_n)$  be a sample from  $P \in \mathcal{P} = \{P_\theta : \theta \in \Theta\}$ , where  $\Theta$  is an open set in  $\mathbb{R}^k$ . Suppose that  $T(X)$  is an estimator with  $\mathbb{E}[T(X)] = g(\theta)$  being a differentiable function of  $\theta$ ;  $P_\theta$  has a p.d.f.  $f_\theta$  w.r.t. a measure  $\nu$  for all  $\theta \in \Theta$ ; and  $f_\theta$  is differentiable as a function of  $\theta$  and satisfies  $\frac{\partial}{\partial \theta} \int h(x)f_\theta(x)d\nu = \int h(x)\frac{\partial}{\partial \theta}f_\theta(x)d\nu, \theta \in \Theta$  for  $h(x) \equiv 1$  and  $h(x) = T(x)$ . Then  $\text{Var}(T(X)) \geq [\frac{\partial}{\partial \theta}g(\theta)]^T [I(\theta)]^{-1} \frac{\partial}{\partial \theta}g(\theta)$ , where  $I(\theta) = \mathbb{E}\{\frac{\partial}{\partial \theta} \log f_\theta(X) [\frac{\partial}{\partial \theta} \log f_\theta(X)]^T\}$  is assumed to be positive definite for any  $\theta \in \Theta$  and is called the Fisher information matrix.

**Proposition 1:** (i) If  $X$  and  $Y$  are independent with the Fisher information matrices  $I_X(\theta)$  and  $I_Y(\theta)$ , respectively, then the Fisher information about  $\theta$  contained in  $(X, Y)$  is  $I_X(\theta) + I_Y(\theta)$ . (ii) Suppose that  $X$  has the p.d.f.  $f_\theta$  that is twice differentiable in  $\theta$  and  $\frac{\partial}{\partial \theta} \int h(x) f_\theta(x) d\nu = \int h(x) \frac{\partial}{\partial \theta} f_\theta(x) d\nu$  holds with  $h(x) \equiv 1$  and  $f_\theta$  replaced by  $\partial f_\theta / \partial \theta$ . Then  $I(\theta) = -\mathbb{E}[\frac{\partial^2}{\partial \theta \partial \theta^T} \log f_\theta(X)]$ .

**Remark 2:** If  $\theta = \psi(\eta)$  and  $\psi$  is differentiable, then the Fisher information that  $X$  contains about  $\eta$  is  $\frac{\partial}{\partial \eta} \psi(\eta) I(\psi(\eta)) [\frac{\partial}{\partial \eta} \psi(\eta)]^T$ . However, the Cramér-Rao lower bound is not affected by any one-to-one reparameterization.

**Proposition 2:** Suppose that the distribution of  $X$  is from an exponential family  $\{f_\theta : \theta \in \Theta\}$ , i.e., the p.d.f. of  $X$  w.r.t. a  $\sigma$ -finite measure is  $f_\theta(x) = \exp\{\eta(\theta)^T T(X) - \xi(\theta)\} c(x)$ , where  $\Theta$  is an open subset of  $\mathbb{R}^k$ . (i) The regularity condition  $\frac{\partial}{\partial \theta} \int h(x) f_\theta(x) d\nu = \int h(x) \frac{\partial}{\partial \theta} f_\theta(x) d\nu$  is satisfied for any  $h$  with  $\mathbb{E}|h(X)| < \infty$  and  $I(\theta) = -\mathbb{E}[\frac{\partial^2}{\partial \theta \partial \theta^T} \log f_\theta(X)]$ . (ii) If  $I(\eta)$  is the Fisher information matrix for the natural parameter  $\eta$ , then the variance-covariance matrix  $\text{Var}(T) = I(\eta)$ . (iii) If  $I(\theta)$  is the Fisher information matrix for the parameter  $\vartheta = \mathbb{E}[T(X)]$ , then  $\text{Var}(T) = [I(\vartheta)]^{-1}$ .

### 3.3 U- and V-statistics

**Definition 1 (U-statistics):** Let  $X_1, \dots, X_n$  be i.i.d. from an unknown population  $P$  in a non-parametric family  $\mathcal{P}$ . If the vector of order statistic is sufficient and complete for  $P \in \mathcal{P}$ , then a symmetric unbiased estimator of an estimable  $\theta$  is the UMVUE of  $\theta$ . In many problems, parameters to be estimated are of the form  $\theta = \mathbb{E}[h(X_1, \dots, X_m)]$  with a positive integer  $m$  and a Borel function  $h$  that is symmetric and satisfies  $\mathbb{E}|h(X_1, \dots, X_m)| < \infty$  for any  $P \in \mathcal{P}$ . An effective way of obtaining an unbiased estimator of  $\theta$  is to use  $U_n = (C_n^m)^{-1} \sum_c h(X_{i_1}, \dots, X_{i_m})$ , where  $\sum_c$  denotes the summation over the  $C_n^m$  combinations of  $m$  distinct elements  $\{i_1, \dots, i_m\}$  from  $\{1, \dots, n\}$ . The statistic is called a U-statistic with kernel  $h$  of order  $m$ .

**Example 1:** Consider the estimation of  $\mu^m$ , where  $\mu = \mathbb{E}X_1$  and  $m$  is an integer  $> 0$ . Using  $h(x_1, \dots, x_m) = x_1 \cdots x_m$ , we obtain the following U-statistic for  $\mu^m$ :  $U_n = (C_n^m)^{-1} \sum_c X_{i_1} \cdots X_{i_m}$ . Consider next the estimation of  $\sigma^2 = \mathbb{E}[(X_1 - X_2)^2/2]$ , we obtain the following U-statistic with kernel  $h(x_1, x_2) = (x_1 - x_2)^2/2$ :  $U_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \frac{(X_i - X_j)^2}{2} = \frac{1}{n-1} (\sum_{i=1}^n X_i^2 - n\bar{X}^2) = S^2$ , which is the sample variance.

