

# Theoretical Machine Learning

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# 1 Introduction

**Outline 1.1** (Main tasks in machine learning) Generation, prediction, decision. Generation:  $X_1, \dots, X_n \sim F$ , infer and analyse  $F$ , unsupervised learning, e.g. GAN, GPT,  $\dots$ . Prediction: data pairs  $(X^{(1)}, Y^{(1)}), \dots, (X^{(n)}, Y^{(n)})$ , input variables  $X^{(i)} \in \mathbb{R}^d$ ,  $f: \mathcal{X} \rightarrow \mathcal{Y}, x \in \mathcal{X}, y \in \mathcal{Y}$ , ascribe, supervised learning. Decision: Reinforcement learning, Agent  $\leftarrow$  action, state, reward  $\rightarrow$  environment.

**Outline 1.2** (Methods for solving tasks) Parameterized/Non-parameterized, frequency(MLE)/Bayesian.

**Outline 1.3** (Modeling error) Supervised: Fix  $X = (X_1, \dots, X_d)^T \in \mathbb{R}^d$ , for regression  $Y \in \mathbb{R}$ , for classification  $Y \in \{0, 1\}$  (also  $\{-1, 1\}, \{1, \dots, M\}, \{0, 1\}^M$ ). Random design for  $X$  (known as generative models):  $Y^{(i)} = g(X^{(i)}, Z^{(i)})$ . Fixed design for  $X$  (known as discriminative models):  $Y^{(i)} = g(x^{(i)}, Z^{(i)})$ . Unsupervised:  $X = g(Z)$  (e.g. factor model:  $X = AZ + \varepsilon, Z \in \mathcal{N}(0, 1), \varepsilon \sim \mathcal{N}(0, \Sigma)$ ).

# 2 Statistical Decision Theory

**Definition 2.1** (Basic concepts) Consider a state space  $\Omega$ , data space  $\mathcal{D}$ , model  $\mathcal{P} = \{p(\theta, x)\}$ , action space  $\mathcal{A}$ . Loss function:  $\mathcal{L}: \Omega \times \mathcal{A} \rightarrow [-\infty, +\infty]$ , measurable, nonnegative. A measurable function  $\delta: \mathcal{D} \rightarrow \mathcal{A}$  is called a nonrandomized decision rule. Risk function is defined as  $\mathcal{R}(\theta, \delta) = \int \mathcal{L}(\theta, \delta(x)) dP_\theta(x) = \mathbb{E}_\theta \mathcal{L}(\theta, \delta(X))$ . Randomized decision: for each  $X = x$ ,  $\delta(x)$  is a probability distribution:  $[A|X = x] \sim \delta_x$ . Risk function for  $\delta$ :  $\mathcal{R}(\theta, \delta) = \mathbb{E}_\theta \mathcal{L}(\theta, A) = \mathbb{E}_\theta \mathbb{E}_a \mathcal{L}(\theta, A|X) = \iint \mathcal{L}(\theta, a) d\delta_x(a) dP_\theta(x)$ .

**Example 2.1** (Parameter estimation)  $\theta \in \Omega, \mathcal{A} = \Omega, \mathcal{L}(\theta, a) = \|\theta - a\|_p^p (p \geq 1) \stackrel{\text{or}}{=} \int \log \frac{P_\theta(x)}{P_a(x)} P_\theta(x) dm(x)$  (KL divergence).  $\mathcal{R} = \text{Var}(a) + \text{bias}^2(a)$ . Bregman loss:  $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$  describe any strictly convex differentiable function. Then  $\mathcal{L}_\phi(\theta, a) = \phi(a) - \phi(\theta) - (\phi - a)^T \nabla \phi(a)$ .

**Example 2.2** (Testing)  $\mathcal{A} = \{0, 1\}$  with action “0” associated with accepting  $H_0: \theta \in \Omega_0$  and “1”:  $H_1: \theta \in \Omega_1$ .  $\delta_x$  is a Bernolli distribution.  $\mathcal{L}(\theta, a) = I\{a = 1, \theta \in \Omega_0\} + I\{a = 0, \theta \in \Omega_1\}$ . Risk  $\mathcal{R}(\theta, \delta) = \mathbb{P}_\theta(A = 1)1_{\theta \in \Omega_0} + \mathbb{P}_\theta(A = 0)1_{\theta \in \Omega_1}$ .

**Definition 2.2** (Admissibility) A decision rule  $\delta$  is called inadmissible if a competing rule  $\delta^*$  such that  $\mathcal{R}(\theta, \delta^*) \leq \mathcal{R}(\theta, \delta)$  for all  $\theta \in \Omega$  and  $\mathcal{R}(\theta, \delta^*) < \mathcal{R}(\theta, \delta)$  for at least one  $\theta \in \Omega$ . Otherwise,  $\delta$  is admissible.

**Definition 2.3** (Bayes rule) The maximum risk  $\bar{\mathcal{R}}(\delta) = \sup_{\theta \in \Omega} \mathcal{R}(\theta, \delta)$  and the Bayes risk  $r(\Lambda, \delta) = \int \mathcal{R}(\theta, \delta) d\Lambda(\theta) = \int \mathcal{L}(\theta, \delta) d\mathbb{P}(x, \theta)$  ( $\Lambda(\theta)$  is a prior). A decision rule that minimizes the Bayes risk is called a Bayes rule, that is,  $\hat{\delta}: r(\Lambda, \hat{\delta}) = \inf_{\delta} r(\Lambda, \delta)$ . Minimax rule  $\delta^*: \sup_{\theta \in \Omega} \mathcal{R}(\theta, \delta^*) = \inf_{\delta} \sup_{\theta \in \Omega} \mathcal{R}(\theta, \delta)$ .

**Theorem 2.1** If risk functions for all decision rules are continuous in  $\theta$ , if  $\delta$  is Bayesian for  $\Lambda$  and has finite integrated risk  $r(\Lambda, \delta) < \infty$ , and if the support of  $\Lambda$  is the whole state space  $\Omega$ , then  $\delta$  is admissible.

**Property 2.1**  $p(\theta|x) = \frac{p_\theta(x)\lambda(\theta)}{\int p_\theta(x)\lambda(\theta)d\theta} := \frac{p_\theta(x)\lambda(\theta)}{m(x)}$ . Define the posterior risk of  $\delta$ :  $r(\delta|X = x) = \int \mathcal{L}(\theta, \delta(x)) d\mathbb{P}(\theta|x)$ . The Bayes risk  $r(\Lambda, \delta)$  satisfies that  $r(\Lambda, \delta) = \int r(\delta|x) dM(x)$ . Let  $\hat{\delta}(x)$  be the value of  $\delta$  that minimizes  $r(\delta|x)$ . Then  $\hat{\delta}$  is the Bayes rule.

**Example 2.3** (Application to supervised learning: regression)  $(X, Y) \in \mathcal{X} \times \mathcal{Y}, f: \mathcal{X} \rightarrow \mathcal{Y}, \mathcal{A} = \Omega = \mathcal{Y}, \mathcal{D} = \mathcal{X}, \delta = f, \mathcal{L}(Y, f(X)) = \|Y - f(X)\|_p^p, p \geq 1$ , risk  $R_f = \iint \mathcal{L}(y, f(x)) d\mathbb{P}(x, y) = \mathbb{E}[\mathcal{L}(Y, f(X))] = \mathbb{E}[\mathbb{E}\mathcal{L}(Y, f(X))|X]$ . When  $p = 2$ ,  $r(f|X = x) = \int \mathcal{L}(y, f(x)) d\mathbb{P}(y|x) = \int |y - f(x)|^2 d\mathbb{P}(y|x)$ . Regression function is  $g(x) := \int y d\mathbb{P}(y|x) \Rightarrow R_f = \mathbb{E}|Y - f(X)|^2 = \mathbb{E}|Y - g(X) + g(X) - f(X)|^2 = \mathbb{E}|Y - g(X)|^2 + \mathbb{E}|g(X) - f(X)|^2 \geq \mathbb{E}|Y - g(X)|^2$ .

**Example 2.4** (Application to supervised learning: pattern classification)  $Y \in \{0, 1\}, p_0 = P(Y = 0), p_1 = P(Y = 1) = 1 - p_0, \mathbb{E}[\mathcal{L}(Y, f(X))] = P(Y \neq f(X))$ . The Bayesian predictor is given by  $f(x) = 1_{\{P(Y=1|X=x) \geq \frac{\mathcal{L}(1,0) - \mathcal{L}(0,0)}{\mathcal{L}(0,1) - \mathcal{L}(1,1)} P(Y=0|X=x)\}}$ .

**Proof**  $\mathbb{E}[\mathcal{L}(Y, f(X))|X = x] = \begin{cases} \mathbb{E}[\mathcal{L}(Y, 0)|X = x] = \mathcal{L}(0, 0)\mathbb{P}(Y = 0|X = x) + \mathcal{L}(1, 0)\mathbb{P}(Y = 1|X = x) \\ \mathbb{E}[\mathcal{L}(Y, 1)|X = x] = \mathcal{L}(0, 1)\mathbb{P}(Y = 0|X = x) + \mathcal{L}(1, 1)\mathbb{P}(Y = 1|X = x) \end{cases}$ , compare the sizes of the two. □

**Property 2.2** (Continuation)  $\mathbb{P}(Y = 1|X = x) = \mathbb{E}(Y|X = x) := g(x)$ ,  $f(x) = 1_{\{g(x) \geq \frac{1}{2}\}}$ . Then  $0 \leq \mathbb{P}(\hat{f}(X) \neq Y) - \mathbb{P}(f(X) \neq Y) \leq 2 \int_{\mathcal{X}} |\hat{g}(x) - g(x)| \mu(dx) \leq 2(\int_{\mathcal{X}} |\hat{g}(x) - g(x)|^2 \mu(dx))^{\frac{1}{2}}$ . In Example 2.4,  $f(x) = 1_{\{\frac{p(x|y=1)}{p(x|y=0)} \geq \frac{p_0(\mathcal{L}(0,1) - \mathcal{L}(0,0))}{p_1(\mathcal{L}(1,0) - \mathcal{L}(1,1))}\}}$ , which takes the same form as the likelihood ratio test (LRT): Likelihood  $L(X) := \frac{p(X|Y=1)}{p(X|Y=0)}$  and  $f(x) = 1_{\{L(x) \geq \eta\}}$ .

**Definition 2.4** (Confusion table) True Positive Rate:  $\text{TPR} = \mathbb{P}(\hat{Y} = 1|Y = 1)$ ; False Negative Rate:  $\text{FNR} = 1 - \text{TPR}$ , type II error; False Positive Rate:  $\text{FPR} = \mathbb{P}(\hat{Y} = 1|Y = 0)$ , type I error; True Negative Rate:  $\text{TNR} = 1 - \text{FPR}$ . Precision:  $\mathbb{P}(Y = 1|\hat{Y} = 1) = \frac{p_1 \text{TPR}}{p_0 \text{FPR} + p_1 \text{TPR}}$ .  $F_1$ -score:  $F_1$  is the harmonic mean of precision and recall, which can be written as  $F_1 = \frac{2 \text{TPR}}{1 + \text{TPR} + \frac{p_0}{p_1} \text{FPR}}$ .

	$Y = 0$	$Y = 1$
$\hat{Y} = 0$	true negative	false negative
$\hat{Y} = 1$	false positive	true positive

**Theorem 2.2** (N-P lemma) Optimization: maximize TPR subject to  $\text{FPR} \leq \alpha, \alpha \in [0, 1]$ . Randomized rule:  $Q$  return 1 with probability  $Q(x)$  and 0 with probability  $1 - Q(x)$ . Maximize  $\mathbb{E}[Q(x)|Y = 1]$  subject to  $\mathbb{E}[Q(x)|Y = 0] \leq \alpha$ . Suppose the likelihood functions  $p(x|y)$  are continuous. Then the optimal predictor is a deterministic LRT.

**Proof** Let  $\eta$  be the threshold for an LRT such that the predictor  $Q_\eta(x) = 1_{\{\alpha(x) \geq \eta\}}$  has  $\text{FPR} = \alpha$ . Such an LRT exists because likelihood functions are continuous. Let  $\beta$  denote the TPR of  $Q_\eta$ . Prove that  $Q_\eta$  is optimal for risk minimization problem corresponding to the loss functions  $\mathcal{L}(0, 1) = \eta \frac{p_1}{p_0}, \mathcal{L}(1, 0) = 1, \mathcal{L}(1, 1) = \mathcal{L}(0, 0) = 0$  since  $\frac{p_0(\mathcal{L}(0,1) - \mathcal{L}(0,0))}{p_1(\mathcal{L}(1,0) - \mathcal{L}(1,1))} = \frac{p_0 \mathcal{L}(0,1)}{p_1 \mathcal{L}(1,0)} = \eta$ . Under these loss functions, the risk of Bayes predictor for  $Q$  is  $\mathcal{R}_Q = p_0 \text{FPR}(Q) \mathcal{L}(0, 1) + p_1(1 - \text{TPR}(Q)) \mathcal{L}(1, 0) = p_1 \eta \text{FPR}(Q) + p_1(1 - \text{TPR}(Q))$ . Now let  $Q$  be any other rule with  $\text{FPR}(Q) \leq \alpha$ ,  $\mathcal{R}_{Q_\eta} = p_1 \eta \alpha + p_1(1 - \beta) \leq p_1 \eta \text{FPR}(Q) + p_1(1 - \text{TPR}(Q)) \leq p_1 \eta \alpha + p_1(1 - \text{TPR}(Q)) \Rightarrow \text{TPR}(Q) \leq \beta$ .  $\square$

**Definition 2.5** (ROC (Receiver operating character) curve)  $y$ -axis is TPR and  $x$ -axis is FPR.

**Proposition 2.1** (1) The points  $(0, 0)$  and  $(1, 1)$  are on the ROC curve; (2) The ROC must lie above the main diagonal; (3) The ROC curve is concave.

**Proof** We only prove (2). Fix  $\alpha \in (0, 1)$  and consider a randomized rule  $\text{TPR} = \text{FPR} = \alpha$ ,  $Q(x) \equiv \alpha$ ; (3): Consider two rules  $(\text{FPR}(\eta_1), \text{TPR}(\eta_1))$  and  $(\text{FPR}(\eta_2), \text{TPR}(\eta_2))$ . Flip a biased coin and use the first rule with probability  $t$  and the second rule with probability  $1 - t$ . Then this yields a randomized rule with  $(\text{FPR}, \text{TPR}) = (t \text{FPR}(\eta_1) + (1 - t) \text{FPR}(\eta_2), t \text{TPR}(\eta_1) + (1 - t) \text{TPR}(\eta_2))$ . Fixing  $\text{FPR} \leq t \text{FPR}(\eta_1) + (1 - t) \text{FPR}(\eta_2)$ ,  $\text{TPR} \geq t \text{TPR}(\eta_1) + (1 - t) \text{TPR}(\eta_2)$ .  $\square$

### 3 Markov Decision Process

**Definition 3.1** (Basic concepts) Five elements: decision epoches, states, actions, transition probabilities and rewards.

(1) Decision epoches: Let  $T$  denote the set of decision epoches, discrete:  $\{1, 2, \dots, N\}$ ; continuous:  $[0, N]$ ;  $N < / = \infty$ : finite or infinite. (2) State and action sets: decision epoch  $t \in T$ , the system occupies a state  $S_t \in \mathcal{S}$ , the decision maker  $a \in \mathcal{A}$ . (3) Reward and transition probabilities:  $t$ , in state  $s$ , choose action  $a$ , (i) the decision maker receives a reward  $r_t(s, a)$ , (ii) the system state at the next decision epoch is determined by the probability distribution  $p_t(\cdot|s, a)$ .

**Definition 3.2** (Decision rules) Prescribe a procedure for action selection in each state at a specified decision epoch. Four cases: (1) Markovian and Deterministic (MD):  $\delta_t : \mathcal{S} \rightarrow \mathcal{A}$ ; (2) M and Randomized (MR):  $\delta_t : \mathcal{S} \rightarrow \Delta(\mathcal{A})(q_{\delta_t(s)}(a))$ ; (3) History-dependent and D (HD):  $h_t = (s_1, a_1, \dots, s_{t-1}, a_{t-1}, s_t) = (h_{t-1}, a_{t-1}, s_t)$ ,  $\mathcal{H}_1 = \mathcal{S}, \mathcal{H}_2 = \mathcal{S} \times \mathcal{A} \times \mathcal{S}, \dots, \delta_t : \mathcal{H}_t \rightarrow \mathcal{A}$ ; (4) HR:  $\delta_t : \mathcal{H}_t \times \Delta(\mathcal{A})$ . A policy  $\pi = (\delta_1, \delta_2, \dots, \delta_{N-1})$  is stationary if  $\delta_1 = \delta_2 = \dots = \delta$  for  $t \in T$ .

**Definition 3.3** Let  $\pi = (\delta_1, \dots, \delta_{N-1})$  in HR and  $R_t := r_t(X_t, Y_t)$  denote the random reward,  $R_N := r_N(X_N)$ ,  $R := (R_1, \dots, R_N)$ . The expected total reward  $U_N^\pi(s) := \mathbb{E}^\pi \{ \sum_{t=1}^{N-1} r_t(X_t, Y_t) + r_N(X_N) | X_1 = s \}$ . Assume  $|r_t(s, a)| \leq M < \infty$  for all  $(s, a) \in \mathcal{S} \times \mathcal{A}$ . Optimal policy:  $U_N^*(s) \geq U_N^\pi(s), s \in \mathcal{S}$ .  $\varepsilon$ -optimal policy:  $U_N^{\pi^*}(s) + \varepsilon > U_N^\pi(s), s \in \mathcal{S}$ . The value of the MDP:  $U_N^*(s) = \sup_{\pi \in \mathcal{D}^{\text{HR}}} U_N^\pi(s), s \in \mathcal{S}$ .

**Property 3.1** (Finite-Horizon Policy Evaluation)  $V_t^\pi(h_t) = \mathbb{E}^\pi\{\sum_{k=t}^{N-1} r_k(X_k, Y_k) + r_N(X_N) | h_t\}$ ,  $V_N^\pi(h_N) = r_N(s)$ ,  $\pi \in \mathcal{D}^{\text{HD}}$ . By the formula of total expectation,

$$V_t^\pi(h_t) = r_t(s_t, \delta_t(h_t)) + \mathbb{E}_{h_t}^\pi V_{t+1}^\pi(h_t, \delta_t(h_t), X_{t+1}) = r_t(s_t, \delta_t(h_t)) + \sum_{j \in \mathcal{S}} V_{t+1}^\pi(h_t, \delta_t(h_t), j) p(j | s_t, \delta_t(h_t)).$$

Consider randomness, i.e.  $\pi \in \mathcal{D}^{\text{HR}}$ ,

$$V_t^\pi(h_t) = \sum_{a \in \mathcal{A}} q_{\delta_t(h_t)}(a) \{r_t(s_t, a) + \sum_{j \in \mathcal{S}} V_{t+1}^\pi(h_t, a, j) p(j | s_t, a)\}.$$

Computational complexity: let  $K = |\mathcal{S}|$ ,  $L = |\mathcal{A}|$ , at decision epoch  $t$ ,  $K^{t+1}L^t$  histories,  $K^2 \sum_{i=0}^{N-1} (KL)^i$  multiplications. If  $\pi \in \mathcal{D}^{\text{MD}}$ ,

$$V_t^\pi(s_t) = r_t(s_t, \delta_t(s_t)) + \sum_{j \in \mathcal{S}} V_{t+1}^\pi(j) p(j | s_t, \delta_t(s_t)),$$

only  $(N-1)K^2$  multiplications. On the other hand, given  $\pi$ , this yields a valid and accurate calculation method for  $U_N^\pi(s)$ .

