

# 高等数学 A I 习题课讲义

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# 1 第 1 次习题课

## 1.1 Questions

1.  $f(x) = |x \sin^3 x| e^{\cos x}$ . Bounded? Monotonic? Even? Odd?
2. Show that  $f(x) = x - [x]$  is bounded and periodic.
3.  $f(x) = \begin{cases} x^2 & x \leq 0 \\ \cos x + \sin x & x > 0 \end{cases}$ . Calculate  $f(-x)$ .
4.  $f(x) = e^{x^2}$  and  $f \circ \phi = 1 - 3x$  and  $\phi(x) \geq 0$ , find  $\phi(x)$  and its domain.
5. Show that  $\lim_{n \rightarrow \infty} \frac{4n^2}{n^2 - n} = 4$ .
6.  $q > 1$ , show that  $\lim_{n \rightarrow \infty} \frac{n^k}{q^n} = 0$ .
7. Calculate  $\lim_{n \rightarrow \infty} \frac{1}{n^2 + n + 1} + \frac{2}{n^2 + n + 2} + \cdots + \frac{n}{n^2 + n + n}$ .
8. Let  $x_1 > 0$  and  $x_{n+1} = \frac{3(1+x_n)}{3+x_n}$ . Calculate  $\lim_{n \rightarrow \infty} x_n$ .
9. Show that  $\lim_{n \rightarrow \infty} n^{1/n} = 1$ .
10. Calculate  $\lim_{n \rightarrow \infty} \sqrt[n]{2^n + 3^n}$ .
11. Show that  $\lim_{n \rightarrow \infty} \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} = \frac{\pi^2}{6}$ .
12.  $\sum_{n=1}^{\infty} b_n = \infty$ ,  $\frac{a_n}{b_n} \rightarrow 0$ ,  $a_n, b_n > 0$ , show that  $\sum_{n=1}^{\infty} \frac{a_n}{b_n} = 0$ .
13. (Stolz)  $0 < b_n \uparrow +\infty$ ,  $a_n > 0$ , if  $(a_n - a_{n-1})/(b_n - b_{n-1}) \rightarrow L$  then  $a_n/b_n \rightarrow L$ .

## 1.2 Solutions

1. Note that  $f(2k\pi + \frac{\pi}{4}) = \frac{\sqrt{2}}{4}(2k\pi + \frac{\pi}{4})e^{\frac{1}{\sqrt{2}}} \rightarrow \infty$ , so  $f(x)$  is not bounded. And  $f(k\pi) \equiv 0$  and  $f(x) \neq 0$ , otherwise, so  $f(x)$  is not monotonous. Obviously it is odd.
2. It is relatively trivial to show that  $f(x)$  has the period 1 and  $|f(x)| \leq 1$  holds for all  $x \in \mathbb{R}$ .
3.  $f(-x) = \begin{cases} x^2 & x \geq 0 \\ \cos x - \sin x & x < 0 \end{cases}$ .
4.  $f(\phi) = e^{\phi^2} = 1 - 3x \Rightarrow \phi = \sqrt{\log(1 - 3x)}$  and its domain is  $\log(1 - 3x) \geq 0 \Leftrightarrow x \leq 0$ .
5.  $|\frac{4n^2}{n^2 - n} - 4| = \frac{4}{n-1}$ , when  $n \geq \frac{4}{\epsilon} + 1$ , the difference is smaller than  $\epsilon$ .
6. Note that  $q^n = (1 + q - 1)^n \geq C_n^{k+1}(q - 1)^{k+1} = a_{k+1}n^{k+1} + a_k n^k + \cdots + a_0$  is a higher-order cumulant than  $n^k$ .
7. Use squeeze theorem.  $(*) \geq \frac{\sum_{i=1}^n i}{(n+1)^2} = \frac{n(n+1)}{2(n+1)^2} \rightarrow \frac{1}{2}$ ,  $(*) \leq \frac{\sum_{i=1}^n i}{n^2 + n} = \frac{n(n+1)}{2(n+n)} \rightarrow \frac{1}{2}$ .
8.  $x_n$  is bound to be monotonous in this kind of problem. First we assume the limit exists and calculate the answer: let  $n \rightarrow \infty$ , we have  $a = \frac{3(1+a)}{3+a} \Rightarrow a = \sqrt{3}$ . Then using recurrence formula, it is relatively simple to show by induction that if  $0 < x_1 < \sqrt{3}$  then  $0 < x_n < x_{n+1} < \sqrt{3}$  if  $x_1 > \sqrt{3}$  then  $x_n > x_{n+1} > \sqrt{3}$ . This means the limit  $\lim x_n$  exists. Let  $n \rightarrow \infty$  in both two sides, and we will get the answer.
9.  $n^{1/n} > (n+1)^{1/(n+1)} \Leftrightarrow n > (1 + 1/n)^n$  hold for  $n \geq 3$ , which means  $a_n$  is monotonically decreasing. And we have proved that  $n^{1/n} < 1 + \epsilon \Leftrightarrow n < (1 + \epsilon)^n$  hold for large  $n$ . Use squeeze theorem.
10.  $3 \leq \sqrt[n]{2^n + 3^n} \leq \sqrt[n]{2 \times 3^n} \rightarrow 3$ .
11. 单调上升性显然. 由于  $\frac{1}{n^2} < \frac{1}{(n-1)n}$ , 从而有上界 2.
12. Use truncation.  $\forall \epsilon > 0, \exists m, \forall M > m, |a_M/b_M| < \epsilon/2$ . Then  $\sum a_n / \sum b_n = \sum_{i=1}^m a_i / \sum b_n + \sum_{i=m+1}^n a_i / \sum b_n := I_1 + I_2$ . When  $n$  is large enough,  $I_1 < \epsilon/2$ ;  $I_2 < \epsilon$  hold for all  $n$ . Thus when  $n$  is large enough,  $I_1 + I_2 < \epsilon$ .
13. 用定义.  $(L - \epsilon)(b_n - b_{n-1}) \leq (a_n - a_{n-1}) \leq (L + \epsilon)(b_n - b_{n-1}), \forall n > N \Rightarrow (L - \epsilon)(b_n - b_N) \leq a_n - a_N \leq (L + \epsilon)(b_n - b_N) \Rightarrow L - \epsilon < \frac{a_n - a_N}{b_n - b_N} < L + \epsilon$ . Then  $|\frac{a_n - a_N}{b_n - b_N} - \frac{a_n}{b_n}| \leq |\frac{(a_n - a_N)b_N}{(b_n - b_N)b_n} + \frac{a_N b_N}{(b_n - b_N)b_n} + \frac{a_N}{b_n - b_N}| \leq (L + \epsilon)\frac{b_N}{b_n} + \frac{a_N b_N}{(b_n - b_N)b_n} + \frac{a_N}{b_n - b_N} \rightarrow 0$  when  $n$  is large enough.

