

Stochastic Processes

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Contents

1	Review of Martingales	2
2	Markov Chains	3
3	Ergodic Theorem	4
4	Brownian Motion	6
5	Stochastic Analysis	10

1 Review of Martingales

- $(X_n)_{n \geq 0}$ is L^2 -bounded martingale $\Rightarrow X_n$ converges in L^2 .
- $(X_n)_{n \geq 0}$ is L^1 -bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n \geq 0}$ is L^p -bounded martingale for $p > 1$, then X_n converges in $L^{p'}$ for $p' \in [1, p)$.
- Statement is false when $p = 1$. Example: $\Omega = [0, 1)$, $\mathcal{F}_n = \sigma\{\frac{i}{2^n}, \frac{i+1}{2^n}\}_{i=0}^{2^n-1}$, $X_n(\omega) := \begin{cases} 2^n & \omega \in [0, \frac{1}{2^n}) \\ 0 & \text{otherwise} \end{cases}$.
- Let $p > 1$ and $(X_n)_{n \geq 0}$ be L^p bounded martingale w.r.t. \mathcal{F}_n . Then $\exists X \in L^p(\Omega, \mathcal{F}_\infty, P)$ s.t. $X_n \rightarrow X$ in L^p and a.s. and $X_n = \mathbb{E}(X | \mathcal{F}_n)$.
- Let $(Z_n)_{n \geq 0}$ be a nonnegative sub-martingale and $Z_n^* = \sup_{0 \leq k \leq n} Z_k$, then $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda} \mathbb{E}(Z_n 1_{\{Z_n^* > \lambda\}}) \leq \frac{1}{\lambda} \mathbb{E} Z_n$.
Corollary: $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda^p} \mathbb{E}(Z_n^p 1_{\{Z_n^* > \lambda\}}) \leq \frac{1}{\lambda^p} \mathbb{E}(Z_n^p)$.
- Doob's maximal inequality: Let $p > 1$, $\exists C = C_p$ s.t. \forall martingale $(X_n)_{n \geq 0}$, we have $\mathbb{E}|X_n^*|^p \leq C_p \mathbb{E}|X_n|^p$ where $|X_n^*| = \sup_{0 \leq k \leq n} |X_k|$.
- If $(X_n)_{n \geq 0}$ is a martingale with $\sup_n \mathbb{E}(|X_n| \log(1 + |X_n|)) < +\infty$, then X_n converges in L^1 .
Proof $\mathbb{E}|X_n^*| = \int_0^{+\infty} \mathbb{P}(|X_n^*| > \lambda) d\lambda \leq 1 + \int_1^{+\infty} \frac{1}{\lambda} (\int_{|X_n^*| > \lambda} |X_n| d\mathbb{P}) d\lambda = 1 + \int |X_n| 1_{X_n^* > 1} (\int_1^{X_n^*} \frac{1}{\lambda} d\lambda) d\mathbb{P} \Rightarrow \mathbb{E}|X_n^*| \leq 1 + \mathbb{E}(|X_n| \log(X_n^* \vee 1)) \Rightarrow \mathbb{E}(X_n^* \vee 1) \leq 2 + \mathbb{E}(|X_n| \log(X_n^* \vee 1))$. Since $x \log y \leq 10^{10}(2+x) \log(2+x) + \frac{y}{2}$ when x, y are large enough (insight: if $y \gg x^2$ then $x \log y \leq \frac{y}{2}$; else $x \log y \leq 10^{10}(2+x) \log(2+x)$), $\mathbb{E} X_n^* \leq 10^{100} [1 + \mathbb{E}(|X_n| + 2) \log(|X_n| + 2)]$. Then use dominated convergence theorem. \square
- Two probability measures \mathbb{P} and \mathbb{Q} on (Ω, \mathcal{F}) , $\mathbb{Q} \ll \mathbb{P}$ on \mathcal{F}_n for every n and $M_n = \frac{d\mathbb{Q}|_{\mathcal{F}_n}}{d\mathbb{P}|_{\mathcal{F}_n}}$. $(M_n)_{n \geq 0}$ is a \mathbb{P} -martingale w.r.t. $(\mathcal{F}_n)_{n \geq 0}$. $\mathbb{Q} \ll \mathbb{P}$ on \mathcal{F}_∞ if and only if $M_n \rightarrow M$ in L^1 . $\mathbb{Q}(A) = \int_A M d\mathbb{P} + \mathbb{Q}(A \cap \{M = +\infty\})$.
Proof Sufficiency. $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F} = \mathcal{F}_\infty$, thus let $Z = \frac{d\mathbb{Q}|_{\mathcal{F}}}{d\mathbb{P}|_{\mathcal{F}}}$, we need to show M_n converges to Z in L^1 . $\forall A \in \mathcal{F}_n$, $\int_A M_n d\mathbb{P} = \mathbb{Q}(A) = \int_A Z d\mathbb{P} \Rightarrow M_n = \mathbb{E}(Z | \mathcal{F}_n)$. Thus M_n is uniformly integrable, thus converges in L^1 .
Necessity. Suppose $M_n \rightarrow M$ a.s. and in L^1 . We need to show $M_n = \mathbb{E}(M | \mathcal{F}_n)$ and $M = \frac{d\mathbb{Q}}{d\mathbb{P}}$. It suffices to show $\mathbb{Q}(A) = \int_A M d\mathbb{P}$ for all $A \in \cup_n \mathcal{F}_n$. Suppose $A \in \mathcal{F}_N$. Then $\mathbb{Q}(A) = \int_A M_N d\mathbb{P} = \int_A M_{N+k} d\mathbb{P} \rightarrow \int_A M d\mathbb{P}$. By $\pi - \lambda$ theorem we can get the desired result.
- Special situation: Suppose $\mathbb{P} \perp \mathbb{Q}$ on \mathcal{F} ($\exists E$ s.t. $\mathbb{P}(E) = 1, \mathbb{Q}(E^c) = 1$) and $\mathbb{P} \ll \mathbb{Q}$ on \mathcal{F}_n . Then $\frac{1}{M_n}$ converges \mathbb{Q} -a.s. Let $\mathbb{R} = \frac{1}{2}(\mathbb{P} + \mathbb{Q})$, $\mathbb{P}, \mathbb{Q} \ll \mathbb{R}$ on \mathcal{F} , $\frac{d\mathbb{P}|_{\mathcal{F}_n}}{d\mathbb{R}|_{\mathcal{F}_n}} = \frac{2}{1+M_n} \rightarrow \frac{2M}{1+M}$ in $L^1(\mathbb{R})$, $\frac{d\mathbb{Q}}{d\mathbb{R}} = \frac{2M_n}{1+M_n} \rightarrow \frac{2}{1+M}$ in $L^1(\mathbb{R})$. Then $\mathbb{Q}(A) = \mathbb{Q}(A \cap E^c) = \int_{A \cap E^c} \frac{2M}{1+M} d\mathbb{R} = \int_A \frac{2M}{1+M} 1_{E^c} d\mathbb{R} \stackrel{\mathbb{P}(E^c)=0}{=} 2\mathbb{R}(A \cap E^c) = 2 \int_A 1_{E^c} d\mathbb{R} \Rightarrow \frac{2M}{1+M} 1_{E^c} = 2 \cdot 1_{E^c} \Rightarrow M = +\infty$ on $E^c \Rightarrow \mathbb{Q}(M = +\infty) = 1$. Similarly $\mathbb{P}(M = 0) = \mathbb{Q}(M = +\infty) = 1$.
General situation: $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2$, $\mathbb{Q}_1 \ll \mathbb{P}$, $\mathbb{Q}_2 \perp \mathbb{P}$ on \mathcal{F} . Therefore we can decompose M_n as $M_n = Y_n + Z_n$ where $Y_n \rightarrow Y$ in $L^1(\mathbb{P})$ and $Z_n \rightarrow 0$ \mathbb{P} -a.s. $\mathbb{Q}_1(A) = \int_A Y d\mathbb{P} = \int_A M d\mathbb{P}$. $\mathbb{Q}_2(A) = \mathbb{Q}_2(A \cap \{Z = +\infty\})$. Since $Z = 0$ \mathbb{P} -a.s., $M < +\infty$ \mathbb{P} -a.s. and $\mathbb{Q}_2(M = +\infty) = 1$, we have $\mathbb{Q}_2(A) = \mathbb{Q}(A \cap \{Z = +\infty\}) = \mathbb{Q}_2(A \cap \{M = +\infty\}) = \mathbb{Q}(A \cap \{M = +\infty\})$. To sum up, $\mathbb{Q}(A) = \int_A M d\mathbb{P} + \mathbb{Q}(A \cap \{M = +\infty\})$. \square
- Statement is false if $M_n \not\rightarrow M$ in L^1 . Example: $\Omega = \{\omega = (\omega_1, \dots, \omega_n, \dots) \in \{\pm 1\}^{\mathbb{N}}\}$, $X_n(\omega) = \omega_n$. X_n 's are i.i.d. under \mathbb{P} and \mathbb{Q} , but $\mathbb{P}(X_n = 1) = \frac{1}{2}, \mathbb{P}(X_n = -1) = \frac{1}{2}, \mathbb{Q}(X_n = 1) = \frac{1}{3}, \mathbb{Q}(X_n = -1) = \frac{2}{3}$. $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. $\mathbb{P}(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k = 0) = 1, \mathbb{Q}(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k = -\frac{1}{3}) = 1$.
- Monotone class theorem for functions: Suppose \mathcal{A} as a π -system and \mathcal{H} be a class of functions from Ω to \mathbb{R} s.t. (1) $1_A \in \mathcal{H}$ for every $A \in \mathcal{A}$, (2) if $f, g \in \mathcal{H}$ then $af + bg \in \mathcal{H}$, (3) if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ then $f \in \mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$ -measurable functions are in \mathcal{H} .
- Let $(Y_n)_{n \geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E} Y_k = 1$. Then $M_n = \prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n \equiv 1$. Otherwise $M_n \rightarrow 0$ a.s.
Proof Note that $\frac{1}{n} \log M_n = \frac{1}{n} \sum_{k=1}^n \log Y_k \rightarrow \mathbb{E} \log Y$ a.s. If $\mathbb{E} \log Y = 0$ then by Jensen's inequality we have $Y_n \equiv 1$ which means M_n converges in L^1 . If $\mathbb{E} \log Y < 0$ then $M_n \rightarrow 0$ a.s. \square

- Kakutani's theorem: $M_n = \prod_{k=1}^n Y_k$, $Y_k \geq 0$ are independent, $\mathbb{E}Y_k = 1$, $\lambda_k = \mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k \lambda_k > 0$, then $M_n \rightarrow M$ in L^1 ; (2) If $\prod_k \lambda_k = 0$, then $M_n \rightarrow 0$ a.s.

Proof Let $Z_n = \prod_{k=1}^n \frac{\sqrt{Y_k}}{\lambda_k}$. Then Z_n is a martingale and has an a.s. limit Z , and $M_n = (\prod_{k=1}^n \lambda_k)^2 Z_n^2$. If $\prod_k \lambda_k > 0$, then Z_n is L^2 bounded and then convergence in L^2 , which implies $M_n \rightarrow M$ in L^1 . If $\prod_k \lambda_k = 0$, it is obvious that $M_n \rightarrow 0$ a.s. \square

- Martingale LLN: Let $(M_n)_{n \geq 0}$ be a martingale s.t. $\sum_{k=1}^{+\infty} \frac{\mathbb{E}(M_k - M_{k-1})^2}{k^2} < +\infty$. Then $\frac{M_n}{n} \rightarrow 0$ a.s.

Proof Let $Y_n = \sum_{k=1}^n \frac{X_k}{k}$. Then $(Y_n)_{n \geq 0}$ is an L^2 bounded martingale, thus $Y_n \rightarrow Y$ a.s. Then use Kronecker's lemma. \square

- Martingale CLT: Let $(M_n)_{n \geq 0}$ be a martingale with $M_0 = 0$ and $\sigma_n^2 = \sum_{k=1}^n \mathbb{E}X_k^2 = \mathbb{E}\langle M \rangle_n$. Assume that $\frac{1}{\sigma_n^2} \max_{1 \leq k \leq n} (\mathbb{E}X_k^2) \rightarrow 0$, $\frac{1}{\sigma_n^2} \sum_{k=1}^n \mathbb{E}(X_k^2 1_{\{|X_k| > \epsilon \sigma_n\}} | \mathcal{F}_{k-1}) \xrightarrow{P} 0$ for all $\epsilon > 0$, $\frac{1}{\sigma_n^2} \langle M \rangle_n \xrightarrow{P} 1$. Then $\frac{M_n}{\sigma_n} \Rightarrow \mathcal{N}(0, 1)$.

