Stochastic Processes

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2023 年 3 月 14 日

目录

1 Review of Martingales

- $(X_n)_{n\geq 0}$ is L^2-bounded martingale $\Rightarrow X_n$ converges in L^2.
- $(X_n)_{n\geq 0}$ is L^1-bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n\geq 0}$ is L^p-bounded martingale for $p > 1$, then X_n converges in $L^{p'}$ for $p' \in [1, p)$.
- Statement is false when $p = 1$. Example: $\Omega = [0, 1), \mathscr{F}_n = \sigma\{[\frac{i}{2^n}, \frac{i+1}{2^n})\}_{i=0}^{2^n-1}, X_n(\omega) := \begin{cases} 2^n & \omega \in [0, \frac{1}{2^n}) \\ 0 & \text{otherwise} \end{cases}$.
- Let $p > 1$ and $(X_n)_{n\geq 0}$ be L^p bounded martingale w.r.t. $\mathscr{F}_n$. Then $\exists X \in L^p(\Omega, \mathscr{F}_\infty, P)$ s.t. $X_n \to X$ in L^p and a.s. and $X_n = \mathbb{E}(X|\mathscr{F}_n)$.
- Let $(Z_n)_{n\geq 0}$ be a nonnegative sub-martingale and $Z_n^* = \sup_{0\leq k\leq n} Z_k$, then $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda}\mathbb{E}(Z_n 1_{\{Z_n^*>\lambda\}}) \leq \frac{1}{\lambda}\mathbb{E}Z_n$. Corollary: $\mathbb{P}(Z_n^* > \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p 1_{\{Z_n^*>\lambda\}}) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p)$.
- Doob's maximal inequality: Let $p > 1, \exists C = C_p$ s.t. $\forall$ martingale $(X_n)_{n\geq 0}$, we have $\mathbb{E}|X_n^*|^p \leq C_p\mathbb{E}|X_n|^p$ where $|X_n^*| = \sup_{0\leq k\leq n} \sup |X_k|$.
- If $(X_n)_{n\geq 0}$ is a martingale with $\sup_n \mathbb{E}(|X_n| \log(1 + |X_n|)) < +\infty$, then X_n converges in L^1.

  Proof $\mathbb{E}|X_n^*| = \int_0^{+\infty} \mathbb{P}(X_n^* > \lambda)\mathrm{d}\lambda \leq 1 + \int_1^{+\infty} \frac{1}{\lambda}(\int_{|X_n^*>\lambda|} |X_n|\mathrm{d}\mathbb{P})\mathrm{d}\lambda = 1 + \int |X_n| 1_{X_n^*>1}(\int_1^{X_n^*} \frac{1}{\lambda}\mathrm{d}\lambda)\mathrm{d}\mathbb{P} \Rightarrow \mathbb{E}|X_n^*| \leq 1 + \mathbb{E}(|X_n| \log(X_n^* \vee 1)) \Rightarrow \mathbb{E}(X_n^* \vee 1) \leq 2 + \mathbb{E}(|X_n| \log(X_n^* \vee 1))$. Since $x \log y \leq 10^{10}(2+x) \log(2+x) + \frac{y}{2}$ when x, y are large enough (insight: if $y >> x^2$ then $x \log y \leq \frac{y}{2}$; else $x \log y \leq 10^{10}(2 + x) \log(2 + x)$), $\mathbb{E}X_n^* \leq 10^{100}[1 + \mathbb{E}(|X_n| + 2) \log(|X_n| + 2)]$. Then use dominated convergence theorem. □

- Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathscr{F})$, $\mathbb{Q} << \mathbb{P}$ on $\mathscr{F}_n$ for every n and $M_n = \frac{\mathrm{d}\mathbb{Q}|_{\mathscr{F}_n}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_n}}$. $(M_n)_{n\geq 0}$ is a $\mathbb{P}$-martingale w.r.t. $(\mathscr{F}_n)_{n\geq 0}$. $\mathbb{Q} << \mathbb{P}$ on $\mathscr{F}_\infty$ if and only if $M_n \to M$ in L^1. $\mathbb{Q}(A) = \int_A M\mathrm{d}\mathbb{P} + \mathbb{Q}(A \cap \{M = +\infty\})$.