**Theorem 3.1** (The Bellman Equations) Let  $V_t^*(h_t) = \sup_{\pi \in \mathcal{D}^{\text{HR}}} V_t^\pi(h_t)$ . The optimality equations:

$$V_t(h_t) = \sup_{a \in \mathcal{A}} \{r_t(s_t, a) + \sum_{j \in \mathcal{S}} V_{t+1}(h_t, a, j) p_t(j | s_t, a)\} \text{ for } t = 1, 2, \dots, N-1 \text{ and } h_t = (h_{t-1}, a_{t-1}, s_t) \in \mathcal{H}_t.$$

For  $t = N$ ,  $V_N(h_N) = r_N(s_N)$ . Suppose  $V_t$  is a solution and  $V_N$  satisfies  $V_N(h_N) = r_N(s_N)$ . Then  $V_t(h_t) = V_t^*(h_t)$  for all  $h_t \in \mathcal{H}_t$ ,  $t = 1, \dots, N$  and  $V_1(s_1) = V_1^*(s_1) = U_N^*(s_1)$  for all  $s_1 \in \mathcal{S}$ .

**Proof** We divide the proof into two parts.

Step 1: Prove  $V_n(h_n) \geq V_n^*(h_n)$  for all  $h_n \in \mathcal{H}_n$ . By induction: For  $t = N$ ,  $V_N(h_N) = r_N(s_N) = V_N^*(h_N)$  for all  $h_N, \pi$ . Now assume that  $V_t(h_t) \geq V_t^*(h_t)$  for all  $h_t \in \mathcal{H}_t$  for  $t = n+1, \dots, N$ . Let  $\pi' = (\delta'_1, \dots, \delta'_{N-1})$  be an arbitrary policy in  $\mathcal{D}^{\text{HR}}$ . On the one hand, for  $t = n$ , it is trivial that

$$\begin{aligned} V_n(h_n) &= \sup_{a \in \mathcal{A}} \{r_n(s_n, a) + \sum_{j \in \mathcal{S}} p(j | s_n, a) V_{n+1}(h_n, a, j)\} \geq \sup_{a \in \mathcal{A}} \{r_n(s_n, a) + \sum_{j \in \mathcal{S}} p_n(j | s_n, a) V_{n+1}^*(h_n, a, j)\} \\ &\geq \sup_{a \in \mathcal{A}} \{r_n(s_n, a) + \sum_{j \in \mathcal{S}} p_n(j | s_n, a) V_{n+1}^{\pi'}(h_n, a, j)\} \geq V_n^{\pi'}(h_n). \end{aligned}$$

Step 2: Prove that for any  $\varepsilon > 0$ , there exists a  $\pi \in \mathcal{D}^{\text{HD}}$  such that

$$V_n^{\pi'}(h_n) + (N-n)\varepsilon \geq V_n(h_n) \Rightarrow V_n^*(h_n) + (N-n)\varepsilon \geq V_n^{\pi'}(h_n) + (N-n)\varepsilon \geq V_n(h_n) \geq V_n^*(h_n).$$

Construct a policy  $\pi' = (\delta'_1, \dots, \delta'_{N-1})$  by choosing  $\delta'_n(h_n)$  to satisfy

$$r_n(s_n, \delta'_n(h_n)) + \sum_{j \in \mathcal{S}} p_n(j | s_n, \delta'_n(h_n)) V_{n+1}(h_n, \delta'_n(h_n), j) + \varepsilon \geq V_n(h_n).$$

By induction: For  $t = N$ ,  $V_N^{\pi'}(h_N) = V_N(h_N)$ . Assume  $V_t^{\pi'}(h_t) + (N-t)\varepsilon \geq V_t(h_t)$  for  $t = n+1, \dots, N$ . For  $t = n$ ,

$$V_n^{\pi'}(h_n) = r_n(s_n, \delta'_n(h_n)) + \sum_{j \in \mathcal{S}} p_n(j | s_n, \delta'_n(h_n)) V_{n+1}^{\pi'}(h_n, \delta'_n(h_n), j) \geq V_n(h_n) - (N-n)\varepsilon. \quad \square$$

**Remark 3.1** The equations yield that  $\delta_t^*(h_t) \in \arg \max_{a \in \mathcal{A}} \{r_t(s_t, a) + \sum_{j \in \mathcal{S}} p_t(s_t, a) V_{t+1}^*(h_t, a, j)\}$ , which means it is HD, i.e.  $U_N^*(s) = \sup_{\pi \in \mathcal{D}^{\text{HR}}} U_N^\pi(s) = \sup_{\pi \in \mathcal{D}^{\text{HD}}} U_N^\pi(s) \stackrel{?}{=} \sup_{\pi \in \mathcal{D}^{\text{MD}}} U_N^\pi(s)$ . We will answer “?” in the following theorem.

**Theorem 3.2** Let  $V_t^*, t = 1, \dots, N$  be solutions of Bellman Equations. Then (a) For each  $t = 1, \dots, N$ ,  $V_t^*(h_t)$  depends on  $h_t$  only through  $s_t$ ; (b) For any  $\varepsilon > 0$ , there exists an  $\varepsilon$ -optimal policy which is D and M; (c) Maximum can be achieved, it is optimal, which is MD.

**Proof** We only prove (a). By induction,  $V_N^*(h_N) = V_N^*(h_{N-1}, a_{N-1}, s) = r_N(s)$  for all  $h_{N-1} \in \mathcal{H}_{N-1}$ . Assume (a) is valid for  $t = n + 1, \dots, N$ . Then  $V_n^*(h_n) = \sup_{a \in \mathcal{A}} \{r_t(s_t, a) + \sum_{j \in \mathcal{S}} p_t(j|s_t, a) V_{t+1}^*(j)\} = V_n^*(s_t)$ .  $\square$

**Definition 3.4** (Backward Induction (Dynamic Programming) Algorithm) 1. Set  $t = N$  and  $V_N^*(s_N) = r_N(s_N)$  for all  $s_N \in \mathcal{S}$ ; 2. Substitute  $t - 1$  for  $t$  and compute  $V_t^*(s_t)$  for each  $s_t \in \mathcal{S}$  according to

$$V_t^*(s_t) = \max_{a \in \mathcal{A}} \{r_t(s_t, a) + \sum_{j \in \mathcal{S}} p_t(j|s_t, a) V_{t+1}^*(j)\},$$

and set  $\mathcal{A}_{s_t} = \arg \max_{a \in \mathcal{A}} \{r_t(s_t, a) + \sum_{j \in \mathcal{S}} p_t(j|s_t, a) V_{t+1}^*(j)\}$ ; 3. If  $t = 1$ , stop. Otherwise return to Step 2.

**Remark 3.2** (1) At time  $t$ , specialized  $\mathcal{S}_t$  and  $\mathcal{A}_s$ , special structure for  $r_t$  and  $p_t$ ; (2)  $K = |\mathcal{S}|$  and  $L = |\mathcal{A}|$ , at each  $t$ , only  $(N - 1)LK^2$  multiplications, ease computation and storage cost (because there are  $(L^K)^{N-1}$  DM policies).

**Definition 3.5** (Infinite-Horizon MDPs) Assumptions: Stationary reward and transition probabilities, i.e.  $r_t(s, a) \equiv r(s, a)$ ,  $p_t(j|s, a) \equiv p(j|s, a)$ ; Bounded rewards, i.e.  $|r(s, a)| \leq M < \infty$  for all  $a \in \mathcal{A}$  and  $s \in \mathcal{S}$ ; Discounting coefficient  $\lambda, 0 \leq \lambda < 1$ ; Discrete state space  $\mathcal{S}$ . The expected total reward of policy  $\pi = (\delta_1, \delta_2, \dots) \in \mathcal{D}^{\text{HR}}$ :

$$U^\pi(s) = \lim_{N \rightarrow +\infty} \mathbb{E}_s^\pi \left\{ \sum_{t=1}^N \lambda^{t-1} r(X_t, Y_t) \right\} = \mathbb{E}_s^\pi \left\{ \sum_{t=1}^{+\infty} \lambda^{t-1} r(X_t, Y_t) \right\}.$$

We say that a policy  $\pi^*$  is optimal when  $U^{\pi^*}(s) \geq U^\pi(s)$  for each  $s \in \mathcal{S}$  and all  $\pi \in \mathcal{D}^{\text{HR}}$ . Define the value of the MDP  $U^*(s) = \sup_{\pi \in \mathcal{D}^{\text{HR}}} U^\pi(s)$ . Let  $U_\nu^\pi(s)$  denote the expected reward obtained by using  $\pi$  when the horizon  $\nu$  is random. Then  $U_\nu^\pi(s) = \mathbb{E}_s^\pi \left\{ \mathbb{E}_{\nu \sim P} \sum_{t=1}^\nu r(X_t, Y_t) \right\}$ .

**Theorem 3.3** Suppose  $\nu$  has a GD( $\lambda$ ), i.e.  $\mathbb{P}(\nu = n) = \lambda^{n-1}(1 - \lambda)$ . Then  $U^\pi(s) = U_\nu^\pi(s)$  for all  $s \in \mathcal{S}$ .

**Proof**  $\mathbb{E}_\nu^\pi(s) = \mathbb{E}_s^\pi \left\{ \sum_{n=1}^{+\infty} \sum_{t=1}^n r(X_t, Y_t) (1 - \lambda) \lambda^{n-1} \right\} = \mathbb{E}_s^\pi \left\{ \sum_{t=1}^{+\infty} \sum_{n=t}^{+\infty} r(X_t, Y_t) (1 - \lambda) \lambda^{n-1} \right\} = \mathbb{E}_s^\pi \left\{ \sum_{t=1}^{+\infty} \lambda^{t-1} r(X_t, Y_t) \right\}$ .  $\square$

**Theorem 3.4** Suppose  $\pi \in \mathcal{D}^{\text{HR}}$ , then for each  $s \in \mathcal{S}$ , there exists a  $\pi' \in \mathcal{D}^{\text{MR}}$  for which  $U^{\pi'}(s) = U^\pi(s)$ .

**Proof** Note that

$$U^\pi(s) = \mathbb{E}_s^\pi \left\{ \sum_{t=1}^{+\infty} \lambda^{t-1} r(X_t, Y_t) \right\} = \sum_{t=1}^{+\infty} \sum_{j \in \mathcal{S}} \sum_{a \in \mathcal{A}} \lambda^{t-1} r(j, a) p^\pi(X_t = j, Y_t = a | X_1 = s).$$

Fixing  $s \in \mathcal{S}$ , we only need to check

$$p^\pi(X_t = j, Y_t = a | X_1 = s) = p^{\pi'}(X_t = j, Y_t = a | X_1 = s).$$

For each  $j \in \mathcal{S}$  and  $a \in \mathcal{A}$ , define the randomized Markov decision rule  $\delta'_t$  by

$$q_{\delta'_t(j)}(a) = p^\pi(Y_t = a | X_t = j, X_1 = s).$$

Then

$$p^{\pi'}(Y_t = a | X_t = j) = p^\pi(Y_t = a | X_t = j, X_1 = s).$$

Assume the conclusion holds for  $t = 0, 1, \dots, n - 1$ . Then

$$\begin{aligned} p^{\pi'}(X_n = j, Y_n = a | X_1 = s) &= p^{\pi'}(Y_n = a | X_n = j, X_1 = s) p^{\pi'}(X_n = j | X_1 = s) \\ &= p^\pi(Y_n = a | X_n = j, X_1 = s) p^{\pi'}(X_n = j | X_1 = s). \end{aligned}$$

Then by induction assumption,

$$\begin{aligned} p^\pi(X_n = j | X_1 = s) &= \sum_{k \in \mathcal{S}} \sum_{a \in \mathcal{A}} p^\pi(X_{n-1} = k, Y_{n-1} = a | X_1 = s) p(j|k, a) \\ &= \sum_{k \in \mathcal{S}} \sum_{a \in \mathcal{A}} p^{\pi'}(X_{n-1} = k, Y_{n-1} = a | X_1 = s) p(j|k, a) = p^{\pi'}(X_n = j | X_1 = s) \end{aligned} \quad \square$$

**Proposition 3.1** (Vector expression for MDP) Let  $\delta$  be MD, define  $r_\delta(s)$  and  $p_\delta(j|s)$  by

$$r_\delta(s) := r(s, \delta(s)), p_\delta(j|s) := p(j|s, \delta(s)).$$

Denote  $r_\delta = (r_\delta(1), \dots, r_\delta(|\mathcal{S}|))^T \in \mathbb{R}^{|\mathcal{S}|}$ ,  $p_\delta = (p_\delta)_{(s,j)} = p(j|s, \delta(s))$ . For MR  $\delta$ , define

$$r_\delta(s) = \sum_{a \in \mathcal{A}} q_{\delta(s)}(a) r(s, a), p_\delta(j|s) = \sum_{a \in \mathcal{A}} q_{\delta(s)}(a) p(j|s, a).$$

The  $(s, j)$ -th component of the  $t$ -step transition probability matrix  $p_\pi^t$  satisfies

$$\begin{aligned} p_\pi^t(j|s) &= [p_{\delta_1} p_{\delta_2} \cdots p_{\delta_t}](j|s) = p^\pi(X_{t+1} = j | X_1 = s) \\ \mathbb{E}_s^\pi g(X_t) &= \sum_{j \in \mathcal{S}} p_\pi^{t-1}(j|s) g(j) = (p_\pi^t g)_s \\ U^\pi &= \sum_{t=1}^{+\infty} \lambda^{t-1} p_\pi^{t-1} r_{\delta_t} = r_{\delta_1} + \lambda p_{\delta_1} (r_{\delta_1} + \lambda p_{\delta_2} r_{\delta_2} + \cdots) = r_{\delta_1} + \lambda p_{\delta_1} U^{\pi_1}. \end{aligned}$$

When  $\pi$  is stationary,  $U = r_\delta + \lambda p_\delta U$ .

**Theorem 3.5** Define  $\mathcal{L}U = \sup_{d \in \mathcal{D}^{\text{MD}}} \{r_d + \lambda p_d U\}$ . Suppose there exists a  $U \in \mathcal{U}$  for which (a)  $U \geq \mathcal{L}U$ , then  $U \geq U^*$ ; (b)  $U \leq \mathcal{L}U$ , then  $U \leq U^*$ ; (c)  $U = \mathcal{L}U$ , then  $U = U^*$ .

**Proof** (a) By the given conditions,

$$\begin{aligned} U &\geq \sup_{\delta \in \mathcal{D}^{\text{MR}}} \{r_\delta + \lambda p_\delta U\} \geq r_{\delta_1} + \lambda p_{\delta_1} U \geq r_{\delta_1} + \lambda p_{\delta_1} (r_{\delta_2} + \lambda p_{\delta_2} U) \\ &\geq r_{\delta_1} + \lambda p_{\delta_1} r_{\delta_2} + \cdots + \lambda^{n-1} p_{\delta_1} p_{\delta_2} \cdots p_{\delta_{n-1}} r_{\delta_n} + \lambda^n p_\pi^n U \\ \Rightarrow U - U^\pi &\geq \lambda^n p_\pi^n U - \sum_{k=n}^{+\infty} \lambda^k p_\pi^k r_{\delta_{k+1}} \geq 0. \end{aligned}$$

(b)  $U \leq \mathcal{L}U \Rightarrow U \leq r_d + \lambda p_d U + \varepsilon 1 \Rightarrow (I - \lambda p_d)U \leq r_d + \varepsilon 1 \Rightarrow U \leq (I - \lambda p_d)^{-1}(r_d + \varepsilon 1) = U^\pi + \varepsilon(1 - \lambda)^{-1} 1_{|\mathcal{S}|}$ .

(c) Omitted.  $\square$

**Theorem 3.6** If  $0 \leq \lambda < 1$ ,  $\mathcal{L}$  is a contraction mapping on  $\mathcal{U}$ .

**Proof** Let  $u$  and  $v$  in  $\mathcal{U}$ . For each  $s \in \mathcal{S}$ , assume  $\mathcal{L}v(s) \geq \mathcal{L}u(s)$  and let  $a_s^* = \arg \max_{a \in \mathcal{A}} \{r(s, a) + \sum_{j \in \mathcal{S}} \lambda p(j|s, a) v(j)\}$ .

Then

$$\begin{aligned} 0 \leq \mathcal{L}v(s) - \mathcal{L}u(s) &\leq r(s, a_s^*) + \sum_{j \in \mathcal{S}} \lambda p(j|s, a_s^*) v(j) - r(s, a_s^*) - \sum_{j \in \mathcal{S}} \lambda p(j|s, a_s^*) u(j) \\ &= \lambda \sum_{j \in \mathcal{S}} p(j|s, a_s^*) (v(j) - u(j)) \leq \lambda \sum_{j \in \mathcal{S}} p(j|s, a_s^*) \|u - v\| = \lambda \|u - v\|. \end{aligned} \quad \square$$

## 4 Statistical Learning Theory

**Definition 4.1** (Basic concepts)  $(X, Y) \sim P \in \mathcal{P}$ , definite  $(X_1, Y_1), \dots, (X_n, Y_n)$  i.i.d.,  $\mathcal{D}_n = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$ , risk  $\mathcal{R}_n(f) = \mathbb{E}_{(X, Y) \in \mathcal{D}_n} l(X, Y)$ . An algorithm  $A$  is a mapping from  $\mathcal{D}_n$  to a function  $\mathcal{X} \rightarrow \mathcal{Y}$ . Excess risk:  $\mathcal{R}_P(A(\mathcal{D}_n)) - \mathcal{R}_P^*$ . Expected error:  $\mathbb{E}[\mathcal{R}_P(A(\mathcal{D}_n))]$ . An algorithm is called consistent in expectation for  $P$  iff  $\mathbb{E}[\mathcal{R}_P(A(\mathcal{D}_n))] - \mathcal{R}_P^* \rightarrow 0$ . PAC (probability approximately correct): for a given  $\delta \in (0, 1)$  and  $\varepsilon > 0$ ,  $\mathbb{P}(\mathcal{R}_P(A(\mathcal{D}_n)) - \mathcal{R}_P^* \leq \varepsilon) \geq 1 - \delta$ .

**Definition 4.2** (Consistency)  $g(x) = \mathbb{E}[Y|X = x]$ ,  $g_n(x, \mathcal{D}_n) = g_n(x)$ ,  $\mathbb{E}\{|g_n(X) - Y|^2 | \mathcal{D}_n\} = \int_{\mathbb{R}^d} |g_n(x) - g(x)|^2 \mu(dx) + \mathbb{E}[g(X) - Y]^2$ . A sequence of regression function estimates  $\{g_n\}$  is called (a) weakly consistent for a certain distribution of  $(X, Y)$  if  $\lim_{n \rightarrow +\infty} \mathbb{E}\{\int [g_n(x) - g(x)] \mu(dx)\} = 0$ ; (b) strongly consistent for a certain distribution if  $\lim_{n \rightarrow +\infty} \int [g_n(x) - g(x)]^2 \mu(dx) = 0$  with probability 1; (c) weakly universally consistent if for all distributions of  $(X, Y)$  with  $\mathbb{E}[Y^2] < \infty$ ,  $\dots$ ; (d) strongly universally consistent  $\dots$ .

**Definition 4.3** (Penalized model)  $g_n = \arg \min_f \{\frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_i|^2 + J_n(f)\}$ . Penalized term for  $f$ :

$$J_n(f) = \lambda_n \int |f''(t)|^2 dt \text{ or } J_{n,k}(f) = \lambda_n \int \sum_{t_1, \dots, t_k \in \{1, \dots, d\}} \left| \frac{\partial^k f}{\partial x_{t_1} \cdots \partial x_{t_d}} \right|^2 dt, \dots$$

**Proposition 4.1** (Curse of dimensionality) Let  $X, X_1, \dots, X_n$  i.i.d.  $\mathbb{R}^d$  uniformly distributed in  $[0, 1]^d$ .

$$\begin{aligned} d_\infty(d, n) &= \mathbb{E}\left\{\min_{i=1, \dots, n} \|X - X_i\|_\infty\right\} = \int_0^\infty \mathbb{P}\left\{\min_{i=1, \dots, n} \|X - X_i\|_\infty > t\right\} dt \\ &= \int_0^\infty (1 - \mathbb{P}\left\{\min_{i=1, \dots, n} \|X - X_i\|_\infty < t\right\}) dt. \end{aligned}$$

Since  $\mathbb{P}\{\min_i \|X - X_i\|_\infty < t\} \leq n\mathbb{P}(\|X - X_1\|_\infty \leq t) \leq n(2t)^d$ ,  $d_\infty(d, n) \geq \frac{d}{2(d+1)} n^{-\frac{1}{d}}$ .

