# High-Dimensional Probability

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## Contents



**Reference** High-Dimensional Probability: An Introduction with Applications in Data Science (Roman Vershynin)

# 0 Appetizer

- Convex combination: For $z_1, z_2, \cdots, z_m \in \mathbb{R}^n$, the form of $\sum_{i=1}^m \lambda_i z_i$ with $\lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1$. Convex hull of $T \subset \mathbb{R}^n$: $\mathrm{conv}(T) = \{$convex combinations of $z_1, \cdots, z_m \in T, m \in \mathbb{N}\}$.

- Caratheodory's theorem: Every point in the convex hull of a set $T \subset \mathbb{R}^n$ can be expressed as a convex combination of at most $n+1$ points from $T$.

- Approximate Caratheodory's theorem: Consider $T \subset \mathbb{R}^n$, $\mathrm{diam}(T) = \sup\{\|s-t\|_2, s, t \in T\} < 1$. Then for any $x \in \mathrm{conv}(T)$ and any $k$, one can find points $x_1, x_2, \cdots, x_k \in T$ such that $\|x - \frac{1}{k}\sum_{i=1}^k x_i\|_2 \leq \frac{1}{\sqrt{k}}$ (repetition is allowed).

  *Proof* WLOG assume $\|t\|_2 \leq 1, \forall t \in T$. Fix $x \in \mathrm{conv}(T), x = \sum_{i=1}^m \lambda_i z_i, z_i \in T$. Define $Z, \mathbb{P}(Z = z_i) = \lambda_i, \mathbb{E}Z = x$. Consider i.i.d. $Z_1, Z_2, \cdots$ of $Z$, $\frac{1}{n}\sum_{j=1}^n Z_j \to x$ a.s. $n \to +\infty$. $\mathbb{E}\|x - \frac{1}{k}\sum_{j=1}^k Z_j\|_2^2 = \frac{1}{k^2}\sum_{j=1}^k \mathbb{E}\|Z_j - x\|_2^2 = \frac{1}{k^2}\sum_{j=1}^k (\mathbb{E}\|Z_j\|^2 - \|\mathbb{E}Z_j\|_2^2) \leq \frac{1}{k} \Rightarrow \exists$ a realization of $Z_1, \cdots, Z_k$ such that $\|x - \frac{1}{k}\sum_{j=1}^k Z_j\|_2 \leq \frac{1}{\sqrt{k}}$. $\square$

- Corollary (Covering polytopes by balls): $P$ is a polytope in $\mathbb{R}^n$ with $N$ vertices, $\mathrm{diam}(P) \leq 1$. Then $P$ can be covered by at most $N^{\lceil 1/\epsilon^2 \rceil}$ Euclidean balls of radii $\epsilon > 0$.

# 1 Preliminaries on random variables

- Jensen's inequality: convex $\phi$, $\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X)$. $\Rightarrow \|X\|_{L^p} \leq \|X\|_{L^q}$ for $p \leq q$.

- Minkowski inequality: $p \geq 1, \|X+Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p}$.

- Cauchy-Schwarz inequality: $\mathbb{E}|XY| \leq \|X\|_{L^2}\|Y\|_{L^2}$.

- Holder inequality: $p, q \in (1, +\infty), \frac{1}{p} + \frac{1}{q} = 1$ or $p = 1, q = \infty$, $\mathbb{E}\|XY\| \leq \|X\|_{L^p}\|Y\|_{L^q}$.

- $X \geq 0$, then $\mathbb{E}X = \int_0^{+\infty} \mathbb{P}(X > t)\mathrm{d}t$.

- Markov inequality: $X \geq 0, t > 0, \mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}$.

- LLN: $X_1, \cdots, X_n, \cdots$ i.i.d., $\mathbb{E}X_i = \mu, \mathrm{Var}(X_i) = \sigma^2, S_N = X_1 + X_2 + \cdots + X_N$. Then: (WLLN) $\mathbb{P}(|\frac{S_N}{N} - \mu| > \epsilon) \to 0, \forall \epsilon > 0$; (SLLN) $\mathbb{P}(\frac{S_N}{N} \to \mu, N \to +\infty) = 1$.

- CLT: $Z_N = \frac{S_N - \mathbb{E}S_N}{\sqrt{\mathrm{Var}(S_N)}} \xrightarrow{d} \mathcal{N}(0,1)$.

- $X_{N,i}, 1 \leq i \leq N$ independent $\mathrm{Ber}(p_{N,i}), \max_{i \leq N} p_{N,i} \to 0, S_N = \sum_{i=1}^N X_{N,i}, \mathbb{E}S_N \to \lambda < +\infty$. Then $S_N \xrightarrow{d}$ Poisson$(\lambda)$.

# 2 Concentration of sums of independent random variables

- Question: $N$ times, $\mathbb{P}(\text{head} \geq \frac{3}{4}N) = ?$ Let $S_N$ be the number of heads, $\mathbb{E}S_N = \frac{N}{2}, \mathrm{Var}(S_N) = \frac{N}{4}$. (1) Chebyshev's inequality: $\mathbb{P}(S_N \geq \frac{3}{4}N) \leq \mathbb{P}(|S_N - \frac{N}{2}| \geq \frac{N}{4}) \leq \frac{4}{N}$; (2) $Z_N = \frac{S_N - \frac{N}{2}}{\sqrt{N/4}}$, expect: $\mathbb{P}(Z_N \geq \sqrt{N/4}) \approx \mathbb{P}(g \geq \sqrt{N/4}) \leq \frac{1}{\sqrt{2\pi}}e^{-N/8}$ where $g \sim \mathcal{N}(0,1)$.

- For all $t > 0$, $(\frac{1}{t} - \frac{1}{t^3})\frac{1}{\sqrt{2\pi}}e^{-t^2/2} \leq \mathbb{P}(g \sim \mathcal{N}(0,1) \geq t) \leq \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

- Berry-Esseen bound: $|\mathbb{P}(Z_N \geq t) - \mathbb{P}(g \geq t)| \leq \frac{\rho}{\sqrt{N}}$ where $\rho = \mathbb{E}|X_1 - \mu|^3/\sigma^3$. And in general, no improvement since $\mathbb{P}(S_N = \frac{N}{2}) \sim \frac{1}{\sqrt{N}} \Rightarrow \mathbb{P}(Z_N = 0) \sim \frac{1}{\sqrt{N}}$ but $\mathbb{P}(g = 0) = 0$.

- Hoeffding's inequality: $X_1, \cdots, X_N$ i.i.d. symmetric Bernoulli ($\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$), $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$. Then $\forall t \geq 0, \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) \leq e^{-t^2/2\|a\|_2^2}, \mathbb{P}(|\sum_{i=1}^N a_i X_i| \geq t) \leq 2e^{-t^2/2\|a\|_2^2}$.

  *Proof* WLOG, $\|a\|_2^2 = 1$. For $\lambda > 0, \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) = \mathbb{P}(e^{\lambda \sum a_i X_i} \geq e^{\lambda t}) \leq e^{-\lambda t}\mathbb{E}e^{\lambda \sum_{i=1}^N a_i X_i} = e^{-\lambda t}\prod_{i=1}^N \mathbb{E}e^{\lambda a_i X_i} = e^{-\lambda t}\prod_{i=1}^N \cosh(\lambda a_i) \leq e^{-\lambda t}\prod_{i=1}^N e^{\lambda^2 a_i^2/2} = e^{-\lambda t + \frac{\lambda^2}{2}} \Rightarrow \mathbb{P}(\sum_{i=1}^N a_i X_i \geq t) \leq \inf_{\lambda \geq 0} e^{-\lambda t + \frac{\lambda^2}{2}} = e^{-\frac{t^2}{2}} (\lambda = t)$. $\square$

- Bounded r.v.s: $X_1, \cdots, X_N$ independent, $X_i \in [m_i, M_i]$. Then $\forall t \geq 0, \mathbb{P}(\sum_{i=1}^N (X_i - \mathbb{E}X_i) \geq t) \leq e^{-\frac{2t^2}{\sum_{i=1}^N (M_i - m_i)^2}}$.

- Chernoff's inequality: $X_i \sim \text{Ber}(p_i)$ independent, $S_N = \sum_{i=1}^N X_i, \mu = \mathbb{E}S_N \Rightarrow \forall t > \mu, \mathbb{P}(S_N \geq t) \leq e^{-\mu}(\frac{e\mu}{t})^t$.

  *Proof* $\mathbb{P}(S_N \geq t) \leq e^{-\lambda t} \prod_{i=1}^N \mathbb{E}e^{\lambda X_i}$. $\mathbb{E}e^{\lambda X_i} = e^\lambda p_i + (1 - p_i) = 1 + (e^\lambda - 1)p_i \leq e^{(e^\lambda - 1)p_i} \Rightarrow \mathbb{P}(S_N \geq t) \leq e^{-\lambda t} e^{(e^\lambda - 1)\mu}$. Take $\lambda^* = \log(t/\mu)$. □

- $d = (n-1)p$ is the expected degree. There is an absolute constant $C$ s.t. for $G(n,p)$, $d \geq C \log n$. Then with high prob (for example 0.9), all vertices of $G$ have degrees between $0.9d$ and $1.1d$.

  *Proof* Ex 2.3.5 $\Rightarrow \mathbb{P}(|d_i - d| \geq \delta d) \leq 2e^{-c\delta^2 d}$. Union bound: $\mathbb{P}(\exists i, |d_i - d| \geq \delta d) \leq n \cdot 2e^{-c\delta^2 d} \leq n \cdot 2 \cdots n^{-Cc\delta^2} = 2n^{1-Cc\delta^2} \leq 1 - p^*$ (let $Cc\delta^2 > 1$). □

- Sub-gaussian properties: The following are equivalent: (i) $\mathbb{P}(|X| \geq t) \leq 2e^{-t^2/k_1^2}$ for all $t \geq 0$; (ii) $\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p} \leq k_2\sqrt{p}$ for all $p \geq 1$; (iii) $\mathbb{E}e^{\lambda^2 X^2} \leq e^{k_3^2 \lambda^2}$ for all $\lambda$ s.t. $|\lambda| \leq \frac{1}{k_3}$; (iv) $\mathbb{E}e^{X^2/k_4^2} \leq 2$; (v) $\mathbb{E}e^{\lambda X} \leq e^{k_5^2 \lambda^2}$, for all $\lambda \in \mathbb{R}$ (if $\mathbb{E}X = 0$).

  *Proof* (i) $\Rightarrow$ (ii): WLOG $k_1 = 1$. $\mathbb{E}|X|^p = \int_0^{+\infty} \mathbb{P}(|X| \geq t) pt^{p-1} \mathrm{d}t \leq \int_0^{+\infty} 2e^{-t^2} pt^{p-1} \mathrm{d}t = p\Gamma(\frac{p}{2}) \overset{\Gamma(x) \leq 3x^x \text{ for } x \geq \frac{1}{2}}{\leq} 3p(\frac{p}{2})^{p/2} \Rightarrow \|X\|_p \leq \frac{1}{\sqrt{2}}(3p)^{1/p} p^{1/2} \leq 3\sqrt{p}$.

  (ii) $\Rightarrow$ (iii): WLOG $k_2 = 1$. $\mathbb{E}e^{\lambda^2 X^2} = \mathbb{E}[1 + \sum_{p=1}^{+\infty} \frac{(\lambda^2 X^2)^p}{p!}]$. $\mathbb{E}|X|^{2p} \leq (2p)^p, p! \geq (\frac{p}{e})^p \Rightarrow \mathbb{E}e^{\lambda^2 X^2} \leq 1 + \sum_{p=1}^{+\infty} \frac{(2\lambda^2 p)^p}{(\frac{p}{e})^p} = \frac{1}{1 - 2e\lambda^2}$ (if $2e\lambda^2 < 1$) $\overset{\frac{1}{1-x} \leq e^{2x} \text{ for } x \in [0, \frac{1}{2}]}{\leq} e^{4e\lambda^2}$ (if $2e\lambda^2 \leq 1/2 \Leftrightarrow \lambda \leq \frac{1}{2\sqrt{e}}$).

  (iii) $\Rightarrow$ (iv): trivial.

  (iv) $\Rightarrow$ (i): $\mathbb{P}(|X| \geq t) = \mathbb{P}(e^{X^2} \leq e^{t^2}) \leq e^{-t^2} \mathbb{E}e^{X^2} \leq 2e^{-t^2}$.

  (iii) $\Rightarrow$ (v): WLOG $k_3 = 1$. If $|\lambda| \leq 1$, then $\mathbb{E}e^{\lambda X} \leq \mathbb{E}(\lambda X + e^{\lambda^2 X^2}) = \mathbb{E}e^{\lambda^2 X^2} \leq e^{\lambda^2}$. If $|\lambda| \geq 1$, then $\mathbb{E}e^{\lambda X} \leq e^{\frac{\lambda^2}{2}} \mathbb{E}e^{\frac{X^2}{2}} e^{\frac{\lambda^2}{2}} e^{\frac{1}{2}} \leq e^{\lambda^2}$.

  (v) $\Rightarrow$ (i): mimic the proof of (iv) $\Rightarrow$ (i). □

- Sub-gaussian r.v.: satisfy the above sub-gaussian properties. $\|X\|_{\psi_2} = \inf\{t > 0 : \mathbb{E}e^{X^2/t^2} \leq 2\}$. Thus $\mathbb{P}(|X| \geq t) \leq 2e^{-ct^2/\|X\|_{\psi_2}^2}$; $\|X\|_{L^p} \leq C\|X\|_{\psi_2}\sqrt{p}$; if $\mathbb{E}X = 0$ then $\mathbb{E}e^{\lambda X} \leq e^{C\lambda^2 \|X\|_{\psi_2}^2}$.

- Maximum of sub-gaussians: $K = \max_{i \leq N} \|X_i\|_{\psi_2}$. Then $\mathbb{E}\max_{i \leq N} X_i \leq CK\sqrt{\log N}$.

- Let $X_1, \cdots, X_N$ be independent and mean zero sub-gaussian, then $\|\sum_{i=1}^N X_i\|_{\psi_2}^2 \leq C \cdot \sum_{i=1}^N \|X_i\|_{\psi_2}^2$.

  *Proof* $\forall \lambda \in \mathbb{R}, \mathbb{E}e^{\lambda \sum_{i=1}^N X_i} = \prod_{i=1}^N \mathbb{E}e^{\lambda X_i} \leq \prod_{i=1}^N e^{c\lambda^2 \|X_i\|_{\psi_2}^2} = e^{c\lambda^2 \sum_{i=1}^n \|X_i\|_{\psi_2}^2}$ □

- Centering: $X$ is sub-gaussian $\Rightarrow X - \mathbb{E}X$ is sub-gaussian and $\|X - \mathbb{E}X\|_{\psi_2} \leq C\|X\|_{\psi_2}$.

