# Advanced Theory of Probability

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# 1 Measure Theory

- Fatou's lemma: If $f_n \geq 0$ then $\liminf_{n\to\infty} \int f_n d\mu \geq \int \liminf_{n\to\infty} f_n d\mu$.
- Monotone convergence theorem: If $f_n \geq 0$ and $f_n \uparrow f$ then $\int f_n d\mu \uparrow \int f d\mu$.
- Dominated convergence theorem: If $f_n \to f$ a.e., $|f_n| \leq g$ for all $n$, and $g$ is integrable, then $\int f_n d\mu \to \int f d\mu$.
- Suppose $X_n \to X$ a.s. Let $g(x), h(x)$ be continuous functions with (i) $g(x) \geq 0$ and $g(x) \to \infty$ as $|x| \to \infty$; (ii) $|h(x)|/g(x) \to 0$ as $|x| \to \infty$; (iii) $\mathbb{E}g(X_n) \leq K < \infty$ for all $n$. Then $\mathbb{E}h(X_n) \to \mathbb{E}h(X)$.
- Fubini's theorem: If $f \geq 0$ or $\int |f| d\mu < \infty$, $\int_X \int_Y f(x,y)\mu_2(dy)\mu_1(dx) = \int_{X\times Y} f d\mu = \int_Y \int_X f(x,y)\mu_1(dx)\mu_2(dy)$.

# 2 Laws of Large Numbers

## 2.1 Independence

- Two events $A$ and $B$ are independent if $P(A\cap B) = P(A)P(B)$. Two random variables $X$ and $Y$ are independent if for all $C, D \in \mathbb{R}$, $P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$. Two $\sigma$-fields $\mathcal{F}$ and $\mathcal{G}$ are independent if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$ the events $A$ and $B$ are independent.
- $\sigma$-fields $\mathcal{F}_1, \cdots, \mathcal{F}_n$ are independent if whenever $A_i \in \mathcal{F}_i$ for $i = 1, \cdots, n$, we have $P(\cap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$. Random variables $X_1, \cdots, X_n$ are independent if whenever $B_i \in \mathbb{R}$ for $i = 1, \cdots, n$ we have $P(\cap_{i=1}^n \{X_i \in B_i\}) = \prod_{i=1}^n P(X_i \in B_i)$. Sets $A_1, \cdots, A_n$ are independent if whenever $I \subset \{1, \cdots, n\}$ we have $P(\cap_{i\in I} A_i) = \prod_{i\in I} P(A_i)$.
- A sequence of events $A_1, \cdots, A_n$ with $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$ is called pairwise independent.
- $\pi$-$\lambda$ theorem: If $\mathcal{P}$ is a $\pi$-system and $\mathcal{L}$ is a $\lambda$-system that contains $\mathcal{P}$ then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
- Suppose $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are independent and each $\mathcal{A}_i$ is a $\pi$-system. Then $\sigma(\mathcal{A}_1), \cdots, \sigma(\mathcal{A}_n)$ are independent.
- Suppose $\mathcal{F}_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m(i)$ are independent and let $\mathcal{G}_i = \sigma(\cup_j \mathcal{F}_{i,j})$. Then $\mathcal{G}_1, \cdots, \mathcal{G}_n$ are independent.
- If for $1 \leq i \leq n, 1 \leq j \leq m(i)$, $X_{i,j}$ are independent and $f_i : \mathbb{R}^{m(i)} \to \mathbb{R}$ are measurable then $f_i(X_{i,1}, \cdots, X_{i,m(i)})$ are independent.
- If $X_1, \cdots, X_n$ are independent and have (a) $X_i \geq 0$ for all $i$, or (b) $\mathbb{E}|X_i| < \infty$ for all $i$ then $\mathbb{E}\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n \mathbb{E}X_i$.
- If $X$ and $Y$ are independent, $F(x) = P(X \leq x)$, and $G(y) = P(Y \leq y)$, then $P(X + Y \geq z) = \int F(z-y)dG(y)$.

## 2.2 Weak Laws of Large Numbers

- $L^2$ weak law: Let $X_1, X_2, \cdots$ be uncorrelated random variables with $\mathbb{E}X_i = \mu$ and $\mathrm{var}(X_i) \leq C < \infty$. If $S_n = X_1 + \cdots + X_n$, then as $n \to \infty$, $S_n/n \to \mu$ in $L^2$ and in probability.
- Let $\mu_n = \mathbb{E}[S_n], \sigma_n^2 = \mathrm{var}(S_n)$. If $\sigma_n^2/b_n^2 \to 0$ then $\frac{S_n - \mu_n}{b_n} \to 0$ in probability.
- Truncation: To truncate a random variable $X$ at level $M$ means to consider $\bar{X}_M = X1_{\{|X|\leq M\}}$.
- For each $n$, let $X_{n,k}, 1 \leq k \leq n$ be independent. Let $0 < b_n \to \infty$ and $\bar{X}_{n,k} = X_{n,k}1_{\{|X_{n,k}|\leq b_n\}}$. Suppose that as $n \to \infty$ (1) $\sum_{k=1}^n P(|X_{n,k}| > b_n) \to 0$; (2) $b_n^{-2}\sum_{k=1}^n \mathrm{var}(\bar{X}_{n,k}) \to 0$. If we let $S_n = \sum_{k=1}^n X_{n,k}$ aand $a_n = \sum_{k=1}^n \mathbb{E}[\bar{X}_{n,k}]$, then $\frac{S_n - a_n}{b_n} \to 0$ in probability.

- Let $X_1, X_2, \cdots$ be i.i.d. with $xP(|X_1| > x) \to 0$ as $x \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and let $\mu_n = \mathbb{E}[X_1 1_{\{|X_1| \le n\}}]$. Then $S_n/n - \mu_n \to 0$ in probability.
- If $Y \ge 0$ and $p > 0$ then $\mathbb{E}[Y^p] = \int_0^\infty p y^{p-1} P(Y > y) dy$.
- Let $\{X_i\}_{i=1}^\infty$ be i.i.d. with $\mathbb{E}[|X_i|] < \infty$. Let $S_n = X_1 + \cdots + X_n$ and let $\mu = \mathbb{E}[X_1]$. Then $S_n/n \to \mu$ in probability.
- The distribution of $X$ is infinitely divisible iff for any $n \in \mathbb{N}$, there exists i.i.d. $Y_i$'s such that $X = \sum_{i=1}^n Y_i$.
- The distribution of $X$ is stable if for all $a, b > 0$, and $X_1, X_2$ i.i.d. copies of $X$, $aX_1 + bX_2 \overset{d}{=} cX + d$ for some $c > 0$.

## 2.3 Borel-Cantelli Lemmas

- If $A_n$ is a sequence of subsets of $\Omega$, then we write

$$\limsup A_n = \cap_{n=1}^\infty \cup_{m=n}^\infty A_m = \{\omega : \omega \text{ in infinitely many } A_i\text{'s}\}$$
$$\liminf A_n = \cup_{n=1}^\infty \cap_{m=n}^\infty A_m = \{\omega : \omega \text{ in all but finitely many } A_i\text{'s}\}$$

- $P(\limsup A_n) \ge \limsup P(A_n)$, $P(\liminf A_n) \le \liminf P(A_n)$.
- Borel-Cantelli lemma: If $\sum_i P(A_i) < \infty$, then $P(A_n \text{ i.o.}) = 0$.
- Let $y_n$ be a sequence of elements of a topological space. If every subsequence $y_{n(m)}$ has a further subsubsequence $y_{n(m_k)}$ that converges to $y$, then $y_n \to y$.
- $X_n \to X$ in probability iff for every subsequence $X_{n(m)}$ there is a further subsubsequence $X_{n(m_k)}$ that converges a.s. to $X$.
- If $f$ is continuous and $X_n \to X$ in probability then $f(X_n) \to f(X)$ in probability. If in addition $f$ is bounded then $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$.
- Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}[X_i] = \mu$ and $\mathbb{E}[X_i^4] < \infty$. Then $S_n/n \to \mu$ a.s.
- For events $A_n, n = 1, 2, \cdots$, independent such that $\sum_{n=1}^\infty P(A_n) = \infty$, then $P(A_n \text{ i.o.}) = 1$.
- If $X_1, X_2, \cdots$ are i.i.d. r.v.'s with $\mathbb{E}[X_i] = \infty$, then $P(|X_n| \ge n \text{ i.o.}) = 1$. Let $C = \{\lim S_n/n \text{ exists \& is finite}\}$. Then $P(C) = 0$.
- If $A_1, A_2, \cdots$ are pairwise independent and $\sum_{n=1}^\infty P(A_n) = \infty$ then $\sum_{i=1}^n 1_{A_i} / \sum_{i=1}^n P(A_i) \to 1$ a.s. as $n \to \infty$.
- For a sequence of increasing events $A_n$, $P(A_n \text{ i.o.}) = 1$ iff $\sum_n P(A_n | A_{n-1}^c) = \infty$.

