Stochastic Processes

Lectured by Weijun Xu LaTeXed by Chengxin Gong

2023 年 3 月 9 日

目录

1 Review of Martingales

- $(X_n)_{n\geq 0}$ is L^2-bounded martingale $\Rightarrow X_n$ converges in L^2.
- $(X_n)_{n\geq 0}$ is L^1-bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n\geq 0}$ is L^p-bounded martingale for $p>1$, then X_n converges in $L^{p'}$ for $p'\in[1,p)$.
- Statement is false when $p=1$. Example: $\Omega=[0,1), \mathscr{F}_n=\sigma\{[\frac{i}{2^n},\frac{i+1}{2^n})\}_{i=0}^{2^n-1}, X_n(\omega):=\begin{cases}2^n & \omega\in[0,\frac{1}{2^n})\\ 0 & \text{otherwise}\end{cases}$.
- Let $p>1$ and $(X_n)_{n\geq 0}$ be L^p bounded martingale w.r.t. $\mathscr{F}_n$. Then $\exists X\in L^p(\Omega,\mathscr{F}_\infty,P)$ s.t. $X_n\to X$ in L^p and a.s. and $X_n=\mathbb{E}(X|\mathscr{F}_n)$.
- Doob's maximal inequality: Let $p>1, \exists C=C_p$ s.t. $\forall$ martingale $(X_n)_{n\geq 0}$, we have $\mathbb{E}|X_n^*|^p\leq C_p\mathbb{E}|X_n|^p$ where $|X_n^*|=\sup_{0\leq k\leq n}\sup|X_k|$.
- Let $(Z_n)_{n\geq 0}$ be a nonnegative sub-martingale and $Z_n^*=\sup_{0\leq k\leq n}Z_k$, then $\mathbb{P}(Z_n^*>\lambda)\leq\frac{1}{\lambda}\mathbb{E}(Z_n 1_{\{Z_n^*>\lambda\}})\leq\frac{1}{\lambda}\mathbb{E}Z_n$. Corollary: $\mathbb{P}(Z_n^*>\lambda)\leq\frac{1}{\lambda^p}\mathbb{E}(Z_n^p 1_{\{Z_n^*>\lambda\}})\leq\frac{1}{\lambda^p}\mathbb{E}(Z_n^p)$.
- If $(X_n)_{n\geq 0}$ is a martingale with $\sup_n\mathbb{E}(|X_n|\log(1+|X_n|))<+\infty$, then X_n converges in L^1.
- Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega,\mathscr{F})$, $\mathbb{Q}<<\mathbb{P}$ on $\mathscr{F}_n$ for every n and $M_n=\frac{\mathrm{d}\mathbb{Q}|_{\mathscr{F}_n}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_n}}$. $(M_n)_{n\geq 0}$ is a $\mathbb{P}$-martingale w.r.t. $(\mathscr{F}_n)_{n\geq 0}$. $\mathbb{Q}<<\mathbb{P}$ on $\mathscr{F}_\infty$ if and only if $M_n\to M$ in L^1. $\mathbb{Q}(A)=\int_A M\mathrm{d}\mathbb{P}+\mathbb{Q}(A\cap\{M=+\infty\})$.

