高等数学 A I 习题课讲义

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1 第 1 次习题课

1.1 Questions

1. $f(x) = |x\sin^3 x|e^{\cos x}$. Bounded? Monotonic? Even? Odd?
2. Show that $f(x) = x - [x]$ is bounded and periodic.
3. $f(x) = \begin{cases} x^2 & x \le 0 \\ \cos x + \sin x & x > 0 \end{cases}$. Calculate $f(-x)$.
4. $f(x) = e^{x^2}$ and $f \circ \phi = 1 - 3x$ and $\phi(x) \ge 0$, find $\phi(x)$ and its domain.
5. Show that $\lim_{n\to\infty} \frac{4n^2}{n^2-n} = 4$.
6. $q > 1$, show that $\lim \frac{n^k}{q^n} = 0$.
7. Calculate $\lim_{n\to\infty} \frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \cdots + \frac{n}{n^2+n+n}$.
8. Let $x_1 > 0$ and $x_{n+1} = \frac{3(1+x_n)}{3+x_n}$. Calculate $\lim_{n\to\infty} x_m$.
9. Show that $\lim n^{1/n} = 1$.
10. Calculate $\lim_{n\to\infty} \sqrt[n]{2^n + 3^n}$.
11. Show that $\lim \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \exists$.
12. $\sum_{n=1}^{\infty} b_n = \infty, \frac{a_n}{b_n} \to 0, a_n, b_n > 0$, show that $\frac{\sum_{n=1}^{\infty} a_n}{\sum_{n=1}^{\infty} b_n} = 0$.
13. (Stolz) $0 < b_n \uparrow +\infty, a_n > 0$, if $(a_n - a_{n-1})/(b_n - b_{n-1}) \to L$ then $a_n/b_n \to L$.

1.2 Solutions

1. Note that $f(2k\pi + \frac{\pi}{4}) = \frac{\sqrt{2}}{4}(2k\pi + \frac{\pi}{4})e^{\frac{1}{\sqrt{2}}} \to \infty$, so $f(x)$ is not bounded. And $f(k\pi) \equiv 0$ and $f(x) \ne 0$, otherwise, so $f(x)$ is not monotonous. Obviously it is odd.
2. It is relatively trivial to show that $f(x)$ has the period 1 and $|f(x)| \le 1$ holds for all $x \in \mathbb{R}$.
3. $f(-x) = \begin{cases} x^2 & x \ge 0 \\ \cos x - \sin x & x < 0 \end{cases}$.
4. $f(\phi) = e^{\phi^2} = 1 - 3x \Rightarrow \phi = \sqrt{\log(1-3x)}$ and its domain is $\log(1-3x) \ge 0 \Leftrightarrow x \le 0$.
5. $|\frac{4n^2}{n^2-n} - 4| = \frac{4}{n-1}$, when $n \ge \frac{4}{\epsilon} + 1$, the difference is smaller than ϵ.
6. Note that $q^n = (1+q-1)^n \ge C_n^{k+1}(q-1)^{k+1} = a_{k+1}n^{k+1} + a_k n^k + \cdots + a_0$ is a higher-order cumulant than n^k.
7. Use squeeze theorem. $(*) \ge \frac{\sum_{i=1}^n i}{(n+1)^2} = \frac{n(n+1)}{2(n+1)^2} \to \frac{1}{2}$, $(*) \le \frac{\sum_{i=1}^n i}{n^2+n} = \frac{n(n+1)}{2(n^+n)} \to \frac{1}{2}$.
8. x_n is bound to be monotonous in this kind of problem. First we assume the limit exists and calculate the answer: let $n \to \infty$, we have $a = \frac{3(1+a)}{3+a} \Rightarrow a = \sqrt{3}$. Then usingrecurrence formula, it is relatively simple to show by induction that if $0 < x_1 < \sqrt{3}$ then $0 < x_n < x_{n+1} < \sqrt{3}$ if $x_1 > \sqrt{3}$ then $x_n > x_{n+1} > \sqrt{3}$. This means the limit $\lim x_n$ exists. Let $n \to \infty$ in both two sides, and we will get the answer.
9. $n^{1/n} > (n+1)^{1/(n+1)} \Leftrightarrow n > (1+1/n)^n$ hold for $n \ge 3$, which means a_n is monotonically decreasing. And we have proved that $n^{1/n} < 1 + \epsilon \Leftrightarrow n < (1+\epsilon)^n$ hold for large n. Use squeeze theorem.
10. $3 \le \sqrt[n]{2^n + 3^n} \le \sqrt[n]{2 \times 3^n} \to 3$.
11. 单调上升性显然. 由于 $\frac{1}{n^2} < \frac{1}{(n-1)n}$, 从而有上界 2.
12. Use truncation. $\forall \epsilon > 0, \exists m, \forall M > m, |a_M/b_M| < \epsilon/2$. Then $\sum a_n / \sum b_n = \sum_{i=1^m} a_i / \sum b_n + \sum_{i=m+1}^n a_i / \sum b_n :=$ $I_1 + I_2$. When n is large enough, $I_1 < \epsilon/2$; $I_2 < \epsilon$ hold for all n. Thus when n is large enough, $I_1 + I_2 < \epsilon$.
13. 用定义. $(L-\epsilon)(b_n - b_{n-1}) \le (a_n - a_{n-1}) \le (L+\epsilon)(b_n - b_{n-1}), \forall n > N \Rightarrow (L-\epsilon)(b_n - b_N) \le a_n - a_N \le (L+\epsilon)(b_n - b_N) \Rightarrow L - \epsilon < \frac{a_n - a_N}{b_n - b_N} < L + \epsilon$. Then $|\frac{a_n - a_N}{b_n - b_N} - \frac{a_n}{b_n}| \le |\frac{(a_n - a_N)b_N}{(b_n - b_N)b_n} + \frac{a_N b_N}{(b_n - b_N)b_n} + \frac{a_N}{b_n - b_N}| \le (L+\epsilon)\frac{b_N}{b_n} + \frac{a_N b_N}{(b_n - b_N)b_n} + \frac{a_N}{b_n - b_N}| \to 0$ when n is large enough.

1.3 Supplements (not required!)

作为一个已经学习数学这么多年的北京大学练习生, 我相信你一定关心过下面这个问题: 可导函数和连续函数之间差多少? 事实上, 我们有以下定义和结论:

(1) Open set: We call a set $A \in \mathbb{R}$ open, iff $\forall x \in A$, there exists $\delta > 0$, such that $(x-\delta, x+\delta) \subset A$.

(2) Closed set: We call a set $B \in \mathbb{R}$ closed, iff its complementary set is open.

(3) Define $f^{-1}(A) = \{x : f(x) \in A\}$, where f is a function.

(4) One can show that a function f is continuous iff $\forall$ open set $A \in \mathbb{R}$, $f^{-1}(A)$ is open.

(5) We call $x \in A$ is an interior point of the set A iff $\exists \delta > 0$, such that $(x-\delta, x+\delta) \subset A$.

(6) We call a set A is countable iff there exists a one-to-one map from A to $\mathbb{N}$ or $|A| < \infty$, where $|A| =$ the number of elements in A.

(7) We call x is a limit point of the set A, iff there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset A$ such that $x_i \to x$. Use A' to denote the set made up of all the limit points of A.

(8) We call $\bar{A} = A \cup A'$ is the closure of A.

(9) Then, 对于所有 $[a,b]$ 上的连续函数, 至少存在一点可导的函数构成的集合是无处稠密集的可列并 (第一纲集). 这里, 无处稠密集是指其闭包不存在内点的集合, 连续函数之间的度量定义为 $\rho_{[a,b]}(f,g) = \max_{x\in[a,b]} |f-g|$.

(10) Baire 纲集定理: 闭集 B_n 无内点, 则 $\cup_n B_n$ 也无内点. 由此容易知道第一纲集是没有内点的.

