

# Modern Statistical Modeling

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May 20, 2023

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# 1 Review of Linear Algebra

- Rank of  $A \in \mathbb{R}^{m \times n}$ : max # of linearly independent row/columns. Facts: (i)  $0 \leq \text{rank}(A) \leq \min(m, n)$ ; (ii)  $\text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T) = \text{rank}(A^T A)$ ; (iii)  $\text{rank}(BAC) = \text{rank}(A)$  for nonsingular compatible  $B, C$ .
- Range(column space):  $\mathcal{C}(A) = \{Ax : x \in \mathbb{R}^n\} \subset \mathbb{R}^m$ . Null space:  $\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$ . Facts: (i)  $\text{rank}(A) = \dim \mathcal{C}(A)$ ; (ii)  $\dim \mathcal{C}(A) + \dim \mathcal{N}(A) = n$ ; (iii)  $\mathcal{N}(A) = \mathcal{C}(A^T)^\perp$ ; (iv)  $\mathcal{C}(AA^T) = \mathcal{C}(A)$ .
- Trace of  $A \in \mathbb{R}^{m \times n}$ :  $\text{tr}(A) = \sum_{i=1}^n a_{ii}$ . Facts: (i) linearity:  $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$ ,  $\text{tr}(cA) = c\text{tr}(A)$ ; (ii) cyclic property:  $\text{tr}(AB) = \text{tr}(BA)$ ,  $\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$ ; (iii)  $\text{tr}(A) = \sum_{i=1}^n \lambda_i a_{ij} b_{ij}$ .
- Trace product:  $\langle A, B \rangle = \text{tr}(A^T B) = \text{tr}(AB^T) = \sum_i \sum_j a_{ij} b_{ij}$ . It induces Frobenius norm:  $\|A\|_F = \sqrt{\langle A, A \rangle} = (\sum_{i,j} a_{ij}^2)^{1/2}$ .
- Determinant:  $\det(A)$  or  $|A|$ . Facts: (i)  $\det(cA) = c^n \det(A)$ ; (ii)  $\det(AB) = \det A \det B$ ; (iii)  $\det(A^{-1}) = \det(A)^{-1}$ ; (iv)  $\det(A) = \prod_{i=1}^n \lambda_i$ .
- Three decomposition. (1) For symmetric  $A$ , spectrum(eigen) decomposition:  $A = V\Lambda V^T = \sum_{i=1}^r \lambda_i v_i v_i^T$  where  $V$  is orthogonal ( $V^T V = V V^T = I$ ) and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ . (2) SVD for  $A \in \mathbb{R}^{n \times p}$  of rank  $r$ :  $A = U\Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$  where  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r, 0, \dots, 0)$ ,  $\sigma_1 \geq \dots \geq \sigma_r \geq 0$  and  $\{u_i\}, \{v_i\}$  orthonormal.  $\arg \min_{Y \in \mathbb{R}^{n \times p}, \text{rank}(Y) \leq r} \|X - Y\|_F = \sum_{i=1}^r \sigma_i u_i v_i^T$  (low rank- $r$  approximation). (3) QR decomposition:  $A = QR$  where  $Q$  is orthonormal and  $R$  is upper-triangular. It corresponds to Gram-Schmidt orthogonalization process.
- Idempotent:  $P^T = P$ . Facts: (i) If  $P$  is symmetric, then  $P$  is idempotent of rank  $r$  iff it has  $r$  eigenvalues 1 and  $n - r$  0; (ii) If  $P$  is a projection matrix, then  $\text{tr}(P) = \text{rank}(P)$ .
- Generalized inverses: For  $A \in \mathbb{R}^{m \times n}$ ,  $A^- \in \mathbb{R}^{n \times m}$  is called a generalized inverse of  $A$  if  $AA^-A = A$ . Moore-Penrose inverse  $A^+$  if (i)  $AA^+A = A$ ; (ii)  $A^+AA^+ = A^+$ ; (iii)  $(A^+A)^T = A^+A$ ; (iv)  $(AA^+)^T = AA^+$ . Such  $A^+$  is unique, and  $A^+ = V\Sigma^+U^T = \sum_{i=1}^r \sigma_i^{-1} v_i u_i^T$ .
- **Theorem 1.1**  $P_X = X(X^T X)^- X^T$  is the orthogonal projection onto  $\mathcal{C}(X)$ . [ $P_X$  does not depend on the choice of  $(X^T X)^-$ ]

**Proof**  $\forall v \in \mathbb{R}^n$ , write  $v = x + w$  where  $x \in \mathcal{C}(X), w \in \mathcal{C}(X)^T$ . By definition,  $P_X v = P_X x + P_X w = P_X x + X(X^T X)^- X^T w = P_X x$ . We need to show  $u^T X(X^T X)^- X^T X = u^T X, \forall u \in \mathbb{R}^n$ .

**Lemma 1.1**  $\mathcal{C}(X^T) = \mathcal{C}(X^T X)$ .

**Proof** Use  $\mathcal{C}(X^T X) \subset \mathcal{C}(X^T)$  and  $\text{rank}(X^T X) = \text{rank}(X)$ . □

By the lemma,  $u^T X(X^T X)^- X^T X = z^T X^T X(X^T X)^- X^T X = z^T X^T X = u^T X$ . □

# 2 Review of Probability Theory

- Distribution related to multivariate normal:  $X \sim \mathcal{N}_p(\mu, \Sigma)$ . Moment generating function:  $M_X(t) = \mathbb{E}e^{t^T X} = \exp(t^T \mu + \frac{1}{2} t^T \Sigma t)$ . Characteristic function:  $\phi_X(t) = \mathbb{E}e^{it^T X} = \exp(it^T \mu - \frac{1}{2} t^T \Sigma t)$ . Facts: (i)  $A_{g \times p} X + b_{g \times 1} \sim \mathcal{N}_g(A\mu + b, A\Sigma A^T)$ ; (ii)  $X \sim \mathcal{N}_p(\mu, \Sigma) \Leftrightarrow a^T X \sim \mathcal{N}(a^T \mu, a^T \Sigma a), \forall a \in \mathbb{R}^p$ ; (iii)  $Y_1 = A_1 X + b_1 \perp\!\!\!\perp Y_2 = A_2 X + b_2 \Leftrightarrow \text{Cov}(Y_1, Y_2) = A_1 \Sigma A_2^T = 0$ .
- Noncentral  $\chi^2$ :  $X \sim \mathcal{N}_p(\mu, I_p)$ . Then  $X^T X \sim \chi_p^2(\lambda)$  with noncentral parameter  $\lambda = \mu^T \mu$ . Pdf of  $\chi_p^2(\lambda)$ :  $f(x; p, \lambda) = \sum_{k=0}^{\infty} \frac{e^{-\lambda/2} (\lambda/2)^k}{k!} f(x; p + 2k, 0)$  where  $f_q(x) = f(x; q, 0) = \frac{x^{q/2} e^{-x/2}}{2^{q/2} \Gamma(q/2)} I(x > 0)$ , a  $\text{Poisson}(\frac{\lambda}{2})$ -weighted mixture of  $\chi_{p+2k}^2$ . M.g.f.:  $M_X(t; p, \lambda) = \frac{1}{(1-2it)^{p/2}} \exp(\frac{\lambda t}{1-2it})$ . Ch.f.:  $\Phi_X(t; p, \lambda) = \frac{1}{(1-2it)^{p/2}} \exp(\frac{i\lambda t}{1-2it})$ . Facts: (i)

If  $X \sim \mathcal{N}(\mu, \Sigma)$  then  $(X - \mu)^T \Sigma^{-1} (X - \mu) \sim \chi_p^2$  and  $X^T \Sigma^{-1} X \sim \chi_p^2(\mu^T \Sigma^{-1} \mu)$ ; (ii) Additivity: If  $X \sim \chi_{p_i}^2(\lambda_i)$  independent for  $i = 1, \dots, k$ , then  $\sum_{i=1}^n X_i \sim \chi_{\sum_i p_i}^2(\sum_i \lambda_i)$ ; (iii) Rank deficient: If  $X \sim \mathcal{N}_p(\mu, I_p)$ ,  $A \in \mathbb{R}^{p \times p}$  symmetric, then  $X^T A X \sim \chi_p^2(\lambda)$  with  $\lambda = \mu^T A \mu \Leftrightarrow A$  is idempotent of rank  $r$ ; (iv) If  $X \sim \mathcal{N}_p(\mu, \Sigma)$ ,  $A \in \mathbb{R}^{p \times p}$  symmetric,  $B \in \mathbb{R}^{q \times p}$ , then  $X^T A X \perp\!\!\!\perp B X \Leftrightarrow B \Sigma A = 0_{q \times p}$ ; (v)  $X^T A X \perp\!\!\!\perp X^T B X \Leftrightarrow A \Sigma B = 0_{p \times p}$ .

- **Theorem 2.1** (Cochran)  $X \sim \mathcal{N}_p(\mu, I_p)$ ,  $X^T X = X^T A_1 X + \dots + X^T A_k X \equiv Q_1 + \dots + Q_k$ ,  $A_i \in \mathbb{R}^{p \times p}$  symmetric of rank  $r_i$ . Then  $Q_i \sim \chi_{r_i}^2(\lambda_i)$  independent for  $i = 1, \dots, k \Leftrightarrow p = r_1 + \dots + r_k$ . In this case,  $\lambda_i = \mu^T A_i \mu$  and  $\lambda_1 + \dots + \lambda_k = \mu^T \mu$ .

**Proof** “ $\Leftarrow$ ”: Note that  $\forall i, \exists c_{ij} \in \mathbb{R}^p, j = 1, \dots, r_i$  s.t.  $Q_i = X^T A_i X = \pm (c_{i1}^T X)^2 \pm \dots \pm (c_{ir_i}^T X)^2$ . Let  $C_i = (c_{i1}, \dots, c_{ir_i})$  and  $C_{p \times r} = (C_1, \dots, C_k)^T$ , then  $X^T X = X^T C \Delta C X$ , where  $\Delta$  is  $p \times p$  diagonal with diagonal entries  $\pm 1 \Rightarrow C^T \Delta C = I_p$ . Thus  $C$  is of full rank and hence  $\Delta = (C^T)^{-1} C^{-1} = (C^{-1})^T C^{-1} = (C^{-1})^T C^{-1}$  is positive definite  $\Rightarrow \Delta = I_p$  and  $C^T C = I_p$ .

“ $\Rightarrow$ ”:  $X^T A_i \sim \chi_{r_i}^2(\lambda_i)$  independent  $\Rightarrow X^T X = \sum_i X^T A_i X \sim \chi_{\sum_i r_i}^2(\sum_i \lambda_i) \Rightarrow \sum_i r_i = p$ .  $\square$

- Noncentral  $F$ : If  $Q_1 \sim \chi_p^2(\lambda)$  and  $Q_2 \sim \chi_q^2$  are independent, then  $\frac{Q_1/p}{Q_2/q} \sim F_{p,q}(\lambda)$ .
- Noncentral  $t$ : If  $U_1 \sim \mathcal{N}(\lambda, 1)$  and  $U_2 \sim \chi_q^2$  are independent, then  $T = \frac{U_1}{\sqrt{U_2/q}} \sim t_q(\lambda)$ .

### 3 Prediction and Nearest Neighbor

- Goal: (1) predict  $y$  from  $x$  (“black box”); (2) which variable(s) in  $x$  contributes to the prediction of  $y$  (“ $x^T \beta$ ”), estimation, testing, variable selection.
- Why are prediction and estimation different: (1) model parameters; (2) identifiability ( $f_{\theta_1} \neq f_{\theta_2} \Rightarrow \theta_1 \neq \theta_2$ ).
- Find prediction function  $f: \mathcal{X} \rightarrow \mathcal{Y}$  that minimizes  $\mathbb{E}_{X,Y} \mathcal{L}(f(X), Y) = \mathbb{E}\{\mathbb{E}(\mathcal{L}(f(X), Y) | X)\}$  where loss function  $\mathcal{L}: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$ .
- Optimal predictor conditioned on  $x$ :  $f^*(x) = \arg \min_{f(x) \in \mathcal{Y}} \mathbb{E}\{\mathcal{L}(f(X), Y) | X = x\}$ .
- Regression:  $y$  numerical, squared error ( $L_2$ -loss)  $\mathcal{L}(\hat{y}, y) = (\hat{y} - y)^2$ ,  $\mathbb{E}\{(Y - f(X))^2 | X\} = \{\mathbb{E}(Y | X) - f(X)\}^2 + \mathbb{E}\{(Y - \mathbb{E}(Y | X))^2 | X\} = \text{bias}^2 + \text{variance}$ . Optimal  $f^*(X) = \mathbb{E}(Y | X)$ .
- To model  $f^*$ ,  $\begin{cases} \text{parametric: linear, } f^*(x) = x^T \beta, \beta \in \mathbb{R}^2 \\ \text{nonparametric: infinite dimension, } f^*(x) = m(x), m \text{ satisfying certain smoothness} \end{cases}$ .
- Classification: 0-1 loss  $\mathcal{L}(\hat{y}, y) = I(\hat{y} \neq y)$ ,  $\mathbb{E}\{\mathcal{L}(h(X), Y) | X = x\} = \sum_{j \neq h(x)} P(Y = j | X = x) = 1 - P(Y = h(X) | X = x)$ . Optimal classification (Bayes classifier):  $h^*(x) = \arg \max_{h(x) \in \mathcal{Y}} P(Y = h(X) | X = x)$ .
- A fully nonparametric approach:  $k$  nearest neighbor ( $k$ -NN). Given training data  $\{(x_i, y_i)\}_{i=1}^m$ , use data “around”  $x$  to estimate  $m(x) = \mathbb{E}(Y | X = x)$ . Rationale: “Things that look alike must be alike”. Classification:  $h_{k\text{-NN}}(x) = \text{majority label among } \{y_i, i \in N_k(x)\}$ . Regression:  $m_{k\text{-NN}}(x) = \frac{1}{k} \sum_{i \in N_k(x)} y_i$ .  $k$  controls size of neighbor set.  $k \uparrow$ : effective sample size  $\uparrow$ , variance  $\downarrow$ , heterogeneity  $\uparrow$ , bias  $\uparrow$ .
- Theory for 1-NN: Consider binary classification:  $\mathcal{Y} = \{0, 1\}$ ,  $\mathcal{L}(h(x), y) = I(h(x) \neq y)$ . Assume  $\mathcal{X} \subset [0, 1]^d$ ,  $\rho$  Euclidean distance,  $S = \{(x_i, y_i)\}_{i=1}^n$ .  $\forall x \in \mathcal{X}$ , let  $\pi_1(x), \dots, \pi_n(x)$  be an ordering of  $\{1, \dots, n\}$  with increasing distance to  $x$ .  $\eta(x) = \mathbb{E}(Y = 1 | X = x)$ . Bayes classifier:  $h^*(x) = I(\eta(x) > \frac{1}{2})$ . Assumption on  $\eta$ :  $\eta$  is  $c$ -Lipschitz for some  $c > 0$ . Goal: Derive an upper bound on  $\mathbb{E}_{S \sim \mathcal{D}^n} \mathcal{L}(\hat{h}_S) = \mathbb{E}_{S \sim \mathcal{D}^n} \mathbb{E}_{(x,y) \sim \mathcal{D}} I(\hat{h}_S(x) \neq y)$ .
- **Lemma 3.1** The 1-NN rule  $\hat{h}_S$  satisfies  $\mathbb{E}_{S \sim \mathcal{D}^n} \mathcal{L}(\hat{h}_S) \leq 2\mathcal{L}(h^*) + c \mathbb{E}_{S \sim \mathcal{D}^n, x \sim \mathcal{D}} \|x - x_{\pi_1}(x)\|$ .

