

高等数学 A I 习题课讲义

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1 第 1 次习题课

1.1 Questions

1. $f(x) = |x \sin^3 x| e^{\cos x}$. Bounded? Monotonic? Even? Odd?
2. Show that $f(x) = x - [x]$ is bounded and periodic.
3. $f(x) = \begin{cases} x^2 & x \leq 0 \\ \cos x + \sin x & x > 0 \end{cases}$. Calculate $f(-x)$.
4. $f(x) = e^{x^2}$ and $f \circ \phi = 1 - 3x$ and $\phi(x) \geq 0$, find $\phi(x)$ and its domain.
5. Show that $\lim_{n \rightarrow \infty} \frac{4n^2}{n^2 - n} = 4$.
6. $q > 1$, show that $\lim_{n \rightarrow \infty} \frac{n^k}{q^n} = 0$.
7. Calculate $\lim_{n \rightarrow \infty} \frac{1}{n^2 + n + 1} + \frac{2}{n^2 + n + 2} + \cdots + \frac{n}{n^2 + n + n}$.
8. Let $x_1 > 0$ and $x_{n+1} = \frac{3(1+x_n)}{3+x_n}$. Calculate $\lim_{n \rightarrow \infty} x_n$.
9. Show that $\lim_{n \rightarrow \infty} n^{1/n} = 1$.
10. Calculate $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n}$.
11. Show that $\lim_{n \rightarrow \infty} \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} = \frac{\pi^2}{6}$.
12. $\sum_{n=1}^{\infty} b_n = \infty$, $\frac{a_n}{b_n} \rightarrow 0$, $a_n, b_n > 0$, show that $\sum_{n=1}^{\infty} \frac{a_n}{b_n} = 0$.
13. (Stolz) $0 < b_n \uparrow +\infty$, $a_n > 0$, if $(a_n - a_{n-1})/(b_n - b_{n-1}) \rightarrow L$ then $a_n/b_n \rightarrow L$.

1.2 Solutions

1. Note that $f(2k\pi + \frac{\pi}{4}) = \frac{\sqrt{2}}{4}(2k\pi + \frac{\pi}{4})e^{\frac{1}{\sqrt{2}}} \rightarrow \infty$, so $f(x)$ is not bounded. And $f(k\pi) \equiv 0$ and $f(x) \neq 0$, otherwise, so $f(x)$ is not monotonous. Obviously it is odd.
2. It is relatively trivial to show that $f(x)$ has the period 1 and $|f(x)| \leq 1$ holds for all $x \in \mathbb{R}$.
3. $f(-x) = \begin{cases} x^2 & x \geq 0 \\ \cos x - \sin x & x < 0 \end{cases}$.
4. $f(\phi) = e^{\phi^2} = 1 - 3x \Rightarrow \phi = \sqrt{\log(1 - 3x)}$ and its domain is $\log(1 - 3x) \geq 0 \Leftrightarrow x \leq 0$.
5. $|\frac{4n^2}{n^2 - n} - 4| = \frac{4}{n-1}$, when $n \geq \frac{4}{\epsilon} + 1$, the difference is smaller than ϵ .
6. Note that $q^n = (1 + q - 1)^n \geq C_n^{k+1}(q - 1)^{k+1} = a_{k+1}n^{k+1} + a_k n^k + \cdots + a_0$ is a higher-order cumulant than n^k .
7. Use squeeze theorem. $(*) \geq \frac{\sum_{i=1}^n i}{(n+1)^2} = \frac{n(n+1)}{2(n+1)^2} \rightarrow \frac{1}{2}$, $(*) \leq \frac{\sum_{i=1}^n i}{n^2 + n} = \frac{n(n+1)}{2(n^2 + n)} \rightarrow \frac{1}{2}$.
8. x_n is bound to be monotonous in this kind of problem. First we assume the limit exists and calculate the answer: let $n \rightarrow \infty$, we have $a = \frac{3(1+a)}{3+a} \Rightarrow a = \sqrt{3}$. Then using recurrence formula, it is relatively simple to show by induction that if $0 < x_1 < \sqrt{3}$ then $0 < x_n < x_{n+1} < \sqrt{3}$ if $x_1 > \sqrt{3}$ then $x_n > x_{n+1} > \sqrt{3}$. This means the limit $\lim x_n$ exists. Let $n \rightarrow \infty$ in both two sides, and we will get the answer.
9. $n^{1/n} > (n+1)^{1/(n+1)} \Leftrightarrow n > (1 + 1/n)^n$ hold for $n \geq 3$, which means a_n is monotonically decreasing. And we have proved that $n^{1/n} < 1 + \epsilon \Leftrightarrow n < (1 + \epsilon)^n$ hold for large n . Use squeeze theorem.
10. $3 \leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{2 \times 3^n} \rightarrow 3$.
11. 单调上升性显然. 由于 $\frac{1}{n^2} < \frac{1}{(n-1)n}$, 从而有上界 2.
12. Use truncation. $\forall \epsilon > 0, \exists m, \forall M > m, |a_M/b_M| < \epsilon/2$. Then $\sum a_n / \sum b_n = \sum_{i=1}^m a_i / \sum b_n + \sum_{i=m+1}^n a_i / \sum b_n := I_1 + I_2$. When n is large enough, $I_1 < \epsilon/2$; $I_2 < \epsilon$ hold for all n . Thus when n is large enough, $I_1 + I_2 < \epsilon$.
13. 用定义. $(L - \epsilon)(b_n - b_{n-1}) \leq (a_n - a_{n-1}) \leq (L + \epsilon)(b_n - b_{n-1}), \forall n > N \Rightarrow (L - \epsilon)(b_n - b_N) \leq a_n - a_N \leq (L + \epsilon)(b_n - b_N) \Rightarrow L - \epsilon < \frac{a_n - a_N}{b_n - b_N} < L + \epsilon$. Then $|\frac{a_n - a_N}{b_n - b_N} - \frac{a_n}{b_n}| \leq |\frac{(a_n - a_N)b_N}{(b_n - b_N)b_n} + \frac{a_N b_N}{(b_n - b_N)b_n} + \frac{a_N}{b_n - b_N}| \leq (L + \epsilon)\frac{b_N}{b_n} + \frac{a_N b_N}{(b_n - b_N)b_n} + \frac{a_N}{b_n - b_N} \rightarrow 0$ when n is large enough.