### 1.3 Supplements (not required!)

作为一个已经学习数学这么多年的北京大学练习生, 我相信你一定关心过下面这个问题: 可导函数和连续函数之间差多少? 事实上, 我们有以下定义和结论:

- (1) Open set: We call a set  $A \in \mathbb{R}$  open, iff  $\forall x \in A$ , there exists  $\delta > 0$ , such that  $(x - \delta, x + \delta) \subset A$ .
- (2) Closed set: We call a set  $B \in \mathbb{R}$  closed, iff its complementary set is open.
- (3) Define  $f^{-1}(A) = \{x : f(x) \in A\}$ , where  $f$  is a function.
- (4) One can show that a function  $f$  is continuous iff  $\forall$  open set  $A \in \mathbb{R}$ ,  $f^{-1}(A)$  is open.
- (5) We call  $x \in A$  is an interior point of the set  $A$  iff  $\exists \delta > 0$ , such that  $(x - \delta, x + \delta) \subset A$ .
- (6) We call a set  $A$  is countable iff there exists a one-to-one map from  $A$  to  $\mathbb{N}$  or  $|A| < \infty$ , where  $|A|$  = the number of elements in  $A$ .
- (7) We call  $x$  is a limit point of the set  $A$ , iff there exists a sequence  $\{x_i\}_{i=1}^{\infty} \subset A$  such that  $x_i \rightarrow x$ . Use  $A'$  to denote the set made up of all the limit points of  $A$ .
- (8) We call  $\bar{A} = A \cup A'$  is the closure of  $A$ .
- (9) Then, 对于所有  $[a, b]$  上的连续函数, 至少存在一点可导的函数构成的集合是无处稠密集的可列并 (第一纲集). 这里, 无处稠密集是指其闭包不存在内点的集合, 连续函数之间的度量定义为  $\rho_{[a,b]}(f, g) = \max_{x \in [a,b]} |f - g|$ .
- (10) Baire 纲集定理: 闭集  $B_n$  无内点, 则  $\cup_n B_n$  也无内点. 由此容易知道第一纲集是没有内点的.

## 2 第 2 次习题课

### 2.1 Questions

1. Show that  $\lim_{x \rightarrow 2} \frac{1}{x-1} = 1$ .
2. Calculate  $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$ .
3. Calculate  $\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$ .
4. Calculate  $\lim_{x \rightarrow +\infty} x(\sqrt{x^2 + 1} - x)$ .
5. Calculate  $\lim_{x \rightarrow +\infty} \cos \sqrt{x+1} - \cos \sqrt{x}$ .
6. Calculate  $\lim_{x \rightarrow 0} \frac{\cos x - \cos 3x}{x^2}$ .
7. Calculate  $\lim_{x \rightarrow +\infty} \left(\frac{x^2+1}{x^2-2}\right)^{x^2}$ .
8. 设数列  $a_n \rightarrow 0$  和  $\lim |a_{n+1}/a_n| = a$ . Show that  $a \leq 1$ .
9. Let  $a_n = \sum_{k=1}^n (\sqrt{1 + \frac{k}{n^2}} - 1)$ , calculate  $\lim a_n$ .
10.  $\lim \frac{\sum a_n}{n} \exists$ , show that  $a_n/n \rightarrow 0$ .
11. Show that  $(n!)^{1/n^2} = 1$ .
12.  $a_1 = b, a_2 = c, a_n = \frac{a_{n-1} + a_{n-2}}{2}$ , calculate  $\lim a_n$ .
13. Calculate  $\lim_{x \rightarrow +\infty} \left(\frac{1}{x} \frac{a^x - 1}{a - 1}\right)^{1/x}$ .
14.  $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$ , 且  $|f(x)| \leq \sin x$ . Show that  $|a_1 + 2a_2 + \cdots + na_n| \leq 1$ .
15.  $f(x)$  is bounded in the interval  $(-\delta, \delta)$ .  $\exists a > 1, b > 1$  such that  $f(ax) = bf(x)$ . Show that  $f(x) \rightarrow 0$  as  $x \rightarrow 0$ .

### 2.2 Solutions

1. Trivial.
2. Note that  $|x^2 \sin \frac{1}{x}| \leq |x^2| \rightarrow 0$ .
3.  $\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} \rightarrow 3x^2$ .
4.  $\lim_{x \rightarrow +\infty} x(\sqrt{x^2 + 1} - x) = \lim_{x \rightarrow +\infty} \frac{x}{x + \sqrt{x^2 + 1}} = \lim_{x \rightarrow +\infty} \frac{1}{1 + \sqrt{1 + \frac{1}{x^2}}} \rightarrow 2$ .
5.  $|\cos \sqrt{x+1} - \cos \sqrt{x}| = |2 \sin \frac{\sqrt{x+1} + \sqrt{x}}{2} \sin \frac{\sqrt{x+1} - \sqrt{x}}{2}| \leq |\sin \frac{\sqrt{x+1} - \sqrt{x}}{2}| \leq \frac{\sqrt{x+1} - \sqrt{x}}{2} \rightarrow 0$ .

6.  $\frac{\cos x - \cos 3x}{x^2} = \frac{2 \sin 2x \sin x}{x^2} \sim \frac{2 \times 2x \times x}{x^2} = 4$ .
7. 原式  $= [(1 + \frac{3}{x^2-2})^{\frac{x^2-2}{3}}]^{\frac{3x^2}{x^2-2}} \rightarrow e^3$ .
8. If  $a > 1$ , then  $\exists N, \forall n > N, |a_{n+1}/a_n| > (1+a)/2$ , then  $|a_n| > |a_N|(\frac{1+a}{2})^{n-N} \Rightarrow |a_n| \rightarrow \infty$ .
9. 分子有理化后利用夹逼原理知答案是  $1/4$ .
10.  $a_n/n = \sum a_n/n - \frac{n-1}{n} \sum a_{n-1}/(n-1) \rightarrow 0$ .
11. 使用夹逼定理知  $1 \leq (n!)^{1/n^2} \leq n^{1/n} \rightarrow 1$ . (PLUS: Stirling:  $n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$ ).
12. It is easy to show that  $a_n - a_{n-1} = (-1)^n \frac{a_1 - a_0}{2^{n-1}}$ , so  $a_n - a_0 = (a_1 - a_0)[1 + (-0.5) + \cdots + (-0.5)^{n-1} + (-0.5)^{n-1}] \rightarrow \frac{2}{3}(a_1 - a_0)$ .
13. We have shown  $(a-1)^{1/x} \rightarrow 1, (1/x)^{1/x} \rightarrow 1$ , thus the original limit is equivalent to  $\lim(a^x - 1)^{1/x}$ . Thus if  $a > 1$ ,  $\lim = a$ , if  $0 < a < 1$ ,  $\lim = 1$ .
14.  $|f(x)/\sin x| \leq 1$  and let  $x \rightarrow 0$ .
15.  $x \in (-\delta, \delta), |f(x)| < M; x \in (-\delta/a, \delta/a), |f(x)| = \frac{1}{b}|f(ax)| \leq \frac{M}{b} \Rightarrow x \in (-\delta/a^n, \delta/a^n), |f(x)| \leq \frac{M}{b^n} \Rightarrow f(x) \rightarrow 0$  as  $x \rightarrow 0$ .

## 2.3 Supplements (not required!)

**闭区间套定理:**  $a_n \uparrow, b_n \downarrow, 0 < b_n - a_n \rightarrow 0$ , then  $\exists$  a unique point  $x \in \cap_n [a_n, b_n]$ .

Proof: Let  $x = \lim a_n = \lim b_n$ .