  Proof Sufficiency. $\mathbb{Q} << \mathbb{P}$ on $\mathscr{F} = \mathscr{F}_\infty$, thus let $Z = \frac{\mathrm{d}\mathbb{Q}|_{\mathscr{F}}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}}}$, we need to show M_n converges to Z in L^1. $\forall A \in \mathscr{F}_n, \int_A M_n\mathrm{d}\mathbb{P} = \mathbb{Q}(A) = \int_A Z\mathrm{d}\mathbb{P} \Rightarrow M_n = \mathbb{E}(Z|\mathscr{F}_n)$. Thus M_n is uniformly integrable, thus converges in L^1.
  Necessity. Suppose $M_n \to M$ a.s. and in L^1 We need to show $M_n = \mathbb{E}(M|\mathscr{F}_n)$ and $M = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. It suffices to show $\mathbb{Q}(A) = \int_A M\mathrm{d}\mathbb{P}$ for all $A \in \cup_n\mathscr{F}_n$. Suppose $A \in \mathscr{F}_N$. Then $\mathbb{Q}(A) = \int_A M_N\mathrm{d}\mathbb{P} = \int_A M_{N+k}\mathrm{d}\mathbb{P} \to \int_A M\mathrm{d}\mathbb{P}$. By $\pi - \lambda$ theorem we can get the desired result.
  Special situation: Suppose $\mathbb{P} \perp \mathbb{Q}$ on $\mathscr{F}(\exists E$ s.t. $\mathbb{P}(E) = 1, \mathbb{Q}(E^c) = 1)$ and $\mathbb{P} << \mathbb{Q}$ on $\mathscr{F}_n$. Then $\frac{1}{M_n}$ converges $\mathbb{Q}$-a.s. Let $\mathbb{R} = \frac{1}{2}(\mathbb{P} + \mathbb{Q})$, $\mathbb{P}, \mathbb{Q} << \mathbb{R}$ on $\mathscr{F}$, $\frac{\mathrm{d}\mathbb{P}|_{\mathscr{F}_n}}{\mathrm{d}\mathbb{R}|_{\mathscr{F}_n}} = \frac{2}{1+M_n} \to \frac{2M}{1+M}$ in $L^1(\mathbb{R})$, $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{R}} = \frac{2M_n}{1+M_n} \to \frac{2}{1+M}$ in $L^1(\mathbb{R})$. Then $\mathbb{Q}(A) = \mathbb{Q}(A \cap E^c) = \int_{A\cap E^c} \frac{2M}{1+M}\mathrm{d}\mathbb{R} = \int_A \frac{2M}{1+M} 1_{E^c}\mathrm{d}\mathbb{R} \overset{\mathbb{P}(E^c)=0}{=} 2\mathbb{R}(A \cap E^c) = 2\int_A 1_{E^c}\mathrm{d}\mathbb{R} \Rightarrow \frac{2M}{1+M} 1_{E^c} = 2 \cdot 1_{E^c} \Rightarrow M = +\infty$ on $E^c \Rightarrow \mathbb{Q}(M = +\infty) = 1$. Similarly $\mathbb{P}(M = 0) = \mathbb{Q}(M = +\infty) = 1$.
  General situation: $\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2, \mathbb{Q}_1 << \mathbb{P}, \mathbb{Q}_2 \perp \mathbb{P}$ on $\mathscr{F}$. Therefore we can decompose M_n as $M_n = Y_n + Z_n$ where $Y_n \to Y$ in $L^1(\mathbb{P})$ and $Z_n \to 0$ $\mathbb{P}$-a.s. $\mathbb{Q}_1(A) = \int_A Y\mathrm{d}\mathbb{P} = \int_A M\mathrm{d}\mathbb{P}$. $\mathbb{Q}_2(A) = \mathbb{Q}_2(A \cap \{Z = +\infty\})$. Since $Z = 0$ $\mathbb{P}$-a.s., $M < +\infty$ $\mathbb{P}$-a.s. and $\mathbb{Q}_2(M = +\infty) = 1$, we have $\mathbb{Q}_2(A) = \mathbb{Q}(A \cap \{Z = +\infty\}) = \mathbb{Q}_2(A \cap \{M = +\infty\}) = \mathbb{Q}(A \cap \{M = +\infty\})$. To sum up, $\mathbb{Q}(A) = \int_A M\mathrm{d}\mathbb{P} + \mathbb{Q}(A \cap \{M = +\infty\})$. □