## 2.4 Strong Law of Large Numbers

- Strong law of large numbers: Let $X_1, X_2, \cdots$ be pairwise independent identically distributed random variables with $\mathbb{E}[X_i] < \infty$. Let $\mathbb{E}X_i = \mu$ and $S_n = X_1 + \cdots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$.
- Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}[X^+] = \infty$ and $\mathbb{E}[X^-] < \infty$, then $S_n/n \to \infty$ a.s.
- Let $X_1, X_2, \cdots$ be i.i.d. with $0 < X_i < \infty$, write $T_n = X_1 + \cdots + X_n$ and let $N_t = \sup\{n : T_n \le t\}$. If $\mathbb{E}[X_1] = \mu \le \infty$, then as $t \to \infty$, $N_t/t \to 1/\mu$, a.s.

- If $X_n \to X_\infty$ a.s. and $N(n) \to \infty$ a.s. then $X_{N(n)} \to X_\infty$ a.s. But the analogous result for convergence in probability is false!
- Empirical distribution functions: Let $X_1, X_2, \cdots$ be i.i.d. with distribution $F$ and let $F_n(x) = \frac{\sum_{i=1}^n 1_{X_i \le x}}{n}$. As $n \to \infty$, $\sup_x |F_n(x) - F(x)| \to 0$ a.s.
- Uniform law of large numbers: Suppose $f(x, \theta)$ is continuous in $\theta \in \Theta$ for some compact $\Theta$. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables. If $f$ is continuous at $\theta$ for a.s. all $x \in \mathbb{R}$ aand measurable of $x$ at each $\theta$ and there exists some function $d(x)$ such that $\mathbb{E}[d(X_i)] < \infty$ and for all $\theta \in \Theta$, $|f(x, \theta)| \le d(x)$. Then $\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n f(X_i, \theta)/n - \mathbb{E}[f(X_1, \theta)]| \stackrel{\text{a.s.}}{\to} 0$.

## 2.5 Convergence of Random Series

- Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of random variables. Define $\mathcal{F}'_n = \sigma(X_n, X_{n+1}, \cdots)$ as the information of the future after time $n$. Let $\mathcal{I} = \cap_{n=1}^\infty \mathcal{F}'_n$ be the tail $\sigma$-field, i.e., the information inthe remote future. Intuitively, $A \in \mathcal{T}$ if and only if changing a finite number of values does not affect the occurrence of the event.
- Kolmogorov's 0-1 law: If $X_1, X_2, \cdots, X_n, \cdots$ are independent and $A \in \mathcal{I}$, then $P(A) = 0$ or 1.
- A finite permutation of $\mathbb{N}$ is a map from $\mathbb{N}$ onto $\mathbb{N}$ such that there is a finite $I$ with $\pi(i) = i$ for all $i \ge I$. For $S^{\mathbb{N}}$, associated with its natural product sigma field $\mathcal{F}^N$, and any $\omega = (\omega_1, \omega_2, \cdots)$, let $\pi(\omega) = (\omega_{\pi(1)}, \omega_{\pi(2)}, \cdots)$. An event $A \in \mathcal{F}^{\mathbb{N}}$ is permutable if $\pi^{-1}(A) = A$ for any finite permutation $\pi$. All permutable events form the exchangeable $\sigma$-field, denoted by $\mathcal{E}$. All events in the tail $\sigma$-field $\mathcal{T}$ are permutable.
- Hewitt-Savage 0-1 law: If $X_1, X_2, \cdots,$ are i.i.d. and $B \in \mathcal{E}(\mathbb{R}^N)$. Denote $X = (X_1, X_2, \cdots)$. Then $P(X \in B) = 0$ or 1.
- Kolmogorov's maximal inequality: Suppose $X_1, X_2, \cdots, X_n$ are independent with $\mathbb{E}[X_i] = 0$, $\text{var}(X_i) < \infty$. Let $S_n = X_1 + \cdots + X_n$, then $P(\max_{k \le n} |S_k| \ge x) \le \frac{\text{var}(S_n)}{x^2}$.
- We call a sequence of r.v's $S_1, S_2, \cdots$ a martingale if (i) there is a sequence of $\sigma$-algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ and $S_i \in \mathcal{F}_i$ for all $i$; (ii) $S_i$'s are integrable; (iii) For each $k$, $\mathbb{E}[S_{k+1} | \mathcal{F}_k] = S_k$. If the "=" in (iii) is replaced by $\ge$ (resp. $\le$), then we say that this sequence is a submartingale (resp. supermartingale).
- Second-moment criterion: Suppose $X_1, X_2, \cdots$ are independent and centered (i.e., for all $i$, $\mathbb{E}[X_i] = 0$). If $\sum_{n=1}^\infty \text{var}(X_n) < \infty$, then $P(\sum_{n=1}^\infty X_n(\omega)$ converges$) = 1$.
- Kronecker's lemma: If $a_n \uparrow \infty$ and $\sum_{n=1}^\infty x_n / a_n$ converges, then $a_n^{-1} \sum_{m=1}^n x_m \to 0$.
- Let $X_1, X_2, \cdots$ be i.i.d. random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \sigma^2 < \infty$. Let $S_n = X_1 + \cdots + X_n$. If $\epsilon > 0$, then $S_n / n^{1/2} (\log n)^{1/2+\epsilon} \to 0$ a.s.
- Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[|X_i|^p] < \infty$ where $1 < p < 2$. Write $S_n = X_1 + \cdots + X_n$. Then $S_n / n^{1/p} \to 0$ a.s.
- Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}[X_1] = \infty$ and let $S_n = X_1 + \cdots + X_n$. Let $a_n$ be a sequence of positive numbers with $a_n / n$ increasing. Then $\limsup_{n \to \infty} |S_n| / a_n = 0$ or $\infty$ according as $\sum_n P(|X_1| \ge a_n) < \infty$ or $= \infty$.
- Kolmogorov's three-series theorem: Let $X_1, X_2, \cdots, X_n, \cdots$ be independent random variables. Let $A > 0$ and $Y_i = X_i 1_{|X_i| \le A}$. In order to show that $\sum X_i$ converges a.s., it is necessary and sufficient that (i) $\sum_{n=1}^\infty P(|X_n| > A) < \infty$; (ii) $\sum_{n=1}^\infty \mathbb{E}[Y_n]$ converges; (iii) $\sum_{n=1}^\infty \text{var}(Y_n) < \infty$.

## 2.6 Large Deviations

- Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}[X_1] = \mu$ and let $S_n = X_1 + X_2 + \cdots + X_n$. According to CLT, the typical value of $S_n - n\mu$ is $O(\sqrt{n})$. What about atypical deviations of $S_n - n\mu$? According to WLLN, we know that for any $a > \mu$, $P(S_n > na) \to 0$. We want to discuss the existence and value of the limit: $\lim_{n\to\infty} \frac{1}{n} \log P(S_n > na)$.

- Let $\pi_n = P(S_n \geq na)$. Then $\pi_{n+m} \geq P(S_n \geq na, S_{n+m} - S_n \geq ma) = \pi_n \pi_m$. Let $\gamma_n = \log \pi_n, \gamma_{n+m} \geq \gamma_n + \gamma_m$. As $n \to \infty$ the limit of $\gamma_n$ exists and $\lim_{n\to\infty} \frac{\gamma_n}{n} = \sup_n \frac{\gamma_n}{n}$. We define $\gamma(a) = \lim_{n\to\infty} \gamma_n/n \leq 0$. Then for any distribution and any $n$ and $a$, $P(S_n \geq na) \leq e^{n\gamma(a)}$. We want to show $\gamma(a) < 0$ if $a > \mu$.

- If the moment generating function $\psi(\theta) = \mathbb{E}[\exp(\theta X_1)] < \infty$ for some $\theta > 0$, then $P(S_n \geq na) \leq \exp[n(\log \psi(\theta) - \theta a)]$. Let $\kappa(\theta) = \log \psi(\theta)$. If $a > \mu$, then $a\theta - \kappa(\theta) > 0$ for all sufficiently small $\theta$.

- We will further strengthen our upper bounds by finding the maximum of $\lambda(\theta) = a\theta - \kappa(\theta)$. Let $\theta_+ = \sup\{\theta : \psi(\theta) < \infty\}$ and $\theta_- = \inf\{\theta : \psi(\theta) < \infty\}$. Now since that $\psi(\theta) \in C^\infty$ within $(\theta_-, \theta_+)$, we have $\lambda'(\theta) = a - \frac{\psi'(\theta)}{\psi(\theta)}$. So the maximal point of $\lambda$ must satisfy $\psi'(\theta)/\psi(\theta) = a$. For the existence and uniqueness of such point(s), we introduce a new distribution, and use a trick named "tilting".