1.3 Supplements (not required!)

作为一个已经学习数学这么多年的北京大学练习生, 我相信你一定关心过下面这个问题: 可导函数和连续函数之间差多少? 事实上, 我们有以下定义和结论:

- (1) Open set: We call a set $A \in \mathbb{R}$ open, iff $\forall x \in A$, there exists $\delta > 0$, such that $(x - \delta, x + \delta) \subset A$.
- (2) Closed set: We call a set $B \in \mathbb{R}$ closed, iff its complementary set is open.
- (3) Define $f^{-1}(A) = \{x : f(x) \in A\}$, where f is a function.
- (4) One can show that a function f is continuous iff \forall open set $A \in \mathbb{R}$, $f^{-1}(A)$ is open.
- (5) We call $x \in A$ is an interior point of the set A iff $\exists \delta > 0$, such that $(x - \delta, x + \delta) \subset A$.
- (6) We call a set A is countable iff there exists a one-to-one map from A to \mathbb{N} or $|A| < \infty$, where $|A|$ = the number of elements in A .
- (7) We call x is a limit point of the set A , iff there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset A$ such that $x_i \rightarrow x$. Use A' to denote the set made up of all the limit points of A .
- (8) We call $\bar{A} = A \cup A'$ is the closure of A .
- (9) Then, 对于所有 $[a, b]$ 上的连续函数, 至少存在一点可导的函数构成的集合是无处稠密集的可列并 (第一纲集). 这里, 无处稠密集是指其闭包不存在内点的集合, 连续函数之间的度量定义为 $\rho_{[a,b]}(f, g) = \max_{x \in [a,b]} |f - g|$.
- (10) Baire 纲集定理: 闭集 B_n 无内点, 则 $\cup_n B_n$ 也无内点. 由此容易知道第一纲集是没有内点的.

2 第 2 次习题课

2.1 Questions

1. Show that $\lim_{x \rightarrow 2} \frac{1}{x-1} = 1$.
2. Calculate $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$.
3. Calculate $\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$.
4. Calculate $\lim_{x \rightarrow +\infty} x(\sqrt{x^2 + 1} - x)$.
5. Calculate $\lim_{x \rightarrow +\infty} \cos \sqrt{x+1} - \cos \sqrt{x}$.
6. Calculate $\lim_{x \rightarrow 0} \frac{\cos x - \cos 3x}{x^2}$.
7. Calculate $\lim_{x \rightarrow +\infty} \left(\frac{x^2+1}{x^2-2}\right)^{x^2}$.
8. 设数列 $a_n \rightarrow 0$ 和 $\lim |a_{n+1}/a_n| = a$. Show that $a \leq 1$.
9. Let $a_n = \sum_{k=1}^n (\sqrt{1 + \frac{k}{n^2}} - 1)$, calculate $\lim a_n$.
10. $\lim \frac{\sum a_n}{n} \exists$, show that $a_n/n \rightarrow 0$.
11. Show that $(n!)^{1/n^2} = 1$.
12. $a_1 = b, a_2 = c, a_n = \frac{a_{n-1} + a_{n-2}}{2}$, calculate $\lim a_n$.
13. Calculate $\lim_{x \rightarrow +\infty} \left(\frac{1}{x} \frac{a^x - 1}{a - 1}\right)^{1/x}$.
14. $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, 且 $|f(x)| \leq \sin x$. Show that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.
15. $f(x)$ is bounded in the interval $(-\delta, \delta)$. $\exists a > 1, b > 1$ such that $f(ax) = bf(x)$. Show that $f(x) \rightarrow 0$ as $x \rightarrow 0$.

2.2 Solutions

1. Trivial.
2. Note that $|x^2 \sin \frac{1}{x}| \leq |x^2| \rightarrow 0$.
3. $\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} \rightarrow 3x^2$.
4. $\lim_{x \rightarrow +\infty} x(\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow +\infty} \frac{x}{x + \sqrt{x^2 + 1}} = \lim_{x \rightarrow +\infty} \frac{1}{1 + \sqrt{1 + \frac{1}{x^2}}} \rightarrow 2$.
5. $|\cos \sqrt{x+1} - \cos \sqrt{x}| = |2 \sin \frac{\sqrt{x+1} + \sqrt{x}}{2} \sin \frac{\sqrt{x+1} - \sqrt{x}}{2}| \leq |\sin \frac{\sqrt{x+1} - \sqrt{x}}{2}| \leq \frac{\sqrt{x+1} - \sqrt{x}}{2} \rightarrow 0$.

6. $\frac{\cos x - \cos 3x}{x^2} = \frac{2 \sin 2x \sin x}{x^2} \sim \frac{2 \times 2x \times x}{x^2} = 4$.
7. 原式 $= [(1 + \frac{3}{x^2-2})^{\frac{x^2-2}{3}}]^{\frac{3x^2}{x^2-2}} \rightarrow e^3$.
8. If $a > 1$, then $\exists N, \forall n > N, |a_{n+1}/a_n| > (1+a)/2$, then $|a_n| > |a_N|(\frac{1+a}{2})^{n-N} \Rightarrow |a_n| \rightarrow \infty$.
9. 分子有理化后利用夹逼原理知答案是 $1/4$.
10. $a_n/n = \sum a_n/n - \frac{n-1}{n} \sum a_{n-1}/(n-1) \rightarrow 0$.
11. 使用夹逼定理知 $1 \leq (n!)^{1/n^2} \leq n^{1/n} \rightarrow 1$. (PLUS: Stirling: $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$).
12. It is easy to show that $a_n - a_{n-1} = (-1)^n \frac{a_1 - a_0}{2^{n-1}}$, so $a_n - a_0 = (a_1 - a_0)[1 + (-0.5) + \cdots + (-0.5)^{n-1} + (-0.5)^{n-1}] \rightarrow \frac{2}{3}(a_1 - a_0)$.
13. We have shown $(a-1)^{1/x} \rightarrow 1, (1/x)^{1/x} \rightarrow 1$, thus the original limit is equivalent to $\lim(a^x - 1)^{1/x}$. Thus if $a > 1$, $\lim = a$, if $0 < a < 1$, $\lim = 1$.
14. $|f(x)/\sin x| \leq 1$ and let $x \rightarrow 0$.
15. $x \in (-\delta, \delta), |f(x)| < M; x \in (-\delta/a, \delta/a), |f(x)| = \frac{1}{b}|f(ax)| \leq \frac{M}{b} \Rightarrow x \in (-\delta/a^n, \delta/a^n), |f(x)| \leq \frac{M}{b^n} \Rightarrow f(x) \rightarrow 0$ as $x \rightarrow 0$.

2.3 Supplements (not required!)

闭区间套定理: $a_n \uparrow, b_n \downarrow, 0 < b_n - a_n \rightarrow 0$, then \exists a unique point $x \in \cap_n [a_n, b_n]$.

Proof: Let $x = \lim a_n = \lim b_n$.

有限覆盖定理: Assume $\{I_\lambda\}_{\lambda \in \Lambda}$ is a group of open sets (might be uncountable). If $[a, b] \subset \cup_{\lambda \in \Lambda} I_\lambda$, then $\exists I_1, \dots, I_m \in \{I_\lambda\}$ such that $[a, b] \subset \cup_{i=1}^m I_i$.

Proof. If the claim is wrong, i.e. there does not exist a finite sub-covering, then for $[a, (a+b)/2]$ and $[(a+b)/2, b]$, 至少有一个区间不存在有限子覆盖, 这样一直切半, 由闭区间套, 必然夹出一个点 x . 由于这是开覆盖, 因此存在开集 O_x 使得 $x \in O_x$. 从而由极限知这个开区间迟早会覆盖前面的从某项开始的闭区间列, 这与假设 (不存在有限覆盖) 矛盾.

