High-Dimensional Probability

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Contents

Reference High-Dimensional Probability: An Introduction with Applications in Data Science (Roman Vershynin)

0 Appetizer

- Convex combination: For $z_1, z_2, \cdots, z_m \in \mathbb{R}^n$, the form of $\sum_{i=1}^m \lambda_i z_i$ with $\lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1$. Convex hull of $T \subset \mathbb{R}^n$: $\text{conv}(T) = \{\text{convex combinations of } z_1, \cdots, z_m \in T, m \in \mathbb{N}\}$.

- Caratheodory's theorem: Every point in the convex hull of a set $T \subset \mathbb{R}^n$ can be expressed as a convex combination of at most $n+1$ points from T.

- Approximate Caratheodory's theorem: Consider $T \subset \mathbb{R}^n$, $\text{diam}(T) = \sup\{\|s-t\|_2, s, t \in T\} < 1$. Then for any $x \in \text{conv}(T)$ and any k, one can find points $x_1, x_2, \cdots, x_k \in T$ such that $\|x - \frac{1}{k}\sum_{i=1}^k x_i\|_2 \leq \frac{1}{\sqrt{k}}$ (repetition is allowed).

  Proof WLOG assume $\|t\|_2 \leq 1, \forall t \in T$. Fix $x \in \text{conv}(T), x = \sum_{i=1}^m \lambda_i z_i, z_i \in T$. Define $Z, \mathbb{P}(Z = z_i) = \lambda_i, \mathbb{E}Z = x$. Consider i.i.d. $Z_1, Z_2, \cdots$ of Z, $\frac{1}{n}\sum_{j=1}^n Z_j \to x$ a.s. $n \to +\infty$. $\mathbb{E}\|x - \frac{1}{k}\sum_{j=1}^k Z_j\|_2^2 = \frac{1}{k^2}\sum_{j=1}^k \mathbb{E}\|Z_j - x\|_2^2 = \frac{1}{k^2}\sum_{j=1}^k (\mathbb{E}\|Z_j\|^2 - \|\mathbb{E}Z_j\|_2^2) \leq \frac{1}{k} \Rightarrow \exists$ a realization of $Z_1, \cdots, Z_k$ such that $\|x - \frac{1}{k}\sum_{j=1}^k Z_j\|_2 \leq \frac{1}{\sqrt{k}}$. □

- Corollary (Covering polytopes by balls): P is a polytope in $\mathbb{R}^n$ with N vertices, $\text{diam}(P) \leq 1$. Then P can be covered by at most $N^{\lfloor 1/\epsilon^2 \rfloor}$ Euclidean balls of radii $\epsilon > 0$.

1 Preliminaries on random variables

- Jensen's inequality: convex ϕ, $\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X)$. $\Rightarrow \|X\|_{L^p} \leq \|X\|_{L^q}$ for $p \leq q$.

- Minkowski inequality: $p \geq 1, \|X+Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p}$.

- Cauchy-Schwarz inequality: $\mathbb{E}|XY| \leq \|X\|_{L^2}\|Y\|_{L^2}$.

- Holder inequality: $p, q \in (1, +\infty), \frac{1}{p} + \frac{1}{q} = 1$ or $p = 1, q = \infty$, $\mathbb{E}\|XY\| \leq \|X\|_{L^p}\|Y\|_{L^q}$.

- $X \geq 0$, then $\mathbb{E}X = \int_0^{+\infty} \mathbb{P}(X > t)\mathrm{d}t$.

- Markov inequality: $X \geq 0, t > 0, \mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}$.

- LLN: $X_1, \cdots, X_n, \cdots$ i.i.d., $\mathbb{E}X_i = \mu, \text{Var}(X_i) = \sigma^2, S_N = X_1 + X_2 + \cdots + X_N$. Then: (WLLN) $\mathbb{P}(|\frac{S_N}{N} - \mu| > \epsilon) \to 0, \forall \epsilon > 0$; (SLLN) $\mathbb{P}(\frac{S_N}{N} \to \mu, N \to +\infty) = 1$.

- CLT: $Z_N = \frac{S_N - \mathbb{E}S_N}{\sqrt{\text{Var}(S_N)}} \xrightarrow{d} \mathcal{N}(0,1)$.

- $X_{N,i}, 1 \leq i \leq N$ independent $\text{Ber}(p_{N,i}), \max_{i \leq N} p_{N,i} \to 0, S_N = \sum_{i=1}^N X_{N,i}, \mathbb{E}S_N \to \lambda < +\infty$. Then $S_N \xrightarrow{d} \text{Poisson}(\lambda)$.

2 Concentration of sums of independent random variables

- Question: N times, $\mathbb{P}(\text{head} \geq \frac{3}{4}N) = ?$ Let S_N be the number of heads, $\mathbb{E}S_N = \frac{N}{2}, \text{Var}(S_N) = \frac{N}{4}$. (1) Chebyshev's inequality: $\mathbb{P}(S_N \geq \frac{3}{4}N) \leq \mathbb{P}(|S_N - \frac{N}{2}| \geq \frac{N}{4}) \leq \frac{4}{N}$; (2) $Z_N = \frac{S_N - \frac{N}{2}}{\sqrt{N/4}}$, expect: $\mathbb{P}(Z_N \geq \sqrt{N/4}) \approx \mathbb{P}(g \geq \sqrt{N/4}) \leq \frac{1}{\sqrt{2\pi}}e^{-N/8}$ where $g \sim \mathcal{N}(0,1)$.

- For all $t > 0$, $(\frac{1}{t} - \frac{1}{t^3})\frac{1}{\sqrt{2\pi}}e^{-t^2/2} \leq \mathbb{P}(g \sim \mathcal{N}(0,1) \geq t) \leq \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

- Berry-Esseen bound: $|\mathbb{P}(Z_N \geq t) - \mathbb{P}(g \geq t)| \leq \frac{\rho}{\sqrt{N}}$ where $\rho = \mathbb{E}|X_1 - \mu|^3/\sigma^3$. And in general, no improvement since $\mathbb{P}(S_N = \frac{N}{2}) \sim \frac{1}{\sqrt{N}} \Rightarrow \mathbb{P}(Z_N = 0) \sim \frac{1}{\sqrt{N}}$ but $\mathbb{P}(g = 0) = 0$.

- Hoeffding's inequality: $X_1, \cdots, X_N$ i.i.d. symmetric Bernoulli ($\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$), $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$. Then $\forall t \geq 0, \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) \leq e^{-t^2/2\|a\|_2^2}, \mathbb{P}(|\sum_{i=1}^N a_i X_i| \geq t) \leq 2e^{-t^2/2\|a\|_2^2}$.

  Proof WLOG, $\|a\|_2^2 = 1$. For $\lambda > 0, \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) = \mathbb{P}(e^{\lambda \sum a_i X_i} \geq e^{\lambda t}) \leq e^{-\lambda t}\mathbb{E}e^{\lambda \sum_{i=1}^N a_i X_i} = e^{-\lambda t}\prod_{i=1}^N \mathbb{E}e^{\lambda a_i X_i} = e^{-\lambda t}\prod_{i=1}^N \cosh(\lambda a_i) \leq e^{-\lambda t}\prod_{i=1}^N e^{\lambda^2 a_i^2/2} = e^{-\lambda t + \frac{\lambda^2}{2}} \Rightarrow \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) \leq \inf_{\lambda \geq 0} e^{-\lambda t + \frac{\lambda^2}{2}} = e^{-\frac{t^2}{2}} (\lambda = t)$. □

- Bounded r.v.s: $X_1, \cdots, X_N$ independent, $X_i \in [m_i, M_i]$. Then $\forall t \geq 0, \mathbb{P}(\sum_{i=1}^N (X_i - \mathbb{E}X_i) \geq t) \leq e^{-\frac{2t^2}{\sum_{i=1}^N (M_i - m_i)^2}}$.

- Chernoff's inequality: $X_i \sim \text{Ber}(p_i)$ independent, $S_N = \sum_{i=1}^N X_i, \mu = \mathbb{E}S_N \Rightarrow \forall t > \mu, \mathbb{P}(S_N \geq t) \leq e^{-\mu}(\frac{e\mu}{t})^t$.

  Proof $\mathbb{P}(S_N \geq t) \leq e^{-\lambda t} \prod_{i=1}^N \mathbb{E}e^{\lambda X_i}$. $\mathbb{E}e^{\lambda X_i} = e^\lambda p_i + (1 - p_i) = 1 + (e^\lambda - 1)p_i \leq e^{(e^\lambda - 1)p_i} \Rightarrow \mathbb{P}(S_N \geq t) \leq e^{-\lambda t} e^{(e^\lambda - 1)\mu}$. Take $\lambda^* = \log(t/\mu)$. □

- $d = (n-1)p$ is the expected degree. There is an absolute constant C s.t. for $G(n,p)$, $d \geq C \log n$. Then with high prob (for example 0.9), all vertices of G have degrees between $0.9d$ and $1.1d$.

  Proof Ex 2.3.5 $\Rightarrow \mathbb{P}(|d_i - d| \geq \delta d) \leq 2e^{-c\delta^2 d}$. Union bound: $\mathbb{P}(\exists i, |d_i - d| \geq \delta d) \leq n \cdot 2e^{-c\delta^2 d} \leq n \cdot 2 \cdots n^{-Cc\delta^2} = 2n^{1-Cc\delta^2} \leq 1 - p^*$ (let $Cc\delta^2 > 1$). □

- Sub-gaussian properties: The following are equivalent: (i) $\mathbb{P}(|X| \geq t) \leq 2e^{-t^2/k_1^2}$ for all $t \geq 0$; (ii) $\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p} \leq k_2\sqrt{p}$ for all $p \geq 1$; (iii) $\mathbb{E}e^{\lambda^2 X^2} \leq e^{k_3^2 \lambda^2}$ for all λ s.t. $|\lambda| \leq \frac{1}{k_3}$; (iv) $\mathbb{E}e^{X^2/k_4^2} \leq 2$; (v) $\mathbb{E}e^{\lambda X} \leq e^{k_5^2 \lambda^2}$, for all $\lambda \in \mathbb{R}$ (if $\mathbb{E}X = 0$).