  Proof Sufficiency. *$\mathbb{Q}<<\mathbb{P}$ on $\mathscr{F}=\mathscr{F}_\infty$, thus let $Z=\frac{\mathrm{d}\mathbb{Q}|_{\mathscr{F}}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}}}$, we need to show M_n converges to Z in L^1. $\forall A\in\mathscr{F}_n, \int_A M_n d\mathbb{P}=\mathbb{Q}(A)=\int_A Z d\mathbb{P}\Rightarrow M_n=\mathbb{E}(Z|\mathscr{F}_n)$. Thus M_n is uniformly integrable, thus converges in L^1.*
  Necessity. Suppose $M_n\to M$ a.s. and in L^1 We need to show $M_n=\mathbb{E}(M|\mathscr{F}_n)$ and $M=\frac{d\mathbb{Q}}{d\mathbb{P}}$. It suffices to show $\mathbb{Q}(A)=\int_A M d\mathbb{P}$ for all $A\in\cup_n\mathscr{F}_n$. Suppose $A\in\mathscr{F}_N$. Then $\mathbb{Q}(A)=\int_A M_N d\mathbb{P}=\int_A M_{N+k}d\mathbb{P}\to\int_A M d\mathbb{P}$. By $\pi-\lambda$ theorem we obtain the result.
  Suppose $\mathbb{P}\perp\mathbb{Q}$ on $\mathscr{F}$ (i.e. $\exists E$ s.t. $\mathbb{P}(E)=1,\mathbb{Q}(E^c)=1$) and $\mathbb{P}<<\mathbb{Q}$ on $\mathscr{F}_n$. Then $\frac{1}{M_n}$ converges $\mathbb{Q}$-a.s. Let $\mathbb{R}=\frac{1}{2}(\mathbb{P}+\mathbb{Q})$, $\mathbb{P},\mathbb{Q}<<\mathbb{R}$ on $\mathscr{F}$, $\frac{d\mathbb{P}|_{\mathscr{F}_n}}{d\mathbb{R}|_{\mathscr{F}_n}}=\frac{2}{1+M_n}\to\frac{2M}{1+M}$ in $L^1(\mathbb{R})$, $\frac{d\mathbb{Q}}{d\mathbb{R}}=\frac{2M_n}{1+M_n}\to\frac{2}{1+M}$ in $L^1(\mathbb{R})$. Then $\mathbb{Q}(A)=\mathbb{Q}(A\cap E^c)=\int_{A\cap E^c}\frac{2M}{1+M}d\mathbb{R}=\int_A\frac{2M}{1+M}1_{E^c}d\mathbb{R}\overset{\mathbb{P}(E^c)=0}{=}2\mathbb{R}(A\cap E^c)=2\int_A 1_{E^c}d\mathbb{R}\Rightarrow M=+\infty$ on $E^c\Rightarrow\mathbb{Q}(M=+\infty)=1$. Similarly $\mathbb{P}(M=0)=\mathbb{Q}(M=+\infty)=1$.
  General situation: $\mathbb{Q}=\mathbb{Q}_1+\mathbb{Q}_2,\mathbb{Q}_1<<\mathbb{P},\mathbb{Q}_2\perp\mathbb{P}$ on $\mathscr{F}$. Then we can write $M_n=Y_n+Z_n$ where $Y_n\to Y$ in $L^1(\mathbb{P})$ and $Z_n\to 0$ $\mathbb{P}$-a.s. $\mathbb{Q}_1(A)=\int_A Y d\mathbb{P}=\int_A M d\mathbb{P}$. $\mathbb{Q}_2(A)=\mathbb{Q}_2(A\cap\{Z=+\infty\})$. Since $Z=0$ $\mathbb{P}$-a.s., $M<+\infty$ $\mathbb{P}$-a.s. and $\mathbb{Q}_2(M=+\infty)=1$, we have $\mathbb{Q}_2(A)=\mathbb{Q}(A\cap\{Z=+\infty\})=\mathbb{Q}_2(A\cap\{M=+\infty\})=\mathbb{Q}(A\cap\{M=+\infty\})$. To sum up, $\mathbb{Q}(A)=\int_A M d\mathbb{P}+\mathbb{Q}(A\cap\{M=+\infty\})$. □

- Statement is false if $M_n\not\to M$ in L^1. Example: $\Omega=\{\omega=(\omega_1,\cdots,\omega_n,\cdots)\in\{\pm1\}^{\mathbb{N}}\}, X_n(\omega)=\omega_n$. X_n's are i.i.d. under $\mathbb{P}$ and $\mathbb{Q}$, but $\mathbb{P}(X_n=1)=\frac{1}{2},\mathbb{P}(X_n=-1)=\frac{1}{2},\mathbb{Q}(X_n=1)=\frac{1}{3},\mathbb{Q}(X_n=-1)=\frac{2}{3}$. $\mathscr{F}_n=\sigma(X_1,\cdots,X_n)$. $\mathbb{P}(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n X_k=0)=1,\mathbb{Q}(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n X_k=-\frac{1}{3})=1$.
- Monotone class theorem for functions: Suppose $\mathcal{A}$ us a π-system and $\mathcal{H}$ be a class of functions from Ω to $\mathbb{R}$ s.t. (1) $1_A\in\mathcal{H}$ for every $A\in\mathscr{A}$, (2) if $f,g\in\mathcal{H}$ then $af+bg\in\mathcal{H}$, (3) if $f_n\in\mathcal{H}$ and $f_n\uparrow f$ then $f\in\mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$-measurable functions are in $\mathcal{H}$.
- Let $(Y_n)_{n\geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E}Y_k=1$. Then $M_n=\prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n\equiv 1$. Otherwise $M_n\to 0$ a.s.