**Theorem 1 (Hoeffding's theorem):** For a U-statistic  $U_n$  with  $\mathbb{E}[h(X_1, \dots, X_m)]^2 < \infty$ ,  $\text{Var}(U_n) = (C_n^m)^{-1} \sum_{k=1}^m C_m^k C_{n-m}^{m-k} \zeta_k$ , where  $\zeta_k = \text{Var}(h_k(X_1, \dots, X_k))$ ,  $h_k(x_1, \dots, x_k) = \mathbb{E}[h(X_1, \dots, X_m) | X_1 = x_1, \dots, X_k = x_k] = \mathbb{E}[h(x_1, \dots, x_k, X_{k+1}, \dots, X_m)]$ ,  $\tilde{h}_k = h_k - \mathbb{E}[h(X_1, \dots, X_m)]$ .

**Proposition 1:** (i)  $\frac{m^2}{n} \zeta_1 \leq \text{Var}(U_n) \leq \frac{m}{n} \zeta_m$ ; (ii)  $(n+1)\text{Var}(U_{n+1}) \leq n\text{Var}(U_n)$  for any  $n > m$ ; (iii) For any fixed  $m$  and  $k = 1, \dots, m$ , if  $\zeta_j = 0$  for  $j < k$  and  $\zeta_k > 0$ , then  $\text{Var}(U_n) = \frac{k!(C_m^k)^2 \zeta_k}{n^k} + O(\frac{1}{n^{k+1}})$ .

**Example 2:** Consider  $h(x_1, x_2) = x_1 x_2$ , the U-statistic unbiased for  $\mu^2$ ,  $\mu = \mathbb{E}X_1$ . Note that  $h_1(x_1) = \mu x_1$ ,  $\tilde{h}_1(x_1) = \mu(x_1 - \mu)$ .  $\zeta_1 = \mathbb{E}[\tilde{h}_1(X_1)]^2 = \mu^2 \text{Var}(X_1) = \mu^2 \sigma^2$ ,  $\tilde{h}(x_1, x_2) = x_1 x_2 - \mu^2$ , and  $\zeta_2 = \text{Var}(X_1 X_2) = (\mu^2 + \sigma^2)^2 - \mu^4$ . Thus for  $U_n = (C_n^2)^{-1} \sum_{1 \leq i < j \leq n} X_i X_j$ ,  $\text{Var}(U_n) = (C_n^2)^{-1} (C_2^1 C_{n-2}^1 \zeta_1 + C_2^2 C_{n-2}^0 \zeta_2) = \frac{2}{n(n-1)} [2(n-2)\mu^2 \sigma^2 + (\mu^2 + \sigma^2)^2 - \mu^4] = \frac{4\mu^2 \sigma^2}{n} + \frac{2\sigma^4}{n(n-1)}$ .

**Remark 1 (Asymptotic distributions of U-statistics):** For nonparametric  $\mathcal{P}$ , the exact distribution of  $U_n$  is hard to derive. We study the method of projection, which is particularly effective for studying asymptotic distributions of U-statistics.

**Definition 2:** Let  $T_n$  be a given statistic based on  $X_1, \dots, X_n$ . The projection of  $T_n$  on  $k_n$  random elements  $Y_1, \dots, Y_{k_n}$  is defined to be  $\tilde{T}_n = \mathbb{E}(T_n) + \sum_{i=1}^{k_n} [\mathbb{E}(T_n|Y_i) - \mathbb{E}(T_n)]$ .

**Theorem 2:** Let  $T_n$  be a symmetric statistics with  $\text{Var}(T_n) < \infty$  for every  $n$  and  $\tilde{T}_n$  be the projection of  $T_n$  on  $X_1, \dots, X_n$ . Then  $\mathbb{E}(T_n) = \mathbb{E}(\tilde{T}_n)$  and  $\mathbb{E}(T_n - \tilde{T}_n)^2 = \text{Var}(T_n) - \text{Var}(\tilde{T}_n)$ .

**Example 3:** For a U-statistic  $U_n$ , one can show that  $\tilde{U}_n = \mathbb{E}(U_n) + \frac{m}{n} \sum_{i=1}^n \tilde{h}_1(X_i)$ , where  $\tilde{U}_n$  is the projection of  $U_n$  on  $X_1, \dots, X_n$  and  $\tilde{h}_1(x) = h_1(x) - \mathbb{E}[h(X_1, \dots, X_m)]$ ,  $h_1(x) = \mathbb{E}[h(x, X_2, \dots, X_m)]$ . Hence, if  $\zeta_1 = \text{Var}(\tilde{h}_1(X_i)) > 0$ ,  $\text{Var}(\tilde{U}_n) = m^2\zeta_1/n$  and  $\mathbb{E}(U_n - \tilde{U}_n)^2 = O(n^{-2})$ . If  $\zeta_1 = 0$  but  $\zeta_2 > 0$ , then we can show that  $\mathbb{E}(U_n - \tilde{U}_n)^2 = O(n^{-3})$ . One may derive results for the cases where  $\zeta_2 = 0$ , but the case of either  $\zeta_1 > 0$  or  $\zeta_2 > 0$  is the most interesting case in applications.

**Theorem 3:** Let  $U_n$  be a U-statistic with  $\mathbb{E}[h(X_1, \dots, X_m)]^2 < \infty$ . (i) If  $\zeta_1 > 0$ , then  $\sqrt{n}[U_n - \mathbb{E}(U_n)] \rightarrow_d \mathcal{N}(0, m^2\zeta_1)$ . (ii) If  $\zeta_1 = 0$  but  $\zeta_2 > 0$ , then  $n[U_n - \mathbb{E}(U_n)] \rightarrow_d \frac{m(m-1)}{2} \sum_{j=1}^{\infty} \lambda_j (\chi_{1j}^2 - 1)$ , where  $\chi_{1j}^2$ 's are i.i.d. random variables having the chi-square distribution  $\chi_1^2$  and  $\lambda_j$ 's are some constants (which may depend on  $P$ ) satisfying  $\sum_{j=1}^{\infty} \lambda_j^2 = \zeta_2$ .

**Proposition 2:**  $\mathbb{E}[\frac{m(m-1)}{2} \sum_{j=1}^{\infty} \lambda_j (\chi_{1j}^2 - 1)]^2 = \frac{m^2(m-1)^2}{2} \zeta_2$ .