  *Proof* $\|\mathbb{E}X\|_{\psi_2} \leq C_1|\mathbb{E}X| \leq C_1\mathbb{E}|X| = C_1\|X\|_{L^1} \leq C_1C_2\|X\|_{\psi_2}$. □

- Sub-exponential properties: The following are equivalent: (1) $\mathbb{P}(|X| \geq t) \leq 2e^{-t/k_1}, \forall t \geq 0$; (2) $\|X\|_{L^p} \leq k_2 p, p \geq 1$; (3) $\mathbb{E}e^{\lambda|X|} \leq e^{k_3\lambda}$ for all $0 \leq \lambda \leq \frac{1}{k_3}$; (4) $\mathbb{E}e^{|X|/k_4} \leq 2$; (5) if $\mathbb{E}X = 0, \mathbb{E}e^{\lambda X} \leq e^{k_5^2\lambda^2}$ for $|\lambda| \leq \frac{1}{k_5}$.

  *Proof* (2) $\Rightarrow$ (5): $k_2 = 1, \mathbb{E}e^{\lambda X} = 1 + \sum_{p=2}^{+\infty} \frac{\lambda^p \mathbb{E}X^p}{p!} \leq 1 + \sum_{p=2}^{+\infty} \frac{\lambda^p p^p}{(p/e)^p} = 1 + \frac{(e\lambda)^2}{1 - e\lambda}(|e\lambda| < 1)$. If $|e\lambda| \leq \frac{1}{2}, 1 + \frac{(e\lambda)^2}{1 - e\lambda} \leq 1 + 2e^2\lambda^2 \leq e^{2e^2\lambda^2} \leq e^{4e^2\lambda^2}$, i.e. $k_5 = 2e$.

  (5) $\Rightarrow$ (1): $k_5 = 1, |x|^p \leq p^p(e^x + e^{-x}) \Rightarrow \mathbb{E}|X|^p \leq p^p(\mathbb{E}e^X + \mathbb{E}e^{-X}) \leq 2ep^p$. □

- $\|X\|_{\psi_1} = \inf\{t > 0 : \mathbb{E}e^{|X|/t} \leq 2\}$. $X$ is sub-gaussian $\Leftrightarrow X^2$ is sub-exponential. $\|X^2\|_{\psi_1} = \|X\|_{\psi_2}^2$.

- $X, Y$ are sub-gaussian $\Rightarrow XY$ is sub-exponential and $\|XY\|_{\psi_1} \leq \|X\|_{\psi_2}\|Y\|_{\psi_2}$.

  *Proof* WLOG $\|X\|_{\psi_2} = \|Y\|_{\psi_2} = 1$. $\mathbb{E}e^{XY} \leq \mathbb{E}e^{\frac{X^2+Y^2}{2}} = \mathbb{E}[e^{\frac{X^2}{2}+\frac{Y^2}{2}}] \leq \frac{1}{2}(\mathbb{E}e^{X^2} + \mathbb{E}e^{Y^2}) = 2$. □

- Orlicz function/space: $\psi : [0, +\infty) \to [0, +\infty)$, convex, increasing, $\psi(0) = 0$, $\psi(x) \to +\infty, x \to +\infty$. $\|X\|_\psi := \inf\{t > 0 : \mathbb{E}\psi(|X|/t) \leq 1\}$. $L_\psi := \{X : \|X\|_\psi < +\infty\}$ is Banach space. Examples: (1) $L_p : \psi(x) = x^p, p \geq 1$; (2) $L_{\psi_2} : \psi_2(x) = e^{x^2} - 1, L_\infty \subset L_{\psi_2} \subset L_p$.

- Bernstein's inequality: $X_1, \cdots, X_N$ independent, mean zero and sub-exponential. Then for $t \geq 0, \mathbb{P}(|\sum_{i=1}^N X_i| \geq t) \leq 2e^{-c\min(\frac{t^2}{\sum_{i=1}^N \|X_i\|_{\psi_1}^2}, \frac{t}{\max_i \|X_i\|_{\psi_1}})}$.

  *Proof* $S = \sum_{i=1}^N X_i$. $\mathbb{P}(S \geq t) \leq e^{-\lambda t} \prod_{i=1}^N \mathbb{E}e^{\lambda X_i}$. $\mathbb{E}e^{\lambda X_i} \leq e^{c\lambda^2 \|X_i\|_{\psi_1}^2}$ if $|\lambda| \leq \frac{c}{\max \|X_i\|_{\psi_1}}$. Then $\mathbb{P}(S \geq t) \leq e^{-\lambda t + c\lambda^2\sigma^2}$ where $\sigma^2 := \sum_{i=1}^N \|X_i\|_{\psi_1}^2$. The following is to find the minimum of a quadratic function with the restriction $|\lambda| \leq \frac{c}{\max \|X_i\|_{\psi_1}}$. □

- Corollary 1: $\mathbb{P}(|\sum_{i=1}^N a_i X_i| \geq t) \leq 2e^{-c\min(\frac{t^2}{K^2\|a\|_2^2}, \frac{t}{K\|a\|_\infty})}$ where $K = \max_i \|X_i\|_{\psi_1}$.

- Corollary 2: $|X_i| \leq K$, then $\mathbb{P}(|\sum_{i=1}^N X_i| \geq t) \leq 2\exp(-\frac{t^2/2}{\sigma^2 + Kt/3})$ where $\sigma^2 = \sum_{i=1}^N \mathbb{E}X_i^2$.

# 3 Random vectors in high dimensions

- $X \in \mathbb{R}^n$, independent sub-gaussian coordinate $X_i$, $\mathbb{E}X_i^2 = 1$. Then $\|\|X\|_2 - \sqrt{n}\|_{\psi_2} \leq CK^2, K = \max_i \|X_i\|_{\psi_2}$.

  *Proof* $\mathbb{E}X_i^2 = 1 \Rightarrow K \geq 1$. $\|X_i^2 - 1\|_{\psi_1} \leq C\|X_i^2\|_{\psi_1} = C\|X_i\|_{\psi_2}^2 \leq CK^2$. Bernstein's inequality: $\mathbb{P}(|\frac{1}{n}\|X\|_2^2 - 1| \geq u) = \mathbb{P}(|\frac{1}{n}\sum_{i=1}^n (X_i^2 - 1)| \geq u) \leq 2e^{-cn\min(\frac{u^2}{K^4}, \frac{u}{K^2})} \leq 2e^{-\frac{cn}{K^4}\min(u^2, u)}$. For any $\delta > 0$, $\mathbb{P}(|\frac{1}{\sqrt{n}}\|X\|_2 - 1| \geq \delta) \leq \mathbb{P}(|\frac{1}{n}\|X\|_2^2 - 1| \geq \max(\delta, \delta^2)) \leq 2e^{-\frac{cn}{K^4}\delta^2} \Rightarrow \mathbb{P}(|\,\|X\|_2 - \sqrt{n}| \geq t) \leq 2e^{-ct^2/K^4}$. □

- Isotropy: $\Sigma(X) = \mathbb{E}XX^T = I$. If $\Sigma \neq I_n$, then let $Z = \Sigma^{-1/2}X$. $X$ is isotropic $\Leftrightarrow \mathbb{E}\langle X, x\rangle^2 = \|x\|_2^2$ for any $x \in \mathbb{R}^n$.

  *Proof* $\mathbb{E}\langle X, x\rangle^2 = \mathbb{E}(x^TXX^Tx) = x^T(\mathbb{E}XX^T)x$. $\|x\|_2^2 = x^TI_nx$. $\Rightarrow \mathbb{E}XX^T = I_n$. □

- $X$ is isotropic $\Rightarrow \mathbb{E}\|X\|_2^2 = n$. If $X, Y$ are independent and isotropic $\Rightarrow \mathbb{E}\langle X, Y\rangle^2 = n$.

  *Proof* $\mathbb{E}\|X\|_2^2 = \mathbb{E}(X^TX) = \mathbb{E}(\mathrm{tr}(X^TX)) = \mathrm{tr}(\mathbb{E}XX^T) = n$.
  $\mathbb{E}\langle X, Y\rangle^2 = \mathbb{E}(X^TYY^TX) = \mathbb{E}(\mathrm{tr}(X^TYY^TX)) = \mathbb{E}(\mathrm{tr}(XX^TYY^T)) = \mathrm{tr}((\mathbb{E}XX^T)(\mathbb{E}YY^T)) = n$. □

- Examples: $X \sim U(\sqrt{n}\mathbb{S}^{n-1}), X \sim U(\{-1, 1\}^n), X = (X_1, \cdots, X_n)$ i.i.d., $\mathbb{E}X_i = 0, \mathrm{Var}(X_i) = 1$ are all isotropic.

- $g \sim \mathcal{N}(0, I_n)$, then $\mathbb{P}(|\,\|g\|_2 - \sqrt{n}| \geq t) \leq 2e^{-ct^2}$.

- Frame: $\{u_i\}_{i=1}^N \subset \mathbb{R}^n$, Approximate Parseval's identity: $A\|x\|_2^2 \leq \sum_{i=1}^N \langle u_i, x\rangle^2 \leq B\|x\|_2^2$. $A, B$: frame bounds. $A = B$: tight frame($\Leftrightarrow \sum_{i=1}^N u_iu_i^T = AI_n$) and in this case, $\sum_{i=1}^N \langle u_i, x\rangle u_i = Ax$.

- (a) Tight frame $\{u_i\}_{i=1}^N, A = B, X = \mathrm{Unif}\{u_i, i = 1, 2, \cdots, N\}$, then $(\frac{N}{A})^{1/2}X$ is isotropic. (b) $X$ is isotropic, $\mathbb{P}(X = x_i) = p_i, i = 1, 2, \cdots, N$. Then $u_i = \sqrt{p_i}x_i$ form a tight frame with $A = B = 1$.

- Isotropic convex sets: $X \sim \mathrm{Unif}(K), K \subset \mathbb{R}^n$ convex, bounded, non-empty interior (convex body). Assume $\mathbb{E}X = 0, \Sigma = \mathrm{Cov}(X)$. Then $Z = \Sigma^{-1/2}X$ is isotropic and $Z \sim \mathrm{Unif}(\Sigma^{-1/2}K)$.

- $X \in \mathbb{R}^n$ is sub-gaussian $\Leftrightarrow \forall X \in \mathbb{R}^n, \langle X, x\rangle$ are sub-gaussian. $\|X\|_{\psi_2} = \sup_{x \in \mathbb{S}^{n-1}} \|\langle X, x\rangle\|_{\psi_2}$.

- $X = (X_1, \cdots, X_n)$ independent, mean zero, sub-gaussian coordinate. Then $X$ is sub-gaussian with $\|X\|_{\psi_2} \leq C\max_{i \leq n} \|X_i\|_{\psi_2}$.

  *Proof* $\|\langle X, x\rangle\|_{\psi_2}^2 = \|\sum X_ix_i\|_{\psi_2}^2 \leq C\sum_{i=1}^n x_i^2\|X_i\|_{\psi_2}^2 \leq C\max_{i \leq n}\|X_i\|_{\psi_2}^2$. □

- Gaussian dist: $X \sim \mathcal{N}(0, I_n), \|X\|_{\psi_2} \leq C$.

- Discrete dist: $X \sim \mathrm{Unif}\{\sqrt{n}e_i, i = 1, 2, \cdots, n\}, \|X\|_{\psi_2} \simeq \sqrt{\frac{n}{\log n}}$.

- Uniform dist: $X \sim \mathrm{Unif}\{\sqrt{n}\mathbb{S}^{n-1}\}, \|X\|_{\psi_2} \leq C$.

  *Proof* $g \sim \mathcal{N}(0, I_n), X \overset{d}{=} \frac{\sqrt{n}g}{\|g\|_2}$. $p(t) := \mathbb{P}(|X_1| \geq t) = \mathbb{P}(\frac{|g|}{\|g\|_2} \geq \frac{t}{\sqrt{n}})$. $\|\|g\|_2 - \sqrt{n}\|_{\psi_2} \leq C \Rightarrow \mathbb{P}(\|g\|_2 \leq \frac{\sqrt{n}}{2}) \leq 2e^{-cn}$. Need to show that all one-dimensional marginals $\langle X, x\rangle$ are sub-gaussian. By rotation invariance, we may assume that $x = (1, 0, \cdots, 0)$. Let $\mathcal{E} = \{\|g\|_2 \geq \frac{\sqrt{n}}{2}\} \Rightarrow p(t) \leq \mathbb{P}(\frac{|g|}{\|g\|_2} \geq \frac{t}{\sqrt{n}}, \mathcal{E}) + \mathbb{P}(\mathcal{E}) \leq \mathbb{P}(|g_1| \geq \frac{t}{2}, \mathcal{E}) + 2e^{-cn} \leq 2e^{-t^2/8} + 2e^{-cn} \overset{t \leq \sqrt{n}}{\leq} 4e^{-ct^2}$. □

- Grothendieck's inequality: $A = \{a_{ij}\}_{m \times n}$ of real numbers. Assume $\forall x_i, y_i \in \{-1, 1\}$, we have $|\sum_{i,j} a_{ij}x_iy_j| \leq 1$. Then for any Hilbert space $\mathscr{H}$, any $u_i, v_j \in \mathscr{H}$ satisfying $\|u_i\| = \|v_j\| = 1$, we have $|\sum_{i,j} a_{ij}\langle u_i, v_j\rangle| \leq K$ with $K \leq 1.783$.

*Proof* (1) Reduction. For any $u_i, v_j \in \mathbb{R}^N$ s.t. $\|u_i\|_2 = \|v_j\|_2 = 1, K_{u,v} := \sum_{i,j} a_{ij}\langle u_i, v_j\rangle, K = \sup_{\|u\|_2=\|v\|_2=1} K_{u,v}$.

(2) Introduce randomness. $g \sim \mathcal{N}(0, I_N), U_i = \langle g, u_i\rangle, V_j = \langle g, v_j\rangle, \mathbb{E}U_iV_j = \langle u_i, v_j\rangle$. $K_{u,v} = \mathbb{E}(\sum_{i,j} a_{ij}U_iV_j) \Rightarrow K_{u,v} \le R^2$ if $|U_i| \le R, |V_j| \le R$.

(3) Truncation. Given $R \ge 1, U_i = U_i^- + U_i^+, U_i^- = U_i 1_{\{|U_i|\le R\}}, V_j = V_j^- + V_j^+, |U_i^-| \le R, |V_j^-| \le R$. $\|U_i^+\|_{L^2}^2 \le 2(R + \frac{1}{R})\frac{1}{\sqrt{2\pi}}e^{-R^2/2} < \frac{4}{R^2}(R > 1)$.

(4) Breaking up the sum. $\mathbb{E}\sum a_{ij}U_iV_j = \mathbb{E}\sum a_{ij}U_i^-V_j^- + \mathbb{E}\sum a_{ij}U_i^+V_j^- + \mathbb{E}\sum a_{ij}U_i^-V_j^+ + \mathbb{E}\sum a_{ij}U_i^+V_j^+ := S_1 + S_2 + S_3 + S_4$. $S_1 \le R^2, S_2 \le K \max\|U_i^+\|_2 \max\|V_j^-\|_2 \le K\frac{2}{R}, S_3 \le K\frac{2}{R}, S_4 \le K\frac{4}{R^2}$.

