Stochastic Processes

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Contents

1 Review of Martingales

- $(X_n)_{n\geq 0}$ is L^2-bounded martingale $\Rightarrow X_n$ converges in L^2.
- $(X_n)_{n\geq 0}$ is L^1-bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n\geq 0}$ is L^p-bounded martingale for $p>1$, then X_n converges in $L^{p'}$ for $p' \in [1,p)$.
- Statement is false when $p=1$. Example: $\Omega=[0,1), \mathscr{F}_n=\sigma\{[\frac{i}{2^n},\frac{i+1}{2^n})\}_{i=0}^{2^n-1}, X_n(\omega):=\begin{cases}2^n & \omega\in[0,\frac{1}{2^n})\\ 0 & \text{otherwise}\end{cases}$.
- Let $p>1$ and $(X_n)_{n\geq 0}$ be L^p bounded martingale w.r.t. $\mathscr{F}_n$. Then $\exists X\in L^p(\Omega,\mathscr{F}_\infty,P)$ s.t. $X_n\to X$ in L^p and a.s. and $X_n=\mathbb{E}(X|\mathscr{F}_n)$.
- Let $(Z_n)_{n\geq 0}$ be a nonnegative sub-martingale and $Z_n^*=\sup_{0\leq k\leq n} Z_k$, then $\mathbb{P}(Z_n^*>\lambda)\leq \frac{1}{\lambda}\mathbb{E}(Z_n 1_{\{Z_n^*>\lambda\}})\leq \frac{1}{\lambda}\mathbb{E}Z_n$. Corollary: $\mathbb{P}(Z_n^*>\lambda)\leq\frac{1}{\lambda^p}\mathbb{E}(Z_n^p 1_{\{Z_n^*>\lambda\}})\leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p)$.
- Doob's maximal inequality: Let $p>1, \exists C=C_p$ s.t. $\forall$ martingale $(X_n)_{n\geq 0}$, we have $\mathbb{E}|X_n^*|^p\leq C_p\mathbb{E}|X_n|^p$ where $|X_n^*|=\sup_{0\leq k\leq n}\sup|X_k|$.
- If $(X_n)_{n\geq 0}$ is a martingale with $\sup_n \mathbb{E}(|X_n|\log(1+|X_n|))<+\infty$, then X_n converges in L^1.

  Proof $\mathbb{E}|X_n^*| = \int_0^{+\infty}\mathbb{P}(|X_n^*|>\lambda)\mathrm{d}\lambda \leq 1+\int_1^{+\infty}\frac{1}{\lambda}(\int_{|X_n^*>\lambda|}|X_n|\mathrm{d}\mathbb{P})\mathrm{d}\lambda = 1+\int|X_n|1_{X_n^*>1}(\int_1^{X_n^*}\frac{1}{\lambda}\mathrm{d}\lambda)\mathrm{d}\mathbb{P} \Rightarrow \mathbb{E}|X_n^*| \leq 1+\mathbb{E}(|X_n|\log(X_n^*\vee 1))\Rightarrow \mathbb{E}(X_n^*\vee 1)\leq 2+\mathbb{E}(|X_n|\log(X_n^*\vee 1))$. Since $x\log y\leq 10^{10}(2+x)\log(2+x)+\frac{y}{2}$ when x,y are large enough (insight: if $y>>x^2$ then $x\log y\leq\frac{y}{2}$; else $x\log y\leq 10^{10}(2+x)\log(2+x)$), $\mathbb{E}X_n^*\leq 10^{100}[1+\mathbb{E}(|X_n|+2)\log(|X_n|+2)]$. Then use dominated convergence theorem. □

- Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega,\mathscr{F})$, $\mathbb{Q}<<\mathbb{P}$ on $\mathscr{F}_n$ for every n and $M_n=\frac{\mathrm{d}\mathbb{Q}|_{\mathscr{F}_n}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_n}}$. $(M_n)_{n\geq 0}$ is a $\mathbb{P}$-martingale w.r.t. $(\mathscr{F}_n)_{n\geq 0}$. $\mathbb{Q}<<\mathbb{P}$ on $\mathscr{F}_\infty$ if and only if $M_n\to M$ in L^1. $\mathbb{Q}(A)=\int_A M\mathrm{d}\mathbb{P}+\mathbb{Q}(A\cap\{M=+\infty\})$.