2 第 2 次习题课

2.1 Questions

1. Show that $\lim_{x\to 2} \frac{1}{x-1} = 1$.
2. Calculate $\lim_{x\to 0} x^2 \sin \frac{1}{x}$.
3. Calculate $\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$.
4. Calculate $\lim_{x\to +\infty} x(\sqrt{x^2+1} - x)$.
5. Calculate $\lim_{x\to +\infty} \cos\sqrt{x+1} - \cos\sqrt{x}$.
6. Calculate $\lim_{x\to 0} \frac{\cos x - \cos 3x}{x^2}$.
7. Calculate $\lim_{x\to +\infty} (\frac{x^2+1}{x^2-2})^{x^2}$.
8. 设数列 $a_n \to 0$ and $\lim |a_{n+1}/a_n| = a$. Show that $a \le 1$.
9. Let $a_n = \sum_{k=1}^{n} (\sqrt{1+\frac{k}{n^2}} - 1)$, calculate $\lim a_n$.
10. $\lim \frac{\sum a_n}{n} \exists$, show that $a_n/n \to 0$.
11. Show that $(n!)^{1/n^2} = 1$.
12. $a_1 = b, a_2 = c, a_n = \frac{a_{n-1}+a_{n-2}}{2}$, calculate $\lim a_n$.
13. Calculate $\lim_{x\to +\infty} (\frac{1}{x}\frac{a^x-1}{a-1})^{1/x}$.
14. $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, 且 $|f(x)| \le \sin x$. Show that $|a_1 + 2a_2 + \cdots + na_n| \le 1$.
15. $f(x)$ is bounded in the interval $(-\delta, \delta)$. $\exists a > 1, b > 1$ such that $f(ax) = bf(x)$. Show that $f(x) \to 0$ as $x \to 0$.

2.2 Solutions

1. Trivial.
2. Note that $|x^2 \sin \frac{1}{x}| \le |x^2| \to 0$.
3. $\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^3 + h^3}{h} \to 3x^2$.
4. $\lim_{x\to +\infty} x(\sqrt{x^2+1} - x) = \lim_{x\to +\infty} \frac{x}{x+\sqrt{x^2+1}} = \lim_{x\to +\infty} \frac{1}{1+\sqrt{1+\frac{1}{x^2}}} \to 2$.
5. $|\cos\sqrt{x+1} - \cos\sqrt{x}| = |2\sin\frac{\sqrt{x+1}+\sqrt{x}}{2} \sin\frac{\sqrt{x+1}-\sqrt{x}}{2}| \le |\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}| \le \frac{\sqrt{x+1}-\sqrt{x}}{2} \to 0$.

6. $\frac{\cos x-\cos 3x}{x^2}=\frac{2\sin 2x\sin x}{x^2}\sim\frac{2\times 2x\times x}{x^2}=4$.
7. 原式 $=[(1+\frac{3}{x^2-2})^{\frac{x^2-2}{3}}]^{\frac{3x^2}{x^2-2}}\to e^3$.
8. If $a>1$, then $\exists N,\forall n>N,|a_{n+1}/a_n|>(1+a)/2$, then $|a_n|>|a_N|(\frac{1+a}{2})^{n-N}\Rightarrow|a_n|\to\infty$.
9. 分子有理化后利用夹逼原理知答案是 1/4.
10. $a_n/n=\sum a_n/n-\frac{n-1}{n}\sum a_{n-1}/(n-1)\to 0$.
11. 使用夹逼定理知 $1\le(n!)^{1/n^2}\le n^{1/n}\to 1$. (PLUS: Stirling: $n!\sim(\frac{n}{e})^n\sqrt{2\pi n}$).
12. It is easy to show that $a_n-a_{n-1}=(-1)^n\frac{a_1-a_0}{2^{n-1}}$, so $a_n-a_0=(a_1-a_0)[1+(-0.5)+\cdots+(-0.5)^{n-1}+(-0.5)^{n-1}]\to\frac{2}{3}(a_1-a_0)$.
13. We have shown $(a-1)^{1/x}\to 1,(1/x)^{1/x}\to 1$, thus the original limit is equivalent to $\lim(a^x-1)^{1/x}$. Thus if $a>1$, $\lim=a$, if $0<a<1$, $\lim=1$.
14. $|f(x)/\sin x|\le 1$ and let $x\to 0$.
15. $x\in(-\delta,\delta),|f(x)|<M$; $x\in(-\delta/a,\delta/a),|f(x)|=\frac{1}{b}|f(ax)|\le\frac{M}{b}\Rightarrow x\in(-\delta/a^n,\delta/a^n),|f(x)|\le\frac{M}{b^n}\Rightarrow f(x)\to 0$ as $x\to 0$.

2.3 Supplements (not required!)

闭区间套定理: $a_n\uparrow,b_n\downarrow,0<b_n-a_n\to 0$, then $\exists$ a unique point $x\in\cap_n[a_n,b_n]$.
Proof: Let $x=\lim a_n=\lim b_n$.
有限覆盖定理: Assume $\{I_\lambda\}_{\lambda\in\Lambda}$ is a group of open sets (might be uncountable). If $[a,b]\subset\cup_{\lambda\in\Lambda}I_\lambda$, then $\exists I_1,\cdots,I_m\in\{I_\lambda\}$ such that $[a,b]\subset\cup_{i=1}^m I_i$.
Proof. If the claim is wrong, i.e. there does not exist a finite sub-covering, then for $[a,(a+b)/2]$ and $[(a+b)/2,b]$, 至少有一个区间不存在有限子覆盖, 这样一直切半, 由闭区间套, 必然夹出一个点 x. 由于这是开覆盖, 因此存在开集 O_x 使得 $x\in O_x$. 从而由极限知这个开区间迟早会覆盖前面的从某项开始的闭区间列, 这与假设 (不存在有限覆盖) 矛盾.
聚点原理: $|a_n|<M$, then $\exists\{n_k\}_{k=1}^\infty\subset\mathbb{N}$, such that $a_{n_k}\to a$ as $k\to\infty$. (Bounded sequence must have some subsequence which converges)
我们给出几种证明方法:
(1) 取 M 使得 $\forall n,|x_n|\le M$, 取 $a_1=-M,b_1=M$. 对 $[a_n,b_n]$ 多次迭代, 每次找到 $\frac{a_n+b_n}{2}$, 这个点将当前区间划分为两个子区间. 两个子区间中必然至少有一个含有无穷项. 任取其中一个含有无穷项的区间作为 $[a_{n+1},b_{n+1}]$. 由闭区间套定理, 最终 a_n,b_n 有相同的极限 ξ, 同时 x_n 中有无穷项与 ξ 任意接近. 选取 $x_{n_k}\in[a_i,b_i]$, 则 $x_{n_k}\to\xi$.
(2) 如果不存在这样的子列, 那么 $\forall x\in[a,b]$, $\exists\delta>0$ such that $|(x-\delta,x+\delta)\cap\{x_i\}_{i=1}^n|\le 1$. 这样构造出的开区间集合覆盖了 $[a,b]$, 由有限覆盖定理, 必然存在有限个开区间覆盖整个区间. 而由假设, 对于取出的每个开区间中至多只有原序列中的一个点, 由于开区间的数量为有限个, 可以得出原序列长度也是有限的, 这显然不成立.
柯西收敛: $\forall\epsilon>0,\exists N,\forall n,m>N,|a_n-a_m|<\epsilon$. Equivalent to the previously-defined limit.
Proof. 取 $\epsilon=1$ 以及满足条件的 N, 那么 $\max_{i=1}^N\{|x_i|+1\}$ 给出了整个序列的上下界. 即 $\{x_n\}$ 有界. 取它的一个收敛子列 x_{n_k}, 并记这个极限为 ξ. 从而 $|x_n-\xi|\le|x_{n_k}-\xi|+|x_n-x_{n_k}|\to 0$.
一个结论: 数列收敛到 a 当且仅当其任意子列都收敛到 a.

3 第 3 次习题课

3.1 Questions

1. Calculate $\lim\frac{(2n-1)!!}{(2n)!!}$.
2. $\{x_n\}$ converges and $\{y_n\}$ diverges, show that $\{x_n+y_n\}$ diverges. $\{x_n\}$ and $\{y_n\}$ diverges, is $\{x_n+y_n\}$ or $\{x_ny_n\}$ bound to diverge? If $\{x_ny_n\}$ is infinitesimal, is $\{x_n\}$ or $\{y_n\}$ bound to be infinitesimal?
3. 求极限. $\sqrt{2},\sqrt{2\sqrt{2}},\sqrt{2\sqrt{2\sqrt{2}}},\cdots$.