聚点原理: $|a_n| < M$, then $\exists \{n_k\}_{k=1}^\infty \subset \mathbb{N}$, such that $a_{n_k} \rightarrow a$ as $k \rightarrow \infty$. (Bounded sequence must have some subsequence which converges)

我们给出几种证明方法:

- (1) 取 M 使得 $\forall n, |x_n| \leq M$, 取 $a_1 = -M, b_1 = M$. 对 $[a_n, b_n]$ 多次迭代, 每次找到 $\frac{a_n+b_n}{2}$, 这个点将当前区间划分为两个子区间. 两个子区间中必然至少有一个含有无穷项. 任取其中一个含有无穷项的区间作为 $[a_{n+1}, b_{n+1}]$. 由闭区间套定理, 最终 a_n, b_n 有相同的极限 ξ , 同时 x_n 中有无穷项与 ξ 任意接近. 选取 $x_{n_k} \in [a_i, b_i]$, 则 $x_{n_k} \rightarrow \xi$.
- (2) 如果不存在这样的子列, 那么 $\forall x \in [a, b], \exists \delta > 0$ such that $|(x - \delta, x + \delta) \cap \{x_i\}_{i=1}^n| \leq 1$. 这样构造出的开区间集合覆盖了 $[a, b]$, 由有限覆盖定理, 必然存在有限个开区间覆盖整个区间. 而由假设, 对于取出的每个开区间中至多只有原序列中的一个点, 由于开区间的数量为有限个, 可以得出原序列长度也是有限的, 这显然不成立.

柯西收敛: $\forall \epsilon > 0, \exists N, \forall n, m > N, |a_n - a_m| < \epsilon$. Equivalent to the previously-defined limit.

Proof. 取 $\epsilon = 1$ 以及满足条件的 N , 那么 $\max_{i=1}^N \{|x_i| + 1\}$ 给出了整个序列的上下界. 即 $\{x_n\}$ 有界. 取它的一个收敛子列 x_{n_k} , 并记这个极限为 ξ . 从而 $|x_n - \xi| \leq |x_{n_k} - \xi| + |x_n - x_{n_k}| \rightarrow 0$.

一个结论: 数列收敛到 a 当且仅当其任意子列都收敛到 a .

3 第3次习题课

3.1 Questions

1. Calculate $\lim \frac{(2n-1)!!}{(2n)!!}$.
2. $\{x_n\}$ converges and $\{y_n\}$ diverges, show that $\{x_n + y_n\}$ diverges. $\{x_n\}$ and $\{y_n\}$ diverges, is $\{x_n + y_n\}$ or $\{x_n y_n\}$ bound to diverge? If $\{x_n y_n\}$ is infinitesimal, is $\{x_n\}$ or $\{y_n\}$ bound to be infinitesimal?
3. 求极限. $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$

- 4 (not required). $0 \leq x_{n+m} \leq x_n + x_m$. Show that $\lim_{n \rightarrow \infty} \frac{x_n}{n} \exists$. (This lemma is useful in large deviation theory).
5. $\lim a_n = a$. 求证 $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n p_i a_{n+1-i}}{\sum_{i=1}^n p_i} = a$. 其中 $p_k > 0$ and $\lim_{n \rightarrow \infty} \frac{p_n}{\sum p_n} = 0$.
6. 求极限. $\lim_{x \rightarrow 0} \frac{e^{ax} - e^{bx}}{x}$, $\lim_{x \rightarrow \infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}$, $\lim_{x \rightarrow \infty} (\sin \frac{1}{x} + \cos \frac{1}{x})^x$.
7. $\lim_{x \rightarrow 0} f(x) \exists \Leftrightarrow \lim_{x \rightarrow 0} f(x^3) \exists$, but $\lim_{x \rightarrow 0} f(x) \exists \not\Leftrightarrow \lim_{x \rightarrow 0} f(x^2)$.
8. 存在 $f(x)$ 在 \mathbb{R} 上处处不连续, 但 $|f(x)|$ 处处连续.
9. $a_1, a_2, \dots, a_p > 0$, calculate $\lim_{x \rightarrow 0+0} (\frac{\sum_{i=1}^p a_i^x}{p})^{1/x}$.
10. 求极限. $\lim_{x \rightarrow 0+0} \frac{x \log(1+3x)}{(1-\cos 2\sqrt{x})^2}$, $\lim_{x \rightarrow 0} \frac{2^x - 3^x}{3^x - 4^x}$.
11. 举出一个函数处处不连续, 定义域 $[0, 1]$ 但是值域为区间.
12. $f(x) \in C[a, b]$, $|f(x)|$ 单调, 则 $f(x)$ 单调.
- 13 (not required). $|f(x) - f(y)| \leq k|x - y|$. $0 < k < 1$. Show that $kx - f(x)$ 单调上升 and $\exists c, f(c) = c$.
14. $f(x) \in C[a, b]$, $\forall x, \exists y$, such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that $\exists \xi$, such that $f(\xi) = 0$.
- 15 (not required). $f(x)$ 在 $[a, b]$ 上只有第一类间断点, 证明 $f(x)$ 有界.
16. 设 $f(x) \in C[a, b]$, $f(a)f(b) < 0$. Show that $\forall n = 1, 2, \dots, \exists \{\xi_i\}_{i=1}^n \subset [a, b], \xi_i \neq \xi_j$ such that $\sum_{i=1}^n e^{f(\xi_i)} = n$.
17. 非负函数 $f \in C[0, 1], f(0) = f(1) = 0$. Show that $\forall a \in (0, 1), \exists x_0 \in [0, 1]$ such that $x_0 + a \in [0, 1]$ and $f(x_0) = f(x_0 + a)$. 如果去掉非负条件还对吗?
18. $f_n(x) = x^n + x$. Show that (1) $\forall n, f_n(x) = 1$ 在 $[0.5, 1]$ 中有且仅有一个根 c_n . (2) find $\lim c_n$.
19. 不等于常数的连续周期函数一定有最小正周期. 如果把连续性去掉结论如何?
- 20 (not required). f 在 $[a, b]$ 内处处有极限. 证明 $\forall \epsilon > 0$, 在 $[a, b]$ 中使得 $|\lim_{t \rightarrow x} f(t) - f(x)| > \epsilon$ 的点至多有有限个. (2) $f(x)$ 至多有可列个间断点.