(5) Putting everything together. $K_{u,v} \le R^2 + \frac{4K}{R} + \frac{4K}{R^2} \Rightarrow K \le R^2 + \frac{4K}{R} + \frac{4K}{R^2} \Rightarrow K \le \frac{R^2}{1-\frac{4}{R}-\frac{4}{R^2}}$. □

- Remark: The assumption can be equivalently stated as $|\sum_{i,j} a_{ij}x_iy_j| \le \max_i |x_i| \max_j |y_j|$. The conclusion can be equivalently stated as $|\sum_{i,j} a_{ij}\langle u_i, v_j\rangle| \le K \max_i \|u_i\| \max_j \|v_j\|$.

- Semidefinite programming: $\max\langle A, X\rangle$ s.t. $X \succeq 0, \langle B_i, X\rangle = b_i, A, B_i\ n\times n, b_i$ real number, $\langle A, X\rangle = \mathrm{tr}(A^TX) = \sum_{i,j=1}^n A_{ij}X_{ij}$.

- Semidefinite relaxation: $\max\sum_{i,j=1}^n A_{ij}x_ix_j, x_i = \pm1, A\ n\times n$ symmetric matrix. Relax to $\max\sum_{i,j=1}^n A_{ij}\langle X_i, X_j\rangle$, $\|X_i\|_2 = 1, X_i \in \mathbb{R}^n$.

- $A$ positive semidefinite, $\mathrm{INT}(A) := \max\sum_{i,j=1}^n A_{ij}x_ix_j, x_i = \pm1, \mathrm{SDP}(A) := \max\sum_{i,j=1}^n A_{ij}\langle X_i, X_j\rangle, \|X_i\|_2 = 1$. Then $\mathrm{INT}(A) \le \mathrm{SDP}(A) \le 2K\cdot\mathrm{INT}(A)$.

- Maximum cut: $G = (V, E)$ finite simple, $V \to V_1 + V_2$, cut number of edges crossing between $V_1$ and $V_2$. MAX-CUT($G$): NP-hard. Adjacency matrix $A = \{A_{ij}\}_{n\times n}, A_{ij} = \begin{cases} 1, & i \leftrightarrow j \\ 0, \text{otherwise} \end{cases}$. Partition: $X = (x_i)_{n\times1}, x_i = \pm1$. $\mathrm{CUT}(G, X) = \frac{1}{4}\sum_{i,j=1}^n A_{ij}(1 - x_ix_j)$. $\text{MAX-CUT}(G) = \frac{1}{4}\max\{\sum_{i,j} A_{ij}(1 - x_ix_j) : x_i = \pm1\}$.

- 0.5-approximation algorithm: Partition at random, $\mathbb{E}\mathrm{CUT}(G, X) = 0.5|E| \ge 0.5\text{MAX-CUT}(G)$.

- 0.878-approximation algorithm: $\mathrm{SDP}(G) = \frac{1}{4}\max\{\sum_{i,j=1}^n A_{ij}(1 - \langle X_i, X_j\rangle), x_i \in \mathbb{R}^n, \|X_i\|_2 = 1\}$. $X_1, \cdots, X_n \to x_1, \cdots, x_n : g \sim \mathcal{N}(0, I_n), x_i := \mathrm{sgn}(\langle X_i, g\rangle)$. $\mathbb{E}\mathrm{CUT}(G, X) \ge 0.878\mathrm{SDP}(G) \ge 0.878\text{MAX-CUT}(G)$.

  *Proof* $\mathbb{E}\mathrm{CUT}(G, X) = \frac{1}{4}\sum_{i,j=1}^n A_{ij}(1 - \mathbb{E}x_ix_j)$ and $1 - \mathbb{E}x_ix_j = 1 - \mathbb{E}\mathrm{sgn}\langle g, X_i\rangle\mathrm{sgn}\langle g, X_j\rangle = 1 - \frac{2}{\pi}\arcsin\langle X_i, X_j\rangle \ge 0.878(1 - \langle X_i, X_j\rangle)$. □

- $u, v \in \mathbb{S}^{n-1}, \mathbb{E}\mathrm{sgn}(\langle g, u\rangle)\mathrm{sgn}(\langle g, v\rangle) = \frac{2}{\pi}\arcsin\langle u, v\rangle$.

- There exists a Hilbert space $\mathcal{H}$ and $\phi, \psi : \mathbb{S}^{n-1} \to \mathbb{S}(\mathcal{H})$ s.t. $\frac{2}{\pi}\arcsin\langle\phi(u), \psi(v)\rangle = \beta\langle u, v\rangle$ for all $u, v \in \mathbb{S}^{n-1}$ and $\beta = \frac{2}{\pi}\log(1 + \sqrt{2})$.

  *Proof* $\langle\phi(u), \psi(v)\rangle = \sin(\frac{\beta\pi}{2}\langle u, v\rangle)$. Ex 3.7.6 $\Rightarrow \exists\mathcal{H}, \phi, \psi$. $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots, \sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!}, \|\phi(u)\|_2^2 = \|\psi(u)\|_2^2 = \sinh(\frac{\beta\pi}{2}) = 1$ for all $u \in \mathbb{S}^{n-1} \Rightarrow \beta = \frac{2}{\pi}\log(1 + \sqrt{2})$.

  *Proof of Grothendieck's inequality with* $K \le \frac{1}{\beta} \approx 1.783$ WLOG $u_i, v_j \in \mathbb{S}^{N-1}$, then $\frac{2}{\pi}\arcsin\langle u_i', v_j'\rangle = \beta\langle u_i, v_j\rangle, \mathcal{H} = \mathbb{R}^M, g \sim \mathcal{N}(0, I_M)$. $\beta\sum a_{ij}\langle u_i, v_j\rangle = \sum_{i,j} a_{ij}\frac{2}{\pi}\arcsin\langle u_i', v_j'\rangle = \sum_{i,j} a_{ij}\mathbb{E}\mathrm{sgn}\langle g, u_i'\rangle\mathrm{sgn}\langle g, v_j'\rangle \le 1$. □

# 4 Random matrices

- Singular vector decomposition: $A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V_{n\times n}^T = \sum_{i=1}^n s_iU_iV_i^T, \Sigma = \mathrm{diag}(s_1, s_2, \cdots, s_r), s_i \ge 0$ singular values. $s_i = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^TA)}$. If $A$ is symmetric, $s_i = |\lambda_i(A)|$.

- Courant-Fisher's min-max theorem: $\lambda_i(A) = \max_{\dim E=i}\min_{x\in\mathbb{S}(E)}\langle Ax, x\rangle, s_i(A) = \max_{\dim E=i}\min_{x\in\mathbb{S}(E)}\|Ax\|_2$.

- Operator norm/spectral norm: $\|A\| = \max_{x\in\mathbb{R}^n\setminus\{0\}}\frac{\|Ax\|_2}{\|x\|_2} = \max_{x\in\mathbb{S}^{n-1}}\|Ax\|_2 = s_1(A)$. Or equivalently, $\|A\| = \max_{x\in\mathbb{S}^{n-1}, y\in\mathbb{S}^{m-1}}\langle Ax, y\rangle$.

- $s_n(A) > 0 \Leftrightarrow m \ge n = \mathrm{rank}(A), s_n(A) = \frac{1}{\|A^+\|}$ where $A^+$ is pseudo-inverse (the norm of $A^{-1}$ restriction to the image of $A$).

- Frobenius norm: $\|A\|_F = (\sum_{i=1}^n \sum_{j=1}^n A_{ij}^2)^{\frac{1}{2}} = (\sum_{i=1}^n s_i^2(A))^{\frac{1}{2}}$.
- Low-rank approximation: $\mathrm{rank}(A) = r, k < r, A_k := \sum_{i=1}^k s_i u_i v_i^T$, $\|A - A_k\| = \min_{\mathrm{rank}(A') \le k} \|A - A'\|$ (holds for $\|\cdot\|, \|\cdot\|_F$).
- Approximate isometries: $m\|x\|_2 \le \|Ax\|_2 \le n\|x\|_2$ where $m = s_n(A), n = s_1(A)$, or $s_n\|x-y\|_2 \le \|Ax - Ay\|_2 \le s_1\|x-y\|_2$.
- $A_{m\times n}, \delta > 0$. If $\|A^T A - I_n\| \le \max(\delta, \delta^2)$, then $(1-\delta)\|x\|_2 \le \|Ax\|_2 \le (1+\delta)\|x\|_2$ for all $x$.

  *Proof* WLOG $\|x\|_2 = 1$. $|\|Ax\|_2^2 - 1| = |\langle (A^T A - I_n)x, x\rangle| \le \max(\delta, \delta^2) \Rightarrow \max(|\|Ax\|_2 - 1|, (\|Ax\|_2 - 1)^2) \le \max(\delta, \delta^2) \Rightarrow |\|Ax\|_2 - 1| \le \delta$. □
- $Q_{n\times m}, QQ^T = I_n \Leftrightarrow P = Q^T Q$ is an orthogonal proj in $\mathbb{R}^m$ onto a subspace with dim $n$.
- $\epsilon$-net: $(T, d)$ a metric space, $K \subset T, \epsilon > 0$. $\mathcal{N} \subset K$ is an $\epsilon$-net of $K$ if $\forall x \in K, \exists x_0 \in \mathcal{N}$ s.t. $d(x, x_0) \le \epsilon$. Covering number: smallest $|\mathcal{N}| = |\mathcal{N}(K, d, \epsilon)|$.
- Compactness: $\mathcal{N}(K, d, \epsilon) < +\infty$ for all $\epsilon > 0$.
- $\epsilon$-separated: $\mathcal{P} \subset T$ is $\epsilon$-separated if $d(x, y) > \epsilon$ for all $x, y \in \mathcal{P}$. Packing number: largest $|\mathcal{P}| = |\mathcal{P}(K, d, \epsilon)|$.
- $\mathcal{P}$ is a maximal $\epsilon$-separated subset $\Rightarrow \mathcal{P}$ is a $\epsilon$-net of $K$.
- $\mathcal{P}(K, d, 2\epsilon) \le \mathcal{N}(K, d, \epsilon) \le \mathcal{P}(K, d, \epsilon)$.

  *Proof* The upper bound follows from the previous lemma. For the lower bound, choose an $2\epsilon$-separated subset $\mathcal{P} = \{x_i\}$ in $K$ and an $\epsilon$-net $\mathcal{N} = \{y_j\}$ of $K$. $\forall x_i, \exists y_j \in \mathcal{N}$, s.t. $|x_i - y_j| < \epsilon$. $\forall y_j$, there exists at most a $x_j \in \mathcal{P}$ s.t. $|x_i - y_j| < \epsilon$. □
- Minkowski sum: $A, B \in \mathbb{R}^n, A + B := \{a + b, a \in A, b \in B\}$.
- $K \subset \mathbb{R}^n, \epsilon > 0, \frac{|K|}{|\epsilon B_2^n|} \le \mathcal{N}(K, \epsilon) \le \mathcal{P}(K, \epsilon) \le \frac{|K + \frac{\epsilon}{2} B_2^n|}{|\frac{\epsilon}{2} B_2^n|}$ where $|\cdot|$ denotes the volume in $\mathbb{R}^n$, $B_2^n$ denotes the unit Euclidean ball in $\mathbb{R}^n$.
- Corollary: Let $K = B_2^n$. $|\epsilon B_2^n| = \epsilon^n |K|, |K + \frac{\epsilon}{2} B_2^n| = (1 + \frac{\epsilon}{2})^n |K| \Rightarrow (\frac{1}{\epsilon})^n \le \mathcal{N}(B_2^n, \epsilon) \le (1 + \frac{2}{\epsilon})^n$. $\epsilon \in (0, 1] \Rightarrow (\frac{1}{\epsilon})^n \le \mathcal{N}(B_2^n, \epsilon) \le (\frac{3}{\epsilon})^n$.
- Hamming cubde: $x, y \in \{0, 1\}^n, d_H(x, y) := \#\{i : x(i) \ne y(i)\}$.
- $(T, d)$ a metric space, $K \subset T$, $\mathcal{C}(K, d, \epsilon)$ the smallest number of bits sufficient specify every points $x \in K$ with accuracy $\epsilon$ in the metric $d$. Then $\log_2 \mathcal{N}(K, d, \epsilon) \le \mathcal{C}(K, d, \epsilon) \le \log_2 \mathcal{N}(K, d, \frac{\epsilon}{2})$. $\log_2 \mathcal{N}(K, \epsilon)$ is often called the metric entropy of $K$.

  *Proof* Lower bound. Assume $\mathcal{C}(K, d, \epsilon) \le N$. There exists a transformation of $x \in K$ into bit strings of length $N$. A partition of $K$ into at most $2^N$ subsets.

  Upper bound. Assume $\log_2 \mathcal{N}(K, d, \frac{\epsilon}{2}) \le N$. There exists an $\frac{\epsilon}{2}$-net $\mathcal{N}$ with $|\mathcal{N}| \le 2^N$. To every point $x \in K$, assign a point $x_0 \in \mathcal{N}$ that is closest to $x$. The encoding $x \mapsto x_0$ represents points in $K$ with accuracy $\epsilon$. □
- Error correcting code: Fix integers $k, n$ and $r$. Encoder $\{0, 1\}^k \to \{0, 1\}^n$, Decoder $\{0, 1\}^n \to \{0, 1\}^k$, $D(y) = x$ if $x \in \{0, 1\}^k, y \in \{0, 1\}^n$ and $d_H(E(x), y) \le r$.
- If $\log_2 \mathcal{P}(\{0, 1\}^n, d_H, 2r) \ge k$, then there exists an error correcting code, $k$ bits $\to n$ bits, correct $r$ error.

  *Proof* $\exists \mathcal{P} \in \{0, 1\}^n, |\mathcal{P}| = 2^k$ s.t closed balls centered at $\mathcal{P}$ with radii $r$ are disjoint. $E : \{0, 1\}^k \to \mathcal{N}$ one to one; $D : \{0, 1\}^n \to \{0, 1\}^k$ nearest-neighbor decodes. □
- If $n \ge k + 2r \log_2(\frac{en}{2r})$, then there exists an error correcting code that encodes $k$-bit strings into $n$-bit strings and can correct $r$ errors.

  *Proof* $\mathcal{P}(\{0, 1\}^n, d_H, 2r) \ge \mathcal{N}(\{0, 1\}^n, d_H, 2r) \ge \frac{2^n}{\sum_{k=0}^{2r} C_n^k} \ge 2^n (\frac{2r}{en})^{2r} \ge 2^k$. □

- $A_{m\times n}, \epsilon \in [0,1)$. Then for any $\epsilon$-set $\mathcal{N}$ of $\mathbb{S}^{n-1}$, $\sup_{x\in\mathcal{N}}\|Ax\|_2 \le \|A\| \le \frac{1}{1-\epsilon}\sup_{x\in\mathcal{N}}\|Ax\|_2$.