  Proof Sufficiency. $\mathbb{Q}<<\mathbb{P}$ on $\mathscr{F}=\mathscr{F}_\infty$, thus let $Z=\frac{\mathrm{d}\mathbb{Q}|_{\mathscr{F}}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}}}$, we need to show M_n converges to Z in L^1. $\forall A\in\mathscr{F}_n, \int_A M_n\mathrm{d}\mathbb{P}=\mathbb{Q}(A)=\int_A Z\mathrm{d}\mathbb{P}\Rightarrow M_n=\mathbb{E}(Z|\mathscr{F}_n)$. Thus M_n is uniformly integrable, thus converges in L^1.
  Necessity. Suppose $M_n\to M$ a.s. and in L^1 We need to show $M_n=\mathbb{E}(M|\mathscr{F}_n)$ and $M=\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. It suffices to show $\mathbb{Q}(A)=\int_A M\mathrm{d}\mathbb{P}$ for all $A\in\cup_n\mathscr{F}_n$. Suppose $A\in\mathscr{F}_N$. Then $\mathbb{Q}(A)=\int_A M_N\mathrm{d}\mathbb{P}=\int_A M_{N+k}\mathrm{d}\mathbb{P}\to\int_A M\mathrm{d}\mathbb{P}$. By $\pi-\lambda$ theorem we can get the desired result.
  Special situation: Suppose $\mathbb{P}\perp\mathbb{Q}$ on $\mathscr{F}(\exists E$ s.t. $\mathbb{P}(E)=1,\mathbb{Q}(E^c)=1)$ and $\mathbb{P}<<\mathbb{Q}$ on $\mathscr{F}_n$. Then $\frac{1}{M_n}$ converges $\mathbb{Q}$-a.s. Let $\mathbb{R}=\frac{1}{2}(\mathbb{P}+\mathbb{Q})$, $\mathbb{P},\mathbb{Q}<<\mathbb{R}$ on $\mathscr{F}$, $\frac{\mathrm{d}\mathbb{P}|_{\mathscr{F}_n}}{\mathrm{d}\mathbb{R}|_{\mathscr{F}_n}}=\frac{2}{1+M_n}\to\frac{2M}{1+M}$ in $L^1(\mathbb{R})$, $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{R}}=\frac{2M_n}{1+M_n}\to\frac{2}{1+M}$ in $L^1(\mathbb{R})$. Then $\mathbb{Q}(A)=\mathbb{Q}(A\cap E^c)=\int_{A\cap E^c}\frac{2M}{1+M}\mathrm{d}\mathbb{R}=\int_A\frac{2M}{1+M}1_{E^c}\mathrm{d}\mathbb{R}\overset{\mathbb{P}(E^c)=0}{=}2\mathbb{R}(A\cap E^c)=2\int_A 1_{E^c}\mathrm{d}\mathbb{R}\Rightarrow\frac{2M}{1+M}1_{E^c}=2\cdot 1_{E^c}\Rightarrow M=+\infty$ on $E^c\Rightarrow\mathbb{Q}(M=+\infty)=1$. Similarly $\mathbb{P}(M=0)=\mathbb{Q}(M=+\infty)=1$.
  General situation: $\mathbb{Q}=\mathbb{Q}_1+\mathbb{Q}_2, \mathbb{Q}_1<<\mathbb{P},\mathbb{Q}_2\perp\mathbb{P}$ on $\mathscr{F}$. Therefore we can decompose M_n as $M_n=Y_n+Z_n$ where $Y_n\to Y$ in $L^1(\mathbb{P})$ and $Z_n\to 0$ $\mathbb{P}$-a.s. $\mathbb{Q}_1(A)=\int_A Y\mathrm{d}\mathbb{P}=\int_A M\mathrm{d}\mathbb{P}$. $\mathbb{Q}_2(A)=\mathbb{Q}_2(A\cap\{Z=+\infty\})$. Since $Z=0$ $\mathbb{P}$-a.s., $M<+\infty$ $\mathbb{P}$-a.s. and $\mathbb{Q}_2(M=+\infty)=1$, we have $\mathbb{Q}_2(A)=\mathbb{Q}(A\cap\{Z=+\infty\})=\mathbb{Q}_2(A\cap\{M=+\infty\})=\mathbb{Q}(A\cap\{M=+\infty\})$. To sum up, $\mathbb{Q}(A)=\int_A M\mathrm{d}\mathbb{P}+\mathbb{Q}(A\cap\{M=+\infty\})$. □

- Statement is false if $M_n\not\to M$ in L^1. Example: $\Omega=\{\omega=(\omega_1,\cdots,\omega_n,\cdots)\in\{\pm 1\}^{\mathbb{N}}\}, X_n(\omega)=\omega_n$. X_n's are i.i.d. under $\mathbb{P}$ and $\mathbb{Q}$, but $\mathbb{P}(X_n=1)=\frac{1}{2},\mathbb{P}(X_n=-1)=\frac{1}{2},\mathbb{Q}(X_n=1)=\frac{1}{3},\mathbb{Q}(X_n=-1)=\frac{2}{3}$. $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$. $\mathbb{P}(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n X_k=0)=1,\mathbb{Q}(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n X_k=-\frac{1}{3})=1$.
- Monotone class theorem for functions: Suppose $\mathcal{A}$ us a π-system and $\mathcal{H}$ be a class of functions from Ω to $\mathbb{R}$ s.t. (1) $1_A\in\mathcal{H}$ for every $A\in\mathscr{A}$, (2) if $f,g\in\mathcal{H}$ then $af+bg\in\mathcal{H}$, (3) if $f_n\in\mathcal{H}$ and $f_n\uparrow f$ then $f\in\mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$-measurable functions are in $\mathcal{H}$.
- Let $(Y_n)_{n\geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E}Y_k=1$. Then $M_n=\prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n\equiv 1$. Otherwise $M_n\to 0$ a.s.

  Proof Note that $\frac{1}{n}\log M_n=\frac{1}{n}\sum_{k=1}^n\log Y_k\to\mathbb{E}\log Y$ a.s. If $\mathbb{E}\log Y=0$ then by Jensen's inequality we have $Y_n\equiv 1$ which means M_n converges in L^1. If $\mathbb{E}\log Y<0$ then $M_n\to 0$ a.s. □

- Kakutani's theorem: $M_n = \prod_{k=1}^n Y_k$, $Y_k \geq 0$ are independent, $\mathbb{E}Y_k = 1$, $\lambda_k = \mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k \lambda_k > 0$, then $M_n \to M$ in L^1; (2) If $\prod_k \lambda_k = 0$, then $M_n \to 0$ a.s.

  Proof Let $Z_n = \prod_{k=1}^n \frac{\sqrt{Y_k}}{\lambda_k}$. Then Z_n is a martingale and has an a.s. limit Z, and $M_n = (\prod_{k=1}^n \lambda_k)^2 Z_n^2$. If $\prod_k \lambda_k > 0$, then Z_n is L^2 bounded and then convergence in L^2, which implies $M_n \to M$ in L^1. If $\prod_k \lambda_k = 0$, it is obvious that $M_n \to 0$ a.s. □

- Martingale LLN: Let $(M_n)_{n\geq 0}$ be a martingale s.t. $\sum_{k=1}^{+\infty} \frac{\mathbb{E}(M_k - M_{k-1})^2}{k^2} < +\infty$. Then $\frac{M_n}{n} \to 0$ a.s.