- We now introduce the distribution $F_\theta$ by "reweighting $F$": $F_\theta(x) = \frac{1}{\psi(\theta)} \int_{-\infty}^x e^{y\theta} dF(y)$. By simple calculus, $\int x dF_\theta(x) = \frac{\psi'(\theta)}{\psi(\theta)}$, $\psi''(\theta) = \int x^2 e^{\theta x} dF(x)$, $\frac{d}{d\theta} \frac{\psi'(\theta)}{\psi(\theta)} = \int x^2 dF_\theta(x) - (\int x dF_\theta(x))^2 \geq 0$. If we assume the distribution $F$ is not a point mass at $\mu$, then $\frac{\psi'(\theta)}{\psi(\theta)}$ is strictly increasing and $a\theta - \log\psi(\theta)$ is concave. Since we have $\frac{\psi'(0)}{\psi(0)} = \mu$, this shows that for each $a > \mu$ there is at most one $\theta_a \geq 0$ that solves $a = \frac{\psi'(\theta_a)}{\psi(\theta_a)}$, and this value of $\theta$ maximizes $a\theta - \log\psi(\theta)$. Let $F^n$ be the c.d.f. of $S_n = X_1 + \cdots + X_n$ and $F_\lambda^n$ be the c.d.f. of $S_n^\lambda = X_1^\lambda + \cdots + X_n^\lambda$ where $X_i$ i.i.d. $\sim F$ and $X_i^\lambda$ i.i.d. $\sim F_\lambda = \frac{1}{\psi(\lambda)} \int_{-\infty}^x e^{y\theta} dF(y)$. By induction, $\frac{dF^n}{dF_\lambda^n} = e^{-\lambda x} \psi(\lambda)^n$. Then as $n \to \infty$, $n^{-1} \log P(S_n \geq na) \to -a\theta_a + \log\psi(\theta_a)$.

- Some important information: $\kappa(\theta) = \log\psi(\theta), \kappa'(\theta) = \frac{\psi'(\theta)}{\psi(\theta)}$, $\theta_a$ solves $\kappa'(\theta_a) = a$, $\gamma(a) = \lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq na) = -a\theta_a + \kappa(\theta_a)$.

- Suppose $x_o = \sup\{x : F(x) < 1\} = \infty, \theta_+ < \infty$, and $\psi'(\theta)/\psi(\theta)$ increases to a finite limit $a_0$ as $\theta \uparrow \theta_+$. If $a_0 \leq a < \infty$, $n^{-1} \log P(S_n \geq na) \to -a\theta_+ + \log\psi(\theta_+)$, i.e. $\gamma(a)$ is linear for $a \geq a_0$.

- Suppose $x_o = \sup\{x : F(x) < 1\} < \infty$ and $F$ has no mass at $x_o$. Then $\psi(\theta) < \infty$ for all $\theta > 0$ and $\psi'(\theta)/\psi(\theta) \to x_o$ as $\theta \to \infty$.

- Now, we have shown the decaying asymptotic for all possible situations:

$$
\begin{cases}
\text{If } x_o < \infty : \begin{cases} a < x_o : \text{exponential, rate} = \theta_a \\ a = x_o : \text{exponential if } P(X_1 = x_o) > 0, 0 \text{ otherwise} \\ a > x_o : 0 \end{cases} \\
\text{If } x_o = \infty : \begin{cases} \text{If } \theta_+ = \infty : \text{exponential, rate} = \theta_a \\ \text{If } \theta_+ < \infty : \begin{cases} \text{If } \psi'(\theta)/\psi(\theta) \to \infty \text{ as } \theta \to \theta_+ : \text{exponential, rate} = \theta_a \\ \text{If } \psi'(\theta)/\psi(\theta) \to a_0 \text{ as } \theta \to \theta_+ : \begin{cases} a < a_0 : \text{exponential, rate} = \theta_a \\ a \geq a_0 : \text{exponential, rate} = \theta_+ \end{cases} \end{cases} \end{cases}
\end{cases}
$$

- Cramér's theorem: Let $I(a)$ be the Legendre transform of $\log\psi(\cdot)$: $I(a) := \sup_{\theta\in\mathbb{R}}(\theta a - \log\psi(\theta))$. Then for any closed set $F$, $\limsup_{n\to\infty} n^{-1} \log P(\frac{S_n}{n} \in F) \leq -\inf_{x\in F} I(x)$; for any open set $G$, $\liminf_{n\to\infty} n^{-1} \log P(\frac{S_n}{n} \in G) \geq -\inf_{x\in G} I(x)$.

- Intuition behind the tilting: Why do we want to introduce the measure $F_\theta$? Intuitively, the new measure is like a "distorting mirror" – it "distorts" our view on how each event is likely to happen. So, when we want to estimate a rare event $A$ under $P$, suppose (1) we caan construct a new measure $Q$ such that $Q[A]$ is easily calculable, e.g., $Q[A] \approx 1$; (2) we have a uniform lower bound of the R-N derivative $dP/dQ \geq c$ on $A$. Then we can conclude that $P[A] = \int_A \frac{dP}{dQ} dQ \geq cQ[A]$.
- Let $\Sigma = \{a_1, \cdots\}$ stand for a finite-size alphabet. Let $M_1(\Sigma)$ be the space of all probability measures on $\Sigma$. The entropy of some $\nu \in M_1(\Sigma)$ is $H(\nu) := -\sum_{i=1}^{|\Sigma|} \nu(a_i)\log(\nu(a_i))$. The relative entropy of $\nu$ with respect to some other $\mu \in M_1(\Sigma)$ is $H(\nu|\mu) := \sum_{i=1}^{|\Sigma|} \nu(a_i) \log \frac{\nu(a_i)}{\mu(a_i)}$.
- Let $Y_i$ be i.i.d. r.v.'s, $\mu \in M_1(\Sigma)$. For $n \geq 1$, write $Y = (Y_1, \cdots, Y_n)$ and call $L_n^Y \in M_1(\Sigma)$ be the empirical frequency of $Y$. Let $T_n(\nu)$ be the set of $y$ a sequence of $n$ letters whose empirical measure is $\nu$.
- If $y \in T_n(\nu)$, then $P_\mu(Y = y) = e^{-n(H(\nu)+H(\nu|\mu))}$. In particular, if $y \in T_n(\mu)$, then $P_\mu(Y = y) = e^{-nH(\mu)}$.
- For every possible empirical measure $\nu$ of $n$ letters, $(n+1)^{-|\Sigma|}e^{nH(\nu)} \leq |T_n(\nu)| \leq e^{nH(\nu)}$.
- For every possible empirical measure $\nu$ of $n$ letters, $(n+1)^{-|\Sigma|}e^{nH(\nu|\mu)} \leq P_\mu(L_n^T = \nu) \leq e^{nH(\nu|\mu)}$.
- Sanov's theorem: For every set $\Gamma \subset M_1(\Sigma)$, $-\inf_{\nu\in\Gamma^\circ} H(\nu|\mu) \leq \liminf \frac{1}{n}\log P_\mu(L_n^Y \in \Gamma) \leq \limsup \frac{1}{n} \log P_\mu(L_n^Y \in \Gamma) \leq -\inf_{\nu\in\Gamma} H(\nu|\mu)$.

### 2.7 Percolation

- Fix $p \in [0, 1]$ and consider the $d$-dimensional lattice $\mathbb{Z}^d$. Assign to each edge $e \in \mathbb{E}$ an independent Bernoulli r.v. $I(e)$ with parameter $p$. If $I(e) = 1$, we say that this edge is open, otherwise closed. Consider the connected components of open egdes, then for any $p \in [0, 1]$, $P_p(A) = 0$ or 1 where $A = \{\exists \text{ infinite open clusters}\}$.
- If $A$ is translation-invariant, then $P(A) = 0$ or 1.
- Actually we can go further and show that for any $N = 0, 1, \cdots, \infty$, $P_p[A(N)] = 0$ or 1, where $A(N) = \{\exists N \text{ infinite open clusters}\}$. Or even further: for $N = 2, 3, \cdots$ and $N = \infty$, $P_p[A(N)] = 0$.
- Let $p_c = p_c(d) = \sup\{p : P_p(A) = 0\}$. Then one can show that $1/3 \leq p_c(2) \leq 2/3$. More generally, $p_c(1) = 1$ and for $d \geq 2$, $1/(2d-1) \leq p_c(d) \leq p_c(2)(= 1/2)$.
- By knowledge of Galton-Watson tree and the analogy between $\mathbb{Z}^d$ and $2d$-regular tree in high dimensions, we can take an educated guess that $p_c(d) \sim \frac{1}{2d}$ as $d \to \infty$.