3.2 Solutions

1. $\frac{(2n-1)!!}{(2n)!!} < \frac{(2n)!!}{(2n+1)!!}$ (use $\frac{i}{i+1} < \frac{i+1}{i+2}$), then $x_n^2 < \frac{1}{2n+1}$ which means $x_n \rightarrow 0$.
2. If $\{x_n\}$ and $\{x_n + y_n\}$ converge then $\{x_n + y_n - x_n = y_n\}$ converges. $x_n = (-1)^n, y_n = (-1)^n, \{x_n\}$ and $\{y_n\}$ diverge but $\{x_n + y_n\}, \{x_n y_n\}$ converge. $x_{2k} = 1, x_{2k-1} = \frac{1}{2k-1}, y_{2k} = \frac{1}{2k}, y_{2k-1} = 1$. $\{x_n\}$ or $\{y_n\}$ are not infinitesimal but $\{x_n y_n\}$ is infinitesimal.
3. $\lim x_n = \sqrt{2 \lim x_{n-1}} \Rightarrow \lim x_n = 2$.
4. 考虑 $\{\frac{x_n}{n}\}$ 的下确界 α . $\exists n$ such that $x_n/n < a + \epsilon$. Let $\max_{i=1}^n \{x_i\} = M$. Then $\frac{x_m}{m} \leq \frac{x_n}{m} + \frac{x_{m-n}}{n} \leq \frac{2x_n}{m} + \frac{x_{m-2n}}{m}$ (assume $m = kn + b$) $\leq \dots \leq \frac{kx_n}{m} + \frac{x_b}{m} \leq \frac{kx_n}{kn+b} + \frac{M}{m} \leq \frac{x_n}{n} + \frac{M}{m}$. Choose enough large m such that $\frac{M}{m} < \epsilon$. Then $\frac{x_m}{m} < a + 2\epsilon$.
5. WLOG let $a = 0$. Let $\sup_n \{a_n\} = M$. $\forall \epsilon > 0, \exists N_1, \forall n \geq N_1$, s.t. $|a_n| < \epsilon$; $\exists N_2, \forall n \geq N_2$, s.t. $p_n / \sum p_n < \epsilon / N_1$. Let $n > N_1 + N_2$, $|\sum_{i=1}^n a_i p_{n+1-i} / \sum_{i=1}^n p_i| \leq |\sum_{i=1}^{N_1} p_i a_{n+1-i} / \sum_{i=1}^n p_i| + |\sum_{i=n-N_1+1}^n p_i a_{n+1-i} / \sum_{i=1}^n p_i| < \epsilon + \frac{\epsilon}{N_1} \times N_1 \times M = (M+1)\epsilon$.
6. $(e^{ax} - e^{bx})/x = a \cdot \frac{e^{ax}-1}{ax} + b \frac{1-e^{bx}}{b} \rightarrow a - b, \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}+\sqrt{x}}} = \frac{\sqrt{1+\sqrt{1/x}}}{\sqrt{1+\sqrt{1/x}+\sqrt{1/x^3+1}}} \rightarrow 0.5, (\sin 1/x + \cos 1/x)^x = [(1 + \cos 1/x + \sin 1/x - 1)^{\frac{1}{\cos 1/x + \sin 1/x - 1}}]^{x \cos 1/x + \sin 1/x - 1} = (1/x = t) = e^{(\cos t + \sin t - 1)/t} = e^1$.
7. Trivial.
8. $f(x) = 1_{\mathbb{Q}} - 1_{\mathbb{R} \setminus \mathbb{Q}}$.
9. $(\frac{\sum_{i=1}^p a_i^x}{p})^{1/x} = [1 + \frac{\sum_{i=1}^p (a_i^x - 1)}{p}]^{1/x} = \{[1 + \frac{\sum_{i=1}^p (a_i^x - 1)}{p}]^{\frac{p}{\sum_{i=1}^p (a_i^x - 1)}}\}^{\frac{\sum_{i=1}^p (a_i^x - 1)}{px}} \rightarrow e^{\frac{\sum_{i=1}^p \log a_i}{p}} = (a_1 a_2 \dots a_p)^{1/p}$.
10. $\lim_{x \rightarrow 0+0} \frac{x \log(1+3x)}{(1-\cos 2\sqrt{x})^2} \sim \frac{x \cdot 3x}{(2x)^2} = \frac{3}{4}, \lim_{x \rightarrow 0} \frac{2^x - 3^x}{3^x - 4^x} = \frac{(2/3)^x - 1}{1 - (4/3)^x} \sim \frac{x \log(2/3)}{-x \log(4/3)} = \frac{\log 3 - \log 2}{\log 4 - \log 3}$.
11. $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ x + 0.5 & x \in [0, 0.5] \& x \in \mathbb{R} \setminus \mathbb{Q} \\ x - 0.5 & x \in [0.5, 1] \& x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$
12. 只需注意到如果 $f(x_0) = 0$, then for any $x \in [a, x_0], f(x) = 0$.

13. 第一问是定义, 第二问用 Cauchy 收敛准则, 证明 $x, f(x), f(f(x)), \dots$ 是柯西列.
14. If there does not exist ξ such that $f(\xi) = 0$. Then WLOG let $f(x) > 0$. Let $x_0 = \arg \min f(x)$. Then there does not exist some y such that $|f(y)| \leq \frac{1}{2}f(x)$. This is contradictory.
15. 第一类间断点 \Rightarrow 每一点都有一个邻域有界 \Rightarrow 所有邻域构成开覆盖, 必有有限子覆盖, 有限个有界总能找到最大的那个.
16. $\exists \eta > 0$, such that $f([a, b]) \supset [-\eta, \eta]$, which further means $e^{f([a, b])} \supset [e^{-\eta}, e^{\eta}] \supset [1 - \epsilon, 1 + \epsilon]$ (\exists a relatively small $\epsilon > 0$). Thus when n is odd, choose $e^{f(\xi_1)} = 1, e^{f(\xi_2)} = 1 - \epsilon/2, e^{f(\xi_3)} = 1 + \epsilon/2, e^{f(\xi_4)} = 1 - \epsilon/3, e^{f(\xi_5)} = 1 + \epsilon/3, \dots$. When n is even, choose $e^{f(\xi_1)} = 1 - \epsilon/2, e^{f(\xi_2)} = 1 + \epsilon/2, e^{f(\xi_3)} = 1 - \epsilon/3, e^{f(\xi_4)} = 1 + \epsilon/3, \dots$.
17. Let $g(x) = f(x+a) - f(x)$. $g(0) \geq 0, g(1-a) \leq 0$, 用介值定理. 去掉非负条件不对, 比如说 $f(x) = \sin(2\pi x), a = 0.7$.
18. (1) Note that $f_n \uparrow \in [0.5, 1]$ 且 $f(0.5) < 1, f(1) > 1$, 使用介值定理. (2) 由于 $\forall \epsilon, \exists N$, such that $\forall n > N, (1 - \epsilon)^n + 1 - \epsilon < 1$. 由于 $f(1 - \epsilon) < 1 = f(c_n)$ and $f_n \uparrow \Rightarrow c_n > 1 - \epsilon$. 由极限定义知 $c_n \rightarrow 1$.
19. 反证法. Assume $f(a) \neq f(b)$, 考虑正周期序列 $T_n \rightarrow 0$, then $(b - a) \div T_n = S_n \cdots m_n$, 其中 $0 \leq m_n < T_n \rightarrow 0$. Thus $a + S_n T_n \rightarrow b, f(a) = f(a + S_n T_n) \rightarrow f(b)$ (连续性) $\Rightarrow f(a) = f(b)$. 矛盾. 把连续性去掉则结论不对, 比如说 Dirichlet 函数.
20. (1) 如果集合有无穷多个元素那一定有聚点 (有界序列必有收敛子列). 从而 $x_n \rightarrow x$. 考虑 y_n 使得 $|y_n - x_n| < 1/n$, 且 $|f(y_n) - f(x_n)| > \epsilon$ (这是集合的定义). 那么 $y_n \rightarrow x, f$ 在 x 的极限何在? (极限存在当且仅当任意趋于其的数列极限均相等, 而这里 $\lim f(y_n)$ 显然与 $\lim f(x_n)$ 不同). (2) 记 (1) 中集合为 A_ϵ . 注意到间断点集合可以写成 $\cup A_{1/n}$. 可列个有限元素集合的并元素一定是可列个的.