**Definition 3 (V-statistics):** Let  $X_1, \dots, X_n$  be i.i.d. from  $P$ . For every U-statistic  $U_n$  as an estimator  $\theta = \mathbb{E}[h(X_1, \dots, X_m)]$ , there is a closely related V-statistic defined by  $V_n = \frac{1}{n^m} \sum_{i_1=1}^n \dots \sum_{i_m=1}^n h(X_{i_1}, \dots, X_{i_m})$ . As an estimator of  $\theta$ ,  $V_n$  is biased; but the bias is small asymptotically. For a fixed  $n$ ,  $V_n$  may be better than  $U_n$  in terms of the mse.

**Proposition 3:** (i) Assume that  $\mathbb{E}|h(X_{i_1}, \dots, X_{i_m})| < \infty$  for all  $1 \leq i_1 \leq \dots \leq i_m \leq m$ . Then the bias of  $V_n$  satisfies  $b_{V_n}(P) = O(n^{-1})$ . (ii) Assume that  $\mathbb{E}[h(X_{i_1}, \dots, X_{i_m})]^2 < \infty$  for all  $1 \leq i_1 \leq \dots \leq i_m \leq m$ . Then the variance of  $V_n$  satisfies  $\text{Var}(V_n) = \text{Var}(U_n) + O(n^{-2})$ .

**Theorem 4:** Let  $V_n$  be a V-statistic with  $\mathbb{E}[h(X_{i_1}, \dots, X_{i_m})]^2 < \infty$  for all  $1 \leq i_1 \leq \dots \leq i_m \leq m$ . (i) If  $\zeta_1 = \text{Var}(h_1(X_1)) > 0$ , then  $\sqrt{n}(V_n - \theta) \rightarrow_d \mathcal{N}(0, m^2\zeta_1)$ . (ii) If  $\zeta_1 = 0$  but  $\zeta_2 = \text{Var}(h_2(X_1, X_2)) > 0$ , then  $n(V_n - \theta) \rightarrow_d \frac{m(m-1)}{2} \sum_{j=1}^{\infty} \lambda_j \chi_{1j}^2$ .

### 3.4 Construction of unbiased or approximately unbiased estimators and method of moments

**Definition 1 (Survey samples from a finite population):** Let  $\mathcal{P} = \{1, \dots, N\}$  be a finite population of interest. For each  $i \in \mathcal{P}$ , let  $y_i$  be a value of interest associated with unit  $i$ . Let  $s = \{i_1, \dots, i_n\}$  be a subset of distinct elements of  $\mathcal{P}$ , which is a sample selected with selection probability  $p(s)$ , where  $p$  is known. The value  $y_i$  is observed iff  $i \in s$ .  $Y = \sum_{j=1}^N y_j$  is the unknown population total of interest. Define  $\pi_i$  = probability that  $i \in s, i = 1, \dots, N$ .

**Theorem 1:** (i) (Horvitz-Thompson). If  $\pi_i > 0$  for  $i = 1, \dots, N$  and  $\pi_i$  is known when  $i \in s$ , then  $\hat{Y}_{ht} = \sum_{i \in s} y_i / \pi_i$  is an unbiased estimator of the population total  $Y$ . (ii) Define  $\pi_{ij}$  = probability that  $i \in s$  and  $j \in s, i = 1, \dots, N, j = 1, \dots, N$ . Then  $\text{Var}(\hat{Y}_{ht}) = \sum_{i=1}^N \sum_{j=i+1}^N (\pi_i \pi_j - \pi_{ij}) (\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j})^2$ .

**Remark 1 (Deriving asymptotically unbiased estimators):** An exactly unbiased estimator may not exist, or is hard to obtain. We often derive asymptotically unbiased estimators. Functions of sample means are popular estimators.

**Remark 2 (Functions of unbiased estimators):** If the parameter to be estimated is  $\vartheta = g(\theta)$  with a vector-valued parameter  $\theta$  and  $U_n$  is a vector of unbiased estimators of components of  $\theta$ ,

then  $T_n = g(U_n)$  is often asymptotically unbiased for  $\vartheta$ . Note that  $\mathbb{E}(T_n) = \mathbb{E}g(U_n)$  may not exist. Assume that  $g$  is differentiable and  $c_n(U_n - \theta) \rightarrow_d Y$ . Then  $\text{amse}_{T_n}(P) = \mathbb{E}\{[\nabla g(\theta)]^T Y\}^2 / c_n^2$ . Hence,  $T_n$  has a good performance in terms of amse if  $U_n$  is optimal in terms of mse.

**Definition 2 (Method of moments):** Consider a parametric problem where  $X_1, \dots, X_n$  are i.i.d. random variables from  $P_\theta, \theta \in \Theta \subset \mathbb{R}^k$ , and  $\mathbb{E}|X_1|^k < \infty$ . Let  $\mu_j = \mathbb{E}X_1^j$  be the  $j$ th moment of  $P$  and let  $\hat{\mu}_j = \frac{1}{n} \sum_{i=1}^n X_i^j$  be the  $j$ th sample moment, which is an unbiased estimator of  $\mu_j, j = 1, \dots, k$ . Typically,  $\mu_j = h_j(\theta), j = 1, \dots, k$ , for some functions  $h_j$  on  $\mathbb{R}^k$ . By substituting  $\mu_j$ 's on the left-hand side by the sample moments  $\hat{\mu}_j$ , we obtain a moment estimator  $\hat{\theta}$ , i.e.  $\hat{\theta}$  satisfies  $\hat{\mu}_j = h_j(\hat{\theta}), j = 1, \dots, k$ . This method of deriving estimators is called the method of moments.

**Example 1:** Let  $X_1, \dots, X_n$  be i.i.d. from a population  $P_\theta$  indexed by the parameter  $\theta = (\mu, \sigma^2)$ , where  $\mu = \mathbb{E}X_1 \in \mathbb{R}$  and  $\sigma^2 = \text{Var}(X_1) \in (0, \infty)$ . Since  $\mathbb{E}X_1 = \mu$  and  $\mathbb{E}X_1^2 = \sigma^2 + \mu^2$ , setting  $\hat{\mu}_1 = \mu$  and  $\hat{\mu}_2 = \sigma^2 + \mu^2$  we obtain the moment estimator  $\hat{\theta} = (\bar{X}, \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2)$ .

