Stochastic Processes

Lectured by Weijun Xu LaTeXed by Chengxin Gong

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目录

1 Review of Martingales

- $(X_n)_{n\geq 0}$ is L^2-bounded martingale $\Rightarrow X_n$ converges in L^2.
- $(X_n)_{n\geq 0}$ is L^1-bounded martingale $\Rightarrow X_n$ converges a.s.
- (1) + (2): If $(X_n)_{n\geq 0}$ is L^p-bounded martingale for $p>1$, then X_n converges in $L^{p'}$ for $p' \in [1,p)$.
- Statement is false when $p=1$. Example: $\Omega = [0,1), \mathscr{F}_n = \sigma\{[\frac{i}{2^n}, \frac{i+1}{2^n})\}_{i=0}^{2^n-1}, X_n(\omega) := \begin{cases} 2^n & \omega \in [0, \frac{1}{2^n}) \\ 0 & \text{otherwise} \end{cases}$.
- Let $p>1$ and $(X_n)_{n\geq 0}$ be L^p bounded martingale w.r.t. $\mathscr{F}_n$. Then $\exists X \in L^p(\Omega, \mathscr{F}_\infty, P)$ s.t. $X_n \to X$ in L^p and a.s. and $X_n = \mathbb{E}(X|\mathscr{F}_n)$.
- Doob's maximal inequality: Let $p>1, \exists C = C_p$ s.t. $\forall$ martingale $(X_n)_{n\geq 0}$, we have $\mathbb{E}|X_n^*|^p \leq C_p \mathbb{E}|X_n|^p$ where $|X_n^*| = \sup_{0\leq k\leq n} \sup |X_k|$.
- Let $(Z_n)_{n\geq 0}$ be a nonnegative sub-martingale and $Z_n^* = \sup_{0\leq k\leq n} Z_k$, then $P(Z_n^* > \lambda) \leq \frac{1}{\lambda}\mathbb{E}(Z_n 1_{\{Z_n^*>\lambda\}}) \leq \frac{1}{\lambda}\mathbb{E}Z_n$. Corollary: $P(Z_n^* > \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p 1_{\{Z_n^*>\lambda\}}) \leq \frac{1}{\lambda^p}\mathbb{E}(Z_n^p)$.
- If $(X_n)_{n\geq 0}$ is a martingale with $\sup_n \mathbb{E}(|X_n|\log(1+|X_n|)) < +\infty$, then X_n converges in L^1.
- Two probability measures P and Q on $(\Omega, \mathscr{F})$, $Q << P$ on $\mathscr{F}_n$ for every n and $M_n = \frac{dQ|_{\mathscr{F}_n}}{dP|_{\mathscr{F}_n}}$. $(M_n)_{n\geq 0}$ is a P-martingale w.r.t. $(\mathscr{F}_n)_{n\geq 0}$. $Q << P$ on $\mathscr{F}_\infty$ if and only if $M_n \to M$ in L^1. $Q(A) = \int_A M dP + Q(A \cap \{M = +\infty\})$.
- Statement is false if $M_n \not\to M$ in L^1. Example: $\Omega = \{\omega = (\omega_1, \cdots, \omega_n, \cdots) \in \{\pm 1\}^{\mathbb{N}}\}, X_n(\omega) = \omega_n$. X_n's are i.i.d. under P and Q, but $P(X_n = 1) = \frac{1}{2}, P(X_n = -1) = \frac{1}{2}, Q(X_n = 1) = \frac{1}{3}, Q(X_n = -1) = \frac{2}{3}$. $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$. $P(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k = 0) = 1, Q(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n X_k = -\frac{1}{3}) = 1$.
- Monotone class theorem for functions: Suppose $\mathcal{A}$ us a π-system and $\mathcal{H}$ be a class of functions from Ω to $\mathbb{R}$ s.t. (1) $1_A \in \mathcal{H}$ for every $A \in \mathscr{A}$, (2) if $f, g \in \mathcal{H}$ then $af + bg \in \mathcal{H}$, (3) if $f_n \in \mathcal{H}$ and $f_n \uparrow f$ then $f \in \mathcal{H}$. Then all nonnegative $\sigma(\mathcal{A})$-measurable functions are in $\mathcal{H}$.
- Let $(Y_n)_{n\geq 0}$ be i.i.d., nonnegative r.v.'s with $\mathbb{E}Y_k = 1$. Then $M_n = \prod_{k=1}^n Y_k$ converges in L^1 iff $Y_n \equiv 1$. Otherwise $M_n \to 0$ a.s.
- Kakutani's theorem: $M_n = \prod_{k=1}^n Y_k$, $Y_k \geq 0$ are independent, $\mathbb{E}Y_k = 1$, $\lambda_k = \mathbb{E}\sqrt{Y_k}$. (1) If $\prod_k \lambda_k > 0$, then $M_n \to M$ in L^1; (2) If $\prod_k \lambda_k = 0$, then $M_n \to 0$ a.s.