**Theorem 4.1** (No-Free lunch theorem) Let  $\{a_n\}$  be a sequence of positive numbers converging to 0. For every sequence of regression estimates, there exists a distribution of  $(X, Y)$  such that  $X$  is uniformly distributed on  $[0, 1]$ ,  $Y = g(X)$ ,  $g$  is  $\pm 1$  valued, and  $\limsup_{n \rightarrow +\infty} \frac{\mathbb{E}\|g_n - g\|^2}{a_n} \geq 1$ .

**Proof** Let  $\{p_j\}$  be a probability distribution and let  $A = \{A_j\}$  be a partition of  $[0, 1]$  such that  $A_j$  is an interval of length  $p_j$ . Consider regression function indexed by a parameter  $c = (c_1, c_2, \dots)$  with  $c_j \in \{\pm 1\}$ . Define  $g^{(c)} : [0, 1] \rightarrow \{-1, 1\}$  by  $g^{(c)}(x) = c_j$  iff  $x \in A_j$  and  $Y = g^{(c)}(X)$ . For  $x \in A_j$ , define  $\bar{g}_n(x) = \frac{1}{p_j} \int_{A_j} g_n(z) \mu(dz)$  to be the projection of  $g_n$  on  $A$ . Then

$$\begin{aligned} \int_{A_j} |g_n(x) - g^{(c)}(x)|^2 \mu(dx) &= \int_{A_j} |g_n(x) - \bar{g}_n(x)|^2 \mu(dx) + \int_A |\bar{g}_n(x) - g^{(c)}(x)|^2 \mu(dx) \\ &\geq \int_A |\bar{g}_n(x) - g^{(c)}(x)|^2 \mu(dx). \end{aligned}$$

Set  $\hat{c}_{nj} = \begin{cases} 1 & \text{if } \int_{A_j} g_n(z) \mu(dz) \geq 0 \\ -1 & \text{otherwise} \end{cases}$ . For  $x \in A_j$ , if  $\hat{c}_{nj} = 1$  and  $c_j = -1$ , then  $\bar{g}_n(x) \geq 0$  and  $g^{(c)}(x) = -1$ , implying  $|\bar{g}_n(x) - g^{(c)}(x)| \geq 1$ ; if  $\hat{c}_{nj} = -1$  and  $c_j = 1$ , then  $\bar{g}_n(x) < 0$  and  $g^{(c)}(x) = 1$ , also implying  $|\bar{g}_n(x) - g^{(c)}(x)| \geq 1$ . Therefore,

$$\begin{aligned} \int_A |\bar{g}_n(x) - g^{(c)}(x)|^2 \mu(dx) &\geq 1_{\{\hat{c}_{nj} \neq c_j\}} \int_{A_j} 1 \mu(dx) \geq 1_{\{\hat{c}_{nj} \neq c_j\}} p_j \geq 1_{\{\hat{c}_{nj} \neq c_j\}} 1_{\{\mu_n(A_j)=0\}} p_j \\ \Rightarrow \mathbb{E} \left\{ \int |g_n(x) - g^{(c)}(x)|^2 \mu(dx) \right\} &\geq \sum_{j=1}^{+\infty} \mathbb{P}(\hat{c}_{nj} \neq c_j, \mu_n(A_j) = 0) p_j := R_n(c). \end{aligned}$$

Now we randomize  $c$ . Let  $C_1, C_2, \dots$  be a sequence of i.i.d. random variables independent of  $X_1, X_2, \dots$  which satisfy  $\mathbb{P}(C_1 = 1) = \mathbb{P}(C_1 = -1) = \frac{1}{2}$ . Thus

$$\begin{aligned} \mathbb{E} R_n(C) &= \sum_{j=1}^{+\infty} \mathbb{E} \mathbb{P}(\hat{C}_{nj} \neq C_j, \mu_n(A_j) = 0) p_j \stackrel{\text{total expectation}}{=} \sum_{j=1}^{+\infty} \mathbb{E} \{1_{\{\mu_n(A_j)=0\}} \mathbb{P}(\hat{C}_{nj} \neq C_j | X_1, \dots, X_n)\} p_j \\ &= \frac{1}{2} \sum_{j=1}^{+\infty} \mathbb{P}(\mu_n(A_j) = 0) p_j = \frac{1}{2} \sum_{j=1}^{+\infty} (1 - p_j)^n p_j. \end{aligned}$$

On the other hand,

$$R_n(c) \leq \sum_{j=1}^{+\infty} \mathbb{P}(\mu_n(A_j) = 0) p_j = \sum_{j=1}^{+\infty} (1 - p_j)^n p_j \Rightarrow \frac{R_n(c)}{\mathbb{E} R_n(C)} \leq 2.$$

By Fatou's lemma,

$$\mathbb{E} \left\{ \limsup_{n \rightarrow +\infty} \frac{R_n(C)}{\mathbb{E} R_n(C)} \right\} \geq \limsup_{n \rightarrow +\infty} \left\{ \frac{R_n(C)}{\mathbb{E} R_n(C)} \right\} = 1,$$

which implies that there exists  $c \in C$  such that

$$\limsup_{n \rightarrow +\infty} \frac{R_n(C)}{\mathbb{E} R_n(C)} \geq 1 \Rightarrow \limsup_{n \rightarrow +\infty} \frac{\mathbb{E} \left\{ \int |g_n(x) - g(x)|^2 \mu(dx) \right\}}{\frac{1}{2} \sum_{j=1}^{+\infty} (1 - p_j)^n p_j} \geq 1.$$

Let  $\{a_n\}$  be a sequence of positive numbers converging to 0 with  $\frac{1}{2} \geq a_1 \geq a_2 \geq \dots$ , then there exists a probability  $\{p_j\}$  such that  $\sum_{j=1}^{+\infty} (1 - p_j)^n p_j \geq a_n, \forall n$ . □



**Definition 4.4** (Minimax lower bounds) (a) The sequence of positive numbers  $a_n$  is called the lower minimax rate of convergence for the  $\mathcal{P}$  if  $\liminf_{n \rightarrow +\infty} \inf_{g_n} \sup_{P \in \mathcal{P}} \frac{\mathbb{E}\|g_n - g\|^2}{a_n} = c_1 > 0$ . (b)  $a_n$  is called optimal rate of convergence for the class  $\mathcal{P}$  if it is a lower minimax rate of convergence and there is an estimate  $g_n$  such that  $\limsup_{n \rightarrow +\infty} \sup_{P \in \mathcal{P}} \frac{\mathbb{E}\|g_n - g\|^2}{a_n} = c_n < \infty$ .

**Definition 4.5** (Smoothness) Let  $q = k + \beta$  for some  $k \in \mathbb{N}$  and  $0 < \beta \leq 1$  and let  $\rho > 0$ . A function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is called  $(q, \rho)$ -smooth if for every  $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \mathbb{N}, \sum_{i=1}^d \alpha_i = k$ , the partial derivative  $\frac{\partial^k f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$  exists and satisfies  $\left| \frac{\partial^k f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}(x) - \frac{\partial^k f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}(z) \right| \leq \rho \|x - z\|^\beta$ . Let  $\mathcal{F}^{(q, \rho)}$  be the set of all  $(q, \rho)$ -smooth functions  $f$ . Let  $\mathcal{P}^{(q, \rho)}$  be the class of distributions  $(X, Y)$  such that (i)  $X$  is uniformly distributed on  $[0, 1]^d$ ; (ii)  $Y = g(X) + N$ , where  $X \perp\!\!\!\perp N$ , and  $N$  is standard normal; (iii)  $g \in \mathcal{F}^{q, \rho}$ .

**Lemma 4.1** Let  $u$  be an  $l$ -dimensional real vector, let  $C$  be a zero means random variables taking values in  $\{-1, 1\}$  and let  $N$  be an  $l$ -dimensional standard normal independent of  $C$ . Set  $Z = Cu + N$ . Then the error probability of the Bayesian decision for  $C$  based on  $Z$  is  $\mathcal{R}^* = \min_{g: \mathbb{R}^l \rightarrow \mathbb{R}} \mathbb{P}(g(Z) \neq C) = \Phi(-\|u\|)$ .

**Proof**  $\mathbb{P}(C = 1) = \mathbb{P}(C = -1) = \frac{1}{2}$ ,  $\mathbb{P}(Z|C = 1) = \mathcal{N}(u, I)$ ,  $\mathbb{P}(Z|C = -1) = \mathcal{N}(-u, I)$ . By the Bayes formula,

$$\mathbb{P}(C = 1|Z = z) = \frac{\mathbb{P}(C = 1)\mathbb{P}(Z|C = 1)}{\mathbb{P}(C = 1)\mathbb{P}(Z|C = 1) + \mathbb{P}(C = -1)\mathbb{P}(Z|C = -1)} = \frac{1}{1 + \exp\left(\frac{\|Z - u\|^2}{2} - \frac{\|Z + u\|^2}{2}\right)} = \frac{1}{1 + \exp(-2Z^T u)}.$$

Therefore, the optimal Bayes decision is  $g^*(Z) = \text{sgn}(Z^T u)$ , and the risk is

$$\begin{aligned} \mathcal{R}^* &= \mathbb{P}(g^*(Z) \neq C) = \mathbb{P}(Z^T u < 0, C = 1) + \mathbb{P}(Z^T u > 0, C = -1) \\ &= \mathbb{P}(\|u\|^2 + u^T N < 0, C = 1) + \mathbb{P}(-\|u\|^2 + u^T N > 0, C = -1) \\ &= \frac{1}{2}\mathbb{P}(u^T N \leq -\|u\|^2) + \frac{1}{2}\mathbb{P}(u^T N > \|u\|^2) = \Phi(-\|u\|). \end{aligned} \quad \square$$

**Theorem 4.2** For the class  $\mathcal{P}^{(q, \rho)}$ , the sequence  $a_n = n^{-\frac{2q}{2q+d}}$  is a lower minimax rate of convergence. In particular,

$$\liminf_{n \rightarrow \infty} \inf_{g_n} \sup_{P_{(X, Y)} \in \mathcal{P}^{(q, \rho)}} \frac{\mathbb{E}\|g_n - g\|^2}{\rho^{\frac{2d}{2q+d}} n^{-\frac{2q}{2q+d}}} \geq c_1 > 0.$$

**Proof** Step 1: Construct an auxiliary function  $g^{(c)}(x)$ . Set  $M_n = \lceil (\rho^2 n)^{\frac{1}{2q+d}} \rceil$ . Partition  $[0, 1]^d$  into  $M_n^d$  cubes  $\{A_{n,j}\}$  of side length  $\frac{1}{M_n}$  and with centers  $\{a_{n,j}\}$ . Choose a function  $\bar{f} : \mathbb{R}^d \rightarrow \mathbb{R}$  such that the support of  $\bar{f}$  is a subset of  $[-\frac{1}{2}, \frac{1}{2}]^d$ ,  $\int \bar{f}^2(x) dx > 0$  and  $\bar{f} \in \mathcal{F}^{(q, 2^{\beta-1})}$ . Define  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $f = \rho \bar{f}$ . Let  $c_n = (c_{n,1}, \dots, c_{n,M_n^d}) \in \mathcal{C}_n$  take values in  $\{\pm 1\}$ . Define  $g^{(c_n)}(x) = \sum_{j=1}^{M_n^d} c_{n,j} f_{n,j}(x)$  where  $f_{n,j}(x) = M_n^{-q} f(M_n(x - a_{n,j}))$ .

Step 2: Show that  $g^{(c_n)} \in \mathcal{F}^{(q, \rho)}$ . Let  $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_i \in \mathbb{N}, \sum_{j=1}^d \alpha_j = k$  and  $D^\alpha = \frac{\partial^k}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$ . If  $x, z \in A_{n,j}$ ,

$$|D^\alpha g^{(c_n)}(x) - D^\alpha g^{(c_n)}(z)| = |c_{n,j}| |D^\alpha f_{n,j}(x) - D^\alpha f_{n,j}(z)| \leq \rho \|x - z\|^\beta.$$

If  $x \in A_{n,i}, z \in A_{n,j}$ , choose  $\bar{x}, \bar{z}$  on the line between  $x$  and  $z$  such that  $\bar{x}$  is on the boundary of  $A_{n,i}$  and  $\bar{z}$  is on the boundary of  $A_{n,j}$ . Then

$$\begin{aligned} |D^\alpha g^{(c_n)}(x) - D^\alpha g^{(c_n)}(z)| &\leq |c_{n,i} D^\alpha f_{n,i}(x)| + |c_{n,j} D^\alpha f_{n,j}(z)| \\ &= |c_{n,i}| |D^\alpha f_{n,i}(x) - D^\alpha f_{n,i}(\bar{x})| + |c_{n,j}| |D^\alpha f_{n,j}(z) - D^\alpha f_{n,j}(\bar{z})| \\ &\leq \rho 2^{\beta-1} (\|x - \bar{x}\|^\beta + \|z - \bar{z}\|^\beta) = \rho 2^\beta \left( \frac{\|x - \bar{x}\|^\beta}{2} + \frac{\|z - \bar{z}\|^\beta}{2} \right) \\ &\leq \rho 2^\beta \left( \frac{\|x - \bar{x}\|}{2} + \frac{\|z - \bar{z}\|}{2} \right)^\beta \leq \rho \|x - z\|^\beta. \end{aligned}$$

Step 3: Prove that

$$\liminf_{n \rightarrow +\infty} \inf_{g_n} \sup_{Y = g^{(c)}(X) + N, c \in \mathcal{C}_n} \frac{M_n^{2q}}{\rho^2} \mathbb{E}\|g_n - g^{(c)}\|^2 > 0.$$



$\{f_{n,j}\}$  forms a set of orthogonal basis. Let  $g_n$  be an arbitrary estimate, and the projection  $\bar{g}_n$  of  $g_n$  to  $\{g^{(c)} : c \in \mathcal{C}_n\}$  is given by  $\bar{g}_n = \sum_{j=1}^{M_n} \tilde{c}_{n,j} f_{n,j}(x)$ . Then

$$\begin{aligned} \|g_n - g^{(c)}\|^2 &= \|g_n - \bar{g}_n\|^2 + \|g_n - g^{(c)}\|^2 \geq \|\bar{g}_n - g^{(c)}\|^2 = \sum_{j=1}^{M_n} \int_{A_{n,j}} (\tilde{c}_{n,j} f_{n,j}(x) - c_{n,j} f_{n,j}(x))^2 dx \\ &= \sum_{j=1}^{M_n} \int_{A_{n,j}} (\tilde{c}_{n,j} - c_{n,j})^2 f_{n,j}^2(x) dx = \int f^2(x) dx \sum_{j=1}^{M_n} (\tilde{c}_{n,j} - c_{n,j})^2 \frac{1}{M_n^{2q+d}}. \end{aligned}$$

Define  $\bar{c}_{n,j} = \text{sgn}(\tilde{c}_{n,j})$ , then

$$|\tilde{c}_{n,j} - c_{n,j}| \geq \frac{|\bar{c}_{n,j} - c_{n,j}|}{2} \Rightarrow \|g_n - g^{(c)}\|^2 \geq \int f^2(x) dx \frac{1}{4} \frac{1}{M_n^{2q+d}} \sum_{j=1}^{M_n} (\bar{c}_{n,j} - c_{n,j})^2 = \frac{\rho^2}{M_n^{2q}} \int \bar{f}^2(x) dx \frac{1}{M_n^d} \sum_{j=1}^{M_n} 1_{\{\bar{c}_{n,j} \neq c_{n,j}\}}.$$

Step 4: Prove that

$$\liminf_{n \rightarrow +\infty} \inf_{\bar{c}_n} \sup_{c_n} \frac{1}{M_n^d} \sum_{j=1}^{M_n} \mathbb{P}(\bar{c}_{n,j} \neq c_{n,j}) > 0.$$

Now we randomize  $c_n$ . Let  $c_{n,1}, \dots, c_{n,M_n^d}$  be i.i.d. random variables independent of  $(X_1, N_1), \dots, (X_n, N_n)$ ,  $\mathbb{P}(C_{n,1} = 1) = \mathbb{P}(C_{n,1} = -1) = \frac{1}{2}$ .  $\bar{c}_{n,j}$  can be interpreted as a decision on  $C_{n,j}$  using  $\mathcal{D}_n$ . Let  $\bar{C}_{n,j} = 1$  if  $\mathbb{P}(\bar{C}_{n,j} = 1 | \mathcal{D}_n) \geq \frac{1}{2}$ . Therefore,

$$\begin{aligned} \inf_{\bar{c}_n} \sup_{c_n} \frac{1}{M_n^d} \sum_{j=1}^{M_n} \mathbb{P}(\bar{c}_{n,j} \neq c_{n,j}) &\geq \inf_{\bar{c}_n} \frac{1}{M_n^d} \sum_{j=1}^{M_n} \mathbb{P}(\bar{c}_{n,j} \neq C_{n,j}) \geq \frac{1}{M_n^d} \sum_{j=1}^{M_n} \mathbb{P}(\bar{C}_{n,j} \neq C_{n,j}) \\ &= \mathbb{P}(\bar{C}_{n,1} \neq C_{n,1}) = \mathbb{E}\{\mathbb{P}(\bar{C}_{n,1} \neq C_{n,1} | X_1, \dots, X_n)\}. \end{aligned}$$

Let  $X_{i_1}, \dots, X_{i_t}$  be those  $X_i \in A_{n,1}$ ,  $(Y_{i_1}, \dots, Y_{i_t}) = C_{n,1}(f_{n,1}(X_{i_1}), \dots, f_{n,1}(X_{i_t})) + (N_{i_1}, \dots, N_{i_t})$ . By lemma 4.1,

$$\begin{aligned} \mathbb{E}\{\mathbb{P}(\bar{C}_{n,1} \neq C_{n,1} | X_1, \dots, X_n)\} &= \mathbb{E}\Phi\left(-\sqrt{\sum_{r=1}^t f_{n,1}^2(X_{i_r})}\right) = \mathbb{E}\Phi\left(-\sqrt{\sum_{i=1}^n f_{n,1}^2(X_i)}\right) \\ &\geq \Phi\left(-\sqrt{\mathbb{E} \sum_{i=1}^n f_{n,1}^2(X_i)}\right) \geq \Phi\left(-\sqrt{\int f^2(x) dx}\right) > 0. \end{aligned} \quad \square$$

## 5 Uniform Laws of Large Numbers

**Definition 5.1** (Background) Set  $Z = (X, Y)$ ,  $Z_i = (X_i, Y_i)$ ,  $g_f(x, y) = |f(x) - y|^2$  for  $f \in \mathcal{F}_n$ ,  $G_n = \{g_f : f \in \mathcal{F}_n\}$ , consider the limit  $\lim_{n \rightarrow +\infty} \sup_{g \in \mathcal{G}_n} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) \right|$ .

**Lemma 5.1** (Hoeffding's inequality) For  $g : \mathbb{R}^d \rightarrow [0, B]$ , the following inequalities hold:

$$\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon\right) \leq 2e^{-\frac{2n\varepsilon^2}{B^2}} \Rightarrow \mathbb{P}\left(\sup_{g \in \mathcal{G}_n} \left|\frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon\right) \leq 2|\mathcal{G}_n|e^{-\frac{2n\varepsilon^2}{B^2}}.$$

For finite class  $\mathcal{G}$  satisfying  $\sum_{n=1}^{+\infty} |\mathcal{G}_n|e^{-\frac{2n\varepsilon^2}{B^2}} < \infty$  for all  $\varepsilon > 0$ , by Borel-Cantelli lemma,

$$\mathbb{P}\left(\sup_{g \in \mathcal{G}_n} \left|\frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\}\right| > \varepsilon \text{ i.o.}\right) = 0$$

**Definition 5.2** (Covering number) Let  $\varepsilon > 0$  and  $\mathcal{G}$  be a set of functions  $\mathbb{R}^d \rightarrow \mathbb{R}$ . Every finite collection of functions  $g_1, \dots, g_N : \mathbb{R}^d \rightarrow \mathbb{R}$  with the property that for every  $g \in \mathcal{G}$  there is a  $j = j(g) \in [N]$  such that  $\|g - g_j\|_\infty < \varepsilon$  is called an  $\varepsilon$ -cover of  $\mathcal{G}$  w.r.t.  $\|\cdot\|_\infty$ . Let  $\mathcal{N}(\varepsilon, \mathcal{G}, \|\cdot\|_\infty)$  or  $\mathcal{N}_\infty(\varepsilon, \mathcal{G})$  be the smallest  $\varepsilon$ -cover of  $\mathcal{G}$  w.r.t.  $\|\cdot\|_\infty$ .