## 4 Estimation in Parametric Models

### 4.1 Bayesian approach

**Definition 1 (Bayesian approach):**  $X$  is from a population in a parametric family  $\mathcal{P} = P_\theta : \theta \in \Theta$ , where  $\theta \in \mathbb{R}^k$  for a fixed integer  $k \geq 1$ .  $\theta$  is viewed as a realization of a random vector  $\theta \in \Theta$  whose prior distribution is  $\Pi$ . Prior distribution: past experience, past data, or a statistician's belief (subjective). Sample  $X \in \mathcal{X}$ : from  $P_\theta = P_{x|\theta}$ , the conditional distribution of  $X$  given  $\theta$ . Posterior distribution: updated prior distribution using observed  $X = x$ .

**Theorem 1 (Bayes formula):** Assume  $\mathcal{P} = \{P_{x|\theta} : \theta \in \Theta\}$  is dominated by a  $\sigma$ -finite measure  $\nu$  and  $f_\theta(x) = dP_{x|\theta}/d\nu$  is a Borel function on  $(\mathcal{X} \times \Theta, \sigma(\mathcal{B}_\mathcal{X} \times \mathcal{B}_\Theta))$ . Let  $\Pi$  be a prior distribution on  $\Theta$ . Suppose that  $m(x) = \int_\Theta f_\theta(x) d\Pi > 0$ . (i) The posterior distribution  $P_{\theta|x} \ll \Pi$  and  $dP_{\theta|x}/d\Pi = f_\theta(x)/m(x)$ . (ii) If  $\Pi \ll \lambda$  and  $d\Pi/d\lambda = \pi(\theta)$  for a  $\sigma$ -finite measure  $\lambda$ , then  $dP_{\theta|x}/d\lambda = f_\theta(x)\pi(\theta)/m(x)$ .

**Definition 2 (Bayes action):** Let  $\mathcal{A}$  be an action space in a decision problem and  $L(\theta, a) \geq 0$  be a loss function. For any  $x \in \mathcal{X}$ , a Bayes action w.r.t.  $\Pi$  is any  $\delta(x) \in \mathcal{A}$  such that  $\mathbb{E}[L(\theta, \delta(x))|X = x] = \min_{a \in \mathcal{A}} \mathbb{E}[L(\theta, a)|X = x]$  where the expectation is w.r.t. the posterior distribution  $P_{\theta|x}$ .

**Definition 3 (Conjugate prior):** An interesting phenomenon is that the prior and the posterior are in the same parametric family of distributions. Such a prior is called a conjugate prior.

**Definition 4 (Generalized Bayes action):** The minimization in Definition 4.1 is the same as the minimizing  $\int_\Theta L(\theta, \delta(x)) f_\theta(x) d\Pi = \min_{a \in \mathcal{A}} \int_\Theta L(\theta, a) f_\theta(x) d\Pi$ . This is still defined even if  $\Pi$  is not a probability measure but a  $\sigma$ -finite measure on  $\Theta$ , in which case  $m(x)$  may not be finite. If  $\Pi(\Theta) \neq 1$ ,  $\Pi$  is called an improper prior.  $\delta(x)$  is called a generalized Bayes action.

**Definition 5 (Hyperparameters and empirical Bayes):** A Bayes action depends on the chosen prior with a vector  $\xi$  of parameters called hyperparameters. If the hyperparameters  $\xi$  is unknown, one way to solve the problem is to estimate  $\xi$  using some historical data; the resulting Bayes action is called an empirical Bayes action. If there is no historical data, we may estimate  $\xi$  using data  $x$  and the resulting Bayes action is also called an empirical Bayes action. The simplest empirical Bayes method is to



estimate  $\xi$  by viewing  $x$  as a “sample” from the marginal distribution  $P_{x|\xi}(A) = \int_{\Theta} P_{x|\theta}(A) d\Pi_{\theta|\xi}$ ,  $A \in \mathcal{B}_X$ , where  $\Pi_{\theta|\xi}$  is a prior depending on  $\xi$  or from the marginal p.d.f.  $m(x) = \int_{\Theta} f_{\theta}(x) d\Pi$ , if  $P_{x|\theta}$  has a p.d.f.  $f_{\theta}$ . The method of moments can be applied to estimate  $\xi$ .

**Example 1:** Let  $X = (X_1, \dots, X_n)$  and  $X_i$ 's be i.i.d. with an unknown mean  $\mu \in \mathbb{R}$  and a known variance  $\sigma^2$ . Assume the prior  $\Pi_{\mu|\xi}$  has mean  $\mu_0$  and variance  $\sigma_0^2$ ,  $\xi = (\mu_0, \sigma_0^2)$ . To obtain a moment estimate of  $\xi$ , we need to calculate  $\int_{\mathbb{R}^n} x_1 m(x) dx$  and  $\int_{\mathbb{R}^n} x_1^2 m(x) dx$ ,  $x = (x_1, \dots, x_n)$ . These two integrals can be obtained without knowing  $m(x)$ . Note that  $\int_{\mathbb{R}^n} x_1 m(x) dx = \int_{\Theta} \int_{\mathbb{R}^n} x_1 f_{\mu}(x) dx d\Pi_{\mu|\xi} = \int_{\mathbb{R}} \mu d\Pi_{\mu|\xi} = \mu_0$  and  $\int_{\mathbb{R}^n} x_1^2 m(x) dx = \int_{\Theta} \int_{\mathbb{R}^n} x_1^2 f_{\mu}(x) dx d\Pi_{\mu|\xi} = \sigma^2 + \int_{\mathbb{R}} \mu^2 d\Pi_{\mu|\xi} = \sigma^2 + \mu_0^2 + \sigma_0^2$ . Thus, by viewing  $x_1, \dots, x_n$  as a sample from  $m(x)$ , we obtain the moment estimates  $\hat{\mu}_0 = \bar{x}$  and  $\hat{\sigma}_0^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 - \sigma^2$ , where  $\bar{x}$  is the sample mean of  $x_i$ 's.