  Proof (i) $\Rightarrow$ (ii): WLOG $k_1 = 1$. $\mathbb{E}|X|^p = \int_0^{+\infty} \mathbb{P}(|X| \geq t) pt^{p-1} \text{d}t \leq \int_0^{+\infty} 2e^{-t^2} pt^{p-1} \text{d}t = p\Gamma(\frac{p}{2}) \overset{\Gamma(x) \leq 3x^x \text{ for } x \geq \frac{1}{2}}{\leq} 3p(\frac{p}{2})^{p/2} \Rightarrow \|X\|_p \leq \frac{1}{\sqrt{2}}(3p)^{1/p} p^{1/2} \leq 3\sqrt{p}$.

  (ii) $\Rightarrow$ (iii): WLOG $k_2 = 1$. $\mathbb{E}e^{\lambda^2 X^2} = \mathbb{E}[1 + \sum_{p=1}^{+\infty} \frac{(\lambda^2 X^2)^p}{p!}]$. $\mathbb{E}|X|^{2p} \leq (2p)^p, p! \geq (\frac{p}{e})^p \Rightarrow \mathbb{E}e^{\lambda^2 X^2} \leq 1 + \sum_{p=1}^{+\infty} \frac{(2\lambda^2 p)^p}{(\frac{p}{e})^p} = \frac{1}{1 - 2e\lambda^2}$ (if $2e\lambda^2 < 1$) $\overset{\frac{1}{1-x} \leq e^{2x} \text{ for } x \in [0, \frac{1}{2}]}{\leq} e^{4e\lambda^2}$ (if $2e\lambda^2 \leq 1/2 \Leftrightarrow \lambda \leq \frac{1}{2\sqrt{e}}$).

  (iii) $\Rightarrow$ (iv): trivial.

  (iv) $\Rightarrow$ (i): $\mathbb{P}(|X| \geq t) = \mathbb{P}(e^{X^2} \leq e^{t^2}) \leq e^{-t^2} \mathbb{E}e^{X^2} \leq 2e^{-t^2}$.

  (iii) $\Rightarrow$ (v): WLOG $k_3 = 1$. If $|\lambda| \leq 1$, then $\mathbb{E}e^{\lambda X} \leq \mathbb{E}(\lambda X + e^{\lambda^2 X^2}) = \mathbb{E}e^{\lambda^2 X^2} \leq e^{\lambda^2}$. If $|\lambda| \geq 1$, then $\mathbb{E}e^{\lambda X} \leq e^{\frac{\lambda^2}{2}} \mathbb{E}e^{\frac{X^2}{2}} e^{\frac{\lambda^2}{2}} e^{\frac{1}{2}} \leq e^{\lambda^2}$.

  (v) $\Rightarrow$ (i): mimic the proof of (iv) $\Rightarrow$ (i). □

- Sub-gaussian r.v.: satisfy the above sub-gaussian properties. $\|X\|_{\psi_2} = \inf\{t > 0 : \mathbb{E}e^{X^2/t^2} \leq 2\}$. Thus $\mathbb{P}(|X| \geq t) \leq 2e^{-ct^2/\|X\|_{\psi_2}}$; $\|X\|_{L^p} \leq C\|X\|_{\psi_2}\sqrt{p}$; if $\mathbb{E}X = 0$ then $\mathbb{E}e^{\lambda X} \leq e^{C\lambda^2 \|X\|_{\psi_2}^2}$.

- Let $X_1, \cdots, X_N$ be independent and mean zero sub-gaussian, then $\|\sum_{i=1}^N X_i\|_{\psi_2}^2 \leq C \cdot \sum_{i=1}^N \|X_i\|_{\psi_2}^2$.

  Proof $\forall \lambda \in \mathbb{R}, \mathbb{E}e^{\lambda \sum_{i=1}^N X_i} = \prod_{i=1}^N \mathbb{E}e^{\lambda X_i} \leq \prod_{i=1}^N e^{c\lambda^2 \|X_i\|_{\psi_2}^2} = e^{c\lambda^2 \sum_{i=1}^n \|X_i\|_{\psi_2}^2}$ □

- Centering: X is sub-gaussian $\Rightarrow X - \mathbb{E}X$ is sub-gaussian and $\|X - \mathbb{E}X\|_{\psi_2} \leq C\|X\|_{\psi_2}$.

  Proof $\|\mathbb{E}X\|_{\psi_2} \leq C_1|\mathbb{E}X| \leq C_1\mathbb{E}|X| = C_1\|X\|_{L^1} \leq C_1 C_2 \|X\|_{\psi_2}$. □

- Sub-exponential properties: The following are equivalent: (1) $\mathbb{P}(|X| \geq t) \leq 2e^{-t/k_1}, \forall t \geq 0$; (2) $\|X\|_{L^p} \leq k_2 p, p \geq 1$; (3) $\mathbb{E}e^{\lambda|X|} \leq e^{k_3\lambda}$ for all $0 \leq \lambda \leq \frac{1}{k_3}$; (4) $\mathbb{E}e^{|X|/k_4} \leq 2$; (5) if $\mathbb{E}X = 0, \mathbb{E}e^{\lambda X} \leq e^{k_5^2 \lambda^2}$ for $|\lambda| \leq \frac{1}{k_5}$.

  Proof (2) $\Rightarrow$ (5): $k_2 = 1, \mathbb{E}e^{\lambda X} = 1 + \sum_{p=2}^{+\infty} \frac{\lambda^p \mathbb{E}X^p}{p!} \leq 1 + \sum_{p=2}^{+\infty} \frac{\lambda^p p^p}{(p/e)^p} = 1 + \frac{(e\lambda)^2}{1 - e\lambda}(|e\lambda| < 1)$. If $|e\lambda| \leq \frac{1}{2}, 1 + \frac{(e\lambda)^2}{1 - e\lambda} \leq 1 + 2e^2\lambda^2 \leq e^{2e^2\lambda^2} \leq e^{4e^2\lambda^2}$, i.e. $k_5 = 2e$.

  (5) $\Rightarrow$ (1): $k_5 = 1, |x|^p \leq p^p(e^x + e^{-x}) \Rightarrow \mathbb{E}|X|^p \leq p^p(\mathbb{E}e^X + \mathbb{E}e^{-X}) \leq 2ep^p$. □

- $\|X\|_{\psi_1} = \inf\{t > 0 : \mathbb{E}e^{|X|/t} \leq 2\}$. X is sub-gaussian $\Leftrightarrow X^2$ is sub-exponential. $\|X^2\|_{\psi_1} = \|X\|_{\psi_2}^2$.

- X, Y are sub-gaussian $\Rightarrow XY$ is sub-exponential and $\|XY\|_{\psi_1} \leq \|X\|_{\psi_2}\|Y\|_{\psi_2}$.

  Proof WLOG $\|X\|_{\psi_2} = \|Y\|_{\psi_2} = 1$. $\mathbb{E}e^{XY} \leq \mathbb{E}e^{\frac{X^2+Y^2}{2}} = \mathbb{E}[e^{\frac{X^2}{2} + \frac{Y^2}{2}}] \leq \frac{1}{2}(\mathbb{E}e^{X^2} + \mathbb{E}e^{Y^2}) = 2$. □

- Orlicz function/space: $\psi : [0, +\infty) \to [0, +\infty)$, convex, increasing, $\psi(0) = 0$, $\psi(x) \to +\infty, x \to +\infty$. $\|X\|_\psi := \inf\{t > 0 : \mathbb{E}\psi(|X|/t) \leq 1\}$. $L_\psi := \{X : \|X\|_\psi < +\infty\}$ is Banach space. Examples: (1) $L_p : \psi(x) = x^p, p \geq 1$; (2) $L_{\psi_2} : \psi_2(x) = e^{x^2} - 1, L_\infty \subset L_{\psi_2} \subset L_p$.

- Bernstein's inequality: $X_1, \cdots, X_N$ independent, mean zero and sub-exponential. Then for $t \geq 0, \mathbb{P}(|\sum_{i=1}^N X_i| \geq t) \leq 2e^{-c\min(\frac{t^2}{\sum_{i=1}^N \|X_i\|_{\psi_1}^2}, \frac{t}{\max_i \|X_i\|_{\psi_1}})}$.

Proof $S=\sum_{i=1}^N X_i$. $\mathbb{P}(S\geq t)\leq e^{-\lambda t}\prod_{i=1}^N \mathbb{E}e^{\lambda X_i}$. $\mathbb{E}e^{\lambda X_i}\leq e^{c\lambda^2\|X_i\|_{\psi_1}^2}$ if $|\lambda|\leq \frac{c}{\max\|X_i\|_{\psi_1}}$. Then $\mathbb{P}(S\geq t)\leq e^{-\lambda t+c\lambda^2\sigma^2}$ where $\sigma^2:=\sum_{i=1}^N\|X_i\|_{\psi_1}^2$. The following is to find the minimum of a quadratic function with the restriction $|\lambda|\leq\frac{c}{\max\|X_i\|_{\psi_1}}$. □

- Corollary 1: $\mathbb{P}(|\sum_{i=1}^N a_iX_i|\geq t)\leq 2e^{-c\min(\frac{t^2}{K^2\|a\|_2^2},\frac{t}{K\|a\|_\infty})}$ where $K=\max_i\|X_i\|_{\psi_1}$.
- Corollary 2: $|X_i|\leq K$, then $\mathbb{P}(|\sum_{i=1}^N X_i|\geq t)\leq 2\exp(-\frac{t^2/2}{\sigma^2+Kt/3})$ where $\sigma^2=\sum_{i=1}^N\mathbb{E}X_i^2$.

3 Random vectors in high dimensions

- $X\in\mathbb{R}^n$, independent sub-gaussian coordinate X_i, $\mathbb{E}X_i^2=1$. Then $\|\|X\|_2-\sqrt{n}\|_{\psi_2}\leq CK^2, K=\max_i\|X_i\|_{\psi_2}$.