- Statement is false if $M_n \not\to M$ in L^1. Example: $\Omega = \{\omega = (\omega_1, \cdots, \omega_n, \cdots) \in \{\pm 1\}^{\mathbb{N}}\}, X_n(\omega) = \omega_n$. X_n's are i.i.d. under $\mathbb{P}$ and $\mathbb{Q}$, but $\mathbb{P}(X_n = 1) = \frac{1}{2}, \mathbb{P}(X_n = -1) = \frac{1}{2}, \mathbb{Q}(X_n = 1) = \frac{1}{3}, \mathbb{Q}(X_n = -1) = \frac{2}{3}$. $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$. $\mathbb{P}(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k = 0) = 1, \mathbb{Q}(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k = -\frac{1}{3}) = 1$.
- Monotone class theorem for functions: Suppose $\mathcal{A}$ us a π-system and $\mathcal{H}$ be a class of functions from Ω to $\mathbb{R}$ s.t. (1) $1_A \in \mathcal{H}$ for every $A \in \mathscr{A}$, (2) if $f, g \in \mathcal{H}$ then $af + bg \in \mathcal{H}$, (3) if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ then $f \in \mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$-measurable functions are in $\mathcal{H}$.
- Let $(Y_n)_{n\geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E}Y_k = 1$. Then $M_n = \prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n \equiv 1$. Otherwise $M_n \to 0$ a.s.

  Proof Note that $\frac{1}{n}\log M_n = \frac{1}{n}\sum_{k=1}^n \log Y_k \to \mathbb{E}\log Y$ a.s. If $\mathbb{E}\log Y = 0$ then by Jensen's inequality we have $Y_n \equiv 1$ which means M_n converges in L^1. If $\mathbb{E}\log Y < 0$ then $M_n \to 0$ a.s. □

- Kakutani's theorem: $M_n = \prod_{k=1}^n Y_k$, $Y_k \geq 0$ are independent, $\mathbb{E}Y_k = 1$, $\lambda_k = \mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k \lambda_k > 0$, then $M_n \to M$ in L^1; (2) If $\prod_k \lambda_k = 0$, then $M_n \to 0$ a.s.

  Proof Let $Z_n = \prod_{k=1}^n \frac{\sqrt{Y_k}}{\lambda_k}$. Then Z_n is a martingale and has an a.s. limit Z, and $M_n = (\prod_{k=1}^n \lambda_k)^2 Z_n^2$. If $\prod_k \lambda_k > 0$, then Z_n is L^2 bounded and then convergence in L^2, which implies $M_n \to M$ in L^1. If $\prod_k \lambda_k = 0$, it is obvious that $M_n \to 0$ a.s. □

- Martingale LLN: Let $(M_n)_{n\geq 0}$ be a martingale s.t. $\sum_{k=1}^{+\infty} \frac{\mathbb{E}(M_k - M_{k-1})^2}{k^2} < +\infty$. Then $\frac{M_n}{n} \to 0$ a.s.

  Proof Let $Y_n = \sum_{k=1}^n \frac{X_k}{k}$. Then $(Y_n)_{n\geq 0}$ is an L^2 bounded martingale, thus $Y_n \to Y$ a.s. Then use Kronecker's lemma. □

- Martingale CLT: Let $(M_n)_{n\geq 0}$ be a martingale with $M_0 = 0$ and $\sigma_n^2 = \sum_{k=1}^n \mathbb{E}X_k^2 = \mathbb{E}\langle M\rangle_n$. Assume that $\frac{1}{\sigma_n^2}\max_{1\leq k\leq n}(\mathbb{E}X_k^2) \to 0$, $\frac{1}{\sigma_n^2}\sum_{k=1}^n \mathbb{E}(X_k^2 1_{\{|X_k|>\epsilon\sigma_n\}}|\mathscr{F}_{k-1}) \xrightarrow{p} 0$ for all $\epsilon > 0$, $\frac{1}{\sigma_n^2}\langle M\rangle_n \xrightarrow{p} 1$. Then $\frac{M_n}{\sigma_n} \Rightarrow \mathcal{N}(0,1)$.