**Definition 6 (Hierarchical Bayes):** Instead of estimating hyperparameters, in the hierarchical Bayes approach we put a prior on hyperparameters. Let  $\Pi_{\theta|\xi}$  be a prior with a hyperparameter vector  $\xi$  and let  $\Lambda$  be a prior on  $\Xi$ , the range of  $\xi$ . Then the “marginal” prior for  $\theta$  is defined by  $\Pi(B) = \int_{\Xi} \Pi_{\theta|\xi}(B) d\Lambda(\xi)$ ,  $B \in \mathcal{B}_{\Theta}$ . If the second-stage prior  $\Lambda$  also depends on some unknown hyperparameters, then one can go on to consider a third-stage prior. In most applications, however, two-stage priors are sufficient, since misspecifying a second-stage prior is much less serious than misspecifying a first-stage prior.

**Example 2:** If  $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$  with a known  $\sigma^2$ , the prior  $\pi(\mu|\xi)$  is the p.d.f. of  $\mathcal{N}(\xi, \sigma_0^2)$  with a known  $\sigma_0^2$ , and the prior of  $\xi$  is  $\mathcal{N}(\mu_0, \tau^2)$  with a known  $\mu_0$  and  $\tau^2$ , then the marginal prior p.d.f. of  $\mu$  is  $\mathcal{N}(\mu_0, \sigma_0^2 + \tau^2)$ .

## 4.2 Bayes rule and computation

**Theorem 1 (Admissibility of Bayes rule)** In a decision problem, let  $\delta(x)$  be a Bayes rule w.r.t. a prior  $\Pi$ . (i) If  $\delta(X)$  is a unique Bayes rule, then  $\delta(X)$  is admissible. (ii) If  $\Theta$  is countable set, the Bayes risk  $r_{\delta}(\Pi) < \infty$ , and  $\Pi$  gives positive probability to each  $\theta \in \Theta$ , then  $\delta(X)$  is admissible. (iii) Let  $\mathcal{E}$  be the class of decision rules having continuous risk functions. If  $\delta(X) \in \mathcal{E}$ ,  $r_{\delta}(\Pi) < \infty$ , and  $\Pi$  gives positive probability to any open subset of  $\Theta$ , then  $\delta(X)$  is  $\mathcal{E}$ -admissible.

**Theorem 2:** Suppose that  $\Theta$  is an open set of  $\mathbb{R}^k$ . In a decision problem, let  $\mathcal{E}$  be the class of decision rules having continuous risk functions. A decision rule  $T \in \mathcal{E}$  is  $\mathcal{E}$ -admissible if there exists a sequence  $\{\Pi_j\}$  of priors such that (a) the generalized Bayes risks  $r_T(\Pi_j)$  are finite for all  $j$ ; (2) for any  $\theta_0 \in \Theta$  and  $\eta > 0$ ,  $\lim_{j \rightarrow \infty} \frac{r_T(\Pi_j) - r_j^*(\Pi_j)}{\Pi_j(O_{\theta_0, \eta})} = 0$ , where  $r_j^*(\Pi_j) = \inf_{T \in \mathcal{E}} r_T(\Pi_j)$  and  $O_{\theta_0, \eta} = \{\theta \in \Theta : \|\theta - \theta_0\| < \eta\}$  with  $\Pi_j(O_{\theta_0, \eta}) < \infty$  for all  $j$ .

**Proposition 1 (Bayes estimators are biased):** If  $\delta(X)$  is a Bayes estimator of  $\vartheta = g(\theta)$  under the squared error loss, then  $\delta(X)$  is not unbiased except in the trivial case where  $r_{\delta}(\Pi) = 0$ .

**Theorem 3:** Suppose that  $X$  has a p.d.f.  $f_{\theta}(x)$  w.r.t. a  $\sigma$ -finite measure  $\nu$ . Suppose that  $\theta = (\theta_1, \theta_2)$ ,  $\theta_j \in \Theta_j$ , and that the prior has a p.d.f.  $\pi(\theta) = \pi_{\theta_1|\theta_2}(\theta_1) \pi_{\theta_2}(\theta_2)$  where  $\pi_{\theta_2}(\theta_2)$  is a p.d.f. w.r.t. a  $\sigma$ -finite measure  $\nu_2$  on  $\Theta_2$  and for any given  $\theta_2$ ,  $\pi_{\theta_1|\theta_2}(\theta_1)$  is a p.d.f. w.r.t. a  $\sigma$ -finite measure  $\nu_1$  on  $\Theta_1$ . Suppose further that if  $\theta_2$  is given, the Bayes estimator of  $h(\theta_1) = g(\theta_1, \theta_2)$  under the squared error loss is  $\delta(X, \theta_2)$ . Then the Bayse estimator of  $g(\theta_1, \theta_2)$  under the squared error loss is  $\delta(X)$  with  $\delta(x) = \int_{\Theta_2} \delta(x, \theta_2) p_{\theta_2|x}(\theta_2) d\nu_2$  where  $p_{\theta_2|x}(\theta_2)$  is the posterior p.d.f. of  $\theta_2$  given  $X = x$ .

**Remark 1:** Often, Bayes actions or estimators have to be computed numerically. Typically we need to compute  $\mathbb{E}_p(g) = \int_{\Theta} g(\theta)p(\theta)d\nu$  with some function  $g$ , where  $p(\theta)$  is a p.d.f. w.r.t. a  $\sigma$ -finite measure  $\nu$  on  $(\Theta, \mathcal{B}_{\Theta})$  and  $\Theta \subset \mathbb{R}^k$ . There are many numerical methods for computing integrals  $\mathbb{E}_p(g)$ .

**Definition 1 (The simple Monte Carlo method):** Generate i.i.d.  $\theta^{(1)}, \dots, \theta^{(m)}$  from a p.d.f.  $h(\theta) > 0$  w.r.t.  $\nu$ . By the SLLN, as  $m \rightarrow \infty$ ,  $\hat{\mathbb{E}}_p(g) = \frac{1}{m} \sum_{j=1}^m \frac{g(\theta^{(j)})p(\theta^{(j)})}{h(\theta^{(j)})} \rightarrow_{\text{a.s.}} \int_{\Theta} \frac{g(\theta)p(\theta)}{h(\theta)} h(\theta)d\nu = \mathbb{E}_p(g)$ .