4 (not required). $0 \le x_{n+m} \le x_n + x_m$. Show that $\lim \frac{x_n}{n} \exists$. (This lemma is useful in large deviation theory).
5. $\lim a_n = a$. 求证 $\lim \frac{\sum_{i=1}^n p_i a_{n+1-i}}{\sum_{i=1}^n p_i} = a$. 其中 $p_k > 0$ and $\lim \frac{p_n}{\sum p_n} = 0$.
6. 求极限. $\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{x}, \lim_{x\to\infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}, \lim_{x\to\infty} (\sin \frac{1}{x} + \cos \frac{1}{x})^x$.
7. $\lim_{x\to 0} f(x)\exists \Leftrightarrow \lim_{x\to 0} f(x^3)\exists$, but $\lim_{x\to 0} f(x)\exists \not\Leftarrow \lim_{x\to 0} f(x^2)$.
8. 存在 $f(x)$ 在 $\mathbb{R}$ 上处处不连续, 但 $|f(x)|$ 处处连续.
9. $a_1, a_2, \cdots, a_p > 0$, calculate $\lim_{x\to 0+0} (\frac{\sum_{i=1}^p a_i^x}{p})^{1/x}$.
10. 求极限. $\lim_{x\to 0+0} \frac{x \log(1+3x)}{(1-\cos 2\sqrt{x})^2}, \lim_{x\to 0} \frac{2^x - 3^x}{3^x - 4^x}$.
11. 举出一个函数处处不连续, 定义域 $[0, 1]$ 但是值域为区间.
12. $f(x) \in C[a, b]$, $|f(x)|$ 单调, 则 $f(x)$ 单调.
13 (not required). $|f(x) - f(y)| \le k|x - y|$. $0 < k < 1$. Show that $kx - f(x)$ 单调上升 and $\exists c, f(c) = c$.
14. $f(x) \in C[a, b]$, $\forall x, \exists y$, such that $|f(y)| \le \frac{1}{2}|f(x)|$. Show that $\exists \xi$, such that $f(\xi) = 0$.
15 (not required). $f(x)$ 在 $[a, b]$ 上只有第一类间断点, 证明 $f(x)$ 有界.
16. 设 $f(x) \in C[a, b], f(a)f(b) < 0$. Show that $\forall n = 1, 2, \cdots, \exists\{\xi_i\}_{i=1}^n \subset [a, b], \xi_i \ne \xi_j$ such that $\sum_{i=1}^n e^{f(\xi_i)} = n$.
17. 非负函数 $f \in C[0, 1], f(0) = f(1) = 0$. Show that $\forall a \in (0, 1), \exists x_0 \in [0, 1]$ such that $x_0 + a \in [0, 1]$ and $f(x_0) = f(x_0 + a)$. 如果去掉非负条件还对吗?
18. $f_n(x) = x^n + x$. Show that (1) $\forall n$ $f_n(x) = 1$ 在 [0.5,1] 中有且仅有一个根 c_n. (2) find $\lim c_n$.
19. 不等于常数的连续周期函数一定有最小正周期. 如果把连续性去掉结论如何?
20 (not required). f 在 $[a, b]$ 内处处有极限. 证明 $\forall \epsilon > 0$, 在 $[a, b]$ 中使得 $|\lim_{t\to x} f(t) - f(x)| > \epsilon$ 的点至多有有限个. (2) $f(x)$ 至多有可列个间断点.

3.2 Solutions

1. $\frac{(2n-1)!!}{(2n)!!} < \frac{(2n)!!}{(2n+1)!!}$ (use $\frac{i}{i+1} < \frac{i+1}{i+2}$), then $x_n^2 < \frac{1}{2n+1}$ which means $x_n \to 0$.
2. If $\{x_n\}$ and $\{x_n + y_n\}$ converge then $\{x_n + y_n - x_n = y_n\}$ converges. $x_n = (-1)^n, y_n = (-1)^n$, $\{x_n\}$ and $\{y_n\}$ diverge but $\{x_n + y_n\}$, $\{x_n y_n\}$ converge. $x_{2k} = 1, x_{2k-1} = \frac{1}{2k-1}, y_{2k} = \frac{1}{2k}, y_{2k-1} = 1$. $\{x_n\}$ or $\{y_n\}$ are not infinitesimal but $\{x_n y_n\}$ is infinitesimal.
3. $\lim x_n = \sqrt{2 \lim x_{n-1}} \Rightarrow \lim x_n = 2$.
4. 考虑 $\{\frac{x_n}{n}\}$ 的下确界 α. $\exists n$ such that $x_n/n < a + \epsilon$. Let $\max_{i=1}^n\{x_i\} = M$. Then $\frac{x_m}{m} \le \frac{x_n}{m} + \frac{x_{m-n}}{n} \le \frac{2x_n}{m} + \frac{x_{m-2n}}{m}$ (assume $m = kn + b$) $\le \cdots \le \frac{kx_n}{m} + \frac{x_b}{m} \le \frac{kx_n}{kn+b} + \frac{M}{m} \le \frac{x_n}{n} + \frac{M}{m}$. Choose enough large m such that $\frac{M}{m} < \epsilon$. Then $\frac{x_m}{m} < a + 2\epsilon$.
5. WLOG let $a = 0$. Let $\sup_n\{a_n\} = M$. $\forall \epsilon > 0, \exists N_1, \forall n \ge N_1$, s.t. $|a_n| < \epsilon$; $\exists N_2, \forall n \ge N_2$, s.t. $p_n/\sum_n p_n < \epsilon/N_1$. Let $n > N_1 + N_2$, $|\sum_{i=1}^n a_i p_{n+1-i}/\sum_{i=1}^n p_i| \le |\sum_{i=1}^{n-N_1} p_i a_{n+1-i}/\sum_{i=1}^n p_i| + |\sum_{i=n-N_1+1}^n p_i a_{n+1-i}/\sum_{i=1}^n p_i| < \epsilon + \frac{\epsilon}{N_1} \times N_1 \times M = (M + 1)\epsilon$.
6. $(e^{ax} - e^{bx})/x = a \cdot \frac{e^{ax}-1}{ax} + b\frac{1-e^{bx}}{b} \to a - b, \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \frac{\sqrt{1+\sqrt{1/x}}}{\sqrt{1+\sqrt{1/x+\sqrt{1/x^3}}}+1} \to 0.5, (\sin 1/x + \cos 1/x)^x = [(1 + \cos 1/x + \sin 1/x - 1)^{\frac{1}{\cos 1/x + \sin 1/x - 1}}]^{x \cos 1/x + \sin 1/x - 1} = (1/x = t) = e^{(\cos t + \sin t - 1)/t} = e^1$.
7. Trivial.
8. $f(x) = 1_{\mathbb{Q}} - 1_{\mathbb{R}\backslash\mathbb{Q}}$.
9. $(\frac{\sum_{i=1}^p a_i^x}{p})^{1/x} = [1 + \frac{\sum_{i=1}^p (a_i^x - 1)}{p}]^{1/x} = \{[1 + \frac{\sum_{i=1}^p (a_i^x-1)}{p}]^{\frac{p}{\sum_{i=1}^p (a_i^x - 1)}}\}^{\frac{\sum_{i=1}^p (a_i^x-1)}{px}} \to e^{\frac{\sum_{i=1}^p \log a_i}{p}} = (a_1 a_2 \cdots a_p)^{1/p}$.
10. $\lim_{x\to 0+0} \frac{x \log(1+3x)}{(1-\cos 2\sqrt{x})^2} \sim \frac{x \cdot 3x}{(2x)^2} = \frac{3}{4}, \lim_{x\to 0} \frac{2^x - 3^x}{3^x - 4^x} = \frac{(2/3)^x - 1}{1-(4/3)^x} \sim \frac{x \log(2/3)}{-x \log(4/3)} = \frac{\log 3 - \log 2}{\log 4 - \log 3}$.
11. $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ x + 0.5 & x \in [0, 0.5] \& x \in \mathbb{R}\backslash\mathbb{Q} \\ x - 0.5 & x \in [0.5, 1] \& x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.
12. 只需注意到如果 $f(x_0) = 0$, then for any $x \in [a, x_0]$, $f(x) = 0$.