**Theorem 5.1** For  $n \in \mathbb{N}$ , let  $\mathcal{G}_n$  be a set of functions  $g : \mathbb{R}^d \rightarrow [0, B]$  and let  $\varepsilon > 0$ . Then

$$\mathbb{P} \left( \sup_{g \in \mathcal{G}_n} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\} \right| > \varepsilon \right) \leq 2\mathcal{N}_\infty \left( \frac{\varepsilon}{3}, \mathcal{G}_n \right) \exp \left( -\frac{2n\varepsilon^2}{9B^2} \right).$$

**Proof** Let  $\mathcal{G}_{n, \frac{\varepsilon}{3}}$  be an  $\frac{\varepsilon}{3}$ -cover of  $\mathcal{G}_n$  w.r.t.  $\|\cdot\|_\infty$  of minimal cardinality. Fix  $g \in \mathcal{G}_n$ , there exists  $\bar{g} \in \mathcal{G}_{n, \frac{\varepsilon}{3}}$  such that  $\|g - \bar{g}\|_\infty < \frac{\varepsilon}{3}$ . Then

$$\begin{aligned} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) \right| &\leq \left| \frac{1}{n} \sum_{i=1}^n (g(Z_i) - \bar{g}(Z_i)) \right| + \left| \frac{1}{n} \sum_{i=1}^n \bar{g}(Z_i) - \mathbb{E}\{\bar{g}(Z)\} \right| + |\mathbb{E}\bar{g}(Z) - \mathbb{E}g(Z)| \\ &\leq \frac{2\varepsilon}{3} + \left| \frac{1}{n} \sum_{i=1}^n \bar{g}(Z_i) - \mathbb{E}\{\bar{g}(Z)\} \right|, \\ \Rightarrow \mathbb{P} \left( \sup_{g \in \mathcal{G}_n} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\} \right| > \varepsilon \right) &\leq \mathbb{P} \left( \sup_{g \in \mathcal{G}_{n, \frac{\varepsilon}{3}}} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}\{g(Z)\} \right| > \frac{\varepsilon}{3} \right) \end{aligned}$$

Then use Hoeffding's inequality. □

**Definition 5.3** Let  $\varepsilon > 0$ ,  $\mathcal{G}$  be a set of functions  $\mathbb{R}^d \rightarrow \mathbb{R}$ ,  $1 \leq p < \infty$ , and  $\nu$  be a probability measure on  $\mathbb{R}^d$ . (a) Every finite collection of functions  $g_1, \dots, g_N : \mathbb{R}^d \rightarrow \mathbb{R}$  with the property that for every  $g \in \mathcal{G}$  there is a  $j = j(g) \in [N]$  such that  $\|g - g_j\|_{L_p(\nu)} < \varepsilon$  is called a  $\varepsilon$ -cover of  $\mathcal{G}$ . Similarly define  $\mathcal{N}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)})$ . (b) Let  $Z^{1:n} = (Z_1, \dots, Z_n) \subset \mathbb{R}^d$  and  $\nu_n$  be the corresponding empirical measure, then  $\|f\|_{L_p(\nu_n)} := \left\{ \frac{1}{n} \sum_{i=1}^n |f(Z_i)|^p \right\}^{\frac{1}{p}}$  and similarly define  $\mathcal{N}_p(\varepsilon, \mathcal{G}, Z^{1:n})$ .

**Definition 5.4** (Packing number) (a) Every finite collection of functions  $g_1, \dots, g_N \in \mathcal{G}$  with  $\|g_j - g_k\|_{L_p(\nu)} \geq \varepsilon$  for all  $1 \leq j < k \leq N$  is called  $\varepsilon$ -packing of  $\mathcal{G}$  with  $\|\cdot\|_{L_p(\nu)}$ . The largest  $\varepsilon$ -packing is denoted as  $\mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)})$ . Similarly define  $\mathcal{M}(\varepsilon, \mathcal{G}, Z^{1:n})$ .

**Property 5.1** (Covering number v.s. packing number)

$$\begin{aligned} \mathcal{M}(2\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}) &\leq \mathcal{N}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}) \leq \mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}), \\ \mathcal{M}(2\varepsilon, \mathcal{G}, Z^{1:n}) &\leq \mathcal{N}(\varepsilon, \mathcal{G}, Z^{1:n}) \leq \mathcal{M}(\varepsilon, \mathcal{G}, Z^{1:n}). \end{aligned}$$

**Theorem 5.2** Let  $\mathcal{F}$  be a set of functions  $\mathbb{R}^d \rightarrow \mathbb{R}$ . Assume that  $\mathcal{F}$  is a linear vector space of dimension  $D$ . Then for arbitrary  $R > 0, \varepsilon > 0$ , and  $z_1, \dots, z_n \in \mathbb{R}^d$ ,

$$\mathcal{N}_2 \left( \varepsilon, \left\{ f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n |f(z_i)|^2 \leq R^2 \right\}, Z^{1:n} \right) \leq \left( \frac{4R + \varepsilon}{\varepsilon} \right)^D.$$

**Definition 5.5** Let  $\mathcal{A}$  be a class of subsets of  $\mathbb{R}^d$  and  $n \in \mathbb{N}$ . For  $z_1, \dots, z_n \in \mathbb{R}^d$ , define  $s(\mathcal{A}, \{z_1, \dots, z_n\}) = |\{A \cap \{z_1, \dots, z_n\} : A \in \mathcal{A}\}|$ .

**Definition 5.6** Let  $\mathcal{G}$  be a subset of  $\mathbb{R}^d$  of size  $n$ . We say  $\mathcal{A}$  shatters  $\mathcal{G}$  if  $s(\mathcal{A}, \mathcal{G}) = 2^n$ . The  $n$ th shatter coefficient of  $\mathcal{A}$  is  $S(\mathcal{A}, n) = \max_{\{z_1, \dots, z_n\} \subset \mathbb{R}^d} s(\mathcal{A}, \{z_1, \dots, z_n\})$ , the maximum number of different subsets of  $n$  points that can be picked out by set from  $\mathcal{A}$ .

**Definition 5.7** (VC dimension) Let  $\mathcal{A}$  be a class of subsets of  $\mathbb{R}^d$  with  $\mathcal{A} \neq \emptyset$ . The VC dimension  $V_{\mathcal{A}}$  of  $\mathcal{A}$  is defined by  $V_{\mathcal{A}} = \sup\{n \in \mathbb{N}, S(\mathcal{A}, n) = 2^n\}$ .

**Proposition 5.1**  $S(\mathcal{A}, n) \leq \sum_{i=0}^{V_{\mathcal{A}}} \binom{n}{i}$ .

**Theorem 5.3** Let  $\mathcal{G}$  be a set of functions  $g : \mathbb{R}^d \rightarrow [0, B]$ . For any  $n \in \mathbb{N}$  and  $\varepsilon > 0$ ,

$$\mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n g(Z) - \mathbb{E}[g(Z)] \right| > \varepsilon \right\} \leq 8\mathbb{E}\mathcal{N}_1 \left( \frac{\varepsilon}{8}, \mathcal{G}, Z^{1:n} \right) \exp \left( -\frac{n\varepsilon^2}{128B^2} \right).$$

**Proof** Step 1: Symmetrization. Let  $Z^{1:n}$  be i.i.d. samples from the same distribution and independent of  $Z^{1:n}$  and  $g^* \in \mathcal{G}$  be a function such that  $\left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) \right| > \varepsilon$  if there exists such one. Otherwise, let  $g^*$  be an arbitrary function in  $\mathcal{G}$ .  $g^*(z)$  depends on  $Z^{1:n}$  and  $\mathbb{P} \left\{ \left| \mathbb{E}[g^*(Z)|Z^{1:n}] - \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) \right| > \frac{\varepsilon}{2} \middle| Z^{1:n} \right\} \leq \frac{\text{Var}(g^*(Z)|Z^{1:n})}{n(\frac{\varepsilon}{2})^2} \leq \frac{B^2/4}{n\varepsilon^2/4} = \frac{B^2}{n\varepsilon^2} \leq \frac{1}{2}$  holds for  $n \geq \frac{2B^2}{\varepsilon^2}$ . Thus we have

$$\begin{aligned} \mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right| > \frac{\varepsilon}{2} \right\} &\geq \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) \right| > \frac{\varepsilon}{2} \right\} \\ &\geq \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] \right| > \varepsilon, \left| \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] \right| \leq \frac{\varepsilon}{2} \right\} \\ &= \mathbb{E} \left\{ 1_{\left\{ \left| \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] \right| > \varepsilon \right\}} \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] \right| \leq \frac{\varepsilon}{2} \middle| Z^{1:n} \right) \right\} \\ &\geq \frac{1}{2} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] \right| > \varepsilon \right\} \end{aligned}$$

$$\text{Therefore, } 2\mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right| > \frac{\varepsilon}{2} \right\} \geq \mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}[g(Z)] \right| > \varepsilon \right\}.$$

Step 2: Introduction of additive randomness by random signs. Let  $U_1, \dots, U_n$  be independent and uniformly distributed over  $\{-1, 1\}$  and independent  $Z^{1:n}$  and  $Z'^{1:n}$ .

$$\begin{aligned} \mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n [g(Z_i) - g(Z'_i)] \right| > \frac{\varepsilon}{2} \right\} &= \mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n U_i [g(Z_i) - g(Z'_i)] \right| > \frac{\varepsilon}{2} \right\} \\ &\leq \mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) \right| > \frac{\varepsilon}{4} \right\} + \mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n U_i g(Z'_i) \right| > \frac{\varepsilon}{4} \right\} \\ &= 2\mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) \right| > \frac{\varepsilon}{4} \right\} \end{aligned}$$

Step 3: Conditioning and introduction of a covering on  $Z^{1:n}$ . Let  $\mathcal{G}_{\frac{\varepsilon}{8}}$  be an  $L_1$   $\frac{\varepsilon}{8}$ -cover of  $\mathcal{G}$  in  $Z^{1:n}$ . Fix  $g \in \mathcal{G}$ , then there exists  $\bar{g} \in \mathcal{G}_{\frac{\varepsilon}{8}}$  s.t.  $\frac{1}{n} \sum_{i=1}^n |g(Z_i) - \bar{g}(Z_i)| < \frac{\varepsilon}{8}$ .  $\left| \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) \right| = \left| \frac{1}{n} \sum_{i=1}^n U_i \bar{g}(Z_i) + \frac{1}{n} \sum_{i=1}^n U_i [g(Z_i) - \bar{g}(Z_i)] \right| \leq \left| \frac{1}{n} \sum_{i=1}^n U_i \bar{g}(Z_i) \right| + \frac{\varepsilon}{8}$ . Thus

$$\mathbb{P} \left\{ \exists g \in \mathcal{G} : \left| \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) \right| > \frac{\varepsilon}{4} \right\} \leq \mathbb{P} \left\{ \exists g \in \mathcal{G}_{\frac{\varepsilon}{8}} : \left| \frac{1}{n} \sum_{i=1}^n U_i \bar{g}(Z_i) \right| > \frac{\varepsilon}{8} \right\} \leq |\mathcal{G}_{\frac{\varepsilon}{8}}| \max_{g \in \mathcal{G}_{\frac{\varepsilon}{8}}} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) \right| > \frac{\varepsilon}{8} \right\}$$

Step 4: Application of Hoeffding's inequality:  $|U_i g(Z_i)| \leq B \Rightarrow \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) \right| > \frac{\varepsilon}{8} \right\} \leq 2 \exp \left( -\frac{2n(\frac{\varepsilon}{8})^2}{(2B)^2} \right) = 2 \exp \left( -\frac{n\varepsilon^2}{128B^2} \right)$ .  $\square$

**Theorem 5.4** Let  $\mathcal{G}$  be a class of functions  $g : \mathbb{R}^d \rightarrow [0, B]$  with  $V_{\mathcal{G}^+} \geq 2$  where  $\mathcal{G}^+ := \{(z, t) \in \mathbb{R}^d \times \mathbb{R} : t \leq g(z), g \in \mathcal{G}\}$ . Let  $p \geq 1$ ,  $\nu$  be a probability measure on  $\mathbb{R}^d$  and  $0 < \varepsilon < \frac{B}{4}$ . Then

$$\mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}) \leq 3 \left( \frac{2eB^p}{\varepsilon^p} \log \frac{3eB^p}{\varepsilon^p} \right)^{V_{\mathcal{G}^+}}.$$

**Proof** Step 1: Set  $p = 1$ . Relate  $\mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)})$  to a shatter coefficient of  $\mathcal{G}^+$ . Set  $m = \mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)})$  and let  $\bar{\mathcal{G}} = \{g_1, \dots, g_m\}$  be a  $\varepsilon$ -packing of  $\mathcal{G}$  w.r.t.  $\|\cdot\|_{L_p(\nu)}$ . Let  $Q_1, \dots, Q_K \in \mathbb{R}^d$  be  $K$  independent r.v.'s with common  $\nu$ . Generate  $K$  independent r.v.'s  $T_1, \dots, T_K$  uniformly distributed on  $[0, B]$ . Denote  $R_i = (Q_i, T_i), i = 1, \dots, K, \mathcal{G}_f = \{(x, t) : t \leq f(x)\}$  for  $f : \mathbb{R}^d \rightarrow [0, B]$ . Then

$$S(\mathcal{G}^+, K) = \max_{\{z_1, \dots, z_K\} \in \mathbb{R}^d \times \mathbb{R}} s(\mathcal{G}^+, \{z_1, \dots, z_K\}) \geq \mathbb{E} s(\mathcal{G}_+, \{R_1, \dots, R_K\}) \geq \mathbb{E} s(\{\mathcal{G}_f : f \in \mathcal{G}\}, \{R_1, \dots, R_K\})$$

$$\begin{aligned}
 &\geq \mathbb{E}s(\{\mathcal{G}_f : f \in \mathcal{G}, \mathcal{G}_f \cap R^{1:K} \neq \mathcal{G}_g \cap R^{1:K} \text{ for all } g \in \bar{\mathcal{G}}, g \neq f\}, R^{1:K}) \\
 &= \mathbb{E} \left\{ \sum_{f \in \bar{\mathcal{G}}} 1_{\{\mathcal{G}_f \cap R^{1:K} \neq \mathcal{G}_g \cap R^{1:K} \text{ for all } g \in \bar{\mathcal{G}}, g \neq f\}} \right\} = \sum_{f \in \bar{\mathcal{G}}} \mathbb{P}(\mathcal{G}_f \cap R^{1:K} \neq \mathcal{G}_g \cap R^{1:K} \text{ for all } g \in \bar{\mathcal{G}}, g \neq f) \\
 &= \sum_{f \in \bar{\mathcal{G}}} (1 - \mathbb{P}(\exists g \in \bar{\mathcal{G}}, g \neq f, \mathcal{G}_f \cap R^{1:K} = \mathcal{G}_g \cap R^{1:K})) \geq \sum_{f \in \bar{\mathcal{G}}} \left( 1 - m \max_{g \in \bar{\mathcal{G}}, g \neq f} \mathbb{P}(\mathcal{G}_f \cap R^{1:K} = \mathcal{G}_g \cap R^{1:K}) \right).
 \end{aligned}$$

For  $f, g \in \bar{\mathcal{G}}, f \neq g$ ,

$$\mathbb{P}(\mathcal{G}_f \cap R^{1:K} = \mathcal{G}_g \cap R^{1:K}) = \mathbb{P}(\mathcal{G}_f \cap \{R_1\} = \mathcal{G}_g \cap \{R_1\})^K,$$

and

$$\begin{aligned}
 \mathbb{P}(\mathcal{G}_f \cap \{R_1\} = \mathcal{G}_g \cap \{R_1\}) &= 1 - \mathbb{P}(\mathcal{G}_f \cap \{R_1\} \neq \mathcal{G}_g \cap \{R_1\}) = 1 - \mathbb{E}[\mathbb{P}(\mathcal{G}_f \cap \{R_1\} \neq \mathcal{G}_g \cap \{R_1\} | Q_1)] \\
 &= 1 - \mathbb{E}[\mathbb{P}(f(Q_1) < T \leq g(Q_1) \text{ or } g(Q_1) < T \leq f(Q_1) | Q_1)] = 1 - \mathbb{E} \left[ \frac{|f(Q_1) - g(Q_1)|}{B} \right] \\
 &= 1 - \frac{1}{B} \int |f(x) - g(x)| \nu(dx) \leq 1 - \frac{\varepsilon}{B} \Rightarrow \mathbb{P}(\mathcal{G}_f \cap \{R_1\} = \mathcal{G}_g \cap \{R_1\})^K \leq \left( 1 - \frac{\varepsilon}{B} \right)^K \leq \exp \left( -\frac{\varepsilon K}{B} \right) \\
 \Rightarrow S(\mathcal{G}^+, K) &\geq m \left( 1 - m \exp \left( -\frac{\varepsilon K}{B} \right) \right).
 \end{aligned}$$

Set  $K = \left\lfloor \frac{B}{\varepsilon} \log(2m) \right\rfloor$ . Then

$$1 - m \exp \left( -\frac{\varepsilon K}{B} \right) \geq 1 - m \exp \left( -\frac{\varepsilon}{B} \left( \frac{B}{\varepsilon} \log(2m) - 1 \right) \right) = 1 - \frac{1}{2} \exp \left( \frac{\varepsilon}{B} \right) \geq 1 - \frac{1}{2} \exp \left( \frac{1}{4} \right) \geq \frac{1}{3} \Rightarrow m \leq 3S(\mathcal{G}_+, K).$$

Step 2: Relate  $S(\mathcal{G}_+, K)$  to  $V_{\mathcal{G}_+}$ . Set  $K = \lfloor \frac{B}{\varepsilon} \log(2\mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)})) \rfloor \leq V_{\mathcal{G}_+} \Rightarrow \mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}) \leq \frac{\varepsilon}{2} \exp(V_{\mathcal{G}_+}) \leq 3 \left( \frac{2eB}{\varepsilon} \log \frac{3eB}{\varepsilon} \right)^{V_{\mathcal{G}_+}}$ . In the case  $K > V_{\mathcal{G}_+}$ , use the following lemma:

**Lemma 5.2** Let  $\mathcal{A} \in \mathbb{R}^d$  and  $V_{\mathcal{A}} < \infty$ . Then  $\forall n \in \mathbb{N}, S(\mathcal{A}, n) \leq (n+1)^{V_{\mathcal{A}}}$  and  $\forall n \geq V_{\mathcal{A}}, S(\mathcal{A}, n) \leq \left( \frac{en}{V_{\mathcal{A}}} \right)^{V_{\mathcal{A}}}$ .

$$\text{Then } \mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}) \leq 3 \left( \frac{eK}{V_{\mathcal{G}_+}} \right)^{V_{\mathcal{G}_+}} \leq 3 \left( \frac{eB}{\varepsilon V_{\mathcal{G}_+}} \log(2\mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)})) \right)^{V_{\mathcal{G}_+}}.$$

Step 3: Setting  $a = \frac{eB}{\varepsilon}$  and  $b = V_{\mathcal{G}_+}$ ,  $\mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}) := x \leq 3 \left( \frac{a}{b} \log(2x) \right)^b \Rightarrow x \leq 3(2a \log(3a))^b$ .