2 Markov Chains

- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain on a discrete space S. $\mathbb{P}^x$: law of $(X_n)_{n\geq 0}$ conditioned on $X_0 = x$. $\mathbb{P}(X_{n+1} \in A|\mathscr{F}_n) = \mathbb{P}^{X_n}(X_1 \in A) = \mathbb{P}(X_1 \in A|X_0 = X_n)$. $\mathbb{E}^x$: expectation under $\mathbb{P}^x$. $\mathbb{P}^x(X_1 = y) = p(x,y)$.

- For every $f : S \to \mathbb{R}$ bounded, define $(\mathcal{P}f)(x) = \sum_{y\in S} p(x,y)f(y) = \mathbb{E}^x(f(X_1))$, $(\mathcal{L}f)(x) = \sum_{y\in S} p(x,y)f(y) - f(x)$. $\mathcal{L} = \mathcal{P} - \mathrm{id}$, the generator.

- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with generator $\mathcal{L}$. Then for every bounded $f : S \to \mathbb{R}$, $M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1}(\mathcal{L}f)(X_k)$ is a martingale. Conversely, let $(X_n)_{n\geq 0}$ be a process and $\mathcal{L}$ be an operator on $\mathcal{B}(S)$ s.t. M_n^f is a martingale for every f, then $(X_n)_{n\geq 0}$ is a Markov chain with generator $\mathcal{L}$.

- Given operator $\mathcal{L}$ on $\mathcal{B}(S)$, we say $f : S \to \mathbb{R}$ is (1) harmonic for $\mathcal{L}$ if $\mathcal{L}f = 0$; (2) sub-harmonic for $\mathcal{L}$ if $\mathcal{L}f \geq 0$; (3) super-harmonic for $\mathcal{L}$ if $\mathcal{L}f \leq 0$.

- Let f be the generator of a Markov chain $(X_n)_{n\geq 0}$. Then f is (sub-/super-)harmonic $\Leftrightarrow$ $f(X_n)_{n\geq 0}$ is a (sub-/super-) martingale.

- f is (sub-/super-)harmonic on $D \subset S$ if $\mathcal{L}f \geq / \leq / = 0$ on D. Let $\tau = \inf\{k \geq 0 : X_k \in D^c\}$, then $(f(X_{n\wedge\tau}))_{n\geq 0}$ is a (sub-/super)martingale.

- Maximum principle: Let $(X_n)_{n\geq 0}$ be a Markov chain and $D \subset S$ s.t. the stopping time $\tau = \inf\{k \geq 0, X_k \in D^c\}$ is a.s. finite. If f is bounded and sub-harmonic on D, then $\sup_{x\in D} f(x) \leq \sup_{x\in D^c} f(x)$.

  Proof f is sub-harmonic implies $(f(X_{n\wedge\tau}))$ is a sub-martingale, hence for $x \in D$ we have $f(x) \leq \mathbb{E}^x f(X_{n\wedge\tau}|) \to \mathbb{E}^x(f(X_\tau)) \leq \sup_{x\in D^c} f(x)$. □

- $A \subset S, \tau_A = \sup\{k \geq 0 : X_k \in A\}$. (1) $u(x) = \mathbb{P}^x(\tau_A < +\infty) \Rightarrow \begin{cases} \mathcal{L}u = 0 & \text{on } A^c \\ u = 1 & \text{on } A \end{cases}$. (2) $u(x) = \mathbb{P}(\tau_A < \tau_B) \Rightarrow$ $\begin{cases} \mathcal{L}u = 0 & \text{on } (A\cup B)^c \\ u = 1 & \text{on } A \\ u = 0 & \text{on } B. \end{cases}$. (3) $u(x) = \mathbb{E}^x[\tau_A] \Rightarrow \begin{cases} \mathcal{L}u = -1 & \text{on } A^c \\ u = 0 & \text{on } A \end{cases}$.