**Proof**  $\mathbb{E}_S \mathcal{L}(\hat{h}_S) = \mathbb{E}_{S_x \sim \mathcal{D}_x^n, x \sim \mathcal{D}_x, y \sim \eta(x), y' \sim \eta(\pi_1(x))} P(y \neq y')$ . Note that  $P(y \neq y') = \eta(x')(1 - \eta(x)) + (1 - \eta(x'))\eta(x) = (\eta - \eta + \eta')(1 - \eta) + (1 - \eta + \eta - \eta')\eta = 2\eta(1 - \eta) + (\eta - \eta')(2\eta - 1)$ . Since  $\eta$  is  $c$ -Lipschitz and  $|2\eta - 1| \leq 1$ ,  $P(y \neq y') \leq 2\eta(1 - \eta) + c\|x - x'\|$ . Substituting back,  $\mathbb{E}_S \mathcal{L}(\hat{h}_S) \leq 2\mathbb{E}_x \eta(x)(1 - \eta(x)) + c\mathbb{E}_{S,x} \|x - x_{\pi_1(x)}\|$ . The Bayes error  $\mathcal{L}(h^*) = \mathbb{E}_x \{\eta(x) \wedge (1 - \eta(x))\} \geq \mathbb{E}_x (\eta(x)(1 - \eta(x)))$ .  $\square$

- **Lemma 3.2** Let  $C_1, \dots, C_r$  be a collection of subsets of  $\mathcal{X}$ . Then  $\mathbb{E}_{S \sim \mathcal{D}^n} \{\sum_{i: C_i \cap S = \emptyset} P(C_i)\} \leq \frac{r}{ne}$  (“probability of subsets that not hit by  $S$ ”).

**Proof** By linearity,  $\mathbb{E}_S \{\sum_{i: C_i \cap S = \emptyset} P(C_i)\} = \sum_{i=1}^r P(C_i) \mathbb{E}_S I(C_i \cap S = \emptyset) = \sum_{i=1}^r P(C_i) P(C_i \cap S = \emptyset)$ . Note that  $P(C_i \cap S = \emptyset) = (1 - P(C_i))^n \leq e^{-nP(C_i)}$ . Thus, LHS  $\leq \sum_{i=1}^r P(C_i) e^{-nP(C_i)} \leq r \max P(C_i) e^{-nP(C_i)} \leq \frac{r}{ne}$ .  $\square$

- **Theorem 3.1** (Generalization upper bound for 1-NN)  $\mathbb{E}_S \mathcal{L}(\hat{h}_S) \leq 2\mathcal{L}(h^*) + 2c\sqrt{dn}^{-\frac{1}{d+1}}$ .

**Proof** Take  $C_i$  of the form  $\{x : x_j \in [(\alpha_j - 1)/T, \alpha_j/T], \forall j\}$ , where  $\alpha_1, \dots, \alpha_d \in \{1, \dots, T\}^d$ .

Case 1: If  $x, x' \in C_i$  for some  $i$ , then  $\|x - x'\| \leq \sqrt{d}\epsilon$ .

Case 2: Otherwise,  $\|x - x'\| \leq \sqrt{d}$ .

Hence,  $\mathbb{E}_{S,x} \|x - x_{\pi_1(x)}\| \leq \mathbb{E}_S \{P(\cup_{i: C_i \cap S \neq \emptyset} C_i) \sqrt{d}\epsilon + P(\cup_{i: C_i \cap S = \emptyset} C_i) \sqrt{d}\} \leq \sqrt{d}(\epsilon + \frac{r}{ne})$ . Since  $r = (\frac{1}{\epsilon})^d, \dots \leq \sqrt{d}(\epsilon + \frac{1}{\epsilon^d ne})$ . Matching the two terms gives  $\epsilon = (\frac{1}{ne})^{\frac{1}{d+1}}$  and the optimal bound  $2\sqrt{d}(ne)^{-\frac{1}{d+1}} \leq 2\sqrt{dn}^{-\frac{1}{d+1}}$ .  $\square$

- **Theorem 3.2** (Generalization upper bound for  $k$ -NN)  $\mathbb{E}_S \mathcal{L}(\hat{h}_S) \leq (1 + \sqrt{\frac{8}{k}}) \mathcal{L}(h^*) + (6c\sqrt{d} + k)n^{-\frac{1}{d+1}}$ .

**Remark 3.1**  $k$  is called regularization parameter/hyperparameter and the optimal  $k \sim n^d$ .

**Remark 3.2** Exponential dependence on  $d$ : “curse of dimensionality”.

- **Theorem 3.3** (Lower bound)  $\forall c > 1$  and any learning rule  $h$ ,  $\exists$  a distribution over  $[0, 1]^d \times \{0, 1\}$  s.t.  $\eta(x)$  is  $c$ -Lipschitz, the Bayes error is 0, but for  $n < (c+1)^d/2$ ,  $\mathbb{E} \mathcal{L}(h) > \frac{1}{4}$  (i.e. minimax bound  $\inf_h \sup_y \mathbb{E} \mathcal{L}(h) \geq Cn^{-\frac{1}{d+1}}$ ).

**Hint** Let  $G_c^d$  be the regular grid on  $[0, 1]^d$  with distance  $1/c$  between points. Then any  $\eta : G_c^d \rightarrow \{0, 1\}$  is  $c$ -Lipschitz. Then use the following theorem.  $\square$

- **Theorem 3.4** (No free-lunch theorem) Let  $A$  be any learning rule for binary classification with 0-1 loss over  $\mathcal{X}^d$  and  $n < |\mathcal{X}|/2$ . Then  $\exists$  distribution  $D$  over  $\mathcal{X} \times \{0, 1\}$  s.t.  $\mathbb{E} \mathcal{L}(A) \geq \frac{1}{4}$ . Furthermore, with prob  $\geq \frac{1}{7}$ ,  $\mathcal{L}(A_S) \geq \frac{1}{8}$ .

## 4 Linear Regression

- $Y_{n \times 1} = X_{n \times p} \beta_{p \times 1} + \epsilon_{n \times 1}$ ,  $\mathbb{E}(\epsilon|X) = 0$ ,  $\text{Var}(\epsilon) = \sigma^2 I_n$  and  $X$  fixed.
- Least squares estimator (LSE) solves the normal equation  $X^T X \hat{\beta} = X^T Y$ ,  $\hat{\beta} = (X^T X)^{-1} X^T Y$ .
- ANOVA:  $y_{ij} = \mu + \alpha_j + \epsilon_{ij}, i = 1, \dots, n_j, j = 1, \dots, J$ .  $\sum_j n_j = n, \sum_j \alpha_j = 0$ .
- **Definition 4.1**  $\theta$  is estimable if  $\exists$  an unbiased estimator of  $\theta$ .  $c^T \beta$  is linearly estimable if  $\exists l \in \mathbb{R}^n$  s.t.  $\mathbb{E}(l^T Y) = c^T \beta, \forall \beta \in \mathbb{R}^p \Leftrightarrow c = X^T l \in \mathcal{C}(X^T)$ .
- **Theorem 4.1** (1) If  $c^T \hat{\beta}$  is unique, then  $c \in \mathcal{C}(X^T X) = \mathcal{C}(X^T)$ .  
 (2) If  $c \in \mathcal{C}(X^T)$ , then  $c^T \hat{\beta}$  is unique and unbiased for  $c^T \beta$ .  
 (3) If  $c^T \beta$  is estimable and  $\epsilon \sim \mathcal{N}_n(0, \sigma^2 I_n)$ , then  $c \in \mathcal{C}(X^T)$ .

**Proof** (1) Let  $b \in \mathcal{C}(X^T X)^\perp$  be arbitrary, then  $X^T Y = X^T X \hat{\beta} = X^T X(\hat{\beta} + b) \Rightarrow c^T \hat{\beta} = c^T(\hat{\beta} + b) \Rightarrow c^T b = 0$ .  
 (2)  $c = X^T l$  for some  $l \in \mathbb{R}^n$ , then  $c^T \hat{\beta} = l^T X^T \hat{\beta} = l^T X^T (X^T X)^{-1} X^T Y = l^T P_X Y$  is unique.  $\mathbb{E}(c^T \hat{\beta}) = l^T P_X \mathbb{E} Y = l^T P_X X \beta = l^T X \beta = c^T \beta$ .

(3) If  $\exists$  an estimator  $T(X, Y)$  unbiased for  $c^T \beta$ , then  $c^T \beta = \int T(X, y) \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\{-\frac{1}{2\sigma^2} \|y - X\beta\|^2\} dy$ . Differentiate with  $\beta$ ,  $c = X^T \int \frac{y - X\beta}{(2\pi\sigma^2)^{\frac{n}{2}} \sigma^2} T(X, y) \exp\{-\frac{1}{2\sigma^2} \|y - X\beta\|^2\} dy$ .  $\square$

**Remark 4.1**  $A\beta$  with  $A \in \mathbb{R}^{q \times p}$  is estimable iff  $\mathcal{C}(A^T) \subset \mathcal{C}(X^T) \Leftrightarrow A = A_* X$  for some  $A_* \in \mathbb{R}^{q \times n}$ . In particular,  $\beta$  is estimable iff  $X$  has full column.

- Ordinary least squares:  $\hat{\beta} = (X^T X)^{-1} X^T Y$ .
- **Proposition 4.1** For any estimable  $A\beta$  and  $B\beta$ ,  $\text{Cov}(A\hat{\beta}, B\hat{\beta}) = \sigma^2 A(X^T X)^{-1} B^T$ ,  $\text{Var}(A\hat{\beta}) = \sigma^2 A(X^T X)^{-1} A^T$ .

**Proof**  $\exists A_*$  and  $B_*$  s.t.  $A = A_* X, B = B_* X$ . Since  $\hat{Y} = X\hat{\beta} = X(X^T X)^{-1} X^T Y = P_X Y$ , we have  $\text{Var}(\hat{Y}) = P_X \text{Var}(Y) P_X^T = \sigma^2 P_X$ . Hence  $\text{Cov}(A\hat{\beta}, B\hat{\beta}) = \text{Cov}(A_* \hat{Y}, B_* \hat{Y}) = A_* \text{Var}(\hat{Y}) B_*^T = \sigma^2 A_* P_X B_*^T = A(X^T X)^{-1} B^T$ .  $\square$

- **Theorem 4.2** (Gauss-Markov) If  $c^T \beta$  is estimable, then  $c^T \hat{\beta}$  has the minimum variance among all linear unbiased estimates. (Best Linear Unbiased Estimator, BLUE)

**Proof** Let  $l^T Y$  be an unbiased estimator of  $c^T \beta$ . Hence,  $c = X^T l$ , so that  $c^T \hat{\beta} = l^T X \hat{\beta} = l^T \hat{Y}$ . Thus,  $\text{Var}(l^T Y) - \text{Var}(c^T \hat{\beta}) = l^T [\text{Var}(Y) - \text{Var}(\hat{Y})] l = \sigma^2 l^T (I - P_X) l \geq 0$ .  $\square$

- Residual  $\hat{\epsilon} = Y - \hat{Y} = (I - P_X)Y \in \mathcal{C}(X)^\perp$ ,  $\mathbb{E}(\hat{\epsilon} | (I - P_X)\mathbb{E}Y) = (I - P_X)X\beta = 0$ ,  $\text{Var}(\hat{\epsilon}) = \sigma^2 (I - P_X)^2 = \sigma^2 (I - P_X)$ ,  $\text{Cov}(\hat{\epsilon}, \hat{Y}) = \text{Cov}((I - P_X)Y, P_X Y) = (I - P_X)(\sigma^2 I)P_X = 0$ .
- Residual sum of squares (RSS):  $\|\hat{\epsilon}\|^2 = \hat{\epsilon}^T \hat{\epsilon} = Y^T (I - P_X) Y$ .  $\mathbb{E}(\text{RSS}) = \mathbb{E} \text{tr}(\hat{\epsilon} \hat{\epsilon}^T) = \text{tr}(\mathbb{E}(\hat{\epsilon} \hat{\epsilon}^T)) = \text{tr}\{(I - P_X)\sigma^2\} = \sigma^2(n - \text{rank}(X))$ .  $\hat{\sigma}^2 = \frac{\text{RSS}}{n-r}$  is an unbiased estimator of  $\sigma^2$ .
- Restricted LSE:  $Y = X\beta + \epsilon$ ,  $\mathbb{E}\epsilon = 0$ ,  $\text{Var}(\epsilon) = \sigma^2 I$ ,  $\text{rank}(X) = r$ ,  $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$ ,  $\beta = \begin{pmatrix} \beta_1^T & \beta_2^T \end{pmatrix}^T$ .  $H_0 : \beta_2 = \beta_2^*$  vs  $\beta_2 \neq \beta_2^*$ .  $\beta_2$  is estimable  $\Rightarrow \text{rank}(X_2) = s$ ,  $\text{rank}(X_1) = r - s$  and  $\mathcal{C}(X_1) \cap \mathcal{C}(X_2) = \{0\}$ .