Step 4: Let  $1 < p < \infty$ . Then for any  $g_j, g_k \in \mathcal{G}$ ,

$$\|g_j - g_k\|_{L_p(\nu)}^p \leq B^{p-1} \|g_j - g_k\|_{L_1(\nu)} \Rightarrow \mathcal{M}(\varepsilon, \mathcal{G}, \|\cdot\|_{L_p(\nu)}) \leq \mathcal{M} \left( \frac{\varepsilon^p}{B^{p-1}}, \mathcal{G}, \|\cdot\|_{L_p(\nu)} \right). \quad \square$$

**Theorem 5.5** (ULLN) Let  $\mathcal{G}$  be a class of functions  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $G : \mathbb{R}^d \rightarrow \mathbb{R}, G(x) = \sup_{g \in \mathcal{G}} |g(x)|$  be an envelope of  $\mathcal{G}$ . Assume  $\mathbb{E}G(Z) < \infty$  and  $V_{\mathcal{G}^+} < \infty$ . Then

$$\sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) \right| \rightarrow 0 \text{ a.s. as } n \rightarrow +\infty$$

**Proof** For  $L > 0$ , set  $\mathcal{G}_L := \{g \cdot 1_{\{G \leq L\}} : g \in \mathcal{G}\}$ . For  $g \in \mathcal{G}$ ,

$$\begin{aligned}
 \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) \right| &\leq \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z_i) 1_{\{G(Z_i) \leq L\}} \right| \\
 &\quad + \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) 1_{\{G(Z_i) \leq L\}} - \mathbb{E}\{g(Z) 1_{G(Z) \leq L}\} \right| + |\mathbb{E}\{g(Z) 1_{G(Z) \leq L}\} - \mathbb{E}\{g(Z)\}| \\
 &= \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) 1_{\{G(Z_i) > L\}} \right| + \mathbb{E}|g(Z)| 1_{\{G(Z) > L\}} + \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) 1_{\{G(Z_i) \leq L\}} - \mathbb{E}\{g(Z) 1_{G(Z) \leq L}\} \right|
 \end{aligned}$$

Since  $\mathbb{P}(\sup_{g \in \mathcal{G}_L} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) \right| > \varepsilon) \leq 8\mathbb{E} \left\{ \mathcal{M}_1 \left( \frac{\varepsilon}{8}, \mathcal{G}_L, Z^{1:n} \right) \exp \left( -\frac{n\varepsilon^2}{128(2L)^2} \right) \right\}$ , use the B-C lemma.  $\square$

## 6 Least Square Estimates: Consistency and Convergence Rate

**Definition 6.1** (Notation)  $\mathbb{E}\{(m(X) - Y)^2\} = \inf_f \mathbb{E}\{(f(X) - Y)^2\} \Rightarrow m(X) = \mathbb{E}[Y|X]$ . Define

$$m_n = \arg \min_{f \in \mathcal{F}_n} \frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_i|^2, m^* = \arg \min_{f \in \mathcal{F}_n} \mathbb{E}\{(f(X) - Y)^2\}.$$

**Theorem 6.1** Let  $\mathcal{F}_n$  be a class of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  depending on the data  $\mathcal{D}_n = \{(X_1, Y_1), \dots, (X_n, Y_n)\}$ . Then

$$\int |m_n(x) - m(x)|^2 \nu(dx) \leq 2 \sup_{f \in \mathcal{F}_n} \left| \frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_i|^2 - \mathbb{E}\{(f(X) - Y)^2\} \right| + \inf_{f \in \mathcal{F}_n} \int |f(x) - m(x)|^2 \nu(dx).$$

**Proof** We do the following decomposition:

$$\begin{aligned} \int |m_n(x) - m(x)|^2 \nu(dx) &= \mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \mathbb{E}[|m(X) - Y|^2] \\ &= \left\{ \mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \inf_{f \in \mathcal{F}_n} \mathbb{E}|f(X) - Y|^2 \right\} + \left\{ \inf_{f \in \mathcal{F}_n} \mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2 \right\} \\ &:= I_1 + I_2. \end{aligned}$$

$$I_1 \leq 2 \sup_{f \in \mathcal{F}_n} \left| \frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_i|^2 - \mathbb{E}|f(X) - Y|^2 \right|. \quad I_2 = \inf_{f \in \mathcal{F}_n} \int (f(x) - m(x))^2 \nu(dx). \quad \square$$

**Proposition 6.1** (Method of Sieves) Let  $\psi_1, \psi_2, \dots, \mathbb{R}^d \rightarrow \mathbb{R}$  be bounded functions such that  $|\psi_j(x)| \leq 1$ . Assume the set of functions  $\cup_{k=1}^{+\infty} \left\{ \sum_{j=1}^k a_j \psi_j(x) : a_1, \dots, a_k \in \mathbb{R} \right\}$  is dense in  $L_2(\mu)$  for any probability measure  $\mu$  on  $\mathbb{R}^d$ . Define the regression function estimate  $m_n$  as a function minimizing the empirical  $L_2$  risk  $\frac{1}{n} \sum_{i=1}^n (f(X_i) - Y_i)^2$  over the function form  $f(x) = \sum_{j=1}^{k_n} a_j \psi_j(x)$  with  $\sum_{j=1}^{k_n} |a_j| \leq \beta_n$ . If  $\mathbb{E}(Y^2) < \infty$  and  $k_n$  and  $\beta_n$  satisfy  $k_n \rightarrow \infty, \beta_n \rightarrow \infty, \frac{k_n \beta_n^4 \log \beta_n}{n} \rightarrow 0$  and  $\frac{\beta_n^4}{n^{1-\delta}} \rightarrow 0$  for some  $\delta > 0$ , then  $\int (m_n(x) - m(x))^2 \mu(dx) \rightarrow 0$  with probability 1.

**Proposition 6.2** Consider  $\mathcal{F}_n = \left\{ \sum_{j=1}^{k_n} a_j \psi_j(x) : \sum_{j=1}^{k_n} |a_j| \leq \beta_n \right\}$  and  $\widetilde{\mathcal{F}}_n = \left\{ \sum_{j=1}^{k_n} a_j \psi_j(x) : a_j \in \mathbb{R} \right\}$ . Step 1: derive  $\widetilde{m}_n$  by using  $\widetilde{\mathcal{F}}_n$ . Step 2: Truncation of  $\widetilde{m}_n$ ,  $m_n(x) = T_{\beta_n} \widetilde{m}_n(x)$  where  $T_L u = \begin{cases} u, & \text{if } |u| \leq L \\ L \operatorname{sgn}(u), & \text{otherwise} \end{cases}$ . (a) If  $\mathbb{E}(Y^2) < \infty$  and  $k_n$  and  $\beta_n$  satisfy  $k_n \rightarrow \infty, \beta_n \rightarrow \infty, \frac{k_n \beta_n^4 \log \beta_n}{n} \rightarrow 0$ , then  $\mathbb{E} \left\{ \int (m_n(x) - m(x))^2 \mu(dx) \right\} \rightarrow 0$ . (b) If adding the extra condition  $\frac{\beta_n^4}{n^{1-\delta}} \rightarrow 0$  for some  $\delta > 0$ , then  $\int (m_n(x) - m(x))^2 \mu(dx) \rightarrow 0$  a.s.

**Proposition 6.3** Let  $\widetilde{F}_n = \widetilde{F}_n(\mathcal{D}_n)$  be a class of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . If  $|Y| \leq \beta_n$  a.s., then

$$\int (m_n(x) - m(x))^2 \mu(dx) \leq 2 \sup_{f \in T_{\beta_n} \widetilde{F}_n} \left| \frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_i|^2 - \mathbb{E}|f(X) - Y|^2 \right| + \inf_{f \in \widetilde{F}_n, \|f\|_\infty \leq \beta_n} \int |f(x) - m(x)|^2 \mu(dx)$$

**Theorem 6.2** Let  $\widetilde{\mathcal{F}}_n = \widetilde{\mathcal{F}}_n(\mathcal{D}_n)$  be a class of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $Y_L = T_L Y, Y_{i,L} = T_L Y_i$ . (a) If

$$\begin{aligned} \lim_{n \rightarrow +\infty} \beta_n = \infty, \quad \lim_{n \rightarrow +\infty} \inf_{f \in \widetilde{F}_n, \|f\|_\infty \leq \beta_n} \int |f(x) - m(x)|^2 \mu(dx) &= 0 \text{ a.s.,} \\ \lim_{n \rightarrow +\infty} \sup_{f \in T_{\beta_n} \widetilde{F}_n} \left| \frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_{i,L}|^2 - \mathbb{E}(f(X) - Y_L)^2 \right| &= 0 \text{ a.s. for all } L > 0, \end{aligned}$$

then  $\lim_{n \rightarrow +\infty} \int |m_n(x) - m(x)|^2 \mu(dx) = 0$  a.s. (b) If  $\beta_n \rightarrow +\infty, \mathbb{E}\{\sim\} \rightarrow 0, \mathbb{E}\{\sim\} \rightarrow 0$ , then  $\mathbb{E}\{\sim\} \rightarrow 0$ .

**Definition 6.2** (Piecewise polynomial partition estimate)  $\mathcal{P}_n = \{A_{n,1}, A_{n,2}, \dots\}$  be a partition of  $\mathbb{R}^d$ ,

$$\hat{m}_n(x) := \frac{\sum_{i=1}^n Y_i I_{\{X_i \in A_n(x)\}}}{\sum_{i=1}^n I_{\{X_i \in A_n(x)\}}}$$

where  $A_n(x)$  denotes the cell  $A_{n,j} \in \mathcal{P}_n$  which contains  $x$ .

**Theorem 6.3** Let  $\mathcal{F}$  be a class of function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  bounded in absolute value by  $B$ . Let  $\varepsilon > 0$ . Then

$$\mathbb{P}\{\exists f \in \mathcal{F} \text{ s.t. } \|f\|_2 - 2\|f\|_n > \varepsilon\} \leq \mathbb{E} \mathcal{N}_2 \left( \frac{\sqrt{2}}{24} \varepsilon, \mathcal{F}, X^{1:2n} \right) \exp \left( -\frac{n\varepsilon^2}{288B^2} \right)$$

where  $\|f\|_n^2 = \frac{1}{n} \sum_{i=1}^n |f(X_i)|^2$ .

**Proof** Step 1: Replace  $L_2(\mu)$  norm by the empirical norm. Let  $\tilde{X}^{1:n} = (X_{n+1}, \dots, X_{2n})$  be a ghost sample of i.i.d. r.v.'s as  $X$  and independent of  $X^{1:n}$ . Define  $\|f\|_{n'}^2 = \frac{1}{n} \sum_{i=n+1}^{2n} |f(X_i)|^2$ . Let  $f^*$  be a function  $f \in \mathcal{F}$  such that  $\|f\|_2 - 2\|f\|_n > \varepsilon$  if there exists any such function, and let  $f^*$  be an arbitrary function in  $\mathcal{F}$  if such a function does not exist. Then

$$\begin{aligned} \mathbb{P}\{2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2 | X^{1:n}\} &\geq \mathbb{P}\{4\|f^*\|_{n'}^2 + \frac{\varepsilon^2}{4} > \|f^*\|_2^2 | X^{1:n}\} = 1 - \mathbb{P}\{4\|f^*\|_{n'}^2 + \frac{\varepsilon^2}{4} \leq \|f^*\|_2^2 | X^{1:n}\} \\ &= 1 - \mathbb{P}\{3\|f^*\|_2^2 + \frac{\varepsilon^2}{4} \leq 4(\|f^*\|_2^2 - \|f^*\|_{n'}^2) | X^{1:n}\} \geq 1 - \frac{16 \text{Var} \left( \frac{1}{n} \sum_{i=n+1}^{2n} |f^*(X_i)|^2 \middle| X^{1:n} \right)}{(3\|f^*\|_2^2 + \frac{\varepsilon^2}{4})^2} \\ &\geq 1 - \frac{\frac{16}{n} B^2 \|f^*\|_2^2}{(3\|f^*\|_2^2 + \frac{\varepsilon^2}{4})^2} \geq 1 - \frac{\frac{16}{3} \frac{B^2}{n}}{3\|f^*\|_2^2 + \frac{\varepsilon^2}{4}} \geq 1 - \frac{64}{3\varepsilon^2} \frac{B^2}{n} \geq \frac{2}{3} \text{ for } n \geq \frac{64B^2}{\varepsilon^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}\{\exists f \in \mathcal{F} : \|f\|_{n'} - \|f\|_n > \frac{\varepsilon}{4}\} &\geq \mathbb{P}\{2\|f^*\|_{n'} - 2\|f^*\|_n > \frac{\varepsilon}{2}\} \geq \mathbb{P}\{2\|f^*\|_{n'} + \frac{\varepsilon}{2} - 2\|f^*\|_n > \varepsilon, 2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2\} \\ &\geq \mathbb{P}\{\|f^*\|_2 - 2\|f^*\|_n > \varepsilon, 2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2\} \\ &= \mathbb{E}\{1_{\{\|f^*\|_2 - 2\|f^*\|_n > \varepsilon\}} \mathbb{P}\{2\|f^*\|_{n'} + \frac{\varepsilon}{2} > \|f^*\|_2 | X^{1:n}\}\} \\ &\geq \frac{2}{3} \mathbb{P}\{\|f^*\|_2 - 2\|f^*\|_n > \varepsilon\} = \frac{2}{3} \mathbb{P}\{\exists f \in \mathcal{F} : \|f\|_2 - 2\|f\|_n > \varepsilon\}. \end{aligned}$$

This proves  $\mathbb{P}\{\exists f \in \mathcal{F} : \|f\|_2 - 2\|f\|_n > \varepsilon\} \leq \frac{3}{2} \mathbb{P}\{\exists f \in \mathcal{F} : \|f\|_{n'} - \|f\|_n > \frac{\varepsilon}{4}\}$ .

Step 2: Introduction of additional randomness. Let  $U_1, \dots, U_n$  be independent and uniformly distributed on  $\{-1, 1\}$  and independent of  $X_1, \dots, X_{2n}$ . Set  $Z_i = \begin{cases} X_{i+n} & \text{if } U_i = 1 \\ X_i & \text{if } U_i = -1 \end{cases}$  and  $Z_{i+n} = \begin{cases} X_i & \text{if } U_i = 1 \\ X_{i+n} & \text{if } U_i = -1 \end{cases}$ . Then

$$\begin{aligned} \mathbb{P}\left\{\exists f \in \mathcal{F} : \|f\|_{n'} - \|f\|_n > \frac{\varepsilon}{4}\right\} &= \mathbb{P}\left\{\exists f \in \mathcal{F} : \left(\frac{1}{n} \sum_{i=n+1}^{2n} |f(X_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n} \sum_{i=1}^n |f(X_i)|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{4}\right\} \\ &= \mathbb{P}\left\{\exists f \in \mathcal{F} : \left(\frac{1}{n} \sum_{i=n+1}^{2n} |f(Z_i)|^2\right)^{\frac{1}{2}} - \left(\frac{1}{n} \sum_{i=1}^n |f(Z_i)|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{4}\right\} \end{aligned}$$

Step 3: Conditioning and introduction of a cover. Let  $\mathcal{G} = \{g_j : j = 1, \dots, \mathcal{N}_2(\frac{\sqrt{2}}{24} \varepsilon, \mathcal{F}, X^{1:2n})\}$  be a  $\frac{\sqrt{2}}{24} \varepsilon$ -cover of  $\mathcal{F}$  w.r.t.  $\|\cdot\|_{2n}$  of minimal size.  $\|f\|_{2n}^2 = \frac{1}{2n} \sum_{i=1}^{2n} |f(X_i)|^2$ . Fix  $f \in \mathcal{F}$ ,  $\|f - g\|_{2n} \leq \frac{\sqrt{2}}{24} \varepsilon$ . Then

$$\begin{aligned} &\left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |f(Z_i)|^2 \right\}^{\frac{1}{2}} - \left\{ \frac{1}{n} \sum_{i=1}^n |f(Z_i)|^2 \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |f(Z_i)|^2 \right\}^{\frac{1}{2}} - \left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right\}^{\frac{1}{2}} + \left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right\}^{\frac{1}{2}} - \left\{ \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right\}^{\frac{1}{2}} + \left\{ \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right\}^{\frac{1}{2}} - \left\{ \frac{1}{n} \sum_{i=1}^n |f(Z_i)|^2 \right\}^{\frac{1}{2}} \\ &\leq \left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |f(Z_i) - g(Z_i)|^2 \right\}^{\frac{1}{2}} + \left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right\}^{\frac{1}{2}} - \left\{ \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right\}^{\frac{1}{2}} + \left\{ \frac{1}{n} \sum_{i=1}^n |g(Z_i) - f(Z_i)|^2 \right\}^{\frac{1}{2}} \\ &\leq 2\sqrt{2}\|f - g\|_{2n} + \left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right\}^{\frac{1}{2}} - \left\{ \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{6} + \left\{ \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right\}^{\frac{1}{2}} - \left\{ \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right\}^{\frac{1}{2}} \end{aligned}$$

In this way,

$$\begin{aligned}
 & \mathbb{P} \left\{ \exists f \in \mathcal{F} : \left( \frac{1}{n} \sum_{i=n+1}^{2n} |f(Z_i)|^2 \right)^{\frac{1}{2}} - \left( \frac{1}{n} \sum_{i=1}^n |f(Z_i)|^2 \right)^{\frac{1}{2}} > \frac{\varepsilon}{4} \middle| X^{1:2n} \right\} \\
 & \leq \mathbb{P} \left\{ \exists g \in \mathcal{G} : \left( \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right)^{\frac{1}{2}} - \left( \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right)^{\frac{1}{2}} > \frac{\varepsilon}{12} \middle| X^{1:2n} \right\} \\
 & \leq |\mathcal{G}| \max_{g \in \mathcal{G}} \mathbb{P} \left\{ \left( \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right)^{\frac{1}{2}} - \left( \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right)^{\frac{1}{2}} > \frac{\varepsilon}{12} \middle| X^{1:2n} \right\}
 \end{aligned}$$

Step 4: Application of Hoeffding's inequality.

$$\begin{aligned}
 \left( \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right)^{\frac{1}{2}} - \left( \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right)^{\frac{1}{2}} & \leq \left| \frac{\frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 - \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2}{\left( \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right)^{\frac{1}{2}} + \left( \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right)^{\frac{1}{2}}} \right| \\
 & \leq \frac{\left| \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 - \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right|}{\left( \frac{1}{n} \sum_{i=1}^{2n} |g(Z_i)|^2 \right)^{\frac{1}{2}}} = \frac{\left| \frac{1}{n} \sum_{i=1}^n U_i |g(X_i)|^2 - \frac{1}{n} \sum_{i=1}^n U_i |g(X_{i+n})|^2 \right|}{\left( \frac{1}{n} \sum_{i=1}^{2n} |g(Z_i)|^2 \right)^{\frac{1}{2}}}
 \end{aligned}$$

Then

$$\begin{aligned}
 \mathbb{P} \left\{ \left( \frac{1}{n} \sum_{i=n+1}^{2n} |g(Z_i)|^2 \right)^{\frac{1}{2}} - \left( \frac{1}{n} \sum_{i=1}^n |g(Z_i)|^2 \right)^{\frac{1}{2}} > \frac{\varepsilon}{12} \middle| X^{1:2n} \right\} & \leq 2 \exp \left( - \frac{2n^2 \frac{\varepsilon^2}{144} \left( \frac{1}{n} \sum_{i=1}^{2n} |g(X_i)|^2 \right)}{\sum_{i=1}^n 4(|g(X_i)|^2 - |g(X_{i+n})|^2)^2} \right) \\
 & \leq 2 \exp \left( - \frac{2n^2 \frac{\varepsilon^2}{144} \left( \frac{1}{n} \sum_{i=1}^{2n} |g(X_i)|^2 \right)}{\sum_{i=1}^n 4B^2(|g(X_i)|^2 + |g(X_{i+n})|^2)} \right) \\
 & = \exp \left( - \frac{n\varepsilon^2}{288B^2} \right). \quad \square
 \end{aligned}$$

**Theorem 6.4** Assume  $\sigma^2 = \sup_{x \in \mathbb{R}^d} \text{Var}(Y|X = x) < \infty$ . Let  $k_n = k_n(x_1, \dots, x_n)$  be the vector space dimension of  $\mathcal{F}_n$ . Then

$$\mathbb{E} \{ \|\widetilde{m}_n - m\|_n^2 | X^{1:n} \} \leq \frac{\sigma^2 k_n}{n} + \min_{f \in \mathcal{F}_n} \|f - m\|_n^2.$$