**有限覆盖定理:** Assume  $\{I_\lambda\}_{\lambda \in \Lambda}$  is a group of open sets (might be uncountable). If  $[a, b] \subset \cup_{\lambda \in \Lambda} I_\lambda$ , then  $\exists I_1, \dots, I_m \in \{I_\lambda\}$  such that  $[a, b] \subset \cup_{i=1}^m I_i$ .

Proof. If the claim is wrong, i.e. there does not exist a finite sub-covering, then for  $[a, (a+b)/2]$  and  $[(a+b)/2, b]$ , 至少有一个区间不存在有限子覆盖, 这样一直切半, 由闭区间套, 必然夹出一个点  $x$ . 由于这是开覆盖, 因此存在开集  $O_x$  使得  $x \in O_x$ . 从而由极限知这个开区间迟早会覆盖前面的从某项开始的闭区间列, 这与假设 (不存在有限覆盖) 矛盾.

**聚点原理:**  $|a_n| < M$ , then  $\exists \{n_k\}_{k=1}^\infty \subset \mathbb{N}$ , such that  $a_{n_k} \rightarrow a$  as  $k \rightarrow \infty$ . (Bounded sequence must have some subsequence which converges)

我们给出几种证明方法:

- (1) 取  $M$  使得  $\forall n, |x_n| \leq M$ , 取  $a_1 = -M, b_1 = M$ . 对  $[a_n, b_n]$  多次迭代, 每次找到  $\frac{a_n+b_n}{2}$ , 这个点将当前区间划分为两个子区间. 两个子区间中必然至少有一个含有无穷项. 任取其中一个含有无穷项的区间作为  $[a_{n+1}, b_{n+1}]$ . 由闭区间套定理, 最终  $a_n, b_n$  有相同的极限  $\xi$ , 同时  $x_n$  中有无穷项与  $\xi$  任意接近. 选取  $x_{n_k} \in [a_i, b_i]$ , 则  $x_{n_k} \rightarrow \xi$ .
- (2) 如果不存在这样的子列, 那么  $\forall x \in [a, b], \exists \delta > 0$  such that  $|(x - \delta, x + \delta) \cap \{x_i\}_{i=1}^n| \leq 1$ . 这样构造出的开区间集合覆盖了  $[a, b]$ , 由有限覆盖定理, 必然存在有限个开区间覆盖整个区间. 而由假设, 对于取出的每个开区间中至多只有原序列中的一个点, 由于开区间的数量为有限个, 可以得出原序列长度也是有限的, 这显然不成立.

**柯西收敛:**  $\forall \epsilon > 0, \exists N, \forall n, m > N, |a_n - a_m| < \epsilon$ . Equivalent to the previously-defined limit.

Proof. 取  $\epsilon = 1$  以及满足条件的  $N$ , 那么  $\max_{i=1}^N \{|x_i| + 1\}$  给出了整个序列的上下界. 即  $\{x_n\}$  有界. 取它的一个收敛子列  $x_{n_k}$ , 并记这个极限为  $\xi$ . 从而  $|x_n - \xi| \leq |x_{n_k} - \xi| + |x_n - x_{n_k}| \rightarrow 0$ .

一个结论: 数列收敛到  $a$  当且仅当其任意子列都收敛到  $a$ .

## 3 第3次习题课

### 3.1 Questions

1. Calculate  $\lim \frac{(2n-1)!!}{(2n)!!}$ .
2.  $\{x_n\}$  converges and  $\{y_n\}$  diverges, show that  $\{x_n + y_n\}$  diverges.  $\{x_n\}$  and  $\{y_n\}$  diverges, is  $\{x_n + y_n\}$  or  $\{x_n y_n\}$  bound to diverge? If  $\{x_n y_n\}$  is infinitesimal, is  $\{x_n\}$  or  $\{y_n\}$  bound to be infinitesimal?
3. 求极限.  $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$

- 4 (not required).  $0 \leq x_{n+m} \leq x_n + x_m$ . Show that  $\lim_{n \rightarrow \infty} \frac{x_n}{n} \exists$ . (This lemma is useful in large deviation theory).
5.  $\lim a_n = a$ . 求证  $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n p_i a_{n+1-i}}{\sum_{i=1}^n p_i} = a$ . 其中  $p_k > 0$  and  $\lim_{n \rightarrow \infty} \frac{p_n}{\sum p_n} = 0$ .
6. 求极限.  $\lim_{x \rightarrow 0} \frac{e^{ax} - e^{bx}}{x}$ ,  $\lim_{x \rightarrow \infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}$ ,  $\lim_{x \rightarrow \infty} (\sin \frac{1}{x} + \cos \frac{1}{x})^x$ .
7.  $\lim_{x \rightarrow 0} f(x) \exists \Leftrightarrow \lim_{x \rightarrow 0} f(x^3) \exists$ , but  $\lim_{x \rightarrow 0} f(x) \exists \not\Leftrightarrow \lim_{x \rightarrow 0} f(x^2)$ .
8. 存在  $f(x)$  在  $\mathbb{R}$  上处处不连续, 但  $|f(x)|$  处处连续.
9.  $a_1, a_2, \dots, a_p > 0$ , calculate  $\lim_{x \rightarrow 0+0} (\frac{\sum_{i=1}^p a_i^x}{p})^{1/x}$ .
10. 求极限.  $\lim_{x \rightarrow 0+0} \frac{x \log(1+3x)}{(1-\cos 2\sqrt{x})^2}$ ,  $\lim_{x \rightarrow 0} \frac{2^x - 3^x}{3^x - 4^x}$ .
11. 举出一个函数处处不连续, 定义域  $[0, 1]$  但是值域为区间.
12.  $f(x) \in C[a, b]$ ,  $|f(x)|$  单调, 则  $f(x)$  单调.
- 13 (not required).  $|f(x) - f(y)| \leq k|x - y|$ .  $0 < k < 1$ . Show that  $kx - f(x)$  单调上升 and  $\exists c, f(c) = c$ .
14.  $f(x) \in C[a, b]$ ,  $\forall x, \exists y$ , such that  $|f(y)| \leq \frac{1}{2}|f(x)|$ . Show that  $\exists \xi$ , such that  $f(\xi) = 0$ .
- 15 (not required).  $f(x)$  在  $[a, b]$  上只有第一类间断点, 证明  $f(x)$  有界.
16. 设  $f(x) \in C[a, b]$ ,  $f(a)f(b) < 0$ . Show that  $\forall n = 1, 2, \dots, \exists \{\xi_i\}_{i=1}^n \subset [a, b], \xi_i \neq \xi_j$  such that  $\sum_{i=1}^n e^{f(\xi_i)} = n$ .
17. 非负函数  $f \in C[0, 1], f(0) = f(1) = 0$ . Show that  $\forall a \in (0, 1), \exists x_0 \in [0, 1]$  such that  $x_0 + a \in [0, 1]$  and  $f(x_0) = f(x_0 + a)$ . 如果去掉非负条件还对吗?
18.  $f_n(x) = x^n + x$ . Show that (1)  $\forall n, f_n(x) = 1$  在  $[0.5, 1]$  中有且仅有一个根  $c_n$ . (2) find  $\lim c_n$ .
19. 不等于常数的连续周期函数一定有最小正周期. 如果把连续性去掉结论如何?
- 20 (not required).  $f$  在  $[a, b]$  内处处有极限. 证明  $\forall \epsilon > 0$ , 在  $[a, b]$  中使得  $|\lim_{t \rightarrow x} f(t) - f(x)| > \epsilon$  的点至多有有限个. (2)  $f(x)$  至多有可列个间断点.