# 3 Central Limit Theorems

### 3.1 The De Moivre-Laplace Theorem

- Central Limit Theorem: Let $X_1, X_2, \cdots$ be i.i.d. with mean $\mu$ and variance $\sigma^2 \in (0, \infty)$. Write $S_n = X_1+\cdots+X_n$, then $\frac{S_n-\mu n}{\sqrt{n}\sigma} \Rightarrow \mathcal{N}(0, 1)$.
- Before discussing the central limit theorem in full generality, we first see a special example for Bernoulli random variables. Let $X_1, X_2, \cdots$ be i.i.d. random variables such that $P(X_1 = 1) = P(X_1 = -1) = 1/2$ and write $S_n = X_1 + \cdots + X_n$. For integers $|k| \leq n$, $P(S_{2n} = 2k) = C_{2n}^{n+k}2^{-2n}$ since $(S_{2n} + 2n)/2 \sim \text{Binomial}(2n, 1/2)$.

- Local central limit theorem: If $2k/\sqrt{2n} \to x$, then $\lim_{n\to\infty}(\pi n)^{1/2}e^{x^2/2}P(S_{2n}=2k)=1$.
- The De Moivre-Laplace Theorem: For $a<b$, $P(a \le S_n/\sqrt{n} \le b) \to \int_a^b (2\pi)^{-1/2}e^{-x^2/2}dx$.

### 3.2 Weak Convergence

- A sequence of distribution function $F_n$ is said to converge weakly to a limit $F$, denoted by $F_n \Rightarrow F$, if $F_n(y) \to F(y)$ at every point of continuity of $F$, i.e. every $y \in \mathbb{R}$ such that $F(\cdot)$ is continuous at $y$.
- A sequence of random variables $X_n$ is said to converge weakly or converge in distribution / law to a limit $X_\infty$ if their distribution functions $F_n$ converges weakly.
- Skorokhod's representation theorem: If $F_n \Rightarrow F$ then there are random variables $Y_n, 1 \le n < \infty$ and $Y$ with living in the same probability space such that $Y_n \sim F_n, Y \sim F$ and $Y_n \to Y$ a.s.
- $X_n \Rightarrow X$ if and only if for every bounded continuous function $g$ we have $\mathbb{E}g(X_n) \to \mathbb{E}g(X)$.
- Continuous mapping theorem: Let $g$ be a measurable function and $D_g = \{x : g \text{ is discontinuous at } x\}$. If $X_n \Rightarrow X$, and $P(X \in D_g)=0$, then $g(X_n) \Rightarrow g(X)$.
- Portmantean theorem: The following statements are equivalent: (1) $X_n \Rightarrow X$; (2) $G$ open, $\liminf_{n\to\infty} P(X_n \in G) \ge P(X \in G)$; (3) $G$ closed, $\limsup_{n\to\infty} P(X_n \in G) \le P(X \in G)$; (4) If $P(X \in \partial A)=0$, then $\lim_{n\to\infty} P(X_n \in A) = P(X \in A)$.
- Helly's selection theorem: For every sequence $F_n$ of distribution functions, there is a subsequence $F_{n(k)}$ and a right continuous nondecreasing function $F$ so that at all points of continuity $y$ of $F$, $\lim_{k\to\infty} F_{n(k)}(y) = F(y)$.
- Every subsequential limit of the sequence $F_n$ is the distribution function of a probability measure iff the sequence is tight, i.e., for all $\epsilon > 0$, there is an $M_\epsilon$ so that $\limsup_{n\to\infty}[1-F_n(M_\epsilon)+F_n(-M_\epsilon)] \le \epsilon$.
- If there is a function $\phi \ge 0$ so that $\phi(x) \to \infty$ as $|x| \to \infty$ and $C = \sup_n \int \phi(x)dF_n(x) < \infty$, then $F_n$ is tight.

### 3.3 Characteristic Functions

- If $X$ is a r.v., we define its Characteristic function (ch.f.) by $\phi(t) := \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)]$.
- All characteristic functions have the following properties: (i) $\phi(0)=1$; (ii) $\phi(-t)=\overline{\phi(t)}$; (iii) $|\phi(t)| = |\mathbb{E}e^{itX}| \le \mathbb{E}|e^{itX}| = 1$; (iv) $|\phi(t+h)-\phi(t)| \le \mathbb{E}|e^{itX}-1|$, so $\phi(t)$ is uniformly continuous on $\mathbb{R}$; (v) $\mathbb{E}e^{it(aX+b)} = e^{itb}\phi(at)$.
- If $X_1$ and $X_2$ are independent and have ch.f.'s $\phi_1$ and $\phi_2$. Then $X_1+X_2$ has ch.f. $\phi_1 \cdot \phi_2$.
- Stein's Lemma: If $X, Y$ are jointly Gaussian, then for differentiable $g : \mathbb{R} \to \mathbb{R}$, as long as the expectations are well-defined, $\mathrm{cov}(g(X),Y) = \mathrm{cov}(X,Y)\mathbb{E}[g'(X)]$.
- If $F_1, \cdots, F_n$ have ch.f. $\phi_1, \cdots, \phi_n$ and $\lambda_i \ge 0, 1 \le i \le n$ have $\lambda_1 + \cdots + \lambda_n = 1$. Then $\sum \lambda_i F_i$ has ch.f. $\sum \lambda_i \phi_i$.
- The inversion formula: If $a<b$, then $\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi(t)dt = \mu(a,b) + \frac{1}{2}\mu(\{a,b\})$.
- If $\int |\phi(t)|dt < \infty$, then $\mu$ has bounded continuous density $f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\phi(t)dt$.
- Continuity theorem: Let $\mu_n, 1 \le n \le \infty$ be probability measures with ch.f. $\phi_n$. (i) If $\mu_n \Rightarrow \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all $t$. (ii) If $\phi_n(t) \to \phi(t)$ for all $t$, and $\phi(t)$ is continuous at 0. Then $\{\mu_n\}_{n=1}^\infty$ is tight and has a weak limit with ch.f. $\phi$.

- Let $\mu$ be a probability measure and $\phi$ be its ch.f. Then $\mu(\{x : |x| \geq 2u^{-1}\}) \leq u^{-1} \int_{-u}^{u} [1 - \phi(t)] dt$.
- If $\int |x|^n \mu(dx) < \infty$, then its ch.f. $\phi$ has a continuous derivative of order $n$ given by $\phi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx)$. In particular, $\phi^{(n)}(0) = \mathbb{E}[(iX)^n]$.
- However, if a characteristic function $\phi_X$ has a $k$-th derivative at zero, then the random variable $X$ has all moments up to $k$ if $k$ is even, but only up to $(k-1)$ if $k$ is odd.
- $|e^{ix} - \sum_{m=0}^{n} \frac{(ix)^m}{m!}| \leq \min(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!})$.
- If $\mathbb{E}|X|^2 < \infty$, then $\phi(t) = 1 + it\mathbb{E}X - t^2 \mathbb{E}|X|^2/2 + o(t^2)$.
- If $\limsup_{h \downarrow 0} \frac{\phi(h) - 2\phi(0) + \phi(-h)}{h^2} > -\infty$, then $\mathbb{E}[X^2] < \infty$.
- Given $\phi$ and $x_1, \cdots, x_n \in \mathbb{R}$, we can consider the matrix with $(i, j)$ entry given by $\phi(x_i - x_j)$. Call $\phi$ positive definite if this matrix is always positive semi-definite Hermitian.
- Bochner's theorem: A function from $\mathbb{R}$ to $\mathbb{C}$ which is continuous at origin with $\phi(0) = 1$ is a ch.f. of some probability measure on $\mathbb{R}$ if and only if it is positive definite.
- Pólya's theorem: If $\phi$ is real-valued, even and continuous such that (i) $\phi(0) = 1$; (ii) $\phi$ is convex for $t > 0$; (iii) $\phi(\infty) = 0$; then $\phi(t)$ is the ch.f. of a distribution symmetric about 0.