3.3 Supplements (not required!)

有界性定理: $f(x) \in C[a, b]$, then $f(x)$ is bounded.

Proof. If not bounded, then choose x_n such that $f(x_n) \rightarrow \infty$, then there exists $\{x_{n_k}\} \subset \{x_n\}$ which converges to some x . This means $f(x) = \infty$, which is contradictory.

最值定理: $f(x) \in C[a, b]$, then $\arg \max f(x) \exists$.

Proof. 找一个数列 $\{x_n\}$ 使得 $f(x_n) \rightarrow \max f(x)$. 利用有界数列必有收敛子列和 $f(x)$ 的连续性.

介值定理: $f(x_1) > 0, f(x_2) < 0, f(x) \in C[x_1, x_2], \exists x_0$ such that $f(x_0) = 0$.

Proof. Use Lebesgue method. Let $x_0 = \sup\{x : f(x) > 0\}$. 利用连续性知如果 $f(x_0) > 0$ 则 x_0 不是上界, 如果 $f(x_0) < 0$ 则有更好的上确界.

4 第 4 次习题课

4.1 Questions

- $f(x) \in C(\mathbb{R}), \lim_{x \rightarrow \infty} f(x) = +\infty$. Show that $\arg \min_{x \in \mathbb{R}} f(x) \exists$.
- Show that $\cos x = 1/x$ has infinite positive roots.
- $f(x) \in C[a, b], x_1, x_2, \dots, x_n \in [a, b]$. Show that $\exists \xi \in [a, b]$ such that $f(\xi) = \frac{1}{n} \sum_{i=1}^n f(x_i)$.
- $f(x) = |x|^{1/4} + |x|^{1/2} - 0.5 \cos x$. How many roots in \mathbb{R} ?
- $f(x) \in C[0, 2], f(0) = f(2)$, show that $\exists x_1, x_2 \in [0, 2]$ such that $|x_1 - x_2| = 1$ and $f(x_1) = f(x_2)$.
- $f(x) = \lim_{n \rightarrow \infty} \frac{x^{n+2} - x^{-n}}{x^n + x^{-n-1}}$. Discuss its continuity.
- $f(x) \in C(\mathbb{R}), f(x+y) = f(x) + f(y)$. Calculate $f(x)$.
- $f(x)$ 连续, $|f(x)|$ 连续否?
- $f(x) \in C[0, 1], 0 \leq f(x) \leq 1$, show that $\exists t \in [0, 1]$ such that $f(t) = t$.
- (not required!) $f(x) \uparrow \in C[0, 1], 0 \leq f(x) \leq 1$, show the same proposition.
- $f(x)$ 在 $x = 3$ 连续, $\lim_{x \rightarrow 3} \frac{f(x)}{x-3} = 2$. 求 $f'(3)$.

12. $f(x) \in D(\{x_0\})$, calculate $\lim_{x \rightarrow 0} \frac{f(x_0+x)-f(x_0-x)}{f(x)}$.
13. 证明奇函数导数是偶函数, 偶函数导数是奇函数.
14. 求导数 $y = \sqrt[3]{2+3x^3}$, $y = \arcsin \frac{1}{x^2}$, $y = \log(\arctan 5x) + \log(1-x)$, $y = e^{\sin^2 x} + \sqrt{\cos x} 2^{\sqrt{\cos x}}$.
15. $f(x) = x|x(x-2)|$, 求 $f'(x)$.
16. $f(x), x \in [-1, 1], x \leq f(x) \leq x^2 + x$, show that $f'(0) = 1$.
17. 求导数. $e^{xy} = 3x^2y$, $\arctan y/x = \log \sqrt{x^2 + y^2}$.
18. 求导数. $f(x)^{g(x)}, x^{x^x}$.
19. 求 n 阶导数. $\frac{x^n}{1-x}, \sin^4 x + \cos^4 x$.
20. 求 0 处的 n 阶导数. $\arcsin^2 x$.
21. 求极限. $\lim_{x \rightarrow +\infty} \sqrt{x^2 + x + 1} - x, \lim_{n \rightarrow \infty} n(\sqrt[n]{n} - 1), \lim_{x \rightarrow 0} (1+2x)^{\frac{(x+1)^2}{x}}$.
22. $f([a, b]) \subset [a, b], |f(x) - f(y)| \leq |x - y|, x_{n+1} = \frac{1}{2}(x_n + f(x_n))$, 证明 $\forall x_1 \in [a, b]$, 则 x_n 收敛.