  *Proof* Fix $x \in \mathbb{S}^{n-1}$, $\|A\| = \|Ax\|_2$. $\exists x_0 \in \mathcal{N}, \|x-x_0\|_2 \le \epsilon$, $\|Ax - Ax_0\|_2 \le \|A\|\|x-x_0\|_2 \le \epsilon\|A\| \Rightarrow \|Ax_0\|_2 \ge \|Ax\|_2 - \|A(x-x_0)\|_2 \ge \|A\| - \epsilon\|A\|$. □

- $A_{m\times n} = \{A_{ij}\}$, $A_{ij}$ independent mean zero sub-gaussian, $K = \max_{i,j}\|A_{ij}\|_{\psi_2}$. Then for any $t > 0$, $\mathbb{P}(\|A\| \le CK(\sqrt{m}+\sqrt{n}+t)) \ge 1 - 2e^{-t^2}$.

  *Proof* Step 1: Approximation. Choose $\epsilon = 1/4$ and $\epsilon$-net $\mathcal{N}$ of $\mathbb{S}^{n-1}$, $\epsilon$-net $\mathcal{M}$ of $\mathbb{S}^{m-1}$ with $|\mathcal{N}| \le 9^n, |\mathcal{M}| \le 9^m$. Ex 4.4.3 $\Rightarrow \|A\| \le 2\max_{x\in\mathcal{N},y\in\mathcal{M}}\langle Ax,y\rangle$.

  Step 2: Concentration. $\langle Ax,y\rangle = \sum_{i,j}A_{ij}x_iy_j, \|\langle Ax,y\rangle\|_{\psi_2}^2 \le C\sum_{i,j}\|A_{ij}\|_{\psi_2}^2x_i^2y_j^2 \le CK^2 \Rightarrow \mathbb{P}(\langle Ax,y\rangle \ge u) \le 2e^{-cu^2/K^2}$.

  Step 3: Union bound. $\mathbb{P}(\max_{x\in\mathcal{N},y\in\mathcal{M}}\langle Ax,y\rangle \ge u) \le \sum_{x\in\mathcal{N},y\in\mathcal{M}}\mathbb{P}(\langle Ax,y\rangle \ge u) \le 9^{n+m}2e^{-cu^2/K^2}$. Take $u = CK(\sqrt{m}+\sqrt{n}+t), u^2 \ge C^2K^2(m+n+t^2)$. $C$ sufficiently large s.t. $cu^2/K^2 \ge 3(n+m+t^2)$. □

- $A_{n\times n}$ symmetric, $A_{ij}, i \le j$ independent mean zero sub-gaussian. Then for $t \ge 0, \mathbb{P}(\|A\| \le CK(\sqrt{n}+t)) \ge 1 - 4e^{-t^2}$.

  *Proof* $A = \underbrace{A^+ + A^-}_{\text{upper + lower triangular matrix}}$, $\mathbb{P}(\|A^+\| \le CK(\sqrt{n}+t)) \ge 1-2e^{-4t^2} \ge 1-2e^{-t^2}, \mathbb{P}(\|A^-\| \le CK(\sqrt{n}+t)) \ge 1-2e^{-4t^2} \ge 1-2e^{-t^2}, \mathbb{P}(\|A\| \le CK(\sqrt{n}+t)) \ge \mathbb{P}(\|A^+\| \le \frac{C}{2}K(\sqrt{n}+t), \|A^-\| \le \frac{C}{2}K(\sqrt{n}+t)) \ge 2(1-2e^{-t^2}) - 1 = 1 - 4e^{-t^2}$. □

- Stochastic block model (SBM): $G(n,p,q), p > q$, $n$ vertices, two community of size $n/2$, $x, y \in$ same community $\Rightarrow \mathbb{P}(x \sim y) = p$, otherwise $\mathbb{P}(x\sim y) = q$. $A = \{A_{ij}\}, A_{ij} = 1$ if $i \sim j$ otherwise 0. $A = \mathbb{E}A + R := D + R, \|D\| = \frac{p+q}{2}\cdot n, \mathbb{P}(\|R\| \le C\sqrt{n}) \ge 1 - 4e^{-n}$.

- Weyl's inequality: Symmetric matrices $S$ and $T$ with same dim, $\max_i|\lambda_i(S) - \lambda_i(T)| \le \|S-T\|$.

- Davis-Kahan: Fix $i$, $\min_{j\ne i}|\lambda_i(S)-\lambda_j(S)| = \delta > 0$. Then $\sin\angle(v_i(S), v_i(T)) \le \frac{2\|S-T\|}{\delta} \Rightarrow \exists\theta \in \{-1,1\}, \|v_i(S) - \theta v_i(T)\|_2 \le \frac{\|S-T\|}{\delta}\cdot 2^{3/2}$.

- Spectual clustering: Recall SBM $A = D+R$ and let $S = D, T = A = D+R$ in Davis-Kahan. $\delta = \min(\lambda_2, \lambda_2 - \lambda_1) = \min(\frac{p-q}{2}, q)n := \mu n$. $\mathbb{P}(\|R\| = \|T-S\| \le C\sqrt{n}) \ge 1 - 4e^{-n} \Rightarrow \exists\theta\in\{\pm1\}, \|v_2(D) - \theta v_2(A)\| \le \frac{C}{\mu\sqrt{n}}$. Let $u_2(D) = (1,1,\cdots,1,-1,-1,\cdots,-1) \Rightarrow \|u_2(D) - \theta u_2(A)\| \le \frac{C}{\mu} \Rightarrow \sum_{j=1}^n|u_2(D)_j - \theta u_2(A)_j|^2 \le \frac{C}{\mu^2}$. Thus the number of disagreeing signs between $u_2(D)$ and $u_2(A)$ must be bounded by $\frac{C}{\mu^2}$.

- $A_{m\times n}$, rows $A_i$ independent mean zero sub-gaussian, isotropic. Then for any $t \ge 0$, $\sqrt{m} - CK^2(\sqrt{n}+t) \le s_n(A) \le s_1(A) \le \sqrt{m} + CK^2(\sqrt{n}+t)$ with prob $\ge 1 - 2e^{-t^2}$. Here $K = \max_i\|A_i\|_{\psi_2}$.

  *Proof* Only need to prove $\|\frac{1}{m}A^TA - I_n\| \le \epsilon := K^2\max\{\delta,\delta^2\}, \delta = C(\frac{\sqrt{n}}{\sqrt{m}} + \frac{t}{\sqrt{m}})$.

  Step 1: Approximation. Find an $\frac14$-net $\mathcal{N}$ of the unit space $\mathbb{S}^{n-1}, |\mathcal{N}| \le 9^n$. $\|\frac1m A^TA - I_n\| \le 2\max_{x\in\mathcal{N}}|\langle(\frac1m A^TA - I_n)x, x\rangle| = 2\max_{x\in\mathcal{N}}|\frac1m\|Ax\|_2^2 - 1|$.

  Step 2: Concentration. $X_i := \langle A_i, x\rangle$ independent, mean zero, $\|X_i\|_{\psi_2} \le K$, $\mathbb{E}X_i^2 = 1$. $\mathbb{P}(|\frac1m\|Ax\|_2^2 - 1| \ge \frac{\epsilon}{2}) \le 2e^{-c_1\delta^2 m} \le 2e^{-c_1C^2(n+t^2)}$.

  Step 3: Union bound. $\mathbb{P}(\max_{x\in\mathcal{N}}|\frac1m\|Ax\|_2^2 - 1| \ge \frac{\epsilon}{2}) \le 9^n\cdot 2e^{-c_1C^2(n+t^2)} \le 2e^{-t^2}$. □

- $X \in \mathbb{R}^n$ sub-gaussian. $\mathbb{E}X = 0, \Sigma = \mathbb{E}XX^T, X_i \overset{\text{d}}{=} X$ i.i.d., $\Sigma_m = \frac1m\sum_{i=1}^m X_iX_i^T$. Assume there exists $K \ge 1$ s.t. $\|\langle X,x\rangle\|_{\psi_2}^2 \le K^2\|\langle X,x\rangle\|_{L^2}^2$. Then for $m$, $\mathbb{E}\|\Sigma_m - \Sigma\| \le CK^2(\sqrt{\frac{n}{m}} + \frac{n}{m})\|\Sigma\|$.

  *Proof* $Z_i = \Sigma^{-1/2}X_i, Z = \Sigma^{-1/2}X, \mathbb{E}Z_iZ_i^T = I_n, \|Z\|_{\psi_2} \le K, \|Z_i\|_{\psi_2} \le K$. Then $\|\Sigma_m - \Sigma\| = \|\Sigma^{1/2}R_m\Sigma^{1/2}\| \le \|R_m\|\|\Sigma\|$ where $R_m = \frac1m\sum_{i=1}^m Z_iZ_i^T - I$. Consider an $m\times n$ random matrix $A$ whose rows are $Z_i^T$. $\mathbb{E}\|R_m\| = \mathbb{E}\|\frac1m A^TA - I\| \le CK^2(\sqrt{\frac{n}{m}} + \frac{n}{m})$. □

# 5 Concentration without independence

- $(X, d_X) \xrightarrow{f} (Y, d_Y), d_Y(f(u), f(v)) \le L\cdot d_x(u,v), \forall u, v \in X$. The infimum of all $L$ in this definition is called the Lipschitz norm of $f$ and is denoted $\|f\|_{\text{Lip}}$.

- $\epsilon > 0, A_\epsilon = A + \epsilon B_2^n, A \subset \mathbb{R}^n, \min_A$ volume of $A_\epsilon$ with colume $A$ fixed is achieved when $A$ is a ball.

- $\sigma_{n-1}(A)$ normalized area on $\mathbb{S}^{n-1}$, $\epsilon > 0$. With given $\sigma_{n-1}(A)$, $\min_A \sigma_{n-1}(A_\epsilon)$ is achieved when $A$ is a spherical cap.

- $A \subset \sqrt{n}\mathbb{S}^{n-1}$. If $\sigma(A) \geq \frac{1}{2}$, then $\forall t \geq 0, \sigma(A_t) \geq 1 - 2e^{-ct^2}$.

  *Proof* Let $H = \{x \in \sqrt{n}\mathbb{S}^{n-1}, x_1 \leq 0\}$. $\sigma(A) \geq \frac{1}{2}\sigma(H)$. Thm 5.1.6 $\Rightarrow \sigma(A_t) \geq \sigma(H_t) = \mathbb{P}(X \in H_t)$. $H_t \supset \{x \in \sqrt{n}\mathbb{S}^{n-1}, x_1 \leq \frac{t}{\sqrt{2}}\} \Rightarrow \sigma(H_t) \geq \mathbb{P}(X_1 \leq \frac{t}{\sqrt{2}})$. $\|X_1\|_{\psi_2} \leq C \Rightarrow \mathbb{P}(X_1 \leq \frac{t}{\sqrt{2}}) \geq 1 - 2e^{-ct^2}$. □

- $X \sim \text{Unif}(\sqrt{n}\mathbb{S}^{n-1}), f : \sqrt{n}\mathbb{S}^{n-1} \to \mathbb{R}$. Then $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq C\|f\|_{\text{Lip}}$.

  *Proof* WLOG $\|f\|_{\text{Lip}} = 1$, $\mathbb{P}(f(X) \geq M) \geq \frac{1}{2}, \mathbb{P}(f(X) \leq M) \geq \frac{1}{2}$. $A := \{x \in \sqrt{n}\mathbb{S}^{n-1} : f(x) \leq M\}$. $\mathbb{P}(X \in A) \geq \frac{1}{2} \Rightarrow \mathbb{P}(A_t) \geq 1 - 2e^{-ct^2} \Rightarrow \mathbb{P}(f(X) \leq M + t) \geq 1 - 2e^{-ct^2}$. By centering, $f(X) - \mathbb{E}f(X) = f(X) - M - (\mathbb{E}f(X) - M)$ is sub-gaussian. □

- $X \sim \mathcal{N}(0, I_n), \gamma_n(A) = \mathbb{P}(X \in A), \epsilon > 0, \gamma_n(A)$ given, half spaces minimize $\gamma_n(A_\epsilon)$.

- $X \sim \mathcal{N}(0, I_n), f : \mathbb{R}^n \to \mathbb{R}, \|f\|_{\text{Lip}} < \infty$. Then $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq C\|f\|_{\text{Lip}}$.

- Hamming cube, $d(x,y) = \frac{1}{n}|\{i : x_i \neq y_i\}|, \mathbb{P}(A) = \frac{|A|}{2^n}$. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- $S_n : n!$ permutation of $n$ symbols. $d(\pi, \rho) = \frac{1}{n}|\{i : \pi(i) \neq \rho(i)\}|, \mathbb{P}(A) = \frac{|A|}{n!}$. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- Special orthogonal group SO$(n)$, determinant $= 1$, $d = \|\cdot\|_F$, $\mathbb{P}$ is uniform measure. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- $G_{n,m}$ all $m$-dim subspaces of $\mathbb{R}^n$ ($\simeq \mathcal{P}_{G_{n,m}}$ orthogonal projections), $d(E, F) = \|\mathcal{P}_E - \mathcal{P}_F\|$, $\mathbb{P}$ is uniform measure. A random subspace $E$ can be constructed by computing the column span (i.e. the image) of a random $n \times m$ Gaussian random matrix $G$ with i.i.d. $\mathcal{N}(0,1)$ entries. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{n}}$.

- A random vector $X$ in $\mathbb{R}^n$ with density $p(x) = e^{-U(x)}$, Hess $U(x) \succeq \kappa I_n$. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq \frac{C\|f\|_{\text{Lip}}}{\sqrt{\kappa}}$.

- $X = (X_1, \cdots, X_n)$ independent coordinates, $|X_i| \leq 1$ a.s., $f$ convex and Lipschitz. $\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq C\|f\|_{\text{Lip}}$.

- $E \sim \text{Unif}(G_{n,m}), z \in \mathbb{R}^n, \epsilon > 0$. Then (a) $(\mathbb{E}\|P_E z\|_2^2)^{\frac{1}{2}} = \sqrt{\frac{m}{n}}\|z\|_2$; (b) $\mathbb{P}(\left|\|P_E z\|_2 - \sqrt{\frac{m}{n}}\|z\|_2\right| \leq \epsilon\sqrt{\frac{m}{n}}\|z\|_2) \geq 1 - 2e^{-c\epsilon^2 m}$.

  *Proof* (a): WLOG $\|z\|_2 = 1$. Rotational invariance: $\mathbb{P}(E \in A) = \mathbb{P}(U(E) \in A)$ where $U$ is $n \times n$ orthogonal $\Rightarrow$ The dist. of $P_E z$ is the same if we fix $E$, $z \in \text{Unif}(\mathbb{S}^{n-1})$. WLOG $Pz = (z_1, \cdots, z_m, 0, \cdots, 0)$. $\mathbb{E}\|Pz\|_2^2 = m\mathbb{E}z_i^2 = \frac{m}{n}$.

