

# Advanced Theory of Probability

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## 1 Review of Measure Theory

- Fatou's lemma: If  $f_n \geq 0$  then  $\liminf_{n \rightarrow \infty} \int f_n d\mu \geq \int \liminf_{n \rightarrow \infty} f_n d\mu$ .
- Monotone convergence theorem: If  $f_n \geq 0$  and  $f_n \uparrow f$  then  $\int f_n d\mu \uparrow \int f d\mu$ .
- Dominated convergence theorem: If  $f_n \rightarrow f$  a.e.,  $|f_n| \leq g$  for all  $n$ , and  $g$  is integrable, then  $\int f_n d\mu \rightarrow \int f d\mu$ .
- Suppose  $X_n \rightarrow X$  a.s. Let  $g(x), h(x)$  be continuous functions with (i)  $g(x) \geq 0$  and  $g(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ ; (ii)  $|h(x)|/g(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ ; (iii)  $\mathbb{E}g(X_n) \leq K < \infty$  for all  $n$ . Then  $\mathbb{E}h(X_n) \rightarrow \mathbb{E}h(X)$ .
- Fubini's theorem: If  $f \geq 0$  or  $\int |f| d\mu < \infty$ ,  $\int_X \int_Y f(x, y) \mu_2(dy) \mu_1(dx) = \int_{X \times Y} f d\mu = \int_Y \int_X f(x, y) \mu_1(dx) \mu_2(dy)$ .

## 2 Laws of Large Numbers

### 2.1 Independence

- Two events  $A$  and  $B$  are independent if  $P(A \cap B) = P(A)P(B)$ . Two random variables  $X$  and  $Y$  are independent if for all  $C, D \in \mathbb{R}$ ,  $P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$ . Two  $\sigma$ -fields  $\mathcal{F}$  and  $\mathcal{G}$  are independent if for all  $A \in \mathcal{F}$  and  $B \in \mathcal{G}$  the events  $A$  and  $B$  are independent.
- $\sigma$ -fields  $\mathcal{F}_1, \dots, \mathcal{F}_n$  are independent if whenever  $A_i \in \mathcal{F}_i$  for  $i = 1, \dots, n$ , we have  $P(\cap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$ . Random variables  $X_1, \dots, X_n$  are independent if whenever  $B_i \in \mathbb{R}$  for  $i = 1, \dots, n$  we have  $P(\cap_{i=1}^n \{X_i \in B_i\}) = \prod_{i=1}^n P(X_i \in B_i)$ . Sets  $A_1, \dots, A_n$  are independent if whenever  $I \subset \{1, \dots, n\}$  we have  $P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$ .
- A sequence of events  $A_1, \dots, A_n$  with  $P(A_i \cap A_j) = P(A_i)P(A_j)$  for all  $i \neq j$  is called pairwise independent.
- $\pi$ - $\lambda$  theorem: If  $\mathcal{P}$  is a  $\pi$ -system and  $\mathcal{L}$  is a  $\lambda$ -system that contains  $\mathcal{P}$  then  $\sigma(\mathcal{P}) \subset \mathcal{L}$ .
- Suppose  $\mathcal{A}_1, \dots, \mathcal{A}_n$  are independent and each  $\mathcal{A}_i$  is a  $\pi$ -system. Then  $\sigma(\mathcal{A}_1), \dots, \sigma(\mathcal{A}_n)$  are independent.
- Suppose  $\mathcal{F}_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m(i)$  are independent and let  $\mathcal{G}_i = \sigma(\cup_j \mathcal{F}_{i,j})$ . Then  $\mathcal{G}_1, \dots, \mathcal{G}_n$  are independent.
- If for  $1 \leq i \leq n, 1 \leq j \leq m(i)$ ,  $X_{i,j}$  are independent and  $f_i : \mathbb{R}^{m(i)} \rightarrow \mathbb{R}$  are measurable then  $f_i(X_{i,1}, \dots, X_{i,m(i)})$  are independent.
- If  $X_1, \dots, X_n$  are independent and have (a)  $X_i \geq 0$  for all  $i$ , or (b)  $\mathbb{E}|X_i| < \infty$  for all  $i$  then  $\mathbb{E}(\prod_{i=1}^n X_i) = \prod_{i=1}^n \mathbb{E}X_i$ .
- If  $X$  and  $Y$  are independent,  $F(x) = P(X \leq x)$ , and  $G(y) = P(Y \leq y)$ , then  $P(X + Y \geq z) = \int F(z - y) dG(y)$ .