2 Markov Chains

- Let $(X_n)_{n \geq 0}$ be a homogeneous Markov chain on a discrete space S . \mathbb{P}^x : law of $(X_n)_{n \geq 0}$ conditioned on $X_0 = x$. $\mathbb{P}(X_{n+1} \in A | \mathcal{F}_n) = \mathbb{P}^{X_n}(X_1 \in A) = \mathbb{P}(X_1 \in A | X_0 = X_n)$. \mathbb{E}^x : expectation under \mathbb{P}^x . $\mathbb{P}^x(X_1 = y) = p(x, y)$.
- For every $f : S \rightarrow \mathbb{R}$ bounded, define $(\mathcal{P}f)(x) = \sum_{y \in S} p(x, y)f(y) = \mathbb{E}^x(f(X_1))$, $(\mathcal{L}f)(x) = \sum_{y \in S} p(x, y)f(y) - f(x)$. $\mathcal{L} = \mathcal{P} - \text{id}$, the generator.
- Let $(X_n)_{n \geq 0}$ be a homogeneous Markov chain with generator \mathcal{L} . Then for every bounded $f : S \rightarrow \mathbb{R}$, $M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (\mathcal{L}f)(X_k)$ is a martingale. Conversely, let $(X_n)_{n \geq 0}$ be a process and \mathcal{L} be an operator on $\mathcal{B}(S)$ s.t. M_n^f is a martingale for every f , then $(X_n)_{n \geq 0}$ is a Markov chain with generator \mathcal{L} .
- Given operator \mathcal{L} on $\mathcal{B}(S)$, we say $f : S \rightarrow \mathbb{R}$ is (1) harmonic for \mathcal{L} if $\mathcal{L}f = 0$; (2) sub-harmonic for \mathcal{L} if $\mathcal{L}f \geq 0$; (3) super-harmonic for \mathcal{L} if $\mathcal{L}f \leq 0$.
- Let f be the generator of a Markov chain $(X_n)_{n \geq 0}$. Then f is (sub-/super-)harmonic $\Leftrightarrow f(X_n)_{n \geq 0}$ is a (sub-/super-) martingale.
- f is (sub-/super-)harmonic on $D \subset S$ if $\mathcal{L}f \geq / \leq / = 0$ on D . Let $\tau = \inf\{k \geq 0 : X_k \in D^c\}$, then $(f(X_{n \wedge \tau}))_{n \geq 0}$ is a (sub-/super)martingale.
- Maximum principle: Let $(X_n)_{n \geq 0}$ be a Markov chain and $D \subset S$ s.t. the stopping time $\tau = \inf\{k \geq 0, X_k \in D^c\}$ is a.s. finite. If f is bounded and sub-harmonic on D , then $\sup_{x \in D} f(x) \leq \sup_{x \in D^c} f(x)$.

Proof f is sub-harmonic implies $(f(X_{n \wedge \tau}))$ is a sub-martingale, hence for $x \in D$ we have $f(x) \leq \mathbb{E}^x(f(X_{n \wedge \tau})) \rightarrow \mathbb{E}^x(f(X_\tau)) \leq \sup_{x \in D^c} f(x)$. \square

- $A \subset S, \tau_A = \sup\{k \geq 0 : X_k \in A\}$. (1) $u(x) = \mathbb{P}^x(\tau_A < +\infty) \Rightarrow \begin{cases} \mathcal{L}u = 0 & \text{on } A^c \\ u = 1 & \text{on } A \end{cases}$. (2) $u(x) = \mathbb{P}(\tau_A < \tau_B) \Rightarrow$

$$\begin{cases} \mathcal{L}u = 0 & \text{on } (A \cup B)^c \\ u = 1 & \text{on } A \\ u = 0 & \text{on } B. \end{cases} \quad (3) \quad u(x) = \mathbb{E}^x[\tau_A] \Rightarrow \begin{cases} \mathcal{L}u = -1 & \text{on } A^c \\ u = 0 & \text{on } A \end{cases}.$$

- Any nonnegative solution v to $\begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = 1 & \text{on } A \end{cases}$ satisfies $v \geq u$. Furthermore, if $u \equiv 1$, then \exists 1 bounded solution

$$\text{to } \begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = f & \text{on } A \end{cases} \quad \text{with } v(x) = \mathbb{E}^x(f(X_{\tau_A})).$$

Proof Let $v(x)$ be a non-negative solution, then $v(X_{n \wedge \tau_A})_{n \geq 0}$ is a martingale. $v(x) = \mathbb{E}^x v(X_{n \wedge \tau_A}) = \mathbb{E}^x v(X_{n \wedge \tau_A}) 1_{\tau_A < \infty} + \mathbb{E}^x v(X_{n \wedge \tau_A}) 1_{\tau_A = \infty} \geq \mathbb{E} v(X_{n \wedge \tau_A}) 1_{\tau_A < \infty}$. Let $n \rightarrow \infty$ and by Fatou's lemma, we have $v(x) \geq \mathbb{E}^x v(X_{\tau_A}) 1_{\tau_A < \infty} = \mathbb{P}^x(\tau_A < \infty) = u(x)$. If $u(x) \equiv 1$ and $v(x)$ is bounded, then by bounded convergence theorem, $v(x) = \mathbb{E}^x v(X_{n \wedge \tau_A}) \rightarrow \mathbb{E}^x v(X_{\tau_A}) = \mathbb{E}^x f(X_{\tau_A})$. \square

- Doob's h -transform: Let h be nonnegative, harmonic with $h(x_0) = 1$ for some $x_0 \in S$. Then $(h(X_n))_{n \geq 0}$ is a martingale with $\mathbb{E}^{\mathbb{P}^{x_0}}(h(X_n)) = 1$. Then \exists ! measure \mathbb{Q}^h on \mathcal{F}_∞ s.t. $\frac{d\mathbb{Q}^h}{d\mathbb{P}^{x_0}}|_{\mathcal{F}_n} = h(X_n), \forall n \geq 0$. $\mathbb{Q}^h(X_0 = x_0) = 1$, $(X_n)_{n \geq 0}$ never visits the set $D = \{x : h(x) = 0\}$. Under \mathbb{Q}^h , $(X_n)_{n \geq 0}$ is again a Markov chain on $S \setminus D$ with transition probability $q(x, y) = \frac{p(x, y)h(y)}{h(x)}$ (or equivalently, $(\mathcal{L}^h f)(x) = \frac{1}{h(x)}(\mathcal{L}(hf))(x)$).

Proof The first two props are trivial. $\mathbb{Q}(X_{n+1} = y | \mathcal{F}_n) = \frac{\mathbb{Q}(X_{n+1} = y, X_n = x_n, \dots, X_0 = x_0)}{\mathbb{Q}(X_n = x_n, \dots, X_0 = x_0)} = \frac{\int_{\{X_{n+1} = y, X_n = x_n, \dots, X_0 = x_0\}} h(X_{n+1}) d\mathbb{P}^{x_0}}{\int_{\{X_n = x_n, \dots, X_0 = x_0\}} h(X_n) d\mathbb{P}^{x_0}} = \frac{h(y) \mathbb{P}^{x_0}(X_{n+1} = y, X_n = x_n, \dots, X_0 = x_0)}{h(x_n) \mathbb{P}^{x_0}(X_n = x_n, \dots, X_0 = x_0)} = \frac{h(y)p(x_n, y)}{h(x_n)}$. Next we show $M_n^f := f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (\mathcal{L}^h f)(X_k)$ is a \mathbb{Q} -martingale for any bounded f . Let $Z_n = \mathbb{E}^{\mathbb{Q}} f(X_{n+1}) | \mathcal{F}_n$. $\forall A \in \mathcal{F}_n$, $\int_A Z_n h(X_n) d\mathbb{P}^{x_0} = \int_A Z_n d\mathbb{Q} = \int_A f(X_{n+1}) d\mathbb{Q} = \int_A f(X_{n+1}) h(X_{n+1}) d\mathbb{P}^{x_0} = \mathbb{E}^{\mathbb{P}^{x_0}} [\mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1}) h(X_{n+1}) 1_A | \mathcal{F}_n)] = \mathbb{E}^{\mathbb{P}^{x_0}} [1_A \mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1}) h(X_{n+1}) | \mathcal{F}_n)] = \int_A \mathcal{P}(hf)(X_n) d\mathbb{P}^{x_0}$. Thus $Z_n = \frac{\mathcal{P}(hf)(X_n)}{h(X_n)}$ only depends on X_n , i.e. $(X_n)_{n \geq 0}$ is a MC on \mathbb{Q} with generator \mathcal{L}^h . \square

- An irreducible Markov chain $(X_n)_{n \geq 0}$ (1) is transient if $\exists x$ and $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty | X_0 = x) < 1$; (2) is recurrent if \exists a finite set $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty) = 1$ for all $x \in S$. (3) is positive recurrent if \exists a finite set $A \subset S$ s.t. $\mathbb{E}(\tau_A) < \infty$ for all $x \in S$.
- Foster-Lyapunov criterion: An irreducible MC on a countable state space S (1) is transient iff $\exists v : S \rightarrow \mathbb{R}^+$ and $A \subset S$ non-empty s.t. $\mathcal{L}v \leq 0$ on A^c and $v(x) < \inf_{y \in A} v(y)$ for some $x \in A^c$; (2) is recurrent iff $\exists v : S \rightarrow \mathbb{R}^+$ s.t. $\mathcal{L}v \leq 0$ on A^c where A is a finite set and $\{x : v(x) \leq N\}$ is finite for every N ; (3) is positive recurrent iff $\exists v : S \rightarrow \mathbb{R}^+$, $A \subset S$ finite, $\exists \epsilon > 0$ s.t. $\mathcal{L}v \leq -\epsilon$ on A^c and $\sum_{y \in S} p(x, y) v(y) < +\infty$ for all $x \in A$.

Proof (1) $v(X_{n \wedge \tau_A})_{n \geq 0}$ is a super-martingale, hence $v(x) \geq \mathbb{E}v(X_{n \wedge \tau_A}) \geq \mathbb{E}v(X_{n \wedge \tau_A}) 1_{\tau_A < \infty}$. Let $n \rightarrow \infty$ we know $v(x) \geq \mathbb{E}v(X_{\tau_A} 1_{\tau_A < \infty}) \geq (\inf_{y \in A} v(y)) \mathbb{P}^x(\tau_A < \infty) \Rightarrow \mathbb{P}^x(\tau_A < \infty) < \frac{v(x)}{\inf_{y \in A} v(y)} < 1$. (2) On $\{\tau_A = \infty\}$, $\limsup_{n \rightarrow \infty} v(X_{n \wedge \tau_A}) = +\infty$ a.s. Since $(v(X_{n \wedge \tau_A}))_{n \geq 0}$ is a nonnegative super-martingale, hence converges a.s., therefore $\lim_{n \rightarrow \infty} v(X_{n \wedge \tau_A}) = +\infty$ a.s. Note that $v(x) \geq \mathbb{E}v(X_{n \wedge \tau_A}) 1_{\tau_A = \infty}$. Since LHS is a finite number, we have $\mathbb{P}^x(\tau_A = \infty) = 0$. (3) $\mathbb{E}v(X_{n \wedge \tau_A}) | \mathcal{F}_{n-1} \leq v(X_{(n-1) \wedge \tau_A}) - \epsilon 1_{\tau_A \geq n}$. Taking expectation on the both sides, $\mathbb{E}v(X_{n \wedge \tau_A}) \leq \mathbb{E}v(X_{(n-1) \wedge \tau_A}) - \epsilon \mathbb{P}^x(\tau_A \geq n) \leq \dots \leq v(x) - \epsilon \sum_{k=1}^n \mathbb{P}^x(\tau_A \geq k) \Rightarrow \mathbb{E}^x \tau_A = \sum_{k=1}^{\infty} \mathbb{P}^x(\tau_A \geq k) \leq \frac{v(x)}{\epsilon} < \infty$.

Conversely, (1) Let $v(x) = \mathbb{P}^x(\tau_A < \infty)$. (2) Let $u(x) = \mathbb{P}^x(\tau_B < \tau_A)$. We have shown that if $x \in (A \cup B)^c$ then $\mathcal{L}u \leq 0$. When $x \in B$, $(\mathcal{L}u)(x) = \sum_{y \in S} p(x, y) u(y) - 1 \leq 0$. Take $B_N \downarrow \emptyset$ s.t. B_N^c is finite for every N . Via a diagonal argument $\Rightarrow \exists$ subsequence $\{N_k\}$ s.t. $v(x) := \sum_{k \geq 1} \mathbb{P}^x(\tau_{B_{N_k}} < \tau_A) < +\infty$ for every $x \in S$. (3) Let $v(x) = \mathbb{E}^x(\tau_A)$. \square

- e.g. $h(x) = \frac{\mathbb{P}^x(\tau_A < \tau_B)}{\mathbb{P}^{x_0}(\tau_A < \tau_B)}$ is harmonic on $(A \cup B)^c$ with $h(x_0) = 1$ ($x_0 \in (A \cup B)^c$). Then $\forall x, y \in (A \cup B)^c, q(x, y) = \frac{h(y)p(x, y)}{h(x)} = \frac{\mathbb{P}^y(\tau_A < \tau_B)p(x, y)}{\mathbb{P}^x(\tau_A < \tau_B)} = \frac{\mathbb{P}^x(X_1 = y, \tau_A < \tau_B)}{\mathbb{P}^x(\tau_A < \tau_B)} = \mathbb{P}^x(X_1 = y | \tau_A < \tau_B)$.
- e.g. \mathbb{P} is simple symmetric random walk on \mathbb{Z} starting from $X_0 = 0$. Question: what is the law of $(X_n)_{n \geq 0}$ conditioned on $X_n \geq 0$ for all n ? Let $\tau_k = \inf\{n \geq 0, X_n = k\}$. On $\{\tau_N < \tau_{-1}\}$, $\frac{h(y)}{h(x)} = \frac{\mathbb{P}^y(\tau_N < \tau_{-1})}{\mathbb{P}^x(\tau_N < \tau_{-1})} = \frac{y+1}{x+1}$. Thus $q_N(x, y) = \frac{1}{2} \frac{y+1}{x+1}, |x - y| = 1, x \in \{0, \dots, N-1\} \Rightarrow q(x, y) = \frac{1}{2} \frac{y+1}{x+1}, x \geq 0, |x - y| = 1$.