**Remark 2:** The simple Monte Carlo method may not work well because (i) the convergence of  $\hat{\mathbb{E}}_p(g)$  is very slow when  $k$  (the dimension of  $\Theta$ ) is large; (ii) generating a random vector from some  $k$ -dimensional distribution may be difficult, if not impossible.

**Remark 3 (More sophisticated MCMC methods):** Different from the simple Monte Carlo in two aspects: (i) generating random vectors can be done using distributions whose dimensions are much lower than  $k$ ; (ii)  $\theta^{(1)}, \dots, \theta^{(m)}$  are not independent, but form a homogeneous Markov chain.

**Definition 2 (Gibbs sampler):** Let  $y = (y_1, y_2, \dots, y_d)$ .  $y_j$ 's may be vectors with different dimensions. At step  $t = 1, 2, \dots$ , given  $y^{(t-1)}$ , generate  $y_1^{(t)}$  from  $P(y_2^{(t-1)}, \dots, y_d^{(t-1)} | y_1^{(t-1)})$ ,  $\dots$ ,  $y_j^{(t)}$  from  $P(y_1^{(t)}, \dots, y_{j-1}^{(t)}, y_{j+1}^{(t-1)}, \dots, y_k^{(t-1)} | y_j^{(t-1)})$ ,  $\dots$ ,  $y_k^{(t)}$  from  $P(y_1^{(t)}, \dots, y_{k-1}^{(t)} | y_k^{(t-1)})$ .

### 4.3 Minimality and admissibility

**Definition 1 (Minimax estimator):** An estimator  $\delta$  is minimax if  $\sup_{\theta} R_{\delta}(\theta) = \inf_T \sup_{\theta} R_T(\theta)$ .

**Remark 1:** A minimax estimator can be very conservative and unsatisfactory. It tries to do as well as possible in the worst case. A unique minimax estimator is admissible, since any estimator better than a minimax estimator is also minimax.

**Theorem 1 (Minimality of a Bayes estimator):** Let  $\Pi$  be a proper prior on  $\Theta$  and  $\delta$  be a Bayes estimator of  $\theta$  w.r.t.  $\Pi$ . Suppose  $\delta$  has constant risk on  $\Theta_{\Pi}$ . If  $\Pi(\Theta_{\Pi}) = 1$ , then  $\delta$  is minimax. If, in addition,  $\delta$  is the unique Bayes estimator w.r.t.  $\Pi$ , then it is the unique minimax estimator.

**Theorem 2:** Let  $\Pi_j, j = 1, 2, \dots$  be a sequence of priors and  $r_j$  be the Bayes risk of a Bayes estimator of  $\theta$  w.r.t.  $\Pi_j$ . Let  $T$  be a constant risk estimator of  $\theta$ . If  $\liminf_j r_j \geq R_T$ , then  $T$  is minimax.

**Example 1:** Let  $X_1, \dots, X_n$  be i.i.d. components having the  $\mathcal{N}(\mu, \sigma^2)$  distribution with an known  $\mu = \theta \in \mathbb{R}$  and a known  $\sigma^2$ . If the prior is  $\mathcal{N}(\mu_0, \sigma_0^2)$ , then the posterior of  $\theta$  given  $X = x$  is  $\mathcal{N}(\mu_*(x), c^2)$  with  $\mu_*(x) = \frac{\sigma^2}{n\sigma_0^2 + \sigma^2} \mu_0 + \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2} \bar{X}$  and  $c^2 = \frac{\sigma_0^2}{\sigma^2} n\sigma_0^2 + \sigma^2$ . We now show that  $\bar{X}$  is minimax under the squared error loss. For any decision rule  $T$ ,  $\sup_{\theta \in \mathbb{R}} R_T(\theta) \geq \int_{\mathbb{R}} R_T(\theta) d\Pi(\theta) \geq \int_{\mathbb{R}} R_{\mu_*}(\theta) d\Pi(\theta) = \mathbb{E}\{[\theta - \mu_*(X)]^2\} = \mathbb{E}\{\mathbb{E}\{[\theta - \mu_*(X)]^2 | X\}\} = \mathbb{E}(c^2) = c^2$ . Since this result is true for any  $\sigma_0^2 > 0$  and  $c^2 \rightarrow \sigma^2/n$  as  $\sigma_0^2 \rightarrow \infty$ ,  $\sup_{\theta \in \mathbb{R}} R_T(\theta) \geq \frac{\sigma^2}{n} = \sup_{\theta \in \mathbb{R}} R_{\bar{X}}(\theta)$  where the equality holds because the risk of  $\bar{X}$  under the squared error loss is  $\sigma^2/n$  and independent of  $\theta = \mu$ . Thus,  $\bar{X}$  is minimax.

**Theorem 3:** Let  $\Theta_0$  be a subset of  $\Theta$  and  $T$  be a minimax estimator of  $\theta$  when  $\Theta_0$  is the parameter space. Then  $T$  is minimax estimator if  $\sup_{\theta \in \Theta} R_T(\theta) = \sup_{\theta \in \Theta_0} R_T(\theta)$ .

**Theorem 4 (Admissibility in one-parameter exponential families):** Suppose that  $X$  has the p.d.f.  $c(\theta)e^{\theta T(x)}$  w.r.t. a  $\sigma$ -finite measure  $\nu$ , where  $T(x)$  is real-valued and  $\theta \in (\theta_-, \theta_+) \subset \mathbb{R}$ . Consider the estimation of  $\theta = \mathbb{E}[T(X)]$  under the squared error loss. Let  $\lambda \geq 0$  and  $\gamma$  be known constants

and let  $T_{\lambda,\gamma}(X) = (T + \gamma\lambda)/(1 + \lambda)$ . Then a sufficient condition for the admissibility of  $T_{\lambda,\gamma}$  is that  $\int_{\theta_0}^{\theta_+} \frac{e^{-\gamma\lambda\theta}}{[c(\theta)]^\lambda} d\theta = \int_{\theta_-}^{\theta_0} \frac{e^{-\gamma\lambda\theta}}{[c(\theta)]^\lambda} d\theta = \infty$ , where  $\theta_0 \in (\theta_-, \theta_+)$ .