13. 第一问是定义, 第二问用 Cauchy 收敛准则, 证明 $x, f(x), f(f(x)), \cdots$ 是柯西列.
14. If there does not exists ξ such that $f(\xi)=0$. Then WLOG let $f(x)>0$. Let $x_0=\arg\min f(x)$. Then there does not exist some y such that $|f(y)| \le \frac{1}{2} f(x)$. This is contradictory.
15. 第一类间断点 $\Rightarrow$ 每一点都有一个邻域有界 $\Rightarrow$ 所有邻域构成开覆盖, 必有有限子覆盖, 有限个有界总能找到最大的那个.
16. $\exists \eta>0$, such that $f([a,b]) \supset [-\eta,\eta]$, which further means $e^{f([a,b])} \supset [e^{-\eta}, e^{\eta}] \supset [1-\epsilon, 1+\epsilon]$($\exists$ a relatively small $\epsilon>0$). Thus when n is odd, choose $e^{f(\xi_1)}=1, e^{f(\xi_2)}=1-\epsilon/2, e^{f(\xi_3)}=1+\epsilon/2, e^{f(\xi_4)}=1-\epsilon/3, e^{f(\xi_5)}=1+\epsilon/3, \cdots$. When n is even, choose $e^{f(\xi_1)}=1-\epsilon/2, e^{f(\xi_2)}=1+\epsilon/2, e^{f(\xi_3)}=1-\epsilon/3, e^{f(\xi_4)}=1+\epsilon/3, \cdots$.
17. Let $g(x)=f(x+a)-f(x)$. $g(0)\ge 0, g(1-a)\le 0$, 用介值定理. 去掉非负条件不对, 比如说 $f(x)=\sin(2\pi x), a=0.7$.
18. (1) Note that $f_n \uparrow \in [0.5,1]$ 且 $f(0.5)<1, f(1)>1$, 使用介值定理. (2) 由于 $\forall \epsilon, \exists N$, such that $\forall n>N, (1-\epsilon)^n + 1-\epsilon<1$. 由于 $f(1-\epsilon)<1=f(c_n)$ and $f_n \uparrow \Rightarrow c_n>1-\epsilon$. 由极限定义知 $c_n \to 1$.
19. 反证法. Assume $f(a)\ne f(b)$, 考虑正周期序列 $T_n \to 0$, then $(b-a)\div T_n = S_n \cdots m_n$, 其中 $0 \le m_n < T_n \to 0$. Thus $a+S_nT_n \to b$, $f(a)=f(a+S_nT_n) \to f(b)$ (连续性) $\Rightarrow f(a)=f(b)$. 矛盾. 把连续性去掉则结论不对, 比如说 Dirichlet 函数.
20. (1) 如果集合有无穷多个元素那一定有聚点 (有界序列必有收敛子列). 从而 $x_n \to x$. 考虑 y_n 使得 $|y_n-x_n|<1/n$, 且 $|f(y_n)-f(x_n)|>\epsilon$(这是集合的定义). 那么 $y_n \to x$, f 在 x 的极限何在?(极限存在当且仅当任意趋于其的数列极限均相等, 而这里 $\lim f(y_n)$ 显然与 $\lim f(x_n)$ 不同). (2) 记 (1) 中集合为 A_ϵ. 注意到间断点集合可以写成 $\cup A_{1/n}$. 可列个有限元素集合的并元素一定是可列个的.

3.3 Supplements (not required!)

有界性定理: $f(x)\in C[a,b]$, then $f(x)$ is bounded.
Proof. If not bounded, then choose x_n such that $f(x_n)\to\infty$, then there exists $\{x_{n_k}\}\subset\{x_n\}$ which converges to some x. This means $f(x)=\infty$, which is contradictory.
最值定理: $f(x)\in C[a,b]$, then $\arg\max f(x) \exists$.
Proof. 找一个数列 $\{x_n\}$ 使得 $f(x_n)\to \max f(x)$. 利用有界数列必有收敛子列和 $f(x)$ 的连续性.
介值定理: $f(x_1)>0, f(x_2)<0, f(x)\in C[x_1,x_2], \exists x_0$ such that $f(x_0)=0$.
Proof. Use Lebesgue method. Let $x_0=\sup\{x: f(x)>0\}$. 利用连续性知如果 $f(x_0)>0$ 则 x_0 不是上界, 如果 $f(x_0)<0$ 则有更好的上确界.

4 第 4 次习题课

4.1 Questions

1. $f(x)\in C(\mathbb{R})$, $\lim_{x\to\infty} f(x)=+\infty$. Show that $\arg\min_{x\in\mathbb{R}} f(x)\exists$.
2. Show that $\cos x = 1/x$ has infinite positive roots.
3. $f(x)\in C[a,b], x_1, x_2, \cdots, x_n \in [a,b]$. Show that $\exists \xi\in[a,b]$ such that $f(\xi)=\frac{1}{n}\sum_{i=1}^n f(x_i)$.
4. $f(x)=|x|^{1/4}+|x|^{1/2}-0.5\cos x$. How many roots in $\mathbb{R}$?
5. $f(x)\in C[0,2], f(0)=f(2)$, show that $\exists x_1, x_2\in[0,2]$ such that $|x_1-x_2|=1$ and $f(x_1)=f(x_2)=$.
6. $f(x)=\lim_{n\to\infty}\frac{x^{n+2}-x^{-n}}{x^n+x^{-n-1}}$. Discuss its continuity.
7. $f(x)\in C(\mathbb{R}), f(x+y)=f(x)+f(y)$. Calculate $f(x)$.
8. $f(x)$ 连续, $|f(x)|$ 连续否?
9. $f(x)\in C[0,1], 0\le f(x)\le 1$, show that $\exists t\in[0,1]$ such that $f(t)=t$.
10. (not required) $f(x)\uparrow\in C[0,1], 0\le f(x)\le 1$, show the same proposition.
11. $f(x)$ 在 $x=3$ 连续, $\lim_{x\to 3}\frac{f(x)}{x-3}=2$. 求 $f'(3)$.

12. $f(x)\in D(\{x_0\})$, calculate $\lim_{x\to 0}\frac{f(x_0+x)-f(x_0-x)}{f(x)}$.
13. 证明奇函数导数是偶函数, 偶函数导数是奇函数.
14. 求导数 $y=\sqrt[3]{2+3x^3}, y=\arcsin\frac{1}{x^2}, y=\log(\arctan 5x)+\log(1-x), y=e^{\sin^2 x}+\sqrt{\cos x}2^{\sqrt{\cos x}}$.
15. $f(x)=x|x(x-2)|$, 求 $f'(x)$.
16. $f(x), x\in[-1,1], x\le f(x)\le x^2+x$, show that $f'(0)=1$.
17. 求导数. $e^{xy}=3x^2y, \arctan y/x=\log\sqrt{x^2+y^2}$.
18. 求导数. $f(x)^{g(x)}, x^{x^x}$.
19. 求 n 阶导数. $\frac{x^n}{1-x}, \sin^4 x+\cos^4 x$.
20. 求 0 处的 n 阶导数. $\arcsin^2 x$.
21. 求极限. $\lim_{x\to+\infty}\sqrt{x^2+x+1}-x, \lim_{n\to\infty}n(\sqrt[n^2]{n}-1), \lim_{x\to 0}(1+2x)^{\frac{(x+1)^2}{x}}$.
22. $f([a,b])\subset[a,b], |f(x)-f(y)|\le|x-y|$, $x_{n+1}=\frac{1}{2}(x_n+f(x_n))$, 证明 $\forall x_1\in[a,b]$, 则 x_n 收敛.