## 3.2 Solutions

1.  $\frac{(2n-1)!!}{(2n)!!} < \frac{(2n)!!}{(2n+1)!!}$  (use  $\frac{i}{i+1} < \frac{i+1}{i+2}$ ), then  $x_n^2 < \frac{1}{2n+1}$  which means  $x_n \rightarrow 0$ .
2. If  $\{x_n\}$  and  $\{x_n + y_n\}$  converge then  $\{x_n + y_n - x_n = y_n\}$  converges.  $x_n = (-1)^n, y_n = (-1)^n, \{x_n\}$  and  $\{y_n\}$  diverge but  $\{x_n + y_n\}, \{x_n y_n\}$  converge.  $x_{2k} = 1, x_{2k-1} = \frac{1}{2k-1}, y_{2k} = \frac{1}{2k}, y_{2k-1} = 1$ .  $\{x_n\}$  or  $\{y_n\}$  are not infinitesimal but  $\{x_n y_n\}$  is infinitesimal.
3.  $\lim x_n = \sqrt{2 \lim x_{n-1}} \Rightarrow \lim x_n = 2$ .
4. 考虑  $\{\frac{x_n}{n}\}$  的下确界  $\alpha$ .  $\exists n$  such that  $x_n/n < a + \epsilon$ . Let  $\max_{i=1}^n \{x_i\} = M$ . Then  $\frac{x_m}{m} \leq \frac{x_n}{m} + \frac{x_{m-n}}{n} \leq \frac{2x_n}{m} + \frac{x_{m-2n}}{m}$  (assume  $m = kn + b$ )  $\leq \dots \leq \frac{kx_n}{m} + \frac{x_b}{m} \leq \frac{kx_n}{kn+b} + \frac{M}{m} \leq \frac{x_n}{n} + \frac{M}{m}$ . Choose enough large  $m$  such that  $\frac{M}{m} < \epsilon$ . Then  $\frac{x_m}{m} < a + 2\epsilon$ .
5. WLOG let  $a = 0$ . Let  $\sup_n \{a_n\} = M$ .  $\forall \epsilon > 0, \exists N_1, \forall n \geq N_1$ , s.t.  $|a_n| < \epsilon$ ;  $\exists N_2, \forall n \geq N_2$ , s.t.  $p_n / \sum p_n < \epsilon / N_1$ . Let  $n > N_1 + N_2$ ,  $|\sum_{i=1}^n a_i p_{n+1-i} / \sum_{i=1}^n p_i| \leq |\sum_{i=1}^{N_1} p_i a_{n+1-i} / \sum_{i=1}^n p_i| + |\sum_{i=n-N_1+1}^n p_i a_{n+1-i} / \sum_{i=1}^n p_i| < \epsilon + \frac{\epsilon}{N_1} \times N_1 \times M = (M+1)\epsilon$ .
6.  $(e^{ax} - e^{bx})/x = a \cdot \frac{e^{ax}-1}{ax} + b \frac{1-e^{bx}}{b} \rightarrow a - b, \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}+\sqrt{x}}} = \frac{\sqrt{1+\sqrt{1/x}}}{\sqrt{1+\sqrt{1/x}+\sqrt{1/x^3+1}}} \rightarrow 0.5, (\sin 1/x + \cos 1/x)^x = [(1 + \cos 1/x + \sin 1/x - 1)^{\frac{1}{\cos 1/x + \sin 1/x - 1}}]^{x \cos 1/x + \sin 1/x - 1} = (1/x = t) = e^{(\cos t + \sin t - 1)/t} = e^1$ .
7. Trivial.
8.  $f(x) = 1_{\mathbb{Q}} - 1_{\mathbb{R} \setminus \mathbb{Q}}$ .
9.  $(\frac{\sum_{i=1}^p a_i^x}{p})^{1/x} = [1 + \frac{\sum_{i=1}^p (a_i^x - 1)}{p}]^{1/x} = \{[1 + \frac{\sum_{i=1}^p (a_i^x - 1)}{p}]^{\frac{p}{\sum_{i=1}^p (a_i^x - 1)}}\}^{\frac{\sum_{i=1}^p (a_i^x - 1)}{px}} \rightarrow e^{\frac{\sum_{i=1}^p \log a_i}{p}} = (a_1 a_2 \dots a_p)^{1/p}$ .
10.  $\lim_{x \rightarrow 0+0} \frac{x \log(1+3x)}{(1-\cos 2\sqrt{x})^2} \sim \frac{x \cdot 3x}{(2x)^2} = \frac{3}{4}, \lim_{x \rightarrow 0} \frac{2^x - 3^x}{3^x - 4^x} = \frac{(2/3)^x - 1}{1 - (4/3)^x} \sim \frac{x \log(2/3)}{-x \log(4/3)} = \frac{\log 3 - \log 2}{\log 4 - \log 3}$ .
11.  $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ x + 0.5 & x \in [0, 0.5] \& x \in \mathbb{R} \setminus \mathbb{Q} \\ x - 0.5 & x \in [0.5, 1] \& x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$
12. 只需注意到如果  $f(x_0) = 0$ , then for any  $x \in [a, x_0], f(x) = 0$ .

13. 第一问是定义, 第二问用 Cauchy 收敛准则, 证明  $x, f(x), f(f(x)), \dots$  是柯西列.
14. If there does not exist  $\xi$  such that  $f(\xi) = 0$ . Then WLOG let  $f(x) > 0$ . Let  $x_0 = \arg \min f(x)$ . Then there does not exist some  $y$  such that  $|f(y)| \leq \frac{1}{2}f(x)$ . This is contradictory.
15. 第一类间断点  $\Rightarrow$  每一点都有一个邻域有界  $\Rightarrow$  所有邻域构成开覆盖, 必有有限子覆盖, 有限个有界总能找到最大的那个.
16.  $\exists \eta > 0$ , such that  $f([a, b]) \supset [-\eta, \eta]$ , which further means  $e^{f([a, b])} \supset [e^{-\eta}, e^{\eta}] \supset [1 - \epsilon, 1 + \epsilon]$  ( $\exists$  a relatively small  $\epsilon > 0$ ). Thus when  $n$  is odd, choose  $e^{f(\xi_1)} = 1, e^{f(\xi_2)} = 1 - \epsilon/2, e^{f(\xi_3)} = 1 + \epsilon/2, e^{f(\xi_4)} = 1 - \epsilon/3, e^{f(\xi_5)} = 1 + \epsilon/3, \dots$ . When  $n$  is even, choose  $e^{f(\xi_1)} = 1 - \epsilon/2, e^{f(\xi_2)} = 1 + \epsilon/2, e^{f(\xi_3)} = 1 - \epsilon/3, e^{f(\xi_4)} = 1 + \epsilon/3, \dots$ .
17. Let  $g(x) = f(x+a) - f(x)$ .  $g(0) \geq 0, g(1-a) \leq 0$ , 用介值定理. 去掉非负条件不对, 比如说  $f(x) = \sin(2\pi x), a = 0.7$ .
18. (1) Note that  $f_n \uparrow \in [0.5, 1]$  且  $f(0.5) < 1, f(1) > 1$ , 使用介值定理. (2) 由于  $\forall \epsilon, \exists N$ , such that  $\forall n > N, (1 - \epsilon)^n + 1 - \epsilon < 1$ . 由于  $f(1 - \epsilon) < 1 = f(c_n)$  and  $f_n \uparrow \Rightarrow c_n > 1 - \epsilon$ . 由极限定义知  $c_n \rightarrow 1$ .
19. 反证法. Assume  $f(a) \neq f(b)$ , 考虑正周期序列  $T_n \rightarrow 0$ , then  $(b - a) \div T_n = S_n \cdots m_n$ , 其中  $0 \leq m_n < T_n \rightarrow 0$ . Thus  $a + S_n T_n \rightarrow b, f(a) = f(a + S_n T_n) \rightarrow f(b)$  (连续性)  $\Rightarrow f(a) = f(b)$ . 矛盾. 把连续性去掉则结论不对, 比如说 Dirichlet 函数.
20. (1) 如果集合有无穷多个元素那一定有聚点 (有界序列必有收敛子列). 从而  $x_n \rightarrow x$ . 考虑  $y_n$  使得  $|y_n - x_n| < 1/n$ , 且  $|f(y_n) - f(x_n)| > \epsilon$  (这是集合的定义). 那么  $y_n \rightarrow x, f$  在  $x$  的极限何在? (极限存在当且仅当任意趋于其的数列极限均相等, 而这里  $\lim f(y_n)$  显然与  $\lim f(x_n)$  不同). (2) 记 (1) 中集合为  $A_\epsilon$ . 注意到间断点集合可以写成  $\cup A_{1/n}$ . 可列个有限元素集合的并元素一定是可列个的.