  Proof Let $Y_n = \sum_{k=1}^n \frac{X_k}{k}$. Then $(Y_n)_{n\geq 0}$ is an L^2 bounded martingale, thus $Y_n \to Y$ a.s. Then use Kronecker's lemma. □

- Martingale CLT: Let $(M_n)_{n\geq 0}$ be a martingale with $M_0 = 0$ and $\sigma_n^2 = \sum_{k=1}^n \mathbb{E}X_k^2 = \mathbb{E}\langle M\rangle_n$. Assume that $\frac{1}{\sigma_n^2} \max_{1\leq k\leq n}(\mathbb{E}X_k^2) \to 0$, $\frac{1}{\sigma_n^2}\sum_{k=1}^n \mathbb{E}(X_k^2 1_{\{|X_k| > \epsilon\sigma_n\}}|\mathscr{F}_{k-1}) \xrightarrow{p} 0$ for all $\epsilon > 0$, $\frac{1}{\sigma_n^2}\langle M\rangle_n \xrightarrow{p} 1$. Then $\frac{M_n}{\sigma_n} \Rightarrow \mathcal{N}(0,1)$.

2 Markov Chains

- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain on a discrete space S. $\mathbb{P}^x$: law of $(X_n)_{n\geq 0}$ conditioned on $X_0 = x$. $\mathbb{P}(X_{n+1} \in A|\mathscr{F}_n) = \mathbb{P}^{X_n}(X_1 \in A) = \mathbb{P}(X_1 \in A|X_0 = X_n)$. $\mathbb{E}^x$: expectation under $\mathbb{P}^x$. $\mathbb{P}^x(X_1 = y) = p(x,y)$.

- For every $f : S \to \mathbb{R}$ bounded, define $(\mathcal{P}f)(x) = \sum_{y\in S} p(x,y)f(y) = \mathbb{E}^x(f(X_1))$, $(\mathcal{L}f)(x) = \sum_{y\in S} p(x,y)f(y) - f(x)$. $\mathcal{L} = \mathcal{P} - \mathrm{id}$, the generator.

- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with generator $\mathcal{L}$. Then for every bounded $f : S \to \mathbb{R}$, $M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1}(\mathcal{L}f)(X_k)$ is a martingale. Conversely, let $(X_n)_{n\geq 0}$ be a process and $\mathcal{L}$ be an operator on $\mathcal{B}(S)$ s.t. M_n^f is a martingale for every f, then $(X_n)_{n\geq 0}$ is a Markov chain with generator $\mathcal{L}$.

- Given operator $\mathcal{L}$ on $\mathcal{B}(S)$, we say $f : S \to \mathbb{R}$ is (1) harmonic for $\mathcal{L}$ if $\mathcal{L}f = 0$; (2) sub-harmonic for $\mathcal{L}$ if $\mathcal{L}f \geq 0$; (3) super-harmonic for $\mathcal{L}$ if $\mathcal{L}f \leq 0$.

- Let f be the generator of a Markov chain $(X_n)_{n\geq 0}$. Then f is (sub-/super-)harmonic $\Leftrightarrow$ $f(X_n)_{n\geq 0}$ is a (sub-/super-) martingale.

- f is (sub-/super-)harmonic on $D \subset S$ if $\mathcal{L}f \geq / \leq / = 0$ on D. Let $\tau = \inf\{k \geq 0 : X_k \in D^c\}$, then $(f(X_{n\wedge\tau}))_{n\geq 0}$ is a (sub-/super)martingale.

- Maximum principle: Let $(X_n)_{n\geq 0}$ be a Markov chain and $D \subset S$ s.t. the stopping time $\tau = \inf\{k \geq 0, X_k \in D^c\}$ is a.s. finite. If f is bounded and sub-harmonic on D, then $\sup_{x\in D} f(x) \leq \sup_{x\in D^c} f(x)$.

  Proof f is sub-harmonic implies $(f(X_{n\wedge\tau}))$ is a sub-martingale, hence for $x \in D$ we have $f(x) \leq \mathbb{E}^x(f(X_{n\wedge\tau})) \to \mathbb{E}^x(f(X_\tau)) \leq \sup_{x\in D^c} f(x)$. □

- $A \subset S, \tau_A = \sup\{k \geq 0 : X_k \in A\}$. (1) $u(x) = \mathbb{P}^x(\tau_A < +\infty) \Rightarrow \begin{cases} \mathcal{L}u = 0 & \text{on } A^c \\ u = 1 & \text{on } A \end{cases}$. (2) $u(x) = \mathbb{P}(\tau_A < \tau_B) \Rightarrow$ $\begin{cases} \mathcal{L}u = 0 & \text{on } (A\cup B)^c \\ u = 1 & \text{on } A \\ u = 0 & \text{on } B. \end{cases}$. (3) $u(x) = \mathbb{E}^x[\tau_A] \Rightarrow \begin{cases} \mathcal{L}u = -1 & \text{on } A^c \\ u = 0 & \text{on } A \end{cases}$.

- Any nonnegative solution v to $\begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = 1 & \text{on } A \end{cases}$ satisfies $v \geq u$. Furthermore, if $u \equiv 1$, then $\exists 1$ bounded solution to $\begin{cases} \mathcal{L}v = 0 & \text{on } A^c \\ v = f & \text{on } A \end{cases}$ with $v(x) = \mathbb{E}^x(f(X_{\tau_A}))$.