4.2 Solutions

1. It is easy to show that $\exists X>0, \forall|x|>X, f(x)>f(0)$. Then $\arg\min_{x\in[-X,X]}f(x)=\arg\min_{x\in\mathbb{R}}f(x)$, 由最值定理知存在性.
2. Let $f(x)=\cos x-1/x$, then $f(2k\pi)>0, f(2k\pi+\frac{\pi}{2})<0$, 由介质定理立得.
3. Note that $\min f(x)\le\frac{1}{n}\sum_{i=1}^n f(x_i)\le\max f(x)$. 使用介质定理.
4. Note that $f(x)$ is even. Then $\forall x>1$, $f(x)>0$. And $f(x)$ is monotonically increasing in $[0,1]$. $f(0)<0, f(1)>0\Rightarrow$ one root. So in $\mathbb{R}$ two roots.
5. Let $g(x)=f(x+1)-f(x)$. Then $g(0)g(1)\le 0\Rightarrow\exists x\in[0,1]$ such that $g(x)=0$.
6. $f(x)=\begin{cases}x^2 & |x|>1\\ -x & 0<|x|<1\end{cases}$.
7. $f(n)=f(1)+f(n-1)=2f(1)+f(n-2)=\cdots=nf(1), f(1)=f(1/n)+f((n-1)/n)=2f(2/n)+f((n-2)/n)=\cdots=nf(1/n)\Rightarrow f(m/n)=mf(1/n)=m/n\times f(1)$. 有理数点满足 $f(x)=xf(1)$, 无理数点用有理数逼近就可以了.
8. Note that $||f(x)|-|f(y)||\le|f(x)-f(y)|$. 因此连续.
9. Let $g(t)=f(t)-t$, $g(0)\ge 0, g(1)\le 1$, 利用介质定理.
10. Use Lebesgue method. Let $x_0=\sup_x\{f(x)>x\}$. Show that $f(x_0)=x_0$. If $f(x_0)>x_0$ then $\forall x_1, x_0<x_1<f(x_0)$, $f(x_1)\ge f(x_0)>x_1$. This means x_0 is not an upper bound. If $f(x_0)<x_0$ then for all $x_1, f(x_0)<x_1<x_0$, $f(x_1)\le f(x_0)<x_1$. This also means x_0 is not an upper bound. Thus $f(x_0)=x_0$.
11. $f(3)=f(x)/(x-3)\times(x-3)\sim 2\times(x-3)=0$ as $x\to 3$. Thus $f'(3)=\lim\frac{f(x)-f(3)}{x-3}=2$.
12. $\frac{f(x_0+x)-f(x_0-x)}{x}=\frac{f(x_0+x)-f(x_0)}{x}+\frac{f(x_0)-f(x_0-x)}{x}\to 2f'(x_0)$.
13. 直接用定义.
14. $y'=\frac{3x^2\sqrt[3]{2+3x^3}}{2+3x^3}, y'=\frac{-2}{x\sqrt{x^4-1}}, y'=\frac{5}{\arctan 5x\times(1+25x^2)}+\frac{1}{x-1}, y'=e^{\sin^2 x}\sin 2x-\frac{\sin x}{2\sqrt{\cos x}}2^{\sqrt{\cos x}}(1+\sqrt{\cos x}\log 2)$.
15. $f'(x)=\begin{cases}3x^2-4x & x<0 \text{ or } x>2\\ 4x-3x^2 & 0\le x<2\\ \nexists & x=2\end{cases}$.
16. 用导数定义 + 夹逼原理.
17. 两边同时求导数, $y'=\frac{y(2-xy)}{x(xy-1)}, y'=\frac{x+y}{x-y}$.
18. 方法都是写成指数函数, $e^{g\log f}, e^{e^{x\log x}\log x}$. 结果是 $f^g(g'\log f+\frac{f}{g}f'), x^{x^x}(x^x(1+\log x)\log x+x^{x-1})$.
19. $\frac{x^n}{1-x}=\frac{x^n-x^{n-1}+x^{n-1}-x^{n-2}+\cdots+x-1+1}{1-x}=-(x^{n-1}+\cdots+x+1)+\frac{1}{1-x}$, 因此 n 阶导数是 $\frac{n!}{(1-x)^{n+1}}$. 第二个用倍角公式写出来是 $1-0.5\sin^2 2x$. $y'=-\sin 4x$. 由课上已知结论知是 $y^{(n)}=-4^{n-1}\sin(4x+\frac{n-1}{2}\pi)$.

20. $f'(x)=2\arcsin x/\sqrt{1-x^2}$, 从而 $(1-x^2)f'(x)^2=4f(x)$. 两边求导 $-2xf'(x)^2+2(1-x^2)f'(x)f''(x)=4f'(x)\Rightarrow -xf'(x)+(1-x^2)f''(x)=2$. 两边求 $n-2$ 次导数, 并带入 $x=0$, 利用 Leibniz 公式知道 $f^{(n)}(0)=(n-2)^2f^{(n-2)}(0)$. 然后再手动把 $f'(0),f''(0)$ 算出来用递推就可以了.
21. $\sqrt{x^2+x+1}-x=\frac{x+1}{\sqrt{x^2+x+1}+x}=\frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1}\to 0.5, n(\sqrt[n^2]{n}-1)=n(e^{\log n/n^2}-1)\sim n\log n/n^2\to 0, (1+2x)^{\frac{(x+1)^2}{x}}=[(1+2x)^{1/(2x)}]^{2(x+1)^2}\to e^2$.
22. 回忆: 这种题一定是单调数列. 容易验证数列是良定义的, 即不会跑出区间 $[a,b]$ 外. 如果 $x_n\geq x_{n-1}$, 有 $x_{n+1}=\frac{1}{2}(f(x_n)+x_n)$ (利用 $f(x_n)-f(x_{n-1})\geq x_{n-1}-x_n$) $\geq\frac{1}{2}(f(x_{n-1})+x_{n-1})=x_n$. 从而如果 $x_2\geq x_1$, 则这成为单调上升有界数列, 必收敛. 同理若 $x_{n-1}\geq x_n$ 也可以推出 $x_n\geq x_{n+1}$.

4.3 Supplements (not required!)

See https://wqgcx.github.io/courses/analysis1.pdf.

5 第 5 次习题课

5.1 Questions

Hint: If you find it difficult to calculate $\lim_{x\to a}\frac{f(x)}{g(x)}$ but simple to calculate $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$, then by L'Hospital, $\lim_{x\to a}\frac{f(x)}{g(x)}=L$.
1. 求出闭区间 $[-1,1]$ 上的一元函数 $f(x)=x^{\frac{2}{3}}-(x^2-1)^{\frac{1}{3}}$ 达到最小值的所有 $[-1,1]$ 上的点.
2. 考虑函数 $f(x)=\begin{cases}x^m\sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$, 其中 m 为正整数. 在 $x\neq 0$ 处, 求 $f'(x)$ 和 $f''(x)$. 求 m 满足的条件, 使得 $f(x)$ 有连续的二阶导函数.
3. 设 $f(x)=\begin{cases}\frac{\log(1+x)}{x}+\frac{x}{2} & x>0\\ a & x=0\\ \frac{\sin bx}{x}+cx & x<0\end{cases}$ 在 $x=0$ 处可导, 确定常数 a,b,c, 的值 (You may want to use the hint mentioned above).
4. $y=e^{-x^2}$, 求 $y^{(4)}|_{x=0}$.
5. $y=\frac{x}{2}\sqrt{a^2-x^2}+\arccos\frac{x}{a}$, 求 $\frac{dy}{dx}$.
6. $y^2\tan(x+y)-\sin(x-y)=0$, 求 $\frac{dy}{dx}$.
7. $y=x^{a^a}+a^{x^a}+a^{a^x}$, 求 $\frac{dy}{dx}$.
8. 求函数 $f(x)=x^{\arcsin x}(0<x<1)$ 的导函数 $f'(x)$.
9. 求函数 $f(x)=\arctan x$ 在 $x=0$ 点的 3 阶导数 $f'''(0)$.
10. 设 $f(x)=\frac{1}{x^2-4}$, 求 $f^{(n)}(x), n\in\mathbb{N}$.
11. 求方程 $y^2+2\log y=x^4$ 所确定的函数 $y=f(x)$ 的二阶导数.
12. 判断下列结论是否正确.
(1) 设 $f(x)$ 在 x_0 处可导, 且 $f'(x_0)>0$, 那么
(1.1) $f(x)$ 在 x_0 点一定连续.
(1.2) $f(x)$ 在 x_0 点的某个邻域内一定连续.
(1.3) $f(x)$ 在 x_0 点的某个邻域内一定单调上升.
(2) $f(x)$ 在 x_0 点二阶可导, 那么
(2.1) $f(x)$ 在 x_0 点一定连续.
(2.2) $f(x)$ 在 x_0 的某个邻域内一定连续.
13. 设 $f(x)=e^{x(x-1)\cdots(x-2021)}$, 求 $f'(2021)$.