  Proof $\mathbb{E}X_i^2=1\Rightarrow K\geq 1$. $\|X_i^2-1\|_{\psi_1}\leq C\|X_i^2\|_{\psi_1}=C\|X_i\|_{\psi_2}^2\leq CK^2$. Bernstein's inequality: $\mathbb{P}(|\frac{1}{n}\|X\|_2^2-1|\geq u)=\mathbb{P}(|\frac{1}{n}\sum_{i=1}^n(X_i^2-1)|\geq u)\leq 2e^{-cn\min(\frac{u^2}{K^4},\frac{u}{K^2})}\leq 2e^{-\frac{cn}{K^4}\min(u^2,u)}$. For any $\delta>0$, $\mathbb{P}(|\frac{1}{\sqrt{n}}\|X\|_2-1|\geq\delta)\leq\mathbb{P}(|\frac{1}{n}\|X\|_2^2-1|\geq\max(\delta,\delta^2))\leq 2e^{-\frac{cn}{K^4}\delta^2}\Rightarrow\mathbb{P}(|\|X\|_2-\sqrt{n}|\geq t)\leq 2e^{-ct^2/K^4}$. □

- Isotropy: $\Sigma(X)=\mathbb{E}XX^T=I$. If $\Sigma\neq I_n$, then let $Z=\Sigma^{-1/2}X$. X is isotropic $\Leftrightarrow\mathbb{E}\langle X,x\rangle^2=\|x\|_2^2$ for any $x\in\mathbb{R}^n$.

  Proof $\mathbb{E}\langle X,x\rangle^2=\mathbb{E}(x^TXX^Tx)=x^T(\mathbb{E}XX^T)x$. $\|x\|_2^2=x^TI_nx$. $\Rightarrow\mathbb{E}XX^T=I_n$. □

- X is isotropic $\Rightarrow\mathbb{E}\|X\|_2^2=n$. If X,Y are independent and isotropic $\Rightarrow\mathbb{E}\langle X,Y\rangle^2=n$.

  Proof $\mathbb{E}\|X\|_2^2=\mathbb{E}(X^TX)=\mathbb{E}(\mathrm{tr}(X^TX))=\mathrm{tr}(\mathbb{E}XX^T)=n$.

  $\mathbb{E}\langle X,Y\rangle^2=\mathbb{E}(X^TYY^TX)=\mathbb{E}(\mathrm{tr}(X^TYY^TX))=\mathbb{E}(\mathrm{tr}(XX^TYY^T))=\mathrm{tr}((\mathbb{E}XX^T)(\mathbb{E}YY^T))=n$. □

- Examples: $X\sim U(\sqrt{n}\mathbb{S}^{n-1}), X\sim U(\{-1,1\}^n), X=(X_1,\cdots,X_n)$ i.i.d., $\mathbb{E}X_i=0, \mathrm{Var}(X_i)=1$ are all isotropic.
- $g\sim\mathcal{N}(0,I_n)$, then $\mathbb{P}(|\|g\|_2-\sqrt{n}|\geq t)\leq 2e^{-ct^2}$.
- Frame: $\{u_i\}_{i=1}^N\subset\mathbb{R}^n$, Approximate Parseval's identity: $A\|x\|_2^2\leq\sum_{i=1}^N\langle u_i,x\rangle^2\leq B\|x\|_2^2$. A,B: frame bounds. $A=B$: tight frame($\Leftrightarrow\sum_{i=1}^N u_iu_i^T=AI_n$) and in this case, $\sum_{i=1}^N\langle u_i,x\rangle u_i=Ax$.
- (a) Tight frame $\{u_i\}_{i=1}^N, A=B, X=\mathrm{Unif}\{u_i, i=1,2,\cdots,N\}$, then $(\frac{N}{A})^{1/2}X$ is isotropic. (b) X is isotropic, $\mathbb{P}(X=x_i)=p_i, i=1,2,\cdots,N$. Then $u_i=\sqrt{p_i}x_i$ form a tight frame with $A=B=1$.
- Isotropic convex sets: $X\sim\mathrm{Unif}(K), K\subset\mathbb{R}^n$ convex, bounded, non-empty interior (convex body). Assume $\mathbb{E}X=0,\Sigma=\mathrm{Cov}(X)$. Then $Z=\Sigma^{-1/2}X$ is isotropic and $Z\sim\mathrm{Unif}(\Sigma^{-1/2}K)$.
- $X\in\mathbb{R}^n$ is sub-gaussian $\Leftrightarrow\forall X\in\mathbb{R}^n,\langle X,x\rangle$ are sub-gaussian. $\|X\|_{\psi_2}=\sup_{x\in\mathbb{S}^{n-1}}\|\langle X,x\rangle\|_{\psi_2}$.
- $X=(X_1,\cdots,X_n)$ independent, mean zero, sub-gaussian coordinate. Then X is sub-gaussian with $\|X\|_{\psi_2}\leq C\max_{i\leq n}\|X_i\|_{\psi_2}$.

  Proof $\|\langle X,x\rangle\|_{\psi_2}^2=\|\sum X_ix_i\|_{\psi_2}^2\leq C\sum_{i=1}^n x_i^2\|X_i\|_{\psi_2}^2\leq C\max_{i\leq n}\|X_i\|_{\psi_2}^2$. □

- Gaussian dist: $X\sim\mathcal{N}(0,I_n),\|X\|_{\psi_2}\leq C$.
- Discrete dist: $X\sim\mathrm{Unif}\{\sqrt{n}e_i, i=1,2,\cdots,n\},\|X\|_{\psi_2}\simeq\sqrt{\frac{n}{\log n}}$.
- Uniform dist: $X\sim\mathrm{Unif}\{\sqrt{n}\mathbb{S}^{n-1}\},\|X\|_{\psi_2}\leq C$.

  Proof $g\sim\mathcal{N}(0,I_n), X\overset{d}{=}\frac{\sqrt{n}g}{\|g\|_2}$. $p(t):=\mathbb{P}(|X_1|\geq t)=\mathbb{P}(\frac{|g|}{\|g\|_2}\geq\frac{t}{\sqrt{n}})$. $\|\|g\|_2-\sqrt{n}\|_{\psi_2}\leq C\Rightarrow\mathbb{P}(\|g\|_2\leq\frac{\sqrt{n}}{2})\leq 2e^{-cn}$. Need to show that all one-dimensional marginals $\langle X,x\rangle$ are sub-gaussian. By rotation invariance, we may assume that $x=(1,0,\cdots,0)$. Let $\mathcal{E}=\{\|g\|_2\geq\frac{\sqrt{n}}{2}\}\Rightarrow p(t)\leq\mathbb{P}(\frac{|g|}{\|g\|_2}\geq\frac{t}{\sqrt{n}},\mathcal{E})+\mathbb{P}(\mathcal{E})\leq\mathbb{P}(|g_1|\geq\frac{t}{2},\mathcal{E})+2e^{-cn}\leq 2e^{-t^2/8}+2e^{-cn}\overset{t\leq\sqrt{n}}{\leq}4e^{-ct^2}$. □

- Grothendieck's inequality: $A=\{a_{ij}\}_{m\times n}$ of real numbers. Assume $\forall x_i,y_i\in\{-1,1\}$, we have $|\sum_{i,j}a_{ij}x_iy_j|\leq 1$. Then for any Hilbert space $\mathscr{H}$, any $u_i,v_j\in\mathscr{H}$ satisfying $\|u_i\|=\|v_j\|=1$, we have $|\sum_{i,j}a_{ij}\langle u_i,v_j\rangle|\leq K$ with $K\leq 1.783$.

Proof (1) Reduction. For any $u_i, v_j \in \mathbb{R}^N$ s.t. $\|u_i\|_2 = \|v_j\|_2 = 1, K_{u,v} := \sum_{i,j} a_{ij}\langle u_i, v_j\rangle, K = \sup_{\|u\|_2=\|v\|_2=1} K_{u,v}$.

(2) Introduce randomness. $g \sim \mathcal{N}(0, I_N), U_i = \langle g, u_i\rangle, V_j = \langle g, v_j\rangle, \mathbb{E}U_iV_j = \langle u_i, v_j\rangle$. $K_{u,v} = \mathbb{E}(\sum_{i,j} a_{ij}U_iV_j) \Rightarrow K_{u,v} \leq R^2$ if $|U_i| \leq R, |V_j| \leq R$.

(3) Truncation. Given $R \geq 1, U_i = U_i^- + U_i^+, U_i^- = U_i 1_{\{|U_i|\leq R\}}, V_j = V_j^- + V_j^+, |U_i^-| \leq R, |V_j^-| \leq R$. $\|U_i^+\|_{L^2}^2 \leq 2(R + \frac{1}{R})\frac{1}{\sqrt{2\pi}}e^{-R^2/2} < \frac{4}{R^2}(R > 1)$.

(4) Breaking up the sum. $\mathbb{E}\sum a_{ij}U_iV_j = \mathbb{E}\sum a_{ij}U_i^-V_j^- + \mathbb{E}\sum a_{ij}U_i^+V_j^- + \mathbb{E}\sum a_{ij}U_i^-V_j^+ + \mathbb{E}\sum a_{ij}U_i^+V_j^+ := S_1 + S_2 + S_3 + S_4$. $S_1 \leq R^2, S_2 \leq K\max\|U_i^+\|_2 \max\|V_j^-\|_2 \leq K\frac{2}{R}, S_3 \leq K\frac{2}{R}, S_4 \leq K\frac{4}{R^2}$.

(5) Putting everything together. $K_{u,v} \leq R^2 + \frac{4K}{R} + \frac{4K}{R^2} \Rightarrow K \leq R^2 + \frac{4K}{R} + \frac{4K}{R^2} \Rightarrow K \leq \frac{R^2}{1-\frac{4}{R}-\frac{4}{R^2}}$. □

- Remark: The assumption can be equivalently stated as $|\sum_{i,j} a_{ij}x_iy_j| \leq \max_i |x_i| \max_j |y_j|$. The conclusion can be equivalently stated as $|\sum_{i,j} a_{ij}\langle u_i, v_j\rangle| \leq K\max_i \|u_i\| \max_j \|v_j\|$.

- Semidefinite programming: $\max\langle A, X\rangle$ s.t. $X \succeq 0, \langle B_i, X\rangle = b_i, A, B_i\ n\times n, b_i$ real number, $\langle A, X\rangle = \mathrm{tr}(A^TX) = \sum_{i,j=1}^n A_{ij}X_{ij}$.

- Semidefinite relaxation: $\max\sum_{i,j=1}^n A_{ij}x_ix_j, x_i = \pm 1, A\ n\times n$ symmetric matrix. Relax to $\max\sum_{i,j=1}^n A_{ij}\langle X_i, X_j\rangle$, $\|X_i\|_2 = 1, X_i \in \mathbb{R}^n$.