**Theorem 5:** Assume that  $X$  has the p.d.f. as described in Theorem 4 with  $\theta_- = -\infty$  and  $\theta_+ = \infty$ . (i) As an estimator of  $\theta = \mathbb{E}(T)$ ,  $T(X)$  is admissible under the squared error loss and the loss  $(a - \theta)^2/\text{Var}(T)$ . (ii)  $Y$  is the unique minimax estimator of  $\theta$  under the loss  $(a - \theta)^2/\text{Var}(T)$ .

**Example 2:** Let  $X_1, \dots, X_n$  be i.i.d. from  $\mathcal{N}(0, \sigma^2)$  with an unknown  $\sigma^2 > 0$  and let  $Y = \sum_{i=1}^n X_i^2$ . Consider the estimation of  $\sigma^2$ . The risk of  $Y/(n+2)$  is a constant under the loss  $(a - \sigma^2)^2/\sigma^4$ . We now apply Theorem 4 to show that  $Y/(n+2)$  is admissible. Note that the joint p.d.f. of  $X_i$ 's is of the form  $c(\theta)e^{\theta T(x)}$  with  $\theta = -n/(4\sigma^2)$ ,  $c(\theta) = (-2\theta/n)^{n/2}$ ,  $T(X) = 2Y/n$ ,  $\theta_- = -\infty$  and  $\theta_+ = 0$ . By Theorem 4,  $T_{\lambda,\gamma} = (T + \gamma\lambda)/(1 + \lambda)$  is admissible under the squared error loss if, for some  $c > 0$ ,  $\int_{-\infty}^{-c} e^{-\gamma\lambda\theta} \left(\frac{-2\theta}{n}\right)^{-n\lambda/2} d\theta = \int_0^c e^{\gamma\lambda\theta} \theta^{-n\lambda/2} d\theta = \infty$ . This means  $T_{\lambda,\gamma}$  is admissible if  $\gamma = 0$  and  $\lambda = 2/n$ , or if  $\gamma > 0$  and  $\lambda \geq 2/n$ . In particular,  $2Y/(n+2)$  is admissible for estimating  $\mathbb{E}(T) = 2\mathbb{E}(Y)/n = 2\sigma^2$ , under the squared error loss. It is easy to see that  $Y/(n+2)$  is then an admissible estimator of  $\sigma^2$  under the squared error loss and the loss  $(a - \sigma^2)^2/\sigma^4$ . Hence  $Y/(n+2)$  is minimax under the loss  $(a - \sigma^2)^2/\sigma^4$ .

#### 4.4 Simultaneous estimation and shrinkage estimators

**Definition 1 (Simultaneous estimation):** Estimation of a  $p$ -vector  $\vartheta$  of parameters (functions of  $\theta$ ) under the decision theory approach.

**Remark 1 (Difference from estimating  $\vartheta$  component-by-component):** A single loss function  $L(\vartheta, a)$ , instead of  $p$  loss functions.

**Definition 2 (Squared error loss):** A natural generalization of the squared error loss is  $L(\theta, a) = \|a - \theta\|^2 = \sum_{i=1}^p (a_i - \theta_i)^2$ .

**Definition 3 (James-Stein estimator):** We start with the simple case where  $X$  is from  $\mathcal{N}_p(\theta, I_p)$  with an unknown  $\theta \in \mathbb{R}^p$ . James and Stein proposed the following class of estimators of  $\theta$  having smaller risks than  $X$  when the squared error loss is used and  $p \geq 3$ :  $\delta_c = X - \frac{p-2}{\|X-c\|^2}(X-c)$ , where  $c \in \mathbb{R}^p$  is fixed and the choice of  $c$  is discussed later.

**Definition 4 (Extended James-Stein estimators):** For the purpose of generalizing the results to more complicated situations, we consider the following extension of the James-Stein estimator:  $\delta_{c,r} = X - \frac{r(p-2)}{\|X-c\|^2}(X-c)$ , where  $c \in \mathbb{R}^p$  and  $r \in \mathbb{R}$  are known.

**Motivation 1 (Shrink the observation toward a given point  $c$ ):** Suppose it were thought a priori likely, though not certain, that  $\theta = c$ . Then we might first test a hypothesis  $H_0 : \theta = c$  and estimate  $\theta$  by  $c$  if  $H_0$  is accepted and by  $X$  otherwise. The best rejection region has the form  $\|X - c\|^2 > t$  for some constant  $t > 0$  so that we might estimate  $\theta$  by  $I_{(t,\infty)}(\|X - c\|^2)X + [1 - I_{(t,\infty)}(\|X - c\|^2)]c$ .  $\delta_{c,r}$  is a smoothed version of this estimator, since, for some function  $\psi$ ,  $\delta_{c,r} = \psi(\|X - c\|^2)X + [1 - \psi(\|X - c\|^2)]c$ . Any estimator having this form is called a shrinkage estimator.

**Motivation 2 (Empirical Bayes estimator):** A Bayes estimator of  $\theta$  is of the form  $\delta = (1 - B)X + Bc$ , where  $c$  is the prior mean of  $\theta$  and  $B$  involves prior variances.  $1 - B$  is “estimated” by  $\psi(\|X - c\|^2)$ .  $\delta_{c,r}$  can be viewed as an empirical Bayes estimator.

**Theorem 1 (Risks of shrinkage estimators):** Suppose that  $X$  is from  $\mathcal{N}_p(\theta, I_p)$  with  $p \geq 3$ . Then,

under the squared error loss, the risks of the following shrinkage estimators of  $\theta$ ,  $\delta_{c,r} = X - \frac{r(p-2)}{\|X-c\|^2}(X-c)$ , where  $c \in \mathbb{R}^p$  and  $r \in \mathbb{R}$  are known, are given by  $R_{\delta_{c,r}}(\theta) = p - (2r - r^2)(p-2)^2\mathbb{E}(\|X-c\|^{-2})$ .

**Remark 2:** The risk of  $\delta_{c,r}$  is smaller than  $p$ , the risk of  $X$  for every value of  $\theta$  when  $p \geq 3$  and  $0 < r < 2$ .  $\delta = \delta_{c,1}$  is better than any  $\delta_{c,r}$  with  $r \neq 1$ .