3 Ergodic Theorem

- Basic setup: a measurable map $T : (\Omega, \mathcal{F}) \rightarrow (\Omega, \mathcal{F})$. Examples: (1) circle rotations: $\Omega = \mathbb{R}/\mathbb{Z}, T : x \mapsto x + \alpha$; (2) doubling map: $\Omega = \mathbb{R}/\mathbb{Z}, x \mapsto 2x$; (3) shift map: $\Omega = S^{\mathbb{N}}, (T\omega)_n = \omega_{n+1}$.
- Let $T : (\Omega, \mathcal{F}) \rightarrow (\Omega, \mathcal{F})$ measurable and \mathbb{P} be a probability measure on (Ω, \mathcal{F}) . We say T is measure-preserving if $\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$ (or $\mathbb{P} \circ T^{-1} = \mathbb{P}$).
- Question: what if we define by $\mathbb{P}(T(A)) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$ instead? $\mathbb{P} \circ T = \mathbb{P} \Rightarrow \mathbb{P} \circ T^{-1} = \mathbb{P}$ while the converse proposition is false.
- $(X_n)_{n \geq 0}$ be i.i.d. $\sim \mu$. We can build $(\Omega, \mathcal{F}, \mathbb{P})$ and $X_n : \Omega \rightarrow \mathbb{R}$ measurable s.t. $(X_n)_{n \geq 0}$ i.i.d. $\sim \mu$ under \mathbb{P} : (1) $\Omega = \mathbb{R}^{\mathbb{N}} = \{\omega : \omega = (\omega_0, \omega_1, \dots)\}$; (2) $X_n(\omega) = \omega_n$; (3) $\mathcal{F} = \sigma(X_0, X_1, \dots, X_n, \dots)$; (4) $\mathbb{P} = \mu^{\otimes \mathbb{N}}$. It is easy to show that the shift map is measure-preserving: \mathcal{F} is generated by sets of the form $A = \{\omega_{k_1} \in I_1, \dots, \omega_{k_N} \in I_N\}$, $T^{-1}(A) = \{\omega : (T\omega)_{k_1} \in I_1, \dots, (T\omega)_{k_N} \in I_N\} = \{\omega : \omega_{k_1+1} \in I_1, \dots, \omega_{k_N+1} \in I_N\}$. Key: the only thing used is that $(X_{k_1}, \dots, X_{k_N}) \stackrel{\text{law}}{=} (X_{k_1+1}, \dots, X_{k_N+1})$ for every N and every k_1, \dots, k_N .

- A sequence of random variables is stationary if $(X_n)_{n \in J} \stackrel{\text{law}}{=} (X_{n+k})_{n \in J}$ for all k and finite set J .
 - Let $T : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, \mathbb{P})$ be measure-preserving and $X : \Omega \rightarrow \mathbb{R}$ be measurable. Then $X_n(\omega) := X(T^n \omega)$ defines a stationary sequence.
- Proof* It suffices to show that for every N , every $I_1, \dots, I_N \subset \mathbb{R}$ and every $k_1 < k_2 < \dots < k_N$, we have $\mathbb{P}(X_{k_1} \in I_1, \dots, X_{k_N} \in I_N) = \mathbb{P}(X_{k_1+1} \in I_1, \dots, X_{k_N+1} \in I_N)$. $\mathbb{P}(\{\omega : X_{k_1}(\omega) \in I_1, \dots, X_{k_N}(\omega) \in I_N\}) = \mathbb{P}(T^{-1}\{\omega : X_{k_1}(\omega) \in I_1, \dots, X_{k_N}(\omega) \in I_N\}) = \mathbb{P}(\{\omega : X_{k_1}(T\omega) \in I_1, \dots, X_{k_N}(T\omega) \in I_N\}) = \mathbb{P}(\{\omega : X_{k_1+1}(\omega) \in I_1, \dots, X_{k_N+1}(\omega) \in I_N\})$. \square
- Let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be a measure-preserving system. (1) A set $A \in \mathcal{F}$ is invariant if $\mathbb{P}(A \Delta T^{-1}(A)) = 0$. (2) A random variable $X : \Omega \rightarrow \mathbb{R}$ is invariant if $X = X \circ T$ \mathbb{P} -a.e.
 - The collection of invariant sets $\mathcal{I} = \{A \in \mathcal{F} : A \text{ is invariant}\}$ is a σ -algebra and $X : \Omega \rightarrow \mathbb{R}$ is invariant iff it is \mathcal{I} -measurable.
 - We say $T : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, \mathbb{P})$ measurable-preserving is ergodic if $\mathbb{P}(A) = 0$ or 1 for all $A \in \mathcal{I}$.
 - Let $T : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, \mathbb{P})$ be measure preserving and $f \in L^p(p \geq 1)$. Then $\frac{1}{N} \sum_{k=0}^{N-1} f \circ T^k \rightarrow \mathbb{E}(f|\mathcal{I})$ a.s. and in L^p . In particular, $\mathbb{E}(f|\mathcal{I}) = \mathbb{E}f$ if T is ergodic.

Proof We first show **convergence in L^p** .

Lemma 1 If $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is a measure-preserving system and $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $\int_{\Omega} X d\mathbb{P} = \int_{\Omega} X \circ T d\mathbb{P}$. In fact, $\|X\|_{L^p} = \|X \circ T\|_{L^p}, p \in [1, +\infty]$.

Proof Take $X = 1_A$. LHS = $\mathbb{P}(A) = \mathbb{P}(T^{-1}(A)) = \int_{\Omega} 1_A(T\omega) d\mathbb{P}$. \square

Let $\mathcal{U}_T : L^p(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow L^p(\Omega, \mathcal{F}, \mathbb{P})$ be defined by $(\mathcal{U}_T f)(\omega) := f(T\omega)$ (or $\mathcal{U}_T f = f \circ T$).

For $p = 2$, $\mathcal{U}_T : L^2 \rightarrow L^2$ is an isometry in the sense that $\langle f, g \rangle = \langle \mathcal{U}_T f, \mathcal{U}_T g \rangle$. LHS = $\frac{1}{N} \sum_{k=0}^{N-1} \mathcal{U}_T^k f, f = \mathbb{E}(f|\mathcal{I}) + (f - \mathbb{E}(f|\mathcal{I})) \Rightarrow$ LHS = $\underbrace{\mathbb{E}(f|\mathcal{I}) + \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{U}_T^k (f - \mathbb{E}(f|\mathcal{I}))}_{\stackrel{?}{=} (\mathcal{U}_T - \text{Id})g} \langle \mathcal{U}_T^* - \text{Id} \rangle g \rangle < \epsilon$.

Lemma 2 Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be an isometry. If $Af = f$, then $A^* f = f$.

Proof $\langle A^* f, g \rangle = \langle f, Ag \rangle = \langle Af, Ag \rangle = \langle f, g \rangle$. \square

Proposition 1 $\mathcal{H} = \text{Ker}(A^*) \oplus \overline{\text{Im}(A)}$.

Proof We show that $\text{Ker}(A^*) = (\text{Im}(A))^{\perp}$. (i) $f \in \text{Ker}(A^*) \Rightarrow A^* f = 0 \Rightarrow \langle f, Ag \rangle = \langle A^* f, g \rangle = 0$. (ii) $f \in (\text{Im}(A))^{\perp} \Rightarrow \langle f, Ag \rangle = 0$ for all $g \in \mathcal{H} \Rightarrow \langle A^* f, g \rangle = 0$ for all $g \in \mathcal{H} \Rightarrow A^* f = 0$. \square

$\mathcal{H} = L^2(\omega, \mathcal{F}, \mathbb{P}) = \text{Ker}(\mathcal{U}_T^* - \text{Id}) + \overline{\text{Im}(\mathcal{U}_T - \text{Id})} \Rightarrow \forall f \in \mathcal{H}, \forall \epsilon > 0, \exists g, h \in \mathcal{H}$ s.t. $\|h\|_{L^2} < \epsilon$ and $f = \mathbb{E}(f|\mathcal{I}) + (\mathcal{U}_T - \text{Id})g + h \Rightarrow$ $\frac{1}{N} \sum_{k=0}^{N-1} \mathcal{U}_T^k f = \mathbb{E}(f|\mathcal{I}) + \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} (\mathcal{U}_T^k g - g)}_{\|\cdot\|_{L^2} \leq \frac{2}{N} \|g\|_{L^2} \rightarrow 0} + \underbrace{\frac{1}{N} \sum_{k=0}^{N-1} \mathcal{U}_T^k h}_{\|\cdot\|_{L^2} < \epsilon} \Rightarrow \limsup_{N \rightarrow \infty} \|\frac{1}{N} \sum_{k=0}^{N-1} \mathcal{U}_T^k f - \mathbb{E}(f|\mathcal{I})\|_{L^2} < \epsilon$.

For $p \neq 2$, let $S_N f = \sum_{k=0}^{N-1} f \circ T^k$ and $A_N f = \frac{1}{N} S_N f$.

(1) If $f \in L^{\infty}$, then $\|A_N f\|_{L^{\infty}} \leq \|f\|_{L^{\infty}}, \|A_N f - \mathbb{E}(f|\mathcal{I})\|_{L^2} \rightarrow 0 \Rightarrow A_N f \rightarrow \mathbb{E}(f|\mathcal{I})$ in L^p for every $p \in [1, +\infty)$ (for $p \geq 2$, $\|A_N f - \mathbb{E}(f|\mathcal{I})\|_{L^p}^p \leq \|f\|_{L^{\infty}}^{p-2} \|A_N f - \mathbb{E}(f|\mathcal{I})\|_{L^2}^2$; for $1 \leq p < 2$, $\|A_N f - \mathbb{E}(f|\mathcal{I})\|_{L^p}^p \leq \|A_N f - \mathbb{E}(f|\mathcal{I})\|_{L^2}^p \|1\|_{L^2}^{2-p}$).

(2) If $f \in L^p(p \geq 1)$, then $\forall \epsilon > 0, \exists g \in L^{\infty}$ s.t. $\|f - g\|_{L^p} < \epsilon$,

$$\|A_N f - \mathbb{E}(f|\mathcal{I})\|_{L^p} \leq \underbrace{\|A_N(f - g)\|_{L^p}}_{< \epsilon} + \underbrace{\|A_N g - \mathbb{E}(g|\mathcal{I})\|_{L^p}}_{\rightarrow 0 \text{ as } N \rightarrow +\infty} + \underbrace{\|\mathbb{E}(g - f|\mathcal{I})\|_{L^p}}_{< \epsilon} \Rightarrow \forall \epsilon > 0, \limsup_{N \rightarrow \infty} \|A_N f - \mathbb{E}(f|\mathcal{I})\|_{L^p} < 2\epsilon.$$

We next show **convergence a.s.**

Maximum ergodic theorem $f \in L^1(\Omega, \mathcal{F}, \mathbb{P}), S_n = \sum_{k=0}^{n-1} f \circ T^k, M_n = \max\{S_1, \dots, S_n\}$. Then $\int_{\{M_n \geq 0\}} f(\omega) d\mathbb{P} \geq 0$.