  Proof Let $v(x)$ be a non-negative solution, then $v(X_{n\wedge\tau_A})_{n\geq 0}$ is a martingale. $v(x) = \mathbb{E}^x v(X_{n\wedge\tau_A}) = \mathbb{E}^x v(X_{n\wedge\tau_A})1_{\tau_A<\infty} + \mathbb{E}^x v(X_{n\wedge\tau_A})1_{\tau_A=\infty} \geq \mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A<\infty}$. Let $n \to \infty$ and by Fatou's lemma, we have $v(x) \geq \mathbb{E}^x v(X_{\tau_A})1_{\tau_A<\infty} = \mathbb{P}^x(\tau_A < \infty) = u(x)$. If $u(x) \equiv 1$ and $v(x)$ is bounded, then by bounded convergence theorem, $v(x) = \mathbb{E}^x v(X_{n\wedge\tau_A}) \to \mathbb{E}^x v(X_{\tau_A}) = \mathbb{E}^x f(X_{\tau_A})$.□

- Doob's h-transform: Let h be nonnegative, harmonic with $h(x_0)=1$ for some $x_0 \in S$. Then $(h(X_n))_{n\geq 0}$ is a martingale with $\mathbb{E}^{\mathbb{P}^{x_0}}(h(X_n))=1$. Then $\exists 1$ measure $\mathbb{Q}^h$ on $\mathscr{F}_\infty$ s.t. $\frac{\mathrm{d}\mathbb{Q}^h}{\mathrm{d}\mathbb{P}^{x_0}|_{\mathscr{F}_n}} = h(X_n), \forall n \geq 0$. $\mathbb{Q}^h(X_0=x_0)=1$, $(X_n)_{n\geq 0}$ never visits the set $D=\{x : h(x)=0\}$. Under $\mathbb{Q}^h$, $(X_n)_{n\geq 0}$ is again a Markov chain on $S\backslash D$ with transition probability $q(x,y)=\frac{p(x,y)h(y)}{h(x)}$ (or equivalently, $(\mathcal{L}^h f)(x) = \frac{1}{h(x)}(\mathcal{L}(hf))(x)$).

  Proof The first two props are trivial. $\mathbb{Q}(X_{n+1}=y|\mathscr{F}_n) = \frac{\mathbb{Q}(X_{n+1}=y, X_n=x_n,\cdots,X_0=x_0)}{\mathbb{Q}(X_n=x_n,\cdots,X_0=x_0)} = \frac{\int_{\{X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0\}} h(X_{n+1})\mathrm{d}\mathbb{P}^{x_0}}{\int_{\{X_n=x_n,\cdots,X_0=x_0\}} h(X_n)\mathrm{d}\mathbb{P}^{x_0}} = \frac{h(y)\mathbb{P}^{x_0}(X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0)}{h(x_n)\mathbb{P}^{x_0}(X_n=x_n,\cdots,X_0=x_0)} = \frac{h(y)p(x_n,y)}{h(x_n)}$. Next we show $M_n^f := f(X_n)-f(X_0)-\sum_{k=0}^{n-1}(\mathcal{L}^h f)(X_k)$ is a $\mathbb{Q}$-martingale for any bounded f. Let $Z_n = \mathbb{E}^{\mathbb{Q}} f(X_{n+1})|\mathscr{F}_n$. $\forall A \in \mathscr{F}_n$, $\int_A Z_n h(X_n)\mathrm{d}\mathbb{P}^{x_0} = \int_A Z_n \mathrm{d}\mathbb{Q} = \int_A f(X_{n+1})\mathrm{d}\mathbb{Q} = \int_A f(X_{n+1})h(X_{n+1})\mathrm{d}\mathbb{P}^{x_0} = \mathbb{E}^{\mathbb{P}^{x_0}}[\mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1})1_A|\mathscr{F}_n)] = \mathbb{E}^{\mathbb{P}^{x_0}}[1_A \mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1})|\mathscr{F}_n)] = \int_A \mathcal{P}(hf)(X_n)\mathrm{d}\mathbb{P}^{x_0}$. Thus $Z_n = \frac{\mathcal{P}(hf)(X_n)}{h(X_n)}$ only depends on X_n, i.e. $(X_n)_{n\geq 0}$ is a MC on $\mathbb{Q}$ with generator $\mathcal{L}^h$. □

- An irreducible Markov chain $(X_n)_{n\geq 0}$ (1) is transient if $\exists x$ and $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty | X_0 = x) < 1$; (2) is recurrent if $\exists$ a finite set $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty) = 1$ for all $x \in S$. (3) is positive recurrent if $\exists$ a finite set $A \subset S$ s.t. $\mathbb{E}(\tau_A) < \infty$ for all $x \in S$.

- Foster-Lyapunov criterion: An irreducible MC on a countable state space S (1) is transient iff $\exists v : S \to \mathbb{R}^+$ and $A \subset S$ non-empty s.t. $\mathcal{L}v \leq 0$ on A^c and $v(x) < \inf_{y\in A} v(y)$ for some $x \in A^c$; (2) is recurrent iff $\exists v : S \to \mathbb{R}^+$ s.t. $\mathcal{L}v \leq 0$ on A^c where A is a finite set and $\{x : v(x) \leq N\}$ is finite for every N; (3) is positive recurrent iff $\exists v : S \to \mathbb{R}^+$, $A \subset S$ finite, $\exists \epsilon > 0$ s.t. $\mathcal{L}v \leq -\epsilon$ on A^c and $\sum_{y\in S} p(x,y)V(y) < +\infty$ for all $x \in A$.