**Proof** Denote  $\mathbb{E}^* \{ \cdot \} = \mathbb{E} \{ \cdot | X^{1:n} \}$ . Then

$$\begin{aligned}
 \mathbb{E}^* \{ \|\widetilde{m}_n - m\|_n^2 \} & = \mathbb{E}^* \left\{ \frac{1}{n} \sum_{i=1}^n |\widetilde{m}_n(X_i) - m(X_i)|^2 \right\} \\
 & = \mathbb{E}^* \left\{ \frac{1}{n} \sum_{i=1}^n |\widetilde{m}_n(X_i) - \mathbb{E}^*(\widetilde{m}_n(X_i)) + \mathbb{E}^*(\widetilde{m}_n(X_i)) - m(X_i)|^2 \right\} \\
 & = \mathbb{E}^* \left\{ \frac{1}{n} \sum_{i=1}^n |\widetilde{m}_n(X_i) - \mathbb{E}^*(\widetilde{m}_n(X_i))|^2 \right\} + \mathbb{E}^* \{ |\mathbb{E}^*(\widetilde{m}_n(X_i)) - m(X_i)|^2 \} \\
 & = \mathbb{E}^* \{ \|\widetilde{m}_n - \mathbb{E}^*(\widetilde{m}_n)\|_n^2 \} + \|\mathbb{E}^*(\widetilde{m}_n) - m\|_n^2.
 \end{aligned}$$

Write that  $\widetilde{m}_n = \sum_{j=1}^{k_n} a_j f_{j,n}$  where  $f_{1,n}, \dots, f_{k_n,n}$  is a basis of  $\mathcal{F}_n$ , and  $a = (a_j)_{j=1, \dots, k_n}$  satisfies that  $\frac{1}{n} B^T B a = \frac{1}{n} B^T Y$ ,  $B = (f_{j,n}(X_i))_{1 \leq i \leq n, 1 \leq j \leq k_n}$  and  $Y = (Y_1, \dots, Y_n)^T$ . Then

$$\mathbb{E}^* \{ \widetilde{m}_n \} = \sum_{j=1}^{k_n} \mathbb{E}^* \{ a_j \} f_{j,n} \text{ and } \frac{1}{n} B^T B \mathbb{E}^* a = \frac{1}{n} B^T \mathbb{E}^* Y = \frac{1}{n} B^T (m(X_1), \dots, m(X_n))^T$$



$$\Rightarrow \|\mathbb{E}^*(\widetilde{m}_n) - m\|_n^2 = \min_{f \in \mathcal{F}_n} \|f - m\|_n^2.$$

Choose a complete orthonormal system  $f_1, \dots, f_k$  in  $\mathcal{F}_n$  w.r.t. the empirical scalar product  $\langle \cdot, \cdot \rangle_n$  where  $\langle f, g \rangle_n = \frac{1}{n} \sum_{i=1}^n f(X_i)g(X_i)$ ,  $k \leq k_n$ . We remind our readers that such a system depends on  $X_1, \dots, X_n$ . Then, on  $\{X_1, \dots, X_n\}$ ,  $\text{span}\{f_1, \dots, f_k\} \subset \mathcal{F}_n$ ,  $\widetilde{m}_n(x) = f(x)^T \frac{1}{n} B^T Y$  where  $B = (f_j(X_i))_{1 \leq j \leq n, 1 \leq j \leq k}$ ,  $B^T B = I$ . Therefore,

$$\begin{aligned} \mathbb{E}^*\{|\widetilde{m}_n(x) - \mathbb{E}^*(\widetilde{m}_n(x))|^2\} &= \mathbb{E}^*\left\{\left|f(x)^T \frac{1}{n} B^T Y - f(x)^T \frac{1}{n} B^T (m(X_1), \dots, m(X_n))^T\right|^2\right\} \\ &= f(x)^T \frac{1}{n} B^T (\mathbb{E}^*\{(Y_i - m(X_i))(Y_j - m(X_j))^T\}) \frac{1}{n} B f(x) \\ &\Rightarrow \mathbb{E}^*\{\|\widetilde{m}_n - \mathbb{E}^*(\widetilde{m}_n)\|_n^2\} \leq \frac{1}{n^2} f^T B^T \sigma^2 I B f = \frac{\sigma^2}{n} \sum_{j=1}^k \|f_j\|_n^2 = \frac{\sigma^2}{n} k \leq \frac{\sigma^2}{n} k_n. \end{aligned} \quad \square$$

**Theorem 6.5** Assume  $\sigma^2 = \sup_{x \in \mathbb{R}^d} \text{Var}(Y|X = x) < \infty$  and  $\|m\|_\infty = \sup_{x \in \mathbb{R}^d} |m(x)| \leq L \in \mathbb{R}_+$ ,  $m_n(\cdot) = T_L \widetilde{m}_n(\cdot)$ .

Then

$$\mathbb{E} \int |m_n(x) - m(x)|^2 \mu(dx) \leq C \cdot \max\{\sigma^2, L^2\} \frac{\log(n) + 1}{n} k_n + 8 \inf_{f \in \mathcal{F}_n} \int |f(x) - m(x)|^2 \mu(dx).$$

**Proof** First we note that

$$\begin{aligned} \int |m_n(x) - m(x)|^2 \mu(dx) &= (\|m_n - m\|_2 - 2\|m_n - m\|_n + 2\|m_n - m\|_n)^2 \\ &\leq (\max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\} + 2\|m_n - m\|_n)^2 \\ &\leq 2(\max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\})^2 + 8\|m_n - m\|_n^2. \end{aligned}$$

On the one hand,

$$\begin{aligned} \mathbb{E}\{8\|m_n - m\|_n^2\} &\leq 8\mathbb{E}\{\mathbb{E}\{\|\widetilde{m}_n - m\|_n^2 | X_1, \dots, X_n\}\} \\ &\leq 8\sigma^2 \frac{k_n}{n} + 8\mathbb{E}\left\{\min_{f \in \mathcal{F}_n} \|f - m\|_n^2\right\} \\ &\leq 8\sigma^2 \frac{k_n}{n} + 8 \inf_{f \in \mathcal{F}_n} \mathbb{E}\|f - m\|_n^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{P}(2 \max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\} > u) &\leq \mathbb{P}\left(\exists f \in T_L \mathcal{F}_n : \|f - m\|_2 - 2\|f - m\|_n > \sqrt{\frac{u}{2}}\right) \\ &\leq 3\mathbb{E} \mathcal{N}_2\left(\frac{\sqrt{u}}{24}, \mathcal{F}_n, X^{1:2n}\right) \exp\left(-\frac{nu}{576(2L)^2}\right) \\ &\leq 9(12en)^{2(k_n+1)} \exp\left(-\frac{nu}{2304L^2}\right) \\ \Rightarrow \mathbb{E}(2 \max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\}) &\leq u + \int_u^\infty \mathbb{P}(2 \max\{\|m_n - m\|_2 - 2\|m_n - m\|_n, 0\} > t) dt \\ \left(\text{take } u \geq \frac{576L^2}{n}\right) &\leq CL^2 \frac{\log(n) + 1}{n} k_n. \end{aligned}$$

Combine these two bounds together.  $\square$

**Property 6.1** (Nonlinear LSE)  $|Y| \leq L \leq \beta_n$  a.s.,  $m_n(\cdot) = T_{\beta_n} \widetilde{m}_n(\cdot)$ ,  $\widetilde{m}_n(\cdot) = \arg \min_{f \in \mathcal{F}_n} \frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_i|^2$ . We do the following decomposition:

$$\begin{aligned} \int |m_n(x) - m(x)|^2 \mu(dx) &= \left\{ \mathbb{E}\{|m_n(X) - Y|^2 | \mathcal{D}_n\} - \mathbb{E}|m(X) - Y|^2 - \frac{2}{n} \sum_{i=1}^n [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \right\} \\ &\quad + \frac{2}{n} \sum_{i=1}^n [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2]. \end{aligned}$$

On the one hand,

$$\mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \right\} \leq \mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n |\widetilde{m}_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2 \right\}$$

$$\begin{aligned}
 &\leq \mathbb{E} \left\{ \inf_{f \in \mathcal{F}_n} \frac{1}{n} \sum_{i=1}^n [|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \right\} \\
 &\leq \inf_{f \in \mathcal{F}_n} \mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n [|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \right\} \\
 &= \inf_{f \in \mathcal{F}_n} \{ \mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2 \} \\
 &= \inf_{f \in \mathcal{F}_n} \int |f(x) - m(x)|^2 \mu(dx)
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 &\mathbb{P} \left\{ \mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \mathbb{E}|m(X) - Y|^2 - \frac{2}{n} \sum_{i=1}^n [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] > \varepsilon \right\} \\
 &= \mathbb{P} \left\{ \mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \mathbb{E}|m(X) - Y|^2 - \frac{1}{n} \sum_{i=1}^n [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] > \frac{\varepsilon}{2} + \frac{1}{2} [\mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \mathbb{E}|m(X) - Y|^2] \right\} \\
 &\leq \mathbb{P} \left\{ \exists f \in T_{\beta_n} \mathcal{F}_n : \mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2 - \frac{1}{n} \sum_{i=1}^n [|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] > \frac{\varepsilon}{2} + \frac{1}{2} [\mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2] \right\}.
 \end{aligned}$$

Set  $Z = (X, Y)$ ,  $Z_i = (X_i, Y_i)$ ,  $g(Z) = |f(X) - Y|^2 - |m(X) - Y|^2$ . We can rewrite the above equation as

$$\mathbb{P} \left\{ \mathbb{E}g(Z) - \frac{1}{n} \sum_{i=1}^n g(Z_i) > \frac{\varepsilon}{2} + \frac{1}{2} \mathbb{E}g(Z) \right\}.$$

Since  $|g(Z)| = |(f(X) + m(X) - 2Y)(f(X) - m(X))| \leq 4\beta_n |f(X) - m(X)|$ ,  $\sigma^2 := \text{Var}(g(Z)) \leq \mathbb{E}g(Z)^2 \leq 16\beta_n^2 \mathbb{E}|f(X) - m(X)|^2 = 16\beta_n^2 (\mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2)$ , the above equation is upper-bounded by

$$\mathbb{P} \left\{ \mathbb{E}g(Z) - \frac{1}{n} \sum_{i=1}^n g(Z_i) > \frac{\varepsilon}{2} + \frac{1}{2} \frac{\text{Var}(g(Z))}{16\beta_n^2} \right\} \stackrel{\text{Bernstein's inequality}}{\leq} \exp \left( - \frac{n[\frac{\varepsilon}{2} + \frac{\sigma^2}{32\beta_n^2}]^2}{2\sigma^2 + 2\frac{8\beta_n^2}{3}[\frac{\varepsilon}{2} + \frac{\sigma^2}{32\beta_n^2}]} \right) \leq \exp \left( - \frac{1}{128 + \frac{32}{3}\beta_n^2} \frac{n\varepsilon}{\beta_n^2} \right).$$

**Theorem 6.6** Let  $n \in \mathbb{N}$  and  $1 \leq L < \infty$ . Assume  $|Y| \leq L$  a.s. Let estimate  $m_n$  be defined by minimization of the empirical  $l_2$  risk over a set of functions  $\mathcal{F}_n$  and truncation at  $L$ . Then one has

$$\mathbb{E} \int |m_n(x) - m(x)|^2 \mu(dx) \leq \frac{c_1}{n} + \frac{(c_2 + c_3 \log n) V_{\mathcal{F}_n^+}}{n} + 2 \inf_{f \in \mathcal{F}_n} \int |f(x) - m(x)|^2 \mu(dx)$$

**Proof** We first introduce a theorem/lemma.

**Theorem 6.7** **Lemma 6.1** Assume  $|Y| \leq B$  a.s. and  $B \geq 1$ . Let  $\mathcal{F}$  be a set of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and  $|f(x)| \leq B$ . Then for any  $n \geq 1$ ,  $\alpha, \beta > 0$  and  $0 < \varepsilon \leq \frac{1}{2}$ ,

$$\begin{aligned}
 &\mathbb{P} \left\{ \exists f \in \mathcal{F} : \mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2 - \frac{1}{n} \sum_{i=1}^n [|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \right. \\
 &\quad \left. \geq \varepsilon(\alpha + \beta + \mathbb{E}|f(X) - Y|^2 - \mathbb{E}|m(X) - Y|^2) \right\} \\
 &\leq 14 \sup_{X^{1:n}} \mathcal{N}_1 \left( \frac{\beta\varepsilon}{20B}, \mathcal{F}, X^{1:n} \right) \exp \left( - \frac{\varepsilon^2(1 - \varepsilon)\alpha n}{214(1 + \varepsilon)B^2} \right).
 \end{aligned}$$

Now let's return to the original **Theorem 6.6**.

$$\begin{aligned}
 \int |m_n(x) - m(x)|^2 \mu(dx) &= \left\{ \mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \mathbb{E}[|m(X) - Y|^2] - 2 \left( \frac{1}{n} \sum_{i=1}^n |m_n(X_i) - Y_i|^2 - \frac{1}{n} \sum_{i=1}^n |m(X_i) - Y_i|^2 \right) \right\} \\
 &\quad + 2 \left\{ \frac{1}{n} \sum_{i=1}^n |m_n(X_i) - Y_i|^2 - \frac{1}{n} \sum_{i=1}^n |m(X_i) - Y_i|^2 \right\} \\
 &:= T_{1,n} + T_{2,n}.
 \end{aligned}$$

Since

$$\mathbb{E}(T_{2,n}) \leq 2 \inf_{f \in \mathcal{F}_n} \int |f(x) - m(x)|^2 \mu(dx), \quad \mathbb{E}(T_{1,n}) = \int_0^\infty \mathbb{P}(T_{1,n} > t) dt \leq \varepsilon + \int_\varepsilon^\infty \mathbb{P}(T_{1,n} > t) dt$$

and

$$\begin{aligned}
 \mathbb{P}(T_{1,n} > t) &= \mathbb{P} \left\{ \mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \mathbb{E}[|m(X) - Y|^2] - \frac{1}{n} \sum_{i=1}^n [|m_n(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \right. \\
 &\quad \left. \geq \frac{1}{2} \left( \frac{t}{2} + \frac{t}{2} + \mathbb{E}[|m_n(X) - Y|^2 | \mathcal{D}_n] - \mathbb{E}[|m(X) - Y|^2] \right) \right\} \\
 &\leq \mathbb{P} \left\{ \exists f \in T_L \mathcal{F}_n : \mathbb{E}[f(X) - Y]^2 - \mathbb{E}[m(X) - Y]^2 - \frac{1}{n} \sum_{i=1}^n [|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2] \right. \\
 &\quad \left. \geq \frac{1}{2} \left( \frac{t}{2} + \frac{t}{2} + \mathbb{E}[f(X) - Y]^2 - \mathbb{E}[m(X) - Y]^2 \right) \right\} \\
 &\stackrel{\text{(by lemma 6.1)}}{\leq} 14 \sup_{X^{1:n}} \mathcal{N}_1 \left( \frac{1}{80Ln}, T_L \mathcal{F}_n, X^{1:n} \right) \exp \left( -\frac{nt}{24 \cdot 214L^4} \right) \\
 &\leq 3(480eL^2n)^{2V_{(T_L \mathcal{F}_n)^+}} \exp \left( -\frac{nt}{24 \cdot 214L^4} \right).
 \end{aligned}$$

Plug this bound into the integral in the previous expectation bound,

$$\mathbb{E}(T_{1,n}) \leq \varepsilon + \frac{24 \cdot 214L^4}{n} 42(480eL^2n)^{2V_{\mathcal{F}_n^+}} \exp \left( -\frac{n\varepsilon}{24 \cdot 214L^4} \right).$$

□

**Lemma 6.2** Let  $V_1, \dots, V_n$  i.i.d. r.v.'s,  $0 \leq V_i \leq B$ ,  $0 < \alpha < 1$  and  $\nu > 0$ . Then

$$\mathbb{P} \left\{ \frac{\left| \frac{1}{n} \sum_{i=1}^n V_i - \mathbb{E}V_1 \right|}{\nu + \frac{1}{n} \sum_{i=1}^n V_i + \mathbb{E}V_1} > \alpha \right\} \leq \mathbb{P} \left\{ \frac{\left| \frac{1}{n} \sum_{i=1}^n V_i - \mathbb{E}V_1 \right|}{\nu + \mathbb{E}V_1} > \alpha \right\} < \frac{B}{4\alpha^2\nu n}.$$

**Proof** The first inequality is trivial. For the second, note that

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n (V_i - \mathbb{E}V_1) \right| > \alpha n(\nu + \mathbb{E}V_1) \right\} \leq \frac{\mathbb{E} \left| \sum_{i=1}^n (V_i - \mathbb{E}V_1) \right|^2}{[\alpha n(\nu + \mathbb{E}V_1)]^2} = \frac{\text{Var}(V_1)}{n\alpha^2(\nu + \mathbb{E}V_1)^2} \leq \frac{\mathbb{E}V_1(B - \mathbb{E}V_1)}{n\alpha^2(\nu + \mathbb{E}V_1)^2}$$

where the last inequality holds since

$$\text{Var}(V_1) = \mathbb{E}\{(V_1 - \mathbb{E}V_1)(V_1 - \mathbb{E}V_1) = \mathbb{E}V_1(V_1 - \mathbb{E}V_1) \leq \mathbb{E}V_1(B - \mathbb{E}V_1)\}.$$

In addition,

$$\frac{\mathbb{E}V_1(B - \mathbb{E}V_1)}{n\alpha^2(\nu + \mathbb{E}V_1)^2} \leq \max_{x \in [0, B]} \frac{x(B - x)}{n\alpha^2(\nu + x)^2} = \frac{B^2}{4\alpha^2\nu n(B + \nu)} < \frac{B}{4\alpha^2\nu n}.$$

□

**Theorem 6.8** Let  $B \geq 1$  and  $G$  be a set of functions  $g : \mathbb{R}^d \rightarrow [0, B]$ . Let  $Z_1, \dots, Z_n$  be i.i.d.  $\mathbb{R}^d$ -valued r.v.'s. Assume  $\alpha > 0$ ,  $0 < \varepsilon < 1$  and  $n \geq 1$ . Then

$$\mathbb{P} \left\{ \sup_{g \in \mathcal{G}} \frac{\frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z)}{\alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \mathbb{E}g(Z)} > \varepsilon \right\} \leq 4\mathbb{E} \mathcal{N}_1 \left( \frac{2\varepsilon}{5}, G, Z^{1:n} \right) \exp \left( -\frac{3\varepsilon^2 \alpha n}{40B} \right).$$

**Proof** Step 1: Replace the expectation with empirical mean. Ghost sample  $Z'_{1:n} = (Z'_1, \dots, Z'_n)$  i.i.d. Let  $g^*$  be a function  $g \in \mathcal{G}$  such that

$$\frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) > \varepsilon \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \mathbb{E}g(Z) \right)$$

if there exists any such function. Otherwise, let  $g^*$  be an arbitrary function in  $G$ .  $g^*$  depends on  $Z^{1:n}$ . Since

$$\begin{aligned}
 &\frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}g(Z) > \varepsilon \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \mathbb{E}g(Z) \right) \text{ and } \frac{1}{n} \sum_{i=1}^n g(Z'_i) - \mathbb{E}g(Z) \leq \frac{\varepsilon}{4} \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(Z'_i) + \mathbb{E}g(Z) \right) \\
 &\Rightarrow \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i) > \frac{3}{4}\varepsilon\alpha + \frac{\varepsilon}{n} \sum_{i=1}^n g(Z_i) - \frac{\varepsilon}{n} \sum_{i=1}^n g(Z'_i) + \frac{3\varepsilon}{4}\mathbb{E}g(Z) \\
 &\Leftrightarrow \left( 1 - \frac{5}{8}\varepsilon \right) \left( \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right) > \frac{3}{8}\varepsilon \left( 2\alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right) + \frac{3\varepsilon}{4}\mathbb{E}g(Z)
 \end{aligned}$$

$$\Rightarrow \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i) > \frac{3\varepsilon}{8} \left( 2\alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right),$$