**Proof**  $\exists C \in \mathbb{R}^{q \times n}$  s.t.  $(0_{s \times (p-s)}, I_s) = CX = (CX_1, CX_2)$ . Hence  $\text{rank}(X_2) = s$  and  $\text{rank}(X_1) = r - s$ . If  $X_1 b_1 = X_2 b_2$  then  $b_2 = CX_1 b_1 = 0$ .  $\square$

- Under  $H_0 : \beta_2 = \beta_2^*$ ,  $Y = X_1 \beta_1 + X_2 \beta_2 + \epsilon$  becomes  $Y - X_2 \beta_2^* = X_1 \beta_1 + \epsilon$ . Restricted normal equation:  $X_1^T X_1 \tilde{\beta}_1 = X_1^T (Y - X_2 \beta_2^*)$ .  $\mathcal{C}(X_1) \subset \mathcal{C}(X) \Rightarrow P_{X_1} P_X = P_{X_1}$ . Since  $P_X Y = \hat{Y} = X \hat{\beta} = X_1 \hat{\beta}_1 + X_2 \hat{\beta}_2$ , we have  $X_1 \tilde{\beta}_1 = P_{X_1} (Y - X_2 \beta_2^*) = P_{X_1} (P_X Y - X_2 \beta_2^*) = P_{X_1} (X_1 \hat{\beta}_1 + X_2 (\hat{\beta}_2 - \beta_2^*)) = X_1 \hat{\beta}_1 + P_{X_1} X_2 (\hat{\beta}_2 - \beta_2^*)$ . Let  $\tilde{Y} = X_1 \tilde{\beta}_1 + X_2 \beta_2^*$  the fitted value of the restricted model.  $\hat{Y} - \tilde{Y} = X_1 \hat{\beta}_1 + X_2 \hat{\beta}_2 - [X_1 \hat{\beta}_1 + P_{X_1} X_2 (\hat{\beta}_2 - \beta_2^*)] - X_2 \beta_2^* = (I - P_{X_1}) X_2 (\hat{\beta}_2 - \beta_2^*)$ .
- **Theorem 4.3**  $\mathcal{C}(Z_2) = \mathcal{C}(X_1)^\perp \cap \mathcal{C}(X)$ , where  $Z_2 = (I - P_{X_1})X_2 = X_2 - P_{X_1} X_2$ .

**Proof**  $\mathcal{C}(Z_2) \subset \mathcal{C}(I - P_{X_1}) = \mathcal{C}(X_1)^\perp$ . Since  $\mathcal{C}(P_{X_1} X_2) \subset \mathcal{C}(X_1)$ ,  $\mathcal{C}(Z_2) = \mathcal{C}(X_2 - P_{X_1} X_2) \subset \mathcal{C}(X)$ . Conversely, if  $X = X_1 b_1 + X_2 b_2 \in \mathcal{C}(X)$  and  $X \perp \mathcal{C}(X_1)$ , then  $X = (I - P_{X_1})X = (I - P_{X_1})X_2 b_2 \in \mathcal{C}(Z_2)$ .  $\square$

**Corollary 4.1**  $P_{Z_2} = P_X - P_{X_1}$ .

- Now  $\hat{Y} - \tilde{Y} = (I - P_{X_1})[X_2(\hat{\beta}_2 - \beta_2^*) + X_1 \hat{\beta}_1] = (I - P_{X_1})(P_X Y - X_2 \beta_2^*) = (I - P_{X_1})P_X (Y - X_2 \beta_2^*) = P_{Z_2}(Y - X_2 \beta_2^*)$ . In view of  $\mathbb{R}^n = \mathcal{C}(X)^\perp \oplus \mathcal{C}(X)$ ,  $Y - \tilde{Y} = (Y - \hat{Y}) + (\hat{Y} - \tilde{Y})$ .  $\text{RSS}_{H_0} = \|Y - \tilde{Y}\|^2 = \|Y - \hat{Y}\|^2 + \|\hat{Y} - \tilde{Y}\|^2$ ,  $\text{RSS} = \|Y - \hat{Y}\|^2 = \|(I - P_X)Y\|^2 = \|(I - P_X)(Y - X_2 \beta_2^*)\|^2$ .  $\text{RSS}_{H_0} - \text{RSS} = \|\hat{Y} - \tilde{Y}\|^2 = \|Z_2(\hat{\beta}_2 - \beta_2^*)\|^2 = \|P_{Z_2}(Y - X_2 \beta_2^*)\|^2$ . By Cochran's theorem,  $\text{RSS}_{H_0} - \text{RSS} \sim \chi_s^2(\lambda)$  with  $\lambda = \|P_{Z_2}(X\beta - X_2 \beta_2^*)\|^2$ .
- Wald's statistics:  $(\hat{\theta} - \theta_0) \text{Var}(\hat{\theta})^{-1} (\hat{\theta} - \theta_0)$ . Since  $\beta_2$  is estimable,  $\exists C \in \mathbb{R}^{s \times n}$ ,  $(0_{s \times p-s}, I_s) = CX = (CX_1, CX_2) \Rightarrow CP_{X_1} = CX_1(X_1^T X_1)^{-1} X_1^T = 0$ ,  $CZ_2 = C(I_n - P_{X_1})X_2 = CX_2 - CP_{X_1} X_2 = I_s \Rightarrow Z_2$  has full column rank.  $\hat{\beta}_2 = (0, I)\hat{\beta} = CX\hat{\beta} = CP_X Y = C(P_{X_1} + P_{Z_2})Y = CP_{Z_2} Y$ . Thus,  $\text{Var}(\hat{\beta}_2) = \text{Var}(CP_{Z_2} Y) = CP_{Z_2} \sigma^2 I_n P_{Z_2} C^T = \sigma^2 CZ_2(Z_2^T Z_2)^{-1} Z_2^T C^T = \sigma^2(Z_2^T Z_2)^{-1}$ .  $(\hat{\beta}_2 - \beta_2^*) \text{Var}(\hat{\beta}_2)^{-1} (\hat{\beta}_2 - \beta_2^*) = \|Z_2(\hat{\beta}_2 - \beta_2^*)\|^2 / \sigma^2 = \frac{\text{RSS}_{H_0} - \text{RSS}}{\sigma^2}$ .

- Inference:  $H = (h_1, \dots, h_s) \in \mathbb{R}^{p \times s}, \xi = \mathbb{R}^s$ . General linear hypothesis:  $H_0 : H^T \beta = \xi$  ( $s$  constraints). Assume (1)  $\mathcal{C}(H) \subset \mathcal{C}(X^T)$ , so that  $H^T \beta$  is estimable; (2)  $H$  has full column rank,  $s = \text{rank}(H) \leq \text{rank}(X) = r \leq p$ .
- Reparameterization: Choose  $A \in \mathbb{R}^{p \times (p-s)}$  s.t.  $\mathcal{C}(A) = \mathcal{C}(H)^\perp$ . Let  $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A^T \beta \\ H^T \beta \end{pmatrix}$  and  $\tilde{X} = X \begin{pmatrix} A^T \\ H^T \end{pmatrix}^{-1} = (\tilde{X}_1, \tilde{X}_2)$ . The reparameterized model  $Y = \tilde{X}\theta + \epsilon$ . Since  $\mathcal{C}(\tilde{X}^T) = \mathcal{C}((A, H)^{-1}X^T) \supset \mathcal{C}((A, H)^{-1}H) = \mathcal{C}\left(\begin{pmatrix} 0 \\ I_s \end{pmatrix}\right)$ ,  $\theta_2$  is estimable.  $\hat{\theta}$  solves the normal equation  $\tilde{X}^T \tilde{X} \hat{\theta} = \tilde{X}^T Y$ . Under  $H_0$ ,  $\tilde{Y} = \tilde{X}_1 \tilde{\theta}_1 + \tilde{X}_2 \xi = \tilde{X}_1 \hat{\theta}_1 + P_{\tilde{X}_1} \tilde{X}_2 (\hat{\theta}_2 - \xi) + \tilde{X}_2 \xi$ ,  $\text{RSS}_{H_0} - \text{RSS} = \|Y - \tilde{Y}\|^2 - \|Y - \hat{Y}\|^2 = \|\hat{Y} - \tilde{Y}\|^2 = \sigma^2 (\hat{\theta}_2 - \xi)^T \text{Var}(\hat{\theta}_2)^{-1} (\hat{\theta}_2 - \xi)$ . Substituting into the original model,  $\hat{\theta}_2 = H^T \hat{\beta}$ ,  $\text{Var}(\hat{\theta}_2) = \sigma^2 H^T (X^T X)^{-1} H$ . Since  $\mathbb{E}(X^T A X) = \text{tr}(A \Sigma) + \mu^T A \mu$  where  $\mu = \mathbb{E}X$ ,  $\Sigma = \text{Var}(X)$ ,  $\mathbb{E}\|\hat{Y} - \tilde{Y}\|^2 / \sigma^2 = \text{tr}(\text{Var}(\hat{\theta}_2)^{-1} \text{Var}(\hat{\theta}_2)) + (H^T \beta - \xi)^T \text{Var}(H^T \beta)^{-1} (H^T \beta - \xi)$ .  $Y - \hat{Y} = (I_n - P_{\tilde{X}})(Y - \tilde{X}_2 \xi)$ ,  $\hat{Y} - \tilde{Y} = \tilde{Z}_2 (H^T \hat{\beta} - \xi) = P_{\tilde{Z}_2} (Y - \tilde{X}_2 \xi)$ . By Cochran's thm,  $\frac{\|Y - \hat{Y}\|^2}{\sigma^2} \sim \chi_{n-r}^2$  and  $\frac{\|\hat{Y} - \tilde{Y}\|^2}{\sigma^2} \sim \chi_s^2(\lambda)$  are independent with  $\lambda = (H^T \beta - \xi)^T \text{Var}(H^T \beta)^{-1} (H^T \beta - \xi)$ . Hence,  $\frac{(\text{RSS}_{H_0} - \text{RSS})/s}{\text{RSS}/(n-r)} \sim F_{s, n-r}(\lambda)$ .
- Let  $\gamma = H^T \beta$  and  $\gamma_0 = \xi$ . Test  $H_0 : \gamma = \gamma_0$  can be regarded as a weighted distance between  $\hat{\gamma}$  and  $\gamma_0$ . To see this, let  $\hat{\gamma} = H^T \hat{\beta} \sim \mathcal{N}_s(\gamma, \sigma^2 D)$  where  $D = H^T (X^T X)^{-1} H$  and  $\hat{\sigma}^2 = \frac{\text{RSS}}{n-r}$ . Under  $H_0$ , (1)  $s = 1$ :  $Z = \frac{\hat{\gamma} - \gamma_0}{\hat{\sigma} \sqrt{D}} \sim \mathcal{N}(0, 1)$  if  $\sigma^2$  is known;  $T = \frac{\hat{\gamma} - \gamma_0}{\hat{\sigma} / \sqrt{D}} \sim t_{n-r}$  if  $\sigma^2$  is unknown. Confidence interval:  $\hat{\gamma} \pm t_{n-r, \alpha/2} \hat{\sigma} \sqrt{D}$ . (2)  $s \geq 1$ : Mahalanobis distance  $\|\hat{\gamma} - \gamma_0\|_{(\sigma^2 D)^{-1}} = \sqrt{(\hat{\gamma} - \gamma_0)^T (\sigma^2 D)^{-1} (\hat{\gamma} - \gamma_0)}$ ,  $\|\hat{\gamma} - \gamma_0\|_{(\sigma^2 D)^{-1}}^2 = (\hat{\gamma} - \gamma_0)^T (\sigma^2 D)^{-1} (\hat{\gamma} - \gamma_0) \sim \chi_s^2(\lambda)$  where  $\lambda = (\gamma - \gamma_0)^T D^{-1} (\gamma - \gamma_0) / \sigma^2$ . Thus  $\mathbb{E}(\hat{\gamma} - \gamma_0)^T D^{-1} (\hat{\gamma} - \gamma_0) / s = (s + \lambda) \sigma^2 / s = (1 + \lambda/s) \sigma^2 \geq \sigma^2$  with equality holding just when  $\gamma = \gamma_0$ . One may reject  $H_0$  if  $(\hat{\gamma} - \gamma_0)^T D^{-1} (\hat{\gamma} - \gamma_0) / (s \sigma^2)$  is large. If  $\sigma^2$  is unknown, replacing  $\sigma^2$  with  $\hat{\sigma}^2$  yields  $\frac{(\hat{\gamma} - \gamma_0)^T D^{-1} (\hat{\gamma} - \gamma_0)}{s \hat{\sigma}^2} = \frac{\|\hat{Y} - \tilde{Y}\|^2 / s}{\|Y - \hat{Y}\|^2 / (n-r)} \sim F_{s, n-r}(\lambda)$ , where  $\lambda = 0$  iff  $H_0$  is true.
- Multiple testing: Simultaneous confidence intervals of level  $1 - \alpha$ .
- Bonferroni: Replace  $\alpha$  by  $\alpha/m$ :  $P(E_j) = 1 - \alpha_j, j = 1, \dots, m$ , then  $P(\cap_j E_j) = 1 - P(\cup_j E_j^c) \geq 1 - \sum_j P(E_j) = 1 - \sum_j \alpha_j = 1 - \alpha$ .
- Scheffé's method: Consider  $Y = X\beta + \epsilon, \epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$ ,  $\text{rank}(X) = r$  and test for  $u^T \gamma, \forall u \in \mathbb{R}^s$ , where  $\gamma = H^T \beta$  is estimable and  $H$  is of full column rank.  $\hat{\gamma} = H^T \hat{\beta} \sim \mathcal{N}_s(\gamma, \sigma^2 D)$  where  $D = H^T (X^T X)^{-1} H$ ,  $\hat{\sigma}^2 = \frac{\text{RSS}}{n-r} \sim \sigma^2 \chi_{n-r}^2$ . For any fixed  $u \in \mathbb{R}^s$ , an  $(1 - \alpha)$  CI for  $u^T \gamma : u^T \hat{\gamma} \pm t_{n-r, \frac{\alpha}{2}} \hat{\sigma} \sqrt{u^T D u}$ . Now allow  $u \in \mathbb{R}^s$  to vary arbitrarily. Since  $\sup_{u \neq 0} \frac{|u^T \hat{\gamma} - u^T \gamma|^2}{u^T D u} \stackrel{v=D^{-\frac{1}{2}}u}{=} \sup_{v \neq 0} \frac{|v^T D^{-\frac{1}{2}}(\hat{\gamma} - \gamma)|^2}{v^T v} \stackrel{\text{Cauchy-Schwarz}}{=} (\hat{\gamma} - \gamma)^T D^{-1} (\hat{\gamma} - \gamma)$ ,  $P(\sup_{u \neq 0} \frac{|u^T \hat{\gamma} - u^T \gamma|^2}{s \hat{\sigma}^2 u^T D u} \leq F_{s, n-r, \alpha}) = 1 - \alpha$ . Simultaneous CIs for  $u^T \gamma, \forall u \in \mathbb{R}^s : u^T \hat{\gamma} \pm \hat{\sigma} \sqrt{s F_{s, n-r, \alpha} u^T D u}$ . (Bonferroni:  $t_{n-r, \alpha/(2m)}$ )
- Tukey's method: Consider  $y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$  i.i.d.,  $j = 1, \dots, m, i = 1, \dots, k$  and test for  $\alpha_i - \alpha_{i'}, \forall i, i' = 1, \dots, k$ . If  $Z_1, \dots, Z_n \sim \mathcal{N}(0, 1), R^2 \sim \chi_v^2$ , then  $\frac{Z_{(n)} - Z_{(1)}}{\sqrt{R^2/v}} \sim q_{n,v}$  (studentized range distribution). Thus  $\frac{\sqrt{m}}{\hat{\sigma}} \max_{i, i'} \{\bar{y}_i - \bar{y}_{i'} - (\alpha_i - \alpha_{i'})\} = \frac{\{\max_i \frac{\sqrt{m}(\bar{y}_i - \mu - \alpha_i)}{\hat{\sigma}} - \min_i \frac{\sqrt{m}(\bar{y}_i - \mu - \alpha_i)}{\hat{\sigma}}\}}{\sqrt{\frac{\text{RSS}/\sigma^2}{n-k}}} \sim q_{k, n-k}$ . Simultaneous CIs:  $\bar{y}_i - \bar{y}_{i'} \pm \frac{\hat{\sigma}}{\sqrt{m}} q_{k, n-k, \alpha}$ . (Bonferroni:  $t_{n-k, \alpha/[k(k-1)]}$ , Scheffé:  $\sqrt{k F_{k, n-k, \alpha}}$ , Tukey:  $q_{k, n-k, \alpha} / \sqrt{2}$  (the best/shortest length))