Proof $M_{n-1}(T\omega) = \max\{S_1(T\omega), \dots, S_{n-1}(T\omega)\} = \max\{S_2(\omega), \dots, S_n(\omega)\} - f(\omega) \Rightarrow \max\{0, M_{n-1}(T\omega)\} = M_n(\omega) - f(\omega) \Rightarrow f(\omega) = M_n(\omega) - \max\{0, M_{n-1}(T\omega)\}$. $\int_{\{M_n > 0\}} f d\mathbb{P} = \int_{\{M_n > 0\}} M_n d\mathbb{P} - \int_{\{M_n > 0\}} \max\{0, M_{n-1}(T\omega)\} d\mathbb{P} = \int_{\{M_n > 0\}} M_n d\mathbb{P} - \int_{\{M_n > 0\} \cap \{M_{n-1} \circ T > 0\}} M_{n-1} \circ T d\mathbb{P} \Rightarrow \int_{\{M_n > 0\}} f d\mathbb{P} \geq \int_{\{M_n \geq 0\}} M_n d\mathbb{P} - \int_{\{\dots\}} M_n \circ T d\mathbb{P} = \int_{\{M_n \geq 0\}} M_n d\mathbb{P} - \int_{T\{\dots\}} M_n d\mathbb{P} \geq 0$. \square

Corollary 1 $\mathbb{P}(\omega : \sup_{n \geq 1} (A_n f)(\omega) > \lambda) \leq \frac{\mathbb{E}|f|}{\lambda}$.

Proof Let $E_N = \{\omega : \sup_{1 \leq n \leq N} (A_n f)(\omega) > \lambda\} = \{\omega : \sup_{1 \leq n \leq N} (A_n(f - \lambda))(\omega) > 0\} = \{\omega : \sup_{1 \leq n \leq N} (S_n(f - \lambda))(\omega) > 0\}$. $E_N \uparrow E = \{\omega : \sup_{n \geq 1} (A_n f)(\omega) > \lambda\}$. $\int_{E_N} (f - \lambda) d\mathbb{P} \geq 0 \Rightarrow \mathbb{P}(E_N) \leq \frac{\int_{E_N} f d\mathbb{P}}{\lambda} \leq \frac{\mathbb{E}|f|}{\lambda} \Rightarrow \mathbb{P}(E) \leq \frac{\mathbb{E}|f|}{\lambda}$. \square

Goal: $f \in L^1$ (for finite measure \mathbb{P} , $L^p \subset L^1$), need to show $\frac{1}{N} \sum_{k=0}^{N-1} f \circ T^k \rightarrow \mathbb{E}(f|\mathcal{I})$ a.s.

(1) If $f \in L^2$ is \mathcal{I} -measurable, then $A_N f = f = \mathbb{E}(f|\mathcal{I})$ a.s.

(2) If $f = (\mathcal{U}_T - \text{Id})g$ for some $g \in L^\infty$, then $(A_N f)(\omega) = \frac{1}{N}(g(T^N \omega) - g(\omega)) \leq \frac{2\|g\|_{L^\infty}}{N} \rightarrow 0$. Check $\mathbb{E}((\mathcal{U}_T - \text{Id})g|\mathcal{I}) = 0 : \forall A \in \mathcal{I}, \int_A (g \circ T - g)d\mathbb{P} = \int_{T^{-1}(A)} g \circ T d\mathbb{P} - \int_A g d\mathbb{P} = \int_A g d\mathbb{P} - \int_A g d\mathbb{P} = 0$.

(3) $\Lambda = \{f = \mathbb{E}(f_0|\mathcal{I}) + (\mathcal{U}_T - \text{Id})g : f_0 \in L^2, g \in L^\infty\}$ is dense in L^1 . If $f \in L^1$, then $\exists f_j \in \Lambda$ s.t. $f_j \rightarrow f$ in L^1 . We need to show $\mathbb{P}(\limsup_{N \rightarrow \infty} |A_N f - \mathbb{E}(f|\mathcal{I})| > \epsilon) = 0$. $|A_N f - \mathbb{E}(f|\mathcal{I})| \leq |A_N(f - f_j)| + \underbrace{|A_N f_j - \mathbb{E}(f_j|\mathcal{I})|}_{\rightarrow 0 \text{ a.s.}} + |\mathbb{E}(f_j - f|\mathcal{I})| \Rightarrow$

$$\mathbb{P}(\limsup_{N \rightarrow \infty} |A_N f - \mathbb{E}(f|\mathcal{I})| > \epsilon) \leq \mathbb{P}(\limsup_{N \rightarrow +\infty} |A_N(f - f_j)| > \frac{\epsilon}{2}) + \mathbb{P}(|\mathbb{E}(f_j - f|\mathcal{I})| > \frac{\epsilon}{2}) \leq \frac{2\mathbb{E}|f_j - f|}{\epsilon} + \frac{2\mathbb{E}|f_j - f|}{\epsilon} \rightarrow 0. \quad \square$$

- Kingman's subadditive ergodic theorem: Let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be a measure-preserving space and $\{g_n\} \in L^1$ subadditive in the sense that $g_{n+m} \leq g_n + g_m \circ T^n$ for every n, m . Then (1) $\lim_{n \rightarrow \infty} \frac{\mathbb{E}(g_n)}{n} \rightarrow \inf_{k \geq 1} \frac{\mathbb{E}(g_k)}{k}$ (possibly $-\infty$); (2) $\frac{g_n}{n}$ convergence a.s. to F where F is \mathcal{I} -measurable and $\mathbb{E}F = \inf_{k \geq 1} \frac{\mathbb{E}(g_k)}{k}$; (3) If $\mathbb{E}F > -\infty$, then the convergence is also in L^1 .

Proof Recall an elementary version. If $\{a_n\} \in \mathbb{R}$ s.t. $a_{n+m} \leq a_n + a_m, \forall n, m$, then $\frac{a_n}{n} \rightarrow \inf_{k \geq 1} \frac{a_k}{k}$ as $n \rightarrow \infty$.

We assume $g_n \leq 0$.

(1) $H(\omega) := \liminf_{n \rightarrow \infty} \frac{g_n(\omega)}{n}$. Claim $H = H \circ T$. $g_{n+1}(\omega) \leq g_1(\omega) + g_n(T\omega) \Rightarrow H \leq H \circ T$. T measure-preserving $\Rightarrow H \stackrel{\text{law}}{=} H \circ T$. Then we must have $H = H \circ T$ \mathbb{P} -a.s.

(2) Now need to show for every $\epsilon > 0$, we have $\limsup_{n \rightarrow \infty} \frac{g_n}{n} < H + \epsilon$ \mathbb{P} -a.s. Let $n_i = \sum_{j=1}^i k_j$ and $n_M = n$. Then $g_n(\omega) \leq g_{k_1}(\omega) + g_{n-k_1}(T^{k_1}\omega) \leq g_{k_1}(\omega) + g_{k_2}(T^{k_1}\omega) + g_{n-k_1-k_2}(T^{n_2}\omega) \leq \dots \Rightarrow g_n(\omega) \leq \sum_{j=0}^{M-1} g_{k_{j+1}}(T^{n_j}\omega)$ (hope $g_{k_{j+1}}(T^{n_j}\omega) \leq k_{j+1}(H(\omega) + \epsilon)$). Fix $k > 0$, define $A_k = \{\omega : \frac{g_l(\omega)}{l} < H(\omega) + \epsilon \text{ for some } 1 \leq l \leq k\}$, $B_k = \{\omega : \frac{g_l(\omega)}{l} \geq H(\omega) + \epsilon \text{ for every } 1 \leq l \leq k\}$. If $\exists 1 \leq l \leq k \wedge (n - 1)$ s.t. $\frac{g_l(\omega)}{l} < H(\omega) + \epsilon$, then let $k_1 := \inf\{l : \frac{g_l(\omega)}{l} < H(\omega) + \epsilon\}$, otherwise let $k_1 = 1$. If $\exists 1 \leq l \leq k \wedge (n - n_p)$ s.t. $\frac{g_l(T^{n_p}\omega)}{l} < H(\omega) + \epsilon$, then $k_{p+1} := \inf\{l : \frac{g_l(T^{n_p}\omega)}{l} < H(\omega) + \epsilon\}$, otherwise let $k_{p+1} = 1$. Let $\Lambda(\omega) = \{0 \leq j \leq M(\omega) - 1 : g_{k_{j+1}}(T^{n_j}\omega) < k_{j+1}(\omega)(H(\omega) + \epsilon)\} \Rightarrow g_n(\omega) \leq \sum_{j \in \Lambda(\omega)} g_{k_{j+1}}(T^{n_j}\omega) \leq \sum_{j \in \Lambda(\omega)} k_{j+1}(H(\omega) + \epsilon) \Rightarrow g_n(\omega) < n\epsilon + H(\omega) \sum_{j \in \Lambda(\omega)} k_{j+1} \Rightarrow \limsup_{n \rightarrow \infty} \frac{g_n(\omega)}{n} < \epsilon + H(\omega) \liminf_{n \rightarrow \infty} \frac{\sum_{j \in \Lambda(\omega)} k_{j+1}}{n}$. $\sum_{j \in \Lambda} k_{j+1} \geq n - k - \sum_{j=0}^{M-1} 1_{B_k}(T^{n_j}\omega) \Rightarrow \frac{\sum_{j \in \Lambda} k_{j+1}}{n} \geq 1 - \frac{k}{n} - \frac{1}{n} \sum_{j=0}^{M-1} 1_{B_k}(T^{n_j}\omega) \Rightarrow \liminf_{n \rightarrow \infty} \frac{\sum_{j \in \Lambda} k_{j+1}}{n} \geq 1 - \mathbb{E}(1_{B_k}|\mathcal{I}) \Rightarrow \limsup_{n \rightarrow \infty} \frac{g_n(\omega)}{n} < \epsilon + H(\omega)(1 - \mathbb{E}(1_{B_k}|\mathcal{I}))$. Let $k \rightarrow \infty$, $B_k \downarrow \emptyset \Rightarrow \mathbb{E}(1_{B_k}|\mathcal{I}) \rightarrow 0$ a.s., thus RHS $\rightarrow \epsilon + H(\omega)$.

(3) Let $g_n^{(\lambda)} = \max\{-\lambda n, g_n\}$. Then $\{g_n^{(\lambda)}\}$ is subadditive and we have $\frac{g_n^{(\lambda)}}{n} \rightarrow F^{(\lambda)}$ a.s. and in L^1 (by uniform boundedness). $\mathbb{E}F^{(\lambda)} = \inf_{k \geq 1} \frac{\mathbb{E}g_k^{(\lambda)}}{k}$ and $F^{(\lambda)} = \max\{F, -\lambda\}$. Then $\mathbb{E}F = \inf_{\lambda > 0} \mathbb{E}F^{(\lambda)} = \inf_{\lambda > 0} \inf_{k \geq 1} \frac{\mathbb{E}g_k^{(\lambda)}}{k} = \inf_{k \geq 1} \inf_{\lambda > 0} \frac{\mathbb{E}g_k^{(\lambda)}}{k} = \inf_{k \geq 1} \frac{\mathbb{E}g_k}{k}$.

For general subadditive $\{g_n\}$, define $\tilde{g}_n = g_n - \sum_{k=0}^{n-1} g_1 \circ T^k$ which is negative and subadditive, and $\frac{g_n}{n} = \frac{\tilde{g}_n}{n} + \frac{1}{n} \sum_{k=0}^{n-1} g_1 \circ T^k$. Convergence of the first term has been proved and convergence of the next term is by the standard ergodic theorem. \square

4 Brownian Motion

- A one-dimensional B.M. (on $[0, T]$) is a real-valued process $(B_t)_{t \in [0, T]}$ s.t. (1) For every $0 < t_1 < \dots < t_n = T$, the r.v.'s $B_{t_1} - B_0, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are independent $\mathcal{N}(0, t_i - t_{i-1})$; (2) With probability 1, the sample path $t \mapsto B_t$ is continuous.
- A real-valued stochastic process $(X_t)_{t \in I}$ is a map $X : I \times \Omega \rightarrow \mathbb{R}$ s.t. (i) For every $\omega \in \Omega$, $X(\omega)$ is a real-valued function on I ; (ii) For every $t \in I$, X_t is a random variable.
- Construction of stochastic processes: \mathcal{F} is the smallest σ -algebra s.t. $X_t : \Omega \rightarrow \mathbb{R}$ is measurable for every $t \in I$. Finite dimensional distributions (f.d.d.) are laws of $(X_{t_1}, \dots, X_{t_n})$. Natural to take $\Omega = \mathbb{R}^I$. \mathcal{F} is generated by cylinder sets $\{\omega : \omega|_J \in A, J \text{ finite}, A \subset \mathbb{R}^{|J|}\}$. For every finite index set $J = (t_1, \dots, t_n) \in I^n$, need to specify the f.d.d. $Q_J := \text{Law}(X_{t_1}, \dots, X_{t_n})$. We are given $\{Q_J\}_{J \text{ finite}}$. We say the family of f.d.d. $\{Q_J\}_{J \text{ finite}}$ is consistent if for every $J' \subset J$, we have $Q_J \circ \pi_{J, J'}^{-1} = Q_{J'}$ where $\pi_{J, J'} : \mathbb{R}^J \rightarrow \mathbb{R}^{J'}$ is the canonical projection.
- Kolmogorov's extension theorem: If the family of f.d.d. is consistent, then \exists probability measure \mathbb{P} on $(\mathbb{R}^I, \mathcal{F})$ s.t. $\mathbb{P} \circ \pi_J^{-1} = Q_J$ for every J finite.