  Proof Note that $\frac{1}{n}\log M_n=\frac{1}{n}\sum_{k=1}^n\log Y_k\to\mathbb{E}\log Y$ a.s. If $\mathbb{E}\log Y=0$ then by Jensen's inequality we have $Y_n\equiv 1$ which means M_n converges in L^1. If $\mathbb{E}\log Y<0$ then $M_n\to 0$ a.s. □

- Kakutani's theorem: $M_n=\prod_{k=1}^n Y_k$, $Y_k\geq 0$ are independent, $\mathbb{E}Y_k=1$, $\lambda_k=\mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k\lambda_k>0$, then $M_n\to M$ in L^1; (2) If $\prod_k\lambda_k=0$, then $M_n\to 0$ a.s.

  Proof Let $Z_n=\prod_{k=1}^n\frac{\sqrt{Y_k}}{\lambda_k}$. Then Z_n is a martingale and has an a.s. limit Z, and $M_n=(\prod_{k=1}^n\lambda_k)^2 Z_n^2$. If $\prod_k\lambda_k>0$, then Z_n is L^2 bounded and then convergence in L^2, which implies $M_n\to M$ in L^1. If $\prod_k\lambda_k=0$, it is obvious that $M_n\to 0$ a.s. □

- Martingale LLN: Let $(M_n)_{n\geq 0}$ be a martingale s.t. $\sum_{k=1}^{+\infty}\frac{\mathbb{E}(M_k-M_{k-1})^2}{k^2}<+\infty$. Then $\frac{M_n}{n}\to 0$ a.s.

  Proof *Let* $Y_n=\sum_{k=1}^n\frac{X_k}{k}$. *Then* $(Y_n)_{n\geq 0}$ *is an* L^2 *bounded martingale, thus* $Y_n\to Y$ *a.s. Then by Kronecker's lemma,* $M_n=\frac{X_1+\cdots+X_n}{n}\to 0$ *a.s.* □

- Martingale CLT: Let $(M_n)_{n\geq 0}$ be a martingale with $M_0=0$ and $\sigma_n^2=\sum_{k=1}^n\mathbb{E}X_k^2=\mathbb{E}\langle M\rangle_n$. Assume that $\frac{1}{\sigma_n^2}\max_{1\leq k\leq n}(\mathbb{E}X_k^2)\to 0$, $\frac{1}{\sigma_n^2}\sum_{k=1}^n\mathbb{E}(X_k^2 1_{\{|X_k|>\epsilon\sigma_n\}}|\mathscr{F}_{k-1})\xrightarrow{p}0$ for all $\epsilon>0$, $\frac{1}{\sigma_n^2}\langle M\rangle_n\xrightarrow{p}1$. Then $\frac{M_n}{\sigma_n}\Rightarrow\mathcal{N}(0,1)$.

2 Markov Chains

- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain on a discrete space S. $\mathbb{P}^x$: law of $(X_n)_{n\geq 0}$ conditioned on $X_0=x$. $\mathbb{P}(X_{n+1}\in A|\mathscr{F}_n)=\mathbb{P}^{X_n}(X_1\in A)=\mathbb{P}(X_1\in A|X_0=X_n)$. $\mathbb{E}^x$: expectation under $\mathbb{P}^x$. $\mathbb{P}^x(X_1=y)=p(x,y)$.

- For every $f:S\to\mathbb{R}$ bounded, define $(\mathcal{P}f)(x)=\sum_{y\in S}p(x,y)f(y)=\mathbb{E}^x(f(X_1))$, $(\mathcal{L}f)(x)=\sum_{y\in S}p(x,y)f(y)-f(x)$. $\mathcal{L}=\mathcal{P}-\mathrm{id}$, the generator.

- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with generator $\mathcal{L}$. Then for every bounded $f:S\to\mathbb{R}$, $M_n=f(X_n)-f(X_0)-\sum_{k=0}^{n-1}(\mathcal{L}f)(X_k)$ is a martingale. Conversely, let $(X_n)_{n\geq 0}$ be a process and $\mathcal{L}$ be an operator on $\mathcal{B}(S)$ s.t. M_n^f is a martingale for every f, then $(X_n)_{n\geq 0}$ is a Markov chain with generator $\mathcal{L}$.

- Given operator $\mathcal{L}$ on $\mathcal{B}(S)$, we say $f:S\to\mathbb{R}$ is (1) harmonic for $\mathcal{L}$ if $\mathcal{L}f=0$; (2) sub-harmonic for $\mathcal{L}$ if $\mathcal{L}f\geq 0$; (3) super-harmonic for $\mathcal{L}$ if $\mathcal{L}f\leq 0$.

- Let f be the generator of a Markov chain $(X_n)_{n\geq 0}$. Then f is (sub-/super-)harmonic $\Leftrightarrow$ $f(X_n)_{n\geq 0}$ is a (sub-/super-) martingale.

- f is (sub-/super-)harmonic on $D\subset S$ if $\mathcal{L}f\geq/\leq/=0$ on D. Let $\tau=\inf\{k\geq 0:X_k\in D^c\}$, then $(f(X_{n\wedge\tau}))_{n\geq 0}$ is a (sub-/super)martingale.

- Maximum principle: Let $(X_n)_{n\geq 0}$ be a Markov chain and $D\subset S$ s.t. the stopping time $\tau=\inf\{k\geq 0,X_k\in D^c\}$ is a.s. finite. If f is bounded and sub-harmonic on D, then $\sup_{x\in D}f(x)\leq\sup_{x\in D^c}f(x)$.

  Proof f *is sub-harmonic implies* $(f(X_{n\wedge\tau}))$ *is a sub-martingale, hence for* $x\in D$ *we have* $f(x)\leq\mathbb{E}^x f(X_{n\wedge\tau}|)\to\mathbb{E}^x(f(X_\tau))\leq\sup_{x\in D^c}f(x)$. □

- $A\subset S,\tau_A=\sup\{k\geq 0:X_k\in A\}$. (1) $u(x)=\mathbb{P}^x(\tau_A<+\infty)\Rightarrow\begin{cases}\mathcal{L}u=0 & \text{on } A^c\\ u=1 & \text{on } A\end{cases}$. (2) $u(x)=\mathbb{P}(\tau_A<\tau_B)\Rightarrow$ $\begin{cases}\mathcal{L}u=0 & \text{on } (A\cup B)^c\\ u=1 & \text{on } A\\ u=0 & \text{on } B.\end{cases}$. (3) $u(x)=\mathbb{E}^x[\tau_A]\Rightarrow\begin{cases}\mathcal{L}u=-1 & \text{on } A^c\\ u=0 & \text{on } A\end{cases}$.

- Any nonnegative solution v to $\begin{cases}\mathcal{L}v=0 & \text{on } A^c\\ v=1 & \text{on } A\end{cases}$ satisfies $v\geq u$. Furthermore, if $u\equiv 1$, then $\exists 1$ bounded solution to $\begin{cases}\mathcal{L}v=0 & \text{on } A^c\\ v=f & \text{on } A\end{cases}$ with $v(x)=\mathbb{E}^x(f(X_{\tau_A}))$.

  Proof *Let* $v(x)$ *be a non-negative solution, then* $v(X_{n\wedge\tau_A})_{n\geq 0}$ *is a martingale.* $v(x)=\mathbb{E}^x v(X_{n\wedge\tau_A})=\mathbb{E}^x v(X_{n\wedge\tau_A})1_{\tau_A<\infty}+\mathbb{E}^x v(X_{n\wedge\tau_A})1_{\tau_A=\infty}\geq\mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A<\infty}$. *Let* $n\to\infty$ *and by Fatou's lemma, we have* $v(x)\geq\mathbb{E}^x v(X_{\tau_A})1_{\tau_A<\infty}=\mathbb{P}^x(\tau_A<\infty)=u(x)$. *If* $u(x)\equiv 1$ *and* $v(x)$ *is bounded, then by bounded convergence theorem,* $v(x)=\mathbb{E}^x v(X_{n\wedge\tau_A})\to\mathbb{E}^x v(X_{\tau_A})=\mathbb{E}^x f(X_{\tau_A})$. □