- Any nonnegative solution v to $\begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = 1 & \text{on } A \end{cases}$ satisfies $v \geq u$. Furthermore, if $u \equiv 1$, then $\exists 1$ bounded solution to $\begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = f & \text{on } A \end{cases}$ with $v(x) = \mathbb{E}^x(f(X_{\tau_A}))$.

  Proof Let $v(x)$ be a non-negative solution, then $v(X_{n\wedge\tau_A})_{n\geq 0}$ is a martingale. $v(x) = \mathbb{E}^x v(X_{n\wedge\tau_A}) = \mathbb{E}^x v(X_{n\wedge\tau_A})1_{\tau_A<\infty} + \mathbb{E}^x v(X_{n\wedge\tau_A})1_{\tau_A=\infty} \geq \mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A<\infty}$. Let $n \to \infty$ and by Fatou's lemma, we have $v(x) \geq \mathbb{E}^x v(X_{\tau_A})1_{\tau_A<\infty} = \mathbb{P}^x(\tau_A < \infty) = u(x)$. If $u(x) \equiv 1$ and $v(x)$ is bounded, then by bounded convergence theorem, $v(x) = \mathbb{E}^x v(X_{n\wedge\tau_A}) \to \mathbb{E}^x v(X_{\tau_A}) = \mathbb{E}^x f(X_{\tau_A})$.□

- Doob's h-transform: Let h be nonnegative, harmonic with $h(x_0)=1$ for some $x_0\in S$. Then $(h(X_n))_{n\geq 0}$ is a martingale with $\mathbb{E}^{\mathbb{P}^{x_0}}(h(X_n))=1$. Then $\exists 1$ measure $\mathbb{Q}^h$ on $\mathscr{F}_\infty$ s.t. $\frac{\mathrm{d}\mathbb{Q}^h}{\mathrm{d}\mathbb{P}^{x_0}|_{\mathscr{F}_n}}=h(X_n),\forall n\geq 0$. $\mathbb{Q}^h(X_0=x_0)=1$, $(X_n)_{n\geq 0}$ never visits the set $D=\{x:h(x)=0\}$. Under $\mathbb{Q}^h$, $(X_n)_{n\geq 0}$ is again a Markov chain on $S\backslash D$ with transition probability $q(x,y)=\frac{p(x,y)h(y)}{h(x)}$ (or equivalently, $(\mathcal{L}^hf)(x)=\frac{1}{h(x)}(\mathcal{L}(hf))(x)$).

Proof The first two props are trivial. $\mathbb{Q}(X_{n+1}=y|\mathscr{F}_n)=\frac{\mathbb{Q}(X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0)}{\mathbb{Q}(X_n=x_n,\cdots,X_0=x_0)}=\frac{\int_{\{X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0\}}h(X_{n+1})\mathrm{d}\mathbb{P}^{x_0}}{\int_{\{X_n=x_n,\cdots,X_0=x_0\}}h(X_n)\mathrm{d}\mathbb{P}^{x_0}}=\frac{h(y)\mathbb{P}^{x_0}(X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0)}{h(x_n)\mathbb{P}^{x_0}(X_n=x_n,\cdots,X_0=x_0)}=\frac{h(y)p(x_n,y)}{h(x_n)}$. Next we show $M_n^f:=f(X_n)-f(X_0)-\sum_{k=0}^{n-1}(\mathcal{L}^hf)(X_k)$ is a $\mathbb{Q}$-martingale for any bounded f. Let $Z_n=\mathbb{E}^{\mathbb{Q}}f(X_{n+1})|\mathscr{F}_n$. $\forall A\in\mathscr{F}_n$, $\int_A Z_nh(X_n)\mathrm{d}\mathbb{P}^{x_0}=\int_A Z_n\mathrm{d}\mathbb{Q}=\int_A f(X_{n+1})\mathrm{d}\mathbb{Q}=\int_A f(X_{n+1})h(X_{n+1})\mathrm{d}\mathbb{P}^{x_0}=\mathbb{E}^{\mathbb{P}^{x_0}}[\mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1})1_A|\mathscr{F}_n)]=\mathbb{E}^{\mathbb{P}^{x_0}}[1_A\mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1})|\mathscr{F}_n)]=\int_A\mathcal{P}(hf)(X_n)\mathrm{d}\mathbb{P}^{x_0}$. Thus $Z_n=\frac{\mathcal{P}(hf)(X_n)}{h(X_n)}$ only depends on X_n, i.e. $(X_n)_{n\geq 0}$ is a MC on $\mathbb{Q}$ with generator $\mathcal{L}^h$. □