## 3.4 Central Limit Theorems

- Central Limit Theorem: Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}[X_1] = \mu, \text{var}(X_1) = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + X_2 + \cdots + X_n$, $\frac{S_n - n\mu}{n^{1/2}\sigma} \Rightarrow \mathcal{N}(0, 1)$.
- The Lindeberg-Feller theorem: For each $n$, let $X_{n,m}, 1 \leq m \leq n$, be independent random variables for each $n$ with $\mathbb{E}[X_{n,m}] = 0$. Suppose (i) $\sum_{m=1}^{n} \mathbb{E}[X_{n,m}^2] \to \sigma^2 > 0$; (ii) For all $\epsilon > 0$, $\lim_{n\to\infty} \sum_{m=1}^{n} \mathbb{E}[X_{n,m}^2 1_{|X_{n,m} > \epsilon|}] = 0$. Then $S_n = X_{n,1} + \cdots + X_{n,n} \Rightarrow \mathcal{N}(0, \sigma^2)$ as $n \to \infty$.
- Converging together lemma: If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, $X_n + Y_n \Rightarrow X + c$. A useful consequence of this result is that if $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$ then $Z_n \Rightarrow X$.
- Lévy's condition for CLT: Let $X_1, X_2, \cdots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. In order that there exist constants $a_n$ and $b_n > 0$ so that $(S_n - a_n)/b_n \Rightarrow \mathcal{N}(0, 1)$, it is necessary and sufficient that $\frac{y^2 P(|X_1| > y)}{\mathbb{E}[X_1^2 1_{|X_1| \leq y}]} \to 0$.
- Chernoff bound: Let $X_i$ be independent Bernoulli r.v's. Write $S_n = X_1 + \cdots + X_n$ and let $\mu = \mathbb{E}[S_n]$. Then for $\delta > 0$, $P(S_n > (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2+\delta}}, P(S_n < (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$.
- Hoeffding's inequality for bounded r.v. Let $X_i$ be independent r.v.'s such that $X_i \in [a_i, , b_i]$ a.s. Write $S_n = X_1 + \cdots + X_n$ and let $\mu = \mathbb{E}[S_n]$. Then for $\delta > 0$, $P(|S_n - \mu| \geq \delta) \leq 2\exp(-\frac{2n^2\delta^2}{\sum_{i=1}^{n}(b_i - a_i)^2})$.
- A random variable is sub-Gaussian, if and only if for some $C < \infty$ and $c > 0$, $P(|X| \geq t) \leq Ce^{-ct^2}$.
- Hoeffding's inequality for sub-Gaussian r.v.'s: Let $X_i$ be independent zero-mean sub-Gaussian r.v.'s. Write $S_n = X_1 + \cdots + X_n$. Then there exists some $c > 0$ such that for any $\delta > 0$, $P(|S_n| \geq \delta) \leq 2\exp(-c\delta^2 / \sum_{i=1}^{n} ||X_i||_{\psi_2})$, where $||X||_{\psi_2} = \inf\{c \geq 0 : \mathbb{E}[e^{X^2/c^2}] \leq 2\}$.
- Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^2] = \sigma^2$, and $\mathbb{E}[|X_i|^3] = \rho < \infty$. Let $\mathcal{N}(x)$ is the distribution of the standard normal distribution, then for all $n \geq 1$ and $x \in \mathbb{R}$, $|F_n(x) - \mathcal{N}(x)| \leq 3\rho/(\sigma^3\sqrt{n})$.

### 3.5 Local Limit Theorems

- A random variable $X$ has a lattice distribution if $\exists b, h > 0$ so that $P(X \in b + h\mathbb{Z}) = 1$. The largest $h$ for which the last statement holds is called the span of the distribution.

- Trichotomy of a random variable: Let $\phi(t)$ be the ch.f. of a random variable $X$. There are only three possibilities: (1) $|\phi(t)| < 1$ for all $t \neq 0$; (2) There is a $\lambda > 0$ so that $|\phi(\lambda)| = 1$ and $|\phi(\lambda)| < 1$ for $0 < t < \lambda$. In this case, $X$ has a lattice distribution with span $2\pi/\lambda$; (3) $|\phi(\lambda)| = 1$ for all $t$. In this case, $X$ is deterministic.

- Let $X_i$ be i.i.d. r.v.'s with $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^2] = \sigma^2 \in (0, \infty)$. Suppose in addition $P(X_i \in b + h\mathbb{Z}) = 1$, i.e. $X_i$ are lattice with span $h$. Let $p_n(x) = P(S_n/\sqrt{n} = x)$ for $x \in \mathcal{L}_n = \{(nb + h\mathbb{Z})/\sqrt{n}\}$, and $n(x)$ be the density of $\mathcal{N}(0, \sigma^2)$. Then $\lim_{n\to\infty} \sup_{x\in\mathcal{L}_n} |\frac{\sqrt{n}}{h} p_n(x) - n(x)| = 0$.

- Let $X_i$ be i.i.d. nonlattice r.v.'s with $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = \sigma^2$. If $x_n/\sqrt{n} \to x$ and $a < b$, $\sqrt{n}P(S_n \in (x_n + a, x_n + b)) \to (b - a)n(x)$.

- Let $p_n^{(d)}(\cdot)$ stand for the $n$-step transition probability for $d$-dimensional simple random walk. Then $p_{2n}^{(d)}(0)$ is monotone decreasing in $d$.

### 3.6 Poisson Convergence

- For each $n$ let $X_{n,m}, 1 \leq m \leq n$ be independent random variables with $P(X_{n,m} = 1) = p_{n,m}, P(X_{n,m} = 0) = 1 - p_{n,m}$. Suppose (i) $\lim_{n\to\infty} \sum_{m=1}^n p_{n,m} = \lambda$; (ii) $\lim_{n\to\infty} \max_{m\leq n} p_{n,m} = 0$. Let $S_n := X_{n,1} + \cdots + X_{n,n}$, then $S_n \Rightarrow \text{Poisson}(\lambda)$.

- $d(\mu, \nu) = ||\mu - \nu||_{\text{TV}}$ defines a metric on the set of probability measures on $\mathbb{Z}$. $||\mu_n - \mu|| \to 0$ if and only if $\mu_n \Rightarrow \mu$.

- The $p$-th Wasserstein distance between two probability measures $\mu$ and $\nu$ on $M$ with $p$-th moment is defined as $W_p(\mu, \nu) = (\inf_{\gamma\in\Gamma(\mu,\nu)} \int_{M\times M} d(x, y)^p d\gamma(x, y))^{1/p}$ where $\Gamma(\mu, \nu)$ is the set of all couplings of $\mu$ and $\nu$. One can show that $W_p$ defines a metric and convergence under $W_p$-metric is equivalent to weak convergence plus convergence of the first $p$-th moment.

- Suppose that $r$ balls are placed at random into $n$ boxes. Then suppose $r/n \to c$, the number of balls in each box is approximately Poisson($c$). Let $X_n$ be the number of empty boxes. Then if $ne^{-r/n} \to \lambda$, $X_n \to \text{Poisson}(\lambda)$.

- Let $X_{n,m}, 1 \leq m \leq n$ be independent random variables with $P(X_{n,m} = 1) = p_{n,m}, P(X_{n,m} \geq 2) = \epsilon_{n,m}$. Suppose $\lim_{n\to\infty} \sum_{m=1}^n p_{n,m} = \lambda, \lim_{n\to\infty} \max_{m\leq n} p_{n,m} = 0, \lim_{n\to\infty} \sum_{m=1}^n \epsilon_{n,m} = 0$. Let $S_n = X_{n,1} + \cdots + X_{n,n}$, then $S_n \Rightarrow \text{Poisson}(\lambda)$.

### 3.7 Poisson Process

- Let $N(s, t)$ be the number of students arriving at a certain dinning hall in the time interval $(s, t]$. Suppose the number of arrivals in intervals that are disjoint are independent, the distribution of $N(s, t)$ only depends on $t - s$, $P(N(0, h) = 1) = \lambda h + o(h)$, $P(N(0, h) \geq 2) = o(h)$. Then $N(0, t)$ has a Poisson distribution with mean $\lambda t$.

- A family of random variables $N_t, t \geq 0$ is called a Poisson process with rate $\lambda$, if (i) for $0 \leq t < s$, $N(s) - N(t) \sim \text{Poisson}(\lambda(s - t))$; (ii) if $0 < t_0 < t_1 < \cdots < t_n, N(t_k) - N(t_{k-1}), 1 \leq k \leq n$ are independent.

- Suppose that between 12:00 and 1:00 cars arrive at the East Gate of PKU according to a Poisson process $N_t$ with rate $\lambda$. Let $Y_i$ be the number of people in the $i$-th vehicle which we assume to be i.i.d. and independent to $N_t$. Then consider $M(t)$ be the total number of visitors within those vehicles by time $t$, i.e. $M(t) = \sum_{i=1}^{N_t} Y_i$ with the convention that $M(t) = 0$ if $N_t = 0$.

- Let $Y_1, Y_2, \cdots$ be i.i.d. r.v.'s; $N$ and independent non-negative interger-valued r.v.; $S = Y_1 + \cdots + Y_N$ with $S = 0$ when $N = 0$. (1) If $\mathbb{E}|Y_i|, \mathbb{E}[N] < \infty$, then $\mathbb{E}[S] = \mathbb{E}[N] \cdot \mathbb{E}[Y_i]$; (2) If $\mathbb{E}[Y_i^2], \mathbb{E}[N^2] < \infty$, then $\mathrm{var}(S) = \mathbb{E}[N]\mathrm{Var}(Y_i) + \mathrm{var}(N)(\mathbb{E}[Y_i])^2$; (iii) If $N \sim \mathrm{Poisson}(\lambda)$, then $\mathrm{var}(S) = \lambda\mathbb{E}[Y_i^2]$.