### 3.3 Supplements (not required!)

**有界性定理:**  $f(x) \in C[a, b]$ , then  $f(x)$  is bounded.

Proof. If not bounded, then choose  $x_n$  such that  $f(x_n) \rightarrow \infty$ , then there exists  $\{x_{n_k}\} \subset \{x_n\}$  which converges to some  $x$ . This means  $f(x) = \infty$ , which is contradictory.

**最值定理:**  $f(x) \in C[a, b]$ , then  $\arg \max f(x) \exists$ .

Proof. 找一个数列  $\{x_n\}$  使得  $f(x_n) \rightarrow \max f(x)$ . 利用有界数列必有收敛子列和  $f(x)$  的连续性.

**介值定理:**  $f(x_1) > 0, f(x_2) < 0, f(x) \in C[x_1, x_2], \exists x_0$  such that  $f(x_0) = 0$ .

Proof. Use Lebesgue method. Let  $x_0 = \sup\{x : f(x) > 0\}$ . 利用连续性知如果  $f(x_0) > 0$  则  $x_0$  不是上界, 如果  $f(x_0) < 0$  则有更好的上确界.

## 4 第 4 次习题课

### 4.1 Questions

- $f(x) \in C(\mathbb{R}), \lim_{x \rightarrow \infty} f(x) = +\infty$ . Show that  $\arg \min_{x \in \mathbb{R}} f(x) \exists$ .
- Show that  $\cos x = 1/x$  has infinite positive roots.
- $f(x) \in C[a, b], x_1, x_2, \dots, x_n \in [a, b]$ . Show that  $\exists \xi \in [a, b]$  such that  $f(\xi) = \frac{1}{n} \sum_{i=1}^n f(x_i)$ .
- $f(x) = |x|^{1/4} + |x|^{1/2} - 0.5 \cos x$ . How many roots in  $\mathbb{R}$ ?
- $f(x) \in C[0, 2], f(0) = f(2)$ , show that  $\exists x_1, x_2 \in [0, 2]$  such that  $|x_1 - x_2| = 1$  and  $f(x_1) = f(x_2)$ .
- $f(x) = \lim_{n \rightarrow \infty} \frac{x^{n+2} - x^{-n}}{x^n + x^{-n-1}}$ . Discuss its continuity.
- $f(x) \in C(\mathbb{R}), f(x+y) = f(x) + f(y)$ . Calculate  $f(x)$ .
- $f(x)$  连续,  $|f(x)|$  连续否?
- $f(x) \in C[0, 1], 0 \leq f(x) \leq 1$ , show that  $\exists t \in [0, 1]$  such that  $f(t) = t$ .
- (not required!)  $f(x) \uparrow \in C[0, 1], 0 \leq f(x) \leq 1$ , show the same proposition.
- $f(x)$  在  $x = 3$  连续,  $\lim_{x \rightarrow 3} \frac{f(x)}{x-3} = 2$ . 求  $f'(3)$ .

12.  $f(x) \in D(\{x_0\})$ , calculate  $\lim_{x \rightarrow 0} \frac{f(x_0+x)-f(x_0-x)}{f(x)}$ .
13. 证明奇函数导数是偶函数, 偶函数导数是奇函数.
14. 求导数  $y = \sqrt[3]{2+3x^3}$ ,  $y = \arcsin \frac{1}{x^2}$ ,  $y = \log(\arctan 5x) + \log(1-x)$ ,  $y = e^{\sin^2 x} + \sqrt{\cos x} 2^{\sqrt{\cos x}}$ .
15.  $f(x) = x|x(x-2)|$ , 求  $f'(x)$ .
16.  $f(x), x \in [-1, 1], x \leq f(x) \leq x^2 + x$ , show that  $f'(0) = 1$ .
17. 求导数.  $e^{xy} = 3x^2y$ ,  $\arctan y/x = \log \sqrt{x^2 + y^2}$ .
18. 求导数.  $f(x)^{g(x)}, x^{x^x}$ .
19. 求  $n$  阶导数.  $\frac{x^n}{1-x}, \sin^4 x + \cos^4 x$ .
20. 求 0 处的  $n$  阶导数.  $\arcsin^2 x$ .