- An irreducible Markov chain $(X_n)_{n\geq 0}$ (1) is transient if $\exists x$ and $A\subset S$ s.t. $\mathbb{P}(\tau_A<\infty|X_0=x)<1$; (2) is recurrent if $\exists$ a finite set $A\subset S$ s.t. $\mathbb{P}(\tau_A<\infty)=1$ for all $x\in S$. (3) is positive recurrent if $\exists$ a finite set $A\subset S$ s.t. $\mathbb{E}(\tau_A)<\infty$ for all $x\in S$.

- Foster-Lyapunov criterion: An irreducible MC on a countable state space S (1) is transient iff $\exists v:S\to\mathbb{R}^+$ and $A\subset S$ non-empty s.t. $\mathcal{L}v\leq 0$ on A^c and $v(x)<\inf_{y\in A}v(y)$ for some $x\in A^c$; (2) is recurrent iff $\exists v:S\to\mathbb{R}^+$ s.t. $\mathcal{L}v\leq 0$ on A^c where A is a finite set and $\{x:v(x)\leq N\}$ is finite for every N; (3) is positive recurrent iff $\exists v:S\to\mathbb{R}^+$, $A\subset S$ finite, $\exists\epsilon>0$ s.t. $\mathcal{L}v\leq-\epsilon$ on A^c and $\sum_{y\in S}p(x,y)V(y)<+\infty$ for all $x\in A$.

Proof (1) $v(X_{n\wedge\tau_A})_{n\geq 0}$ is a super-martingale, hence $v(x)\geq\mathbb{E}v(X_{n\wedge\tau_A}\geq\mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A<\infty})$. Let $n\to\infty$ we know $v(x)\geq\mathbb{E}v(X_{\tau_A}1_{\tau_A<\infty})\geq(\inf_{y\in A}v(y))\mathbb{P}^x(\tau_A<\infty)\Rightarrow\mathbb{P}^x(\tau_A<\infty)<\frac{v(x)}{\inf_{y\in A}v(y)}<1$. (2) On $\{\tau_A=\infty\}$, $\limsup_{n\to\infty}v(X_{n\wedge\tau_A})=+\infty$ a.s. Since $(v(X_{n\wedge\tau_A}))_{n\geq 0}$ is a nonnegative super-martingale, hence converges a.s., therefore $\lim_{n\to\infty}v(X_{n\wedge\tau_A})=+\infty$ a.s. Note that $v(x)\geq\mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A=\infty}$. Since LHS is a finite number, we have $\mathbb{P}^x(\tau_A=\infty)=0$. (3) $\mathbb{E}v(X_{n\wedge\tau_A})|\mathscr{F}_{n-1}\leq v(X_{(n-1)\wedge\tau_A})-\epsilon 1_{\tau_A\geq n}$. Taking expectation on the both sides, $\mathbb{E}v(X_{n\wedge\tau_A})\leq\mathbb{E}v(X_{(n-1)\wedge\tau_A})-\epsilon\mathbb{E}^x1_{\tau_A\geq n}\leq\cdots\leq v(x)-\epsilon\sum_{k=1}^n\mathbb{P}^x(\tau_A\geq k)\Rightarrow\mathbb{E}^x\tau_A=\sum_{k=1}^\infty\mathbb{P}^x(\tau_A\geq k)\leq\frac{v(x)}{\epsilon}<\infty$.