### 2.2 Weak Laws of Large Numbers

- $L^2$  weak law: Let  $X_1, X_2, \dots$  be uncorrelated random variables with  $\mathbb{E}X_i = \mu$  and  $\text{var}(X_i) \leq C < \infty$ . If  $S_n = X_1 + \dots + X_n$ , then as  $n \rightarrow \infty$ ,  $S_n/n \rightarrow \mu$  in  $L^2$  and in probability.
- Let  $\mu_n = \mathbb{E}[S_n], \sigma_n^2 = \text{var}(S_n)$ . If  $\sigma_n^2/b_n^2 \rightarrow 0$  then  $\frac{S_n - \mu_n}{b_n} \rightarrow 0$  in probability.
- Truncation: To truncate a random variable  $X$  at level  $M$  means to consider  $\bar{X}_M = X1_{\{|X| \leq M\}}$ .
- For each  $n$ , let  $X_{n,k}, 1 \leq k \leq n$  be independent. Let  $0 < b_n \rightarrow \infty$  and  $\bar{X}_{n,k} = X_{n,k}1_{\{|X_{n,k}| \leq b_n\}}$ . Suppose that as  $n \rightarrow \infty$  (1)  $\sum_{k=1}^n P(|X_{n,k}| > b_n) \rightarrow 0$ ; (2)  $b_n^{-2} \sum_{k=1}^n \text{var}(\bar{X}_{n,k}) \rightarrow 0$ . If we let  $S_n = \sum_{k=1}^n X_{n,k}$  and  $a_n = \sum_{k=1}^n \mathbb{E}[\bar{X}_{n,k}]$ , then  $\frac{S_n - a_n}{b_n} \rightarrow 0$  in probability.

- Let  $X_1, X_2, \dots$  be i.i.d. with  $xP(|X_1| > x) \rightarrow 0$  as  $x \rightarrow \infty$ . Let  $S_n = X_1 + \dots + X_n$  and let  $\mu_n = \mathbb{E}[X_1 1_{\{|X_1| \leq n\}}]$ . Then  $S_n/n - \mu_n \rightarrow 0$  in probability.
- If  $Y \geq 0$  and  $p > 0$  then  $\mathbb{E}[Y^p] = \int_0^\infty py^{p-1}P(Y > y)dy$ .
- Let  $\{X_i\}_{i=1}^\infty$  be i.i.d. with  $\mathbb{E}[|X_i|] < \infty$ . Let  $S_n = X_1 + \dots + X_n$  and let  $\mu = \mathbb{E}[X_1]$ . Then  $S_n/n \rightarrow \mu$  in probability.
- The distribution of  $X$  is infinitely divisible iff for any  $n \in \mathbb{N}$ , there exists i.i.d.  $Y_i$ 's such that  $X = \sum_{i=1}^n Y_i$ .
- The distribution of  $X$  is stable if for all  $a, b > 0$ , and  $X_1, X_2$  i.i.d. copies of  $X$ ,  $aX_1 + bX_2 \stackrel{d}{=} cX + d$  for some  $c > 0$ .

### 2.3 Borel-Cantelli Lemmas

- If  $A_n$  is a sequence of subsets of  $\Omega$ , then we write

$$\begin{aligned}\limsup A_n &= \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m = \{\omega : \omega \text{ in infinitely many } A_i \text{'s}\} \\ \liminf A_n &= \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m = \{\omega : \omega \text{ in all but finitely many } A_i \text{'s}\}\end{aligned}$$