14. 设 $x = \arcsin\frac{t}{\sqrt{1+t^2}}, y = \arccos\frac{1}{\sqrt{1+t^2}}$, 求 $\frac{dy}{dx}$.
15. 设 $y = \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} - \frac{1}{2\sqrt{2}}\arctan\frac{\sqrt{2}x}{x^2-1}$, 求 $\frac{dy}{dx}$ 并化简.
16. 求积分 $\int\frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)}dx$.
17. 求积分 $\int\frac{2x^2+x+5}{x^4-x^2-6}dx$.
18. 求积分 $\int\frac{\cos^3 x}{\sin x+\cos x}dx$.
19. 求积分 $\int\frac{3+5x}{\sqrt{4x^2-4x+5}}dx$ (Hint: $\int\frac{1}{\sqrt{x^2+a^2}}dx = \log|x+\sqrt{x^2+a^2}|+C$).
20. 设 $y = f(x) = x^3, x = g(t) = t^2, y = f(g(t)) = t^6, \Delta t = 0.1, \Delta x = g(1+0.1) - g(1) = 0.21$.
(1) 当把 t 作为自变量时, 函数 $y = f(g(t))$ 的二阶微分记为 $d_t^2 y$, 函数 $x = g(t)$ 的一阶微分记为 $d_t x$. 计算出: 当 $t = 1, \Delta t = 0.1$ 时, 函数 $y = f(g(t))$ 的二阶微分 $d_t^2 y|_{t=1,\Delta t=0.1}$ 和函数 $x = g(t)$ 的一阶微分 $d_t x|_{t=1,\Delta t=0.1}$.
(2) 当把 x 作为自变量时, 函数 $y = f(x)$ 的二阶微分记为 $d_x^2 y$, x(看作 x 的函数) 的一阶微分记为 $d_x x$. 计算出: 当 $x = 1, \Delta x = 0.21$ 时, 函数 $y = f(x)$ 的二阶微分 $d_x^2 y|_{x=1,\Delta x=0.21}$ 和函数 x(看作 x 的函数) 的一阶微分 $d_x x|_{x=1,\Delta x=0.21}$.
(3) $\frac{d_t^2 y}{(d_t x)^2}|_{t=1,\Delta t=0.1}$ 与 $\frac{d_x^2 y}{(d_x x)^2}|_{x=1,\Delta x=0.21}$ 相等吗?
21. 求极限 $\lim_{x\to 0+0} x^x$.
22. 求极限 $\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{x+\tan x}$.
23. 求极限 $\lim_{n\to\infty}\cos\frac{a}{2}\cos\frac{a}{2^2}\cdots\cos\frac{a}{2^n}$ $(a\in(0,1))$.
24. 求极限 $\lim_{x\to+\infty}\left(\frac{\sqrt{1+x^2}}{x}\right)^{x^2}$.
25. 求极限 $\sqrt{n}(\sqrt[n]{n}-1)$.
26. 设 $\lim_{n\to+\infty}(x_n - x_{n-2}) = 0$, 证明 $\lim_{n\to+\infty}\frac{x_n}{n} = 0$.
27. 设 $f(x)\in C[0,1]$, 如果极限 $\lim_{x\to+\infty}\frac{f(0)+f(1/n)+f(2/n)+\cdots+f(1)}{n} = M$, 其中 M 是 $f(x)$ 在 $[0,1]$ 上的最大值, 则 $f(x)\equiv M$.

5.2 Solutions

1. $f(1) = f(-1) = 1, f'(x) = \frac{2}{3x^{1/3}} - \frac{2x}{3(x^2-1)^{2/3}} = \frac{2[(x^2-1)^{2/3}-x^{4/3}]}{3x^{1/3}(x^2-1)^{2/3}} \geq 0 \Rightarrow 0 < x \leq \frac{\sqrt{2}}{2}$ 或者 $-1 < x \leq -\frac{\sqrt{2}}{2}$. 注意到 $f(0) = 1$. 从而达到最小值的点是 $-1, 0, 1$.
2. $f'(x) = -x^{m-2}\cos\frac{1}{x} + mx^{m-1}\sin\frac{1}{x}, f''(x) = -x^{m-4}\sin\frac{1}{x} - (m-2)x^{m-3}\cos\frac{1}{x} - mx^{m-3}\cos\frac{1}{x} - m(m-1)x^{m-2}\sin\frac{1}{x}$. 要使得 $f''(0)$ 存在需要 $f'(0)\exists, f'(x)\to f'(0)$ 且 $\lim_{x\to 0}\frac{f'(x)-f'(0)}{x}\exists \Rightarrow m \geq 4$, 二阶导函数连续性意味着 $f''(x)\to f''(0) \Rightarrow m \geq 5$.
3. 连续性: $f(0+0) = a \Rightarrow a = 1, b = 1$. $f'_+(0) = 0, f'_-(0) = 0 \Rightarrow c = 0$ (需要用 L'Hospital).
4. $y' = -2xe^{-x^2}, y'' = 4x^2e^{-x^2} - 2e^{-x^2}, y''' = -2x(4x^2-2)e^{-x^2} + 8xe^{-x^2}, y'''' = (-24x^2+12)e^{-x^2} + (16x^4-24x^2)e^{-x^2}$. 从而 $y''''(0) = 12$.
5. $\frac{dy}{dx} = \frac{\sqrt{a^2-x^2}}{2} - \frac{x^2}{2\sqrt{a^2-x^2}} - \frac{1}{\sqrt{a^2-x^2}}$.
6. 两边求导, $2yy'\tan(x+y) + \frac{y^2}{\cos^2(x+y)}(y'+1) + (y'-1)\cos(x-y) = 0 \Rightarrow y' = \frac{\cos^2(x+y)\cos(x-y)-y^2}{\cos^2(x+y)\cos(x-y)+y^2+2y\sin(x+y)\cos(x+y)}$.
7. $y' = a^a x^{a^a-1} + a^{x^a+1}x^{a-1}\log a + a^{a^x+x}(\log a)^2$.
8. $f(x) = e^{\arcsin x\log x}, f'(x) = x^{\arcsin x}(\frac{\log x}{\sqrt{1-x^2}} + \frac{\arcsin x}{x})$.
9. $f'(x) = \frac{1}{1+x^2}, f''(x) = -\frac{2x}{(1+x^2)^2}, f'''(x) = -\frac{2(1+x^2)^2-8x^2(1+x^2)}{(1+x^2)^4} \Rightarrow f'''(0) = -2$.
10. $\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{1}{4}(\frac{1}{x-2} - \frac{1}{x+2}) \Rightarrow f^{(n)}(x) = \frac{(-1)^n n!}{4}(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x+2)^{n+1}})$.
11. 两边求导, $2yy' + 2\frac{y'}{y} = 4x^3 \Rightarrow y^2y' + y' = 2x^3y$, 再求一次, $2y(y')^2 + y^2y'' + y'' = 6x^2y + 2x^3y'$, 利用 $y' = \frac{2x^3y}{y^2+1}$, 得到 $y'' = \frac{6x^2y}{y^2+1} + \frac{4x^6y}{(y^2+1)^2} - \frac{8x^6y^3}{(y^2+1)^3}$.
12. (1.1) 可导一定连续. (1.2)(1.3) 不一定, $f(x) = \begin{cases} x & x\in\mathbb{Q} \\ x+x^2 & x\in\mathbb{R}\backslash\mathbb{Q} \end{cases}$.
(2.1) (2.2) 都是对的.
13. $f'(x) = e^{x(x-1)\cdots(x-2021)}[x(x-1)\cdots(x-2021)]'$, 从而 $f'(2021) = 2021!$.
14. $\frac{dx}{dt} = \frac{1}{1+t^2}, \frac{dy}{dt} = \frac{1}{1+t^2}\mathrm{sgn}(t)$, 注意到 t, x 同号, 因此 $\frac{dy}{dx} = \mathrm{sgn}(x)$.