## 5 Exponential Families

- One parameter exponential families:  $\mathcal{G} = \{g_\eta(y) = e^{\eta y - \psi(\eta)} g_0(y) d\nu(y), \eta \in A, y \in \mathcal{Y}\}$ , or  $\log g_\theta(x) = A(\theta)B(x) + C(\theta) + D(x)$ .  $\eta$ : natural parameter;  $y$ : sufficient statistics;  $\psi(\eta)$ : normalizing function s.t.  $\frac{\int e^{\eta y} g_0(y) d\nu(y)}{e^{\psi(\eta)}} = 1$ ;  $A$ : natural parameter space s.t.  $\int e^{\eta y} g_0(y) d\nu(y) < \infty$ .  $e^{\eta y - \psi(\eta)}$ : exponential tilting, a method of generating an additive distribution family.
- Mean and variance:  $e^{\psi(\eta)} = \int_Y e^{\eta y} g_0(y) d\nu(y)$ , differentiating w.r.t.  $y$ ,  $\psi'(\eta) e^{\psi(\eta)} = \int_Y y e^{\eta y} g_0(y) d\nu(y)$ ,  $[\psi''(\eta) + \psi'(\eta)^2] e^{\psi(\eta)} = \int_Y y^2 e^{\eta y} g_0(y) d\nu(y) \Rightarrow \psi'(\eta) = \mathbb{E}_\eta Y = \mu_\eta, \psi''(\eta) = \mathbb{E}_\eta Y^2 - \mu_\eta^2 = \text{Var}_\eta(Y) = V_\eta$ .

- Cumulants: Let  $\kappa_j, j = 1, 2, \dots$  satisfy  $\psi(\eta) - \psi(\eta_0) = \kappa_1(\eta - \eta_0) + \frac{\kappa_2}{2}(\eta - \eta_0)^2 + \frac{\kappa_3}{3!}(\eta - \eta_0)^3 + \dots$ .  $\psi'''(\eta_0) = \kappa_3 = \mathbb{E}_0(Y - \mu_0)^3$ ,  $\psi''''(\eta_0) = \kappa_4 = \mathbb{E}_0(Y - \mu_0)^4 - 3\kappa_2^2$ . They correspond to central/noncentral moments. Skewness(偏度):  $\gamma = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{\mathbb{E}(Y - \mathbb{E}Y)^3}{(\text{Var}(Y))^{3/2}}$ . Kurtosis(峰度):  $\delta = \frac{\kappa_4}{\kappa_2^2} = \frac{\mathbb{E}(Y - \mathbb{E}Y)^4}{(\text{Var}(Y))^2} - 3$ .
- If  $y \sim g_\eta(\cdot)$  in an exponential family, then  $y \sim [\psi', \psi'^{1/2}, \psi'''/\psi'^{3/2}, \psi''''/\psi'^{2}]$ (expectation, SD, skewness, kurtosis). e.g. Poisson:  $\psi = e^\eta = \mu, \phi' = \dots = \phi'''' = \mu, y \sim [\mu, \sqrt{\mu}, 1/\sqrt{\mu}, 1/\mu]$ .
- **Theorem 5.1**  $P(Y \leq \text{median}(Y)) \approx 0.5 + \frac{1}{6\sqrt{2\pi}}\text{skewness}(Y)$ .
- **Lemma 5.1**  $Y = [y_0, y_1]$ , then  $\mathbb{E}_\eta[-l'_0(y)] = \eta - (g_\eta(y_1) - g_\eta(y_0))$  where  $l_0(y) = \log g_0(y)$  and  $l'_0(y) = \frac{dl_0(y)}{dy}$ .

**Proof** Integration by parts. □

- MLEs in exponential family:  $Y_i \sim g_\eta$  i.i.d. for  $i = 1, \dots, n$ .  $g_\eta^{(n)}(y) = e^{n(\eta\bar{y} - \psi(\eta))} \prod_{i=1}^n g_0(y_i)$ ,  $\eta^{(n)} = n\eta$ ,  $\psi^{(n)}(y) = n\psi(\eta^{(n)}/n)$ . log-likelihood:  $l_\eta(y) = \log g_\eta^{(n)}(y) = n(\eta\bar{y} - \psi(\eta)) + C$ , score:  $l'_\eta(y) = n(\bar{y} - \mu_\eta)$ , score equation:  $l'_\eta(y) = 0 \Rightarrow \mu_{\hat{\eta}} = \bar{y}$ . Since  $\frac{d\mu}{d\eta} = \psi''(\eta) = V_\eta > 0$ , we can solve  $\hat{\eta}$  by  $\hat{\eta} = \psi'^{-1}(\hat{\mu})$ . e.g. (1) Poisson:  $\hat{\eta} = \log(\bar{y})$ ; (2) Binomial:  $\hat{\eta} = \log(\frac{\bar{y}}{1-\bar{y}})$ .
- Fisher information:  $I_\eta^{(n)} = nI_\eta = nV_\eta, I_\mu^{(n)} = nI_\mu = \frac{n}{V_\eta}$ . C-R lower bound:  $\xi = h(\eta)$ , any unbiased estimator  $\bar{\xi}$  of  $\xi$ ,  $\text{Var}(\bar{\xi}) \geq \frac{1}{I_\mu^{(n)}(\xi)} = \frac{(h'(\eta))^2}{nV_\eta}$ . In particular,  $\xi = \mu$ , then  $\text{Var}(\hat{\mu}) \geq \frac{V_\eta}{n}$ .
- Important distributions: (1) Normal:  $\mathcal{N}(\eta, 1), \psi(\eta) = \frac{1}{2}\eta^2, g_0(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$ ; (2) Binomial:  $g_\eta(y) = C_N^y \pi^y (1-\pi)^{N-y} = C_N^y e^{y \log \pi + (N-y) \log(1-\pi)}, y = 0, 1, \dots, N, \eta = \log \frac{\pi}{1-\pi}, \pi = \frac{1}{1+e^{-\eta}} = \frac{e^\eta}{1+e^\eta}, \psi(\eta) = N \log(1+e^\eta)$ ; (3) Gamma( $k, \theta$ )(shape, scale),  $\chi_k^2 = \text{Gamma}(k/2, 2)$ ; (4) Negative Binomial:  $\text{NB}(k, \theta) = \# \text{ tails until } k\text{th head}$ .  $g_\eta(y) = C_{y+k-1}^{k-1} (1-\theta)^y \theta^k = C_{y+k-1}^{k-1} e^{y \log(1-\theta) + k \log \theta}, y = 0, 1, 2, \dots, \theta \in (0, 1), \eta = \log(1-\theta), \psi(\eta) = k \log(1-e^\eta), \mu = k \frac{1-e^\eta}{\theta}, V = \frac{\mu}{\theta}$  (property:  $k \rightarrow \infty, \mu$  fixed,  $Y \rightarrow \text{Poisson}(\mu)$ ).
- Inverse Gaussian:  $W(t)$ : Wiener process with drift  $1/\mu$ .  $W(t) = \frac{1}{\mu}t + B(t)$  and  $W(t) \sim \mathcal{N}(t/\mu, t)$ ,  $\text{Cov}(W(t), W(t+s)) = t$ .  $Y = 1\text{st passage time to } W(t) = 1$ . Density of  $\text{IG}(\mu)$ :  $g(y) = \frac{1}{\sqrt{2\pi y^3}} \exp\{-\frac{(y-\mu)^2}{2\mu^2 y}\} = \frac{1}{\sqrt{2\pi y^3}} \exp(-\frac{y}{2\mu^2} + \frac{1}{\mu} - \frac{1}{2y})$  with  $\eta = -\frac{1}{2\mu^2}, \psi(\eta) = -\sqrt{2\eta}$  belongs to the exponential family.
- Tilted hypergeometric: Consider  $2 \times 2$  talk (Table 1). Counts  $X = (x_1, x_2, x_3, x_4) \sim \text{Multinomial}(N, (\pi_1, \pi_2, \pi_3, \pi_4))$ . Test:  $H_0 : \theta = \log(\frac{\pi_1/\pi_2}{\pi_3/\pi_4}) = 0$ . Under  $H_0$ , conditional distribution of  $x_1$  given  $(r_1, r_2, c_1, c_2)$  is  $g_0(x_1|r_1, r_2, c_1, c_2) = \frac{C_{r_1}^{x_1} C_{r_2}^{c_1-x_1}}{C_N^{c_1}} \sim \text{hypergeometric with } \max(0, c_1 - r_2) \leq x_1 \leq \min(c_1, r_1)$ . When  $H_0$  is not true,  $g_\theta(x_1|r_1, r_2, c_1, c_2) = \frac{g_0(x_1|r_1, r_2, c_1, c_2) e^{\theta x_1} C_N^{c_1}}{C(\theta)}$  belongs to the exponential family with  $C(\theta) = \sum_{x_1} C_{r_1}^{x_1} C_{r_2}^{c_1-x_1} e^{\theta x_1}$ .

Table 1:  $2 \times 2$  talk

	Yes	No	
Male	$x_1$	$x_2$	$r_1$
Female	$x_3$	$x_4$	$r_2$
	$c_1$	$c_2$	$N$

- Deviance (Kullback-Leibler divergence): Generating Euclidean distance to exponential families,  $2\text{KL}(\eta_1, \eta_2) = D(\eta_1, \eta_2) := 2 \int \eta_1(y) \log \frac{\eta_1(y)}{\eta_2(y)} d\nu(y) = 2\mathbb{E}_{\eta_1}[(\eta_1 - \eta_2)y - (\psi(\eta_1) - \psi(\eta_2))] = 2[(\eta_1 - \eta_2)\mu_1 - (\psi(\eta_1) - \psi(\eta_2))]$ . Mutual information:  $D(f(x, y), f(x)f(y))/2$ . Example: (1)  $\mathcal{N}(\mu, 1) : D(\mu_1, \mu_2) = (\mu_1 - \mu_2)^2$ ; (2)  $\text{Poisson}(\mu) : D(\mu_1, \mu_2) = 2\mu_1[\log(\frac{\mu_1}{\mu_2}) - (1 - \frac{\mu_2}{\mu_1})]$ ; (3)  $\text{Binomial}(N, \pi) : D(\pi_1, \pi_2) = 2N[\pi_1 \log(\frac{\pi_1}{\pi_2}) + (1 - \pi_1) \log(\frac{1-\pi_1}{1-\pi_2})]$ .
- **Theorem 5.2** (Hoeffding's formula) For  $g_\eta(y) = e^{\eta y - \psi(\eta)} g_0(y)$ , let  $\hat{\eta}$  be the MLE of  $\eta$  and  $\hat{\mu}$  be the MLE of  $\mu$ . Then  $g_\eta(y) = g_{\hat{\eta}}(y) e^{-D(\hat{\eta}, \eta)/2}, g_\mu(y) = g_{\hat{\mu}}(y) e^{-D(\hat{\mu}, \mu)/2}$ .

**Proof**  $\frac{g_\eta(y)}{g_{\hat{\eta}}(y)} = e^{(\eta - \hat{\eta})y - (\psi(\eta) - \psi(\hat{\eta}))} \stackrel{y \equiv \hat{\mu}}{=} e^{-D(\hat{\eta}, \eta)/2}.$  □

• **Proposition 5.1**  $D(\eta_1, \eta_2) = I_{\eta_1} \times (\eta_2 - \eta_1)^2 + O((\eta_2 - \eta_1)^3).$

**Proof**  $\frac{\partial}{\partial \eta_2} D(\eta_1, \eta_2) = \frac{\partial}{\partial \eta_2} 2[(\eta_1 - \eta_2)\mu_1 - (\psi(\eta_1) - \psi(\eta_2))] = 2(-\mu_1 + \mu_2) \Rightarrow \frac{\partial}{\partial \eta_2} D(\eta_1, \eta_2)|_{\eta_2=\eta_1} = 0.$   $\frac{\partial^2}{\partial \eta_2^2} D(\eta_1, \eta_2) = 2\frac{\partial \mu_2}{\partial \eta_2} \Rightarrow \frac{\partial^2}{\partial \eta_2^2} D(\eta_1, \eta_2)|_{\eta_2=\eta_1} = 2V_{\eta_1}.$  Taylor expansion:  $D(\eta_1, \eta_2) = 2V_{\eta_1} \frac{(\eta_2 - \eta_1)^2}{2} + O((\eta_2 - \eta_1)^3) = I_{\eta_1}(\eta_2 - \eta_1)^2 + O((\eta_2 - \eta_1)^3).$  □

• Deviance residuals: Exponential family analogue of normal residuals  $y - \mu$ :  $\text{sgn}(y - \mu)\sqrt{D(y, \mu)}$ . Let  $y_i \sim g_\mu(\cdot)$  i.i.d. for  $i = 1, \dots, n$ . Define the deviance residual  $R = \text{sgn}(\bar{y} - \mu)\sqrt{nD(\bar{y}, \mu)} = \text{sgn}(\bar{y} - \mu)\sqrt{D^{(n)}(\bar{y}, \mu)}$ . The hope is that  $R$  will be nearly  $\mathcal{N}(0, 1)$ , at least closer to normal than the more obvious “Pearson residual”  $R_p = \frac{\bar{y} - \mu}{\sqrt{V_\mu/n}}$ .