Therefore,

$$\begin{aligned} & \mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i) > \frac{3\varepsilon}{8} \left( 2\alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right) \right\} \\ & \geq \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) > \frac{3\varepsilon}{8} \left( 2\alpha + \frac{1}{n} \sum_{i=1}^n g^*(Z_i) + \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) \right) \right\} \\ & \geq \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] > \varepsilon \left( \alpha + \frac{1}{n} \sum_{i=1}^n g^*(Z_i) + \mathbb{E}[g^*(Z)|Z^{1:n}] \right) \text{ and } \right. \\ & \quad \left. \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] \leq \frac{\varepsilon}{4} \left( \alpha + \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) + \mathbb{E}[g^*(Z)|Z^{1:n}] \right) \right\} \\ & = \mathbb{E} \mathbb{1}_{\left\{ \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] > \varepsilon(\dots) \right\}} \underbrace{\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n g^*(Z'_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] \leq \frac{\varepsilon}{4}(\dots) \right\}}_{> 1 - \frac{B}{4(\frac{\varepsilon}{4})^2 \alpha n} = 1 - \frac{4B}{\varepsilon^2 \alpha n} \geq \frac{1}{2} \text{ for } n > \frac{8B}{\varepsilon^2 \alpha} \text{ by lemma 6.2}} \\ & \geq \frac{1}{2} \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n g^*(Z_i) - \mathbb{E}[g^*(Z)|Z^{1:n}] > \varepsilon \left( \alpha + \frac{1}{n} \sum_{i=1}^n g^*(Z_i) + \mathbb{E}[g^*(Z)|Z^{1:n}] \right) \right\} \\ & = \frac{1}{2} \mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}[g(Z)] > \varepsilon \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \mathbb{E}[g(Z)] \right) \right\} \end{aligned}$$

Step 2: Symmetrization. Let  $U_1, \dots, U_n \stackrel{\text{i.i.d.}}{\sim} U\{-1, 1\}$  independent of  $Z'_{1:n}, Z^{1:n}$ . Therefore,

$$\begin{aligned} & \mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i) > \frac{3\varepsilon}{8} \left( 2\alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right) \right\} \\ & = \mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n U_i [g(Z_i) - \frac{1}{n} \sum_{i=1}^n g(Z'_i)] > \frac{3\varepsilon}{8} \left( 2\alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) + \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right) \right\} \\ & \leq \mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) > \frac{3\varepsilon}{8} \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) \right) \right\} + \mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n U_i g(Z'_i) < -\frac{3\varepsilon}{8} \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(Z'_i) \right) \right\} \\ & = 2\mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n U_i g(Z_i) > \frac{3\varepsilon}{8} \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(Z_i) \right) \right\}. \end{aligned}$$

Step 3: Conditioning and introduction of a covering. Let  $\delta > 0$  and  $\mathcal{G}_\delta$  be an  $L_1$   $\delta$ -cover of  $\mathcal{G}$  on  $z^{1:n}$ . For  $g \in \mathcal{G}$ , there exists a  $\bar{g} \in \mathcal{G}_\delta$  such that  $\frac{1}{n} \sum_{i=1}^n |g(z_i) - \bar{g}(z_i)| < \delta$ . Therefore,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n U_i g(z_i) &= \frac{1}{n} \sum_{i=1}^n U_i g(z_i) - \frac{1}{n} \sum_{i=1}^n U_i \bar{g}(z_i) + \frac{1}{n} \sum_{i=1}^n U_i \bar{g}(z_i) \\ &\leq \frac{1}{n} \sum_{i=1}^n U_i \bar{g}(z_i) + \frac{1}{n} \sum_{i=1}^n |g(z_i) - \bar{g}(z_i)| \leq \frac{1}{n} \sum_{i=1}^n U_i \bar{g}(z_i) + \delta. \end{aligned}$$

On the other hand,  $\frac{1}{n} \sum_{i=1}^n g(z_i) \geq \frac{1}{n} \sum_{i=1}^n \bar{g}(z_i) - \frac{1}{n} \sum_{i=1}^n |g(z_i) - \bar{g}(z_i)| \geq \frac{1}{n} \sum_{i=1}^n \bar{g}(z_i) - \delta$ . Therefore,

$$\begin{aligned} & \mathbb{P} \left\{ \exists g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n U_i g(z_i) > \frac{3\varepsilon}{8} \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(z_i) \right) \right\} \\ & \leq \mathbb{P} \left\{ \exists g \in \mathcal{G}_\delta : \frac{1}{n} \sum_{i=1}^n U_i g(z_i) + \delta > \frac{3\varepsilon}{8} \left( \alpha + \frac{1}{n} \sum_{i=1}^n g(z_i) - \delta \right) \right\} \\ & \leq |\mathcal{G}_\delta| \max_{g \in \mathcal{G}_\delta} \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n U_i g(z_i) > \frac{3\varepsilon\alpha}{8} - \frac{3\varepsilon\delta}{8} - \delta + \frac{3\varepsilon}{8} \frac{1}{n} \sum_{i=1}^n g(z_i) \right\} \end{aligned}$$

$$\left( \text{Take } \delta = \frac{\varepsilon\alpha}{5} \right) \leq |\mathcal{G}_\delta| \max_{g \in \mathcal{G}_\delta} \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n U_i g(z_i) > \frac{\varepsilon\alpha}{10} + \frac{3\varepsilon}{8} \frac{1}{n} \sum_{i=1}^n g(z_i) \right\}$$

Step 4: Application of Hoeffding's inequality. □

**Proof (Theorem 6.7 Lemma 6.1)** Let  $Z = (X, Y)$ ,  $Z_i = (X_i, Y_i)$ ,  $g_f(x, y) = |f(x) - y|^2 - |m(x) - y|^2$  where  $m(x) = \mathbb{E}[Y|X = x]$ . Since  $|f(x)| \leq B$ ,  $|Y| \leq B$  and  $|m(x)| \leq B$ ,  $-4B^2 \leq g_f(x, y) \leq 4B^2$ . Therefore, LHS can be written as

$$\mathbb{P} \left\{ \exists f \in \mathcal{F} : \mathbb{E}g_f(Z) - \frac{1}{n} \sum_{i=1}^n g_f(Z_i) \geq \varepsilon(\alpha + \beta + \mathbb{E}g_f(Z)) \right\}.$$

Step 1: Symmetrization by a ghost sample  $Z'_{1:n} \stackrel{\text{i.i.d.}}{\sim} Z_{1:n}$ . Consider a function  $f_n \in \mathcal{F}$  depending on  $Z_{1:n}$  such that

$$\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] - \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z_i) \geq \varepsilon(\alpha + \beta + \mathbb{E}[g_{f_n}(Z)|Z_{1:n}])$$

if such a function exists in  $\mathcal{F}$ . Since  $\text{Var}(g_{f_n}(Z)|Z_{1:n}) \leq 16B^2\mathbb{E}[g_{f_n}(Z)|Z_{1:n}]$ ,

$$\begin{aligned} & \mathbb{P} \left\{ \mathbb{E}[g_{f_n}(Z)|Z_{1:n}] - \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z'_i) > \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] \middle| Z_{1:n} \right\} \\ & \leq \frac{\text{Var}(g_{f_n}(Z)|Z_{1:n})}{n \left( \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] \right)^2} \leq \frac{16B^2\mathbb{E}[g_{f_n}(Z)|Z_{1:n}]}{n \left( \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] \right)^2} \\ & \leq \frac{16B^2}{\varepsilon^2(\alpha + \beta)n} \leq \frac{1}{8} \left( \text{for } n > \frac{128B^2}{\varepsilon^2(\alpha + \beta)} \right). \end{aligned}$$

That's to say,

$$\mathbb{P} \left\{ \mathbb{E}[g_{f_n}(Z)|Z_{1:n}] - \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z'_i) \leq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] \middle| Z_{1:n} \right\} \geq \frac{7}{8}.$$

Therefore,

$$\begin{aligned} & \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n g_f(Z'_i) - \frac{1}{n} \sum_{i=1}^n g_f(Z_i) \geq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}g_f(Z) \right\} \\ & \geq \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z'_i) - \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z_i) \geq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] \right\} \\ & \geq \mathbb{P} \left\{ \mathbb{E}[g_{f_n}(Z)|Z_{1:n}] - \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z_i) \geq \varepsilon(\alpha + \beta) + \varepsilon\mathbb{E}[g_{f_n}(Z)|Z_{1:n}], \right. \\ & \quad \left. \mathbb{E}[g_{f_n}(Z)|Z_{1:n}] - \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z'_i) \leq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] \right\} \\ & \geq \frac{7}{8} \mathbb{P} \left\{ \mathbb{E}[g_{f_n}(Z)|Z_{1:n}] - \frac{1}{n} \sum_{i=1}^n g_{f_n}(Z_i) \geq \varepsilon(\alpha + \beta) + \varepsilon\mathbb{E}[g_{f_n}(Z)|Z_{1:n}] \right\} \\ & = \frac{7}{8} \mathbb{P} \left\{ \exists f \in \mathcal{F} : \mathbb{E}g_f(Z) - \frac{1}{n} \sum_{i=1}^n g_f(Z_i) \geq \varepsilon(\alpha + \beta + \mathbb{E}g_f(Z)) \right\}. \end{aligned}$$

In other words,

$$\text{LHS} \leq \frac{8}{7} \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n g_f(Z'_i) - \frac{1}{n} \sum_{i=1}^n g_f(Z_i) \geq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}g_f(Z) \right\}.$$

Step 2: Replacement of the expectation by an empirical mean of the ghost sample. That's to say,

$$\begin{aligned} & \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n g_f(Z'_i) - \frac{1}{n} \sum_{i=1}^n g_f(Z_i) \geq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}g_f(Z) \right\} \\ & \leq \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n g_f(Z'_i) - \frac{1}{n} \sum_{i=1}^n g_f(Z_i) \geq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2}\mathbb{E}g_f(Z), \right. \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) - \mathbb{E}g_f^2(Z) \leq \varepsilon \left( \alpha + \beta + \frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) + \mathbb{E}g_f^2(Z) \right), \\
 & \frac{1}{n} \sum_{i=1}^n g_f^2(Z'_i) - \mathbb{E}g_f^2(Z) \leq \varepsilon \left( \alpha + \beta + \frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) + \mathbb{E}g_f^2(Z) \right) \Big\} \\
 & + 2\mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{\frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) - \mathbb{E}g_f^2(Z)}{\alpha + \beta + \frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) + \mathbb{E}g_f^2(Z)} > \varepsilon \right\}.
 \end{aligned}$$

The second inequality follows from the fact that

$$\mathbb{P}(A) = \mathbb{P}((A \cap B \cap C) \cup \bar{B} \cup \bar{C}) \leq \mathbb{P}(A \cap B \cap C) + \mathbb{P}(\bar{B}) + \mathbb{P}(\bar{C}).$$

By theorem 6.8,

$$\mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{\frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) - \mathbb{E}g_f^2(Z)}{\alpha + \beta + \frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) + \mathbb{E}g_f^2(Z)} > \varepsilon \right\} \leq 4\mathbb{E}\mathcal{N}_1 \left( \frac{\alpha + \beta}{5} \varepsilon, \{g_f : f \in \mathcal{F}, Z_{1:n}\} \right) \exp \left( -\frac{\varepsilon^2(\alpha + \beta)n}{40 \times 16B^4} \right).$$

Now we consider the first probability on the RHS. The second inequality inside the probability implies

$$\begin{aligned}
 (1 + \varepsilon)\mathbb{E}g_f^2(Z) & \geq (1 - \varepsilon)\frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) - \varepsilon(\alpha + \beta) \\
 \Leftrightarrow \frac{1}{32B^2}\mathbb{E}g_f^2(Z) & \geq \frac{1 - \varepsilon}{32B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) - \frac{\varepsilon(\alpha + \beta)}{32B^2(1 + \varepsilon)}.
 \end{aligned}$$

Since

$$\mathbb{E}g_f(Z) \geq \frac{1}{16B^2}\mathbb{E}g_f^2(Z) = \frac{2}{32B^2}\mathbb{E}g_f^2(Z),$$

the first probability on the RHS can be bounded by

$$\begin{aligned}
 \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n g_f(Z'_i) - \frac{1}{n} \sum_{i=1}^n g_f(Z_i) \geq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2} \left( \frac{1 - \varepsilon}{32B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) - \frac{\varepsilon(\alpha + \beta)}{32B^2(1 + \varepsilon)} \right. \right. \\
 \left. \left. + \frac{1 - \varepsilon}{32B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(Z'_i) - \frac{\varepsilon(\alpha + \beta)}{32B^2(1 + \varepsilon)} \right) \right\}.
 \end{aligned}$$

Step 3: Additional randomization by random signs. The bound is equal to

$$\begin{aligned}
 \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n U_i[g_f(Z'_i) - g_f(Z_i)] \geq \frac{\varepsilon}{2}(\alpha + \beta) + \frac{\varepsilon}{2} \left( \frac{1 - \varepsilon}{32B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) - \frac{\varepsilon(\alpha + \beta)}{32B^2(1 + \varepsilon)} \right. \right. \\
 \left. \left. + \frac{1 - \varepsilon}{32B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(Z'_i) - \frac{\varepsilon(\alpha + \beta)}{32B^2(1 + \varepsilon)} \right) \right\},
 \end{aligned}$$

which is bounded by

$$2\mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n U_i g_f(Z_i) \right| \geq \frac{\varepsilon}{4}(\alpha + \beta) - \frac{\varepsilon^2(\alpha + \beta)}{64B^2(1 + \varepsilon)} + \frac{\varepsilon(1 - \varepsilon)}{64B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(Z_i) \right\}.$$

Step 4: Conditioning and using covering:

$$\begin{aligned}
 & \mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n U_i g_f(z_i) \right| \geq \frac{\varepsilon(\alpha + \beta)}{4} - \frac{\varepsilon^2(\alpha + \beta)}{64B^2(1 + \varepsilon)} + \frac{\varepsilon(1 - \varepsilon)}{64B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(z_i) \right\} \\
 & \leq \mathbb{P} \left\{ \exists g \in \mathcal{G}_\delta : \left| \frac{1}{n} \sum_{i=1}^n U_i g(z_i) \right| + \delta \geq \frac{\varepsilon(\alpha + \beta)}{4} - \frac{\varepsilon^2(\alpha + \beta)}{64B^2(1 + \varepsilon)} + \frac{\varepsilon(1 - \varepsilon)}{64B^2(1 + \varepsilon)} \left( \frac{1}{n} \sum_{i=1}^n g_f^2(z_i) - 8B^2\delta \right) \right\} \\
 & \leq |\mathcal{G}_\delta| \max_{g \in \mathcal{G}_\delta} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n U_i g(z_i) \right| \geq \frac{\varepsilon(\alpha + \beta)}{4} - \frac{\varepsilon^2(\alpha + \beta)}{64B^2(1 + \varepsilon)} - \delta - \delta \frac{\varepsilon(1 - \varepsilon)}{8(1 + \varepsilon)} + \frac{\varepsilon(1 - \varepsilon)}{64B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g_f^2(z_i) \right\} \\
 & \left( \text{Take } \delta = \frac{\varepsilon\beta}{5} \right) \leq |\mathcal{G}_\delta| \max_{g \in \mathcal{G}_\delta} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n U_i g(z_i) \right| \geq \frac{\varepsilon\alpha}{4} - \frac{\varepsilon^2\alpha}{64B^2(1 + \varepsilon)} + \frac{\varepsilon(1 - \varepsilon)}{64B^2(1 + \varepsilon)}\frac{1}{n} \sum_{i=1}^n g^2(z_i) \right\}.
 \end{aligned}$$

Step 5: Application of Bernstein's inequality:

$$\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n U_i g(z_i) \right| \geq \frac{\varepsilon \alpha}{4} - \frac{\varepsilon^2 \alpha}{64 B^2 (1 + \varepsilon)} + \frac{\varepsilon (1 - \varepsilon)}{64 B^2 (1 + \varepsilon)} \frac{1}{n} \sum_{i=1}^n g^2(z_i) \right\} \leq 2 \exp \left( - \frac{\varepsilon^2 (1 - \varepsilon) \alpha n}{140 B^2 (1 + \varepsilon)} \right).$$

Step 6: Bounding the covering number:

$$\mathcal{N}_1 \left( \frac{\varepsilon \beta}{5}, \{g_f, f \in \mathcal{F}\}, Z^{1:n} \right) \leq \mathcal{N}_1 \left( \frac{\varepsilon \beta}{20 B}, \mathcal{F}, X^{1:n} \right).$$

□

## 7 Advanced Techniques from Empirical Process Theory

**Theorem 7.1** Let  $L \in \mathbb{R}_+$  and  $\varepsilon_1, \dots, \varepsilon_n$  be independent random variables with expectation zero and values in  $[-L, L]$ . Let  $z_1, \dots, z_n \in \mathbb{R}^d$ ,  $R > 0$  and  $\mathcal{F}$  be a class of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  with the property  $\|f\|_n^2 = \frac{1}{n} \sum_{i=1}^n |f(z_i)|^2 \leq R^2 (\forall f \in \mathcal{F})$ . Then  $\sqrt{n} \delta \geq 48 \sqrt{2} L \int_{\frac{\delta}{8L}}^{\frac{R}{2}} (\log \mathcal{N}_2(u, \mathcal{F}, z_{1:n}))^{\frac{1}{2}} du$  and  $\sqrt{n} \delta \geq 36 R L$  imply

$$\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(z_i) \varepsilon_i \right| > \delta \right\} \leq 5 \exp \left( - \frac{n \delta^2}{2304 L^2 R^2} \right).$$

**Proof** For  $R \leq \frac{\delta}{2L}$ ,

$$\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(z_i) \varepsilon_i \right| \leq \sqrt{\frac{1}{n} \sum_{i=1}^n f(z_i)^2} \sqrt{\frac{1}{n} \sum_{i=1}^n \varepsilon_i^2} \leq \sup_{f \in \mathcal{F}} \|f\|_n \sqrt{\frac{1}{n} \sum_{i=1}^n \varepsilon_i^2} \leq R L \leq \delta,$$

so WLOG we assume  $R > \frac{\delta}{2L}$ .