  (b): $f : z \to \|Pz\|_2, \|f\|_{\text{Lip}} = 1 \Rightarrow \|\|Pz\|_2 - \mathbb{E}\|Pz\|_2\|_{\psi_2} \leq \frac{C}{\sqrt{n}}$ □

- Johnson-Lindenstrauss lemma: $\mathcal{X}$ a set of $N$ points in $\mathbb{R}^n, \epsilon > 0, m \geq \frac{C}{\epsilon^2}\log N, E \sim \text{Unif}(G_{n,m}), Q = \sqrt{\frac{n}{m}}\mathcal{P}_E$. Then $\mathbb{P}(|\|Qx - Qy\|_2 - \|x - y\|_2| \leq \epsilon\|x - y\|_2$ for any $x, y \in \mathcal{X}) \geq 1 - 2e^{-c\epsilon^2 m}$.

  *Proof* Let $\mathcal{X} - \mathcal{X} := \{x - y : x, y \in \mathcal{X}\}$. The latest lemma $\Rightarrow \forall z, \mathbb{P}\left((1-\epsilon)\sqrt{\frac{m}{n}}\|z\|_2 \leq \|Pz\|_2 \leq (1+\epsilon)\sqrt{\frac{m}{n}}\|z\|_2\right) \geq 1 - 2e^{-c\epsilon^2 m}$. Union bound: $\mathbb{P}(\cdots$, for any $z \in \mathcal{X} - \mathcal{X}) \geq 1 - N^2 \cdot 2e^{-c\epsilon^2 m} \geq 1 - 2e^{-c'\epsilon^2 m}$. □

- $f : \mathbb{R} \to \mathbb{R}, X = \sum_{i=1}^n \lambda_i u_i u_i^T$, define $f(X) = \sum_{i=1}^n f(\lambda_i) u_i u_i^T$.

- P.S.D. order: $X \succeq 0, X \succeq Y$ if $X - Y \succeq 0$.

- Golden-Thompson inequality: $\text{tr}(e^{A+B}) \leq \text{tr}(e^A e^B)$.

- Lieb's inequality: $H : n \times n$ symmetric matrix, $X$ P.D., $f(X) = \text{tr}(e^{H + \log X})$. Then $f$ is concave.

- $X$ is a random P.D. matrix $\Rightarrow \mathbb{E}f(X) \leq f(\mathbb{E}X)$. $X = e^Z$, $Z$ symmetric. Then $\mathbb{E}\text{tr}(e^{H+Z}) \leq \text{tr}(e^{H + \log \mathbb{E}e^Z})$.

- $X_1, \cdots, X_N$ independent mean zero $n \times n$ symmetric random matrices, $\|X_i\| \leq K$ a.s. for all $i$. Then for $\forall t \geq 0$, $\mathbb{P}(\|\sum_{i=1}^N X_i\| \geq t) \leq 2ne^{-\frac{t^2/2}{\sigma^2 + Kt/3}}$ where $\sigma^2 = \|\sum_{i=1}^N \mathbb{E}X_i^2\|$.

  *Proof* Step 1: Reduction to MGF. $S := \sum_{i=1}^N X_i$. $\|S\| = \max_i |\lambda_i(S)| = \max(\lambda_{\max}(S), \lambda_{\max}(-S))$. $\mathbb{P}(\lambda_{\max}(S) \geq t) \leq e^{-\lambda t}\mathbb{E}e^{\lambda\lambda_{\max}(S)}$. $E := \mathbb{E}e^{\lambda\lambda_{\max}(S)} = \mathbb{E}\lambda_{\max}(e^{\lambda S}) \Rightarrow E \leq \mathbb{E}\text{tr}(e^{\lambda S})$.

  Step 2: Apply Lieb's inequality. $\mathbb{E}\text{tr}(e^{\lambda S}) = \mathbb{E}\text{tr}(e^{\sum_{i=1}^{N-1} \lambda X_i + \lambda X_N}) \leq \mathbb{E}\text{tr}(e^{\sum_{i=1}^{N-1} \lambda X_i + \log \mathbb{E}e^{\lambda X_N}}) \leq \text{tr}(e^{\sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i}})$.

  Step 3: Lemma: $X$ is an $n \times n$ symmetric mean zero random matrix, $\|X\| \leq K$ a.s. Then $e^{\lambda X} \preceq e^{g(\lambda)\mathbb{E}X^2}$ where $g(\lambda) = \frac{\lambda^2/2}{1 - |\lambda|K/3}, |\lambda| < 3/K$.

*Proof* $e^z \le 1 + z + \frac{1}{1-|z|/3}\frac{z^2}{2}$ if $|z| < 3$. $z = \lambda x$. If $|x| \le K, |\lambda| < \frac{3}{K}$, $e^{\lambda x} \le 1 + \lambda x + g(\lambda)x^2$. (b) of Ex. 5.4.5 $\Rightarrow$ If $\|X\| \le K, |\lambda| < 3/K$, $\mathbb{E}e^{\lambda X} \preceq I + g(\lambda)\mathbb{E}X^2$ (since $\mathbb{E}X = 0$) $\preceq e^{g(\lambda)\mathbb{E}X^2}$. ☐

Step 4: $E \le \text{tr}(e^{\sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i}})$. The latest lemma + (g) of Ex.5.4.5 $\Rightarrow \log \mathbb{E}e^{\lambda X_i} \preceq g(\lambda)\mathbb{E}X_i^2 \Rightarrow \sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i} \preceq g(\lambda) \cdot Z$ where $Z := \sum_{i=1}^N \mathbb{E}X_i^2$ and $\sigma^2 = \|Z\|$. (e) of Ex.5.4.5 $\Rightarrow \text{tr}(e^{\sum_{i=1}^N \log \mathbb{E}e^{\lambda X_i}}) \le \text{tr}(e^{g(\lambda Z)}) \Rightarrow E \le \text{tr}(e^{g(\lambda Z)}) \le n\lambda_{\max}(e^{g(\lambda Z)}) = ne^{g(\lambda)\|Z\|} = ne^{g(\lambda)\sigma^2}$. Minimize for $\lambda$ as a function of $t$ with $0 < \lambda < 3/K$. ☐

- $X \in \mathbb{R}^n, \Sigma = \mathbb{E}XX^T, \Sigma_m = \frac{1}{m}\sum_{i=1}^m X_iX_i^T, X_i \overset{\text{i.i.d.}}{\sim} X, \|X\|_2 \le K(\mathbb{E}\|X\|_2^2)^{\frac{1}{2}}$ a.s.. Then $\mathbb{E}\|\Sigma_m - \Sigma\| \le C(\sqrt{\frac{K^2 n\log n}{m}} + \frac{K^2 n \log n}{m})\|\Sigma\|$.

  *Proof* $\mathbb{E}\|X\|_2^2 = \mathbb{E}X^TX = \mathbb{E}\text{tr}(X^TX) = \mathbb{E}\text{tr}(XX^T) = \text{tr}(\Sigma) \Rightarrow \|X\|_2^2 \le K^2\text{tr}(\Sigma)$ a.s.. Ex 5.4.11 $\Rightarrow \mathbb{E}\|\Sigma_m - \Sigma\| = \frac{1}{m}\mathbb{E}\|\sum_{i=1}^m (X_iX_i^T - \Sigma)\| \lesssim \frac{1}{m}(\sigma\sqrt{\log n} + M\log n)$ where $\sigma^2 = \|\sum_{i=1}^m \mathbb{E}(X_iX_i^T - \Sigma)^2\| = m\|\mathbb{E}(XX^T - \Sigma)^2\|$ and $M$ is chosen s.t. $\|XX^T - \Sigma\| \le M$ a.s.. Then $\mathbb{E}(XX^T - \Sigma)^2 = \mathbb{E}(XX^T)^2 - \Sigma^2 \preceq \mathbb{E}(XX^T)^2 = \mathbb{E}(\|X\|_2^2 XX^T) \preceq K^2\text{tr}(\Sigma)\Sigma \Rightarrow \sigma^2 \le K^2 m\text{tr}(\Sigma)\|\Sigma\|$. $\|XX^T - \Sigma\| \le \|X\|_2^2 + \|\Sigma\| \le K^2\text{tr}(\Sigma) + \|\Sigma\| \le 2K^2\text{tr}(\Sigma) := M$ (since $K \ge 1$ and $\|\Sigma\| \le \text{tr}(\Sigma)$). Substitute our bounds for $\sigma^2$ and $M$ into the previous bound $\frac{1}{m}(\sigma\sqrt{\log n} + M\log n)$. ☐

# 6 Quadradic forms, symmetrization, contraction

- $Y \perp Z, \mathbb{E}Z = 0$, then $\mathbb{E}F(Y) \le \mathbb{E}F(Y + Z)$.

  *Proof* $F(y) = F(\mathbb{E}(y + Z)) \le \mathbb{E}F(y + Z) \Rightarrow \mathbb{E}F(Y) = \mathbb{E}(\mathbb{E}(F(Y + \mathbb{E}Z)|Y)) = \mathbb{E}(\mathbb{E}(F(\mathbb{E}(Y + Z))|Y)) \le \mathbb{E}(\mathbb{E}(F(Y + Z)|Y)) = \mathbb{E}F(Y + Z)$. ☐

- Decoupling: $A_{n\times n}$ diagonal-free(i.e. the diagonal entries of $A$ equal zero), $X = (X_1, \cdots, X_n)$ independent mean zero. Then for every convex function $F : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}F(X^TAX) \le \mathbb{E}F(4X^TAX')$ where $X' \overset{\text{d}}{=} X, X' \perp X$.

  *Proof* $\delta_1, \cdots, \delta_n \overset{\text{i.i.d.}}{\sim} \text{Ber}(1, \frac{1}{2}), I = \{i : \delta_i = 1\}, \mathbb{E}\delta_i(1 - \delta_j) = \frac{1}{4}, X^TAX = \sum_{i\ne j} a_{ij}X_iX_j = 4\mathbb{E}_\delta \sum_{i,j} \delta_i(1 - \delta_j)a_{ij}X_iX_j = 4\mathbb{E}_I \sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j \Rightarrow \mathbb{E}_XF(X^TAX) = \mathbb{E}_XF(4\mathbb{E}_I \sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j) \le \mathbb{E}_I\mathbb{E}_XF(4\sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j)$. There exists an $I$ s.t. $\mathbb{E}_XF(4\sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j') = \mathbb{E}_XF(4\sum_{(i,j)\in I\times I^c} a_{ij}X_iX_j) \ge \mathbb{E}F(X^TAX)$. LHS $\le \mathbb{E}_XF(4\sum_{i,j} a_{ij}X_iX_j')$ by the latest lemma since $\mathbb{E}[(\sum_{(i,j)\in I\times I} + \sum_{(i,j)\in I^c\times I^c} + \sum_{(i,j)\in I^c\times I})a_{ij}X_iX_j'|\{X_i, i \in I\}, \{X_j', j \in I^c\}] = 0$. ☐

- $X, X' \sim \mathcal{N}(0, I_n), X \perp X'$, then $\mathbb{E}e^{\lambda X^TAX'} \le e^{C\lambda^2\|A\|_F^2}, |\lambda| \le \frac{c}{\|A\|}$.

  *Proof* $A = \sum_i s_iu_iv_i^T, X^TAX' = \sum_i s_i \underbrace{\langle u_i, X\rangle}_{:=g_i}\underbrace{\langle v_i, X'\rangle}_{:=g_i'}$. $(g_1, \cdots g_n) \perp (g_1', \cdots, g_n') \sim \mathcal{N}(0, I_n) \Rightarrow \mathbb{E}e^{\lambda X^TAX} = \prod_{i=1}^n \mathbb{E}e^{\lambda s_i g_i g_i'} = \prod_{i=1}^n \mathbb{E}e^{\lambda^2 s_i^2 g_i^2/2} \le \prod_{i=1}^n e^{C\lambda^2 s_i^2}(\lambda^2 s_i^2 \le c) \le e^{C\lambda^2\|A\|_F^2}(\lambda^2 \le \frac{c}{\max_i s_i^2} = \frac{c}{\|A\|^2})$. ☐

- $X, X'$ independent sub-gaussian mean zero, $\|X\|_{\psi_2} \le K, \|X'\|_{\psi_2} \le K$. $g, g' \sim \mathcal{N}(0, I_n), g \perp g'$. Then $\mathbb{E}e^{\lambda X^TAX'} \le \mathbb{E}e^{CK^2\lambda g^TAg'}$.

  *Proof* Conditioned on $X'$, $\mathbb{E}_Xe^{\lambda X^AX'} \le e^{C\lambda^2K^2\|AX'\|_2^2}, \mathbb{E}_ge^{\mu g^TAX'} = e^{\frac{\mu^2\|AX'\|_2^2}{2}}$. $\mu = \sqrt{2c}K\lambda \Rightarrow \mathbb{E}_Xe^{\lambda X^TAX'} \le \mathbb{E}_ge^{\sqrt{2c}K\lambda g^TAX'} \Rightarrow \mathbb{E}e^{\lambda X^TAX'} \le \mathbb{E}e^{\sqrt{2c}K\lambda g^TAX'} \le \mathbb{E}e^{2cK^2\lambda g^TAg'}$ ☐

- Hanson-Wright inequality: $X = (X_1, \cdots, X_n)$ independent mean zero sub-gaussian, then $\mathbb{P}(|X^TAX - \mathbb{E}X^TAX| \ge t) \le 2e^{-c\min(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|})}$.

  *Proof* WLOG $K = 1$. $X^TAX = \sum_{i,j} a_{ij}X_iX_j, \mathbb{E}X^TAX = \sum_i a_{ii}\mathbb{E}X_i^2, X^TAX - \mathbb{E}X^TAX = \sum_i a_{ii}(X_i^2 - \mathbb{E}X_i^2) + \sum_{i\ne j} a_{ij}X_iX_j$. $p := \mathbb{P}(X^TAX - \mathbb{E}X^TAX \ge t) \le \mathbb{P}(\sum_i a_{ii}(X_i^2 - \mathbb{E}X_i^2) \ge \frac{t}{2}) + \mathbb{P}(\sum_{i\ne j} a_{ij}X_iX_j \ge \frac{t}{2}) := p_1 + p_2$.

  Step 1: $\|X_i^2 - \mathbb{E}X_i^2\|_{\psi_1} \lesssim 1$. Bernstein $\Rightarrow p_1 \le e^{-c\min(\frac{t^2}{\sum a_{ii}^2}, \frac{t}{\max_i |a_{ii}|})} \le e^{-c\min(\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|})}$.

  Step 2: $S := \sum_{i\ne j} a_{ij}X_iX_j$. $p_2 \le e^{-\frac{\lambda t}{2}}\mathbb{E}e^{\lambda S}, \mathbb{E}e^{\lambda S} \le \mathbb{E}e^{4\lambda X^TAX'} \le \mathbb{E}e^{c_1\lambda g^TAg'} \le e^{C\lambda^2\|A\|_F^2}$(with $\lambda \le \frac{c}{\|A\|}$). ☐

- $B_{m\times n}, X \in \mathbb{R}^n, \{X_i\}$ independent mean-zero, unit-variance, sub-gaussian. Then $\|\|BX\|_2 - \|B\|_F\|_{\psi_2} \le CK^2\|B\|$, $K = \max_i \|X\|_{\psi_2}$.