15. $y'(x)=\frac{1}{x^4+1}$.
16. $\frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)}=\frac{2}{x+1}+\frac{1}{x}+\frac{x}{(x^2+1)}$, 因此积分后是 $2\log|x+1|+\log|x|+\frac{1}{2}\log(x^2+1)+C$.
17. $\frac{2x^2+x+5}{x^4-x^2-6}=\frac{11}{10\sqrt{3}}\frac{1}{x-\sqrt{3}}-\frac{11}{10\sqrt{3}}\frac{1}{x+\sqrt{3}}-\frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{x^2+2}+\frac{1}{5}\frac{x}{x^2-3}-\frac{1}{5}\frac{x}{x^2+2}$, 因此积分后是 $\frac{11}{10\sqrt{3}}\log|x-\sqrt{3}|-\frac{11}{10\sqrt{3}}\log|x+\sqrt{3}|-\frac{1}{5\sqrt{2}}\arctan(\frac{x}{\sqrt{2}})+\frac{1}{10}\log|x^2-3|-\frac{1}{10}\log(x^2+2)+C$.
18. $\frac{\cos^3 x}{\sin x+\cos x}=\frac{1}{\cos^2 x(\tan x+1)(\tan^2 x+1)^2}$, 后面用有理式展开积分.
19. $\frac{3+5x}{\sqrt{(2x-1)^2+4}}=\frac{5(x-0.5)}{2\sqrt{(x-0.5)^2+1}}+\frac{5.5}{2\sqrt{(x-0.5)^2+1}}$, 因此积分后是 $2.5\sqrt{(x-0.5)^2+1}+2.75\log|x-0.5+\sqrt{(x-0.5)^2+1}|+C$.
20. (1) $d_t^2y|_{t=1,\Delta t=0.1}=30t^4(\Delta t)^2|_{t=1,\Delta t=0.01}=0.3, d_tx|_{t=1,\Delta t=0.1}=2t\Delta t|_{t=1,\Delta t=0.1}=0.2$.
(2) $d_x^2y|_{x=1,\Delta x=0.21}=6x(\Delta x)^2=0.2646, d_xx|_{x=1,\Delta x=0.21}=1\Delta_x|_{x=1,\Delta x=0.21}=0.21$.
(3) $(d_tx)^2|_{t=1,\Delta t=0.1}=0.2^2=0.04, (d_xx)^2|_{x=1,\Delta x=0.21}=0.21^2=0.0441, \frac{d_t^2y}{(d_tx)^2}|_{t=1,\Delta t=0.1}=\frac{0.3}{0.04}=7.5\neq 6=\frac{0.2646}{0.0441}=\frac{d_x^2y}{(d_xx)^2}|_{x=1,\Delta x=0.21}$, 因此不相等.
21. $x^x=e^{x\log x}\to e^0=1$.
22. $\sqrt[3]{1+x}-1\sim\frac{1}{3}x, x+\tan x\sim 2x$, 因此极限值为 $\frac{1}{6}$.
23. $\cos\frac{a}{2}\cdots\cos\frac{a}{2^n}\sin\frac{a}{2^n}=\frac{\sin a}{2^n}$, 因此极限值为 $\frac{\sin a}{a}$.
24. $(1+\frac{\sqrt{1+x^2}-x}{x})^{\frac{x}{\sqrt{1+x^2}-x}x(\sqrt{1+x^2}-x)}$, 由于 $x(\sqrt{1+x^2}-x)=\frac{x}{\sqrt{1+x^2}+x}\to 0.5$, 因此原极限为 $\sqrt{e}$.
25. $\sqrt{n}(\sqrt[n]{n}-1)\sim\sqrt{n}(e^{(\log n)/n}-1)\sim\sqrt{n}(\log n)/n\to 0$.
26. 分奇偶讨论.
27. 如果结论不对, 则存在一个长度为 δ 的区间, 在这个区间上 $f(x)\leq M-\epsilon$, 则至少有 $[\delta/n]-1$ 个 $f(i/n)$ 落在这个区间里, 这样一来极限值就会小于等于 $M(1-\delta)+(M-\epsilon)\delta$, 矛盾.

5.3 Supplements (required!)

Riemann-Lebesgue's Lemma: $f\in R[a,b], g\in C[0,T], g(x+T)=g(x), \forall x\in\mathbb{R}$. Then

$$\int_a^b f(x)g(nx)dx=\int_a^b f(x)dx\frac{1}{T}\int_0^T g(x)dx$$

Proof: WLOG $\int_0^T g(x)dx=0$, otherwise let $h(x)=g(x)-\frac{1}{T}\int_0^T g(x)dx$.
Then by the definition of Riemann integral, $\forall\epsilon>0$, there exists some step function $s_\epsilon(x)$, such that $s_\epsilon(x)=\begin{cases}C_1 & a=x_0\leq x<x_1\\ C_2 & x_1\leq x<x_2\\ \cdots & \\ C_m & x_{m-1}\leq x\leq b\end{cases}$ and $\int_a^b|f(x)-s_\epsilon(x)|dx<\epsilon$. Let $M=\sup_{x\in[0,T]}g(x)$. Then
$|\int_a^b f(x)g(nx)dx|=|\int_a^b(f(x)-s_\epsilon(x))g(nx)dx+\int_a^b s_\epsilon(x)g(nx)dx|\leq\int_a^b|f(x)-s_\epsilon(x)|g(nx)dx+|\sum_{i=1}^m C_i\int_{x_{i-1}}^{x_i}g(nx)dx<M\epsilon+\frac{1}{n}\sum_{i=1}^m C_i\int_{nx_{i-1}}^{nx_i}g(x)dx\leq M\epsilon+\frac{1}{n}\sum_{i=1}^m C_iMT$. The last equation uses the fact that $\int_0^T g(x)dx=0$, which further means $\int_c^d g(x)dx=\int_c^{c+T}g(x)dx+\int_{c+T}^{c+2T}g(x)dx+\cdots+\int_{c+kT}^d g(x)dx$ $(c+kT\leq d<c+(k+1)T)=\int_{c+kT}^d g(x)dx\leq MT$. Choose a large enough n, and we can let $\frac{1}{n}\sum_{i=1}^m C_iMT<\epsilon$. #.

6 第 6 次习题课

6.1 Questions

1. 求极限. $\lim_{n\to+\infty}\frac{1}{n}(\sqrt{1+\frac{1}{n}}+\cdots+\sqrt{1+\frac{n}{n}}), \lim_{n\to+\infty}\sum_{k=1}^n\frac{n}{n^2+k^2}, \lim_{n\to\infty}\sum_{i=1}^n(1+\frac{i}{n})\sin\frac{i\pi}{n^2}$.
2. 求导数. $\int_{x^3+1}^{2^{2^x}}\frac{\sin t}{t^4+2}dt, \int_e^{e^{e^x}}\frac{dt}{1+\log t}(x>1)$.
3. 求不定积分. $\int\frac{dx}{\sqrt{a^2-x^2}}, \int\frac{dx}{\sqrt{x^2+a^2}}, \int\frac{dx}{\sqrt{x^2-a^2}}, \int\sqrt{a^2-x^2}dx, \int\sqrt{x^2+a^2}dx, \int\sqrt{x^2-a^2}dx$.
4. 求不定积分. $\int\frac{dx}{x+\sqrt{x^2+x+1}}, \int\sqrt{\tan x}dx, \int\frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx, \int x^2\sqrt{x^2+1}dx, \int\frac{dx}{x(x^3+2)}$.