- Recall the problem of counting the number of cars arriving at the East Gate of PKU. Noting that $Y_i$ now stands for the number of people in each vehicel, $Y_i$ has to take positive integer values. Let $N_t^j$ be the number of cars with exactly $j$ passengers. For $Y_i$ taking value on $1, 2, \cdots, m < \infty$, $N_t^j$ are independent rate $\lambda P(Y_i = j)$ Poisson processes.

- Suppose that in a Poisson process with rate $\lambda$, for a point that lands at time $s$, we keep it with probability $p(s)$. Then the result is an inhomogenous Poisson process with rate $\lambda p(s)$.

- inhomogenous Poisson process as time change of Poisson process: For $p(t)$, and the standard Poisson process $N_t$ with rate $\lambda$, we call $\hat{N}(t) = N(\int_0^t \lambda p(s)ds)$ be the inhomogenous Poisson process with transition rate function $\lambda(t) = \lambda p(t)$.

- Suppose $\lambda$ is $\sigma$-finite, we say a random measure $\mu$ is a Poisson Point Process/Poisson random measure with intensity measure $\lambda$ if (1) for all $B \in \mathcal{S}$, $\mu(B) \sim \mathrm{Poisson}(\lambda(B))$; (2) If $B_1, \cdots, B_n$ be disjoint sets in $\mathcal{S}$, then the random variables $\mu(B_1), \cdots, \mu(B_n)$ are also independent.

- Let $T_n$ be the time of the $n$-th arrival of a Poisson process with rate $\lambda$. Let $U_1, U_2, \cdots, U_n$ be independent uniform on $(0, t)$ and let $(V_k^n)_{k=1,2,\cdots,n}$ be the order statistics of $\{U_1, \cdots, U_n\}$, i.e. $V_k^n$ is the $k$-th smallest number from $(U_1, \cdots, U_n)$. Then, conditioning on $N(t) = n$, the vectors $V = (V_1^n, \cdots, V_n^n)$ and $T = (T_1, \cdots, T_n)$ have the same distribution.

- If $0 < s < t$, then $P(N_s = m | N_t = n) = C_n^m (s/t)^m (1 - s/t)^{n-m}$.

### 3.8 Limit Theorems in $\mathbb{R}^d$

- We say $X_n \Rightarrow X_\infty$ if $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_\infty)]$ for all bounded and continuous $f$.

- General Portmantean Theorem: The following statements are equivalent: (1) $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_\infty)]$ for all bounded and continuous $f$; (2) $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_\infty)]$ for all bounded and Lipschitz-continuous $f$; (3) For all closed sets $K$, $\limsup_{n\to\infty} P(X_n \in K) \le P(X_\infty \in K)$; (4) For all open sets $G$, $\liminf_{n\to\infty} P(X_n \in G) \ge P(X_\infty \in G)$; (5) For all sets $A$ with $P(X_\infty \in \partial A) = 0$, $\lim_{n\to\infty} P(X_n \in A) = P(X_\infty \in A)$; (6) Let $D_f =$ the set of discontinuous of $f$. For all bounded functions $f$ with $P(X_\infty \in D_f) = 0$, we have $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_\infty)]$.

- For distribution $F_n$ and $F$ on $\mathbb{R}^d$, we say that $F_n$ converges weakly to $F$, and write $F_n \Rightarrow F$, if $F_n(x) \to F(x)$ at all continuity points of $F$.

- Distribution function in $\mathbb{R}^d$: (i) Nondecreasing: $x \le y \Rightarrow F(x) \le F(y)$. (ii) $\lim_{x\to\infty} F(x) = 1, \lim_{x_i\to-\infty} F(x) = 0$. (iii) $F$ is right continuous: $\lim_{y\uparrow x} F(y) = F(x)$. (iv) $\triangle_A F \ge 0$ for all rectangles $A$.

- Equivalence of two definitions: On $\mathbb{R}^d$ weak convergence defined in terms of convergence of distribution $F_n \Rightarrow F_\infty$ is equivalent to notion of weak convergence defined for a general metric space.

- Tightness in $\mathbb{R}^d$: A sequence of probability measures $\mu_n$ is said to be tight if for any $\epsilon > 0$, there is an $M < \infty$ such that $\liminf_{n\to\infty} \mu_n([-M, M]^d) \geq 1 - \epsilon$.

- If $\mu_n$ is tight, there is a weakly convergent subsequence.

- The characteristic function of $\vec{X} = (X_1, \cdots, X_d)$ is $\phi(\vec{t}) = \mathbb{E}[\exp(i\vec{t} \cdot \vec{X})]$. If $A = [a_1, b_1] \times \cdots \times [a_d, b_d]$ with $\mu(\partial A) = 0$, then $\mu(A) = \lim_{T\to\infty}(2\pi)^{-d} \int_{[-T,T]^d} \left(\prod_{j=1}^d \psi_j(t_j)\right) \phi(\vec{t})dt$, where $\psi_j(s) = \frac{\exp(-isa_j)-\exp(-isb_j)}{is}$.

- Convergence theorem: Let $X_n, 1 \leq n \leq \infty$ be random vectors with ch.f. $\phi_n$. A necessary and sufficient condition for $F_n$ to converge weakly to a probability distribution $F_\infty$ is that $\phi_n(\vec{t}) \to \phi_\infty(\vec{t})$, which is continuous at 0.

- Cramer-Wold device: A sufficient condition for $X_n \Rightarrow X_\infty$ is that $\vec{\theta} \cdot X_n \Rightarrow \vec{\theta} \cdot X_\infty$ for all $\vec{\theta} \in \mathbb{R}^d$.

- The central limit theorem in $\mathbb{R}^d$: Let $X_1, X_2, \cdots$ be i.i.d. random vectors with $\mathbb{E}X_n = \mu$, and finite covariances $(\Gamma_{i,j})_{m\times m}$. Then $(S_n - n\mu)/n^{1/2} \Rightarrow \chi$, where $\chi$ is a multivariate normal with mean 0 and covariances $(\Gamma_{i,j})_{m\times m}$.

# 4 Martingales

## 4.1 Conditional Expectation

- Existence and uniqueness of conditional expectation: Let $(\Omega, \mathscr{H}, P)$ be a probability space, $X$ be a random variable such that $\mathbb{E}[|X|] < \infty$, $\mathscr{G} \subset \mathscr{H}$ be a sub $\sigma$-algebra of $\mathscr{H}$. Then (1) Existence: $\exists$ r.v. $Y$ such that $Y \in \mathscr{G}, \mathbb{E}[|Y|] < \infty$ and $\forall G \in \mathscr{G}, \mathbb{E}[Y; G] = \mathbb{E}[X; G]$. We call such $Y$ a version of $\mathbb{E}[X|\mathscr{G}]$. (2) Uniqueness: If $Y, \widetilde{Y}$ are versions of $\mathbb{E}[X|\mathscr{G}]$, then $Y = \widetilde{Y}$ a.s.

- Orthogonal projection in $L^2$: If $\mathbb{E}[X^2] < \infty$, then $Y = \mathbb{E}[X|\mathscr{G}]$ is a version of the orthogonal projection of $X$ from $L^2(\Omega, \mathscr{H}, P)$ to $L^2(\Omega, \mathscr{G}, P)$, i.e. $Y$ is the best $G$-measurable predictor of $X$, which minimizes $\mathbb{E}[(Y - X)^2]$.

- Properties of conditional expectation: (1) $Y = \mathbb{E}[X|\mathscr{G}] \Rightarrow \mathbb{E}[Y] = \mathbb{E}[X]$. (2) $X \in \mathscr{G} \Rightarrow \mathbb{E}[X|\mathscr{G}] = X$ a.s. (3) Linearity: $\mathbb{E}[aX_1 + bX_2|\mathscr{G}] = a\mathbb{E}[X_1|\mathscr{G}] + b\mathbb{E}[X_2|\mathscr{G}]$ a.s. (4) Positivity: $X \geq 0 \Rightarrow \mathbb{E}[X|\mathscr{G}] \geq 0$ a.s. (5) Monotone convergence theorem: $0 \leq X_n \uparrow X \Rightarrow \mathbb{E}[X_n|\mathscr{G}] \uparrow \mathbb{E}[X|\mathscr{G}]$ a.s. (6) Fatou's lemma: $X_n \geq 0 \Rightarrow \mathbb{E}[\liminf\limits_{n\to\infty} X_n|\mathscr{G}] \leq \liminf\limits_{n\to\infty} \mathbb{E}[X_n|\mathscr{G}]$ a.s. (7) Dominated convergence theorem: $|X_n(\omega)| \leq V(\omega)$ a.s. $\forall n$, $\mathbb{E}[V] < \infty$, $X_n \to X$ a.s., then $\mathbb{E}[X_n|\mathscr{G}] \to \mathbb{E}[X|\mathscr{G}]$ a.s. (8) If $c(x)$ is convex, $\mathbb{E}[|c(x)|] < \infty$, then $\mathbb{E}[c(x)|\mathscr{G}] \geq c(\mathbb{E}[X|\mathscr{G}])$ a.s. (9) Tower property: If $\mathscr{H} \subset \mathscr{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathscr{G}]\mathscr{H}] = \mathbb{E}[\mathbb{E}[X|\mathscr{H}]|\mathscr{G}] = \mathbb{E}[X|\mathscr{H}]$. (10) If $Z \in \mathscr{G}$ then $\mathbb{E}[ZX|\mathscr{G}] = Z\mathbb{E}[X|\mathscr{G}]$. (11) If $\mathscr{H} \perp\!\!\!\perp \sigma(X, \mathscr{G})$ then $\mathbb{E}[X|\sigma(\mathscr{G}, \mathscr{H})] = \mathbb{E}[X|\mathscr{G}]$ a.s. In particular, if $X \perp\!\!\!\perp \mathscr{H}$, then $\mathbb{E}[X|\mathscr{H}] = \mathbb{E}[X]$ a.s.