  Proof (1) $v(X_{n\wedge\tau_A})_{n\geq 0}$ is a super-martingale, hence $v(x) \geq \mathbb{E}v(X_{n\wedge\tau_A}) \geq \mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A<\infty}$. Let $n \to \infty$ we know $v(x) \geq \mathbb{E}v(X_{\tau_A}1_{\tau_A<\infty}) \geq (\inf_{y\in A} v(y))\mathbb{P}^x(\tau_A < \infty) \Rightarrow \mathbb{P}^x(\tau_A < \infty) < \frac{v(x)}{\inf_{y\in A} v(y)} < 1$. (2) On $\{\tau_A = \infty\}$, $\limsup_{n\to\infty} v(X_{n\wedge\tau_A}) = +\infty$ a.s. Since $(v(X_{n\wedge\tau_A}))_{n\geq 0}$ is a nonnegative super-martingale, hence converges a.s., therefore $\lim_{n\to\infty} v(X_{n\wedge\tau_A}) = +\infty$ a.s. Note that $v(x) \geq \mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A=\infty}$. Since LHS is a finite number, we have $\mathbb{P}^x(\tau_A = \infty) = 0$. (3) $\mathbb{E}v(X_{n\wedge\tau_A})|\mathscr{F}_{n-1} \leq v(X_{(n-1)\wedge\tau_A}) - \epsilon 1_{\tau_A \geq n}$. Taking expectation on the both sides, $\mathbb{E}v(X_{n\wedge\tau_A}) \leq \mathbb{E}v(X_{(n-1)\wedge\tau_A}) - \epsilon\mathbb{E}^x 1_{\tau_A\geq n} \leq \cdots \leq v(x) - \epsilon\sum_{k=1}^n \mathbb{P}^x(\tau_A \geq k) \Rightarrow \mathbb{E}^x\tau_A = \sum_{k=1}^\infty \mathbb{P}^x(\tau_A \geq k) \leq \frac{v(x)}{\epsilon} < \infty$.

  Conversely, (1) Let $v(x) = \mathbb{P}^x(\tau_A < \infty)$. (2) Let $u(x) = \mathbb{P}^x(\tau_B < \tau_A)$. We have shown that if $x \in (A\cup B)^c$ then $\mathcal{L}u \leq 0$. When $x \in B$, $(\mathcal{L}u)(x) = \sum_{y\in S} p(x,y)u(y) - 1 \leq 0$. Take $B_N \downarrow \emptyset$ s.t. B_N^c is finite for every N. Via a diagonal argument $\Rightarrow \exists$ subsequence $\{N_k\}$ s.t. $v(x) := \sum_{k\geq 1} \mathbb{P}^x(\tau_{B_{N_k}} < \tau_A) < +\infty$ for every $x \in S$. (3) Let $v(x) = \mathbb{E}^x(\tau_A)$. □

- e.g. $h(x) = \frac{\mathbb{P}^x(\tau_A<\tau_B)}{\mathbb{P}^{x_0}(\tau_A<\tau_B)}$ is harmonic on $(A\cup B)^c$ with $h(x_0) = 1 (x_0 \in (A\cup B)^c)$. Then $\forall x, y \in (A\cup B)^c, q(x,y) = \frac{h(y)p(x,y)}{h(x)} = \frac{\mathbb{P}^y(\tau_A<\tau_B)p(x,y)}{\mathbb{P}^x(\tau_A<\tau_B)} = \frac{\mathbb{P}^x(X_1=y,\tau_A<\tau_B)}{\mathbb{P}^x(\tau_A<\tau_B)} = \mathbb{P}^x(X_1 = y|\tau_A < \tau_B)$.

- e.g. $\mathbb{P}$ is simple symmetric random walk on $\mathbb{Z}$ starting from $X_0 = 0$. Question: what is the law of $(X_n)_{n\geq 0}$ conditioned on $X_n \geq 0$ for all n? Let $\tau_k = \inf\{n \geq 0, X_n = k\}$. On $\{\tau_N < \tau_{-1}\}$, $\frac{h(y)}{h(x)} = \frac{\mathbb{P}^y(\tau_N<\tau_{-1})}{\mathbb{P}^x(\tau_N<\tau_{-1})} = \frac{y+1}{x+1}$. Thus $q_N(x,y) = \frac{1}{2}\frac{y+1}{x+1}, |x-y| = 1, x \in \{0,\cdots,N-1\} \Rightarrow q(x,y) = \frac{1}{2}\frac{y+1}{x+1}, x \geq 0, |x-y| = 1$.

3 Ergodic Theorem

- Basic setup: a measurable map $T : (\Omega,\mathscr{F}) \to (\Omega,\mathscr{F})$. Examples: (1) circle rotations: $\Omega = \mathbb{R}/\mathbb{Z}, T : x \mapsto x+\alpha$; (2) doubling map: $\Omega = \mathbb{R}/\mathbb{Z}, x \mapsto 2x$; (3) shift map: $\Omega = S^{\mathbb{N}}, (T\omega)_n = \omega_{n+1}$.

- Let $T : (\Omega,\mathscr{F}) \to (\Omega,\mathscr{F})$ measurable and $\mathbb{P}$ be a probability measure on $(\Omega,\mathscr{F})$. We say T is measure-preserving if $\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A)$ for every $A \in \mathscr{F}$ (or $\mathbb{P}\circ T^{-1} = \mathbb{P}$).

- Question: what if we define by $\mathbb{P}(T(A)) = \mathbb{P}(A)$ for every $A \in \mathscr{F}$ instead? $\mathbb{P}\circ T = \mathbb{P} \Rightarrow \mathbb{P}\circ T^{-1} = \mathbb{P}$ while the converse proposition is false.