• **Theorem 5.3**  $R \sim \mathcal{N}(-a_n, (1 + b_n)^2)$  where  $a_n = \frac{\gamma_\mu/6}{\sqrt{n}}$  and  $b_n = \frac{7/36 \gamma_\mu^2 - \delta_\mu}{n}$  (recall  $\gamma_\mu, \delta_\mu$  is skewness and kurtosis of  $g_\mu$ ). The constants  $a_n$  and  $b_n$  are called “Bartlett corrections”. More precisely,  $P(\frac{R+a_n}{1+b_n} > z_\alpha) = \alpha + O(n^{-3/2})$ .

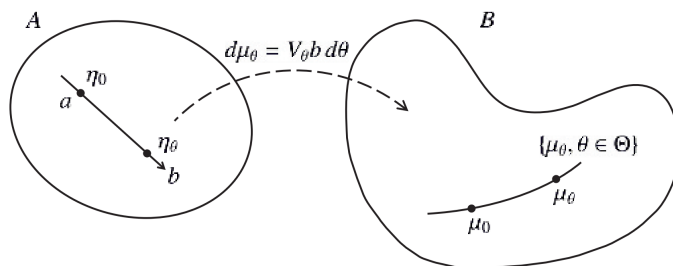
**Corollary 5.1**  $D^{(n)}(\bar{y}, \mu) = R^2 \sim (1 + \frac{5\gamma_\mu^2 - 3\delta_\mu}{12n})\chi_1^2.$

- We wish to approximate the density under  $g_\mu^{(n)}$  of the sufficient statistic  $\hat{\mu} = \bar{y}$ . Normal approximation:  $g_\mu^{(n)}(\hat{\mu}) = \sqrt{\frac{n}{2\pi V_\mu}} e^{-\frac{n(\hat{\mu} - \mu)^2}{2V_\mu}}$ . Saddlepoint approximation:  $g_\mu^{(n)}(\hat{\mu}) = \sqrt{\frac{n}{2\pi \hat{V}}} e^{-nD(\hat{\mu}, \mu)/2}.$
- Lugananni-Rice Formula: Observing  $\bar{y} = \hat{\mu}$ ,  $p$ -value  $\alpha(\mu) = \int_{\hat{\mu}}^\infty g_\mu^{(n)}(t) d\nu(t) \approx 1 - \Phi(R) - \phi(R)(\frac{1}{R} - \frac{1}{Q}) + O(n^{-3/2})$  where  $\Phi$  and  $\phi$  are cdf/pdf of  $\mathcal{N}(0, 1)$ ,  $R = \text{sgn}(\hat{\mu} - \mu)\sqrt{nD(\hat{\mu}, \mu)}$  is the deviance residual, and  $Q = \sqrt{n\hat{V}(\hat{\eta} - \eta)}$  is the crude form of the Pearson residual based on the canonical parameter.
- Transformation:  $\zeta = H(\mu), \hat{\zeta} = H(\hat{\mu}), \hat{\mu}$  the MLE of  $\mu, H'(\mu) = V_\mu^{\delta-1}, 0 \leq \delta \leq 1.$

$\delta =$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1
$\zeta =$	Canonical parameter $\eta$	Normal likelihood	Stabilized variance	Normal density	Expectation parameter $\mu$

Example (when  $\delta = \frac{1}{2}$ ): (1) Poisson( $\mu$ ),  $H'(\mu) = \mu^{-1/2}, H(\mu) = 2\sqrt{\mu}, 2\sqrt{\bar{y}} \sim \mathcal{N}(2\sqrt{\mu}, 1)$ ; (2) Binomial( $N, \pi$ ),  $\hat{\zeta} = 2\sqrt{N} \sin^{-1} \sqrt{\frac{Np+3/8}{N+3/4}}.$

- Multiparameter exponential families: A  $p$ -parameter exponential family  $\mathcal{G} = \{g_\eta(y) : \eta \in A \subset \mathbb{R}^p, y \in \mathcal{Y} \subset \mathbb{R}^p\}$  with  $g_\eta(y) = e^{\eta^T y - \psi(\eta)} g_0(y) d\nu(y), \mu = \mathbb{E}_\eta Y = \psi'(\eta), V = \text{Var}_\eta(Y) = \psi''(\eta), d\mu = V d\eta, d\eta = V^{-1} d\mu.$  Assume  $V$  will be positive definite for all  $\eta$  in  $A = \{\eta : \int_{\mathcal{Y}} e^{\eta^T y} g_0(y) d\nu < \infty\}$ . Let  $B = \{\mu = \mathbb{E}_\eta Y, \eta \in A\}.$
- Facts: (1)  $A$  is convex; (2)  $B \subset \text{convex hull of } \mathcal{Y}$ ; (3)  $\text{Angle}(d\eta, d\mu) < \frac{\pi}{2}$  ( $d\eta^T d\mu = d\eta^T V d\eta > 0$ ).
- Transformation:  $\zeta = h(\eta) = H(\mu) \in \mathbb{R}, \eta, \mu \in \mathbb{R}^p, D = \frac{d\eta}{d\mu} = V^{-1}.$  Then  $H'(\mu) = Dh'(\eta), H''(\mu) = Dh''(\eta)D^T + D_2 h'(\eta)$  where  $D_2 = (\frac{\partial^2 \eta_k}{\partial \mu_i \partial \mu_j})_{i,j,k}.$
- One-parameter subfamilies:  $\eta_\theta = a + b\theta, \theta \in \Theta \subset \mathbb{R}, a, b \in \mathbb{R}^p, \mathcal{F} = \{f_\theta(y) = g_{\eta_\theta}(y) = e^{(a+b\theta)^T y - \psi(a+b\theta)} g_0(y) d\nu, \theta \in \Theta\}.$  Still a one-parameter exponential family, natural parameter  $\theta$ , sufficient statistics  $x = b^T y$ . MLE of  $\theta$  (score equation):  $l'_\theta(\bar{y}) = 0 \Rightarrow b^T(\bar{y} - \mu_\theta) = 0.$





- Stein's least favorable subfamily:  $\zeta = s(\eta) = t(\mu)$ ,  $\zeta_0 = s(\eta_0) = t(\mu_0)$ ,  $s'_0 = \frac{\partial s(\eta)}{\partial \eta}|_{\eta_0}$ ,  $t'_0 = \frac{\partial t(\mu)}{\partial \mu}|_{\mu_0}$ . Define the LFF:  $\eta_\theta = \eta_0 + t'_0 \theta$ ,  $\theta \in \text{neighborhood of } 0$ .



- **Theorem 5.4** The 1-parameter CRLB for estimating  $\zeta$  in LFF evaluated at  $\theta = 0$  is the same as the  $p$ -parameter CRLB for estimating  $\zeta$  in  $\mathcal{G}$  at  $\eta = \eta_0$ , which equals  $t'_0 V_0 t_0$ , where  $V_0$  is the variance evaluated at  $\eta_0$  or  $\mu_0$ .

**Remark 5.1** In other words, the reduction to the LFF does not make it any easier to estimate  $\zeta$ . It can be shown that any choice other than  $b = t'_0$  for the family  $\eta_\theta = \eta_0 + b\theta$  makes the one-parameter CRLB smaller than the  $p$ -parameter CRLB. Stein's construction is useful when some statistical property is easily calculated only in the one-parameter case.

- Examples: (1)  $\mathcal{N}(\lambda, \Gamma) : g(x) = \frac{1}{\sqrt{2\pi\Gamma}} \exp(-\frac{x^2}{2\Gamma} + \frac{\lambda}{\Gamma}x - \frac{\lambda^2}{2\Gamma})$ ,  $\eta = (\lambda/\Gamma, -\frac{1}{2\Gamma})^T$ ,  $y = (x, x^2)^T$ ,  $\mu = (\lambda, \lambda^2 + \Gamma)^T$ ; (2) Beta( $\alpha, \beta$ ) :  $g(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} = \exp\{\alpha \log x + \beta \log(1-x) - \log B(\alpha, \beta)\}$ ,  $\eta = (\alpha, \beta)^T$ ,  $y = (\log x, \log(1-x))^T$ ; (3) Dirichlet( $\alpha_1, \dots, \alpha_p$ ),  $g_\alpha(x) = \frac{1}{B(\alpha)} \prod_{i=1}^p x_i^{\alpha_i-1}$ ,  $B(\alpha) = \frac{\prod_{i=1}^p \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^p \alpha_i)}$ ,  $x \in \mathbb{S}^{p-1}$ ; (4) Graph/Degree model:  $Y_{ij} = I(i=j)$ ,  $\pi_{ij} = P(Y_{ij} = 1) = \frac{e^{\theta_i + \theta_j}}{1 + e^{\theta_i + \theta_j}}$ ,  $\theta_i = \beta^T x_i$  where  $x_i$ 's are optional predictors. Sufficient statistics is degree of node  $i$ . (5) Bradley-Terry model:  $\pi_{ij} = \frac{e^{\theta_i}}{e^{\theta_i} + e^{\theta_j}} = \frac{e^{\theta_i - \theta_j}}{1 + e^{\theta_i - \theta_j}}$ ,  $w_{ij} \sim \text{Binomial}(n_{ij}, \pi_{ij})$ ,  $g_\theta \propto \exp(\sum_{i,j} (\theta_i - \theta_j) w_{ij}) = \exp(\sum_i \theta_i \sum_j w_{ij} - \sum_j \theta_j \sum_i w_{ij}) = \exp\{\sum_i \theta_i [\#\text{win}(i) - \#\text{lose}(i)]\}$ .
- Truncated data:  $y \sim g_\eta(y) = e^{\eta^T y - \psi(\eta)} g_0(y)$ , observed only if  $y$  falls in  $\mathcal{Y}_0 \subset \mathcal{Y}$ . Conditional density:  $g_\eta(y|\mathcal{Y}_0) = \frac{e^{\eta^T y - \psi(\eta)} g_0(y)}{G_\eta(\mathcal{Y}_0)}$ , where  $G_\eta(\mathcal{Y}_0) = \int_{\mathcal{Y}_0} g_\eta(y) dy$ .
- **Lemma 5.2** Partition  $\eta = (\eta_1, \eta_2)$ ,  $y = (y_1, y_2)$ .  $y_1|y_2 \sim g_{\eta_1}(y_1|y_2) = e^{\eta_1^T y_1 - \psi(\eta_1|\eta_2)} dG_0(y_1|y_2)$ ,  $y_2 \sim g_{\eta_1, \eta_2}(y_2) = e^{\eta_2^T y_2 - \psi_{\eta_1}(\eta_2)} dG_{\eta_1, 0}(y_2)$ .

**Proof**  $g_\eta(y_2) = \int_{\mathcal{Y}_1} e^{\eta_1^T y_1 + \eta_2^T y_2 - \psi(\eta)} g_0(y_1|y_2) g_0(y_2) dy_1 = e^{\eta_2^T y_2 - \psi(\eta)} (\int_{\mathcal{Y}_1} e^{\eta_1^T y_1} g_0(y_1|y_2) dy_1) g_0(y_2) \Rightarrow g_\eta(y_1|y_2) = \frac{g_\eta(y)}{g_\eta(y_2)} = \frac{e^{\eta_1^T y_1 + \eta_2^T y_2 - \psi(\eta)} g_0(y)}{e^{\eta_2^T y_2 - \psi(\eta) + \psi(\eta_1|\eta_2)} g_0(y_2)} = e^{\eta_1^T y_1 - \psi(\eta_1|\eta_2)} dG_0(y_1|y_2)$ .  $\square$

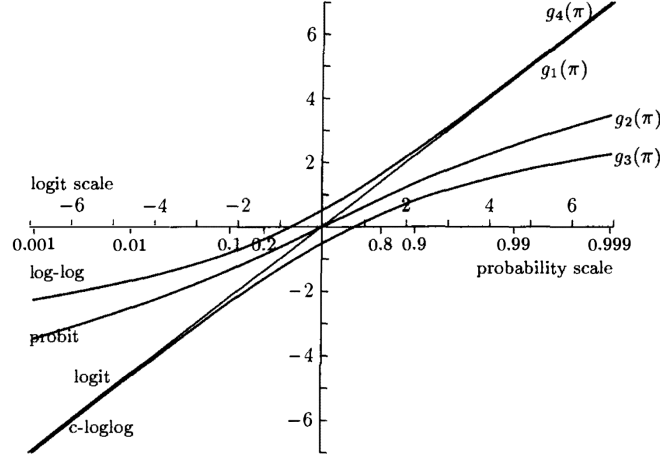
**Remark 5.2** Usually after a transformation  $M \in \mathbb{R}^{p \times p}$  nonsingular,  $\tilde{\eta} = (M^{-1})^T \eta$ ,  $\tilde{y} = My$ .

- Examples: (1) Fisher's exact test for  $2 \times 2$  table (Recall Table 1),  $H_0 : \theta = \log(\frac{\pi_1/\pi_2}{\pi_3/\pi_4}) = 0$ . The natural parameter is  $\eta = (\log \pi_1, \dots, \log \pi_4)$ . Let  $(M^{-1})^T = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ , so that  $M = \frac{1}{4}(M^{-1})^T$ ,  $\tilde{y} = Mx$ ,  $\tilde{y}_1 = \frac{1}{4}(x_1 - x_2 - x_3 + x_4) = x_1 - \frac{r_1}{2} - \frac{c_1}{2} + \frac{N}{4}$ . (2) Wishart statistics:  $x_1, \dots, x_n \sim \mathcal{N}_d(\lambda, \Gamma)$  independent,  $y_1 = \bar{x}$ ,  $y_2 = \frac{1}{n} \sum_{i=1}^n x_i x_i^T$ . Wishart statistics  $W = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})^T = y_2 - y_1 y_1^T$ .  $y_2|y_1$  is in a  $\frac{d(d+1)}{2}$ -dim exponential family. (3) Poisson trick:  $s = (s_1, \dots, s_L)$ ,  $s_l \sim \text{Poisson}(\mu_L)$  independent  $\Rightarrow s|n = \sum_{l=1}^L s_l \sim \text{Multinomial}_L(n, \pi)$  where  $\pi_l = \frac{\mu_l}{\sum_j \mu_j}$ . Conversely, if  $s|n \sim \text{Multinomial}(n, \pi)$  and  $n \sim \text{Poisson}(\mu_+)$ , then  $s_l \sim \text{Poisson}(\mu_+ \pi_l)$  i.i.d.
- Rotational speeds of stars: Bimodal:  $f(x) = w \frac{\phi(x/c_1)}{c_1} + (1-w) \frac{\phi(x/c_2)}{c_2}$ . Two competing candidates for  $\phi(x) : \phi_1(x) = 2xe^{-x^2}$ ,  $\phi_2(x) = 4x^2 e^{-x^2} \pi^{-1/2}$ . We take the bin partitions and set  $y_l$  to be the count and  $\pi_l$  be the probability of bin  $l$ .  $y_l \sim \text{Poisson}(\mu_l)$ ,  $\mu_l = n\pi_l$ . Any choice of  $(w, c_1, c_2)$  produces estimates of  $\pi_l$  and  $\mu_l$ .