For  $s \in \mathbb{N}_+$ , let  $\{f_1^s, \dots, f_{N_s}^s\}$  be a  $\|\cdot\|_n$ -cover of  $\mathcal{F}$  of radius  $\frac{R}{2^s}$  of size  $N_s = \mathcal{N}_2 \left( \frac{R}{2^s}, \mathcal{F}, z_{1:n} \right)$ . Set  $S = \min\{s \geq 1 : \frac{R}{2^s} \leq \frac{\delta}{2L}\}$ . Since

$$\begin{aligned} \left| \frac{1}{n} \sum_{i=1}^n f(z_i) \varepsilon_i \right| &= \left| \frac{1}{n} \sum_{i=1}^n (f(z_i) - f^S(z_i)) \varepsilon_i + \sum_{s=1}^S \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right| \\ &\leq \left| \frac{1}{n} \sum_{i=1}^n (f(z_i) - f^S(z_i)) \varepsilon_i \right| + \sum_{s=1}^S \left| \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right| \\ &\leq \|f - f^S\|_n \cdot \sqrt{\frac{1}{n} \sum_{i=1}^n \varepsilon_i^2} + \sum_{s=1}^S \left| \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right| \\ &\leq \frac{\delta}{2} + \sum_{s=1}^S \left| \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right|, \end{aligned}$$

for  $\eta_1, \dots, \eta_S \geq 0$  satisfy  $\eta_1 + \dots + \eta_S \leq 1$ , we have

$$\begin{aligned} \mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(z_i) \varepsilon_i \right| > \delta \right\} &\leq \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{\delta}{2} + \sum_{s=1}^S \left| \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right| > \frac{\delta}{2} + \frac{\delta}{2} \sum_{s=1}^S \eta_s \right\} \\ &\leq \sum_{s=1}^S \mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right| > \frac{\delta}{2} \eta_s \right\} \\ &\leq \sum_{s=1}^S N_s N_{s-1} \max_{f \in \mathcal{F}} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right| > \frac{\delta}{2} \eta_s \right\}. \end{aligned}$$

For  $s \in \{1, \dots, S\}$  and  $f \in \mathcal{F}$ ,  $(f^s(z_1) - f^{s-1}(z_1)) \varepsilon_1, \dots, (f^s(z_n) - f^{s-1}(z_n)) \varepsilon_n$  are independent, have zero means and take values in  $[-L|f^s(z_i) - f^{s-1}(z_i)|, L|f^s(z_i) - f^{s-1}(z_i)|]$ . Therefore,

$$\frac{1}{n} \sum_{i=1}^n (2L|f^s(z_i) - f^{s-1}(z_i)|)^2 = 4L^2 \|f^s - f^{s-1}\|_n^2 \leq 4L^2 (\|f^s - f\|_n + \|f^{s-1} - f\|_n)^2 \leq 4L^2 \left( \frac{R}{2^s} + \frac{R}{2^{s-1}} \right)^2 = \frac{36R^2 L^2}{2^{2s}}.$$



By Hoeffding's inequality,

$$\begin{aligned} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^n (f^s(z_i) - f^{s-1}(z_i)) \varepsilon_i \right| > \frac{\delta}{2} \eta_s \right\} &\leq 2 \exp \left( - \frac{2n \left( \frac{\eta_s \delta}{2} \right)^2}{36 \frac{R^2 L^2}{2^{2s}}} \right) \\ \Rightarrow \mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(z_i) \varepsilon_i \right| > \delta \right\} &\leq \sum_{s=1}^S 2N_s^2 \exp \left( - \frac{n\delta^2 \eta_s^2 2^{2s}}{72R^2 L^2} \right) = \sum_{s=1}^S 2 \exp \left( 2 \log N_s - \frac{n\delta^2 \eta_s^2 2^{2s}}{72R^2 L^2} \right). \end{aligned}$$

Choose  $\eta_s$  such that

$$2 \log N_s \leq \frac{1}{2} \frac{n\delta^2 \eta_s^2 2^{2s}}{72R^2 L^2} \Leftrightarrow \eta_s \geq \bar{\eta}_s := \frac{12\sqrt{2}RL}{2^s \delta \sqrt{n}} (\log N_s)^{\frac{1}{2}}.$$

More precisely, set

$$\eta_s := \max \left\{ \bar{\eta}_s, \frac{2^{-s} \sqrt{s}}{4} \right\}.$$

Because of

$$\sum_{s=1}^S \frac{2^{-s} \sqrt{s}}{4} \leq \frac{1}{8} \sum_{s=1}^{+\infty} s \left( \frac{1}{2} \right)^{s-1} = \frac{1}{2}$$

and

$$\sum_{s=1}^S \bar{\eta}_s = \sum_{s=1}^S \frac{24\sqrt{2}L}{\delta \sqrt{n}} \frac{R}{2^{s+1}} \left\{ \log \mathcal{N}_2 \left( \frac{R}{2^s}, \mathcal{F}, z_{1:n} \right) \right\}^{\frac{1}{2}} \leq \sum_{s=1}^S \frac{24\sqrt{2}L}{\delta \sqrt{n}} \int_{\frac{R}{2^{s+1}}}^{\frac{R}{2^s}} \left\{ \log \mathcal{N}_2(u, \mathcal{F}, z_{1:n}) \right\}^{\frac{1}{2}} du \leq \frac{1}{2},$$

we have

$$\sum_{s=1}^S \eta_s \leq \sum_{s=1}^S \frac{2^{-s} \sqrt{s}}{4} + \sum_{s=1}^S \bar{\eta}_s \leq 1.$$

Therefore,

$$\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(z_i) \varepsilon_i \right| > \delta \right\} \leq \sum_{s=1}^S 2 \exp \left( - \frac{n\delta^2 \eta_s^2 2^{2s}}{144R^2 L^2} \right) \leq \sum_{s=1}^S 2 \exp \left( - \frac{n\delta^2}{16 \cdot 144R^2 L^2} \cdot s \right) \leq 5 \exp \left( - \frac{n\delta^2}{2304L^2 R^2} \right). \square$$

**Theorem 7.2** (Extension of **Theorem 6.8**) Let  $Z, Z_1, \dots, Z_n$  be i.i.d. with values in  $\mathbb{R}^d$ . Let  $K \geq 1$  and  $\mathcal{F}$  be a class of functions  $f : \mathbb{R}^d \rightarrow [0, K]$ . Let  $0 < \varepsilon < 1$  and  $\alpha > 0$ . Assume that  $\sqrt{n}\varepsilon\sqrt{\alpha} \geq 576\sqrt{K}$  and that for all  $z_1, \dots, z_n \in \mathbb{R}^d$  and all  $\delta \geq \frac{\alpha K}{2}$ ,  $\frac{\sqrt{n}\varepsilon\delta}{192\sqrt{2}K} \geq \int_{\frac{\varepsilon\delta}{32K}}^{\sqrt{\delta}} \left( \log \mathcal{N}_2 \left( u, \left\{ f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n f(z_i) \leq \frac{4\delta}{K} \right\}, z_{1:n} \right) \right)^{\frac{1}{2}} du$ . Then

$$\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \frac{|\mathbb{E}[f(Z)] - \frac{1}{n} \sum_{i=1}^n f(Z_i)|}{\alpha + \mathbb{E}[f(Z)] + \frac{1}{n} \sum_{i=1}^n f(Z_i)} > \varepsilon \right\} \leq 15 \exp \left( - \frac{n\alpha\varepsilon^2}{128 \cdot 2304K} \right).$$

**Proof** Step 1:  $Z'_{1:n} = (Z'_1, \dots, Z'_n)$ ,

$$\begin{aligned} &\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \frac{|\mathbb{E}[f(Z)] - \frac{1}{n} \sum_{i=1}^n f(Z_i)|}{\alpha + \mathbb{E}[f(Z)] + \frac{1}{n} \sum_{i=1}^n f(Z_i)} > \varepsilon \right\} \\ &\leq \frac{100}{99} \mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n f(Z_i) - \frac{1}{n} \sum_{i=1}^n f(Z'_i) \right| > \frac{\varepsilon}{8} \left( 2\alpha + \frac{1}{n} \sum_{i=1}^n (Z_i) + \frac{1}{n} \sum_{i=1}^n f(Z'_i) \right) \right\}. \end{aligned}$$

Step 2:  $U_1, \dots, U_n \stackrel{\text{i.i.d.}}{\sim} U\{-1, 1\}$ ,

$$\leq \frac{100}{99} \cdot 2 \mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n U_i f(Z_i) \right| > \frac{\varepsilon}{8} \left( \alpha + \frac{1}{n} \sum_{i=1}^n f(Z_i) \right) \right\}.$$

Step 3: Peeling,

$$\begin{aligned} &\leq \sum_{k=1}^{+\infty} \mathbb{P} \left\{ \exists f \in \mathcal{F} : 1_{\{k \neq 1\}} 2^{k-1} \alpha \leq \frac{1}{n} \sum_{i=1}^n f(Z_i) < 2^k \alpha, \left| \frac{1}{n} \sum_{i=1}^n U_i f(Z_i) \right| > \frac{\varepsilon}{8} \left( \alpha + \frac{1}{n} \sum_{i=1}^n f(Z_i) \right) \right\} \\ &\leq \sum_{k=1}^{+\infty} \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n f(Z_i) \leq 2^k \alpha, \left| \frac{1}{n} \sum_{i=1}^n U_i f(Z_i) \right| > \frac{\varepsilon}{8} \alpha 2^{k-1} \right\}. \end{aligned}$$

Step 4: Application of Theorem 7.1.  $R^2 = \alpha 2^k K, L = 1, \delta = \frac{\varepsilon}{8} \alpha 2^{k-1}$ .  $\square$

**Theorem 7.3** Let  $Z, Z_1, \dots, Z_n$  be i.i.d. with values in  $\mathbb{R}^d$ . Let  $K_1, K_2 \geq 1$  and  $\mathcal{F}$  be a class of functions  $\mathbb{R}^d \rightarrow \mathbb{R}$  with properties  $|f(z)| \leq K_1 (\forall z \in \mathbb{R}^d)$  and  $\mathbb{E}f(Z)^2 \leq K_2 \mathbb{E}f(Z)$ . Let  $0 < \varepsilon < 1$  and  $\alpha > 0$ . Assume that  $\sqrt{n\varepsilon}\sqrt{1-\varepsilon}\sqrt{\alpha} \geq 288 \max\{2K_1, \sqrt{2K_2}\}$  and that for all  $z_1, \dots, z_n \in \mathbb{R}^d$  and  $\delta \geq \frac{\alpha}{8}$ ,

$$\frac{\sqrt{n\varepsilon}(1-\varepsilon)\delta}{96\sqrt{2}\max\{K_1, 2K_2\}} \geq \int_{\frac{\varepsilon(1-\varepsilon)\delta}{16\max\{K_1, 2K_2\}}}^{\sqrt{\delta}} \left( \log \mathcal{N}_2 \left( u, \left\{ f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n f(z_i)^2 \leq 16\delta \right\}, z_1^n \right) \right)^{\frac{1}{2}} du.$$

Then

$$\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \frac{|\mathbb{E}f(Z) - \frac{1}{n} \sum_{i=1}^n f(Z_i)|}{\alpha + \mathbb{E}f(Z)} > \varepsilon \right\} \leq 60 \exp \left( -\frac{n\alpha\varepsilon^2(1-\varepsilon)}{128 \cdot 2304 \max\{K_1^2, K_2\}} \right).$$

**Proof** Step 1: Ghost samples  $Z'_{1:n} = (Z'_1, \dots, Z'_n)$ ,

$$\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \frac{|\mathbb{E}f(Z) - \frac{1}{n} \sum_{i=1}^n f(Z_i)|}{\alpha + \mathbb{E}f(Z)} > \varepsilon \right\} \leq \frac{10}{9} \mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n f(Z_i) - \frac{1}{n} \sum_{i=1}^n f(Z'_i) \right| > \frac{\varepsilon}{2} \alpha + \frac{\varepsilon}{2} \mathbb{E}f(Z) \right\}.$$

Step 2:

$$\begin{aligned} &\leq \mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n f(Z_i) - \frac{1}{n} \sum_{i=1}^n f(Z'_i) \right| > \frac{\varepsilon\alpha}{4} + \frac{(1-\varepsilon)\varepsilon}{4(1+\varepsilon)K_2} \left( \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 + \frac{1}{n} \sum_{i=1}^n f(Z'_i)^2 \right) \right\} \\ &+ 2\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \frac{\left| \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 - \mathbb{E}f(Z)^2 \right|}{\alpha + \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 + \mathbb{E}f(Z)^2} > \varepsilon \right\}. \end{aligned}$$

Step 3:

$$\mathbb{P} \left\{ \sup_{f \in \mathcal{F}} \frac{\left| \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 - \mathbb{E}f(Z)^2 \right|}{\alpha + \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 + \mathbb{E}f(Z)^2} > \varepsilon \right\} \leq 15 \exp \left( -\frac{n\varepsilon^2\alpha}{128 \cdot 2304 K_1^2} \right).$$

Step 4:  $U_1, \dots, U_n \stackrel{\text{i.i.d.}}{\sim} U\{-1, 1\}$ ,

$$\begin{aligned} &\mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n f(Z_i) - \frac{1}{n} \sum_{i=1}^n f(Z'_i) \right| > \frac{\varepsilon\alpha}{4} + \frac{(1-\varepsilon)\varepsilon}{4(1+\varepsilon)K_2} \left( \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 + \frac{1}{n} \sum_{i=1}^n f(Z'_i)^2 \right) \right\} \\ &\leq 2\mathbb{P} \left\{ \exists f \in \mathcal{F} : \left| \frac{1}{n} \sum_{i=1}^n U_i f(Z_i) \right| > \frac{\varepsilon\alpha}{8} + \frac{(1-\varepsilon)\varepsilon}{4(1+\varepsilon)K_2} \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 \right\}. \end{aligned}$$

Step 5: Peeling,

$$\leq \sum_{k=1}^{+\infty} \mathbb{P} \left\{ \exists f \in \mathcal{F} : \frac{1}{n} \sum_{i=1}^n f(Z_i)^2 \leq 2^k \frac{K_2(1+\varepsilon)\alpha}{2(1-\varepsilon)}, \left| \frac{1}{n} \sum_{i=1}^n U_i f(Z_i) \right| > \frac{\varepsilon\alpha}{8} 2^{k-1} \right\}.$$

Step 6: Application of Theorem 7.1. □

**Definition 7.1** (Piecewise polynomial partitioning estimates)  $X \in [0, 1]$  a.s. and  $|Y| \leq L$  a.s.. The piecewise polynomial partitioning estimate is defined by minimizing the empirical  $L_2$  risk over the set  $\mathcal{F}_{K,M}$  of all piecewise polynomials of degree  $M$  (or less) with respect to an equidistant partition of  $[0, 1]$  into  $K$  intervals. More precisely, set

$$m_{n,(K,M)}(\cdot) = \arg \min_{f \in \mathcal{F}_{K,M}(L+1)} \frac{1}{n} \sum_{i=1}^n |f(X_i) - Y_i|^2$$

where  $\mathcal{F}_{K,M}(L+1) = \{f \in \mathcal{F}_{K,M} : \sup_{x \in [0,1]} |f(x)| \leq L+1\}$ .

**Theorem 7.4** Let  $M \in \mathbb{N}, K \in \mathbb{N}_+, x \in [0, 1]$  and  $L > 0$ . Then

$$\mathbb{E} \int |m_{n,(K,M)}(x) - m(x)|^2 \mu(dx) \leq c_1 \cdot \frac{(M+1)K}{n} + 2 \inf_{f \in \mathcal{F}_{K,M}(L+1)} \int |f(x) - m(x)|^2 \mu(dx).$$

**Proof** Conduct the following decomposition:

$$\int |m_{n,(K,M)}(x) - m(x)|^2 \mu(dx) = \mathbb{E}\{|m_{n,(K,M)}(X) - Y|^2 | \mathcal{D}_n\} - \mathbb{E}\{|m(X) - Y|^2\} := T_{1,n} + T_{2,n}$$

where

$$T_{1,n} = 2 \frac{1}{n} \sum_{i=1}^n \{|m_{n,(K,M)}(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2\},$$

$$T_{2,n} = \mathbb{E}\{|m_{n,(K,M)}(X) - Y|^2 | \mathcal{D}_n\} - \mathbb{E}\{|m(X) - Y|^2\} - T_{1,n}.$$

By definition of the estimate,

$$T_{1,n} = 2 \min_{f \in \mathcal{F}_{K,M}(L+1)} \frac{1}{n} \sum_{i=1}^n \{|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2\} \Rightarrow \mathbb{E}T_{1,n} \leq 2 \inf_{f \in \mathcal{F}_{K,M}(L+1)} \int |f(x) - m(x)|^2 \mu(dx).$$

On the other hand,

$$\mathbb{E}\{T_{2,n}\} \leq \int_0^{+\infty} \mathbb{P}\{T_{2,n} > t\} dt$$

and

$$\begin{aligned} \mathbb{P}\{T_{2,n} > t\} &\leq \mathbb{P}\left\{\exists f \in \mathcal{F}_{K,M}(L+1) : 2\mathbb{E}\{|f(X) - Y|^2 - |m(X) - Y|^2\} - \frac{2}{n} \sum_{i=1}^n \{|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2\} \right. \\ &\quad \left. > t + \mathbb{E}\{|f(X) - Y|^2 - |m(X) - Y|^2\}\right\}. \end{aligned}$$

Then use the following lemma. □

**Lemma 7.1** Let  $(X, Y), (X_1, Y_1), \dots, (X_n, Y_n)$  i.i.d.,  $|Y| \leq L$  a.s.,  $L \geq 1$ . Let  $K \in \mathbb{N}_+$  and  $M \in \mathbb{N}$ . Then for  $\alpha \geq c_3 \frac{(M+1)K}{n}$  ( $c_3$  depends on  $L$ ),

$$\begin{aligned} \mathbb{P}\left\{\exists f \in \mathcal{F}_{K,M}(L+1) : \mathbb{E}\{|f(X) - Y|^2 - |m(X) - Y|^2\} - \frac{1}{n} \sum_{i=1}^n \{|f(X_i) - Y_i|^2 - |m(X_i) - Y_i|^2\} \right. \\ \left. > \frac{1}{2}(\alpha + \mathbb{E}\{|f(X) - Y|^2 - |m(X) - Y|^2\})\right\} \leq 60 \exp\left(-\frac{n\alpha}{128 \cdot 2304 \cdot 800 \cdot L^4}\right). \end{aligned}$$

**Proof** Set  $Z = (X, Y), Z_i = (X_i, Y_i)$ , define  $g_f : \mathbb{R} \times \mathcal{R} \rightarrow \mathbb{R}, g_f(x, y) = (|f(x) - y|^2 - |m(x) - y|^2)1_{\{y \in [-L, L]\}}$ . Set  $\mathcal{G} = \{g_f : f \in \mathcal{F}_{K,M}(L+1)\}$ , then the LHS can be written as

$$\mathbb{P}\left\{\exists g \in \mathcal{G} : \frac{\mathbb{E}g(Z) - \frac{1}{n} \sum_{i=1}^n g(Z_i)}{\alpha + \mathbb{E}g(Z)} > \frac{1}{2}\right\}.$$

Furthermore,

$$\begin{aligned} \mathbb{E}\{g_f(Z)^2\} &= \mathbb{E}\{(|f(X) - Y|^2 - |m(X) - Y|^2)^2\} \leq \mathbb{E}\{|f(X) + m(X) - 2Y| \times |f(X) - Y|\} \\ &\leq (L+1+3L)^2 \mathbb{E}\{|f(X) - Y|^2\} \leq 25L^2 \mathbb{E}\{g_f(Z)\}. \end{aligned}$$

By the definition of piecewise polynomials,  $\mathcal{G}$  is a subset of a linear vector space of dimension

$$D = K \cdot ((2M+1) + (M+1)) + 1.$$

Therefore,

$$\log \mathcal{N}_2\left(u, \left\{g \in \mathcal{G} : \frac{1}{n} \sum_{i=1}^n g(z_i)^2 \leq 16\delta\right\}, Z_{1:n}\right) \leq D \log \frac{16\sqrt{\delta} + u}{u}.$$

Then use Theorem 7.3. □

**Corollary 7.1**  $C, L > 0$  and  $p = k + \beta$  with  $k \in \mathbb{N}$  and  $\beta \in (0, 1]$ . Set  $M = k$  and  $K_n = \lceil C^{\frac{2}{2q+1}} n^{\frac{1}{2q+1}} \rceil$ . Then there exists a constant  $c_2$  which only depends on  $p$  and  $L$  such that for all  $n \geq \max\{C^{\frac{1}{q}}, C^{-2}\}$ ,

$$\mathbb{E} \int |m_{n,(K_n,M)}(x) - m(x)|^2 \mu(dx) \leq c_2 C^{\frac{2}{2q+1}} n^{-\frac{2q}{2q+1}}$$

for every distribution of  $(X, Y)$  with  $X \in [0, 1]$  a.s.,  $|Y| \leq L$  a.s. and  $m$   $(q, C)$ -smooth.

**Proof** There exists a piecewise polynomial  $g \in \mathcal{F}_{K_n, M}$  such that

$$\begin{aligned}
 & \sup_{x \in [0, 1]} |g(x) - m(x)| \leq \frac{1}{2^q k!} \cdot \frac{C}{K_n^q} \\
 \Rightarrow & \sup_{x \in [0, 1]} |g(x)| \leq \sup_{x \in [0, 1]} |m(x)| + \frac{1}{2^q k!} \frac{C}{K_n^q} \leq L + (Cn^{-q})^{\frac{1}{2q+1}} \leq L + 1 \\
 \Rightarrow & g \in \mathcal{F}_{K_n, M}(L + 1) \\
 \Rightarrow & \inf_{f \in \mathcal{F}_{K, M}(L+1)} \int |f(x) - m(x)|^2 \mu(dx) \leq \sup_{x \in [0, 1]} |g(x) - m(x)|^2 \leq \frac{1}{(2^q k!)^2} C^2 \frac{1}{K_n^{2q}}.
 \end{aligned}$$

□