- $(X_n)_{n\geq 0}$ be i.i.d. $\sim \mu$. We can build $(\Omega,\mathscr{F},\mathbb{P})$ and $X_n : \Omega \to \mathbb{R}$ measurable s.t. $(X_n)_{n\geq 0}$ i.i.d. $\sim \mu$ under $\mathbb{P}$: (1) $\Omega = \mathbb{R}^{\mathbb{N}} = \{\omega : \omega = (\omega_0,\omega_1,\cdots)\}$; (2) $X_n(\omega) = \omega_n$; (3) $\mathscr{F} = \sigma(X_0, X_1,\cdots,X_n,\cdots)$; (4) $\mathbb{P} = \mu^{\otimes\mathbb{N}}$. It is easy to show that the shift map is measure-preserving: $\mathscr{F}$ is generated by sets of the form $A = \{\omega_{k_1} \in I_1,\cdots,\omega_{k_N} \in I_N\}$, $T^{-1}(A) = \{\omega : (T\omega)_{k_1} \in I_1,\cdots,(T\omega)_{k_N} \in I_N\} = \{\omega : \omega_{k_1+1} \in I_1,\cdots,\omega_{k_N+1} \in I_N\}$. Key: the only thing used is that $(X_{k_1},\cdots,X_{k_N}) \overset{\text{law}}{=} (X_{k_1+1},\cdots,X_{k_N+1})$ for every N and every $k_1,\cdots,k_N$.

- A sequence of random variables is stationary if $(X_n)_{n\in J} \overset{\text{law}}{=} (X_{n+k})_{n\in J}$ for all k and finite set J.

- Let $T : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ be measure-preserving and $X : \Omega \to \mathbb{R}$ be measurable. Then $X_n(\omega) := X(T^n\omega)$ defines a stationary sequence.

  Proof It suffices to show that for every N, every $I_1, \cdots, I_N \subset \mathbb{R}$ and every $k_1 < k_2 < \cdots < k_N$, we have $\mathbb{P}(X_{k_1} \in I_1, \cdots, X_{k_N} \in I_N) = \mathbb{P}(X_{k_1+1} \in I_1, \cdots, X_{k_N+1} \in I_N)$. $\mathbb{P}(\{\omega : X_{k_1}(\omega) \in I_1, \cdots, X_{k_N}(\omega) \in I_N\}) = \mathbb{P}(T^{-1}\{\omega : X_{k_1}(\omega) \in I_1, \cdots, X_{k_N}(\omega) \in I_N\}) = \mathbb{P}(\{\omega : X_{k_1}(T\omega) \in I_1, \cdots, X_{k_N}(T\omega) \in I_N\}) = \mathbb{P}(\{\omega : X_{k_1+1}(\omega) \in I_1, \cdots, X_{k_N+1}(\omega) \in I_N\})$. □

- Let $(\Omega, \mathscr{F}, \mathbb{P}, T)$ be a measure-preserving system. (1) A set $A \in \mathscr{F}$ is invariant if $\mathbb{P}(A \triangle T^{-1}(A)) = 0$. (2) A random variable $X : \Omega \to \mathbb{R}$ is invariant if $X = X \circ T$ $\mathbb{P}$-a.e.

- The collection of invariant sets $\mathcal{I} = \{A \in \mathscr{F} : A \text{ is invariant}\}$ is a σ-algebra and $X : \Omega \to \mathbb{R}$ is invariant iff it is $\mathcal{I}$-measurable.

- We say $T : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ measurable-preserving is ergodic if $\mathbb{P}(A) = 0$ or 1 for all $A \in \mathcal{I}$.

- Let $T : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ be measure preserving and $f \in L^p(p \geq 1)$. Then $\frac{1}{N}\sum_{k=0}^{N-1} f \circ T^K \to \mathbb{E}(f|\mathcal{I})$ a.s. and in L^p. In particular, $\mathbb{E}(f|\mathcal{I}) = \mathbb{E}f$ if T is ergodic.

  Proof We first show convergence in L^p.

  Lemma 1 If $(\Omega, \mathscr{F}, \mathbb{P}, T)$ is a measure-preserving system and $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$. Then $\int_\Omega X \mathrm{d}\mathbb{P} = \int_\Omega X \circ T \mathrm{d}\mathbb{P}$. In fact, $||X||_{L^p} = ||X \circ T||_{L^p}, p \in [1, +\infty]$.

  Proof Take $X = 1_A$. LHS $= \mathbb{P}(A) = \mathbb{P}(T^{-1}(A)) = \int_\Omega 1_A(T\omega)\mathrm{d}\mathbb{P}$. □

  Let $\mathcal{U}_T : L^p(\Omega, \mathscr{F}, \mathbb{P}) \to L^p(\Omega, \mathscr{F}, \mathbb{P})$ be defined by $(\mathcal{U}_T f)(\omega) := f(T\omega)$ (or $\mathcal{U}_T f = f \circ T$).

  For $p = 2$, $\mathcal{U}_T : L^2 \to L^2$ is an isometry in the sense that $\langle f, g\rangle = \langle \mathcal{U}_T f, \mathcal{U}_T g\rangle$. LHS $= \frac{1}{N}\sum_{k=0}^{n} \mathcal{U}_T^k f$, $f = \mathbb{E}(f|\mathcal{I}) + (f - \mathbb{E}(f|\mathcal{I})) \Rightarrow$ LHS $= \underbrace{\mathbb{E}(f|\mathcal{I})}_{\mathrm{Ker}(\mathcal{U}_T - \mathrm{Id}) \overset{?}{=} \mathrm{Ker}(\mathcal{U}_T^* - \mathrm{Id})} + \frac{1}{N}\sum_{k=0}^{N} \mathcal{U}_T^k(f - \mathbb{E}(f|\mathcal{I}))$. Since $\mathcal{H} = \mathrm{Ker}(A) \oplus \overline{\mathrm{Im}(A^*)}$, $\exists g \in \mathcal{H}$ s.t. $||f - \mathbb{E}(f|\mathcal{I}) - \underbrace{(\mathcal{U}_T^* - \mathrm{Id})g}_{\overset{?}{=}(\mathcal{U}_T - \mathrm{Id})g}|| < \epsilon$.