Proof Let $\mathcal{C} = \{\omega : \omega|_J \in A, J \text{ finite}, A \subset \mathbb{R}^{|J|}\}$. It suffices to construct \mathbb{P} on \mathcal{C} and prove uniqueness. (1) If $E \in \mathcal{C}$, then $E = \{\omega : \omega|_J \in A\}$ for some I and $A \subset \mathbb{R}^{|J|}$, and define $\mathbb{P}(E) = Q_J(A)$. Uniqueness follows immediately (if it is well defined).

Suppose $\exists J'$ and $A' \subset \mathbb{R}^{|J'|}$ s.t. $E = \{\omega : \omega|_J \in A\} = \{\omega : \omega|_{J'} \in A'\}$. Let $J^* = J \cup J'$, then $\{\omega : \omega|_{J^*} \in A \times \mathbb{R}^{J^* \setminus J}\} = \{\omega : \omega|_J \in A\} = \{\omega : \omega|_{J'} \in A'\} = \{\omega : \omega|_{J^*} \in A' \times \mathbb{R}^{J^* \setminus J'}\}$. By consistency, $Q_J(A) = Q_{J^*}(A \times \mathbb{R}^{J^* \setminus J}) = Q_{J^*}(A' \times \mathbb{R}^{J^* \setminus J'}) = Q_{J'}(A')$. (2) We first show finite additivity. Let $E, E' \subset \mathcal{C}$ be disjoint. Then there exist J and $A, A' \subset \mathbb{R}^{|J|}$ disjoint s.t. $E = \{\omega : \omega|_J \in A\}$ and $E' = \{\omega : \omega|_J \in A'\}$. $\mathbb{P}(E \cup E') = Q_J(A \cup A') = Q_J(A) + Q_J(A') = \mathbb{P}(E) + \mathbb{P}(E')$. (3) For countable additivity, it suffices to show that if $E_n \downarrow \emptyset$, then $\mathbb{P}(E_n) \downarrow 0$. Need to show that if $\{E_n\}$ is a sequence of decreasing sets in \mathcal{C} s.t. $\mathbb{P}(E_n) \downarrow \delta > 0$, then $\cap_{n \geq 1} E_n$ is non-empty. We can find $J_1 \subset \dots \subset J_n \subset \dots$ and sets $A_n \in J_n$ with $A_{n+1} \subset \pi_{J_{n+1}, J_n}^{-1}(A_n)$ s.t. $E_n = \{\omega : \omega|_{J_n} \in A_n\}$. For every n , \exists compact $K_n \subset A_n$ s.t. $Q_{J_n}(K_n) > Q_{J_n}(A_n) - \frac{\delta}{2^{n+1}}$. Let $G_n = \pi_{J_n}^{-1}(K_n)$. Consider the set $\cap_{k=1}^N G_k$ in $\Omega = \mathbb{R}^I$. $\mathbb{P}(\cap_{k=1}^N G_k) \geq \mathbb{P}(\cap_{k=1}^N E_k) - \sum_{k=1}^N \mathbb{P}(E_k \setminus G_k) > \delta - \frac{\delta}{2} = \frac{\delta}{2} \Rightarrow$ For every N , $\exists \omega^{(N)} \in \cap_{k=1}^N G_k \Rightarrow \omega^{(N)}|_{J_m} \in K_m$ for every $m \leq N \Rightarrow \exists$ subsequence of $\{\omega^{(N)}|_{J_1}\}_N$ convergent in K_1 and denote the limit by $Z_1 \in K_1 \Rightarrow \exists$ further subsequence $\{\omega^{(N)}\}$ s.t. $\omega^{(N)}|_{J_2} \rightarrow Z_2 \in K_2$ and $Z_2|_{J_1} = Z_1 \Rightarrow \exists$ subsequence $\{\omega^{(m_l)}\}_{k \geq 1}$ s.t. $\omega_{J_n}^{(m_l)} \rightarrow Z_n \in K_n$ and $Z_n|_{J_{n-1}} = Z_{n-1} \Rightarrow \exists Z \in \mathbb{R}^{\mathbb{N}}$ s.t. $Z|_{J_n} = Z_n$. Let $\omega \in \mathbb{R}^I$ be the sample point s.t. $\omega|_{J_n} = Z_n \Rightarrow \omega \in \cap_{k \geq 1} G_k$. \square

- We say two processes $(X_t)_{t \in I}$ and $(Y_t)_{t \in I}$ (1) have the same f.d.d. if $\text{Law}_{\mathbb{P}}(X_{t_1}, \dots, X_{t_n}) = \text{Law}_{\mathbb{P}}(Y_{t_1}, \dots, Y_{t_n})$ for every $t_1, \dots, t_n \in I$; (2) are modifications of each other if for every $t \in I$ we have $\mathbb{P}(X_t = Y_t) = 1$; (3) are indistinguishable if $X(\omega) = Y(\omega)$ for \mathbb{P} a.e. ω . In the following text we set $I = [0, 1]$.

- Kolmogorov's continuity criterion: Let $(X_t)_{t \in [0, 1]}$ be a process s.t. $(\mathbb{E}|X_t - X_s|^p)^{\frac{1}{p}} \leq C|t - s|^\alpha$ where $\alpha p > 1$, C independent of s and t . Then for every $\beta < \alpha - \frac{1}{p}$, \exists modification \tilde{X} of X s.t. $\mathbb{E}(\sup_{s \neq t} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t - s|^\beta})^p < +\infty$.

Proof Step 1: Choose a countable dense subset $D \subset [0, 1]$ and show that $\mathbb{E}[\sup_{s, t \in D, s \neq t} \frac{|X_t - X_s|}{|t - s|^\beta}]^p < +\infty$.

Step 2: $X|_D$ is β -Holder continuous \Rightarrow can extend to \tilde{X} on $[0, 1]$ by $\tilde{X}_t(\omega) = \begin{cases} X_t(\omega) & \text{if } t \in D \\ \lim_{n \rightarrow \infty} X_{t_n}(\omega) & \text{if } t_n \rightarrow t \end{cases}$ and $\|\tilde{X}\|_\beta \leq \|X\|_\beta$.

Step 3: Show that \tilde{X} is a modification of X .

Proof of Step 1: Let $D_n = \{\frac{j}{2^n}, j = 0, 1, \dots, 2^n\}$ and $D = \cup_n D_n$. For $s, t \in D_n$, $\exists 1 \leq m \leq 2^n$ s.t. $\frac{1}{2^{n+1}} < |t - s| \leq \frac{1}{2^m}$. For every n , let s_n, t_n be the points in D_n with smallest distance to s and t . Then (1) $|s_{n+1} - s_n| \leq \frac{1}{2^{n+1}}, |t_{n+1} - t_n| \leq \frac{1}{2^{n+1}}$; (2) $|s_m - t_m| \leq \frac{1}{2^m}$. Note that $X_t = X_{t_N} = \sum_{n=m}^{N-1} (X_{t_{n+1}} - X_{t_n}) + X_{t_m}$, $X_s = X_{s_N} = \sum_{n=m}^{N-1} (X_{s_{n+1}} - X_{s_n}) + X_{s_m} \Rightarrow |X_t - X_s| \leq |X_{t_m} - X_{s_m}| + \sum_{n \geq m} (|X_{t_{n+1}} - X_{t_n}| + |X_{s_{n+1}} - X_{s_n}|)$. Then we have

$$\begin{aligned} |X_{t_{n+1}} - X_{t_n}| &\leq \sup_{0 \leq j \leq 2^{n+1}-1} |X_{\frac{j+1}{2^{n+1}}} - X_{\frac{j}{2^{n+1}}}| \text{ and } |X_{s_{n+1}} - X_{s_n}| \leq \sup_{0 \leq j \leq 2^{n+1}-1} |X_{\frac{j+1}{2^{n+1}}} - X_{\frac{j}{2^{n+1}}}| \\ \Rightarrow |X_t - X_s| &\lesssim 2 \sum_{n \geq m} \sup_{0 \leq j \leq 2^n-1} |X_{\frac{j+1}{2^n}} - X_{\frac{j}{2^n}}| \\ \Rightarrow \frac{|X_t - X_s|}{|t - s|^\beta} &\lesssim 2^{m\beta} \sum_{n \geq m} \sup_{0 \leq j \leq 2^n-1} |X_{\frac{j+1}{2^n}} - X_{\frac{j}{2^n}}| \leq \sum_{n \geq 0} 2^{n\beta} \sup_{0 \leq j \leq 2^n-1} |X_{\frac{j+1}{2^n}} - X_{\frac{j}{2^n}}| \\ \Rightarrow \left\| \sup_{s, t \in D, s \neq t} \frac{|X_t - X_s|}{|t - s|^\beta} \right\|_p &\lesssim \sum_{n \geq 0} 2^{n\beta} \left(\sum_{j=0}^{2^n-1} \mathbb{E} |X_{\frac{j+1}{2^n}} - X_{\frac{j}{2^n}}|^p \right)^{\frac{1}{p}} \lesssim \sum_{n \geq 0} 2^{-(\alpha - \frac{1}{p} - \beta)n} \end{aligned}$$

The remaining details are left for exercise. \square

- In case of B.M., the condition is satisfied for every $p \geq 1$ and $\alpha = \frac{1}{2}$.
- Almost none Brownian path is Hölder- $\frac{1}{2}$ continuous, i.e. $\mathbb{P}(\{\omega : \sup_{s, t \in [0, 1]} \frac{|B_t(\omega) - B_s(\omega)|}{\sqrt{|t - s|}} < +\infty\}) = 0$.

Proof $\sup_{s, t \in [0, 1]} \frac{|B_t - B_s|}{\sqrt{|t - s|}} \geq \sup_n \sup_{0 \leq j \leq n-1} \sqrt{n} |B_{\frac{j+1}{n}} - B_{\frac{j}{n}}| \Rightarrow \mathbb{P}(\sup_{s, t \in [0, 1]} \frac{|B_t - B_s|}{\sqrt{|t - s|}} \leq \lambda) \leq \mathbb{P}(\sup_{0 \leq j \leq n-1} |Z_j| \leq \lambda)$ for every n . Let $n \rightarrow \infty$ and then RHS $\rightarrow 0$. \square

- For every t , almost none Brownian path is Hölder- $\frac{1}{2}$ continuous at t , i.e. $\mathbb{P}(\{\omega : \sup_{|h| \leq 1} \frac{|B_{t+h} - B_t|}{\sqrt{|h|}} < \infty\}) = 0$.
- Almost every Brownian path is Hölder- $\frac{1}{2}$ continuous at some t , i.e. $\mathbb{P}(\{\omega : \exists t \in [0, 1] \text{ s.t. } \sup_{|h| \leq 1} \frac{|B_{t+h} - B_t|}{\sqrt{|h|}} < \infty\}) = 1$.
- $\alpha > \frac{1}{2}$, almost none Brownian path is Hölder- α continuous at any t , i.e. $\mathbb{P}(\{\omega : \exists t \in [0, 1] \text{ s.t. } \sup_{|h| \leq 1} \frac{|B_{t+h} - B_t|}{h^\alpha} < \infty\}) = 0$.
- Let $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$ and $W_t^{(s)} = B_{s+t} - B_s$. For every $s \geq 0$, define $\mathcal{F}_s^+ = \cap_{\epsilon > 0} \mathcal{F}_{s+\epsilon}$. Then $(\mathcal{F}_s^+)_{s \geq 0}$ is right-continuous in the sense that $\mathcal{F}_s^+ = \cap_{\epsilon > 0} \mathcal{F}_{s+\epsilon}^+$.

- Markov property 1 of B.M.: $(W_t^{(s)})_{t \geq 0}$ is a B.M. independent of \mathcal{F}_s .
- Markov property 2 of B.M.: For every $s \geq 0$, $(W_t^s)_{t \geq 0}$ is a B.M. independent of \mathcal{F}_s^+ .

Proof We need to show that $\mathbb{E}(\Phi(W_t^{(s)})1_A) = \mathbb{E}(\Phi(B_t))\mathbb{P}(A)$ for every bounded measurable function $\Phi : C(\mathbb{R}^+, \mathbb{R}) \rightarrow \mathbb{R}$ and every $A \in \mathcal{F}_s^+$. By monotone class theorem, it suffices to prove it for $\Phi = 1_E$, where E ranges over all cylinder sets. Then it suffices to consider Φ that depends on finitely many values $(W_{t_1}^{(s)}, \dots, W_{t_n}^{(s)})$ and is bounded and continuous. Suppose $\Phi(g) = \Phi(g_{t_1}, \dots, g_{t_n})$, and $\mathbb{E}(\Phi(W_{t_1}^{(s)}, \dots, W_{t_n}^{(s)})1_A) = \mathbb{E}(\lim_{\epsilon \rightarrow 0} \Phi(W_{t_1}^{(s+\epsilon)}, \dots, W_{t_n}^{(s+\epsilon)})1_A) = \lim_{\epsilon \rightarrow 0} \mathbb{E}(\Phi(W_{t_1}^{(s+\epsilon)}, \dots, W_{t_n}^{(s+\epsilon)})1_A) = \lim_{\epsilon \rightarrow 0} \mathbb{E}(\Phi(W_{t_1}^{s+\epsilon}, \dots, W_{t_n}^{s+\epsilon}))\mathbb{P}(A) = \mathbb{E}(\Phi(B_{t_1}, \dots, B_{t_n}))\mathbb{P}(A)$. \square

- Blumenthal's 0-1 law: If $A \in \mathcal{F}_0^+$, then $\mathbb{P}(A) = 0$ or 1.