- $P(\limsup A_n) \geq \limsup P(A_n)$ ,  $P(\liminf A_n) \leq \liminf P(A_n)$ .
- Borel-Cantelli lemma: If  $\sum_i P(A_i) < \infty$ , then  $P(A_n \text{ i.o.}) = 0$ .
- Let  $y_n$  be a sequence of elements of a topological space. If every subsequence  $y_{n(m)}$  has a further subsubsequence  $y_{n(m_k)}$  that converges to  $y$ , then  $y_n \rightarrow y$ .
- $X_n \rightarrow X$  in probability iff for every subsequence  $X_{n(m)}$  there is a further subsubsequence  $X_{n(m_k)}$  that converges a.s. to  $X$ .
- If  $f$  is continuous and  $X_n \rightarrow X$  in probability then  $f(X_n) \rightarrow f(X)$  in probability. If in addition  $f$  is bounded then  $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X)]$ .
- Let  $X_1, X_2, \dots$  be i.i.d. with  $\mathbb{E}[X_i] = \mu$  and  $\mathbb{E}[X_i^4] < \infty$ . Then  $S_n/n \rightarrow \mu$  a.s.
- For events  $A_n, n = 1, 2, \dots$ , independent such that  $\sum_{n=1}^\infty P(A_n) = \infty$ , then  $P(A_n \text{ i.o.}) = 1$ .
- If  $X_1, X_2, \dots$  are i.i.d. r.v.'s with  $\mathbb{E}[X_i] = \infty$ , then  $P(|X_n| \geq n \text{ i.o.}) = 1$ . Let  $C = \{\lim S_n/n \text{ exists \& is finite}\}$ . Then  $P(C) = 0$ .
- If  $A_1, A_2, \dots$  are pairwise independent and  $\sum_{n=1}^\infty P(A_n) = \infty$  then  $\sum_{i=1}^n 1_{A_i} / \sum_{i=1}^n P(A_i) \rightarrow 1$  a.s. as  $n \rightarrow \infty$ .
- For a sequence of increasing events  $A_n$ ,  $P(A_n \text{ i.o.}) = 1$  iff  $\sum_n P(A_n | A_{n-1}^c) = \infty$ .

### 2.4 Strong Law of Large Numbers

- Strong law of large numbers: Let  $X_1, X_2, \dots$  be pairwise independent identically distributed random variables with  $\mathbb{E}[X_i] < \infty$ . Let  $\mathbb{E}X_i = \mu$  and  $S_n = X_1 + \dots + X_n$ . Then  $S_n/n \rightarrow \mu$  a.s. as  $n \rightarrow \infty$ .
- Let  $X_1, X_2, \dots$  be i.i.d. with  $\mathbb{E}[X^+] = \infty$  and  $\mathbb{E}[X^-] < \infty$ , then  $S_n/n \rightarrow \infty$  a.s.
- Let  $X_1, X_2, \dots$  be i.i.d. with  $0 < X_i < \infty$ , write  $T_n = X_1 + \dots + X_n$  and let  $N_t = \sup\{n : T_n \leq t\}$ . If  $\mathbb{E}[X_1] = \mu \leq \infty$ , then as  $t \rightarrow \infty$ ,  $N_t/t \rightarrow 1/\mu$ , a.s.

- If  $X_n \rightarrow X_\infty$  a.s. and  $N(n) \rightarrow \infty$  a.s. then  $X_{N(n)} \rightarrow X_\infty$  a.s. But the analogous result for convergence in probability is false!
- Empirical distribution functions: Let  $X_1, X_2, \dots$  be i.i.d. with distribution  $F$  and let  $F_n(x) = \frac{\sum_{i=1}^n 1_{X_i \leq x}}{n}$ . As  $n \rightarrow \infty$ ,  $\sup_x |F_n(x) - F(x)| \rightarrow 0$  a.s.
- Uniform law of large numbers: Suppose  $f(x, \theta)$  is continuous in  $\theta \in \Theta$  for some compact  $\Theta$ . Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables. If  $f$  is continuous at  $\theta$  for a.s. all  $x \in \mathbb{R}$  and measurable of  $x$  at each  $\theta$  and there exists some function  $d(x)$  such that  $\mathbb{E}[d(X_i)] < \infty$  and for all  $\theta \in \Theta$ ,  $|f(x, \theta)| \leq d(x)$ . Then  $\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n f(X_i, \theta) - \mathbb{E}[f(X_1, \theta)]| \xrightarrow{\text{a.s.}} 0$ .