  Lemma 2 Let $A : \mathcal{H} \to \mathcal{H}$ be an isometry. If $Af = f$, then $A^*f = f$.

  Proof $\langle A^*f, g\rangle = \langle f, Ag\rangle = \langle Af, Ag\rangle = \langle f, g\rangle$. □

  Proposition 1 $\mathcal{H} = \mathrm{Ker}(A^*) \oplus \overline{\mathrm{Im}(A)}$.

  Proof We show that $\mathrm{Ker}(A^*) = (\mathrm{Im}(A))^\perp$. (i) $f \in \mathrm{Ker}(A^*) \Rightarrow A^*f = 0 \Rightarrow \langle f, Ag\rangle = \langle A^*f, g\rangle = 0$. (ii) $f \in (\mathrm{Im}(A))^\perp \Rightarrow \langle f, Ag\rangle = 0$ for all $g \in \mathcal{H} \Rightarrow \langle A * f, g\rangle = 0$ for all $g \in \mathcal{H} \Rightarrow A^*f = 0$. □

  $\mathcal{H} = L^2(\omega, \mathscr{F}, \mathbb{P}) = \mathrm{Ker}(\mathcal{U}_T^* - \mathrm{Id}) + \overline{\mathrm{Im}(\mathcal{U}_T - \mathrm{Id})} \Rightarrow \forall f \in \mathscr{H}, \forall \epsilon > 0, \exists g, h \in \mathscr{H}$ s.t. $||h||_{L^2} < \epsilon$ and $f = \mathbb{E}(f|\mathcal{I}) + (\mathcal{U}_T - \mathrm{Id})g + h \Rightarrow \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{U}_T^k f = \mathbb{E}(f|\mathcal{I}) + \underbrace{\frac{1}{N}(\mathcal{U}_T^N g - g)}_{||\cdot||_{L^2} \leq \frac{1}{N}||g||_{L^2} \to 0} + \underbrace{\frac{1}{N}\sum_{k=0}^{N-1} \mathcal{U}_T^k h}_{||\cdot||_{L^2} < \epsilon} \Rightarrow \limsup_{N\to\infty} ||\frac{1}{N}\sum_{k=0}^{N-1} \mathcal{U}_T f - \mathbb{E}(f|\mathcal{I})||_{L^2} < \epsilon$.

  For $p \neq 2$, let $S_N f = \sum_{k=0}^{N-1} f \circ T^k$ and $A_N f = \frac{1}{N} S_N f$.

  (1) If $f \in L^\infty$, then $||A_N f||_{L^\infty} \leq ||f||_{L^\infty}, ||A_N f - \mathbb{E}(f|\mathcal{I})||_{L^2} \to 0 \Rightarrow A_N f \to \mathbb{E}(f|\mathcal{I})$ in L^p for every $p \in [1, +\infty)$ (for $p \geq 2$, $||A_N f - \mathbb{E}(f|\mathcal{I})||_{L^p}^p \leq ||f||_{L^\infty}^{p-2} ||A_N f - \mathbb{E}(f|\mathcal{I})||_{L^2}^2$; for $1 \leq p < 2$, $||A_N f - \mathbb{E}(f|\mathcal{I})||_{L^p}^p \leq ||A_N f - \mathbb{E}(f|\mathcal{I})||_{L^2}^p ||1||_{L^2}^{2-p}$).

  (2) If $f \in L^p(p \geq 1)$, then $\forall \epsilon > 0, \exists g \in L^\infty$ s.t $||f - g||_{L^p} < \epsilon$,

  $||A_N f - \mathbb{E}(f|\mathcal{I})||_{L^p} \leq \underbrace{||A_N(f - g)||_{L^p}}_{<\epsilon} + \underbrace{||A_N g - \mathbb{E}(g|\mathcal{I})||_{L^p}}_{\to 0 \text{ as } N \to +\infty} + \underbrace{||\mathbb{E}(g - f|\mathcal{I})||_{L^p}}_{<\epsilon} \Rightarrow \forall \epsilon > 0, \limsup_{N\to\infty} ||A_N f - \mathbb{E}(f|\mathcal{I})||_{L^p} < 2\epsilon$.

  We next show convergence a.s.

  Maximum ergodic theorem $f \in L^1(\Omega, \mathscr{F}, \mathbb{P}), S_n = \sum_{k=0}^{n-1} f \circ T^k, M_n = \max\{S_1, \cdots, S_n\}$. Then $\int_{\{M_n \geq 0\}} f(\omega)\mathbb{P}(\mathrm{d}\omega) \geq 0$.