- A positive semidefinite, $\mathrm{INT}(A) := \max\sum_{i,j=1}^n A_{ij}x_ix_j, x_i = \pm 1, \mathrm{SDP}(A) := \max\sum_{i,j=1}^n A_{ij}\langle X_i, X_j\rangle, \|X_i\|_2 = 1$. Then $\mathrm{INT}(A) \leq \mathrm{SDP}(A) \leq 2K \cdot \mathrm{INT}(A)$.

- Maximum cut: $G = (V, E)$ finite simple, $V \to V_1 + V_2$, cut number of edges crossing between V_1 and V_2. MAX-CUT(G): NP-hard. Adjacency matrix $A = \{A_{ij}\}_{n\times n}, A_{ij} = \begin{cases} 1, & i \leftrightarrow j \\ 0, \text{otherwise} \end{cases}$. Partition: $X = (x_i)_{n\times 1}, x_i = \pm 1$. $\mathrm{CUT}(G, X) = \frac{1}{4}\sum_{i,j=1}^n A_{ij}(1 - x_ix_j)$. $\text{MAX-CUT}(G) = \frac{1}{4}\max\{\sum_{i,j} A_{ij}(1 - x_ix_j) : x_i = \pm 1\}$.

- 0.5-approximation algorithm: Partition at random, $\mathbb{E}\mathrm{CUT}(G, X) = 0.5|E| \geq 0.5\text{MAX-CUT}(G)$.

- 0.878-approximation algorithm: $\mathrm{SDP}(G) = \frac{1}{4}\max\{\sum_{i,j=1}^n A_{ij}(1 - \langle X_i, X_j\rangle), x_i \in \mathbb{R}^n, \|X_i\|_2 = 1\}$. $X_1, \cdots, X_n \to x_1, \cdots, x_n : g \sim \mathcal{N}(0, I_n), x_i := \mathrm{sgn}(\langle X_i, g\rangle)$. $\mathbb{E}\mathrm{CUT}(G, X) \geq 0.878\mathrm{SDP}(G) \geq 0.878\text{MAX-CUT}(G)$.

  Proof $\mathbb{E}\mathrm{CUT}(G, X) = \frac{1}{4}\sum_{i,j=1}^n A_{ij}(1 - \mathbb{E}x_ix_j)$ and $1 - \mathbb{E}x_ix_j = 1 - \mathbb{E}\mathrm{sgn}\langle g, X_i\rangle\mathrm{sgn}\langle g, X_j\rangle = 1 - \frac{2}{\pi}\arcsin\langle X_i, X_j\rangle \geq 0.878(1 - \langle X_i, X_j\rangle)$. □

- $u, v \in \mathbb{S}^{n-1}, \mathbb{E}\mathrm{sgn}(\langle g, u\rangle)\mathrm{sgn}(\langle g, v\rangle) = \frac{2}{\pi}\arcsin\langle u, v\rangle$.

- There exists a Hilbert space $\mathcal{H}$ and $\phi, \psi : \mathbb{S}^{n-1} \to \mathbb{S}(\mathcal{H})$ s.t. $\frac{2}{\pi}\arcsin\langle\phi(u), \psi(v)\rangle = \beta\langle u, v\rangle$ for all $u, v \in \mathbb{S}^{n-1}$ and $\beta = \frac{2}{\pi}\log(1 + \sqrt{2})$.

  Proof $\langle\phi(u), \psi(v)\rangle = \sin(\frac{\beta\pi}{2}\langle u, v\rangle)$. Ex 3.7.6 $\Rightarrow \exists\mathcal{H}, \phi, \psi$. $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots, \sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!}, \|\phi(u)\|_2^2 = \|\psi(u)\|_2^2 = \sinh(\frac{\beta\pi}{2}) = 1$ for all $u \in \mathbb{S}^{n-1} \Rightarrow \beta = \frac{2}{\pi}\log(1 + \sqrt{2})$.

  Proof of Grothendieck's inequality with $K \leq \frac{1}{\beta} \approx 1.783$ WLOG $u_i, v_j \in \mathbb{S}^{N-1}$, then $\frac{2}{\pi}\arcsin\langle u_i', v_j'\rangle = \beta\langle u_i, v_j\rangle, \mathcal{H} = \mathbb{R}^M, g \sim \mathcal{N}(0, I_M)$. $\beta\sum a_{ij}\langle u_i, v_j\rangle = \sum_{i,j} a_{ij}\frac{2}{\pi}\arcsin\langle u_i', v_j'\rangle = \sum_{i,j} a_{ij}\mathbb{E}\mathrm{sgn}\langle g, u_i'\rangle\mathrm{sgn}\langle g, v_j'\rangle \leq 1$. □

4 Random matrices

- Singular vector decomposition: $A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V_{n\times n}^T = \sum_{i=1}^n s_iU_iV_i^T, \Sigma = \mathrm{diag}(s_1, s_2, \cdots, s_r), s_i \geq 0$ singular values. $s_i = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^TA)}$. If A is symmetric, $s_i = |\lambda_i(A)|$.

- Courant-Fisher's min-max theorem: $\lambda_i(A) = \max_{\dim E=i}\min_{x\in\mathbb{S}(E)}\langle Ax, x\rangle, s_i(A) = \max_{\dim E=i}\min_{x\in\mathbb{S}(E)}\|Ax\|_2$.

- Operator norm/spectral norm: $\|A\| = \max_{x\in\mathbb{R}^n\setminus\{0\}}\frac{\|Ax\|_2}{\|x\|_2} = \max_{x\in\mathbb{S}^{n-1}}\|Ax\|_2 = s_1(A)$. Or equivalently, $\|A\| = \max_{x\in\mathbb{S}^{n-1}, y\in\mathbb{S}^{m-1}}\langle Ax, y\rangle$.

- $s_n(A) > 0 \Leftrightarrow m \geq n = \mathrm{rank}(A), s_n(A) = \frac{1}{\|A^+\|}$ where A^+ is pseudo-inverse (the norm of A^{-1} restriction to the image of A).

- Frobenius norm: $\|A\|_F = (\sum_{i=1}^n \sum_{j=1}^n A_{ij}^2)^{\frac{1}{2}} = (\sum_{i=1}^n s_i^2(A))^{\frac{1}{2}}$.

- Low-rank approximation: $\text{rank}(A) = r, k < r, A_k := \sum_{i=1}^k s_i u_i v_i^T, \|A - A_k\| = \min_{\text{rank}(A') \leq k} \|A - A'\|$ (holds for $\|\cdot\|, \|\cdot\|_F$).

- Approximate isometries: $m\|x\|_2 \leq \|Ax\|_2 \leq n\|x\|_2$ where $m = s_n(A), n = s_1(A)$, or $s_n\|x-y\|_2 \leq \|Ax - Ay\|_2 \leq s_1\|x-y\|_2$.

- $A_{m\times n}, \delta > 0$. If $\|A^TA - I_n\| \leq \max(\delta, \delta^2)$, then $(1-\delta)\|x\|_2 \leq \|Ax\|_2 \leq (1+\delta)\|x\|_2$ for all x.

  Proof WLOG $\|x\|_2 = 1$. $|\|Ax\|_2^2 - 1| = |\langle (A^TA - I_n)x, x\rangle| \leq \max(\delta, \delta^2) \Rightarrow \max(|\|Ax\|_2 - 1|, (\|Ax\|_2 - 1)^2) \leq \max(\delta, \delta^2) \Rightarrow |\|Ax\|_2 - 1| \leq \delta$. □

- $Q_{n\times m}, QQ^T = I_n \Leftrightarrow P = Q^TQ$ is an orthogonal proj in $\mathbb{R}^m$ onto a subspace with dim n.

- ϵ-net: (T, d) a metric space, $K \subset T, \epsilon > 0$. $\mathcal{N} \subset K$ is an ϵ-net of K if $\forall x \in K, \exists x_0 \in \mathcal{N}$ s.t. $d(x, x_0) \leq \epsilon$. Covering number: smallest $|\mathcal{N}| = |\mathcal{N}(K, d, \epsilon)|$.

- Compactness: $\mathcal{N}(K, d, \epsilon) < +\infty$ for all $\epsilon > 0$.

- ϵ-separated: $\mathcal{P} \subset T$ is ϵ-separated if $d(x, y) > \epsilon$ for all $x, y \in \mathcal{P}$. Packing number: largest $|\mathcal{P}| = |\mathcal{P}(K, d, \epsilon)|$.

- $\mathcal{P}$ is a maximal ϵ-separated subset $\Rightarrow \mathcal{P}$ is a ϵ-net of K.

- $\mathcal{P}(K, d, 2\epsilon) \leq \mathcal{N}(K, d, \epsilon) \leq \mathcal{P}(K, d, \epsilon)$.

  Proof The upper bound follows from the previous lemma. For the lower bound, choose an 2ϵ-separated subset $\mathcal{P} = \{x_i\}$ in K and an ϵ-net $\mathcal{N} = \{y_j\}$ of K. $\forall x_i, \exists y_j \in \mathcal{N}$, s.t. $|x_i - y_j| < \epsilon$. $\forall y_j$, there exists at most a $x_j \in \mathcal{P}$ s.t. $|x_i - y_j| < \epsilon$. □

- Minkowski sum: $A, B \in \mathbb{R}^n, A + B := \{a + b, a \in A, b \in B\}$.

- $K \subset \mathbb{R}^n, \epsilon > 0, \frac{|K|}{|\epsilon B_2^n|} \leq \mathcal{N}(K, \epsilon) \leq \mathcal{P}(K, \epsilon) \leq \frac{|K + \frac{\epsilon}{2}B_2^n|}{|\frac{\epsilon}{2}B_2^n|}$ where $|\cdot|$ denotes the volume in $\mathbb{R}^n$, B_2^n denotes the unit Euclidean ball in $\mathbb{R}^n$.

- Corollary: Let $K = B_2^n$. $|\epsilon B_2^n| = \epsilon^n|K|, |K + \frac{\epsilon}{2}B_2^n| = (1 + \frac{\epsilon}{2})^n|K| \Rightarrow (\frac{1}{\epsilon})^n \leq \mathcal{N}(B_2^n, \epsilon) \leq (1 + \frac{2}{\epsilon})^n$. $\epsilon \in (0, 1] \Rightarrow (\frac{1}{\epsilon})^n \leq \mathcal{N}(B_2^n, \epsilon) \leq (\frac{3}{\epsilon})^n$.