## 4.2 Martingales, Almost Sure Convergence

- Filtered spaces: $(\Omega, \mathscr{F}, \{\mathscr{F}_n\}_{n=0}^\infty, P)$ satisfies $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}$ (i.e. $\{\mathscr{F}_n\}_{n=1}^\infty$ is a filtration) and $\sigma(\cup_{i=0}^\infty \mathscr{F}_n) := \mathscr{F}_\infty \subset \mathscr{F}$ (but not necessarily $\mathscr{F}_\infty = \mathscr{F}$). Given a filtration $\{\mathscr{F}_n\}$, if a sequence of r.v.'s $\{X_n\}$ satisfies $X_n \in \mathscr{F}_n$, we say $\{X_n\}$ is adapted to $\{\mathscr{F}_n\}$.

- Martingale: $X = \{X_n\}$ discrete time stochastic process is a martingale if: (1) $\{X_n\}$ is adapted to some filtration $\{\mathscr{F}_n\}$; (2) $\forall n, \mathbb{E}[|X_n|] < \infty$ (but not necessarily $\mathbb{E}[|X_n|] < M < \infty$); (3) $\forall n, \mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$. If "=" in (3) is replaced by "$\geq$" or "$\leq$", then we say $X$ is a submartingale/supermartingale.

- $m < n$, $\{X_n\}$ is martingale/submartingale/supermartingale, $\mathbb{E}[X_n|\mathscr{F}_m] = / \geq / \leq X_m$.

- If $X_n$ is a martingale w.r.t. $\mathscr{F}_n$ and $\phi$ is a convex function with $\mathbb{E}|\phi(X_n)| < \infty$ for all $n$, then $\phi(X_n)$ is a submartingale w.r.t. $\mathscr{F}_n$.

- A process is predictable if $C_n \in \mathscr{F}_{n-1}$.
- You can't beat the system: Let $Y_n = \sum_{k=1}^n C_k(X_k - X_{k-1})$, $C$ is a predictable process. (1) If $C$ is non-negative, $|C_n(\omega)| \leq K, \forall n, \forall \omega$, and $X$ is martingale/supermartingale, then $Y$ is martingale/supermartingale. (2) If $C$ is a bounded predictable process and $X$ is a martingale, then $Y$ is a martingale. (3) In (1) and (2), the boundness condition on $C$ may be replaced by the condition $C_n \in L^2, \forall n$, provided we also insist that $X_n \in L^2, \forall n$.
- Stopping time: $T : \Omega \to \mathbb{Z}_+$, if $\{T \leq n\} \in \mathscr{F}_n, \forall n \leq \infty$.
- If $X$ is a martingale/supermartingale and $T$ is a stopping time, then the stopped process $(X_{T\wedge n})_n$ is a martingale/supermartingale, $\mathbb{E}[X_{T\wedge n}] = / \leq \mathbb{E}[X_0]$.
- Doob's optional stopping theorem: Let $T$ be a stopping time and $X$ be a martingale/supermartingale. Then $X_T$ is integrable and $\mathbb{E}[X_T] = / \leq \mathbb{E}[X_0]$ in each of the following situations: (1) $T$ is bounded; (2) $X$ is bounded and $T$ is a.s. finite; (3) $\mathbb{E}[T] < \infty$, and, for some $K \in \mathbb{R}_+$, $|X_n(\omega) - X_{n-1}(\omega)| \leq K$.
- Define $C_1 := I_{\{X_0 < a\}}$ and, for $n \geq 2, C_n := I_{\{C_{n-1}=1\}} I_{\{X_{n-1} \leq b\}} + I_{\{C_{n-1}=0\}} I_{\{X_{n-1} < a\}}$. $Y_n = \sum_{k=1}^n C_k(X_k - X_{k-1})$. The number $U_N[a,b](\omega)$ of upcrossings of $[a,b]$ made by $n \mapsto X_n(\omega)$ by time $N$ is defined to be the largest $k$ in $\mathbb{Z}_+$ such that we can find $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_k < t_k \leq N$ with $X_{s_i}(\omega) < a, X_{t_i}(\omega) > b, 1 \leq i \leq k$.
- The fundamental inequality (recall that $Y_0(\omega) = 0$) is obvious: $Y_N(\omega) \geq (b-a)U_N[a,b](\omega) - [X_N(\omega) - a]^-$.
- Doob's upcrossing lemma: Let $X$ be a supermartingale. Let $U_N[a,b]$ be the number of upcrossings of $[a,b]$ by time $N$. Then $(b-a)\mathbb{E}U_N[a,b] \leq \mathbb{E}[(X_N - a)^-]$.
- Let $X$ be a supermartingale bounded in $L^1$ in that $\sup_n \mathbb{E}|X_n| < \infty$. Let $a, b \in \mathbb{R}$ with $a < b$. Then, with $U_\infty([a,b]) := \lim_N U_N[a,b]$, $(b-a)\mathbb{E}U_\infty[a,b] \leq |a| + \sup_n \mathbb{E}|X_n| < \infty$ so that $P(U_\infty[a,b] = \infty) = \infty$.
- Doob's forward convergence theorem: Let $X$ be a supermartingale bounded in $L_1$: $\sup_n \mathbb{E}|X_n| < \infty$. Then, almost surely, $X_\infty := \lim_n X_n$ exists and is finite. For definiteness, we define $X_\infty(\omega) := \limsup_n X_n(\omega), \forall \omega$, so that $X_\infty$ is $\mathscr{F}_\infty$ measurable and $X_\infty = \lim_n X_n$, a.s.
- Martingale convergence theorem: If $X_n$ is a submartingale with $\sup \mathbb{E}X_n^+ < \infty$, then as $n \to \infty$, $X_n$ converges a.s. to a limit $X$ with $\mathbb{E}|X| < \infty$.
- If $X_n \geq 0$ is a supermartingale, then as $n \to \infty$, $X_n \to X$ a.s. and $\mathbb{E}X \leq \mathbb{E}X_0$.

### 4.3 Examples

- Doob's decomposition: Any submartingale $X_n, n \geq 0$, can be written in a unique way as $X_n = M_n + A_n$, where $M_n$ is a martingale and $A_n$ is a predictable increasing sequence with $A_0 = 0$.
- Let $X_1, X_2, \cdots$ be a martingale with $|X_{n+1} - X_n| \leq M < \infty$. Let $C = \{\lim_n X_n \text{ exists and is finite}\}, D = \{\limsup_n X_n = +\infty \text{ and } \liminf_n X_n = -\infty\}$. Then $P(C \cup D) = 1$.
- Second Borel-Cantelli lemma: Let $\mathscr{F}_n, n \geq 0$ be a filtration with $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $B_n, n \geq 1$ a sequence of events with $B_n \in \mathscr{F}_n$. Then $\{B_n \text{ i.o.}\} = \{\sum_{n=1}^\infty P(B_n|\mathscr{F}_{n-1}) = \infty\}$.
- Let $\mu, \nu$ be two probability measures on $(\Omega, \mathscr{F})$. Let $\mathscr{F}_n \uparrow \mathscr{F}$ be $\sigma$-fields. Let $\mu_n$ and $\nu_n$ be the restrictions of $\mu$ and $\nu$ to $\mathscr{F}_n$. Suppose $\mu_n << \nu_n$ for all $n$. Let $X_n = d\mu_n/d\nu_n$ and let $X = \limsup_n X_n$. Then $\mu(A) = \int_A X d\nu + \mu(A \cap \{X = \infty\}) := \mu_r(A) + \mu_s(A)$, which gives the Lebesgue decomposition of $\mu$, i.e., $\mu_r << \nu, \mu_s \perp \nu$.