4.2 Solutions

1. It is easy to show that $\exists X > 0, \forall |x| > X, f(x) > f(0)$. Then $\arg \min_{x \in [-X, X]} f(x) = \arg \min_{x \in \mathbb{R}} f(x)$, 由最值定理知存在性.
2. Let $f(x) = \cos x - 1/x$, then $f(2k\pi) > 0, f(2k\pi + \frac{\pi}{2}) < 0$, 由介值定理立得.
3. Note that $\min f(x) \leq \frac{1}{n} \sum_{i=1}^n f(x_i) \leq \max f(x)$. 使用介值定理.
4. Note that $f(x)$ is even. Then $\forall x > 1, f(x) > 0$. And $f(x)$ is monotonically increasing in $[0, 1]$. $f(0) < 0, f(1) > 0 \Rightarrow$ one root. So in \mathbb{R} two roots.
5. Let $g(x) = f(x+1) - f(x)$. Then $g(0)g(1) \leq 0 \Rightarrow \exists x \in [0, 1]$ such that $g(x) = 0$.
6. $f(x) = \begin{cases} x^2 & |x| > 1 \\ -x & 0 < |x| < 1 \end{cases}$.
7. $f(n) = f(1) + f(n-1) = 2f(1) + f(n-2) = \dots = nf(1), f(1) = f(1/n) + f((n-1)/n) = 2f(2/n) + f((n-2)/n) = \dots = nf(1/n) \Rightarrow f(m/n) = mf(1/n) = m/n \times f(1)$. 有理数点满足 $f(x) = xf(1)$, 无理数点用有理数逼近就可以了.
8. Note that $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$. 因此连续.
9. Let $g(t) = f(t) - t, g(0) \geq 0, g(1) \leq 1$, 利用介值定理.
10. Use Lebesgue method. Let $x_0 = \sup_x \{f(x) > x\}$. Show that $f(x_0) = x_0$. If $f(x_0) > x_0$ then $\forall x_1, x_0 < x_1 < f(x_0), f(x_1) \geq f(x_0) > x_1$. This means x_0 is not an upper bound. If $f(x_0) < x_0$ then for all $x_1, f(x_0) < x_1 < x_0, f(x_1) \leq f(x_0) < x_1$. This also means x_0 is not an upper bound. Thus $f(x_0) = x_0$.
11. $f(3) = f(x)/(x-3) \times (x-3) \sim 2 \times (x-3) = 0$ as $x \rightarrow 3$. Thus $f'(3) = \lim_{x \rightarrow 3} \frac{f(x)-f(3)}{x-3} = 2$.
12. $\frac{f(x_0+x)-f(x_0-x)}{x} = \frac{f(x_0+x)-f(x_0)}{x} + \frac{f(x_0)-f(x_0-x)}{x} \rightarrow 2f'(x_0)$.
13. 直接用定义.
14. $y' = \frac{3x^2 \sqrt[3]{2+3x^3}}{2+3x^3}, y' = \frac{-2}{x\sqrt{x^4-1}}, y' = \frac{5}{\arctan 5x \times (1+25x^2)} + \frac{1}{x-1}, y' = e^{\sin^2 x} \sin 2x - \frac{\sin x}{2\sqrt{\cos x}} 2^{\sqrt{\cos x}} (1 + \sqrt{\cos x} \log 2)$.
15. $f'(x) = \begin{cases} 3x^2 - 4x & x < 0 \text{ or } x > 2 \\ 4x - 3x^2 & 0 \leq x < 2 \\ \varnothing & x = 2 \end{cases}$.
16. 用导数定义 + 夹逼原理.
17. 两边同时求导数, $y' = \frac{y(2-xy)}{x(xy-1)}, y' = \frac{x+y}{x-y}$.
18. 方法都是写成指数函数, $e^{g \log f}, e^{e^{x \log x} \log x}$. 结果是 $f^g(g' \log f + \frac{f}{g} f'), x^{x^x}(x^x(1 + \log x) \log x + x^{x-1})$.
19. $\frac{x^n}{1-x} = \frac{x^n - x^{n-1} + x^{n-1} - x^{n-2} + \dots + x - 1 + 1}{1-x} = -(x^{n-1} + \dots + x + 1) + \frac{1}{1-x}$, 因此 n 阶导数是 $\frac{n!}{(1-x)^{n+1}}$. 第二个用倍角公式写出来是 $1 - 0.5 \sin^2 2x. y' = -\sin 4x$. 由课上已知结论知是 $y^{(n)} = -4^{n-1} \sin(4x + \frac{n-1}{2} \pi)$.

20. $f'(x) = 2 \arcsin x / \sqrt{1-x^2}$, 从而 $(1-x^2)f'(x)^2 = 4f(x)$. 两边求导 $-2xf'(x)^2 + 2(1-x^2)f'(x)f''(x) = 4f'(x) \Rightarrow -xf'(x) + (1-x^2)f''(x) = 2$. 两边求 $n-2$ 次导数, 并带入 $x=0$, 利用 Leibniz 公式知道 $f^{(n)}(0) = (n-2)^2 f^{(n-2)}(0)$. 然后再手动把 $f'(0), f''(0)$ 算出来用递推就可以了.

21. $\sqrt{x^2+x+1} - x = \frac{x+1}{\sqrt{x^2+x+1}+x} = \frac{1+1/x}{\sqrt{1+1/x+1/x^2+1}} \rightarrow 0.5, n(\sqrt[n]{n} - 1) = n(e^{\log n/n^2} - 1) \sim n \log n / n^2 \rightarrow 0, (1+2x)^{\frac{(x+1)^2}{x}} = [(1+2x)^{1/(2x)}]^{2(x+1)^2} \rightarrow e^2$.

22. 回忆: 这种题一定是单调数列. 容易验证数列是良定义的, 即不会跑出区间 $[a, b]$ 外. 如果 $x_n \geq x_{n-1}$, 有 $x_{n+1} = \frac{1}{2}(f(x_n) + x_n)$ (利用 $f(x_n) - f(x_{n-1}) \geq x_{n-1} - x_n \geq \frac{1}{2}(f(x_{n-1}) + x_{n-1}) - x_n = x_n - x_{n-1}$). 从而如果 $x_2 \geq x_1$, 则这成为单调上升有界数列, 必收敛. 同理若 $x_{n-1} \geq x_n$ 也可以推出 $x_n \geq x_{n+1}$.

4.3 Supplements (not required!)

See <https://wqgcx.github.io/courses/analysis1.pdf>.

5 第 5 次习题课

5.1 Questions

Hint: If you find it difficult to calculate $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ but simple to calculate $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$, then by L'Hospital, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$.

1. 求出闭区间 $[-1, 1]$ 上的一元函数 $f(x) = x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}}$ 达到最小值的所有 $[-1, 1]$ 上的点.

2. 考虑函数 $f(x) = \begin{cases} x^m \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, 其中 m 为正整数. 在 $x \neq 0$ 处, 求 $f'(x)$ 和 $f''(x)$. 求 m 满足的条件, 使得 $f(x)$ 有连续的二阶导函数.

3. 设 $f(x) = \begin{cases} \frac{\log(1+x)}{x} + \frac{x}{2} & x > 0 \\ a & x = 0 \\ \frac{\sin bx}{x} + cx & x < 0 \end{cases}$ 在 $x=0$ 处可导, 确定常数 a, b, c 的值 (You may want to use the hint mentioned above).

4. $y = e^{-x^2}$, 求 $y^{(4)}|_{x=0}$.

5. $y = \frac{x}{2}\sqrt{a^2 - x^2} + \arccos \frac{x}{a}$, 求 $\frac{dy}{dx}$.

6. $y^2 \tan(x+y) - \sin(x-y) = 0$, 求 $\frac{dy}{dx}$.

7. $y = x^{a^a} + a^{x^a} + a^{a^x}$, 求 $\frac{dy}{dx}$.

8. 求函数 $f(x) = x^{\arcsin x}$ ($0 < x < 1$) 的导函数 $f'(x)$.

9. 求函数 $f(x) = \arctan x$ 在 $x=0$ 点的 3 阶导数 $f'''(0)$.

10. 设 $f(x) = \frac{1}{x^2-4}$, 求 $f^{(n)}(x), n \in \mathbb{N}$.

11. 求方程 $y^2 + 2 \log y = x^4$ 所确定的函数 $y = f(x)$ 的二阶导数.

12. 判断下列结论是否正确.

(1) 设 $f(x)$ 在 x_0 处可导, 且 $f'(x_0) > 0$, 那么

(1.1) $f(x)$ 在 x_0 点一定连续.

(1.2) $f(x)$ 在 x_0 点的某个邻域内一定连续.

(1.3) $f(x)$ 在 x_0 点的某个邻域内一定单调上升.

(2) $f(x)$ 在 x_0 点二阶可导, 那么

(2.1) $f(x)$ 在 x_0 点一定连续.

(2.2) $f(x)$ 在 x_0 的某个邻域内一定连续.

13. 设 $f(x) = e^{x(x-1)\cdots(x-2021)}$, 求 $f'(2021)$.