- Hamming cubde: $x, y \in \{0, 1\}^n, d_H(x, y) := \#\{i : x(i) \neq y(i)\}$.

- (T, d) a metric space, $K \subset T$, $\mathcal{C}(K, d, \epsilon)$ the smallest number of bits sufficient specify every points $x \in K$ with accuracy ϵ in the metric d. Then $\log_2 \mathcal{N}(K, d, \epsilon) \leq \mathcal{C}(K, d, \epsilon) \leq \log_2 \mathcal{N}(K, d, \frac{\epsilon}{2})$. $\log_2 \mathcal{N}(K, \epsilon)$ is often called the metric entropy of K.

  Proof Lower bound. Assume $\mathcal{C}(K, d, \epsilon) \leq N$. There exists a transformation of $x \in K$ into bit strings of length N. A partition of K into at most 2^N subsets.

  Upper bound. Assume $\log_2 \mathcal{N}(K, d, \frac{\epsilon}{2}) \leq N$. There exists an $\frac{\epsilon}{2}$-net $\mathcal{N}$ with $|\mathcal{N}| \leq 2^N$. To every point $x \in K$, assign a point $x_0 \in \mathcal{N}$ that is closest to x. The encoding $x \mapsto x_0$ represents points in K with accuracy ϵ. □

- Error correcting code: Fix integers k, n and r. Encoder $\{0, 1\}^k \to \{0, 1\}^n$, Decoder $\{0, 1\}^n \to \{0, 1\}^k$, $D(y) = x$ if $x \in \{0, 1\}^k, y \in \{0, 1\}^n$ and $d_H(E(x), y) \leq r$.

- If $\log_2 \mathcal{P}(\{0, 1\}^n, d_H, 2r) \geq k$, then there exists an error correcting code, k bits $\to n$ bits, correct r error.

  Proof $\exists \mathcal{P} \in \{0, 1\}^n, |\mathcal{P}| = 2^k$ s.t closed balls centered at $\mathcal{P}$ with radii r are disjoint. $E : \{0, 1\}^k \to \mathcal{N}$ one to one; $D : \{0, 1\}^n \to \{0, 1\}^k$ nearest-neighbor decodes. □

- If $n \geq k + 2r\log_2(\frac{en}{2r})$, then there exists an error correcting code that encodes k-bit strings into n-bit strings and can correct r errors.

  Proof $\mathcal{P}(\{0, 1\}^n, d_H, 2r) \geq \mathcal{N}(\{0, 1\}^n, d_H, 2r) \geq \frac{2^n}{\sum_{k=0}^{2r} C_n^k} \geq 2^n(\frac{2r}{en})^{2r} \geq 2^k$. □

- $A_{m\times n}, \epsilon \in [0,1)$. Then for any ϵ-set $\mathcal{N}$ of $\mathbb{S}^{n-1}$, $\sup_{x\in\mathcal{N}}\|Ax\|_2 \le \|A\| \le \frac{1}{1-\epsilon}\sup_{x\in\mathcal{N}}\|Ax\|_2$.

  Proof Fix $x \in \mathbb{S}^{n-1}$, $\|A\| = \|Ax\|_2$. $\exists x_0 \in \mathcal{N}, \|x-x_0\|_2 \le \epsilon$, $\|Ax - Ax_0\|_2 \le \|A\|\|x-x_0\|_2 \le \epsilon\|A\| \Rightarrow \|Ax_0\|_2 \ge \|Ax\|_2 - \|A(x-x_0)\|_2 \ge \|A\| - \epsilon\|A\|$. □

- $A_{m\times n} = \{A_{ij}\}$, A_{ij} independent mean zero sub-gaussian, $K = \max_{i,j}\|A_{ij}\|_{\psi_2}$. Then for any $t > 0$, $\mathbb{P}(\|A\| \le CK(\sqrt{m}+\sqrt{n}+t)) \ge 1-2e^{-t^2}$.

  Proof Step 1: Approximation. Choose $\epsilon = 1/4$ and ϵ-net $\mathcal{N}$ of $\mathbb{S}^{n-1}$, ϵ-net $\mathcal{M}$ of $\mathbb{S}^{m-1}$ with $|\mathcal{N}| \le 9^n, |\mathcal{M}| \le 9^m$. Ex 4.4.3 $\Rightarrow \|A\| \le 2\max_{x\in\mathcal{N},y\in\mathcal{M}}\langle Ax, y\rangle$.

  Step 2: Concentration. $\langle Ax, y\rangle = \sum_{i,j} A_{ij}x_iy_j$, $\|\langle Ax,y\rangle\|_{\psi_2}^2 \le C\sum_{i,j}\|A_{ij}\|_{\psi_2}^2 x_i^2y_j^2 \le CK^2 \Rightarrow \mathbb{P}(\langle Ax,y\rangle \ge u) \le 2e^{-cu^2/K^2}$.

  Step 3: Union bound. $\mathbb{P}(\max_{x\in\mathcal{N},y\in\mathcal{M}}\langle Ax,y\rangle \ge u) \le \sum_{x\in\mathcal{N},y\in\mathcal{M}}\mathbb{P}(\langle Ax,y\rangle \ge u) \le 9^{n+m}2e^{-cu^2/K^2}$. Take $u = CK(\sqrt{m}+\sqrt{n}+t), u^2 \ge C^2K^2(m+n+t^2)$. C sufficiently large s.t. $cu^2/K^2 \ge 3(n+m+t^2)$. □

- $A_{n\times n}$ symmetric, $A_{ij}, i \le j$ independent mean zero sub-gaussian. Then for $t \ge 0, \mathbb{P}(\|A\| \le CK(\sqrt{n}+t)) \ge 1-4e^{-t^2}$.

  Proof $A = \underbrace{A^+ + A^-}_{\text{upper + lower triangular matrix}}$, $\mathbb{P}(\|A^+\| \le CK(\sqrt{n}+t)) \ge 1-2e^{-4t^2} \ge 1-2e^{-t^2}$, $\mathbb{P}(\|A^-\| \le CK(\sqrt{n}+t)) \ge 1-2e^{-4t^2} \ge 1-2e^{-t^2}$, $\mathbb{P}(\|A\| \le CK(\sqrt{n}+t)) \ge \mathbb{P}(\|A^+\| \le \frac{C}{2}K(\sqrt{n}+t), \|A^-\| \le \frac{C}{2}K(\sqrt{n}+t)) \ge 2(1-2e^{-t^2}) - 1 = 1-4e^{-t^2}$. □

- Stochastic block model (SBM): $G(n,p,q), p > q$, n vertices, two community of size $n/2$, $x, y \in$ same community $\Rightarrow \mathbb{P}(x \sim y) = p$, otherwise $\mathbb{P}(x\sim y) = q$. $A = \{A_{ij}\}, A_{ij} = 1$ if $i \sim j$ otherwise 0. $A = \mathbb{E}A + R := D + R, \|D\| = \frac{p+q}{2}\cdot n, \mathbb{P}(\|R\| \le C\sqrt{n}) \ge 1-4e^{-n}$.

- Weyl's inequality: Symmetric matrices S and T with same dim, $\max_i|\lambda_i(S) - \lambda_i(T)| \le \|S-T\|$.

- Davis-Kahan: Fix i, $\min_{j\ne i}|\lambda_i(S) - \lambda_j(S)| = \delta > 0$. Then $\sin\angle(v_i(S), v_i(T)) \le \frac{2\|S-T\|}{\delta} \Rightarrow \exists\theta \in \{-1,1\}, \|v_i(S) - \theta v_i(T)\|_2 \le \frac{\|S-T\|}{\delta}\cdot 2^{3/2}$.

- Spectual clustering: Recall SBM $A = D+R$ and let $S = D, T = A = D+R$ in Davis-Kahan. $\delta = \min(\lambda_2, \lambda_2 - \lambda_1) = \min(\frac{p-q}{2}, q)n := \mu n$. $\mathbb{P}(\|R\| = \|T-S\| \le C\sqrt{n}) \ge 1-4e^{-n} \Rightarrow \exists\theta \in \{\pm1\}, \|v_2(D) - \theta v_2(A)\| \le \frac{C}{\mu\sqrt{n}}$. Let $u_2(D) = (1,1,\cdots,1,-1,-1,\cdots,-1) \Rightarrow \|u_2(D) - \theta u_2(A)\| \le \frac{C}{\mu} \Rightarrow \sum_{j=1}^n|u_2(D)_j - \theta u_2(A)_j|^2 \le \frac{C}{\mu^2}$. Thus the number of disagreeing signs between $u_2(D)$ and $u_2(A)$ must be bounded by $\frac{C}{\mu^2}$.

- $A_{m\times n}$, rows A_i independent mean zero sub-gaussian, isotropic. Then for any $t \ge 0$, $\sqrt{m} - CK^2(\sqrt{n}+t) \le s_n(A) \le s_1(A) \le \sqrt{m} + CK^2(\sqrt{n}+t)$ with prob $\ge 1-2e^{-t^2}$. Here $K = \max_i\|A_i\|_{\psi_2}$.

  Proof Only need to prove $\|\frac{1}{m}A^TA - I_n\| \le \epsilon := K^2\max\{\delta,\delta^2\}, \delta = C(\frac{\sqrt{n}}{\sqrt{m}} + \frac{t}{\sqrt{m}})$.

  Step 1: Approximation. Find an $\frac{1}{4}$-net $\mathcal{N}$ of the unit space $\mathbb{S}^{n-1}, |\mathcal{N}| \le 9^n$. $\|\frac{1}{m}A^TA - I_n\| \le 2\max_{x\in\mathcal{N}}|\langle(\frac{1}{m}A^TA - I_n)x, x\rangle| = 2\max_{x\in\mathcal{N}}|\frac{1}{m}\|Ax\|_2^2 - 1|$.

  Step 2: Concentration. $X_i := \langle A_i, x\rangle$ independent, mean zero, $\|X_i\|_{\psi_2} \le K$, $\mathbb{E}X_i^2 = 1$. $\mathbb{P}(|\frac{1}{m}\|Ax\|_2^2 - 1| \ge \frac{\epsilon}{2}) \le 2e^{-c_1\delta^2m} \le 2e^{-c_1C^2(n+t^2)}$.

