Stochastic Processes

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2023 年 3 月 9 日

目录

1 Review of Martingales

- $(X_n)_{n\geq 0}$ is L^2-bounded martingale $\Rightarrow X_n$ converges in L^2.
- $(X_n)_{n\geq 0}$ is L^1-bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n\geq 0}$ is L^p-bounded martingale for $p > 1$, then X_n converges in $L^{p'}$ for $p' \in [1, p)$.
- Statement is false when $p = 1$. Example: $\Omega = [0, 1), \mathscr{F}_n = \sigma\{[\frac{i}{2^n}, \frac{i+1}{2^n})\}_{i=0}^{2^n-1}, X_n(\omega) := \begin{cases} 2^n & \omega \in [0, \frac{1}{2^n}) \\ 0 & \text{otherwise} \end{cases}$.
- Let $p > 1$ and $(X_n)_{n\geq 0}$ be L^p bounded martingale w.r.t. $\mathscr{F}_n$. Then $\exists X \in L^p(\Omega, \mathscr{F}_\infty, P)$ s.t. $X_n \to X$ in L^p and a.s. and $X_n = \mathbb{E}(X|\mathscr{F}_n)$.
- Doob's maximal inequality: Let $p > 1, \exists C = C_p$ s.t. $\forall$ martingale $(X_n)_{n\geq 0}$, we have $\mathbb{E}|X_n^*|^p \leq C_p \mathbb{E}|X_n|^p$ where $|X_n^*| = \sup_{0\leq k\leq n} \sup |X_k|$.
- Let $(Z_n)_{n\geq 0}$ be a nonnegative sub-martingale and $Z_n^* = \sup_{0\leq k\leq n} Z_k$, then $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda}\mathbb{E}(Z_n 1_{\{Z_n^* > \lambda\}}) \leq \frac{1}{\lambda}\mathbb{E}Z_n$. Corollary: $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p 1_{\{Z_n^* > \lambda\}}) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p)$.
- If $(X_n)_{n\geq 0}$ is a martingale with $\sup_n \mathbb{E}(|X_n| \log(1 + |X_n|)) < +\infty$, then X_n converges in L^1.
- Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathscr{F})$, $\mathbb{Q} << \mathbb{P}$ on $\mathscr{F}_n$ for every n and $M_n = \frac{\mathrm{d}\mathbb{Q}|_{\mathscr{F}_n}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_n}}$. $(M_n)_{n\geq 0}$ is a $\mathbb{P}$-martingale w.r.t. $(\mathscr{F}_n)_{n\geq 0}$. $\mathbb{Q} << \mathbb{P}$ on $\mathscr{F}_\infty$ if and only if $M_n \to M$ in L^1. $\mathbb{Q}(A) = \int_A M \mathrm{d}\mathbb{P} + \mathbb{Q}(A \cap \{M = +\infty\})$.
- Statement is false if $M_n \not\to M$ in L^1. Example: $\Omega = \{\omega = (\omega_1, \cdots, \omega_n, \cdots) \in \{\pm 1\}^{\mathbb{N}}\}, X_n(\omega) = \omega_n$. X_n's are i.i.d. under $\mathbb{P}$ and $\mathbb{Q}$, but $\mathbb{P}(X_n = 1) = \frac{1}{2}, \mathbb{P}(X_n = -1) = \frac{1}{2}, \mathbb{Q}(X_n = 1) = \frac{1}{3}, \mathbb{Q}(X_n = -1) = \frac{2}{3}$. $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$. $\mathbb{P}(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k = 0) = 1, \mathbb{Q}(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k = -\frac{1}{3}) = 1$.
- Monotone class theorem for functions: Suppose $\mathcal{A}$ us a π-system and $\mathcal{H}$ be a class of functions from Ω to $\mathbb{R}$ s.t. (1) $1_A \in \mathcal{H}$ for every $A \in \mathscr{A}$, (2) if $f, g \in \mathcal{H}$ then $af + bg \in \mathcal{H}$, (3) if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ then $f \in \mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$-measurable functions are in $\mathcal{H}$.
- Let $(Y_n)_{n\geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E}Y_k = 1$. Then $M_n = \prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n \equiv 1$. Otherwise $M_n \to 0$ a.s.
- Kakutani's theorem: $M_n = \prod_{k=1}^n Y_k$, $Y_k \geq 0$ are independent, $\mathbb{E}Y_k = 1$, $\lambda_k = \mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k \lambda_k > 0$, then $M_n \to M$ in L^1; (2) If $\prod_k \lambda_k = 0$, then $M_n \to 0$ a.s.