## 4.2 Solutions

1. It is easy to show that  $\exists X > 0, \forall |x| > X, f(x) > f(0)$ . Then  $\arg \min_{x \in [-X, X]} f(x) = \arg \min_{x \in \mathbb{R}} f(x)$ , 由最值定理知存在性.
2. Let  $f(x) = \cos x - 1/x$ , then  $f(2k\pi) > 0, f(2k\pi + \frac{\pi}{2}) < 0$ , 由介值定理立得.
3. Note that  $\min f(x) \leq \frac{1}{n} \sum_{i=1}^n f(x_i) \leq \max f(x)$ . 使用介值定理.
4. Note that  $f(x)$  is even. Then  $\forall x > 1, f(x) > 0$ . And  $f(x)$  is monotonically increasing in  $[0, 1]$ .  $f(0) < 0, f(1) > 0 \Rightarrow$  one root. So in  $\mathbb{R}$  two roots.
5. Let  $g(x) = f(x+1) - f(x)$ . Then  $g(0)g(1) \leq 0 \Rightarrow \exists x \in [0, 1]$  such that  $g(x) = 0$ .
6.  $f(x) = \begin{cases} x^2 & |x| > 1 \\ -x & 0 < |x| < 1 \end{cases}$ .
7.  $f(n) = f(1) + f(n-1) = 2f(1) + f(n-2) = \dots = nf(1), f(1) = f(1/n) + f((n-1)/n) = 2f(2/n) + f((n-2)/n) = \dots = nf(1/n) \Rightarrow f(m/n) = mf(1/n) = m/n \times f(1)$ . 有理数点满足  $f(x) = xf(1)$ , 无理数点用有理数逼近就可以了.
8. Note that  $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$ . 因此连续.
9. Let  $g(t) = f(t) - t, g(0) \geq 0, g(1) \leq 1$ , 利用介值定理.
10. Use Lebesgue method. Let  $x_0 = \sup_x \{f(x) > x\}$ . Show that  $f(x_0) = x_0$ . If  $f(x_0) > x_0$  then  $\forall x_1, x_0 < x_1 < f(x_0), f(x_1) \geq f(x_0) > x_1$ . This means  $x_0$  is not an upper bound. If  $f(x_0) < x_0$  then for all  $x_1, f(x_0) < x_1 < x_0, f(x_1) \leq f(x_0) < x_1$ . This also means  $x_0$  is not an upper bound. Thus  $f(x_0) = x_0$ .
11.  $f(3) = f(x)/(x-3) \times (x-3) \sim 2 \times (x-3) = 0$  as  $x \rightarrow 3$ . Thus  $f'(3) = \lim_{x \rightarrow 3} \frac{f(x)-f(3)}{x-3} = 2$ .
12.  $\frac{f(x_0+x)-f(x_0-x)}{x} = \frac{f(x_0+x)-f(x_0)}{x} + \frac{f(x_0)-f(x_0-x)}{x} \rightarrow 2f'(x_0)$ .
13. 直接用定义.
14.  $y' = \frac{3x^2 \sqrt[3]{2+3x^3}}{2+3x^3}, y' = \frac{-2}{x\sqrt{x^4-1}}, y' = \frac{5}{\arctan 5x \times (1+25x^2)} + \frac{1}{x-1}, y' = e^{\sin^2 x} \sin 2x - \frac{\sin x}{2\sqrt{\cos x}} 2^{\sqrt{\cos x}} (1 + \sqrt{\cos x} \log 2)$ .
15.  $f'(x) = \begin{cases} 3x^2 - 4x & x < 0 \text{ or } x > 2 \\ 4x - 3x^2 & 0 \leq x < 2 \\ \text{?} & x = 2 \end{cases}$ .
16. 用导数定义 + 夹逼原理.
17. 两边同时求导数,  $y' = \frac{y(2-xy)}{x(xy-1)}, y' = \frac{x+y}{x-y}$ .
18. 方法都是写成指数函数,  $e^{g \log f}, e^{x^{\log x} \log x}$ . 结果是  $f^g(g' \log f + \frac{f}{g} f'), x^{x^x} (x^x (1 + \log x) \log x + x^{x-1})$ .
19.  $\frac{x^n}{1-x} = \frac{x^n - x^{n-1} + x^{n-1} - x^{n-2} + \dots + x - 1 + 1}{1-x} = -(x^{n-1} + \dots + x + 1) + \frac{1}{1-x}$ , 因此  $n$  阶导数是  $\frac{n!}{(1-x)^{n+1}}$ . 第二个用倍角公式写出来是  $1 - 0.5 \sin^2 2x, y' = -\sin 4x$ . 由课上已知结论知是  $y^{(n)} = -4^{n-1} \sin(4x + \frac{n-1}{2}\pi)$ .
20.  $f'(x) = 2 \arcsin x / \sqrt{1-x^2}$ , 从而  $(1-x^2)f'(x)^2 = 4f(x)$ . 两边求导  $-2xf'(x)^2 + 2(1-x^2)f'(x)f''(x) = 4f'(x) \Rightarrow -xf'(x) + (1-x^2)f''(x) = 2$ . 两边求  $n-2$  次导数, 并带入  $x=0$ , 利用 Leibniz 公式知道  $f^{(n)}(0) = (n-2)^2 f^{(n-2)}(0)$ . 然后再手动把  $f'(0), f''(0)$  算出来用递推就可以了.

### 4.3 Supplements (not required!)

See <https://wqgcx.github.io/courses/analysis1.pdf>.

## 5 第 5 次习题课

### 5.1 Questions

Hint: If you find it difficult to calculate  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  but simple to calculate  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ , then by L'Hospital,  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$ .

1. 求出闭区间  $[-1, 1]$  上的一元函数  $f(x) = x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}}$  达到最小值的所有  $[-1, 1]$  上的点.

2. 考虑函数  $f(x) = \begin{cases} x^m \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ , 其中  $m$  为正整数. 在  $x \neq 0$  处, 求  $f'(x)$  和  $f''(x)$ . 求  $m$  满足的条件, 使得  $f(x)$  有连续的二阶导函数.

3. 设  $f(x) = \begin{cases} \frac{\log(1+x)}{x} + \frac{x}{2} & x > 0 \\ a & x = 0 \\ \frac{\sin bx}{x} + cx & x < 0 \end{cases}$  在  $x = 0$  处可导, 确定常数  $a, b, c$  的值 (You may want to use the hint mentioned above).

4.  $y = e^{-x^2}$ , 求  $y^{(4)}|_{x=0}$ .

5.  $y = \frac{x}{2}\sqrt{a^2 - x^2} + \arccos \frac{x}{a}$ , 求  $\frac{dy}{dx}$ .

6.  $y^2 \tan(x+y) - \sin(x-y) = 0$ , 求  $\frac{dy}{dx}$ .

7.  $y = x^{a^a} + a^{x^a} + a^{a^x}$ , 求  $\frac{dy}{dx}$ .

8. 求函数  $f(x) = x^{\arcsin x}$  ( $0 < x < 1$ ) 的导函数  $f'(x)$ .

9. 求函数  $f(x) = \arctan x$  在  $x = 0$  点的 3 阶导数  $f'''(0)$ .

10. 设  $f(x) = \frac{1}{x^2-4}$ , 求  $f^{(n)}(x), n \in \mathbb{N}$ .

11. 求方程  $y^2 + 2 \log y = x^4$  所确定的函数  $y = f(x)$  的二阶导数.

12. 判断下列结论是否正确.

(1) 设  $f(x)$  在  $x_0$  处可导, 且  $f'(x_0) > 0$ , 那么

(1.1)  $f(x)$  在  $x_0$  点一定连续.

(1.2)  $f(x)$  在  $x_0$  点的某个邻域内一定连续.

(1.3)  $f(x)$  在  $x_0$  点的某个邻域内一定单调上升.

(2)  $f(x)$  在  $x_0$  点二阶可导, 那么

(2.1)  $f(x)$  在  $x_0$  点一定连续.

(2.2)  $f(x)$  在  $x_0$  的某个邻域内一定连续.

13. 设  $f(x) = e^{x(x-1)\cdots(x-2021)}$ , 求  $f'(2021)$ .

14. 设  $x = \arcsin \frac{t}{\sqrt{1+t^2}}, y = \arccos \frac{1}{\sqrt{1+t^2}}$ , 求  $\frac{dy}{dx}$ .

15. 设  $y = \frac{1}{4\sqrt{2}} \log \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} - \frac{1}{2\sqrt{2}} \arctan \frac{\sqrt{2}x}{x^2-1}$ , 求  $\frac{dy}{dx}$  并化简.