Conversely, (1) Let $v(x)=\mathbb{P}^x(\tau_A<\infty)$. (2) Let $u(x)=\mathbb{P}^x(\tau_B<\tau_A)$. We have shown that if $x\in(A\cup B)^c$ then $\mathcal{L}u\leq 0$. When $x\in B$, $(\mathcal{L}u)(x)=\sum_{y\in S}p(x,y)u(y)-1\leq 0$. Take $B_N\downarrow\emptyset$ s.t. B_N^c is finite for every N. Via a diagonal argument $\Rightarrow\exists$ subsequence $\{N_k\}$ s.t. $v(x):=\sum_{k\geq 1}\mathbb{P}^x(\tau_{B_{N_k}}<\tau_A)<+\infty$ for every $x\in S$. (3) Let $v(x)=\mathbb{E}^x(\tau_A)$. □

- e.g. $h(x)=\frac{\mathbb{P}^x(\tau_A<\tau_B)}{\mathbb{P}^{x_0}(\tau_A<\tau_B)}$ is harmonic on $(A\cup B)^c$ with $h(x_0)=1(x_0\in(A\cup B)^c)$. Then $\forall x,y\in(A\cup B)^c,q(x,y)=\frac{h(y)p(x,y)}{h(x)}=\frac{\mathbb{P}^y(\tau_A<\tau_B)p(x,y)}{\mathbb{P}^x(\tau_A<\tau_B)}=\frac{\mathbb{P}^x(X_1=y,\tau_A<\tau_B)}{\mathbb{P}^x(\tau_A<\tau_B)}=\mathbb{P}^x(X_1=y|\tau_A<\tau_B)$.

- e.g. $\mathbb{P}$ is simple symmetric random walk on $\mathbb{Z}$ starting from $X_0=0$. Question: what is the law of $(X_n)_{n\geq 0}$ conditioned on $X_n\geq 0$ for all n? Let $\tau_k=\inf\{n\geq 0,X_n=k\}$. On $\{\tau_N<\tau_{-1}\}$, $\frac{h(y)}{h(x)}=\frac{\mathbb{P}^y(\tau_N<\tau_{-1})}{\mathbb{P}^x(\tau_N<\tau_{-1})}=\frac{y+1}{x+1}$. Thus $q_N(x,y)=\frac{1}{2}\frac{y+1}{x+1},|x-y|=1,x\in\{0,\cdots,N-1\}\Rightarrow q(x,y)=\frac{1}{2}\frac{y+1}{x+1},x\geq 0,|x-y|=1$.

3 Ergodic Theorem

- Basic setup: a measurable map $T:(\Omega,\mathscr{F})\to(\Omega,\mathscr{F})$. Examples: (1) circle rotations: $\Omega=\mathbb{R}/\mathbb{Z},T:x\mapsto x+\alpha$; (2) doubling map: $\Omega=\mathbb{R}/\mathbb{Z},x\mapsto 2x$; (3) shift map: $\Omega=S^{\mathbb{N}},(T\omega)_n=\omega_{n+1}$.

- Let $T:(\Omega,\mathscr{F})\to(\Omega,\mathscr{F})$ measurable and $\mathbb{P}$ be a probability measure on $(\Omega,\mathscr{F})$. We say T is measure-preserving if $\mathbb{P}(T^{-1}(A))=\mathbb{P}(A)$ for every $A\in\mathscr{F}$ (or $\mathbb{P}\circ T^{-1}=\mathbb{P}$).

- Question: what if we define by $\mathbb{P}(T(A))=\mathbb{P}(A)$ for every $A\in\mathscr{F}$ instead? $\mathbb{P}\circ T=\mathbb{P}\Rightarrow\mathbb{P}\circ T^{-1}=\mathbb{P}$ while the converse proposition is false.

- $(X_n)_{n\geq 0}$ be i.i.d. $\sim\mu$. We can build $(\Omega,\mathscr{F},\mathbb{P})$ and $X_n:\Omega\to\mathbb{R}$ measurable s.t. $(X_n)_{n\geq 0}$ i.i.d. $\sim\mu$ under $\mathbb{P}$: (1) $\Omega=\mathbb{R}^{\mathbb{N}}=\{\omega:\omega=(\omega_0,\omega_1,\cdots)\}$; (2) $X_n(\omega)=\omega_n$; (3) $\mathscr{F}=\sigma(X_0,X_1,\cdots,X_n,\cdots)$; (4) $\mathbb{P}=\mu^{\otimes\mathbb{N}}$. It is easy to show that the shift map is measure-preserving: $\mathscr{F}$ is generated by sets of the form $A=\{\omega_{k_1}\in I_1,\cdots,\omega_{k_N}\in I_N\}$, $T^{-1}(A)=\{\omega:(T\omega)_{k_1}\in I_1,\cdots,(T\omega)_{k_N}\in I_N\}=\{\omega:\omega_{k_1+1}\in I_1,\cdots,\omega_{k_N+1}\in I_N\}$. Key: the only thing used is that $(X_{k_1},\cdots,X_{k_N})\overset{\text{law}}{=}(X_{k_1+1},\cdots,X_{k_N+1})$ for every N and every $k_1,\cdots,k_N$.