  Step 3: Union bound. $\mathbb{P}(|\frac{1}{m}\|Ax\|_2^2 - 1| \ge \frac{\epsilon}{2}) \le 9^n\cdot 2e^{-c_1C^2(n+t^2)} \le 2e^{-t^2}$. □

- $X \in \mathbb{R}^n$ sub-gaussian. $\mathbb{E}X = 0, \Sigma = \mathbb{E}XX^T, X_i \overset{\text{d}}{=} X$ i.i.d., $\Sigma_m = \frac{1}{m}\sum_{i=1}^m X_iX_i^T$. Assume there exists $K \ge 1$ s.t. $\|\langle X,x\rangle\|_{\psi_2}^2 \le K^2\|\langle X,x\rangle\|_{L^2}^2$. Then for m, $\mathbb{E}\|\Sigma_m - \Sigma\| \le CK^2(\sqrt{\frac{n}{m}} + \frac{n}{m})\|\Sigma\|$.

  Proof $Z_i = \Sigma^{-1/2}X_i, Z = \Sigma^{-1/2}X, \mathbb{E}Z_iZ_i^T = I_n, \|Z\|_{\psi_2} \le K, \|Z_i\|_{\psi_2} \le K$. Then $\|\Sigma_m - \Sigma\| = \|\Sigma^{1/2}R_m\Sigma^{1/2}\| \le \|R_m\|\|\Sigma\|$ where $R_m = \frac{1}{m}\sum_{i=1}^m Z_iZ_i^T - I$. Consider an $m\times n$ random matrix A whose rows are Z_i^T. $\mathbb{E}\|R_m\| = \mathbb{E}\|\frac{1}{m}A^TA - I\| \le CK^2(\sqrt{\frac{n}{m}} + \frac{n}{m})$.□

5 Concentration without independence

- $(X, d_X) \xrightarrow{f} (Y, d_Y), d_Y(f(u), f(v)) \le L\cdot d_x(u,v), \forall u, v \in X$. The infimum of all L in this definition is called the Lipschitz norm of f and is denoted $\|f\|_{\text{Lip}}$.

- $\epsilon > 0, A_\epsilon = A + \epsilon B_2^n, A \subset \mathbb{R}^n, \min_A$ volume of A_ϵ with colume A fixed is achieved when A is a ball.

- $\sigma_{n-1}(A)$ normalized area on $\mathbb{S}^{n-1}$, $\epsilon > 0$. With given $\sigma_{n-1}(A)$, $\min_A \sigma_{n-1}(A_\epsilon)$ is achieved when A is a spherical cap.

- $A \subset \sqrt{n}\mathbb{S}^{n-1}$. If $\sigma(A) \geq \frac{1}{2}$, then $\forall t \geq 0, \sigma(A_t) \geq 1 - 2e^{-ct^2}$.

  Proof Let $H = \{x \in \sqrt{n}\mathbb{S}^{n-1}, x_1 \leq 0\}$. $\sigma(A) \geq \frac{1}{2}\sigma(H)$. Thm 5.1.6 $\Rightarrow \sigma(A_t) \geq \sigma(H_t) = \mathbb{P}(X \in H_t)$. $H_t \supset \{x \in \sqrt{n}\mathbb{S}^{n-1}, x_1 \leq \frac{t}{\sqrt{2}}\} \Rightarrow \sigma(H_t) \geq \mathbb{P}(X_1 \leq \frac{t}{\sqrt{2}})$. $\|X_1\|_{\psi_2} \leq C \Rightarrow \mathbb{P}(X_1 \leq \frac{t}{\sqrt{2}}) \geq 1 - 2e^{-ct^2}$. □

- $X \sim \text{Unif}(\sqrt{n}\mathbb{S}^{n-1}), f : \sqrt{n}\mathbb{S}^{n-1} \to \mathbb{R}$. Then $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq C\|f\|_{\text{Lip}}$.

  Proof WLOG $\|f\|_{\text{Lip}} = 1$, $\mathbb{P}(f(X) \geq M) \geq \frac{1}{2}, \mathbb{P}(f(X) \leq M) \geq \frac{1}{2}$. $A := \{x \in \sqrt{n}\mathbb{S}^{n-1} : f(x) \leq M\}$. $\mathbb{P}(X \in A) \geq \frac{1}{2} \Rightarrow \mathbb{P}(A_t) \geq 1 - 2e^{-ct^2} \Rightarrow \mathbb{P}(f(X) \leq M + t) \geq 1 - 2e^{-ct^2}$. By centering, $f(X) - \mathbb{E}f(X) = f(X) - M - (\mathbb{E}f(X) - M)$ is sub-gaussian. □

- $X \sim \mathcal{N}(0, I_n), \gamma_n(A) = \mathbb{P}(X \in A), \epsilon > 0, \gamma_n(A)$ given, half spaces minimize $\gamma_n(A_\epsilon)$.

- $X \sim \mathcal{N}(0, I_n), f : \mathbb{R}^n \to \mathbb{R}, \|f\|_{\text{Lip}} < \infty$. Then $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq C\|f\|_{\text{Lip}}$.

- Hamming cube, $d(x, y) = \frac{1}{n}|\{i : x_i \neq y_i\}|, \mathbb{P}(A) = \frac{|A|}{2^n}$. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- $S_n : n!$ permutation of n symbols. $d(\pi, \rho) = \frac{1}{n}|\{i : \pi(i) \neq \rho(i)\}|, \mathbb{P}(A) = \frac{|A|}{n!}$. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- Special orthogonal group $\text{SO}(n)$, determinant $= 1$, $d = \|\cdot\|_F$, $\mathbb{P}$ is uniform measure. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- $G_{n,m}$ all m-dim subspaces of $\mathbb{R}^n$ ($\simeq P_{G_{n,m}}$ orthogonal projections), $d(E, F) = \|P_E - P_F\|$, $\mathbb{P}$ is uniform measure. A random subspace E can be constructed by computing the column span (i.e. the image) of a random $n \times m$ Gaussian random matrix G with i.i.d. $\mathcal{N}(0, 1)$ entries. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- A random vector X in $\mathbb{R}^n$ with density $p(x) = e^{-U(x)}$, Hess $U(x) \succeq \kappa I_n$. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{\kappa}}$.

- $X = (X_1, \cdots, X_n)$ independent coordinates, $|X_i| \leq 1$ a.s., f convex and Lipschitz. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq C\|f\|_{\text{Lip}}$.

- $E \sim \text{Unif}(G_{n,m}), z \in \mathbb{R}^n, \epsilon > 0$. Then (a) $(\mathbb{E}\|P_E z\|_2^2)^{\frac{1}{2}} = \sqrt{\frac{m}{n}}\|z\|_2$; (b) $\mathbb{P}(\big|\|P_E z\|_2 - \sqrt{\frac{m}{n}}\|z\|_2\big| \leq \epsilon\sqrt{\frac{m}{n}}\|z\|_2) \geq 1 - 2e^{-c\epsilon^2 m}$.

  Proof (a): WLOG $\|z\|_2 = 1$. Rotational invariance: $\mathbb{P}(E \in A) = \mathbb{P}(U(E) \in A)$ where U is $n \times n$ orthogonal $\Rightarrow$ The dist. of $P_E z$ is the same if we fix E, $z \in \text{Unif}(\mathbb{S}^{n-1})$. WLOG $Pz = (z_1, \cdots, z_m, 0, \cdots, 0)$. $\mathbb{E}\|Pz\|_2^2 = m\mathbb{E}z_i^2 = \frac{m}{n}$.

  (b): $f : z \to \|Pz\|_2, \|f\|_{\text{Lip}} = 1 \Rightarrow \|\|Pz\|_2 - \mathbb{E}\|Pz\|_2\|_{\psi_2} \leq \frac{C}{\sqrt{n}}$ □

- $\mathcal{X}$ a set of N points in $\mathbb{R}^n, \epsilon > 0, m \geq \frac{C}{\epsilon^2}\log N, E \sim \text{Unif}(G_{n,m}), Q = \sqrt{\frac{n}{m}}P_E$. Then $\mathbb{P}(|\|Qx - Qy\|_2 - \|x - y\|_2| \leq \epsilon\|x - y\|_2$ for any $x, y, \in \mathcal{X}) \geq 1 - 2e^{-c\epsilon^2 m}$.

  Proof Let $\mathcal{X} - \mathcal{X} := \{x - y : x, y \in \mathcal{X}\}$. The latest lemma $\Rightarrow \forall z, \mathbb{P}\left((1 - \epsilon)\sqrt{\frac{m}{n}}\|z\|_2 \leq \|Pz\|_2 \leq (1 + \epsilon)\sqrt{\frac{m}{n}}\|z\|_2\right) \geq 1 - 2e^{-c\epsilon^2 m}$. Union bound: $\mathbb{P}(\cdots,$ for any $z \in \mathcal{X} - \mathcal{X}) \geq 1 - N^2 \cdot 2e^{-c\epsilon^2 m} \geq 1 - 2e^{-c'\epsilon^2 m}$. □

- $f : \mathbb{R} \to \mathbb{R}, X = \sum_{i=1}^n \lambda_i u_i u_i^T$, define $f(X) = \sum_{i=1}^n f(\lambda_i) u_i u_i^T$.

- P.S.D. order: $X \succeq 0, X \succeq Y$ if $X - Y \succeq 0$.

- Golden-Thompson inequality: $\text{tr}(e^{A+B}) \leq \text{tr}(e^A e^B)$.

- Lieb's inequality: $H : n \times n$ symmetric matrix, X P.D., $f(X) = \text{tr}(e^{H + \log X})$. Then f is concave.

- X is a random P.D. matrix $\Rightarrow \mathbb{E}f(X) \leq f(\mathbb{E}X)$. $X = e^Z$, Z symmetric. Then $\mathbb{E}\text{tr}(e^{H+Z}) \leq \text{tr}(e^{H + \log \mathbb{E}e^Z})$.

- $X_1, \cdots, X_N$ independent mean zero $n \times n$ symmetric random matrices, $\|X_i\| \leq K$ a.s. for all i. Then for $\forall t \geq 0$, $\mathbb{P}(\|\sum_{i=1}^N X_i\| \geq t) \leq 2ne^{-\frac{t^2/2}{\sigma^2 + Kt/3}}$ where $\sigma^2 = \|\sum_{i=1}^N \mathbb{E}X_i^2\|$.