## 6 Generalized Linear Models

- Data types for response  $y$  : 
$$\left\{ \begin{array}{l} \text{numerical:} \left\{ \begin{array}{l} \text{continuous: Box-Cox transformation:} \left\{ \begin{array}{ll} \frac{x^\lambda - 1}{\lambda}, & \lambda \neq 0 \\ \log x, & \lambda = 0 \end{array} \right. \\ \text{discrete: count} \end{array} \right. \\ \text{categorical:} \left\{ \begin{array}{l} \text{nominal:} \left\{ \begin{array}{l} \text{binary} \\ \text{multinomial} \end{array} \right. \\ \text{ordinal} \end{array} \right. \end{array} \right.$$
- Three components of GLMs: (1) Random: distribution of  $Y$  with  $\mathbb{E}Y = \mu$ ; (2) Systematic:  $\eta = \sum_{j=1}^p x_j \beta_j$ ; (3) Link:  $g(\mu) = \eta$ .
- Example 1 (Dilution assays): density  $\rho_0$ , at the  $x$ -th dilution  $\rho_x = \rho_0 2^{-x}$ ,  $x = 0, 1, 2, \dots$ , proportion of infected plates  $y_x = \frac{r_x}{m_x}$ ,  $Y = I(\text{infected})$ ,  $\mathbb{E}(Y|x) = P(Y = 1|x) = \pi_x$ , # organism on a plate:  $N_x \sim \text{Poisson}(\rho_x v)$ ,  $\pi_x = P(N_x \geq 1) = 1 - e^{-\rho_x v} = 1 - e^{-\rho_0 v 2^{-x}}$ , link function  $g(\pi_x) = \log(-\log(1 - \pi_x)) = \log v + \log \rho_0 - x \log 2$ .
- Example 2 (Dose response): dose level  $x$ , survival rate  $\pi_x$ , cell  $j$ , dose level  $x_j$ ,  $y_j$  survive out of  $m_j$  animals. (1) Probit model:  $\pi_x = \Phi(\alpha + \beta x)$ , where  $\Phi$  is the c.d.f. of  $\mathcal{N}(0, 1)$ , link function  $g = \Phi^{-1}$ . (2) Logistic/Logit model:  $\pi_x = \text{expit}(\alpha + \beta x) = \frac{1}{1 + e^{-(\alpha + \beta x)}}$ , link function  $g(\pi_x) = \text{logit}(\pi_x) = \log \frac{\pi_x}{1 - \pi_x}$ .
- Random component:  $Y$  has a distribution in an exponential family:  $f(y; \theta, \phi) = \exp\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\}$  where  $\phi$  is dispersion parameter. Usually  $a(\phi) = \phi/w_i$ . log-likelihood:  $l(\theta; y) = \frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)$ .  $\frac{\partial l}{\partial \theta} = \frac{y - b'(\theta)}{a(\phi)}$ ,  $\frac{\partial^2 l}{\partial \theta^2} = -\frac{b''(\theta)}{a(\phi)}$ .  $\mathbb{E} \frac{\partial l}{\partial \theta} = 0$ ,  $\mathbb{E}(\frac{\partial l}{\partial \theta})^2 = -\mathbb{E} \frac{\partial^2 l}{\partial \theta^2}$ ,  $\mathbb{E}Y = \mu = b'(\theta)$ ,  $\text{Var}(Y) = a(\phi)b''(\theta)$ .
- Systematic component: predictors  $(x_1, \dots, x_p)$ ,  $\eta = x^T \beta$ .
- Canonical link function:  $g = b'^{-1}(\mu)$  so that  $\eta = g(\mu) = b'^{-1}(b'(\theta)) = \theta$ .
- Goodness of fit: Null model: one parameter,  $\mu$  common mean. Full model:  $n$  parameters, one per observation. Idea: Measure discrepancy between an intermediate model and the full model.
- Assume  $l(y, \phi; y)$ ,  $l(\hat{\mu}, \phi; y)$  maximize log-likelihood over  $\beta$  with fixed  $\phi$ ,  $g_1/g_2$  is full/current model respectively,  $\tilde{\theta}/\hat{\theta} = \theta(y)/\theta(\hat{\mu})$  and  $a_i(\phi) = \phi/w_i$ .  $2\mathbb{E}_{P_n} \log \frac{l(y, \phi; y)}{l(\hat{\mu}, \phi; y)} = 2 \sum_{i=1}^n \frac{w_i}{\phi} [(\tilde{\theta}_i - \hat{\theta}_i)y_i - b(\tilde{\theta}_i) + b(\hat{\theta}_i)] := \frac{D(y, \hat{\mu})}{\phi}$ . Under suitable regularity conditions, if the fitted model is correct,  $D(y, \hat{\mu})/\phi \sim \chi_{n-p}^2$  where  $p$  is the dimension of  $\beta$ .
- Pearson's  $\chi^2$ -statistic:  $\chi^2 = \sum_{i=1}^n \frac{(y_i - \hat{\mu}_i)^2}{V(\hat{\mu}_i)/w_i}$  where  $V(\mu) = b''(b'^{-1}(\mu))$ . Under suitable regularity conditions, if the model is correct,  $\chi^2/\phi \sim \chi_{n-p}^2$ .
- Residuals: (1) Deviance residual:  $r_D = \text{sgn}(y - \hat{\mu})\sqrt{d_i}$  where  $d_i = 2w_i[(\tilde{\theta}_i - \hat{\theta}_i)y_i - b(\tilde{\theta}_i) + b(\hat{\theta}_i)]$ ; (2) Pearson residual:  $r_p = \frac{y - \hat{\mu}}{\sqrt{V(\hat{\mu})/w_i}}$ ; (3) Anscombe residual:  $\delta = \frac{2}{3}, H'(\mu) = V_\mu^{-\frac{1}{3}}, A = \int \frac{d\mu}{V^{1/3}(\mu)}$ . For Poisson distribution,  $A = \frac{3}{2}\mu^{2/3}$ , and we must scale by dividing by the SD of  $A(Y)$ , i.e.  $A'(\mu)\sqrt{V(\mu)} \Rightarrow r_A = \frac{\frac{3}{2}(y^{2/3} - \mu^{2/3})}{\mu^{1/6}}$ .
- Algorithms for fitting GLMs:  $l(\beta)$  log-likelihood,  $u(\beta) = \frac{\partial}{\partial \beta} l(\beta)$ ,  $H(\beta) = \frac{\partial^2}{\partial \beta \partial \beta^T} l(\beta)$ . The MLE of  $\hat{\beta}$  solves the estimating equation.  $0 = u(\hat{\beta}) \approx u(\beta^{(0)}) + H(\beta^{(0)})(\hat{\beta} - \beta^{(0)})$  giving the update  $\beta^{(t+1)} = \beta^{(t)} - H(\beta^{(t)})^{-1}u(\beta^{(t)})$ . Fisher scoring:  $\beta^{(t+1)} = \beta^{(t)} + I(\beta^{(t)})^{-1}u(\beta^{(t)})$  (since  $I(\beta) = -\mathbb{E}H(\beta)$ ). In a GLM,  $l = \sum_{i=1}^n l_i$ ,  $l_i = \frac{y_i \theta_i - b(\theta_i)}{a_i(\phi)} + c_i(y_i, \phi)$ , and  $u_{ir} = \frac{\partial l_i}{\partial \beta_r} = \frac{\partial l_i}{\partial \theta_i} \frac{\partial \theta_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \eta_i} \frac{\partial \eta_i}{\partial \beta_r} = \frac{y_i - \mu_i}{a_i(\phi)} \frac{1}{V(\mu_i)} \frac{1}{g'(\mu_i)} x_{ir} = \frac{(y_i - \mu_i)x_{ir}}{a_i(\phi)V(\mu_i)g'(\mu_i)} = (y - \mu)^T W \frac{d\eta}{d\mu} x_{(r)}$  where  $W = \text{diag}(\frac{1}{a_i(\phi)V(\mu_i)g'(\mu_i)^2})$ . Since  $\text{Cov}(u_r, u_s) = \sum_{i=1}^n \frac{\text{Var}(y_i)x_{ir}x_{is}}{a_i(\phi)^2 V(\mu_i)^2 g'(\mu_i)^2} = \sum_{i=1}^n \frac{x_{ir}x_{is}}{a_i(\phi)V(\mu_i)g'(\mu_i)^2} \Rightarrow I(\beta) = \text{Var}(u(\beta)) = X^T W X$ ,  $u(\beta) = X^T W \frac{d\eta}{d\mu} (y - \mu)$  where  $X = (x_{ir})_{n \times p}$ .  $H(\beta) = -X^T W X + X^T \{\frac{\partial}{\partial \beta^T} (W \frac{d\eta}{d\mu})\}(y - \mu)$ .
- Under what conditions  $-H(\beta) = I(\beta)$ ? Take canonical link  $\eta_i = b^{-1}(\mu_i) = \theta_i$ ,  $V(\mu_i) = b''(\theta_i) = \frac{\partial \mu_i}{\partial \theta_i} = \frac{\partial \mu_i}{\partial \eta_i}$ ,  $w_{ii} = \frac{1}{a_i(\phi)V(\mu_i)g'(\mu_i)^2} = \frac{1}{a_i(\phi)} \frac{\partial \eta_i}{\partial \mu_i} \Rightarrow W \frac{d\eta}{d\mu} = \text{diag}(\frac{1}{a_i(\phi)}) \Rightarrow \frac{\partial}{\partial \beta^T} (W \frac{d\eta}{d\mu}) = 0$ .

- Substituting back,  $\beta^{(t+1)} = \beta^{(t)} + (X^T W^{(t)} X)^{-1} X^T W^{(t)} \frac{d\eta}{d\mu}(y - \mu) = (X^T W^{(t)} X)^{-1} X^T W^{(t)} [X\beta^{(t)} + \frac{d\eta}{d\mu}(y - \mu)] = (X^T W^{(t)} X)^{-1} X^T W^{(t)} [\eta^{(t)} + \frac{d\eta}{d\mu}|_{\mu^{(t)}}(y - \mu^{(t)})]$  (iteratively reweighted least squares).
- Inference about  $\beta$ :  $I(\beta)^{\frac{1}{2}}(\hat{\beta} - \beta) \Rightarrow \mathcal{N}_p(0, I)$ ,  $\widehat{\text{Var}}(\hat{\beta}) = (X^T W(\hat{\beta}) X)^{-1}$ ,  $h(\hat{\beta}) \sim \mathcal{N}(h(\beta), h'(\beta)^T I(\beta)^{-1} h'(\beta))$ ,  $\hat{\eta} = x^T \hat{\beta} \sim \mathcal{N}(x^T \beta, x^T I(\beta)^{-1} x)$ .
- CI for  $x$  that gives rise to a specified mean response  $\mu_0$ :  $\{x : \frac{x^T \hat{\beta} - g(\mu_0)}{\sqrt{x^T I(\hat{\beta})^{-1} x}} < z_{\alpha/2}\}$  (Fieller's method).
- Binary responses:  $g(\pi_i) = \eta_i = x_i^T \beta$ ,  $g : (0, 1) \rightarrow \mathbb{R}$ , link functions:  $g_1 = \log(\frac{\pi}{1-\pi})$ ,  $g_2 = \Phi^{-1}(\pi)$ ,  $g_3 = \log(-\log(1-\pi))$  (complementary log-log),  $g_4 = \log(-\log \pi)$  (log-log). These  $g_i$ 's are from the inverse of the cdfs:  $f_1 = \frac{e^x}{(1+e^x)^2}$  (logistic),  $f_3 = e^{x-e^x}$ , i.e.  $\log X, X \sim \text{Exp}(1)$ ,  $f_4 = e^{-x+e^x}$ , i.e.  $-\log X, X \sim \text{Exp}(1)$  (Gumbel).



- Application: Many epidemiological studies have the goal of comparing distinct groups, e.g., assessing risk factors for some disease. Denote  $D$  = disease status,  $X$  = exposure status.

	$\overline{D}$	$D$	
$\overline{X}$	$\pi_{00}$	$\pi_{01}$	$\pi_{0\cdot}$
$X$	$\pi_{10}$	$\pi_{11}$	$\pi_{1\cdot}$
	$\pi_{\cdot 0}$	$\pi_{\cdot 1}$	1

Sampling probabilities:  $P(D|x) = \frac{e^{\alpha+x^T \beta}}{1+e^{\alpha+x^T \beta}}$ ,  $\pi_0 = P(Z=1|\overline{D})$ ,  $\pi_1 = P(Z=1|D)$  where  $Z$  is indicator of being sampled. This is because  $|D|$  may be much smaller than  $|\overline{D}|$  and we need more data on  $D$  (i.e.  $\pi_0 \gg \pi_1$ ). Then  $P(D|Z=1, x) = \frac{P(Z=1|D, x)P(D|x)}{P(Z=1|D, x)P(D|x) + P(Z=1|\overline{D}, x)P(\overline{D}|x)} = \frac{\pi_0 e^{\alpha+x^T \beta}}{\pi_0 e^{\alpha+x^T \beta} + \pi_1} = \frac{e^{\alpha+x^T \beta + \log(\pi_0/\pi_1)}}{1+e^{\alpha+x^T \beta + \log(\pi_0/\pi_1)}} := \frac{e^{\alpha^*+x^T \beta}}{1+e^{\alpha^*+x^T \beta}}$  by Bayes formula. Thus, the “biased” random sampling of  $D$  and  $\overline{D}$  does not impact the value of  $\beta$ , and only translates  $\alpha$  to  $\alpha + \log(\pi_0/\pi_1)$ . We can conduct logistic regression on the new dataset.