- A sequence of random variables is stationary if $(X_n)_{n\in J} \overset{\text{law}}{=} (X_{n+k})_{n\in J}$ for all k and finite set J.

- Let $T : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ be measure-preserving and $X : \Omega \to \mathbb{R}$ be measurable. Then $X_n(\omega) := X(T^n\omega)$ defines a stationary sequence.

  Proof It suffices to show that for every N, every $I_1, \cdots, I_N \subset \mathbb{R}$ and every $k_1 < k_2 < \cdots < k_N$, we have $\mathbb{P}(X_{k_1} \in I_1, \cdots, X_{k_N} \in I_N) = \mathbb{P}(X_{k_1+1} \in I_1, \cdots, X_{k_N+1} \in I_N)$. $\mathbb{P}(\{\omega : X_{k_1}(\omega) \in I_1, \cdots, X_{k_N}(\omega) \in I_N\}) = \mathbb{P}(T^{-1}\{\omega : X_{k_1}(\omega) \in I_1, \cdots, X_{k_N}(\omega) \in I_N\}) = \mathbb{P}(\{\omega : X_{k_1}(T\omega) \in I_1, \cdots, X_{k_N}(T\omega) \in I_N\}) = \mathbb{P}(\{\omega : X_{k_1+1}(\omega) \in I_1, \cdots, X_{k_N+1}(\omega) \in I_N\})$. $\square$

- Let $(\Omega, \mathscr{F}, \mathbb{P}, T)$ be a measure-preserving system. (1) A set $A \in \mathscr{F}$ is invariant if $\mathbb{P}(A \triangle T^{-1}(A)) = 0$. (2) A random variable $X : \Omega \to \mathbb{R}$ is invariant if $X = X \circ T$ $\mathbb{P}$-a.e.

- The collection of invariant sets $\mathcal{I} = \{A \in \mathscr{F} : A \text{ is invariant}\}$ is a σ-algebra and $X : \Omega \to \mathbb{R}$ is invariant iff it is $\mathcal{I}$-measurable.

- We say $T : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ measurable-preserving is ergodic if $\mathbb{P}(A) = 0$ or 1 for all $A \in \mathcal{I}$.

- Let $T : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ be measure preserving and $X \in L^p (p \geq 1)$. Then $\frac{1}{N} \sum_{k=0}^{N-1} X \circ T^K \to \mathbb{E}(X|\mathcal{I})$ a.s. and in L^p. Furthermore, T is ergodic if and only if $\mathcal{I}$ only consists of sets with measure 0 or 1.

  Proof Before giving the proof, we need some preparation.

  Lemma If $(\Omega, \mathscr{F}, \mathbb{P}, T)$ is a measure-preserving system and $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$. Then $\int_\Omega X \mathrm{d}\mathbb{P} = \int_\Omega X \circ T \mathrm{d}\mathbb{P}$. In fact, $||X||_{L^p} = ||X \circ T||_{L^p}$.

  Proof Take $X = 1_A$. LHS $= \mathbb{P}(A) = \mathbb{P}(T^{-1}(A)) = \int_\Omega 1_A(T\omega)\mathrm{d}\mathbb{P}$. $\square$

  Let $\mathcal{U}_T : L^p(\Omega, \mathscr{F}, \mathbb{P}) \to L^p(\Omega, \mathscr{F}, \mathbb{P})$ be defined by $(\mathcal{U}_T f)(\omega) := f(T\omega)$ (or $\mathcal{U}_T f = f \circ T$). For $p = 2$, $\mathcal{U}_T : L^2 \to L^2$ is an isometry in the sense that $\langle f, g\rangle = \langle \mathcal{U}_T f, \mathcal{U}_T g\rangle$. $\square$