14. 设 $x = \arcsin \frac{t}{\sqrt{1+t^2}}, y = \arccos \frac{1}{\sqrt{1+t^2}}$, 求 $\frac{dy}{dx}$.
15. 设 $y = \frac{1}{4\sqrt{2}} \log \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} - \frac{1}{2\sqrt{2}} \arctan \frac{\sqrt{2}x}{x^2-1}$, 求 $\frac{dy}{dx}$ 并化简.
16. 求积分 $\int \frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)} dx$.
17. 求积分 $\int \frac{2x^2+x+5}{x^4-x^2-6} dx$.
18. 求积分 $\int \frac{\cos^3 x}{\sin x + \cos x} dx$.
19. 求积分 $\int \frac{3+5x}{\sqrt{4x^2-4x+5}} dx$ (Hint: $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log|x + \sqrt{x^2+a^2}| + C$).
20. 设 $y = f(x) = x^3, x = g(t) = t^2, y = f(g(t)) = t^6, \Delta t = 0.1, \Delta x = g(1+0.1) - g(1) = 0.21$.
 (1) 当把 t 作为自变量时, 函数 $y = f(g(t))$ 的二阶微分记为 $d_t^2 y$, 函数 $x = g(t)$ 的一阶微分记为 $d_t x$. 计算出: 当 $t = 1, \Delta t = 0.1$ 时, 函数 $y = f(g(t))$ 的二阶微分 $d_t^2 y|_{t=1, \Delta t=0.1}$ 和函数 $x = g(t)$ 的一阶微分 $d_t x|_{t=1, \Delta t=0.1}$.
 (2) 当把 x 作为自变量时, 函数 $y = f(x)$ 的二阶微分记为 $d_x^2 y$, x (看作 x 的函数) 的一阶微分记为 $d_x x$. 计算出: 当 $x = 1, \Delta x = 0.21$ 时, 函数 $y = f(x)$ 的二阶微分 $d_x^2 y|_{x=1, \Delta x=0.21}$ 和函数 x (看作 x 的函数) 的一阶微分 $d_x x|_{x=1, \Delta x=0.21}$.
 (3) $\frac{d_t^2 y}{(d_t x)^2}|_{t=1, \Delta t=0.1}$ 与 $\frac{d_x^2 y}{(d_x x)^2}|_{x=1, \Delta x=0.21}$ 相等吗?
21. 求极限 $\lim_{x \rightarrow 0+0} x^x$.
22. 求极限 $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x}-1}{x+\tan x}$.
23. 求极限 $\lim_{n \rightarrow \infty} \cos \frac{a}{2} \cos \frac{a}{2^2} \cdots \cos \frac{a}{2^n} \quad (a \in (0, 1))$.
24. 求极限 $\lim_{x \rightarrow +\infty} \left(\frac{\sqrt{1+x^2}}{x} \right)^{x^2}$.
25. 求极限 $\sqrt[n]{n}(\sqrt[n]{n}-1)$.
26. 设 $\lim_{n \rightarrow +\infty} (x_n - x_{n-2}) = 0$, 证明 $\lim_{n \rightarrow +\infty} \frac{x_n}{n} = 0$.
27. 设 $f(x) \in C[0, 1]$, 如果极限 $\lim_{x \rightarrow +\infty} \frac{f(0)+f(1/n)+f(2/n)+\cdots+f(1)}{n} = M$, 其中 M 是 $f(x)$ 在 $[0, 1]$ 上的最大值, 则 $f(x) \equiv M$.

5.2 Solutions

1. $f(1) = f(-1) = 1, f'(x) = \frac{2}{3x^{1/3}} - \frac{2x}{3(x^2-1)^{2/3}} = \frac{2[(x^2-1)^{2/3}-x^{4/3}]}{3x^{1/3}(x^2-1)^{2/3}} \geq 0 \Rightarrow 0 < x \leq \frac{\sqrt{2}}{2}$ 或者 $-1 < x \leq -\frac{\sqrt{2}}{2}$. 注意到 $f(0) = 1$. 从而达到最小值的点是 $-1, 0, 1$.
2. $f'(x) = -x^{m-2} \cos \frac{1}{x} + mx^{m-1} \sin \frac{1}{x}, f''(x) = -x^{m-4} \sin \frac{1}{x} - (m-2)x^{m-3} \cos \frac{1}{x} - mx^{m-3} \cos \frac{1}{x} - m(m-1)x^{m-2} \sin \frac{1}{x}$.
要使得 $f''(0)$ 存在需要 $f'(0) \exists, f'(x) \rightarrow f'(0)$ 且 $\lim_{x \rightarrow 0} \frac{f'(x)-f'(0)}{x} \exists \Rightarrow m \geq 4$, 二阶导函数连续性意味着 $f''(x) \rightarrow f''(0) \Rightarrow m \geq 5$.
3. 连续性: $f(0+0) = a \Rightarrow a = 1, b = 1. f'_+(0) = 0, f'_-(0) = 0 \Rightarrow c = 0$ (需要用 L'Hospital).
4. $y' = -2xe^{-x^2}, y'' = 4x^2e^{-x^2} - 2e^{-x^2}, y''' = -2x(4x^2-2)e^{-x^2} + 8xe^{-x^2}, y'''' = (-24x^2+12)e^{-x^2} + (16x^4-24x^2)e^{-x^2}$.
从而 $y''''(0) = 12$.
5. $\frac{dy}{dx} = \frac{\sqrt{a^2-x^2}}{2} - \frac{x^2}{2\sqrt{a^2-x^2}} - \frac{1}{\sqrt{a^2-x^2}}$.
6. 两边求导, $2yy' \tan(x+y) + \frac{y^2}{\cos^2(x+y)}(y'+1) + (y'-1) \cos(x-y) = 0 \Rightarrow y' = \frac{\cos^2(x+y) \cos(x-y) - y^2}{\cos^2(x+y) \cos(x-y) + y^2 + 2y \sin(x+y) \cos(x+y)}$.
7. $y' = a^a x^{a^a-1} + a^{a+1} x^{a-1} \log a + a^{a+x} (\log a)^2$.
8. $f(x) = e^{\arcsin x \log x}, f'(x) = x^{\arcsin x} \left(\frac{\log x}{\sqrt{1-x^2}} + \frac{\arcsin x}{x} \right)$.
9. $f'(x) = \frac{1}{1+x^2}, f''(x) = -\frac{2x}{(1+x^2)^2}, f'''(x) = -\frac{2(1+x^2)^2-8x^2(1+x^2)}{(1+x^2)^4} \Rightarrow f'''(0) = -2$.
10. $\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{1}{4} \left(\frac{1}{x-2} - \frac{1}{x+2} \right) \Rightarrow f^{(n)}(x) = \frac{(-1)^n n!}{4} \left(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x+2)^{n+1}} \right)$.
11. 两边求导, $2yy' + 2\frac{y'}{y} = 4x^3 \Rightarrow y^2 y' + y' = 2x^3 y$, 再求一次, $2y(y')^2 + y^2 y'' + y'' = 6x^2 y + 2x^3 y'$, 利用 $y' = \frac{2x^3 y}{y^2+1}$, 得到 $y'' = \frac{6x^2 y}{y^2+1} + \frac{4x^6 y}{(y^2+1)^2} - \frac{8x^6 y^3}{(y^2+1)^3}$.
12. (1.1) 可导一定连续. (1.2)(1.3) 不一定, $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ x+x^2 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.
- (2.1) (2.2) 都是对的.
13. $f'(x) = e^{x(x-1)\cdots(x-2021)} [x(x-1)\cdots(x-2021)]'$, 从而 $f'(2021) = 2021!$.
14. $\frac{dx}{dt} = \frac{1}{1+t^2}, \frac{dy}{dt} = \frac{1}{1+t^2} \operatorname{sgn}(t)$, 注意到 t, x 同号, 因此 $\frac{dy}{dx} = \operatorname{sgn}(x)$.