- Doob's h-transform: Let h be nonnegative, harmonic with $h(x_0)=1$ for some $x_0\in S$. Then $(h(X_n))_{n\geq 0}$ is a martingale with $\mathbb{E}^{\mathbb{P}^{x_0}}(h(X_n))=1$. Then $\exists 1$ measure $\mathbb{Q}^h$ on $\mathscr{F}_\infty$ s.t. $\frac{\mathrm{d}\mathbb{Q}^h}{\mathrm{d}\mathbb{P}^{x_0}|_{\mathscr{F}_n}}=h(X_n),\forall n\geq 0$. $\mathbb{Q}^h(X_0=x_0)=1$, $(X_n)_{n\geq 0}$ never visits the set $D=\{x:h(x)=0\}$. Under $\mathbb{Q}^h$, $(X_n)_{n\geq 0}$ is again a Markov chain on $S\backslash D$ with transition probability $q(x,y)=\frac{p(x,y)h(y)}{h(x)}$ (or equivalently, $(\mathcal{L}^h f)(x)=\frac{1}{h(x)}(\mathcal{L}(hf))(x)$).

Proof *The first two statements are obvious. Then by definition, we have* $\mathbb{Q}(X_{n+1} = y|\mathscr{F}_n) = \frac{\mathbb{Q}(X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0)}{\mathbb{Q}(X_n=x_n,\cdots,X_0=x_0)} = \frac{\int_{\{X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0\}} h(X_{n+1})d\mathbb{P}^{x_0}}{\int_{\{X_n=x_n,\cdots,X_0=x_0\}} h(X_n)d\mathbb{P}^{x_0}} = \frac{h(y)\mathbb{P}^{x_0}(X_{n+1}=y,X_n=x_n,\cdots,X_0=x_0)}{h(x_n)\mathbb{P}^{x_0}(X_n=x_n,\cdots,X_0=x_0)} = \frac{h(y)p(x_n,y)}{h(x_n)}$*. Next we show* $M_n^f := f(X_n) - f(X_0) - \sum_{k=0}^{n-1}(\mathcal{L}^h f)(X_k)$ *is a* $\mathbb{Q}$*-martingale for any bounded* f*. Let* $Z_n = \mathbb{E}^{\mathbb{Q}} f(X_{n+1})|\mathscr{F}_n$*.* $\forall A \in \mathscr{F}_n$*,* $\int_A Z_n h(X_n)d\mathbb{P}^{x_0} = \int_A Z_n d\mathbb{Q} = \int_A f(X_{n+1})d\mathbb{Q} = \int_A f(X_{n+1})h(X_{n+1})d\mathbb{P}^{x_0} = \mathbb{E}^{\mathbb{P}^{x_0}}[\mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1})1_A|\mathscr{F}_n)] = \mathbb{E}^{\mathbb{P}^{x_0}}[1_A\mathbb{E}^{\mathbb{P}^{x_0}}(f(X_{n+1})h(X_{n+1})|\mathscr{F}_n)] = \int_A \mathcal{P}(hf)(X_n)d\mathbb{P}^{x_0}$*. Thus* $Z_n = \frac{\mathcal{P}(hf)(X_n)}{h(X_n)}$ *only depends on* X_n*, i.e.* $(X_n)_{n\geq 0}$ *is a MC on* $\mathbb{Q}$ *with generator* $\mathcal{L}^h$*.* □

- An irreducible Markov chain $(X_n)_{n\geq 0}$ (1) is transient if $\exists x$ and $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty|X_0 = x) < 1$; (2) is recurrent if $\exists$ a finite set $A \subset S$ s.t. $\mathbb{P}(\tau_A < \infty) = 1$ for all $x \in S$. (3) is positive recurrent if $\exists$ a finite set $A \subset S$ s.t. $\mathbb{E}(\tau_A) < \infty$ for all $x \in S$.