## Binomial Regression

- $Y_i \sim \text{Binomial}(m_i, \pi_i)$ ,  $i = 1, \dots, n$ . For simplicity,  $m_i = m, \forall i$ . The log-likelihood  $l(\pi; y) = \sum_{i=1}^n [y_i \log \frac{\pi_i}{1-\pi_i} + m \log(1-\pi_i)] + C(y)$ . Under logistic link,  $\log \frac{\pi_i}{1-\pi_i} = x_i^T \beta$ , or  $\pi_i = \frac{e^{x_i^T \beta}}{1+e^{x_i^T \beta}}$ , so that  $l(\beta; y) = \sum_{i=1}^n [y_i x_i^T \beta - m \log(1+e^{x_i^T \beta})]$ . Exponential family has the form  $l(\theta; y) = \sum_{i=1}^n [\frac{y_i \theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i, \phi)]$ , so  $\eta_i = \theta_i = x_i^T \beta$ ,  $b(\theta_i) = m \log(1+e^{x_i^T \beta})$ ,  $a_i(\phi) \equiv 1$ . General likelihood equation  $u(\beta) = X^T W \frac{d\eta}{d\mu}(y - \mu) = 0$  where  $W \frac{d\eta}{d\mu} = \text{diag}(\frac{1}{a_i(\phi)})$  under canonical link. Now  $a_i(\phi) \equiv 1$ , so  $u(\beta) = X^T (y - \mu) = 0$ . The weight matrix  $W = \frac{d\mu}{d\eta} = m \frac{d\pi}{d\eta} = \text{diag}\{m\pi_i(1-\pi_i)\}$ . The working response  $z_i = \eta_i + \frac{d\eta_i}{d\mu_i}(y_i - \mu_i) = \eta_i + \frac{y_i - m_i \pi_i}{m_i} \frac{d\eta_i}{d\pi_i} = \eta_i + \frac{y_i - m_i \pi_i}{m_i \pi_i (1-\pi_i)}$ . Solve  $X^T W X \hat{\beta} = X^T W Z$ .
- **Theorem 6.1** (Wedderburn, 1976) If the link function is log concave and  $0 < y_i < m_i, \forall i$ , then  $\hat{\beta}$  is finite and the log-likelihood has a unique maximum at  $\hat{\beta}$ .

- **Theorem 6.2** (Shao, Ex 4.117) For logistic regression, if  $\sum_{i=1}^n x_i x_i^T$  is positive definite,  $\forall n \geq n_0$ , then the log-likelihood equation has at most one solution when  $n \geq n_0$  and a solution exists with probability  $\rightarrow 1$ .
- Deviance: The fitted log-likelihood  $l(\hat{\pi}; y) = \sum_{i=1}^n [y_i \log(\frac{\hat{\pi}_i}{1-\hat{\pi}_i}) + m_i \log(1-\hat{\pi}_i)] = \sum_{i=1}^n [y_i \log \hat{\pi}_i + (m_i - y_i) \log(1-\hat{\pi}_i)]$ , maximum achievable log-likelihood  $l(\tilde{\pi}; y)$  where  $\tilde{\pi}_i = \frac{y_i}{m_i}$ ,  $D(y, \hat{\pi}) = 2l(\tilde{\pi}; y) - 2l(\hat{\pi}; y) = 2 \sum_{i=1}^n [y_i \log(\frac{y_i}{\hat{\pi}_i}) + (m_i - y_i) \log(\frac{m_i - y_i}{1-\hat{\pi}_i})]$ . Asymptotic properties:  $D(y, \hat{\pi}) \sim \chi_{n-p}^2$  (assumptions: no overdispersion;  $m_i \rightarrow \infty$  with  $n$  fixed). Note that if  $n \rightarrow \infty$  while  $m_i$  fixed,  $D$  is not independent of  $\hat{\pi}$  and large  $D \not\Rightarrow$  poor fit.
- Extrapolation: predict  $x_0$  corresponding to  $\pi_0$ . Using Fieller's method,  $|\frac{\hat{\beta}_0 + \hat{\beta}_1 x_0 - g(\pi_0)}{V(x_0)}| \leq Z_{\alpha/2}$  where  $V(x_0)^2 = \text{Var}(\hat{\beta}_0) + 2x_0 \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) + x_0^2 \text{Var}(\hat{\beta}_1)$ .
- Overdispersion: “nominal” variance:  $m\pi(1-\pi)$ .  $\text{Var}(y) > / < m\pi(1-\pi)$ : over/under dispersion. Mechanism: clustering is the population. Assume  $m$  subjects from  $m/k$  clusters, each of size  $k$ .  $Z_i \sim \text{Binomial}(k, \pi_i)$  and  $Y = Z_1 + \dots + Z_{m/k}$ . If  $\mathbb{E}\pi_i = \pi$  and  $\text{Var}(\pi_i) = \tau^2 \pi(1-\pi)$ , then  $\mathbb{E}Y = \mathbb{E}(\mathbb{E}(Y|\pi)) = \mathbb{E}[k(\pi_1 + \dots + \pi_{m/k})] = m\pi$ ,  $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|\pi)] + \text{Var}[\mathbb{E}(Y|\pi)] = m\pi(1-\pi)(1-\tau^2) + m\tau^2 \pi(1-\pi) = m\pi(1-\pi)[1 + (k-1)\tau^2]$ . Since  $0 \leq \tau^2 \leq 1$  ( $\text{Var}(\pi_i) = \mathbb{E}\pi_i^2 - \pi^2 \leq \mathbb{E}\pi_i - \pi^2 = \pi(1-\pi)$ ),  $1 \leq \sigma^2 := 1 + (k-1)\tau^2 \leq k \leq m$ .
- Estimation of  $\sigma^2$  with overdispersion: Case 1 (with replication): For the same  $x$ -value, observe  $(y_1, m_1), \dots, (y_r, m_r)$ ,  $\hat{\pi} = \frac{\sum_{i=1}^r y_i}{\sum_{i=1}^r m_i}$ ,  $\mathbb{E}[\sum_{j=1}^r \frac{(y_j - m_j \hat{\pi})^2}{m_j}] = (r-1)\sigma^2 \pi(1-\pi)$ ,  $\hat{\sigma}^2 = \frac{1}{r-1} \sum_{j=1}^r \frac{(y_j - m_j \hat{\pi})^2}{m_j \hat{\pi}(1-\hat{\pi})}$  approximately unbiased for  $\sigma^2$ . Case 2 (without replication): Using the fitted  $\hat{\pi}_i$ ,  $\hat{\sigma}^2 = \frac{1}{n-p} \sum_{i=1}^n \frac{(y_i - m_i \hat{\pi}_i)^2}{m_i \hat{\pi}_i(1-\hat{\pi}_i)} \sim \chi_{n-p}^2$ ,  $\text{Var}(\hat{\beta}) \approx \hat{\sigma}^2 (X^T W X)^{-1}$ .

## Poisson Regression

- $Y \sim$  counts of events that occur over a period of time or a region at a constant rate. log-link:  $\log \mu_i = \eta_i = x_i^T \beta$ . Nominal variance  $\text{Var}(y_i) = \mu_i$ . More generally, let  $\text{Var}(y_i) = \sigma^2 \mu_i$ . Over/Under dispersion:  $\sigma^2 > / < 1$ .
- Mechanism for overdispersion: clustered Poisson process.  $Y = Z_1 + \dots + Z_N$ ,  $Z_i$  i.i.d.,  $N \sim \text{Poisson}$  independent of  $Z_i$ .  $\mathbb{E}Y = \mathbb{E}N\mathbb{E}Z$ ,  $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|N)] + \text{Var}(\mathbb{E}(Y|N)) = \mathbb{E}N\text{Var}(Z) + \text{Var}(N)(\mathbb{E}Z)^2 = \mathbb{E}N\mathbb{E}Z^2 (> \mathbb{E}Y$  if  $\mathbb{E}Z^2 > \mathbb{E}Z$ ). Estimation of  $\sigma^2$ :  $\hat{\sigma}^2 = \frac{1}{n-p} \sum_{i=1}^n \frac{(y_i - \hat{\mu}_i)^2}{\hat{\mu}_i}$ .

## Gamma Regression

- Motivation:  $\text{Var}(Y) = \text{Const} - \text{linear}$ ;  $\text{Var}(Y) \propto \mathbb{E}Y - \text{Poisson}$ ;  $\text{Var}(Y) \propto (\mathbb{E}Y)^2 - \text{Gamma}$ .  $\sigma := \frac{\sqrt{\text{Var}(Y)}}{\mathbb{E}Y} = \text{const}$ : coefficient of variance.
- Mechanism: (1) Multiplicative error:  $Y = \mu(1 + \epsilon)$ ,  $\mathbb{E}\epsilon = 0$ ,  $\text{Var}(\epsilon) = \sigma^2$ ,  $\mathbb{E}Y = \mu$ ,  $\text{Var}(Y) = \mu^2 \sigma^2$ . (2) log-transformed additive:  $\log Y = \mu + \epsilon$ ,  $\mathbb{E}\epsilon = 0$ ,  $\text{Var}(\epsilon) = \sigma^2$ ,  $Y = e^\mu e^\epsilon$ ,  $\mathbb{E}Y = e^\mu \mathbb{E}(e^\epsilon)$ ,  $\text{Var}(Y) = e^{2\mu} \text{Var}(e^\epsilon)$ ,  $\frac{\text{Var}(Y)}{(\mathbb{E}Y)^2} = \frac{\text{Var}(e^\epsilon)}{(\mathbb{E}e^\epsilon)^2} \approx \frac{\text{Var}(1+\epsilon)}{(1+\mathbb{E}\epsilon)^2} = \text{Var}(\epsilon) = \sigma^2$ .
- Parameterization of gamma: Gamma( $k, \theta$ ) pdf  $\frac{1}{\Gamma(k)\theta^k} y^{k-1} e^{-y/\theta} dy I(y > 0)$ ,  $\mathbb{E}Y = k\theta$ ,  $\text{Var}(Y) = k\theta^2$ ,  $\sigma^2 = \frac{1}{k}$ ; Gamma( $\mu, \nu$ ) pdf  $\frac{1}{\Gamma(\nu)} (\frac{\nu y}{\mu})^\nu e^{-\nu y/\mu} d(\log y) I(y > 0)$ .
- Choice of link function: (1) Canonical link  $g(\mu) = \frac{1}{\mu}$ . Example: Plants density experiments:  $x$  density, yield per plant  $\propto \frac{1}{\beta_0 x + \beta_1}$ , yield per unit area  $\propto \frac{x}{\beta_0 x + \beta_1}$ ,  $\mu = \eta^{-1} = \frac{x}{\beta_0 + \beta_1}$  or  $\eta = \beta_0 + \frac{\beta_1}{x}$ . (2) log link:  $\eta = \log \mu = x^T \beta$ . (3) identity link:  $\eta = \mu = x^T \beta$ .
- Estimation of  $\sigma^2$ :  $\nu = \frac{1}{\sigma^2}$ ,  $D(y, \hat{\mu}) = 2n[\log \hat{\nu} - \frac{\Gamma'(\hat{\nu})}{\Gamma(\hat{\nu})}]$ .
- Example (Rainfall data): Daily rainfall skewed to the right with a spike around 0. Two stages: (1) wet/dry day: Markov chain and logistic; (2) rainfall on wet days: gamma/log-normal. Stage 1:  $\pi_0(t) = P(\text{day } t \text{ is wet} | \text{day } t-1 \text{ is wet})$ ,  $\pi_1(t) = P(\text{day } t \text{ is wet} | \text{day } t-1 \text{ is dry})$ . logistic model:  $\text{logit}(\pi_i(t)) = \alpha_i + \alpha_{i,1} \sin(\frac{2\pi t}{365}) + \beta_{i,1} \cos(\frac{2\pi t}{365})$ . Stage 2:  $\log(\mu(t)) = \text{const} + \text{harmonic terms}$  where  $\mu(t)$  is mean rainfall on day  $t$  | wet day.

## Categorical Data and Multinomial Regression

- Types of measurement scales:  $\begin{cases} \text{nominal: exchangeable} \\ \text{ordinal: ordered but no measure of distance} \\ \text{interval: numerical scores} \\ \text{cardinal: counts} \end{cases}$
- Ordinal: Response probabilities:  $\pi_1, \dots, \pi_k$ ; cumulative probabilities:  $\gamma_j = \sum_{i=1}^j \pi_i, j = 1, 2, \dots, k-1, \gamma_k \equiv 1$ . Principle: Inferences should not essentially change by combining adjacent categories. Proportional odds model:  $\log(\frac{\gamma_j(x)}{1-\gamma_j(x)}) = \theta_j - \beta^T x, j = 1, 2, \dots, k-1$  (parallel regressions),  $\frac{\text{odds}(Y \leq j | x_1)}{\text{odds}(Y \leq j | x_2)} = e^{-\beta^T(x_1 - x_2)}$ . Why “-”? Latent variable interpretation:  $Z \sim \text{logistic}(\beta^T x, 1), Z - \beta^T x \sim \text{logistic}(0, 1)$ . Let  $Y = j$  whenever  $\theta_{j-1} < Z \leq \theta_j$ . Then  $P(Y \leq j) = P(Z \leq \theta_j) = P(Z - \beta^T x \leq \theta_j - \beta^T x) = \frac{e^{\theta_j - \beta^T x}}{1 + e^{\theta_j - \beta^T x}}$ . Extensions: (1) Nonparallel:  $\text{logit}(\gamma_j(x)) = \theta_j - \beta_j^T x$ ; (2) Scale modeling:  $\text{logit}(\gamma_j(x)) = \frac{\theta_j - \beta^T x}{e^{\tau^T x}}$ .
- Interval: Features: (1) The categories are of interest in themselves and are not chosen arbitrarily; (2) It does not normally make sense to form a new category by amalgamating adjacent categories; (3) difference between scores  $s_j$  is a measure of distance. Modeling strategies: (1) Extend proportional odds model:  $\theta_j = \xi_0 + \xi_1(\frac{s_j + s_{j+1}}{2})$  and replace  $\beta^T x$  by  $\beta^T x + \xi^T x(c_j - \bar{c}), c_j = \frac{s_j + s_{j+1}}{2}$  or  $\text{logit}(\frac{s_j + s_{j+1}}{2})$ ; (2) Model log-probabilities:  $\eta_{ij} = \log \pi_j(x_i), \pi_j(x_i) = \frac{e^{\eta_{ij}}}{\sum_j e^{\eta_{ij}}}, \eta_j(x_i) = \eta_j + \alpha_i$  or  $\eta_j(x_i) = \eta_j + (\beta^T x_i)s_j + \alpha_i, \frac{\pi_j}{\pi_{j'}} \nearrow$  by a factor  $e^{s_j - s_{j'}}$  with a unit change in  $\beta^T x$ ; (3) Model the scores  $\mathbb{E}(S | x_i) = \sum_{j=1}^k \pi_j(x_i)s_j = \beta^T x_i$ .
- Nominal: model  $\pi_j$  or  $\eta_j$ : e.g.  $\eta_j(x_i) = \eta_j(x_0) + \beta_j^T(x_i - x_0) + \alpha_i$ .
- Multinomial: Data  $(y_1, \dots, y_n), y_i = (y_{i1}, \dots, y_{ik}), \sum_j y_{ij} = m_i$  fixed, parameters  $\pi_i, i = 1, \dots, n, \pi_i = (\pi_{i1}, \dots, \pi_{ik})$  s.t.  $\sum_j \pi_{ij} = 1$ . Log-likelihood:  $l(\pi, y) = \sum_{i,j} y_{ij} \log \pi_{ij}$ . Use the method of Lagrange multipliers  $\Rightarrow \mathcal{L}_\lambda(\pi, y) = \sum_{i,j} y_{ij} \log \pi_{ij} - \sum_i \lambda_i (\sum_j \pi_{ij} - 1), \frac{\partial \mathcal{L}}{\partial \pi_{ij}} = \frac{y_{ij}}{\pi_{ij}} - \lambda_i = 0 \Rightarrow m_i = \lambda_i \Rightarrow \pi_{ij} = \frac{y_{ij}}{m_i}$ . Let  $\Sigma_i = m \{\text{diag}(\pi_i) - \pi_i \pi_i^T\}$  (rank  $k-1$ ) and  $\Sigma_i^- = \text{diag}(\frac{1}{m \pi_{ij}})$ . In matrix form,  $\frac{\partial \mathcal{L}}{\partial \pi} = M \Sigma^-(y - \mu) = 0$  where  $\Sigma = \text{diag}(\Sigma_1, \dots, \Sigma_n)$  of rank  $n(k-1)$  and  $M = \text{diag}(\underbrace{m_1, \dots, m_1}_{k \text{ times}}, \dots, \underbrace{m_n, \dots, m_n}_{k \text{ times}})$ . Full score equation w.r.t.  $\beta_r$ :  $\frac{\partial l}{\partial \pi_{ij}} \frac{\partial \pi_{ij}}{\partial \beta_r} = 0$ . GLMs: (1) log-linear:  $\log \pi_{ij} = x_{ij}^* \beta^*$ ; (2) prop odds:  $\text{logit}(\gamma_{ij}) = x_{ij}^{*T} \beta^*$  ( $\gamma_{ij}$ : cumulative probability). Overdispersion:  $\mathbb{E}Y = m\pi, \text{Var}(Y) = \sigma^2 \Sigma, \hat{\sigma}^2 = \frac{X^2}{\text{residual d.f.} = n(k-1) - p}$  where  $X^2$  is Pearson's statistic.