15. $y'(x) = \frac{1}{x^4+1}$.
16. $\frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)} = \frac{2}{x+1} + \frac{1}{x} + \frac{x}{(x^2+1)}$, 因此积分后是 $2\log|x+1| + \log|x| + \frac{1}{2}\log(x^2+1) + C$.
17. $\frac{2x^2+x+5}{x^4-x^2-6} = \frac{11}{10\sqrt{3}}\frac{1}{x-\sqrt{3}} - \frac{11}{10\sqrt{3}}\frac{1}{x+\sqrt{3}} - \frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{x^2+2} + \frac{1}{5}\frac{x}{x^2-3} - \frac{1}{5}\frac{x}{x^2+2}$, 因此积分后是 $\frac{11}{10\sqrt{3}}\log|x-\sqrt{3}| - \frac{11}{10\sqrt{3}}\log|x+\sqrt{3}| - \frac{1}{5\sqrt{2}}\arctan(\frac{x}{\sqrt{2}}) + \frac{1}{10}\log|x^2-3| - \frac{1}{10}\log(x^2+2) + C$.
18. $\frac{\cos^3 x}{\sin x + \cos x} = \frac{1}{\cos^2 x (\tan x + 1)(\tan^2 x + 1)^2}$, 后面用有理式展开积分.
19. $\frac{3+5x}{\sqrt{(2x-1)^2+4}} = \frac{5(x-0.5)}{2\sqrt{(x-0.5)^2+1}} + \frac{5.5}{2\sqrt{(x-0.5)^2+1}}$, 因此积分后是 $2.5\sqrt{(x-0.5)^2+1} + 2.75\log|x-0.5+\sqrt{(x-0.5)^2+1}| + C$.
20. (1) $d_t^2 y|_{t=1, \Delta t=0.1} = 30t^4(\Delta t)^2|_{t=1, \Delta t=0.1} = 0.3, d_t x|_{t=1, \Delta t=0.1} = 2t\Delta t|_{t=1, \Delta t=0.1} = 0.2$.
 (2) $d_x^2 y|_{x=1, \Delta x=0.21} = 6x(\Delta x)^2 = 0.2646, d_x x|_{x=1, \Delta x=0.21} = 1\Delta x|_{x=1, \Delta x=0.21} = 0.21$.
 (3) $(d_t x)^2|_{t=1, \Delta t=0.1} = 0.2^2 = 0.04, (d_x x)^2|_{x=1, \Delta x=0.21} = 0.21^2 = 0.0441, \frac{d_t^2 y}{(d_t x)^2}|_{t=1, \Delta t=0.1} = \frac{0.3}{0.04} = 7.5 \neq 6 = \frac{0.2646}{0.0441} = \frac{d_x^2 y}{(d_x x)^2}|_{x=1, \Delta x=0.21}$, 因此不相等.
21. $x^x = e^{x \log x} \rightarrow e^0 = 1$.
22. $\sqrt[3]{1+x} - 1 \sim \frac{1}{3}x, x + \tan x \sim 2x$, 因此极限值为 $\frac{1}{6}$.
23. $\cos \frac{a}{2} \cdots \cos \frac{a}{2^n} \sin \frac{a}{2^n} = \frac{\sin a}{2^n}$, 因此极限值为 $\frac{\sin a}{a}$.
24. $(1 + \frac{\sqrt{1+x^2}-x}{x})^{\frac{x}{\sqrt{1+x^2}-x}} x(\sqrt{1+x^2}-x)$, 由于 $x(\sqrt{1+x^2}-x) = \frac{x}{\sqrt{1+x^2}+x} \rightarrow 0.5$, 因此原极限为 \sqrt{e} .
25. $\sqrt[n]{n}(\sqrt[n]{n}-1) \sim \sqrt[n]{n}(e^{(\log n)/n}-1) \sim \sqrt[n]{n}(\log n)/n \rightarrow 0$.
26. 分奇偶讨论.
27. 如果结论不对, 则存在一个长度为 δ 的区间, 在这个区间上 $f(x) \leq M - \epsilon$, 则至少有 $[\delta/n] - 1$ 个 $f(i/n)$ 落在这个区间里, 这样一来极限值就会小于等于 $M(1-\delta) + (M-\epsilon)\delta$, 矛盾.

5.3 Supplements (required!)

Riemann-Lebesgue's Lemma: $f \in R[a, b], g \in C[0, T], g(x+T) = g(x), \forall x \in \mathbb{R}$. Then

$$\int_a^b f(x)g(nx)dx = \int_a^b f(x)dx \frac{1}{T} \int_0^T g(x)dx$$

Proof: WLOG $\int_0^T g(x)dx = 0$, otherwise let $h(x) = g(x) - \frac{1}{T} \int_0^T g(x)dx$.

Then by the definition of Riemann integral, $\forall \epsilon > 0$, there exists some step function $s_\epsilon(x)$, such that $s_\epsilon(x) =$

$$\begin{cases} C_1 & a = x_0 \leq x < x_1 \\ C_2 & x_1 \leq x < x_2 \\ \dots & \\ C_m & x_{m-1} \leq x \leq b \end{cases} \quad \text{and } \int_a^b |f(x) - s_\epsilon(x)|dx < \epsilon. \text{ Let } M = \sup_{x \in [0, T]} g(x). \text{ Then}$$

$|\int_a^b f(x)g(nx)dx| = |\int_a^b (f(x) - s_\epsilon(x))g(nx)dx + \int_a^b s_\epsilon(x)g(nx)dx| \leq \int_a^b |f(x) - s_\epsilon(x)|g(nx)dx + |\sum_{i=1}^m C_i \int_{x_{i-1}}^{x_i} g(nx)dx| < M\epsilon + \frac{1}{n} \sum_{i=1}^m C_i \int_{nx_{i-1}}^{nx_i} g(x)dx \leq M\epsilon + \frac{1}{n} \sum_{i=1}^m C_i MT$. The last equation uses the fact that $\int_0^T g(x)dx = 0$, which further means $\int_c^d g(x)dx = \int_c^{c+T} g(x)dx + \int_{c+T}^{c+2T} g(x)dx + \cdots + \int_{c+kT}^d g(x)dx$ ($c+kT \leq d < c+(k+1)T$) $= \int_{c+kT}^d g(x)dx \leq MT$. Choose a large enough n , and we can let $\frac{1}{n} \sum_{i=1}^m C_i MT < \epsilon$. #.

6 第 6 次习题课

6.1 Questions

6.2 Solutions

6.3 Supplements