  Proof Step 1: Reduction to MGF. $S := \sum_{i=1}^N X_i$. $\|S\| = \max_i |\lambda_i(S)| = \max(\lambda_{\max}(S), \lambda_{\max}(-S))$. $\mathbb{P}(\lambda_{\max}(S) \geq t) \leq e^{-\lambda t}\mathbb{E}e^{\lambda\lambda_{\max}(S)}$. $E := \mathbb{E}e^{\lambda\lambda_{\max}(S)} = \mathbb{E}\lambda_{\max}(e^{\lambda S}) \Rightarrow E \leq \mathbb{E}\text{tr}(e^{\lambda S})$.

  Step 2: Apply Lieb's inequality. $\mathbb{E}\text{tr}(e^{\lambda S}) = \mathbb{E}\text{tr}(e^{\sum_{i=1}^{N-1}\lambda X_i + \lambda X_N}) \leq \mathbb{E}\text{tr}(e^{\sum_{i=1}^{N-1}\lambda X_i + \log \mathbb{E}e^{\lambda X_N}}) \leq \text{tr}(e^{\sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i}})$.

  Step 3: Lemma: X is an $n \times n$ symmetric mean zero random matrix, $\|X\| \leq K$ a.s. Then $e^{\lambda X} \preceq e^{g(\lambda)\mathbb{E}X^2}$ where $g(\lambda) = \frac{\lambda^2/2}{1 - |\lambda|K/3}, |\lambda| < 3/K$.

Proof $e^z \le 1 + z + \frac{1}{1-|z|/3}\frac{z^2}{2}$ if $|z| < 3$. $z = \lambda x$. If $|x| \le K, |\lambda| < \frac{3}{K}$, $e^{\lambda x} \le 1 + \lambda x + g(\lambda)x^2$. (b) of Ex. 5.4.5 $\Rightarrow$ If $\|X\| \le K, |\lambda| < 3/K$, $\mathbb{E}e^{\lambda X} \preceq I + g(\lambda)\mathbb{E}X^2$ (since $\mathbb{E}X = 0$) $\preceq e^{g(\lambda)\mathbb{E}X^2}$. □

Step 4: $E \le \mathrm{tr}(e^{\sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i}})$. The latest lemma + (g) of Ex.5.4.5 $\Rightarrow \log \mathbb{E}e^{\lambda X_i} \preceq g(\lambda)\mathbb{E}X_i^2 \Rightarrow \sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i} \preceq g(\lambda)\cdot Z$ where $Z := \sum_{i=1}^N \mathbb{E}X_i^2$ and $\sigma^2 = \|Z\|$. (e) of Ex.5.4.5 $\Rightarrow \mathrm{tr}(e^{\sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i}}) \le \mathrm{tr}(e^{g(\lambda Z)}) \Rightarrow E \le \mathrm{tr}(e^{g(\lambda Z)}) \le n\lambda_{\max}(e^{g(\lambda Z)}) = ne^{g(\lambda)\|Z\|} = ne^{g(\lambda)\sigma^2}$. Minimize for λ as a function of t with $0 < \lambda < 3/K$. □

- $X \in \mathbb{R}^n, \Sigma = \mathbb{E}XX^T, \Sigma_m = \frac{1}{m}\sum_{i=1}^m X_iX_i^T, X_i \overset{\text{i.i.d.}}{\sim} X, \|X\|_2 \le K(\mathbb{E}\|X\|_2^2)^{\frac{1}{2}}$ a.s.. Then $\mathbb{E}\|\Sigma_m - \Sigma\| \le C(\sqrt{\frac{K^2 n \log n}{m}} + \frac{K^2 n\log n}{m})\|\Sigma\|$.

  Proof $\mathbb{E}\|X\|_2^2 = \mathbb{E}X^TX = \mathbb{E}\mathrm{tr}(X^TX) = \mathbb{E}\mathrm{tr}(XX^T) = \mathrm{tr}(\Sigma) \Rightarrow \|X\|_2^2 \le K^2\mathrm{tr}(\Sigma)$ a.s.. Ex 5.4.11 $\Rightarrow \mathbb{E}\|\Sigma_m - \Sigma\| = \frac{1}{m}\mathbb{E}\|\sum_{i=1}^m (X_iX_i^T - \Sigma)\| \lesssim \frac{1}{m}(\sigma\sqrt{\log n} + M\log n)$ where $\sigma^2 = \|\sum_{i=1}^m \mathbb{E}(X_iX_i^T - \Sigma)^2\| = m\|\mathbb{E}(XX^T - \Sigma)^2\|$ and M is chosen s.t. $\|XX^T - \Sigma\| \le M$ a.s.. Then $\mathbb{E}(XX^T - \Sigma)^2 = \mathbb{E}(XX^T)^2 - \Sigma^2 \preceq \mathbb{E}(XX^T)^2 = \mathbb{E}(\|X\|_2^2 XX^T) \preceq K^2\mathrm{tr}(\Sigma)\Sigma \Rightarrow \sigma^2 \le K^2 m\mathrm{tr}(\Sigma)\|\Sigma\|$. $\|XX^T - \Sigma\| \le \|X\|_2^2 + \|\Sigma\| \le K^2\mathrm{tr}(\Sigma) + \|\Sigma\| \le 2K^2\mathrm{tr}(\Sigma) := M$ (since $K \ge 1$ and $\|\Sigma\| \le \mathrm{tr}(\Sigma)$). Substitute our bounds for σ^2 and M into the previous bound $\frac{1}{m}(\sigma\sqrt{\log n} + M\log n)$. □

6 Quadradic forms, symmetrization, contraction

- $Y \perp Z, \mathbb{E}Z = 0$, then $\mathbb{E}F(Y) \le \mathbb{E}F(Y+Z)$.

  Proof $F(y) = F(\mathbb{E}(y+Z)) \le \mathbb{E}F(y+Z) \Rightarrow \mathbb{E}F(Y) = \mathbb{E}(\mathbb{E}(F(Y+\mathbb{E}Z)|Y)) = \mathbb{E}(\mathbb{E}(F(\mathbb{E}(Y+Z))|Y)) \le \mathbb{E}(\mathbb{E}(F(Y+Z)|Y)) = \mathbb{E}F(Y+Z)$. □

- Decoupling: $A_{n\times n}$ diagonal-free(i.e. the diagonal entries of A equal zero), $X = (X_1, \cdots, X_n)$ independent mean zero. Then for every convex function $F: \mathbb{R} \to \mathbb{R}$, $\mathbb{E}F(X^TAX) \le \mathbb{E}F(4X^TAX')$ where $X' \overset{\text{d}}{=} X, X' \perp X$.

  Proof $\delta_1, \cdots, \delta_n \overset{\text{i.i.d.}}{\sim} \mathrm{Ber}(1, \frac{1}{2}), I = \{i : \delta_i = 1\}, \mathbb{E}\delta_i(1-\delta_j) = \frac{1}{4}, X^TAX = \sum_{i\ne j} a_{ij}X_iX_j = 4\mathbb{E}_\delta \sum_{i,j}\delta_i(1-\delta_j)a_{ij}X_iX_j = 4\mathbb{E}_I \sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j \Rightarrow \mathbb{E}_X F(X^TAX) = \mathbb{E}_X F(4\mathbb{E}_I \sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j) \le \mathbb{E}_I\mathbb{E}_X F(4\sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j)$. There exists an I s.t. $\mathbb{E}_X F(4\sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j') = \mathbb{E}_X F(4\sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j) \ge \mathbb{E}F(X^TAX)$. LHS $\le \mathbb{E}_X F(4\sum_{i,j} a_{ij}X_iX_j')$ by the latest lemma since $\mathbb{E}[(\sum_{(i,j)\in I\times I} + \sum_{(i,j)\in I^c\times I^c} + \sum_{(i,j)\in I^c\times I}) a_{ij}X_iX_j' | \{X_i, i\in I\}, \{X_j', j\in I^c\}] = 0$. □

- $X, X' \sim \mathcal{N}(0, I_n), X \perp X'$, then $\mathbb{E}e^{\lambda X^TAX'} \le e^{C\lambda^2\|A\|_F^2}, |\lambda| \le \frac{c}{\|A\|}$.

  Proof $A = \sum_i s_iu_iv_i^T, X^TAX' = \sum_i s_i \underbrace{\langle u_i, X\rangle}_{:=g_i}\underbrace{\langle v_i, X'\rangle}_{:=g_i'}$. $(g_1, \cdots g_n) \perp (g_1', \cdots, g_n') \sim \mathcal{N}(0, I_n) \Rightarrow \mathbb{E}e^{\lambda X^TAX} = \prod_{i=1}^n \mathbb{E}e^{\lambda s_ig_ig_i'} = \prod_{i=1}^n \mathbb{E}e^{\lambda^2 s_i^2 g_i^2/2} \le \prod_{i=1}^n e^{C\lambda^2 s_i^2}(\lambda^2 s_i^2 \le c) \le e^{C\lambda^2\|A\|_F^2}(\lambda^2 \le \frac{c}{\max_i s_i^2} = \frac{c}{\|A\|^2})$. □

- X, X' independent sub-gaussian mean zero, $\|X\|_{\psi_2} \le K, \|X'\|_{\psi_2} \le K$. $g, g' \sim \mathcal{N}(0, I_n), g \perp g'$. Then $\mathbb{E}e^{\lambda X^TAX'} \le \mathbb{E}e^{CK^2\lambda g^TAg'}$.

  Proof Conditioned on X', $\mathbb{E}_X e^{\lambda X^AX'} \le e^{C\lambda^2K^2\|AX'\|_2^2}, \mathbb{E}_g e^{\mu g^TAX'} = e^{\frac{\mu^2\|AX'\|_2^2}{2}}$. $\mu = \sqrt{2c}K\lambda \Rightarrow \mathbb{E}_X e^{\lambda X^TAX'} \le \mathbb{E}_g e^{\sqrt{2c}K\lambda g^TAX'} \Rightarrow \mathbb{E}e^{\lambda X^TAX'} \le \mathbb{E}e^{\sqrt{2c}K\lambda g^TAX'} \le \mathbb{E}e^{2cK^2\lambda g^TAg'}$ □

- Hanson-Wright inequality: $X = (X_1, \cdots, X_n)$ independent mean zero sub-gaussian, then $\mathbb{P}(|X^TAX - \mathbb{E}X^TAX| \ge t) \le 2e^{-c\min(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|})}$.