## 2.5 Convergence of Random Series

- Let  $X_1, X_2, \dots, X_n, \dots$  be a sequence of random variables. Define  $\mathcal{F}'_n = \sigma(X_n, X_{n+1}, \dots)$  as the information of the future after time  $n$ . Let  $\mathcal{I} = \cap_{n=1}^\infty \mathcal{F}'_n$  be the tail  $\sigma$ -field, i.e., the information in the remote future. Intuitively,  $A \in \mathcal{I}$  if and only if changing a finite number of values does not affect the occurrence of the event.
- Kolmogorov's 0-1 law: If  $X_1, X_2, \dots, X_n, \dots$  are independent and  $A \in \mathcal{I}$ , then  $P(A) = 0$  or  $1$ .
- A finite permutation of  $\mathbb{N}$  is a map from  $\mathbb{N}$  onto  $\mathbb{N}$  such that there is a finite  $I$  with  $\pi(i) = i$  for all  $i \geq I$ . For  $S^\mathbb{N}$ , associated with its natural product sigma field  $\mathcal{F}^\mathbb{N}$ , and any  $\omega = (\omega_1, \omega_2, \dots)$ , let  $\pi(\omega) = (\omega_{\pi(1)}, \omega_{\pi(2)}, \dots)$ . An event  $A \in \mathcal{F}^\mathbb{N}$  is permutable if  $\pi^{-1}(A) = A$  for any finite permutation  $\pi$ . All permutable events form the exchangeable  $\sigma$ -field, denoted by  $\mathcal{E}$ . All events in the tail  $\sigma$ -field  $\mathcal{I}$  are permutable.
- Hewitt-Savage 0-1 law: If  $X_1, X_2, \dots$ , are i.i.d. and  $B \in \mathcal{E}(\mathbb{R}^\mathbb{N})$ . Denote  $X = (X_1, X_2, \dots)$ . Then  $P(X \in B) = 0$  or  $1$ .
- Kolmogorov's maximal inequality: Suppose  $X_1, X_2, \dots, X_n$  are independent with  $\mathbb{E}[X_i] = 0$ ,  $\text{var}(X_i) < \infty$ . Let  $S_n = X_1 + \dots + X_n$ , then  $P(\max_{k \leq n} |S_k| \geq x) \leq \frac{\text{var}(S_n)}{x^2}$ .
- We call a sequence of r.v.'s  $S_1, S_2, \dots$  a martingale if (i) there is a sequence of  $\sigma$ -algebras  $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$  and  $S_i \in \mathcal{F}_i$  for all  $i$ ; (ii)  $S_i$ 's are integrable; (iii) For each  $k$ ,  $\mathbb{E}[S_{k+1} | \mathcal{F}_k] = S_k$ . If the "=" in (iii) is replaced by  $\geq$  (resp.  $\leq$ ), then we say that this sequence is a submartingale (resp. supermartingale).
- Second-moment criterion: Suppose  $X_1, X_2, \dots$  are independent and centered (i.e., for all  $i$ ,  $\mathbb{E}[X_i] = 0$ ). If  $\sum_{n=1}^\infty \text{var}(X_n) < \infty$ , then  $P(\sum_{n=1}^\infty X_n(\omega) \text{ converges}) = 1$ .
- Kronecker's lemma: If  $a_n \uparrow \infty$  and  $\sum_{n=1}^\infty x_n/a_n$  converges, then  $a_n^{-1} \sum_{m=1}^n x_m \rightarrow 0$ .
- Let  $X_1, X_2, \dots$  be i.i.d. random variables with  $\mathbb{E}[X_i] = 0$  and  $\mathbb{E}[X_i^2] = \sigma^2 < \infty$ . Let  $S_n = X_1 + \dots + X_n$ . If  $\epsilon > 0$ , then  $S_n/n^{1/2}(\log n)^{1/2+\epsilon} \rightarrow 0$  a.s.
- Let  $X_1, X_2, \dots$  be i.i.d. with  $\mathbb{E}[X_i] = 0$  and  $\mathbb{E}[|X_i|^p] < \infty$  where  $1 < p < 2$ . Write  $S_n = X_1 + \dots + X_n$ . Then  $S_n/n^{1/p} \rightarrow 0$  a.s.
- Let  $X_1, X_2, \dots$  be i.i.d. with  $\mathbb{E}[X_1] = \infty$  and let  $S_n = X_1 + \dots + X_n$ . Let  $a_n$  be a sequence of positive numbers with  $a_n/n$  increasing. Then  $\limsup_{n \rightarrow \infty} |S_n|/a_n = 0$  or  $\infty$  according as  $\sum_n P(|X_1| \geq a_n) < \infty$  or  $= \infty$ .
- Kolmogorov's three-series theorem: Let  $X_1, X_2, \dots, X_n, \dots$  be independent random variables. Let  $A > 0$  and  $Y_i = X_i 1_{|X_i| \leq A}$ . In order to show that  $\sum X_i$  converges a.s., it is necessary and sufficient that (i)  $\sum_{n=1}^\infty P(|X_n| > A) < \infty$ ; (ii)  $\sum_{n=1}^\infty \mathbb{E}[Y_n]$  converges; (iii)  $\sum_{n=1}^\infty \text{var}(Y_n) < \infty$ .