**Remark 3 (The improvement):** To see that  $\delta_c$  may have a substantial improvement over  $X$  in terms of risks, consider the special case where  $\theta = c$ . Since  $\|X-c\|^2$  has the chi-square distribution  $\chi_p^2$  when  $\theta = c$ ,  $\mathbb{E}\|X-c\|^{-2} = (p-2)^{-1}$  and  $R_{\delta_{c,1}}(\theta) = p - (2r - r^2)(p-1)^2\mathbb{E}(\|X-c\|^{-2}) = 2$ . The ratio  $R_X(\theta)/R_{\delta_c}(\theta)$  equals  $p/2$  when  $\theta = c$  and can be substantially larger than 1 near  $\theta = c$  when  $p$  is large.

**Remark 4 (Minimaxity and admissibility of  $\delta_c$ ):** Since  $X$  is minimax,  $\delta_{c,r}$  is minimax provided that  $p \geq 3$  and  $0 < r < 2$ . Unfortunately, the James-Stein estimator  $\delta_c$  with any  $c$  is also inadmissible. It is dominated by  $\delta_c^+ = X - \min\{1, \frac{p-2}{\|X-c\|^2}\}(X-c)$ . This estimator, however, is still inadmissible. Although neither the James-Stein estimator  $\delta_c$  nor  $\delta_c^+$  is admissible, it is found that no substantial improvements over  $\delta_c^+$  are possible.

**Definition 5 (Extension of Theorem 1 to  $\text{Var}(X) = \sigma^2 D$ ):** Consider the case where  $\text{Var}(X) = \sigma^2 D$  with an unknown  $\sigma^2 > 0$  and a known positive definite matrix  $D$ . If  $\sigma^2$  is known, then an extended James-Stein estimator is  $\tilde{\delta}_{c,r} = X - \frac{(p-2)r\sigma^2}{\|D^{-1}(X-c)\|^2}D^{-1}(X-c)$ . Under the squared error loss, the risk of  $\tilde{\delta}_{c,r}$  is  $\sigma^2[\text{tr}(D) - (2r - r^2)(p-2)^2\mathbb{E}(\|D^{-1}(X-c)\|^{-2})]$ . When  $\sigma^2$  is unknown, we assume that there exists a statistic  $S_0^2$  such that  $S_0^2$  is independent of  $X$  and  $S_0^2/\sigma^2$  has the chi-square distribution  $\chi_m^2$ . Replacing  $r\sigma^2$  in  $\tilde{\delta}_{c,r}$  by  $\hat{\sigma}^2 = tS_0^2$  with a constant  $t > 0$  leads to the following extended James-Stein estimator:  $\tilde{\delta}_c = X - \frac{(p-2)\hat{\sigma}^2}{\|D^{-1}(X-c)\|^2}D^{-1}(X-c)$ . From the risk formula for  $\tilde{\delta}_{c,r}$  and the independence of  $\hat{\sigma}^2$  and  $X$ , the risk of  $\tilde{\delta}_c$  is  $R_{\tilde{\delta}_c}(\theta) = \sigma^2\{\text{tr}(D) - [2tm - t^2m(m+2)](p-2)^2\sigma^2\kappa(\theta)\}$ , where  $\theta = (\theta, \sigma^2)$  and  $\kappa(\theta) = \mathbb{E}(\|D^{-1}(X-c)\|^{-2})$ . Replacing  $t$  by  $1/(m+2)$  leads to  $R_{\tilde{\delta}_c}(\theta) = \sigma^2[\text{tr}(D) - m(m+2)^{-1}(p-2)^2\sigma^2\mathbb{E}(\|D^{-1}(X-c)\|^{-2})]$ , which is smaller than  $\sigma^2\text{tr}(D)$  (the risk of  $X$ ) for any fixed  $\theta, p \geq 3$ .

**Example 1:** Consider the general linear model  $X = Z\beta + \epsilon$  with  $\epsilon \sim \mathcal{N}_p(0, \sigma^2)$ ,  $p \geq 3$ , and a full rank  $Z$ . Consider the estimation of  $\theta = \beta$  under the squared error loss. The LSE  $\hat{\beta}$  is from  $\mathcal{N}(\beta, \sigma^2 D)$  with a known matrix  $D = (Z^T Z)^{-1}$ ,  $S_0^2 = \text{SSR}$  is independent of  $\hat{\beta}$ ,  $S_0^2/\sigma^2$  has the chi-square distribution  $\chi_{n-p}^2$ . Hence, from the previous discussion, the risk of the shrinkage estimator  $\hat{\beta} - \frac{(p-2)\hat{\sigma}^2}{\|Z^T Z(\hat{\beta}-c)\|^2}Z^T Z(\hat{\beta}-c)$  is smaller than that of  $\hat{\beta}$  for any  $\beta$  and  $\sigma^2$ , where  $c \in \mathbb{R}^p$  is fixed and  $\hat{\sigma}^2 = \text{SSR}/(n-p+2)$ .

**Definition 6 (Other shrinkage estimators):** From the previous discussion, the James-Stein estimators improve  $X$  substantially when we shrink the observations toward a vector  $c$  that is near  $\theta = \mathbb{E}X$ . One may consider shrinking the observations toward the mean of the observations rather than a given point; that is, one may obtain a shrinkage estimator by replacing  $c$  in  $\delta_{c,r}$  by  $\bar{X}J_p$ , where  $\bar{X} = p^{-1}\sum_{i=1}^p X_i$  and  $J_p$  is the  $p$ -vectors of ones. However, we have to replace the factor  $p-2$  in  $\delta_{c,r}$  by  $p-3$ . This leads to shrinkage estimators  $X - \frac{p-3}{\|X-\bar{X}J_p\|^2}(X-\bar{X}J_p)$  and  $X - \frac{(p-3)\hat{\sigma}^2}{\|D^{-1}(X-\bar{X}J_p)\|^2}D^{-1}(X-\bar{X}J_p)$ . These estimators are better than  $X$  (and, hence, are minimax) when  $p \geq 4$ , under the squared error loss.

**Remark 5:** The idea of shrinkage has been used in problems with high dimensions, e.g. LASSO.