Proof If $A \in \mathcal{F}_0^+$, then $(B_t)_{t \geq 0} \perp\!\!\!\perp A$. On the other hand, $A \in \sigma(B_t, t \geq 0) \Rightarrow A$ is independent of A . \square

- Let $\tau_1 = \inf\{t \geq 0 : B_t > 0\}$, then $\tau_1 = 0$ a.s.

Proof $\{\tau_1 = 0\} = \cap_{n \geq 1} \{\sup_{s \in [0, \frac{1}{n}]} B_s > 0\} \in \mathcal{F}_0^+ \Rightarrow \mathbb{P}(\tau_1 = 0) = 0$ or 1. $\mathbb{P}(\tau_1 = 0) = \lim_{\epsilon \rightarrow 0} \mathbb{P}(\tau_1 \leq \epsilon) \geq \frac{1}{2}$. \square

- Let $\tau_2 = \inf\{t > 0 : B_t = 0\}$. Then $\tau_2 = 0$ a.s.

Proof The prior proposition + symmetry + continuity of B.M. \square

- Strong Markov property: Let τ be a stopping time w.r.t. $(\mathcal{F}_t)_{t \geq 0}$. The process $(1_{\tau < +\infty} W_t^\tau)_{t \geq 0}$ is a B.M. independent of \mathcal{F}_τ under the measure $\mathbb{P}(\cdot | \tau < +\infty)$.

Proof We only prove the case when $\mathbb{P}(\tau < +\infty) = 1$. It suffices to show $\mathbb{E}(\Phi(W_{t_1}^\tau, \dots, W_{t_n}^\tau)1_A) = \mathbb{E}(\Phi(B_{t_1}, \dots, B_{t_n}))\mathbb{P}(A)$ for every continuous and bounded $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $A \in \mathcal{F}_\tau$. Let $\tau_k(\omega) := \frac{j}{k}$ if $\tau(\omega) \in (\frac{j-1}{k}, \frac{j}{k}]$, $\tau_k \rightarrow \tau$ a.s. Bounded convergence $\Rightarrow \mathbb{E}(\Phi(W_{t_1}^{(\tau_k)}, \dots, W_{t_n}^{(\tau_k)})1_A) \rightarrow \mathbb{E}(\Phi(W_{t_1}^{(\tau)}, \dots, W_{t_n}^{(\tau)})1_A)$. Then

$$\begin{aligned} \mathbb{E}(\Phi(W_{t_1}^{(\tau)}, \dots, W_{t_n}^{(\tau)})1_A) &= \lim_{k \rightarrow \infty} \mathbb{E}(\Phi(W_{t_1}^{(\tau_k)}, \dots, W_{t_n}^{(\tau_k)})1_A) \\ &= \lim_{k \rightarrow \infty} \mathbb{E}(\sum_{j \geq 0} \Phi(W_{t_1}^{(\frac{j}{k})}, \dots, W_{t_n}^{(\frac{j}{k})})1_{A \cap \{\frac{j-1}{k} < \tau \leq \frac{j}{k}\}}) \\ &= \lim_{k \rightarrow \infty} \mathbb{E}(\sum_{j \geq 0} \Phi(W_{t_1}^{(\frac{j}{k})}, \dots, W_{t_n}^{(\frac{j}{k})})1_{A \cap \{\frac{j-1}{k} < \tau \leq \frac{j}{k}\}}) \\ &= \lim_{k \rightarrow \infty} \sum_{j \geq 0} \mathbb{E}(\Phi(B_{t_1}, \dots, B_{t_n}))\mathbb{P}(A \cap \{\frac{j-1}{k} < \tau \leq \frac{j}{k}\}) \\ &= \mathbb{E}(\Phi(B_{t_1}, \dots, B_{t_n}))\mathbb{P}(A) \end{aligned} \quad \square$$

- Maximum principle: Let $M_t = \sup_{s \in [0, t]} B_s$. Then $\mathbb{P}(M_T \geq a) = 2\mathbb{P}(B_T \geq a)$ for $a > 0$.

Proof Let $\tau_a = \inf\{t > 0 : B_t = a\}$. Then $\{M_T \geq a\} = \tau_a \leq T$. $\{W_t^{(\tau_a)}\}_{t \geq 0}$ is a B.M. independent of \mathcal{F}_{τ_a} . $\mathbb{P}(M_T \geq a) = \mathbb{P}(M_T \geq a, B_T \geq a) + \mathbb{P}(M_T \geq a, B_T \leq a) = \mathbb{P}(B_T \geq a) + \mathbb{P}(M_T \geq a, B_T \leq a)$. $\mathbb{P}(M_T \geq a, B_T \geq a) = \mathbb{P}(\tau_a \leq T, B_T - B_{\tau_a} \leq 0) = \mathbb{P}(\tau_a \leq T, B_T - B_{\tau_a} \geq 0) = \mathbb{P}(B_T \geq a)$. \square

- Let (\mathcal{X}, d) be a complete, separable metric space with Borel σ -algebra. Let $(\mathbb{P}_n)_{n \geq 0}$ and \mathbb{Q} be probability measure on it. We say \mathbb{P}_n convergences weakly to \mathbb{Q} ($\mathbb{P}_n \Rightarrow \mathbb{Q}$) if for every bounded and continuous $f : \mathcal{X} \rightarrow \mathbb{R}$, we have $\int_{\mathcal{X}} f d\mathbb{P}_n \rightarrow \int_{\mathcal{X}} f d\mathbb{Q}$.
- $\mathbb{P}_n \Rightarrow \mathbb{Q}$ (weakly on $C([0, 1], \mathbb{R})$) iff (1) $\mathbb{P}_n|_J \Rightarrow \mathbb{Q}|_J$ for every finite $J \subset [0, 1]$; (2) $\{\mathbb{P}_n\}_n$ is relatively compact, i.e., every subsequence has a further subsequence that is weakly convergent.
- Let $M = M(\mathcal{X})$ be the set of all probability measure on $(\mathcal{X}, d, \mathcal{B}(\mathcal{X}))$. We say $\Gamma \subset M$ is tight if for every $\epsilon > 0$, \exists compact $K_\epsilon \subset \mathcal{X}$ s.t. $\sup_{\mu \in \Gamma} \mu(\mathcal{X} \setminus K_\epsilon) < \epsilon$.
- Prokhorov's theorem: Let (\mathcal{X}, d) be a complete, separable metric space. Then, $\Gamma \subset M(\mathcal{X})$ is tight if and only if it is relatively compact.
- Arzela-Ascoli: A set $A \subset C([0, 1], \mathbb{R})$ is relatively compact iff (1) $\sup_{f \in A} |f(0)| < +\infty$; (2) $\sup_{f \in A} \text{Osc}_f(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ where $\text{Osc}_f(\delta) = \sup_{|s-t| < \delta} |f(s) - f(t)|$.

- A set $\Gamma \subset M(C[0, 1], \mathbb{R})$ is tight iff (i) $\lim_{\lambda \rightarrow \infty} \sup_{\mu \in \Gamma} \mu(\{\omega : |\omega(0)| \geq \lambda\}) = 0$; (ii) $\forall \epsilon > 0, \lim_{\delta \rightarrow 0} \sup_{\mu \in \Gamma} \mu(\{\omega : \text{Osc}_\omega(\delta) > \epsilon\}) = 0$.

Proof “ \Rightarrow ”: Suppose $\Gamma \subset M$ is tight. Then $\forall \eta > 0, \exists$ compact $K \subset C([0, 1], \mathbb{R})$ s.t. $\mu(K^c) < \eta$ for every $\mu \in \Gamma$. By Arzela-Ascoli, (a) $\sup_{\omega \in K} |\omega(0)| < +\infty$; (b) $\sup_{\omega \in K} \text{Osc}_\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. (a) $\Rightarrow \{\omega : |\omega(0)| > \lambda\} \subset K^c$ for sufficient large λ . (b) $\Rightarrow \forall \epsilon > 0, \{\omega : \text{Osc}_\omega(\delta) > \epsilon\} \subset K^c$ for sufficient small δ . $\Rightarrow \sup_{\mu \in \Gamma} \mu(\{\omega : |\omega(0)| > \lambda\}) < \eta, \sup_{\mu \in \Gamma} \mu(\{\omega : \text{Osc}_\omega(\delta) > \epsilon\}) < \eta$. “ \Leftarrow ”: Suppose $\Gamma \subset C([0, 1], \mathbb{R})$ satisfies (i) and (ii). For every $\eta > 0$, we need find compact $K \subset C([0, 1], \mathbb{R})$ s.t. $\sup_{\mu \in \Gamma} \mu(K^c) < \eta$. By (i), choose $\lambda > 0$ s.t. $\sup_{\mu \in \Gamma} \mu(\{\omega : |\omega(0)| > \lambda\}) < \frac{\eta}{2}$ and define $A_0 = \{\omega : |\omega(0)| \leq \lambda\}$. By (ii), $\forall k \geq 1, \exists \delta_k (\downarrow 0 \text{ as } k \rightarrow \infty)$ s.t. $\sup_{\mu \in \Gamma} \mu(\{\omega : \text{Osc}_\omega(\delta_k) > \frac{1}{k}\}) \leq \frac{\eta}{2^{k+1}}$ and define $A_k = \{\omega : \text{Osc}_\omega(\delta_k) \leq \frac{1}{k}\}$. $E := \bigcap_{k \geq 0} A_k$ is a compact subset of $C([0, 1], \mathbb{R})$ and $\sup_{\mu \in \Gamma} \mu(E^c) \leq 1 - \eta$. \square

- Donsker's invariance principle: $X_i, i = 1, 2, \dots$ are i.i.d. r.v.'s with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$. Define $W^{(n)}(t) := S_{\lfloor nt \rfloor} + \{nt\}(S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor})$ and $\mathbb{P}_n := \mathbb{P} \circ (\frac{W_n}{\sqrt{n}})^{-1}$. Then $\mathbb{P}_n \Rightarrow \text{B.M.}$

Proof We need to show $\mathbb{P}_n(\{\omega : \text{Osc}_\omega(\delta) > \epsilon\}) \rightarrow 0$ as $\delta \rightarrow 0$.

Step 1. $\mathbb{P}_n(\{\omega : \text{Osc}_\omega(\delta) > \epsilon\}) = \mathbb{P}(\text{Osc}_{W^{(n)}}(\delta) > \epsilon\sqrt{n})$. It suffices to show $\forall \epsilon > 0, \limsup_{n \rightarrow \infty} \mathbb{P}(\text{Osc}_{W^{(n)}}(\delta) > \epsilon\sqrt{n}) \rightarrow 0$ as $\delta \rightarrow 0$.

Step 2. $\mathbb{P}(\text{Osc}_{W^{(n)}}(\delta) > \epsilon\sqrt{n}) = \mathbb{P}(\sup_{|s-t| \leq \delta} |W^{(n)}(s) - W^{(n)}(t)| > \epsilon\sqrt{n}) = \mathbb{P}(\sup_{t \in [0, 1-\delta]} \sup_{h \in [0, \delta]} |W^{(n)}(t+h) - W^{(n)}(t)| > \epsilon\sqrt{n}) \leq \mathbb{P}(\sup_k \sup_{h \in [0, 2\delta]} |W^{(n)}(k\delta+h) - W^{(n)}(k\delta)| > \frac{\epsilon\sqrt{n}}{2}) \leq (\frac{1}{\delta} + 1) \sup_{k \leq \frac{1}{\delta} + 1} \mathbb{P}(\sup_{k \in [0, 2\delta]} |W^{(n)}(k\delta+h) - W^{(n)}(k\delta)| > \frac{\epsilon\sqrt{n}}{2})$.