2 Markov Chains

- Let $(X_n)_{n\geq 0}$ be a time-homogeneous Markov chain on a discrete space S. $\mathbb{P}^x$: law of $(X_n)_{n\geq 0}$ conditioned on $X_0 = x$. $\mathbb{P}(X_{n+1} \in A|\mathscr{F}_n) = \mathbb{P}^{X_n}(X_1 \in A) = \mathbb{P}(X_1 \in A|X_0 = X_n)$. $\mathbb{E}^x$: expectation under $\mathbb{P}^x$. $\mathbb{P}^x(X_1 = y) = p(x, y)$.
- For every $f : S \to \mathbb{R}$ bounded, define $(\mathcal{P}f)(x) = \sum_{y\in S} p(x, y)f(y) = \mathbb{E}^x(f(X_1))$, $(\mathcal{L}f)(x) = \sum_{y\in S} p(x, y)f(y) - f(x)$. $\mathcal{L} = \mathcal{P} - \mathrm{id}$, the generator.
- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with generator $\mathcal{L}$. Then for every bounded $f : S \to \mathbb{R}$, $M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1}(\mathcal{L}f)(X_k)$ is a martingale. Conversely, let $(X_n)_{n\geq 0}$ be a process and $\mathcal{L}$ be an operator on $\mathcal{B}(S)$ s.t. M_n^f is a martingale for every f, then $(X_n)_{n\geq 0}$ is a Markov chain with generator $\mathcal{L}$.
- Given operator $\mathcal{L}$ on $\mathcal{B}(S)$, we say $f : S \to \mathbb{R}$ is (1) harmonic for $\mathcal{L}$ if $\mathcal{L}f = 0$; (2) sub-harmonic for $\mathcal{L}$ if $\mathcal{L}f \geq 0$; (3) super-harmonic for $\mathcal{L}$ if $\mathcal{L}f \leq 0$.

- Let f be the generator of a Markov chain $(X_n)_{n\geq 0}$. Then f is (sub-/super-)harmonic $\Leftrightarrow$ $f(X_n)_{n\geq 0}$ is a (sub-/super-) martingale.

- f is (sub-/super-)harmonic on $D\subset S$ if $\mathcal{L}f \geq / \leq / = 0$ on D. Let $\tau = \inf\{k\geq 0 : X_k\in D^c\}$, then $(f(X_{n\wedge\tau}))_{n\geq 0}$ is a (sub-/super)martingale.

- Maximum principle: Let $(X_n)_{n\geq 0}$ be a Markov chain and $D\subset S$ s.t. the stopping time $\tau = \inf\{k\geq 0, X_k\in D^c\}$ is a.s. finite. If f is bounded and sub-harmonic on D, then $\sup_{x\in D} f(x)\leq \sup_{x\in D^c} f(x)$.

- $A\subset S, \tau_A = \sup\{k\geq 0 : X_k\in A\}$. (1) $u(x) = \mathbb{P}^x(\tau_A < +\infty) \Rightarrow \begin{cases}\mathcal{L}u = 0 & \text{on } A^c\\ u = 1 & \text{on } A\end{cases}$. (2) $u(x) = \mathbb{P}(\tau_A < \tau_B) \Rightarrow \begin{cases}\mathcal{L}u = 0 & \text{on } (A\cup B)^c\\ u = 1 & \text{on } A\\ u = 0 & \text{on } B.\end{cases}$. (3) $u(x) = \mathbb{E}^x[\tau_A] \Rightarrow \begin{cases}\mathcal{L}u = -1 & \text{on } A^c\\ u = 0 & \text{on } A\end{cases}$.

- Any nonnegative solution v to $\begin{cases}\mathcal{L}v = 0 & \text{on } A^c\\ v = 1 & \text{on } A\end{cases}$ satisfies $v\geq u$. Furthermore, if $u\equiv 1$, then $\exists 1$ bounded solution to $\begin{cases}\mathcal{L}v = 0 & \text{on } A^c\\ v = f & \text{on } A\end{cases}$ with $v(x) = \mathbb{E}^x(f(X_{\tau_A}))$.