5. 函数 $f(x)$ 在 $[0,1]$ 上有连续的导函数. 证明: 对于任意 $x\in[0,1]$, 有 $|f(x)|\le\int_0^1|f(t)|dt+\int_0^1|f'(t)|dt$, 并写出取等号条件.
6. $x_1>0$, 对于每个正整数 n, 有 $x_{n+1}=\frac{1}{2}(x_n+\frac{1}{x_n})$. 证明 $\lim x_n$ 存在并求之.
7. 设 $x>0$, 定义 $p(x)=\int_0^x\frac{dt}{\sqrt{t^3+2021}}$, 证明方程 $p(x+1)=p(x)+\sin x$ 有无穷个互不相等的正实数解.
8. 设 $f(x)\in R[a,b], \int_a^b f(x)dx>0$, show that $\exists[\alpha,\beta]\subset[a,b]$ such that $f(x)>0, x\in[\alpha,\beta]$.
9 (not required). $f(x)\in R[a,b]$, 是否有 $[f(x)]$ 可积? 其中 $[\cdot]$ 表示向下取整.
10. 设 $f(x)\in C[0,\pi]$ 满足 $\int_0^\pi f(x)\cos dx=\int_0^\pi f(x)\sin xdx=0$, show that $\exists\alpha,\beta\in(0,\pi),\alpha\ne\beta$, such that $f(\alpha)=f(\beta)=0$.
11. 设 $f(x)\in C[a,b]$ 满足 $\forall\phi(x)\in C[a,b]$, 只要 $\int_a^b\phi(x)dx=0$, 就有 $\int_a^b f(x)\phi(x)dx=0$. 证明 $f(x)\equiv C$.
12. $a_n/n^\alpha\to1$, 求 $\lim_{n\to\infty}\frac{1}{n^{1+\alpha}}(a_1+a_2+\cdots+a_n)$.

6.2 Solutions

6.3 Supplements (not required!)

你也许认为生活中很多函数都是可积的, 但是实际上不对!
See https://wqgcx.github.io/courses/analysis2.pdf 了解更多可积性理论.
测度: 我们把满足以下性质的非负集函数 (定义域是集合, 且函数值非负) 叫做测度: $m(\emptyset)=0$, and for disjoint sets $A_1,A_2,\cdots$, $\sum_i m(A_i)=m(\cup A_i)$.
π 系: 一族集合构成的集合 $\mathscr{P}$, 且满足 $\forall A,B\in\mathscr{P}\Rightarrow A\cap B\in\mathscr{P}$.
半环: $\mathscr{P}$ 是 π 系, 若 $A,B\in\mathscr{P},A\supset B$, 则存在有限个两两不交的集合 $C_1,C_2,\cdots,C_k$ 使得 $A\backslash B=\cup_k C_k$.
外测度: 我们把满足以下性质的非负集函数 (定义域是集合, 且函数值非负) 叫做外测度: $m(\emptyset)=0$, and for sets $A_1,A_2,\cdots$, $\sum_i m(A_i)\ge m(\cup A_i)$.
σ-域: 如果 $\emptyset,\Omega\in\mathscr{P}; A\in\mathscr{P}\Rightarrow A^c\in\mathscr{P}; A_1,A_2,\cdots\in\mathscr{P}\Rightarrow\cup_i A_i\in\mathscr{P}$, 则称 $\mathscr{P}$ 是 σ-域.
容易验证所有形如 $(a,b], a,b\in\mathbb{R}$ 的区间构成的集合是半环, 定义 $m((a,b])=b-a$, 这是半环上的外测度. 由测度扩张定理, 这个外测度可以扩张到 $\sigma(\{(a,b]\})$ 上. 利用 Caratheodory 条件可以完备化. 这个测度成为 Lebesgue 测度.
More on Lebesgue measure: Cantor set, fat Cantor set, Cantor-Lebesgue function, etc.
Lebesgue 定理: $f(x)\in R[a,b]$ iff $m(\{x: f(x)\text{在}x\text{处间断}\})=0$, 其中 m 是 Lebesgue 测度.
Proof. "$\Rightarrow$" 对于区域 $[a,b]$ 的任何分割 $a=x_0<x_1<x_2<\cdots<x_n=b$, 定义 $\omega_i=\sup\{|f(x)-f(y)|,x,y\in[x_{i-1},x_i]\}, \Delta_i=|x_i-x_{i-1}|, \Delta=\max\{\Delta_i\}$. 因而 f 是 Riemann 可积等价于 $\lim_{\Delta\to0}\sum_i\omega_i\Delta_i=0$. 再定义 $\omega_\epsilon(f)=\{x:\lim_{\delta\to0}\sup_{y\in[x-\delta,x+\delta]}|f(y)-f(x)|\ge\epsilon\}$. 先假设如果 f 的不连续点集测度为正, 那么存在 ϵ_0 使得 $\omega_{\epsilon_0}(f)>0$. 对任意分割, 我们有 $\sum_i\omega\Delta_i\ge\sum_{[x_{i-1},x_i]\cap\omega_{\epsilon_0}(f)\ne\emptyset}\omega_i\Delta_i\ge\epsilon_0\sum_{[x_{i-1},x_i]\cap\omega_{\epsilon_0}(f)\ne\emptyset}(x_i-x_{i-1})\ge\epsilon_0 m(\omega_\epsilon(f))$. 这表明 f 不是 Riemann 可积的. 因此如果 f 是 Riemann 可积的, 那么不连续点集必定是零测集.
"$\Leftarrow$" 现在我们假设 $\omega_\epsilon(f)$ 是零测集, 我们证明 f 是 Riemann 可积的. 对任意 $\epsilon>0$, 存在闭集 $A_\epsilon\subset[a,b]$ 使得 f 在 A_ϵ 上连续. 对 $x_0\in A_\epsilon$, 存在 $\delta>0$ 使得 $|f(x)-f(y)|<\epsilon,\forall x,y\in(x_0-\delta,x_0+\delta)$. 由于 A_ϵ 是有界闭集, 因此存在有限个开区间 $(x_l-\frac{1}{2}\delta_l,x_l+\frac{1}{2}\delta_l)$ 覆盖住 A_ϵ. 取 $\delta=\min\{\frac{1}{3}\delta_l\}$. 这表明对于任意 $x_0\in A_\epsilon$, 必定有某个 $x_l\in A_\epsilon$, 使得 $x_0\in(x_l-\frac{1}{2}\delta_l,x_l+\frac{1}{2}\delta_l)$. 这表明 $[x_0-\delta,x_0+\delta]\subset(x_l-\delta_l,x_l+\delta_l)$, 因而有 $|f(x)-f(y)|<\epsilon,\forall x,y\in[x_0-\delta,x_0+\delta]$. 取 $[a,b]$ 分割使得 $\Delta<\frac{1}{2}\delta$. 现在我们来考虑 $\sum_i\omega_i\Delta_i$. 如果区间 $[x_{i-1},x_i]$ 与 A_ϵ 的交集非空, 含有某个点 $y_0\in[x_{i-1},x_i]\cap A_\epsilon$, 那么对于任意 $x,y\in[y_0-\delta,y_0+\delta]$ 都有 $|f(x)-f(y)|<\epsilon$. 注意到 $[x_{i-1},x_i]\subset[y_0-\delta,y_0+\delta]$, 故而 $\omega_i<\epsilon$. 这样我们可以估计 $\sum_i\omega_i\Delta_i=\sum_{[x_{i-1},x_i]\cap A_\epsilon\ne\emptyset}\omega_i\Delta_i+\sum_{[x_{i-1},x_i]\cap A_\epsilon=\emptyset}\omega_i\Delta_i\le\epsilon(b-a)+2Mm([a,b]\backslash A_\epsilon)$. 这里 M 为 f 在 $[a,b]$ 上的上界. 这就表明如果 f 的不连续点零测且 f 有界, 则 f 在 $[a,b]$ 上 Riemann 可积.