  Proof WLOG $K = 1$. $X^TAX = \sum_{i,j} a_{ij}X_iX_j, \mathbb{E}X^TAX = \sum_i a_{ii}\mathbb{E}X_i^2, X^TAX - \mathbb{E}X^TAX = \sum_i a_{ii}(X_i^2 - \mathbb{E}X_i^2) + \sum_{i\ne j} a_{ij}X_iX_j$. $p := \mathbb{P}(X^TAX - \mathbb{E}X^TAX \ge t) \le \mathbb{P}(\sum_i a_{ii}(X_i^2 - \mathbb{E}X_i^2) \ge \frac{t}{2}) + \mathbb{P}(\sum_{i\ne j} a_{ij}X_iX_j \ge \frac{t}{2}) := p_1 + p_2$.

  Step 1: $\|X_i^2 - \mathbb{E}X_i^2\|_{\psi_2} \lesssim 1$. Bernstein $\Rightarrow p_1 \le e^{-c\min(\frac{t^2}{\sum a_{ii}^2}, \frac{t}{\max_i |a_i|})} \le e^{-c\min(\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|})}$.

  Step 2: $S := \sum_{i\ne j} a_{ij}X_iX_j$. $p_2 \le e^{-\frac{\lambda t}{2}}\mathbb{E}e^{\lambda S}, \mathbb{E}e^{\lambda S} \le \mathbb{E}e^{4\lambda X^TAX'} \le \mathbb{E}e^{c_1\lambda g^TAg'} \le e^{C\lambda^2\|A\|_F^2}$(with $\lambda \le \frac{c}{\|A\|}$). □

- $B_{m\times n}, X \in \mathbb{R}^n, \{X_i\}$ independent mean-zero, unit-variance, sub-gaussian. Then $\|\|BX\|_2 - \|B\|_F\|_{\psi_2} \le CK^2\|B\|$, $K = \max_i \|X\|_{\psi_2}$.

  Proof $A = B^TB, X^TAX = \|BX\|_2^2, \mathbb{E}X^TAX = \|B\|_F^2, \|A\| = \|B\|^2, \|A\|_F = \|B^TB\|_F \le \|B^T\|\|B\|_F = \|B\|\|B\|_F$.Thus $\forall u \ge 0$, $\mathbb{P}(|\|BX\|_2^2 - \|B\|_F^2| \ge u) \le e^{-\frac{c}{K^4}\min(\frac{u^2}{\|B\|^2\|B\|_F^2}, \frac{u}{\|B\|^2})}$. Let $u = \epsilon\|B\|_F^2, \mathbb{P}(|\|BX\|_2^2 - \|B\|_F^2| \ge \epsilon\|B\|_F^2) \le 2e^{-c\min(\epsilon^2,\epsilon)\frac{\|B\|_F^2}{K^4\|B\|^2}}$. Let $\delta^2 = \min(\epsilon^2, \epsilon)$, then $\epsilon = \max(\delta, \delta^2), |\|BX\| - \|B\|_F| \ge \delta\|B\|_F \Rightarrow |\|BX\|_2^2 - \|B\|_F^2| \ge \epsilon\|B\|_F^2 \Rightarrow \mathbb{P}(|\|BX\|_2 - \|B\|_F| \ge \delta\|B\|_F) \le 2e^{-c\delta^2\frac{\|B\|_F^2}{K^4\|B\|^2}}$. □

- $X_1, X_2, \cdots, X_N$ independent, mean zero in a normed space, $\epsilon_1, \epsilon_2, \cdots, \epsilon_N$ a sequence of independent symmetric Bernoulli random variables.. Then $\frac{1}{2}\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\| \le \mathbb{E}\|\sum_{i=1}^N X_i\| \le 2\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$.

  Proof Upper bound. $X' \perp X, X' \overset{\mathrm{d}}{=} X$. $p = \mathbb{E}\|\sum_{i=1}^N X_i\| \le \mathbb{E}\|\sum_{i=1}^N X_i - \sum_{i=1}^N X_i'\| = \mathbb{E}\|\sum_{i=1}^N \epsilon_i(X_i - X_i')\| \le \mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\| + \mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i'\| = 2\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$. $\square$

- $A_{n\times n}$ symmetric independent mean zero. Then $\mathbb{E}\|A\| \le C\sqrt{\log n}\mathbb{E}\max\|A_i\|_2$ where A_i is i-th row of A.

  Proof $A = \sum_{i\le j} Z_{ij}$ independent mean zero symmetric where $Z_{ij} = \begin{cases} A_{ij}(e_i e_j^T + e_j e_i^T), & i \le j \\ A_{ii} e_i e_i^T & i = j \end{cases} \Rightarrow \mathbb{E}\|A\| \le 2\mathbb{E}\|\sum_{i\le j} \epsilon_{ij} Z_{ij}\|$.

  Ex 5.4.3(a) $\Rightarrow$ Conditioned on $\{Z_{ij}\}, \mathbb{E}_\epsilon\|\sum_{i\le j}\epsilon_{ij}Z_{ij}\| \le C\sqrt{\log n}\|\sum_{i\le j} Z_{ij}^2\|^{\frac{1}{2}} \Rightarrow \mathbb{E}\|\sum_{i\le j}\epsilon_{ij}Z_{ij}\| \le C\sqrt{\log n}\mathbb{E}\|\sum_{i\le j} Z_{ij}^2\|^{\frac{1}{2}}$, $\sum_{i\le j} Z_{ij}^2 = \sum_{i=1}^n(\sum_{j=1}^n A_{ij}^2)e_i e_i^T = \sum_{i=1}^n \|A_i\|_2^2 e_i e_i^T \Rightarrow \|\sum_{i\le j} Z_{ij}^2\|^{\frac{1}{2}} = \max\|A_i\|_2$. $\square$

- $X_{n\times n}, \mathrm{rank}(X) = r << n, Y_{ij} = \delta_{ij}X_{ij}, \delta_{ij} \sim \mathrm{Ber}(p), p = \frac{m}{n^2}$. Let $\|p^{-1}Y - \hat{X}\| = \min_{\mathrm{rank}(A')\le r}\|p^{-1}Y - A'\|$. Then $\mathbb{E}\frac{1}{n}\|\hat{X} - X\|_F \le C\sqrt{\frac{rn\log n}{m}}\|X\|_\infty$.

  Proof Step 1. $\|\hat{X} - X\| \le \|\hat{X} - p^{-1}Y\| + \|p^{-1}Y - X\| \le 2\|p^{-1}Y - X\| = \frac{2}{p}\|Y - pX\|$. $(Y - pX)_{ij} = (\delta_{ij} - p)X_{ij}$ independent mean zero, Ex 6.5.2 $\Rightarrow \mathbb{E}\|Y - pX\| \le C\sqrt{\log n}(\mathbb{E}\max_i \|(Y - pX)_i\|_2 + \mathbb{E}\max_j \|(T - pX)^j\|_2)$. $\|(Y - pX)_i\|_2^2 = \sum_{j=1}^n (\delta_{ij} - p)^2 X_{ij}^2 \le \sum_{j=1}^n (\delta_{ij} - p)^2\|X\|_\infty^2$. Ex 6.6.2 $\Rightarrow \mathbb{E}\max_i \sum_{j=1}^n(\delta_{ij} - p)^2 \le Cpn \Rightarrow \frac{2}{p}\|Y - pX\| \le C\sqrt{\frac{n\log n}{p}}\|X\|_\infty$.

  Step 2. $\mathrm{rank}(X) \le r, \mathrm{rank}(\hat{X}) \le r, \mathrm{rank}(\hat{X} - X) \le 2r$. $\|\hat{X} - X\|_F \le \sqrt{2r}\|\hat{X} - X\| \Rightarrow \mathbb{E}\|\hat{X} - X\|_F \le C\sqrt{\frac{rn\log n}{p}}\|X\|_\infty$. $\square$

- $X_1, \cdots, X_N$ vectors in some normed space, $a = (a_1, \cdot, a_N) \in \mathbb{R}^N$. Then $\mathbb{E}\|\sum_{i=1}^N a_i\epsilon_i X_i\| \le \|a\|_\infty \mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$.

  Proof WLOG $\|a\|_\infty \le 1, f(a) = \mathbb{E}\|\sum_{i=1}^N a_i\epsilon_i X_i\|$ is convex, which implies the maximum of f is attained at the boundary. Thus $f(a) \le f(a^*) = \mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$ with $a_i^* = 1$ or -1. $\square$

- $X_1, \cdots, X_N$ independent mean zero, $g_1, \cdots, g_N \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, $\frac{C}{\sqrt{\log N}}\mathbb{E}\|\sum_{i=1}^N g_i X_i\| \le \mathbb{E}\|\sum_{i=1}^N X_i\| \le 3\mathbb{E}\|\sum_{i=1}^N g_i X_i\|$.

  Proof Upper: $\mathbb{E}\|\sum_{i=1}^N X_i\| \le 2\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\| = 2\sqrt{\frac{\pi}{2}}\mathbb{E}_{X,\epsilon}\|\sum_{i=1}^N \epsilon_i \mathbb{E}_g|g_i|X_i\| \le 2\sqrt{\frac{\pi}{2}}\mathbb{E}\|\sum_{i=1}^N \epsilon_i|g_i|X_i\| = 2\sqrt{\frac{\pi}{2}}\mathbb{E}\|\sum_{i=1}^N g_i X_i\|$.

  Lower: $\mathbb{E}\|\sum_{i=1}^N g_i X_i\| = \mathbb{E}\|\sum_{i=1}^N \epsilon_i g_i X_i\| \le \mathbb{E}_g\mathbb{E}_X(\|g\|_\infty \mathbb{E}_\epsilon\|\sum_{i=1}^N \epsilon_i X_i\|) = \mathbb{E}_g\|g\|_\infty \mathbb{E}_{X,\epsilon}\|\sum_{i=1}^N \epsilon_i X_i\| \le 2\mathbb{E}_g\|g\|_\infty \mathbb{E}_X\|\sum_{i=1}^N X_i\| \le C\sqrt{\log N}\mathbb{E}_X\|\sum_{i=1}^N X_i\|$. $\square$