  *Proof* $A = B^TB, X^TAX = \|BX\|_2^2, \mathbb{E}X^TAX = \|B\|_F^2, \|A\| = \|B\|^2, \|A\|_F = \|B^TB\|_F \le \|B^T\|\|B\|_F = \|B\|\|B\|_F$.Thus $\forall u \ge 0$, $\mathbb{P}(|\|BX\|_2^2 - \|B\|_F^2| \ge u) \le e^{-\frac{c}{K^4}\min(\frac{u^2}{\|B\|^2\|B\|_F^2}, \frac{u}{\|B\|^2})}$. Let $u = \epsilon\|B\|_F^2, \mathbb{P}(|\|BX\|_2^2 - \|B\|_F^2| \ge \epsilon\|B\|_F^2) \le 2e^{-c\min(\epsilon^2,\epsilon)\frac{\|B\|_F^2}{K^4\|B\|^2}}$. Let $\delta^2 = \min(\epsilon^2, \epsilon)$, then $\epsilon = \max(\delta, \delta^2), |\|BX\| - \|B\|_F| \ge \delta\|B\|_F \Rightarrow |\|BX\|_2^2 - \|B\|_F^2| \ge \epsilon\|B\|_F^2 \Rightarrow \mathbb{P}(|\|BX\|_2 - \|B\|_F| \ge \delta\|B\|_F) \le 2e^{-c\delta^2\frac{\|B\|_F^2}{K^4\|B\|^2}}$. ☐

- Symmetrization: $X_1, X_2, \cdots, X_N$ independent, mean zero in a normed space, $\epsilon_1, \epsilon_2, \cdots, \epsilon_N$ a sequence of independent symmetric Bernoulli random variables. Then $\frac{1}{2}\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\| \le \mathbb{E}\|\sum_{i=1}^N X_i\| \le 2\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$.

  *Proof* Upper bound. $X' \perp X, X' \overset{\mathrm{d}}{=} X$. $p = \mathbb{E}\|\sum_{i=1}^N X_i\| \le \mathbb{E}\|\sum_{i=1}^N X_i - \sum_{i=1}^N X_i'\| = \mathbb{E}\|\sum_{i=1}^N \epsilon_i(X_i - X_i')\| \le \mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\| + \mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i'\| = 2\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$. □

- $A_{n\times n}$ symmetric independent mean zero. Then $\mathbb{E}\|A\| \le C\sqrt{\log n}\mathbb{E}\max\|A_i\|_2$ where $A_i$ is $i$-th row of $A$.

  *Proof* $A = \sum_{i\le j} Z_{ij}$ independent mean zero symmetric where $Z_{ij} = \begin{cases} A_{ij}(e_i e_j^T + e_j e_i^T), & i \le j \\ A_{ii} e_i e_i^T & i = j \end{cases} \Rightarrow \mathbb{E}\|A\| \le 2\mathbb{E}\|\sum_{i\le j} \epsilon_{ij} Z_{ij}\|$.

  Ex 5.4.3(a) $\Rightarrow$ Conditioned on $\{Z_{ij}\}, \mathbb{E}_\epsilon\|\sum_{i\le j}\epsilon_{ij}Z_{ij}\| \le C\sqrt{\log n}\|\sum_{i\le j} Z_{ij}^2\|^{\frac{1}{2}} \Rightarrow \mathbb{E}\|\sum_{i\le j}\epsilon_{ij}Z_{ij}\| \le C\sqrt{\log n}\mathbb{E}\|\sum_{i\le j} Z_{ij}^2\|^{\frac{1}{2}}$, $\sum_{i\le j} Z_{ij}^2 = \sum_{i=1}^n(\sum_{j=1}^n A_{ij}^2)e_i e_i^T = \sum_{i=1}^n \|A_i\|_2^2 e_i e_i^T \Rightarrow \|\sum_{i\le j} Z_{ij}^2\|^{\frac{1}{2}} = \max\|A_i\|_2$. □

- Matrix completion: $X_{n\times n}, \operatorname{rank}(X) = r << n, Y_{ij} = \delta_{ij}X_{ij}, \delta_{ij} \sim \operatorname{Ber}(p), p = \frac{m}{n^2}, \hat{X} = \arg\min_{\operatorname{rank}(A')\le r}\|p^{-1}Y - A'\|$. Then $\mathbb{E}\frac{1}{n}\|\hat{X} - X\|_F \le C\sqrt{\frac{rn\log n}{m}}\|X\|_\infty$.

  *Proof* Step 1. $\|\hat{X} - X\| \le \|\hat{X} - p^{-1}Y\| + \|p^{-1}Y - X\| \le 2\|p^{-1}Y - X\| = \frac{2}{p}\|Y - pX\|$. $(Y - pX)_{ij} = (\delta_{ij} - p)X_{ij}$ independent mean zero, Ex 6.5.2 $\Rightarrow \mathbb{E}\|Y - pX\| \le C\sqrt{\log n}(\mathbb{E}\max_i\|(Y - pX)_i\|_2 + \mathbb{E}\max_j\|(T - pX)^j\|_2)$. $\|(Y - pX)_i\|_2^2 = \sum_{j=1}^n(\delta_{ij} - p)^2 X_{ij}^2 \le \sum_{j=1}^n(\delta_{ij} - p)^2\|X\|_\infty^2$. Ex 6.6.2 $\Rightarrow \mathbb{E}\max_i\sum_{j=1}^n(\delta_{ij} - p)^2 \le Cpn \Rightarrow \frac{2}{p}\|Y - pX\| \le C\sqrt{\frac{n\log n}{p}}\|X\|_\infty$.

  Step 2. $\operatorname{rank}(X) \le r, \operatorname{rank}(\hat{X}) \le r, \operatorname{rank}(\hat{X} - X) \le 2r$. $\|\hat{X} - X\|_F \le \sqrt{2r}\|\hat{X} - X\| \Rightarrow \mathbb{E}\|\hat{X} - X\|_F \le C\sqrt{\frac{rn\log n}{p}}\|X\|_\infty$. □

- Contraction principle: $X_1, \cdots, X_N$ vectors in some normed space, $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$. Then $\mathbb{E}\|\sum_{i=1}^N a_i\epsilon_i X_i\| \le \|a\|_\infty\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$.

  *Proof* WLOG $\|a\|_\infty \le 1, f(a) = \mathbb{E}\|\sum_{i=1}^N a_i\epsilon_i X_i\|$ is convex, which implies the maximum of $f$ is attained at the boundary. Thus $f(a) \le f(a^*) = \mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\|$ with $a_i^* = 1$ or $-1$. □

- Symmetrization with gaussians: $X_1, \cdots, X_N$ independent mean zero, $g_1, \cdots, g_N \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, $\frac{C}{\sqrt{\log N}}\mathbb{E}\|\sum_{i=1}^N g_i X_i\| \le \mathbb{E}\|\sum_{i=1}^N X_i\| \le 3\mathbb{E}\|\sum_{i=1}^N g_i X_i\|$.

  *Proof* Upper: $\mathbb{E}\|\sum_{i=1}^N X_i\| \le 2\mathbb{E}\|\sum_{i=1}^N \epsilon_i X_i\| = 2\sqrt{\frac{\pi}{2}}\mathbb{E}_{X,\epsilon}\|\sum_{i=1}^N \epsilon_i\mathbb{E}_g|g_i|X_i\| \le 2\sqrt{\frac{\pi}{2}}\mathbb{E}\|\sum_{i=1}^N \epsilon_i|g_i|X_i\| = 2\sqrt{\frac{\pi}{2}}\mathbb{E}\|\sum_{i=1}^N g_i X_i\|$.

  Lower: $\mathbb{E}\|\sum_{i=1}^N g_i X_i\| = \mathbb{E}\|\sum_{i=1}^N \epsilon_i g_i X_i\| \le \mathbb{E}_g\mathbb{E}_X(\|g\|_\infty\mathbb{E}_\epsilon\|\sum_{i=1}^N \epsilon_i X_i\|) = \mathbb{E}_g\|g\|_\infty\mathbb{E}_{X,\epsilon}\|\sum_{i=1}^N \epsilon_i X_i\| \le 2\mathbb{E}_g\|g\|_\infty\mathbb{E}_X\|\sum_{i=1}^N X_i\| \le C\sqrt{\log N}\mathbb{E}_X\|\sum_{i=1}^N X_i\|$. □

# 7 Random processes

- Basic concepts: $\{X_t\}_{t\in T\subset\mathbb{R}^n}, \mathbb{E}X_t = 0, \forall t \in T, \Sigma(t,s) = \operatorname{Cov}(X_t, X_s) = \mathbb{E}X_t X_s, d(t,s) = \|X_t - X_s\|_{L^2} = (\mathbb{E}(X_t - X_s)^2)^{\frac{1}{2}}$ (increments).

- Gaussian process: $T_0 \subset T, |T_0| < \infty, \{X_t\}_{t\in T_0}$ has normal distribution.

- $Y$ is a mean zero Gaussian r.v. in $\mathbb{R}^n$. Then there exists $t_1, \cdots, t_n \in \mathbb{R}^n$ s.t. $Y \overset{\mathrm{d}}{=} (\langle g, t_i\rangle)_{i=1}^n, g \sim \mathcal{N}(0, I_n)$.

- Gaussian integration by parts: $X \sim \mathcal{N}(0,1)$. Then for $f$ differentiable, $\mathbb{E}f'(X) = \mathbb{E}Xf(X)$.

  *Proof* $f$ has bounded support: $\mathbb{E}f(X) = \int_\mathbb{R} f'(x)\phi(x)dx = -\int_\mathbb{R} f(x)\phi'(x)dx$. General $f$: $f_n \to f$. □

- $X \sim \mathcal{N}(0,\Sigma), f: \mathbb{R}^n \to \mathbb{R}$. Then $\mathbb{E}Xf(X) = \Sigma\cdot\mathbb{E}\nabla f(X)$.

- $X \sim \mathcal{N}(0,\Sigma^X), Y \sim \mathcal{N}(0,\Sigma^Y), X \perp Y, Z(u) = \sqrt{u}X + \sqrt{1-u}Y, u \in [0,1]$. Then $f: \mathbb{R}^n \to \mathbb{R}$ twice-differentiable, $\frac{d}{du}\mathbb{E}f(Z(u)) = \frac{1}{2}\sum_{i,j=1}^n(\Sigma_{ij}^X - \Sigma_{ij}^Y)\mathbb{E}[\frac{\partial^2 f}{\partial x_i\partial x_j}(Z(u))]$.

  *Proof* $\frac{d}{du}\mathbb{E}f(Z(u)) = \frac{1}{2}\sum_{i=1}^n\mathbb{E}\frac{\partial f}{\partial x_i}(Z(u))(\frac{X_i}{\sqrt{u}} - \frac{Y_i}{\sqrt{1-u}})$. $\sum_{i=1}^n\frac{1}{\sqrt{u}}\mathbb{E}X_i\frac{\partial f}{\partial x_i}(Z(u)) := \sum_{i=1}^n\frac{1}{\sqrt{u}}\mathbb{E}X_i g_i(X)$ (conditioned on $Y$) where $g_i(X) := \frac{\partial f}{\partial x_i}(\sqrt{u}X + \sqrt{1-u}Y)$. $\mathbb{E}X_i g_i(X) = \sum_{j=1}^n\Sigma_{ij}^X\mathbb{E}\frac{\partial g_i}{\partial x_j}(X) = \sum_{j=1}^n\Sigma_{ij}^X\mathbb{E}\frac{\partial^2 f}{\partial x_i x_j}(\sqrt{u}X + \sqrt{1-u}Y)\sqrt{u}$. □

- $\mathbb{E}X_i^2 = \mathbb{E}Y_i^2, \mathbb{E}(X_i - X_j)^2 \le \mathbb{E}(Y_i - Y_j)^2, \frac{\partial^2 f}{\partial x_i\partial x_j} \ge 0, \forall i \ne j$. Then $\mathbb{E}f(X) \ge \mathbb{E}f(Y)$.

- Slepian's inequality: Let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ be two mean zero Gaussian processes. Assume $\mathbb{E}X_t^2 = \mathbb{E}Y_t^2, \mathbb{E}(X_t - X_s)^2 \le \mathbb{E}(Y_t - Y_s)^2$. Then for every $t \in \mathbb{R}$, $\mathbb{P}(\sup_{t\in T} X_t \ge t) \le \mathbb{P}(\sup_{t\in T} Y_t \ge t)$ and $\mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t$.

*Proof* Let $f(x) \approx 1_{\{\max x_i < t\}} = \prod_{i=1}^n 1_{\{x_i<t\}}$ and use the latest lemma. $\square$

- Sudakov-Fernique's inequality: Let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ be two mean zero Gaussian processes. Assume $\mathbb{E}(X_t - X_s)^2 \le \mathbb{E}(Y_t - Y_s)^2$. Then $\mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t$.

  *Proof* Let $f(x) = \frac{1}{\beta}\log\sum_{i=1}^n e^{\beta x_i}$. $f(x) \to \max x_i$ as $\beta\to\infty$. $\frac{\partial^2 f}{\partial x_i \partial x_j} \le 0$. $\square$

- $A_{m\times n}$ independent $\mathcal{N}(0,1)$ entries. Then $\mathbb{E}\|A\| \le \sqrt{m} + \sqrt{n}$.

  *Proof* $\max_{u\in\mathbb{S}^{n-1}, v\in\mathbb{S}^{m-1}}\langle Au, v\rangle := \max_{(u,v)\in T} X_{uv}$ where $T = \mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$ and $X_{uv}\sim\mathcal{N}(0,1)$. $\mathbb{E}(X_{uv} - X_{wz})^2 = \mathbb{E}(\langle Au, v\rangle - \langle Aw, z\rangle)^2 = \mathbb{E}(\sum_{i,j} A_{ij}(u_jv_i - w_jz_i))^2 = \sum_{i,j}(u_jv_i - w_jz_i)^2 = \|uv^T - wz^T\|_F \le \|u-w\|_2^2 + \|v-z\|_2^2$. Define $Y_{uv} = \langle g, u\rangle + \langle h, v\rangle, g\sim\mathcal{N}(0, I_n), h\sim\mathcal{N}(0, I_m), g\perp h$. $\mathbb{E}(Y_{uv} - Y_{wz})^2 = \|u-w\|_2^2 + \|v-z\|_2^2$. Then $\mathbb{E}\|A\| = \mathbb{E}\sup_{(u,v)\in T} X_{uv} \le \mathbb{E}_{(u,v)\in T} Y_{uv} = \mathbb{E}\sup_{u\in\mathbb{S}^{n-1}}\langle g, u\rangle + \mathbb{E}\sup_{v\in\mathbb{S}^{m-1}}\langle g, v\rangle = \mathbb{E}\|g\|_2 + \mathbb{E}\|h\|_2 \le \sqrt{n}+\sqrt{m}$. $\square$

- $\mathbb{P}(\|A\| \ge \sqrt{m}+\sqrt{n}+t) \le 2e^{-ct^2}$.