16. 求积分  $\int \frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)} dx$ .

17. 求积分  $\int \frac{2x^2+x+5}{x^4-x^2-6} dx$ .

18. 求积分  $\int \frac{\cos^3 x}{\sin x + \cos x} dx$ .

19. 求积分  $\int \frac{3+5x}{\sqrt{4x^2-4x+5}} dx$  (Hint:  $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log|x + \sqrt{x^2+a^2}| + C$ ).

20. 设  $y = f(x) = x^3, x = g(t) = t^2, y = f(g(t)) = t^6, \Delta t = 0.1, \Delta x = g(1+0.1) - g(1) = 0.21$ .

(1) 当把  $t$  作为自变量时, 函数  $y = f(g(t))$  的二阶微分记为  $d_t^2 y$ , 函数  $x = g(t)$  的一阶微分记为  $d_t x$ . 计算出: 当  $t = 1, \Delta t = 0.1$  时, 函数  $y = f(g(t))$  的二阶微分  $d_t^2 y|_{t=1, \Delta t=0.1}$  和函数  $x = g(t)$  的一阶微分  $d_t x|_{t=1, \Delta t=0.1}$ .



- (2) 当把  $x$  作为自变量时, 函数  $y = f(x)$  的二阶微分记为  $d_x^2 y$ ,  $x$  (看作  $x$  的函数) 的一阶微分记为  $d_x x$ . 计算出: 当  $x = 1, \Delta x = 0.21$  时, 函数  $y = f(x)$  的二阶微分  $d_x^2 y|_{x=1, \Delta x=0.21}$  和函数  $x$  (看作  $x$  的函数) 的一阶微分  $d_x x|_{x=1, \Delta x=0.21}$ .
- (3)  $\frac{d_t^2 y}{(d_t x)^2}|_{t=1, \Delta t=0.1}$  与  $\frac{d_x^2 y}{(d_x x)^2}|_{x=1, \Delta x=0.21}$  相等吗?
21. 求极限  $\lim_{x \rightarrow 0+0} x^x$ .
22. 求极限  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x}-1}{x+\tan x}$ .
23. 求极限  $\lim_{n \rightarrow \infty} \cos \frac{a}{2} \cos \frac{a}{2^2} \cdots \cos \frac{a}{2^n} \quad (a \in (0, 1))$ .
24. 求极限  $\lim_{x \rightarrow +\infty} \left( \frac{\sqrt{1+x^2}}{x} \right)^{x^2}$ .
25. 求极限  $\sqrt[n]{n}(\sqrt[n]{n}-1)$ .
26. 设  $\lim_{n \rightarrow +\infty} (x_n - x_{n-2}) = 0$ , 证明  $\lim_{n \rightarrow +\infty} \frac{x_n}{n} = 0$ .
27. 设  $f(x) \in C[0, 1]$ , 如果极限  $\lim_{n \rightarrow +\infty} \frac{f(0)+f(1/n)+f(2/n)+\cdots+f(1)}{n} = M$ , 其中  $M$  是  $f(x)$  在  $[0, 1]$  上的最大值, 则  $f(x) \equiv M$ .

## 5.2 Solutions

1.  $f(1) = f(-1) = 1, f'(x) = \frac{2}{3x^{1/3}} - \frac{2x}{3(x^2-1)^{2/3}} = \frac{2[(x^2-1)^{2/3}-x^{4/3}]}{3x^{1/3}(x^2-1)^{2/3}} \geq 0 \Rightarrow 0 < x \leq \frac{\sqrt{2}}{2}$  或者  $-1 < x \leq -\frac{\sqrt{2}}{2}$ . 注意到  $f(0) = 1$ . 从而达到最小值的点是  $-1, 0, 1$ .
2.  $f'(x) = -x^{m-2} \cos \frac{1}{x} + mx^{m-1} \sin \frac{1}{x}, f''(x) = -x^{m-4} \sin \frac{1}{x} - (m-2)x^{m-3} \cos \frac{1}{x} - mx^{m-3} \cos \frac{1}{x} - m(m-1)x^{m-2} \sin \frac{1}{x}$ . 要使得  $f''(0)$  存在需要  $f'(0) \exists, f'(x) \rightarrow f'(0)$  且  $\lim_{x \rightarrow 0} \frac{f'(x)-f'(0)}{x} \exists \Rightarrow m \geq 4$ , 二阶导函数连续性意味着  $f''(x) \rightarrow f''(0) \Rightarrow m \geq 5$ .
3. 连续性:  $f(0+0) = a \Rightarrow a = 1, b = 1. f'_+(0) = 0, f'_-(0) = 0 \Rightarrow c = 0$ .
4.  $y' = -2xe^{-x^2}, y'' = 4x^2e^{-x^2} - 2e^{-x^2}, y''' = -2x(4x^2-2)e^{-x^2} + 8xe^{-x^2}, y'''' = (-24x^2+12)e^{-x^2} + (16x^4-24x^2)e^{-x^2}$ . 从而  $y''''(0) = 12$ .
5.  $\frac{dy}{dx} = \frac{\sqrt{a^2-x^2}}{2} - \frac{x^2}{2\sqrt{a^2-x^2}} - \frac{1}{\sqrt{a^2-x^2}}$ .
6. 两边求导,  $2yy' \tan(x+y) + \frac{y^2}{\cos^2(x+y)}(y'+1) + (y'-1) \cos(x-y) = 0 \Rightarrow y' = \frac{\cos^2(x+y) \cos(x-y) - y^2}{\cos^2(x+y) \cos(x-y) + y^2 + 2y \sin(x+y) \cos(x+y)}$ .
7.  $y' = a^a x^{a-1} + a^{x+1} x^{a-1} \log a + a^{a+x} (\log a)^2$ .
8.  $f(x) = e^{\arcsin x \log x}, f'(x) = x^{\arcsin x} \left( \frac{\log x}{\sqrt{1-x^2}} + \frac{\arcsin x}{x} \right)$ .
9.  $f'(x) = \frac{1}{1+x^2}, f''(x) = -\frac{2x}{(1+x^2)^2}, f'''(x) = -\frac{2(1+x^2)^2-8x^2(1+x^2)}{(1+x^2)^4} \Rightarrow f'''(0) = -2$ .
10.  $\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{1}{4} \left( \frac{1}{x-2} - \frac{1}{x+2} \right) \Rightarrow f^{(n)}(x) = \frac{(-1)^n n!}{4} \left( \frac{1}{(x-2)^{n+1}} - \frac{1}{(x+2)^{n+1}} \right)$ .
11. 两边求导,  $2yy' + 2\frac{y'}{y} = 4x^3 \Rightarrow y^2 y' + y' = 2x^3 y$ , 再求一次,  $2y(y')^2 + y^2 y'' + y'' = 6x^2 y + 2x^3 y'$ , 利用  $y' = \frac{2x^3 y}{y^2+1}$ , 得到  $y'' = \frac{6x^2 y}{y^2+1} + \frac{4x^6 y}{(y^2+1)^2} - \frac{8x^6 y^3}{(y^2+1)^3}$ .
12. (1.1) 可导一定连续. (1.2)(1.3) 不一定,  $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ x+x^2 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ .
- (2.1) (2.2) 都是对的.
13.  $f'(x) = e^{x(x-1)\cdots(x-2021)} [x(x-1)\cdots(x-2021)]'$ , 从而  $f'(2021) = 2021!$ .
14.  $\frac{dx}{dt} = \frac{1}{1+t^2}, \frac{dy}{dt} = \frac{1}{1+t^2} \operatorname{sgn}(t)$ , 注意到  $t, x$  同号, 因此  $\frac{dy}{dx} = \operatorname{sgn}(x)$ .
15.  $y'(x) = \frac{1}{x^4+1}$ .
16.  $\frac{4x^3+2x^2+3x+1}{x(x+1)(x^2+1)} = \frac{2}{x+1} + \frac{1}{x} + \frac{x}{(x^2+1)}$ , 因此积分后是  $2 \log|x+1| + \log|x| + \frac{1}{2} \log(x^2+1) + C$ .
17.  $\frac{2x^2+x+5}{x^4-x^2-6} = \frac{11}{10\sqrt{3}} \frac{1}{x-\sqrt{3}} - \frac{11}{10\sqrt{3}} \frac{1}{x+\sqrt{3}} - \frac{1}{5\sqrt{2}} \frac{\sqrt{2}}{x^2+2} + \frac{1}{5} \frac{x}{x^2-3} - \frac{1}{5} \frac{x}{x^2+2}$ , 因此积分后是  $\frac{11}{10\sqrt{3}} \log|x-\sqrt{3}| - \frac{11}{10\sqrt{3}} \log|x+\sqrt{3}| - \frac{1}{5\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{10} \log|x^2-3| - \frac{1}{10} \log(x^2+2) + C$ .
18.  $\frac{\cos^3 x}{\sin x + \cos x} = \frac{1}{\cos^2 x (\tan x + 1)(\tan^2 x + 1)^2}$ , 后面用有理式展开积分.
19.  $\frac{3+5x}{\sqrt{(2x-1)^2+4}} = \frac{5(x-0.5)}{2\sqrt{(x-0.5)^2+1}} + \frac{5.5}{2\sqrt{(x-0.5)^2+1}}$ , 因此积分后是  $2.5\sqrt{(x-0.5)^2+1} + 2.75 \log|x-0.5+\sqrt{(x-0.5)^2+1}| + C$ .
20. (1)  $d_t^2 y|_{t=1, \Delta t=0.1} = 30t^4(\Delta t)^2|_{t=1, \Delta t=0.01} = 0.3, d_t x|_{t=1, \Delta t=0.1} = 2t\Delta t|_{t=1, \Delta t=0.1} = 0.2$ .
- (2)  $d_x^2 y|_{x=1, \Delta x=0.21} = 6x(\Delta x)^2 = 0.2646, d_x x|_{x=1, \Delta x=0.21} = 1\Delta x|_{x=1, \Delta x=0.21} = 0.21$ .