## Quasi-Likelihood Estimation

- For a GLM, the inference depends on the assumed distribution for  $y_i$  only through the mean  $\mu_i$  and the variance function  $V(\cdot)$ . In addition, a GLM specifies the independence of observations, which is not indispensable.
- More generally, suppose  $\text{Var}(y) = \sigma^2 V(\mu)$ . Independent:  $V(\mu) = \text{diag}\{V_1(\mu), \dots, V_n(\mu)\} = \{V_1(\mu_1), \dots, V_n(\mu_n)\}$ . Dependent:  $V(\mu)$  nondiagonal. Quasi-score function:  $u(\beta) = D^T V^{-1}(y - \mu) / \sigma^2$  where  $D = \frac{\partial \mu}{\partial \beta^T} = (\frac{\partial \mu_i}{\partial \beta_r})_{n \times p}$ . Facts:  $\mathbb{E}u(\beta) = 0, \text{Var}(\beta) = D^T V^{-1} D / \sigma^2, \frac{\partial}{\partial \beta^T} u(\beta) = \frac{1}{\sigma^2} [\frac{\partial}{\partial \beta^T} (D^T V^{-1})(y - \mu) - D^T V^{-1} D], \mathbb{E}[\frac{\partial}{\partial \beta^T} u(\beta)] = -D^T V^{-1} D / \sigma^2$ . Quasi-information matrix:  $I(\beta) = \text{Var}(u(\beta)) = -\mathbb{E} \frac{\partial}{\partial \beta^T} u(\beta) = D^T V^{-1} D / \sigma^2$ .
- Independent:  $\frac{\partial u(\beta)}{\partial \beta^T}$  diagonal. Dependent:  $\frac{\partial u_r(\beta)}{\partial \beta_s} \neq \frac{\partial u_s(\beta)}{\partial \beta_r}$  in general. Thm: symmetric  $\frac{\partial u(\beta)}{\partial \beta^T} \Leftrightarrow$  the line integral (quasi-log-likelihood)  $Q(\mu; y, \gamma(s)) = \frac{1}{\sigma^2} \int_{s_0}^{s_1} (y - \gamma)^T V(\gamma)^{-1} d\gamma(s)$  along a path  $\gamma: [s_0, s_1] \rightarrow \mathbb{R}^n$  from  $\gamma(s_0) = y$  to  $\gamma(s_1) = \mu$  is path-independent. Independent:  $Q(\mu, y) = \sum_{i=1}^n Q_i(\mu_i, y_i) = \sum_{i=1}^n \int_{y_i}^{\mu_i} \frac{y_i - t}{\sigma^2 V_i(t)} dt$ , quasi-deviance:  $D(y, \mu) = 2[Q(y, y) - Q(\mu, y)] \sigma^2 = 2 \sum_{i=1}^n \int_{\mu_i}^{y_i} \frac{y_i - t}{V_i(t)} dt$ . Dependent: Solve  $\sigma^2 u(\beta) = D^T V^{-1}(y - \mu) = 0$  (GEE).
- Fisher-scoring:  $\beta^{(t+1)} = \beta^{(t)} + I(\beta^{(t)})^{-1} u(\beta^{(t)}) = \beta^{(t)} + [D(\beta^{(t)}) V(\beta^{(t)})^{-1} D(\beta^{(t)})]^{-1} D(\beta^{(t)})^T V(\beta^{(t)})^{-1} (y - \mu(\beta^{(t)})) = (X^T W^{(t)} X)^{-1} X^T W^{(t)} Z^{(t)}$  where  $Z^{(t)} = X \beta^{(t)} + \frac{d\eta}{d\mu}|_{\mu(\beta^{(t)})} (y - \mu(\beta^{(t)}))$  and  $W^{(t)} = (\frac{d\eta}{d\mu}|_{\mu(\beta^{(t)})})^{-1} V(\beta^{(t)})^{-1} (\frac{d\eta}{d\mu}|_{\mu(\beta^{(t)})})^{-1}$ .

- Asymptotics: Under regularity conditions,  $\hat{\beta}$  is consistent and asymptotic normal with variance  $\text{Var}(\hat{\beta}) = I(\beta)^{-1} = \sigma^2(D^T V^{-1} D)^{-1}$ .
- Optimality: Estimating function  $G(\beta; y)$  if  $\mathbb{E}G(\beta; y) = 0, \forall \beta$ . Linear estimating function  $h(\beta) = H^T(y - \mu(\beta))$ . Let  $\tilde{\beta}$  solve  $h(\beta) = 0$ . Taylor expansion  $\Rightarrow 0 = h(\tilde{\beta}) = h(\beta) + [\frac{\partial H^T}{\partial \beta}(y - \mu(\beta)) - H^T D](\tilde{\beta} - \beta)$ . Take expectations  $\Rightarrow 0 = h(\beta) - H^T D(\tilde{\beta} - \beta) \Rightarrow \tilde{\beta} \approx \beta + (H^T D)^{-1} h(\beta)$ . Thus,  $\mathbb{E}\tilde{\beta} \approx \beta$ ,  $\text{Var}(\tilde{\beta}) \approx (H^T D)^{-1} H^T V H (D^T H)^{-1}$ . Since  $\text{Var}(\hat{\beta}) = \sigma^2(D^T V^{-1} D)^{-1}$ ,  $\text{Var}(\hat{\beta})^{-1} - \text{Var}(\tilde{\beta})^{-1} \approx \sigma^{-2} D^T V^{-1} D - \sigma^{-2} D^T H (H^T V H)^{-1} H^T D = \sigma^{-2} D^T V^{-1/2} (I - P_{V^{1/2} H}) V^{-1/2} D \succeq 0 \Rightarrow \text{Var}(\hat{\beta})$  is minimal.

## Longitudinal Data & GLMMs

- Longitudinal study: individuals (subjects) are measured repeatedly over time.
- Questions: (1) average time course of response change; (2) degree of heterogeneity across individuals; (3) factors that predict response change.
- Two perspectives: (1) Marginal/population-averaged: linear, normal,  $\mathbb{E}Y_{ij} = \beta_0 + \beta_1 t_{ij}$ ; (2) Subject-specific:  $Y_{ij} = \beta_{0i} + \beta_{1i} t_{ij} + \epsilon_{ij}, \beta_i = \beta + b_i, \mathbb{E}b_i = 0 \Rightarrow \mathbb{E}(Y_{ij}|b_i) = \beta_0 + b_{0i} + (\beta_1 + b_{1i}) t_{ij}, \mathbb{E}Y_{ij} = \beta_0 + \beta_1 t_{ij}$ .
- GLMMs:  $Y_{ij}|b_i = f_{y_{ij}|b_i}(y_{ij}|b_i) = \exp\{\frac{y_{ij}\theta_{ij} - \psi(\theta_{ij})}{\phi} + c(y_{ij}, \phi)\}$  i.i.d.,  $\mathbb{E}(Y_{ij}|b_i) = \mu_i = \psi'(\theta_{ij}), \text{Var}(Y_{ij}|b_i) = \phi\psi''(\theta_{ij}), g(\mu_{ij}) = \eta_{ij} = x_{ij}^T \beta + z_{ij}^T b_i$ . Typically  $b_i \sim \mathcal{N}(0, D)$ .
- Mean of  $Y_{ij}$ :  $\mathbb{E}Y_{ij} = \mathbb{E}[\mathbb{E}(Y_{ij}|b_i)] = \mathbb{E}\mu_{ij} = \mathbb{E}[g^{-1}(x_{ij}^T \beta + z_{ij}^T b_i)]$ . Example:  $g(\mu) = \log \mu, h(\mu) = g^{-1}(\mu) = e^\mu, z_{ij} \equiv 1, b_i \sim \mathcal{N}(0, \sigma_b^2)$ . Then  $\mathbb{E}Y_{ij} = e^{x_{ij}^T \beta} e^{\sigma_b^2/2}$ .
- Variance of  $Y_{ij}$ :  $\text{Var}(Y_{ij}) = \text{Var}[\mathbb{E}(Y_{ij}|b_i)] + \mathbb{E}[\text{Var}(Y_{ij}|b_i)] = \text{Var}(\mu_{ij}) + \mathbb{E}[\phi V(\mu_{ij})] = \text{Var}[g^{-1}(x_{ij}^T \beta + z_{ij}^T b_i)] + \mathbb{E}[\phi V(g^{-1}(x_{ij}^T \beta + z_{ij}^T b_i))]$ . Example (cont'd):  $Y_{ij}|b_i \sim \text{Poisson}(\mu_{ij})$ , so that  $\mathbb{E}(Y_{ij}|b_i) = \text{Var}(Y_{ij}|b_i) = \mu_{ij}$ . Then  $\text{Var}(Y_{ij}) = \text{Var}(\mu_{ij}) + \mathbb{E}\mu_{ij}$ . Still assume  $b_i \sim \mathcal{N}(0, \sigma_b^2)$ . Then  $\text{Var}(Y_{ij}) = \text{Var}(e^{x_{ij}^T \beta + b_i}) + \mathbb{E}(e^{x_{ij}^T \beta + b_i}) = e^{2x_{ij}^T \beta} (e^{2\sigma_b^2} - e^{\sigma_b^2}) + e^{x_{ij}^T \beta} e^{\sigma_b^2/2} = e^{x_{ij}^T \beta + \sigma_b^2/2} [e^{x_{ij}^T \beta} e^{\sigma_b^2/2} (e^{\sigma_b^2} - 1) + 1] > \mathbb{E}Y_{ij}$  (overdispersed).
- Estimation of GLMMs: (1) Maximum likelihood: log-likelihood  $l(\beta, D) = \log \prod_{i=1}^N \int \varphi(b_i; 0, D) \prod_{j=1}^{n_i} f(y_{ij}|b_i) db_i$ , or more generally,  $\log \int \prod_{i,j} f_{y_{ij}|b_i}(y_{ij}|b_i) f_{b_i}(b_i) db_i$ . Let  $b_i = Qv_i$ , where  $v_i \sim \mathcal{N}_q(0, I_q)$  and  $D = QQ^T$  (Cholesky decomposition), then  $l(\beta, D) = \log \prod_i \int_{\mathbb{R}} \phi(v_{i1}) \cdots \int_{\mathbb{R}} \phi(v_{i1}) \prod_j f(y_{ij}|v_i) dv_i$ . Approximate 1-D integral by Gauss-Hermite quadrature:  $\int_{\mathbb{R}} e^{-u^2} f(u) du \approx \sum_{k=1}^K w_k f(u_k)$  where  $u_k$  are roots of Hermite polynomials  $H_k(u)$  and  $w_k = \frac{2^{k-1} k! \sqrt{\pi}}{k^2 (H_{k-1}(u_k))^2}$ . Back to our integral,  $\int_{\mathbb{R}} \phi(v_{i1}) \prod_j f(y_{ij}|v_i) dv_{i1} \approx \sum_k \frac{w_k}{\sqrt{\pi}} \prod_j f(y_{ij}|(\sqrt{2}u_k, v_{i2}, \dots, v_{iq}))$ . Adaptive GH: For simplicity,  $\int_{\mathbb{R}} \phi(v_i) \prod_j f(y_{ij}|v_i) dv_i \propto$  posterior density of  $v_i|y_{ij}$ . Approximate  $f(v_i|y_{ij})$  by  $\phi(v_i; \mu_i, \tau_i^2)$  and write  $\int_{\mathbb{R}} \varphi(v_i; \mu_i, \tau_i^2) [\frac{\phi(v_i) \prod_j f(y_{ij}|v_i)}{\varphi(v_i; \mu_i, \tau_i^2)}] dv_i \approx w_{ik} \prod_j f(y_{ij}|u_{ik})$ .  $u_{ik}$  &  $w_{ik}$  are subject-specific,  $u_{ik} = \mu_i + \tau_i u_k, w_{ik} = \sqrt{2\pi} \tau_i e^{u_k^2/2} \phi(\mu_i + \tau_i u_k) w_k$ . (2) Quasi-likelihood: Similar to IRLS, use current estimates  $(\beta^k, D^k, V^k, b_i^k)$  to linearize the model for  $y_{ij}$ :  $y_{ij} \approx h(\eta_{ij}^k) + x_{ij}^T (\beta - \beta^k) h'(\eta_{ij}^k) + z_{ij}^T (b_i - b_i^k) h'(\eta_{ij}^k) + \epsilon_{ij}$  where  $\text{Var}(\epsilon_{ij}) = \phi V(\mu_{ij}^k)$ . Let the offset  $o_{ij} = h(\eta_{ij}^k) - h'(\eta_{ij}^k) x_{ij}^T \beta^k - h'(\eta_{ij}^k) z_{ij}^T b_i^k$ , total residual  $\xi_{ij} = h'(\eta_{ij}^k) z_{ij}^T b_i + \epsilon_{ij} \Rightarrow y_{ij} + o_{ij} + h'(\eta_{ij}^k) x_{ij}^T \beta + \xi_{ij}$  (just an OLS).