- Doob's h-transform: Let h be nonnegative, harmonic with $h(x_0) = 1$ for some $x_0\in S$. Then $(h(X_n))_{n\geq 0}$ is a martingale with $\mathbb{E}^{\mathbb{P}^{x_0}}(h(X_n)) = 1$. Then $\exists 1$ measure $\mathbb{Q}^h$ on $\mathscr{F}_\infty$ s.t. $\frac{\mathrm{d}\mathbb{Q}^h}{\mathrm{d}\mathbb{P}^{x_0}|_{\mathscr{F}_n}} = h(X_n), \forall n\geq 0$. $\mathbb{Q}^h(X_0 = x_0) = 1$, $(X_n)_{n\geq 0}$ never visits the set $D = \{x : h(x) = 0\}$. Under $\mathbb{Q}^h$, $(X_n)_{n\geq 0}$ is again a Markov chain on $S\backslash D$ with transition probability $q(x,y) = \frac{p(x,y)h(y)}{h(x)}$ (or equivalently, $(\mathcal{L}^h f)(x) = \frac{1}{h(x)}(\mathcal{L}(hf))(x)$).

- An irreducible Markov chain $(X_n)_{n\geq 0}$ (1) is transient if $\exists x$ and $A\subset S$ s.t. $\mathbb{P}(\tau_A < \infty | X_0 = x) < 1$; (2) is recurrent if $\exists$ a finite set $A\subset S$ s.t. $\mathbb{P}(\tau_A < \infty) = 1$ for all $x\in S$. (3) is positive recurrent if $\exists$ a finite set $A\subset S$ s.t. $\mathbb{E}(\tau_A) < \infty$ for all $x\in S$.

- Foster-Lyapunov criterion: An irreducible MC on a countable state space S (1) is transient iff $\exists v : S\to\mathbb{R}^+$ and $A\subset S$ non-empty s.t. $\mathcal{L}v\leq 0$ on A^c and $v(x) < \inf_{y\in A} v(y)$ for some $x\in A^c$; (2) is recurrent iff $\exists v : S\to\mathbb{R}^+$ s.t. $\mathcal{L}v\leq 0$ on A^c where A is a finite set and $\{x : v(x)\leq N\}$ is finite for every N; (3) is positive recurrent iff $\exists v : S\to\mathbb{R}^+$, $A\subset S$ finite and $\epsilon > 0$ s.t. $\mathcal{L}v\leq -\epsilon$ on A^c.

- e.g. $h(x) = \frac{\mathbb{P}^x(\tau_A<\tau_B)}{\mathbb{P}^{x_0}(\tau_A<\tau_B)}$ is harmonic on $(A\cup B)^c$ with $h(x_0) = 1 (x_0\in (A\cup B)^c)$. Then $\forall x,y\in (A\cup B)^c, q(x,y) = \frac{h(y)p(x,y)}{h(x)} = \frac{\mathbb{P}^y(\tau_A<\tau_B)p(x,y)}{\mathbb{P}^x(\tau_A<\tau_B)} = \frac{\mathbb{P}^x(X_1=y,\tau_A<\tau_B)}{\mathbb{P}^x(\tau_A<\tau_B)} = \mathbb{P}^x(X_1 = y|\tau_A < \tau_B)$.

- e.g. $\mathbb{P}$ is simple symmetric random walk on $\mathbb{Z}$ starting from $X_0 = 0$. Question: what is the law of $(X_n)_{n\geq 0}$ conditioned on $X_n\geq 0$ for all n? Let $\tau_k = \inf\{n\geq 0, X_n = k\}$. On $\{\tau_N < \tau_{-1}\}$, $\frac{h(y)}{h(x)} = \frac{\mathbb{P}^y(\tau_N<\tau_{-1})}{\mathbb{P}^x(\tau_N<\tau_{-1})} = \frac{y+1}{x+1}$. Thus $q_N(x,y) = \frac{1}{2}\frac{y+1}{x+1}, |x-y| = 1, x\in\{0,\cdots,N-1\} \Rightarrow q(x,y) = \frac{1}{2}\frac{y+1}{x+1}, x\geq 0, |x-y| = 1$.

3 Ergodic Theorem

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