(3)  $(d_t x)^2|_{t=1, \Delta t=0.1} = 0.2^2 = 0.04$ ,  $(d_x x)^2|_{x=1, \Delta x=0.21} = 0.21^2 = 0.0441$ ,  $\frac{d_t^2 y}{(d_t x)^2}|_{t=1, \Delta t=0.1} = \frac{0.3}{0.04} = 7.5 \neq 6 = \frac{0.2646}{0.0441} = \frac{d_x^2 y}{(d_x x)^2}|_{x=1, \Delta x=0.21}$ , 因此不相等.

21.  $x^x = e^{x \log x} \rightarrow e^0 = 1$ .

22.  $\sqrt[3]{1+x} - 1 \sim \frac{1}{3}x$ ,  $x + \tan x \sim 2x$ , 因此极限值为  $\frac{1}{6}$ .

23.  $\sin \frac{a}{2^n} \cos \frac{a}{2} \cdots \cos \frac{a}{2^n} = \frac{\sin a}{2^n}$ , 因此极限值为  $\frac{\sin a}{a}$ .

24.  $(1 + \frac{\sqrt{1+x^2}-x}{x})^{\frac{x}{\sqrt{1+x^2}-x}x(\sqrt{1+x^2}-x)}$ , 由于  $x(\sqrt{1+x^2}-x) = \frac{x}{\sqrt{1+x^2}+x} \rightarrow 0.5$ , 因此原极限为  $\sqrt{e}$ .

25.  $\sqrt{n}(\sqrt[n]{n}-1) \sim \sqrt{n}(e^{(\log n)/n}-1) \sim \sqrt{n}(\log n)/n \rightarrow 0$ .

26. 分奇偶讨论.

27. 如果结论不对, 则存在一个长度为  $\delta$  的区间, 在这个区间上  $f(x) \leq M - \epsilon$ , 则至少有  $[\delta/n] - 1$  个  $f(i/n)$  落在这个区间里, 这样一来极限值就会小于等于  $M(1-\delta) + (M-\epsilon)\delta$ , 矛盾.

### 5.3 Supplements (required!)

Riemann-Lebesgue's Lemma:  $f \in R[a, b]$ ,  $g \in C[0, T]$ ,  $g(x+T) = g(x)$ ,  $\forall x \in \mathbb{R}$ . Then

$$\int_a^b f(x)g(nx)dx = \int_a^b f(x)dx \frac{1}{T} \int_0^T g(x)dx$$

Proof: WLOG  $\int_0^T g(x)dx = 0$ , otherwise let  $h(x) = g(x) - \frac{1}{T} \int_0^T g(x)dx$ .

Then by the definition of Riemann integral,  $\forall \epsilon > 0$ , there exists some step function  $s_\epsilon(x)$ , such that  $s_\epsilon(x) =$

$$\begin{cases} C_1 & a = x_0 \leq x < x_1 \\ C_2 & x_1 \leq x < x_2 \\ \dots & \\ C_m & x_{m-1} \leq x \leq b \end{cases} \quad \text{and } \int_a^b |f(x) - s_\epsilon(x)|dx < \epsilon. \text{ Let } M = \sup_{x \in [0, T]} g(x). \text{ Then}$$

$|\int_a^b f(x)g(nx)dx| = |\int_a^b (f(x) - s_\epsilon(x))g(nx)dx + \int_a^b s_\epsilon(x)g(nx)dx| \leq \int_a^b |f(x) - s_\epsilon(x)|g(nx)dx + |\sum_{i=1}^m C_i \int_{x_{i-1}}^{x_i} g(nx)dx| < M\epsilon + \frac{1}{n} \sum_{i=1}^m C_i \int_{nx_{i-1}}^{nx_i} g(x)dx \leq M\epsilon + \frac{1}{n} \sum_{i=1}^m C_i MT$ . The last equation uses the fact that  $\int_0^T g(x)dx = 0$ , which further means  $\int_c^d g(x)dx = \int_c^{c+T} g(x)dx + \int_{c+T}^{c+2T} g(x)dx + \cdots + \int_{c+kT}^d g(x)dx$  ( $c+kT \leq d < c+(k+1)T$ )  $= \int_{c+kT}^d g(x)dx \leq MT$ .

Choose a large enough  $n$ , and we can let  $\frac{1}{n} \sum_{i=1}^m C_i MT < \epsilon$ . #.

## 6 第 6 次习题课

### 6.1 Questions

### 6.2 Solutions

### 6.3 Supplements