2 Markov Chains

- Let $(X_n)_{n\geq 0}$ be a time-homogeneous Markov chain on a discrete space S. P^x : probability measure of $(X_n)_{n\geq 0}$ conditioned on $X_0 = x$. $P(X_{n+1} \in A|\mathscr{F}_n) = P^{X_n}(X_1 \in A) = P(X_1 \in A|X_0 = X_n)$. $\mathbb{E}^x$: expectation under P^x. $P^x(X_1 = y) = p(x,y)$.
- For every $f : S \to \mathbb{R}$ bounded, define $(Pf)(x) = \sum_{y\in S} p(x,y)f(y) = \mathbb{E}^x(f(X_1))$, $(Lf)(x) = \sum_{y\in S} p(x,y)f(y) - f(x)$. $L = P - \text{id}$, the generator.
- Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with generator L. Then for every bounded $f : S \to \mathbb{R}$, $M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1}(Lf)(X_k)$ is a martingale. Conversely, let $(X_n)_{n\geq 0}$ be a process and L be an operator on $\mathcal{B}(\mathcal{S})$ s.t. M_n^g is a martingale for every f, then $(X_n)_{n\geq 0}$ is a Markov chain with generator L.

- Given operator L on $\mathcal{B}(\mathcal{S})$, we say $f : S \to \mathbb{R}$ is (1) harmonic for L if $Lf = 0$; (2) sub-harmonic for L if $Lf \geq 0$; (3) super-harmonic for L if $Lf \leq 0$.
- Let f be the generator of a Markov chain $(X_n)_{n\geq 0}$. Then f is (sub-/super-)harmonic $\Leftrightarrow$ $f(X_n)_{n\geq 0}$ is a (sub-/super-) martingale.
- f is (sub-/super-)harmonic on $D \subset S$ if $Lf \leq / \geq / = 0$ on D. Let $\tau = \inf\{k \geq 0 : X_k \in D^c\}$, then $(f(X_{n\wedge\tau}))_{n\geq 0}$ is a (sub-/super)martingale.
- Maximum principle: Let $(X_n)_{n\geq 0}$ be a Markov chain and $D \subset S$ s.t. the stopping time $\tau = \inf\{k \geq 0, X_k \in D^c\}$ is a.s. finite. If f is bounded and sub-harmonic on D, then $\sup_{x\in D} f(x) \leq \sup_{x\in D^c} f(x)$.
- $A \subset S, \tau_A = \sup\{k \geq 0 : X_k \in A\}, u(x) = P^x(\tau_A < +\infty)$. Then $u(x) = 1$ for $x \in A$, $u(x) = \sum_{y\in S} p(x,y)u(y) = (Pu)(x)$, $\Rightarrow \begin{cases} Lu = 0 & \text{on } A^c \\ u = 1 & \text{on } A \end{cases}$.
- Any nonnegative solution v to $\begin{cases} Lv = 0 & \text{on } A^c \\ v = 1 & \text{on } A \end{cases}$ satisfies $v \geq u$. Furthermore, if $u \equiv 1$, then $\exists 1$ bounded solution to $\begin{cases} Lv = 0 & \text{on } A^c \\ v = f & \text{on } A \end{cases}$ with $v(x) = \mathbb{E}^x(f(X_{\tau_A}))$.
- Doob's h-transform: Let h be positive, harmonic with $h(x_0) = 1$ for some $x_0 \in S \Rightarrow (h(X_n))_{n\geq 0}$ is a martingale with $\mathbb{E}^{P^{x_0}}(h(X_n)) = 1$. Then $\exists 1$ measure Q^h on $\mathscr{F}_\infty$ s.t. $\frac{dQ^h}{dP^{x_0}|_{\mathscr{F}_n}} = h(X_n), \forall n \geq 0$. $Q^h(X_0 = x_0) = 1$, $(X_n)_{n\geq 0}$ never visits the set $D = \{x : h(x) = 0\}$. Under Q^h, $(X_n)_{n\geq 0}$ is again a Markov chain on $S\backslash D$ with transition probability $q(x,y) = \frac{p(x,y)h(y)}{h(x)}$ (or equivalently, $(L^h f)(x) = \frac{1}{h(x)}(L(hf))(x)$).