Step 3. We need to show $\forall \epsilon > 0, \frac{1}{\delta} \limsup_{n \rightarrow \infty} \sup_{t \in [0, 1]} \mathbb{P}(\sup_{h \in [0, \delta]} |W^{(n)}(t+h) - W^{(n)}(t)| > \epsilon\sqrt{n}) \rightarrow 0$. $W^{(n)}(t+h) - W^{(n)}(t) = S_{\lfloor n(t+h) \rfloor} + \{n(t+h)\}X_{\lfloor n(t+h) \rfloor} - S_{\lfloor nt \rfloor} - \{nt\}X_{\lfloor nt \rfloor} \Rightarrow |W^{(n)}(t+h) - W^{(n)}(t)| \leq |S_{\lfloor n(t+h) \rfloor} - S_{\lfloor nt \rfloor}| + |X_{\lfloor n(t+h) \rfloor}| + |X_{\lfloor nt \rfloor}| \Rightarrow \mathbb{P}(\sup_{h \in [0, \delta]} |W^{(n)}(t+h) - W^{(n)}(t)| > \epsilon\sqrt{n}) \leq \mathbb{P}(\sup_{h \in [0, \delta]} |S_{\lfloor n(t+h) \rfloor} - S_{\lfloor nt \rfloor}| > \frac{\epsilon\sqrt{n}}{3}) + O_n(1)$. $\mathbb{P}(\sup_{0 \leq k \leq n\delta} |S_{\lfloor nt \rfloor + k} - S_{\lfloor nt \rfloor}| > \frac{\epsilon\sqrt{n}}{3}) \lesssim \frac{\mathbb{E}(S_{\lfloor nt \rfloor + \lfloor n\delta \rfloor} - S_{\lfloor nt \rfloor})^2}{\epsilon^2 n} \sim \frac{\delta}{\epsilon^2}$, which means that the maximal inequality is not enough if we only have finite second moment.

Step 4. We need to show $\frac{1}{\delta} \limsup_{n \rightarrow \infty} \mathbb{P}(\sup_{0 \leq k \leq n\delta} |S_k| > \epsilon\sqrt{n}) = 0$. Let $\tau = \inf\{k \geq 1, |S_k| > \epsilon\sqrt{n}\}$. $\mathbb{P}(\max_{1 \leq k \leq \lfloor n\delta \rfloor} |S_k| > \epsilon\sqrt{n}) = \mathbb{P}(\max_{1 \leq k \leq \lfloor n\delta \rfloor} |S_k| > \epsilon\sqrt{n}, |S_{\lfloor n\delta \rfloor}| > \frac{\epsilon\sqrt{n}}{2}) + \mathbb{P}(\max_{1 \leq k \leq \lfloor n\delta \rfloor} |S_k| > \epsilon\sqrt{n}, |S_{\lfloor n\delta \rfloor}| \leq \frac{\epsilon\sqrt{n}}{2})$. The first term $\leq \mathbb{P}(|S_{\lfloor n\delta \rfloor}| > \frac{\epsilon\sqrt{n}}{2}) \lesssim e^{-\frac{1}{\delta}}$ (by CLT). The second term $= \sum_{k=1}^{\lfloor n\delta \rfloor} \mathbb{P}(|S_{\lfloor n\delta \rfloor}| \leq \frac{\epsilon\sqrt{n}}{2} | \tau = k) \mathbb{P}(\tau = k) \leq \sum_{k=1}^{\lfloor n\delta \rfloor} \mathbb{P}(|S_{\lfloor n\delta \rfloor - k}| > \frac{\epsilon\sqrt{n}}{2}) \mathbb{P}(\tau = k) \leq \sum_{k=1}^{\lfloor n\delta \rfloor} \frac{4(\lfloor n\delta \rfloor - k)}{\epsilon^2 n} \mathbb{P}(\tau = k) \leq \frac{4\delta}{\epsilon^2} \mathbb{P}(\max_{1 \leq k \leq \lfloor n\delta \rfloor} |S_k| > \epsilon\sqrt{n})$. If $\delta < \frac{\epsilon^2}{8} \Rightarrow \mathbb{P}(\max_{1 \leq k \leq \lfloor n\delta \rfloor} |S_k| > \epsilon\sqrt{n}) \leq 2\mathbb{P}(|S_{\lfloor n\delta \rfloor}| > \frac{\epsilon\sqrt{n}}{2})$. \square

- $(\mathcal{P}_t f)(x) = \mathbb{E}^x(f(X_t)), (\mathcal{L}f)(x) = \lim_{h \downarrow 0} \frac{(\mathcal{P}_h f)(x) - f(x)}{h}, \mathcal{P}_{t+s} = \mathcal{P}_t \circ \mathcal{P}_s, \mathcal{P}_t \circ \mathcal{L} = \mathcal{L} \circ \mathcal{P}_t$. For B.M., $p_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|y|^2}{2t}}$, therefore $(\mathcal{L}f)(x) = \lim_{t \downarrow 0} \frac{1}{t} (\mathbb{E}^x(f(B_t)) - f(x)) = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}} p_t(y) [f(x-y) - f(x)] dy = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}} p_t(y) (-f'(x)y + \frac{1}{2}f''(x)y^2 + O(|y|^3)) \sim \lim_{t \downarrow 0} (\frac{1}{t} \int_{\mathbb{R}} p_t(y)y^2 dy) \cdot \frac{1}{2}f''(x) = \frac{1}{2}f''(x) \Rightarrow \mathcal{L} = \frac{1}{2}\Delta$.

- Feynman-Kac formula: Suppose $v : \mathbb{R}^d \rightarrow \mathbb{R}$ is bounded and continuous and let $u(t, x) = \mathbb{E}^x[f(B_t)e^{\int_0^t v(B_s)ds}]$ where B is d -dim B.M. Then u satisfies the PDE
$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u + vu \\ u(0, \cdot) = f(\cdot) \end{cases}$$

Proof $e^{\int_0^t v(B_s)ds} = \sum_{n \geq 0} \frac{1}{n!} (\int_0^t v(B_s)ds)^n$

$$\begin{aligned} \frac{1}{n!} \int \cdots \int_{[0, 1]^n} v(B_{s_1}) \cdots v(B_{s_n}) ds_1 \cdots ds_n &= \int \cdots \int_{0 < s_1 < \cdots < s_n < t} v(B_{s_1}) \cdots v(B_{s_n}) ds_1 \cdots ds_n \\ &= \int \cdots \int_{0 < s_1 < \cdots < s_n < t} v(B_{t-s_1}) \cdots v(B_{t-s_n}) ds_1 \cdots ds_n \end{aligned}$$

Denote the region $\Delta_n(t) = \{0 < s_1 < \cdots < s_n < t\}$, then $u(t, x) = \sum_{n \geq 0} \int \cdots \int_{\Delta_n(t)} \mathbb{E}^x(f(B_t)v(B_{t-s_1}) \cdots v(B_{t-s_n})) ds_1 \cdots ds_n :=$

$$\sum_{n \geq 0} I_n(t, x). \quad I_0(t, x) = \mathbb{E}^x(f(B_t)) = (\mathcal{P}_t f)(x) \Rightarrow \begin{cases} \partial_t I_0 = \frac{1}{2}\Delta I_0 \\ I_0(0, \cdot) = f \end{cases}$$

$$\begin{aligned} I_1(t, x) &= \int_0^t \mathbb{E}^x(f(B_t)v(B_{t-s})) ds = \int_0^t \mathbb{E}^x(v(B_{t-s})\mathbb{E}^x(f(B_t)|\mathcal{F}_{t-s})) ds = \int_0^t \mathbb{E}^x(v(B_{t-s})(\mathcal{P}_s f)(B_{t-s})) ds = \int_0^t (\mathcal{P}_{t-s} v \mathcal{P}_s f)(x) ds \\ &\Rightarrow \partial_t I_1 = v \mathcal{P}_t f + \frac{1}{2}\Delta I_1 = \frac{1}{2}\Delta I_1 + v I_0, I_1(0, \cdot) = 0. \end{aligned}$$

$$\begin{aligned} I_n(t, x) &= \int \cdots \int_{0 < s_1 < \cdots < s_n < t} \mathbb{E}^x(f(B_t)v(B_{t-s_1}) \cdots v(B_{t-s_n})) ds_1 \cdots ds_n = \int_0^t \mathbb{E}^x(v(B_{t-s})\mathbb{E}^x(f(B_t)|\mathcal{F}_{t-s})) ds \\ &= \int_0^t \left(\int \cdots \int_{\Delta_{n-1}(s_n)} (\mathcal{P}_{t-s_n} v \mathcal{P}_{s_n-s_{n-1}} v \cdots \mathcal{P}_{s_2-s_1} v \mathcal{P}_{s_1} f)(x) ds_1 \cdots ds_{n-1} \right) ds_n \\ &\Rightarrow \partial_t I_n = v I_{n-1} + \frac{1}{2}\Delta I_n, I_n(0, \cdot) = 0. \end{aligned}$$

\square

- Let $\xi_t = \frac{1}{t} \int_0^t 1_{\mathbb{R}^+}(B_s) ds$, then $\mathbb{P}(\xi_t \leq x) = \int_0^x \frac{1}{\pi \sqrt{y(1-y)}} dy = \frac{2}{\pi} \arcsin(\sqrt{x})$.

$$\text{Proof } \xi_t = \frac{1}{t} \int_0^t 1_{\mathbb{R}^+}(B_s) ds = \int_0^1 1_{\mathbb{R}^+}(B_{t \cdot \frac{s}{t}}) d\frac{s}{t} = \int_0^1 1_{\mathbb{R}^+}(\sqrt{t} \frac{B_{st}}{\sqrt{t}}) ds = \int_0^1 1_{\mathbb{R}^+}(\frac{B_{st}}{\sqrt{t}}) ds \stackrel{\text{Law}}{=} \int_0^1 1_{\mathbb{R}^+}(B_s) ds := \xi.$$

$$\text{Let } u(t, x) = \mathbb{E}x(e^{-\sigma t \xi}) \Rightarrow \begin{cases} \partial_t u = \frac{1}{2} \partial_x^2 u - \sigma 1_{\mathbb{R}^+} u \\ u(0, \cdot) = 1 \end{cases}. \text{ Define } g(x) = \int_0^{+\infty} e^{-\lambda t} u(t, x) dt \Rightarrow \frac{1}{2} g'' = (\lambda + \sigma 1_{\mathbb{R}^+}) g - 1 \Rightarrow g''(x) = \begin{cases} 2(\lambda + \sigma)g(x) - 2, & x \geq 0 \\ 2\lambda g(x) - 2, & x \leq 0 \end{cases} \Rightarrow g(x) = \begin{cases} B e^{-\sqrt{2(\lambda+\sigma)}x} + \frac{1}{\lambda+\sigma}, & x \geq 0 \\ C e^{\sqrt{2\lambda}x} + \frac{1}{\lambda}, & x \leq 0 \end{cases}. g(0) \text{ and } g'(0) \text{ well-defined} \Rightarrow B = \frac{\sqrt{\lambda+\sigma}-\sqrt{\lambda}}{\sqrt{\lambda(\lambda+\sigma)}}, C = -\frac{\sqrt{\lambda+\sigma}-\sqrt{\lambda}}{\lambda\sqrt{\lambda+\sigma}}. g(0) = \frac{1}{\sqrt{\lambda(\lambda+\sigma)}} = \mathbb{E}(\frac{1}{\lambda+\sigma\xi}) \text{ for every } \lambda, \sigma > 0 \text{ (take } \lambda = 1) \Rightarrow \mathbb{E}(\frac{1}{1+\sigma\xi}) = \frac{1}{\sqrt{1+\sigma}} \text{ for every } \sigma > 0. \text{ Power expansion} \\ \Rightarrow \sum_{n \geq 0} (-1)^n \mathbb{E}(\xi^n) \sigma^n = \sum_{n \geq 0} (-\sigma)^n \int_0^1 \frac{x^n}{\pi \sqrt{x(1-x)}} dx \quad \square$$

- Law of iterated logarithm: $\limsup_{h \rightarrow 0} \frac{B_h}{\sqrt{2h \log \log(\frac{1}{h})}} = 1$ a.s., $\liminf_{h \rightarrow 0} \frac{B_h}{\sqrt{2h \log \log(\frac{1}{h})}} = -1$ a.s.

$$\text{Proof } \text{Since } W_t = tB_{1/t} \text{ is again a standard B.M., it is equivalent to prove } \limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1 \text{ a.s.}$$

Step 1. Let $\Psi(t) = \sqrt{2t \log \log t}$ and $t_n = \gamma^n$ ($\gamma > 1$). We want to show $\limsup_{n \rightarrow \infty} \frac{B_{t_n}}{\Psi(t_n)} \leq 1$ a.s.

$$\mathbb{P}(\frac{B_{t_n}}{\Psi(t_n)} > \alpha) = \mathbb{P}(\frac{B_{t_n}}{\sqrt{t_n}} > \sqrt{2}\alpha\sqrt{\log \log t_n}) \sim (C + o_n(1)) \frac{1}{\sqrt{\log n}} (\frac{1}{n})^{\alpha^2} \Rightarrow \sum_n \mathbb{P}(\frac{B_{t_n}}{\Psi(t_n)} > \alpha) = \begin{cases} < +\infty, & \text{if } \alpha > 1 \\ = +\infty, & \text{if } \alpha \leq 1 \end{cases}. \text{ Borel-Cantelli} \\ \Rightarrow \mathbb{P}(\frac{B_{t_n}}{\Psi(t_n)} > 1 + \epsilon \text{ i.o.}) = 0.$$

For arbitrary t , assume $r^n < t < r^{n+1}$. Then $\frac{B_t}{\Psi(t)} = \frac{\Psi(r^n)}{\Psi(t)} \frac{B_{r^n}}{\Psi(r^n)} + \frac{\Psi(r^n)}{\Psi(t)} \frac{B_t - B_{r^n}}{\Psi(r^n)}$. The first term ≤ 1 a.s. Then need to show $\limsup_{n \rightarrow +\infty} \sup_{t \in [r^n, r^{n+1}]} \frac{B_t - B_{r^n}}{\Psi(r^n)} = \epsilon$ a.s. for every $\epsilon > 0$. $\mathbb{P}(\sup_{t \in [r^n, r^{n+1}]} \frac{B_t - B_{r^n}}{\Psi(r^n)} \geq \epsilon) = \mathbb{P}(\sup_{t \in [r^n, r^{n+1}]} (B_t - B_{r^n}) \geq \epsilon \Psi(r^n)) = 2\mathbb{P}(B_{r^{n+1}} - B_{r^n} \geq \epsilon \Psi(r^n)) = 2\mathbb{P}(\frac{B_{r^{n+1}} - B_{r^n}}{\sqrt{r^n(r-1)}} \geq \epsilon \frac{\Psi(r^n)}{\sqrt{r^n(r-1)}}) \sim (\log(r^n(r-1)))^{-\frac{\epsilon^2}{r-1}} \sim n^{-\frac{\epsilon^2}{r-1}}$. Take r close enough to 1 s.t. $\frac{\epsilon^2}{r-1} > 1$.