  Proof $M_{n-1}(T\omega) = \max\{S_1(T\omega), \cdots, S_{n-1}(T\omega)\} = \max\{S_2(\omega), S_n(\omega)\} - f(\omega) \Rightarrow \max\{0, M_{n-1}(T\omega)\} = M_n(\omega) - f(\omega) \Rightarrow f(\omega) = M_n(\omega) - \max\{0, M_{n-1}(T\omega)\}$. $\int_{\{M_n>0\}} f\mathrm{d}\mathbb{P} = \int_{\{M_n>0\}} M_n \mathrm{d}\mathbb{P} - \int_{\{M_n>0\}} \max\{0, M_{n-1}(T\omega)\}\mathrm{d}\mathbb{P} = \int_{\{M_n>0\}} M_n \mathrm{d}\mathbb{P} - \int_{\{M_n>0\}\cap\{M_{n-1}\circ T>0\}} M_{n-1} \circ T \mathrm{d}\mathbb{P} \Rightarrow \int_{\{M_n>0\}} f\mathrm{d}\mathbb{P} \geq \int_{\{M_n\geq 0\}} M_n \mathrm{d}\mathbb{P} - \int_{\{\cdots\}} M_n \circ T\mathrm{d}\mathbb{P} = \int_{\{M_n \geq 0\}} M_n \mathrm{d}\mathbb{P} - \int_{T\{\cdots\}} M_n \mathrm{d}\mathbb{P} \geq 0$. □

  Corollary 1 $\mathbb{P}(\omega : \sup_{n\geq 1}(A_n f)(\omega) > \lambda) \leq \frac{\mathbb{E}|f|}{\lambda}$.

  Proof Let $E_N = \{\omega : \sup_{1\leq n\leq N}(A_n f)(\omega) > \lambda\} = \{\omega : \sup_{1\leq n\leq N}(A_n(f - \lambda))(\omega) > 0\} = \{\omega : \sup_{1\leq n\leq N}(S_n(f - \lambda))(\omega) > 0\}$. $E_N \uparrow E = \{\omega : \sup_{n\geq 1}(A_n f)(\omega) > \lambda\}$. $\int_{E_n}(f - \lambda)\mathrm{d}\mathbb{P} \geq 0 \Rightarrow \mathbb{P}(E_n) \leq \frac{\int_{E_n} f\mathrm{d}\mathbb{P}}{\lambda} \leq \frac{\mathbb{E}|f|}{\lambda} \Rightarrow \mathbb{P}(E) \leq \frac{\mathbb{E}|f|}{\lambda}$. □

Goal: $f \in L^1$ (for finite measure $\mathbb{P}$, $L^p \subset L^1$), need to show $\frac{1}{N}\sum_{k=0}^{N-1} f \circ T^k \to \mathbb{E}(f|\mathcal{I})$ a.s.

(1) If $f \in L^2$ is $\mathcal{I}$-measurable, then $A_N f = f = \mathbb{E}(f|\mathcal{I})$ a.s.

(2) If $f = (\mathcal{U}_T - \mathrm{Id})g$ for some $g \in L^\infty$, then $(A_N f)(\omega) = \frac{1}{N}(g(T^N\omega) - g(\omega)) \leq \frac{2||g||_{L^\infty}}{N} \to 0$. Check $\mathbb{E}((\mathcal{U}_T - \mathrm{Id})g|\mathcal{I}) = 0 : \forall A \in \mathcal{I}, \int_A (g \circ T - g)\mathrm{d}\mathbb{P} = \int_{T^{-1}(A)} g \circ T \mathrm{d}\mathbb{P} - \int_A g\mathrm{d}\mathbb{P} = \int_A g\mathrm{d}\mathbb{P} - \int_A g\mathrm{d}\mathbb{P} = 0$.

(3) $\Lambda = \{f = \mathbb{E}(f_0|\mathcal{I}) + (\mathcal{U}_T - \mathrm{Id})g : f_0 \in L^2, g \in L^\infty\}$ is dense in L^1. If $f \in L^1$, then $\exists f_j \in \Lambda$ s.t. $f_j \to f$ in L^1. We need to show $\mathbb{P}(\limsup_{N\to\infty} |A_N f - \mathbb{E}(f|\mathcal{I})| > \epsilon) = 0$. $|A_N f - \mathbb{E}(f|\mathcal{I})| \leq |A_N(f - f_j)| + \underbrace{|A_N f_j - \mathbb{E}(f_j|\mathcal{I})|}_{\to 0 \text{ a.s.}} + |\mathbb{E}(f_j - f|\mathcal{I})| \Rightarrow$ $\mathbb{P}(\limsup_{N\to\infty} |A_N f - \mathbb{E}(f|\mathcal{I})| > \epsilon) \leq \mathbb{P}(\limsup_{N\to+\infty} |A_N(f - f_j)| > \frac{\epsilon}{2}) + \mathbb{P}(|\mathbb{E}(f_j - f|\mathcal{I})| > \frac{\epsilon}{2}) \leq \frac{2\mathbb{E}|f_j - f|}{\epsilon} + \frac{2\mathbb{E}|f_j - f|}{\epsilon} \to 0$. □

- Kingman's subadditive ergodic theorem: Let $(\Omega, \mathscr{F}, \mathbb{P}, T)$ be a measure-preserving space and $\{g_n\} \in L^1$ subadditive in the sense that $g_{n+m} \leq g_n + g_m \circ T^n$ for every n, m. Then (1) $\lim_{n\to\infty} \frac{\mathbb{E}(g_n)}{n} \to \inf_{k\geq 1} \frac{\mathbb{E}(g_k)}{k}$ (possibly $-\infty$); (2) $\frac{g_n}{n}$ convergence a.s. to F where F is $\mathcal{I}$-measurable and $\mathbb{E}F = \inf_{k\geq 1} \frac{\mathbb{E}(g_k)}{k}$; (3) If $\mathbb{E}F > -\infty$, then the convergence is also in L^1.

Proof Recall an elementary version. If $\{a_n\} \in \mathbb{R}$ s.t. $a_{n+m} \leq a_n + a_m, \forall n, m$, then $\frac{a_n}{n} \to \inf_{k\geq 1} \frac{a_k}{k}$ as $n \to \infty$. □