- Kakutani dichotomy for infinite product measures: Let $\mu, \nu$ be two probability measures on sequence space $(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}})$ that make the coordinates $\xi_n(\omega) = \omega_n$ independent. Let $F_n(x) = \mu(\xi_n \le x), G_n(x) = \nu(\xi_n \le x)$. Suppose $F_n << G_n$ and let $q_n = dF_n/dG_n > 0, G_n$-a.s. Let $\mathscr{F}_n = \sigma(\xi_m : m \le n)$, let $\mu_n, \nu_n$ be the restrictions of $\mu$ and $\nu$ to $\mathscr{F}_n$, and let $X_n = \frac{d\mu_n}{d\nu_n} = \prod_{m=1}^n q_m$. Then $X_n \to X, \nu$-a.s. $\sum_{m=1}^\infty \log(q_m) > \infty$ is a tail event, so the Kolmogorov 0-1 law implies $\nu(X = 0) \in \{0, 1\}$ and it follows that either $\mu << \nu$ or $\mu \perp \nu$.

- $\mu << \nu$ or $\mu \perp \nu$, according as $\prod_{m=1}^\infty \int \sqrt{q_m} dG_m > 0$ or $= 0$.

## 4.4 Doob's Inequality, Convergence in $L^p$, $p > 1$

- If $X_n$ is a submartingale and $N$ is a stopping time with $P(N \le k) = 1$, then $\mathbb{E}X_0 \le \mathbb{E}X_N \le \mathbb{E}X_k$.

- Doob's inequality: Let $X_m$ be a submartingale, $\bar{X}_n = \max_{0\le m\le n} X_m^+, \lambda > 0$ and $A = \{\bar{X}_n \ge \lambda\}$. Then $\lambda P(A) \le \mathbb{E}X_n 1_A \le \mathbb{E}X_n^+$.

- $L^p$ maximum inequality: If $X_n$ is a submartingale, then for $1 < p < \infty$, $\mathbb{E}(\bar{X}_n^p) \le (\frac{p}{p-1})^p \mathbb{E}(X_n^+)^p$. Consequently, if $Y_n$ is a martingale and $Y_n^* = \max_{0\le m\le n} |Y_m|$, $\mathbb{E}|Y_n^*|^p \le (\frac{p}{p-1})^p \mathbb{E}(|Y_n|^p)$.

- $L^p$ convergence theorem: If $X_n$ is a martingale with $\sup \mathbb{E}|X_n|^p < \infty$ where $p > 1$, then $X_n \to X$ a.s. and in $L^p$.

## 4.5 Square Integrable Martingales

- In this subsection, we will suppose $X_n$ is a martingale with $X_0 = 0$ and $\mathbb{E}X_n^2 < \infty$ for all $n$.

- Let $X_n^2 = M_n + A_n$ be the Doob decomposition of $X_n^2$. Then $X_n$ is $L^2$-bounded iff $\mathbb{E}A_\infty = \sum_{n=1}^\infty \mathbb{E}(X_n - X_{n-1})^2 < \infty$.

- $\mathbb{E}(\sup_m |X_m|^2) \le 4\mathbb{E}A_\infty$.

- $\lim_{n\to\infty} X_n$ exists and is finite a.s. on $\{A_\infty < \infty\}$.

- Let $f \ge 1$ be increasing with $\int_0^\infty f(t)^{-2} dt < \infty$. Then $X_n/f(A_n) \to 0$ a.s. on $\{A_\infty = \infty\}$.

- Second Borel-Cantelli Lemma: Suppose $B_n$ is adapted to $\mathscr{F}_n$ and $p_n = P(B_n|\mathscr{F}_{n-1})$. $\sum_{m=1}^n 1_{B(m)} / \sum_{m=1}^n p_m \to 1$ a.s. on $\{\sum_{m=1}^\infty p_m = \infty\}$.

- $\mathbb{E}(\sup_n |X_n|) \le 3\mathbb{E}A_\infty^{1/2}$.

## 4.6 Uniform Integrability, Convergence in $L^1$

- $\{X_i\}_{i\in I}$ is uniformly integrable if $\lim_{M\to\infty}(\sup_{i\in I} \mathbb{E}(|X_i|; |X_i| > M)) = 0$.

- Given a probability space $(\Omega, \mathscr{F}_0, P)$ and an $X \in L^1$, then $\{\mathbb{E}(X|\mathscr{F}) : \mathscr{F} \text{ is a } \sigma\text{-field} \subset \mathscr{F}_0\}$ is uniformly integrable.

- Let $\phi \ge 0$ be any function with $\phi(x)/x \to \infty$ as $x \to \infty$. If $\mathbb{E}\phi(|X_i|) \le C$ for all $i \in I$, then $\{X_i, i \in I\}$ is uniformly integrable.

- Suppose that $\mathbb{E}|X_n| < \infty$ for all $n$. If $X_n \to X$ in probability, then the following are equivalent: (i) $\{X_n : n \ge 0\}$ is uniformly integrable. (ii) $X_n \to X$ in $L^1$. (iii) $\mathbb{E}|X_n| \to \mathbb{E}|X| < \infty$.

- For a submartingale, the following are equivalent: (i) It is uniformly integrable. (ii) It converges a.s. and in $L^1$. (iii) It converges in $L^1$.
- If a martingale $X_n \to X$ in $L^1$, then $X_n = \mathbb{E}(X|\mathscr{F}_n)$.
- For a martingale, the following are equivalent: (i) It is uniformly integrable. (ii) It converges a.s. and in $L^1$. (iii) It converges in $L^1$. (iv) There is an integrable random variable $X$ so that $X_n = \mathbb{E}(X|\mathscr{F}_n)$.
- Suppose $\mathscr{F}_n \uparrow \mathscr{F}_\infty$, i.e., $\mathscr{F}_n$ is an increasing sequence of $\sigma$-fields and $\mathscr{F}_\infty = \sigma(\cup_n \mathscr{F}_n)$. As $n \to \infty$, $\mathbb{E}(X|\mathscr{F}_n) \to \mathbb{E}(X|\mathscr{F}_\infty)$ a.s. and in $L^1$.
- Lévy's 0-1 law: If $\mathscr{F}_n \uparrow \mathscr{F}_\infty$ and $A \in \mathscr{F}_\infty$, then $\mathbb{E}(1_A|\mathscr{F}_n) \to 1_A$ a.s.

## 4.7 Backwards Martingales

- A backwards martingale is a martingale indexed by the negative integers, i.e., $X_n, n \leq 0$, adapted to an increasing sequence of $\sigma$-fields $\mathscr{F}_n$ with $\mathbb{E}(X_{n+1}|\mathscr{F}_n) = X_n$ for $n \leq -1$.
- $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in $L^1$.
- If $X_{-\infty} = \lim_{n\to-\infty} X_n$ and $\mathscr{F}_{-\infty} = \cap_n \mathscr{F}_n$, then $X_{-\infty} = \mathbb{E}(X_0|\mathscr{F}_{-\infty})$.
- A sequence $X_1, X_2, \cdots$ is said to be exchangeable if for each $n$ and permutation $\pi$ of $\{1, \cdots, n\}$, $(X_1, \cdots, X_n)$ and $(X_{\pi(1)}, \cdots, X_{\pi(n)})$ have the same distribution. If $X_1, X_2, \cdots$ are exchangeable then conditional on $\mathcal{E}$(exchangeable $\sigma$-field), $X_1, X_2, \cdots$ are independent and identically distributed.
- If $X_1, X_2, \cdots$ are exchangeable and take values in $\{0, 1\}$, then there is a probability distribution on $[0, 1]$ so that $P(X_1 = 1, \cdots, X_k = 1, X_{k+1} = 0, \cdots, X_n = 0) = \int_0^1 \theta^k (1-\theta)^{n-k} dF(\theta)$.

## 4.8 Optional Stopping Theorems

- If $X_n$ is a uniformly integrable submartingale, then for any stopping time $N$, $X_{N\wedge n}$ is uniformly integrable.
- If $\mathbb{E}|X_N| < \infty$ and $X_n 1_{(N>n)}$ is uniformly integrable, then $X_{N\wedge n}$ is uniformly integrable and hence $\mathbb{E}X_0 \leq \mathbb{E}X_N$.
- If $X_n$ is a uniformly integrable submartingale, then for any stopping time $N \leq \infty$, we have $\mathbb{E}X_0 \leq \mathbb{E}X_N \leq \mathbb{E}X_\infty$, where $X_\infty = \lim_n X_n$.
- If $X_n$ is a nonnegative supermartingale and $N \leq \infty$ is a stopping time, then $\mathbb{E}X_0 \leq \mathbb{E}X_N$, where $X_\infty = \lim_n X_n$.