Step 2. We want to show $\limsup_{t \rightarrow +\infty} \frac{B_t}{\Psi(t)} \geq 1$ a.s. Take $t_n = \gamma^n$. Need to show for every $\epsilon > 0$, $\exists r > 1$ s.t. $\limsup_{n \rightarrow +\infty} \frac{B_{r^n}}{\Psi(r^n)} > 1 - \epsilon$ a.s. $\mathbb{P}(\frac{B_{r^{n+1}} - B_{r^n}}{\Psi(r^{n+1})}) = \mathbb{P}(\frac{B_{r^{n+1}} - B_{r^n}}{\sqrt{r^n(r-1)}} > \alpha \sqrt{2 \log \log(r^n(r-1))}) \sim e^{-\alpha^2 \log \log(r^n(r-1))} \frac{1}{\sqrt{\log \log(r^n(r-1))}} \sim \frac{1}{\sqrt{\log n}} n^{-\alpha^2} \Rightarrow \mathbb{P}(B_{r^{n+1}} - B_{r^n} > \Psi(r^n(r-1)) \text{ i.o.}) = 1. \frac{B_{r^{n+1}}}{\Psi(r^{n+1})} = \frac{\Psi(r^n)}{\Psi(r^{n+1})} \frac{B_{r^n}}{\Psi(r^n)} + \frac{\Psi(r^n(r-1))}{\Psi(r^{n+1})} \frac{B_{r^{n+1}} - B_{r^n}}{\Psi(r^n(r-1))} \Rightarrow \limsup_{n \rightarrow +\infty} \frac{B_{r^{n+1}}}{\Psi(r^{n+1})} \geq -\frac{1}{\sqrt{r}} + \sqrt{\frac{r-1}{r}}$ arbitrary close to 1 for sufficient large r . \square

- Lévy's construction of B.M. (based on Gaussianity): Let $\{Z_t, t \text{ dyadic}\}$ be i.i.d. $\mathcal{N}(0, 1)$ on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $B_0(t) = tZ_1, B_1(\frac{1}{2}) = \frac{1}{2}(B_0(0) + B_0(1)) + \frac{1}{2}Z_{\frac{1}{2}}, B_2(\frac{1}{4}) = \frac{1}{2}(B_1(0) + B_1(\frac{1}{2})) + \frac{1}{4}Z_{\frac{1}{4}}, B_2(\frac{3}{4}) = \frac{1}{2}(B_1(\frac{1}{2}) + B_1(1)) + \frac{1}{4}Z_{\frac{3}{4}}$. We will show that for every $p > 1$ and $\alpha < \frac{1}{2}$, $\{B_N\}$ is Cauchy on $L_w^p(C^\alpha[0, 1], \mathbb{R})$.

Proof We define the functions $\{h_n^{(k)}, n \geq 0, 0 \leq k \leq 2^{n-1}, \text{even}\}$. $h_0^{(0)} = 1$ on $[0, 1]$, $h_n^{(k)} = 2^{\frac{n-1}{2}} (1_{[\frac{2k}{2^n}, \frac{2k+1}{2^n}]} - 1_{[\frac{2k+1}{2^n}, \frac{2k+2}{2^n}]})$. The collection $\{h_n^{(k)}\}$ is an orthogonal basis of $L^2([0, 1], \mathbb{R})$. Let $\{Z_n^{(k)}, k \leq 2^n - 1, \text{even}, n \geq 0\}$ be i.i.d. $\mathcal{N}(0, 1)$ and define $B_N(t) = \sum_{n=0}^N \sum_{k=0}^{2^n-1} Z_n^{(k)} \int_0^t h_n^{(k)}(r) dr$. $\{B_N\}$ has the same law as the piecewise linear construction metioned above. We will show now for $p > 1, \alpha < \frac{1}{2}$, $\{B_N\}$ is Cauchy in $L_w^p(C^\alpha[0, 1], \mathbb{R})$. In other words, we need to show $(\mathbb{E} \|B_N - B_{N-1}\|_{C^\alpha}^p)^{\frac{1}{p}}$ decays fast enough as $N \rightarrow +\infty$ where $\|B_N - B_{N-1}\|_{C^\alpha} = \sup_{t \in [0, 1]} |B_N(t) - B_{N-1}(t)| + \sup_{s, t \in [0, 1]} \frac{|(B_N(t) - B_N(s)) - (B_{N-1}(t) - B_{N-1}(s))|}{|t - s|^\alpha}$. The first term is dominated by the second term. $|(B_N(t) - B_N(s)) - (B_{N-1}(t) - B_{N-1}(s))| = |\sum_{k=0}^{2^N-1} Z_N^{(k)} \int_s^t h_N^{(k)}(r) dr| := (*)$. If $|t - s| \leq \frac{1}{2^N}$, $(*) \lesssim 2^{\frac{N}{2}} |t - s| \sup_{0 \leq t \leq 2^{N-1}} |Z_N^{(k)}|$ (at most one integral is nonzero); otherwise, $(*) \lesssim 2^{-\frac{N}{2}} \sup_{0 \leq t \leq 2^{N-1}} |Z_N^{(k)}|$ (at most two integrals are nonzero). Thus $\|B_N - B_{N-1}\|_{C^\alpha} \lesssim \begin{cases} 2^{\frac{N}{2}} |t - s|^{1-\alpha} \sup_{0 \leq k \leq 2^{N-1}} |Z_N^{(k)}| \lesssim^{\mathbb{E}} 2^{-N(1-\alpha)} 2^{\frac{N}{2}} = 2^{-(\frac{1}{2}-\alpha)N}, & |t - s| \leq \frac{1}{2^N} \\ 2^{-\frac{N}{2}} |t - s|^{-\alpha} \sup_{0 \leq t \leq 2^{N-1}} |Z_N^{(k)}| \lesssim^{\mathbb{E}} 2^{-\frac{N}{2}} 2^{\alpha N} \lesssim 2^{-(\frac{1}{2}-\alpha)N}, & \text{otherwise} \end{cases} \square$

5 Stochastic Analysis

What is stochastic analysis?

Itô's goal: describing diffusions as continuous-time Markov process.

- (1) Kolmogorov's way via PDEs.
- (2) Itô's: stochastic calculus and SDEs.
- (3) Malliavin: stochastic calculus of variations.

Discrete space-time Markov chains: $(X_n)_{n \geq 0}$ Markov chain on countable \mathcal{S} . $p(x, y) := \mathbb{P}(X_{n+1} = y | X_n = x) \rightarrow$ can calculate $p^{(n)}(x, y)$. $\{p^{(n)}(x, \cdot)\}$ is an evolution of probability densities on \mathcal{S} .

More “probabilistic view”: naturally induces ONE probability measure on the “path space”: $\mathcal{S}^{\mathbb{N}} := \{\omega = (\omega_0, \omega_1, \dots), \omega_k \in \mathcal{S}\}$.

Moving to continuous setting: (1) describing Markov process $(X_t)_{t \geq 0}$ on \mathbb{R}^d ; (2) want to specify the transition probability $\{p_t(x, \cdot)\}_{t \geq 0}$ s.t. $\mathbb{P}(X_t \in A | X_0 = x) = \int_A p_t(x, y) dy$.

(Wiener, Levy) construct B.M. in \mathbb{R}^d : $p_t(x, y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \rightarrow$ can be realised as ONE probability measure on $C^\alpha([0, 1], \mathbb{R}^d)$ for every $\alpha < \frac{1}{2}$ (but NOT $\alpha = \frac{1}{2}$).

Assumptions on infinitesimal behavior (of mean and variance): (1) $\mathbb{E}(X_{t+h}^{(i)} - X_t^{(i)} | X_t) = b^{(i)}(X_t)h + o(t)$; (2) $\mathbb{E}((X_{t+h}^{(i)} - X_t^{(i)})(X_{t+h}^{(j)} - X_t^{(j)}) | X_t) = a^{i,j}(X_t)h + o(h)$; (3) $\mathbb{E}(\text{product of } \geq 3 \text{ terms} | X_t) = o(h)$.

(1) + (2) + (3) $\Rightarrow \mathbb{E}(f(X_{t+h}) | X_t) = f(X_t) + (\mathcal{L}f)(X_t) \cdot h + o(h)$ where $\mathcal{L} = \frac{1}{2} \sum_{i,j} a^{i,j} \partial_i \partial_j + \sum_j b^j \partial_j$ and $(\mathcal{L}f)(x) = \frac{1}{2} \sum_{i,j} a^{i,j}(x) (\partial_i \partial_j f)(x) + \sum_j b^j(x) (\partial_j f)(x) = \lim_{h \rightarrow 0} \frac{1}{h} (\mathbb{E}(f(X_h) | X_0 = x) - f(x))$.

$p_{t+h}(x, y) = \int p_h(x, z) p_t(z, y) dz = \mathbb{E}^x(p_t(X_h, y)) \Rightarrow \frac{1}{h} [p_{t+h}(x, y) - p_t(x, y)] = \frac{1}{h} [\mathbb{E}^x(p_t(X_h, y)) - p_t(x, y)] \Rightarrow \partial_t p_t = \mathcal{L}_x p_t$ (backward equation).

$p_{t+h}(x, y) = \int p_t(x, z) p_h(z, y) dz$. $\langle p_t(x, \cdot), l \rangle := \int p_t(x, z) l(z) dz$. $\langle p_{t+h}(x, \cdot), l \rangle = \int p_t(x, z) \int p_h(z, y) l(y) dy dz \Rightarrow \frac{1}{h} \langle p_{t+h}(x, \cdot) - p_t(x, \cdot), l \rangle = \int p_t(x, z) \frac{1}{h} (\mathbb{E}^z(l(X_h)) - l(z)) dz \rightarrow \langle p_t(x, \cdot), \mathcal{L}l \rangle = \langle \mathcal{L}_y^* p_t(x, \cdot), l \rangle \Rightarrow \partial_t p_t = \mathcal{L}_y^* p_t$ (forward equation).

Observation: $(X_t)_{t \geq 0}$ has the same law as $(Y_t)_{t \geq 0}$: $Y_{t+h} = Y_t + b(Y_t)h + \underbrace{\sigma(Y_t)}_{\mathbb{R}^{d \times m}} \underbrace{(B_{t+h} - B_t)}_{m\text{-dim B.M.}}$. And $dY_t = b(Y_t)dt + \sigma(Y_t)dB_t$, $Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s$.

Problem: the map $B \mapsto Y$ initially defined on smooth functions B is NOT continuous on $C^\alpha([0, 1]; \mathbb{R}^m)$.

A classical thm from harmonic analysis: The product (defined on smooth functions) $(f, g) \mapsto fg$ extends continuously as a bilinear map $C^\alpha \times C^\beta \rightarrow C^{\alpha \wedge \beta}$ if and only if $\alpha + \beta > 0$.

Recall $Y_t = Y_0 + \int_0^t b(Y_s)ds + \int_0^t \sigma(Y_s)dB_s$. (1) The input “ B ” is usually equipped with a probability measure that gives good cancellations. (2) The integrand $\sigma(Y_t)$ has a special structure (or relationship with B) in the sense that Y is the solution to the equation.

$(B_t)_{t \geq 0}$ is a standard B.M. Z is adapted to \mathcal{F}_t (generated by B). Then $\sum_j Z_{s_j} (B_{s_{j+1}} - B_{s_j})$ converges a.s. and define the limit to be $\int Z_s dB_s$.