- Foster-Lyapunov criterion: An irreducible MC on a countable state space S (1) is transient iff $\exists v : S \to \mathbb{R}^+$ and $A \subset S$ non-empty s.t. $\mathcal{L}v \leq 0$ on A^c and $v(x) < \inf_{y\in A} v(y)$ for some $x \in A^c$; (2) is recurrent iff $\exists v : S \to \mathbb{R}^+$ s.t. $\mathcal{L}v \leq 0$ on A^c where A is a finite set and $\{x : v(x) \leq N\}$ is finite for every N; (3) is positive recurrent iff $\exists v : S \to \mathbb{R}^+$, $A \subset S$ finite, $\exists \epsilon > 0$ s.t. $\mathcal{L}v \leq -\epsilon$ on A^c and $\sum_{y\in S} p(x,y)V(y) < +\infty$ for all $x \in A$.

  Proof *(1)* $v(X_{n\wedge\tau_A})_{n\geq 0}$ *is a super-martingale, hence* $v(x) \geq \mathbb{E}v(X_{n\wedge\tau_A} \geq \mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A<\infty})$*. Let* $n \to \infty$ *we know* $v(x) \geq \mathbb{E}v(X_{\tau_A}1_{\tau_A<\infty}) \geq (\inf_{y\in A} v(y))\mathbb{P}^x(\tau_A < \infty) \Rightarrow \mathbb{P}^x(\tau_A < \infty) < \frac{v(x)}{\inf_{y\in A} v(y)} < 1$*. (2) On* $\{\tau_A = \infty\}$, $\limsup_{n\to\infty} v(X_{n\wedge\tau_A}) = +\infty$ *a.s. Since* $(v(X_{n\wedge\tau_A}))_{n\geq 0}$ *is a nonnegative super-martingale, hence converges a.s., therefore* $\lim_{n\to\infty} v(X_{n\wedge\tau_A}) = +\infty$ *a.s. Note that* $v(x) \geq \mathbb{E}v(X_{n\wedge\tau_A})1_{\tau_A=\infty}$*. Since LHS is a finite number, we have* $\mathbb{P}^x(\tau_A = \infty) = 0$*. (3)* $\mathbb{E}v(X_{n\wedge\tau_A})|\mathscr{F}_{n-1} \leq v(X_{(n-1)\wedge\tau_A}) - \epsilon 1_{\tau_A\geq n}$*. Taking expectation on the both sides,* $\mathbb{E}v(X_{n\wedge\tau_A}) \leq \mathbb{E}v(X_{(n-1)\wedge\tau_A}) - \epsilon\mathbb{E}^x 1_{\tau_A\geq n} \leq \cdots \leq v(x) - \epsilon\sum_{k=1}^n \mathbb{P}^x(\tau_A \geq k) \Rightarrow \mathbb{E}^x\tau_A = \sum_{k=1}^\infty \mathbb{P}^x(\tau_A \geq k) \leq \frac{v(x)}{\epsilon} < \infty$*.* □

- e.g. $h(x) = \frac{\mathbb{P}^x(\tau_A<\tau_B)}{\mathbb{P}^{x_0}(\tau_A<\tau_B)}$ is harmonic on $(A\cup B)^c$ with $h(x_0) = 1(x_0 \in (A\cup B)^c)$. Then $\forall x, y \in (A\cup B)^c, q(x,y) = \frac{h(y)p(x,y)}{h(x)} = \frac{\mathbb{P}^y(\tau_A<\tau_B)p(x,y)}{\mathbb{P}^x(\tau_A<\tau_B)} = \frac{\mathbb{P}^x(X_1=y,\tau_A<\tau_B)}{\mathbb{P}^x(\tau_A<\tau_B)} = \mathbb{P}^x(X_1 = y|\tau_A < \tau_B)$.

- e.g. $\mathbb{P}$ is simple symmetric random walk on $\mathbb{Z}$ starting from $X_0 = 0$. Question: what is the law of $(X_n)_{n\geq 0}$ conditioned on $X_n \geq 0$ for all n? Let $\tau_k = \inf\{n \geq 0, X_n = k\}$. On $\{\tau_N < \tau_{-1}\}$, $\frac{h(y)}{h(x)} = \frac{\mathbb{P}^y(\tau_N<\tau_{-1})}{\mathbb{P}^x(\tau_N<\tau_{-1})} = \frac{y+1}{x+1}$. Thus $q_N(x,y) = \frac{1}{2}\frac{y+1}{x+1}, |x-y| = 1, x \in \{0,\cdots,N-1\} \Rightarrow q(x,y) = \frac{1}{2}\frac{y+1}{x+1}, x \geq 0, |x-y| = 1$.

3 Ergodic Theorem

-