  *Proof* $A\sim\mathcal{N}(0, I_{nm}), f(A) = \|A\| \le \|A\|_2 \Rightarrow \|f\|_{\text{Lip}} \le 1 \Rightarrow \|f(A) - \mathbb{E}f(A)\|_{\psi_2} \le C$. $\square$

- Sudakov's minoration inequality: $\{X_t\}_{t\in T}$ mean zero Gaussian process. $\forall\epsilon>0, \mathbb{E}\sup_{t\in T} X_t \ge C\epsilon\sqrt{\log\mathcal{N}(T, d, \epsilon)}$.

  *Proof* Assume $\mathcal{N}(T,d,\epsilon) = N < \infty$. Let $\mathscr{N}$ be a maximal $\epsilon$-separated subset of $T$. $|\mathscr{N}| \ge N$. It suffices to show $\mathbb{E}\sup_{t\in\mathscr{N}} X_t \ge C\epsilon\sqrt{\log N}$. $Y_t := \frac{\epsilon}{\sqrt{2}}g_t, g_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. $\mathbb{E}(X_t - X_s)^2 = d(t,s)^2 \ge \epsilon^2 = \mathbb{E}(Y_t - Y_s)^2 \Rightarrow \mathbb{E}\sup_{t\in\mathscr{N}} X_t \ge \mathbb{E}\sup_{t\in\mathscr{N}} Y_t = C\epsilon\sqrt{\log N}$. $\square$

- $X_t = \langle g, t\rangle, g\sim\mathcal{N}(0, I_n), d(s,t) = \|t-s\|_2, \mathbb{E}\sup_{t\in T}\langle g, t\rangle \ge C\epsilon\sqrt{\log\mathcal{N}(T,\epsilon)}$.

- $P$ a polytope in $\mathbb{R}^n$ with $N$ vertices, diameter is bounded by 1. Then for $\epsilon>0, \mathcal{N}(P,\epsilon) \le N^{c/\epsilon^2}$.

  *Proof* $x_1, x_2, \cdots, x_N$ vertices of $P$, $\mathbb{E}\sup_{t\in P}\langle g, t\rangle = \mathbb{E}\sup_{i\le N}\langle g, x_i\rangle \le C\sqrt{\log N}$. $\square$

- Gaussian width: $w(T) := \mathbb{E}\sup_{x\in T}\langle g, x\rangle, g\sim\mathcal{N}(0, I_n)$.

- Properties of Gaussian width: (a) $w(T) < \infty \Leftrightarrow T$ is bounded; (b) For every orthogonal matrix $U$ and vector $y$, $w(UT+y) = w(T)$; (c) $w(\text{conv}(T)) = w(T)$; (d) $w(T+S) = w(T) + w(S), w(aT) = |a|w(T)$; (e) $w(T) = \frac{1}{2}w(T-T) = \frac{1}{2}\mathbb{E}\sup_{x,y\in T}\langle g, x-y\rangle$; (f) $\frac{1}{\sqrt{2\pi}}\text{diam}(T) \le w(T) \le \frac{\sqrt{n}}{2}\text{diam}(T)$.

  *Proof* (e): $w(T) = \frac{1}{2}(w(T)+w(T)) = \frac{1}{2}(w(T)+w(-T)) \overset{\text{(d)}}{=} \frac{1}{2}w(T-T)$.

  (f): Lower bound. Fix $x,y\in T, x-y, y-x\in T-T, w(T) \ge \frac{1}{2}\mathbb{E}\max(\langle x-y, g\rangle, \langle y-x, g\rangle) = \frac{1}{2}\mathbb{E}|\langle x-y, g\rangle| = \sqrt{\frac{1}{2\pi}}\|x-y\|_2$.

  Upper bound. $w(T) = \frac{1}{2}\mathbb{E}\sup_{x,y\in T}\langle g, x-y\rangle \le \frac{1}{2}\mathbb{E}\sup_{x,y\in T}\|g\|_2\|x-y\|_2 = \frac{1}{2}\mathbb{E}\|g\|_2\text{diam}(T)$ and $\mathbb{E}\|g\|_2 \le (\mathbb{E}\|g\|^2)^{\frac{1}{2}} = \sqrt{n}$. $\square$

- Spherical width: $w_{\mathbb{S}}(T) = \mathbb{E}\sup_{x\in T}\langle\theta, x\rangle, \theta\sim\text{Unif}(\mathbb{S}^{n-1})$.

- $(\sqrt{n}-C)w_{\mathbb{S}}(T) \le w(T) \le (\sqrt{n}+C)w_{\mathbb{S}}(T)$.

  *Proof* $g = \|g\|_2\cdot\frac{g}{\|g\|_2} := r\cdot\theta, r\perp\theta$. $w(T) = \mathbb{E}\sup_{x\in T}\langle r\theta, x\rangle = \mathbb{E}r\mathbb{E}\sup_{x\in T}\langle\theta, x\rangle = \mathbb{E}\|g\|_2 w_{\mathbb{S}}(T)$. Ex 3.1.4 $\Rightarrow |\mathbb{E}\|g\|_2 - \sqrt{n}| \le C$. $\square$

- Squared version of the Gaussian width: $h(T)^2 = \mathbb{E}\sup_{t\in T}\langle g, t\rangle^2, g\sim\mathcal{N}(0, I_n)$.

- Stable dimension: bounded $T\subset\mathbb{R}^n, d(T) := \frac{h(T-T)^2}{\text{diam}^2(T)} \asymp \frac{w^2(T)}{\text{diam}^2(T)}$.

- $d(T) \le \dim(T)$.

  *Proof* Let $\dim(T) = k$ and $T\subset\mathbb{R}^k$. $h(T-T)^2 = \mathbb{E}\sup_{x,y\in T}\langle g, x-y\rangle^2$. $x - y = \text{diam}(T)\cdot z$ for some $z\in B_2^k$. $\therefore h(T-T)^2 \le \text{diam}^2(T)\mathbb{E}\sup_{z\in B_2^k}\langle g, z\rangle^2 = \text{diam}^2(T)\mathbb{E}\|g\|_2^2 = \text{diam}^2(T)\cdot k$. $\square$

- Stable rank: $A_{m\times n}, r(A) := \frac{\|A\|_F^2}{\|A\|^2} = d(AB_2^n) \le \text{rank}(A) = \dim(AB_2^n)$.

- Gaussian complexity: $\gamma(T) := \mathbb{E}\sup_{x\in T}|\langle g, x\rangle|, g\sim\mathcal{N}(0, I_n)$.

- $T\subset\mathbb{R}^n, \mathcal{P}$ projection onto $E\sim\text{Unif}(G_{n,m})$. $\forall m\le n$, with probability at least $1-2e^{-m}$, $\text{diam}(\mathcal{P}T) \le C[w_{\mathbb{S}}(T) + \sqrt{\frac{m}{n}}\text{diam}(T)]$.

*Proof* Step 1: Approximation. WLOG $\mathrm{diam}(T) \le 1$. $Q_{n\times n}$: choosing the first $m$ rows of $U_{n\times n} \sim \mathrm{Unif}(O(n))$. Then $\|\mathcal{P}x\|_2 \overset{\mathrm{d}}{=} \|Qx\|_2, \forall x \in \mathbb{R}^n$. $Q^T z \sim \mathrm{Unif}(\mathbb{S}^{n-1}), \forall z \in \mathbb{S}^{m-1}$. $\mathrm{diam}(\mathcal{P}T) \overset{\mathrm{d}}{=} \mathrm{diam}(QT) = \sup_{x\in T-T} \|Qx\|_2 = \sup_{x\in T-T} \max_{z\in\mathbb{S}^{m-1}} \langle Qx, z\rangle = \sup_{x\in T-T} \max_{z\in\mathbb{S}^{m-1}} \langle x, Q^T z\rangle$. Choose an $\frac{1}{2}$-net $\mathcal{N}$ of $\mathbb{S}^{m-1}$, $|\mathcal{N}| \le 5^m \Rightarrow \mathrm{diam}(QT) \le 2\max_{z\in\mathcal{N}} \sup_{x\in T-T} \langle x, Q^T z\rangle$.

Step 2: Concentration. For $z \in \mathcal{N}$, $\mathbb{E}\sup_{x\in T-T} \langle Q^T z, x\rangle = w_{\mathbb{S}}(T-T) = 2w_{\mathbb{S}}(T)$. The function $f : Q^T z \to \sup_{x\in T-T} \langle Q^T z, x\rangle$ is Lipschitz on $\mathbb{S}^{n-1} \Rightarrow \langle Q^T z, x\rangle$ is sub-gaussian $\Rightarrow \mathbb{P}(\sup_{x\in T-T} \langle Q^T z, x\rangle \ge 2w_{\mathbb{S}}(T) + t) \le 2e^{-cnt^2}$.

Step 3: Union bound. $\mathbb{P}(\max_{z\in\mathcal{N}} \sup_{x\in T-T} \langle Q^T z, x\rangle \ge 2w_{\mathbb{S}}(T) + t) \le 5^m 2e^{-cnt^2} \le 2e^{-m}$ ($t = C\sqrt{\frac{m}{n}}$ and $C$ large enough). □

- Phase transition: Equivalently write it as $\mathrm{diam}(\mathcal{P}T) \le C\max(w_{\mathbb{S}}(T), \sqrt{\frac{m}{n}}\mathrm{diam}(T))$. Set $w_{\mathbb{S}}(T) = \sqrt{\frac{m}{n}}\mathrm{diam}(T) \Rightarrow m = \frac{(\sqrt{n}w_{\mathbb{S}}(T))^2}{\mathrm{diam}^2(T)} \asymp \frac{w^2(T)}{\mathrm{diam}^2(T)} \asymp d(T)$. That is, if $m > d(T), \mathrm{diam}(\mathcal{P}T) \le C\sqrt{\frac{m}{n}}\mathrm{diam}(T)$; if $m < d(T), \mathrm{diam}(\mathcal{P}T) \le Cw_{\mathbb{S}}(T)$.

- Random matrix $G_{m\times n}$ with independent $\mathcal{N}(0,1)$ entries. $\forall m \le n$, with probability at least $1-2e^{-m}$, $\mathrm{diam}(GT) \le C[w(T) + \sqrt{m}\mathrm{diam}(T)]$.

# 8 Chaining

- Sub-gaussian increments: $\{X_t\}_{t\in T}, (T,d)$. Exist $K \ge 0$, s.t.$\|X_t - X_s\|_{\psi_2} \le Kd(t,s)$ for $t, s \in T$.

- $\{X_t\}_{t\in T}, \mathbb{E}X_t = 0, (T,d)$, sub-gaussian increments. Then $\mathbb{E}\sup_{t\in T} X_t \le CK\sum_{k\in\mathbb{Z}} 2^{-k}\sqrt{\log\mathcal{N}(T,d,2^{-k})}$.

*Proof* Step 1: Chaining setup. WLOG assume $K = 1$ and $T$ is finite. $\epsilon_k = 2^{-k}, k \in \mathbb{Z}$. Choose an $\epsilon_k$-net $T_k$ of $T$ so that $|T_k| = \mathcal{N}(T,d,\epsilon_k)$. $T$ is finite $\Rightarrow \exists$ small enough $\kappa \in \mathbb{Z}$ and large enough $K \in \mathbb{Z}$ s.t. $T_\kappa = \{t_0\}, T_k = T$. For a point $t \in T$, $\pi_k(t)$: a closest point in $T_k$. Then $d(t, \pi_k(t)) \le \epsilon_k$. $\mathbb{E}X_{t_0} = 0 \Rightarrow \mathbb{E}\sup_{t\in T} X_t = \mathbb{E}\sup_{t\in T}(X_t - X_{t_0})$. Since $X_t - X_{t_0} = \sum_{k=\kappa+1}^K (X_{\pi_k(t)} - X_{\pi_{k-1}(t)})$, $\mathbb{E}\sup_{t\in T}(X_t - X_{t_0}) \le \sum_{k=\kappa+1}^K \mathbb{E}\sup_{t\in T}(X_{\pi_k(t)} - X_{\pi_{k-1}(t)})$.

Step 2: Control the increments. $\|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\|_{\psi_2} \le d(\pi_k(t), \pi_{k-1}(t)) \le \epsilon_k + \epsilon_{k-1} \le 2\epsilon_{k-1}$. Ex 2.5.10 $\Rightarrow \mathbb{E}\sup_{t\in T}(X_{\pi_k(t)} - X_{\pi_{k-1}(t)}) \le C \cdot 2\epsilon_{k-1} \cdot \sqrt{\log(|T_k| \cdot |T_{k-1}|)} \le C \cdot 2\epsilon_{k-1}\sqrt{2\log|T_k|}$.

Step 3: Summing up the increments. $\mathbb{E}\sup_{t\in T}(X_t - X_{t_0}) \le C\sum_{k=\kappa+1}^K \epsilon_{k-1}\sqrt{\log|T_k|}$. □

- Dudley's integral inequality: $\mathbb{E}\sup_{t\in T} X_t \le CK\int_0^\infty \sqrt{\log\mathcal{N}(T,d,\epsilon)}d\epsilon$.

*Proof* $2^{-k} = 2\int_{2^{-k-1}}^{2^{-k}} d\epsilon$. $\sum_{k\in\mathbb{Z}} 2^{-k}\sqrt{\log\mathcal{N}(T,d,2^{-k})} \le 2\sum_{k\in\mathbb{Z}}\int_{2^{-k-1}}^{2^{-k}} \sqrt{\log\mathcal{N}(T,k,\epsilon)}d\epsilon$. □

- $\mathbb{E}\sup_{t,s\in T}|X_t - X_s| \le CK\int_0^\infty \sqrt{\log\mathcal{N}(T,d,\epsilon)}d\epsilon$.

- $T \subset \mathbb{R}^n, w(T) \le C\int_0^{+\infty} \sqrt{\log\mathcal{N}(T,\epsilon)}d\epsilon$.

- $T \subset \mathbb{R}^n, s(T) := \sup_{\epsilon\ge 0} \epsilon\sqrt{\log\mathcal{N}(T,\epsilon)}$. Then $cs(T) \le w(T) \le Cs(T)\log n$.

- Empirical process: $f \in \mathscr{F}, f : w \to \mathbb{R}, (\Omega, \Sigma, \mu)$. $X$ is a random point in $\Omega$. $X \sim \mu$. $X_1, \cdots, X_n$ i.i.d. copies of $X$. $X_f := \frac{1}{n}\sum_{i=1}^n f(X_i) - \mathbb{E}f(X)$ empirical process indexed by $\mathscr{F}$.

- $X_i \in [0,1], \mathscr{F} := \{f : [0,1] \to \mathbb{R}, \|f\|_{\mathrm{Lip}} \le L\}$. Then $\mathbb{E}\sup_{f\in\mathscr{F}} |\frac{1}{n}\sum_{i=1}^n f(X_i) - \mathbb{E}f(X)| \le \frac{CL}{\sqrt{n}}$.

*Proof* WLOG $\mathscr{F} = \{f : [0,1] \to [0,1], \|f\|_{\mathrm{Lip}} \le 1\}$.

Step 1: Check sub-gaussian increments. Let $Z_i := (f-g)(X_i) - \mathbb{E}(f-g)(X)$. Then $\forall f, g \in \mathscr{F}, \|X_f - X_g\|_{\psi_2} = \frac{1}{n}\|\sum_{i=1}^n Z_i\|_{\psi_2} \lesssim \frac{1}{n}(\sum_{i=1}^n \|Z_i\|_{\psi_2}^2)^{\frac{1}{2}}$. $\|Z_i\|_{\psi_2} \lesssim \|(f-g)(X_i)\|_{\psi_2} \lesssim \|f-g\|_\infty$.

Step 2: Apply Dudley's inequality. $\mathbb{E}\sup_{f\in\mathscr{F}}|X_f| \le \frac{1}{\sqrt{n}}\int_0^1 \sqrt{\log\mathcal{N}(\mathscr{F}, \|\cdot\|_\infty, \epsilon)}d\epsilon \overset{\text{Ex 8.2.6}}{\le} \frac{1}{\sqrt{n}}\int_0^1 \sqrt{\frac{C}{\epsilon}\log\frac{C}{\epsilon}}d\epsilon \lesssim \frac{1}{\sqrt{n}}$. □

- VC dimension: $\mathscr{F} = \{f : \Omega \to \{0,1\}\}$. Shattered $\Lambda \subset \Omega$, any $g : \Lambda \to \{0,1\}$ can be obtained by $f \in \mathscr{F}$ restricted on $\Lambda$. $\mathrm{VC}(\mathscr{F}) = \max_\Lambda |\Lambda|$.

- $|\mathscr{F}| \le |\{\Lambda \subset \Omega, \Lambda \text{ is shattered by } \mathscr{F}\}|$.

*Proof* $|\Omega| = 1$, trivial. If $|\Omega| = n+1, \Omega = \Omega_0 \cup \{x_0\}, |\Omega_0| = n$. $\mathscr{F}_0 = \{f \in \mathscr{F} : f(x_0) = 0\}, \mathscr{F}_1 = \{f \in \mathscr{F} : f(x_0) = 1\}, S(\mathscr{F}) = |\{\Lambda \subset \Omega : \Lambda \text{ is shattered by } \mathscr{F}\}|$. Then $S(\mathscr{F}_0) \ge |\mathscr{F}_0|, S(\mathscr{F}_1) \ge |\mathscr{F}_1|, |\mathscr{F}| = |\mathscr{F}_0| + |\mathscr{F}_1|$.

(1) $\Lambda$ is shattered by $\mathscr{F}_0(\mathscr{F}_1)$ but not by $\mathscr{F}_1(\mathscr{F}_0)$. Then $\Lambda_0 \in \mathscr{F}_0(\mathscr{F}_1), \notin \mathscr{F}_1(\mathscr{F}_0)$.

(2) $\Lambda$ is shattered by $\mathscr{F}_0$ and $\mathscr{F}_1$. Replace it with $\Lambda \cup \{x_0\}$. □

- $|\Omega| = n$. Then $|\mathscr{F}| \le \sum_{k=0}^d C_n^k \le (\frac{en}{d})^d$ where $d = \mathrm{VC}(\mathscr{F})$.

- Dimension reduction: $|\mathscr{F}| = N$, a class of boolean functions. Assume $\|f-g\|_{L^2(\mu)} > \epsilon$ for $f, g \in \mathscr{F}$. Then there exists $n \le C\epsilon^{-4}\log N$ and $\Omega_n \subset \Omega, |\Omega_n| = n$ s.t. $\mu_n$ is uniform probability mass on $\Omega_n$, $\|f-g\|_{L^2(\mu_n)} \ge \frac{\epsilon}{2}$ for all $f, g \in \mathscr{F}$.

  *Proof* $X_1, X_2, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mu$. Denote $h := (f-g)^2, \Omega_n = \{X_1, \cdots, X_n\}$. $\|f-g\|^2_{L^2(\mu_n)} - \|f-g\|^2_{L^2(\mu)} = \frac{1}{n}\sum_{i=1}^n (h(X_i) - \mathbb{E}h(X))$. $\|h(X_i) - \mathbb{E}h(X)\|_{\psi_2} \lesssim \|h(X_i)\|_{\psi_2} \lesssim \|h(X_i)\|_\infty \le 1 \Rightarrow \mathbb{P}(\|f-g\|^2_{L^2(\mu_0)} - \|f-g\|^2_{L^2(\mu)} > \frac{\epsilon^2}{4}) \le 2e^{-cn\epsilon^4}$. Therefore, $\|f-g\|^2_{L^2(\mu_n)} \ge \frac{3}{4}\epsilon^2$ hold for all $f, g \in \mathscr{F}$ with prob $\ge 1 - 2N^2 e^{-cn\epsilon^4}$. $n = C\epsilon^{-4}\log N$ sufficiently large. □

- $\forall \epsilon \in (0,1), \mathcal{N}(\mathscr{F}, L^2(\mu), \epsilon) \le (\frac{2}{\epsilon})^{Cd}$.

  *Proof* Choose $N \ge \mathcal{N}(\mathscr{F}, L^2(\mu), \epsilon)$ $\epsilon$-separated functions in $\mathscr{F}$. $|\Omega_n| = n \le C\epsilon^{-4}\log N$ s.t. $\mathscr{F}|_{\Omega_n} := \mathscr{F}_n$ is still $\frac{\epsilon}{2}$-separated in $L^2(\mu_n)$. $N \le (\frac{en}{d_n})^{d_n} \le (\frac{C\epsilon^{-4}\log N}{d_n})^{d_n}$ where $d_n = \mathrm{VC}(\mathscr{F}_n) \Rightarrow N \le (C\epsilon^{-4})^{2d_n}$. □

- $\mathrm{VC}(\mathscr{F}) \ge 1$. Let $X_1, \cdots, X_n \in \Omega \sim \mu$. Then $\mathbb{E}\sup_{f\in\mathscr{F}} |\frac{1}{n}\sum_{i=1}^n f(X_i) - \mathbb{E}f(X)| \le C\sqrt{\frac{\mathrm{VC}(\mathscr{F})}{n}}$.

  *Proof* Ex 8.3.24 $\Rightarrow \mathbb{E}\sup_{f\in\mathscr{F}} |\frac{1}{n}\sum_{i=1}^n f(X_i) - \mathbb{E}f(X)| \le \frac{2}{\sqrt{n}}\mathbb{E}\sup_{f\in\mathscr{F}}|Z_f|$ where $Z_f = \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i f(X_i)$ and $\epsilon_i \overset{\text{i.i.d.}}{\sim}$ symmetric Bernoulli. Conditioned on $(X_i)$, $\|Z_f - Z_g\|_{\psi_2} = \frac{1}{\sqrt{n}}\|\sum_{i=1}^n \epsilon_i (f-g)(X_i)\|_{\psi_2} \lesssim [\frac{1}{n}\sum_{i=1}^n (f-g)^2(X_i)]^{\frac{1}{2}} = \|f-g\|_{L^2(\mu_n)}$. Applying Dudley's inequality, $\frac{2}{\sqrt{n}}\mathbb{E}\sup_{f\in\mathscr{F}} Z_f \lesssim \frac{1}{\sqrt{n}}\mathbb{E}_X\int_0^1 \sqrt{\log\mathcal{N}(\mathscr{F}, L^2(\mu_n), \epsilon)}d\epsilon \lesssim \frac{1}{\sqrt{n}}\mathbb{E}_X\int_0^1 \sqrt{\mathrm{VC}(\mathscr{F})\log\frac{2}{\epsilon}}d\epsilon \lesssim \sqrt{\frac{\mathrm{VC}(\mathscr{F})}{n}}$. □

- $\mathcal{R}(f_n^*) - \mathcal{R}(f^*) \le 2\sup_{f\in\mathscr{F}}|\mathcal{R}_n(f) - \mathcal{R}(f)|$.

  *Proof* $\epsilon = \sup_{f\in\mathscr{F}}|\mathcal{R}_n(f) - \mathcal{R}(f)|$. $\mathcal{R}(f_n^*) \le \mathcal{R}_n(f_n^*) + \epsilon \le \mathcal{R}_n(f^*) + \epsilon \le \mathcal{R}(f^*) + 2\epsilon$. □

- For two-class classification, $\mathrm{VC}(\mathscr{F}) \ge 1$. Then $\mathbb{E}\mathcal{R}(f_n^*) \le \mathcal{R}(f^*) + C\sqrt{\frac{\mathrm{VC}(\mathscr{F})}{n}}$ where $\mathcal{R}(\cdot)$ is the MSE risk.

  *Proof* Only to show $\mathbb{E}\sup_{f\in\mathscr{F}}|\mathcal{R}_n(f) - \mathcal{R}(f)| \lesssim \sqrt{\frac{\mathrm{VC}(\mathscr{F})}{n}}$. LHS $= \frac{1}{n}\sum_{i=1}^n [l(x_i) - \mathbb{E}l(x)]$ where $l = (f-T)^2$ is Boolean. Let $\mathscr{L} = \{(f-T)^2 : f \in \mathscr{F}\}$. Dudley's inequality $\Rightarrow$ LHS $\lesssim \frac{1}{\sqrt{n}}\mathbb{E}_X\int_0^1\sqrt{\log\mathcal{N}(\mathscr{L}, L^2(\mu_n), \epsilon)}d\epsilon \overset{\text{Ex 8.4.6}}{\le} \frac{1}{\sqrt{n}}\mathbb{E}_X\int_0^1\sqrt{\log\mathcal{N}(\mathscr{F}, L^2(\mu_n), \epsilon)}d\epsilon$. □

- Talagrand's $\gamma_2$ functional: $(T, d)$ metric space, $(T_k)_{k=1}^\infty (T_k \subset T)$ admissiable sequence iff $|T_0| = 1, |T_k| \le 2^{2^k}, \forall k$. $\gamma_2(T, d) := \inf_{(T_k)}\sup_{t\in T}\sum_{k=0}^\infty 2^{k/2}d(t, T_k)$.

- $\{X_t\}_{t\in T}$ mean zero sub-gaussian increments. Then $\mathbb{E}\sup_{t\in T} X_t \le CK\gamma_2(T, d)$.

  *Proof* Step 1: Chaining setup. WLOG $K = 1, |T| < \infty$. Let $(T_k)$ be an admissiable sequence, $T_0 = \{t_0\}, t_0 = \pi_0(t) \to \pi_1(t) \to \cdots \to \pi_k(t) = t, d(t, \pi_k(t)) = d(t, T_k)$. Then $X_t - X_{t_0} = \sum_{k=1}^K (X_{\pi_k(t)} - X_{\pi_{k-1}}(t))$.

  Step 2: Controling the increments. Fix $k$ and $t$, for $u \ge 0$, $\mathbb{P}(|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}| \le Cu2^{k/2}d(\pi_k(t), \pi_{k-1}(t))) \ge 1 - 2e^{-8u^2 2^k}$. Unfix $t$ and $k$, $|T_k||T_{k-1}| \le |T_k|^2 \le 2^{2^{k+1}}$. Let $A = \{|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}| \le Cu2^{k/2}d(\pi_k(t), \pi_{k-1}(t))$ for $\forall k, t\}$. Then $\mathbb{P}(A) \ge 1 - \sum_{k=1}^\infty 2^{2^{k+1}} 2e^{-8u^2 2^k} \ge 1 - 2e^{-u^2}$ if $u > C'$.

  Step 3: Summing up the increments. In event $A$, $\sup_{t\in T}|X_t - X_{t_0}| \le C_1 u\gamma_2(T, d) \Rightarrow \|\sup_{t\in T}|X_t - X_{t_0}|\|_{\psi_2} \le C_2\gamma_2(T, d)$. □

- $\{X_t\}_{t\in T}$ mean zero Gaussian process on $T$, $d(t, s) = \|X_t - X_s\|_{L^2}$. Then $c\gamma_2(T, d) \le \mathbb{E}\sup_{t\in T} X_t \le C\gamma_2(T, d)$.

- Talagrand's comparison inequality: $\{X_t\}_{t\in T}$ mean zero, $\{Y_t\}_{t\in T}$ mean zero Gaussian, $\forall t, s \in T, \|X_t - X_s\|_{\psi_2} \le K\|Y_t - Y_s\|_{L^2} \Rightarrow \mathbb{E}\sup_{t\in T} X_t \le CK\mathbb{E}\sup_{t\in T} Y_t$.

- $A_{m\times n}, A_{ij}$ independent mean zero sub-gaussian, $T \subset \mathbb{R}^n, S \subset \mathbb{R}^m$. Then $\mathbb{E}\sup_{x\in T, y\in S}\langle Ax, y\rangle \le CK[w(T)\mathrm{rad}(S) + w(S)\mathrm{rad}(T)]$ where $K = \max_{ij}\|A_{ij}\|_{\psi_2}, \mathrm{rad}(T) := \sup_{x\in T}\|x\|_2$.

  *Proof* WLOG $K = 1$. $X_{uv} := \langle Au, v\rangle, u \in T, v \in S$. Then $\|X_{uv} - X_{wz}\|_{\psi_2} = \|\sum_{i,j} A_{ij}(u_i v_j - w_i z_j)\|_{\psi_2} \le (\sum_{i,j}\|A_{ij}(u_i v_j - w_i z_j)\|^2_{\psi_2})^{\frac{1}{2}} \le (\sum_{i,j}\|u_i v_j - w_i z_j\|_2^2)^{\frac{1}{2}} = \|uv^T - wz^T\|_F \le \|(u-w)v^T\|_F + \|w(v-z)^T\|_F = \|u-w\|_2\|v\|_2 + \|v-z\|_2\|w\|_2 \le \|u-w\|_2\mathrm{rad}(S) + \|v-z\|_2\mathrm{rad}(T)$. Let $Y_{uv} = \langle g, u\rangle\mathrm{rad}(S) + \langle h, v\rangle\mathrm{rad}(T)$ where $g \sim \mathcal{N}(0, I_n), h \sim \mathcal{N}(0, I_m)$. $\|Y_{uv} - Y_{wz}\|_2^2 = \|u-w\|_2^2\mathrm{rad}(S)^2 + \|v-z\|_2^2\mathrm{rad}(T)^2 \Rightarrow \|X_{uv} - X_{wz}\|_{\psi_2} \lesssim \|Y_{uv} - Y_{wz}\|_2$. Applying the comparison inequality, $\mathbb{E}\sup_{u\in T, v\in S} X_{uv} \lesssim \mathbb{E}\sup_{u\in T, v\in S} Y_{uv} = \mathbb{E}\sup_{u\in T}\langle g, u\rangle\mathrm{rad}(S) + \mathbb{E}\sup_{v\in S}\langle h, s\rangle\mathrm{rad}(T) = w(T)\mathrm{rad}(S) + w(S)\mathrm{rad}(T)$. □