## 2.6 Large Deviations

- Let  $X_1, X_2, \dots$  be i.i.d. with  $\mathbb{E}[X_1] = \mu$  and let  $S_n = X_1 + X_2 + \dots + X_n$ . According to CLT, the typical value of  $S_n - n\mu$  is  $O(\sqrt{n})$ . What about atypical deviations of  $S_n - n\mu$ ? According to WLLN, we know that for any  $a > \mu$ ,  $P(S_n > na) \rightarrow 0$ . We want to discuss the existence and value of the limit:  $\lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n > na)$ .
- Let  $\pi_n = P(S_n \geq na)$ . Then  $\pi_{n+m} \geq P(S_n \geq na, S_{n+m} - S_n \geq ma) = \pi_n \pi_m$ . Let  $\gamma_n = \log \pi_n$ ,  $\gamma_{n+m} \geq \gamma_n + \gamma_m$ . As  $n \rightarrow \infty$  the limit of  $\gamma_n$  exists and  $\lim_{n \rightarrow \infty} \frac{\gamma_n}{n} = \sup_n \frac{\gamma_n}{n}$ . We define  $\gamma(a) = \lim_{n \rightarrow \infty} \gamma_n/n \leq 0$ . Then for any distribution and any  $n$  and  $a$ ,  $P(S_n \geq na) \leq e^{n\gamma(a)}$ . We want to show  $\gamma(a) < 0$  if  $a > \mu$ .
- If the moment generating function  $\psi(\theta) = \mathbb{E}[\exp(\theta X_1)] < \infty$  for some  $\theta > 0$ , then  $P(S_n \geq na) \leq \exp[n(\log \psi(\theta) - a\theta)]$ . Let  $\kappa(\theta) = \log \psi(\theta)$ . If  $a > \mu$ , then  $a\theta - \kappa(\theta) > 0$  for all sufficiently small  $\theta$ .
- We will further strengthen our upper bounds by finding the maximum of  $\lambda(\theta) = a\theta - \kappa(\theta)$ . Let  $\theta_+ = \sup\{\theta : \psi(\theta) < \infty\}$  and  $\theta_- = \inf\{\theta : \psi(\theta) < \infty\}$ . Now since that  $\psi(\theta) \in C^\infty$  within  $(\theta_-, \theta_+)$ , we have  $\lambda'(\theta) = a - \frac{\psi'(\theta)}{\psi(\theta)}$ . So the maximal point of  $\lambda$  must satisfy  $\psi'(\theta)/\psi(\theta) = a$ . For the existence and uniqueness of such point(s), we introduce a new distribution, and use a trick named “tilting”.
- We now introduce the distribution  $F_\theta$  by “reweighting  $F$ ”:  $F_\theta(x) = \frac{1}{\psi(\theta)} \int_{-\infty}^x e^{y\theta} dF(y)$ . By simple calculus,  $\int x dF_\theta(x) = \frac{\psi'(\theta)}{\psi(\theta)}$ ,  $\psi''(\theta) = \int x^2 e^{\theta x} dF(x)$ ,  $\frac{d}{d\theta} \frac{\psi'(\theta)}{\psi(\theta)} = \int x^2 dF_\theta(x) - (\int x dF_\theta(x))^2 \geq 0$ . If we assume the distribution  $F$  is not a point mass at  $\mu$ , then  $\frac{\psi'(\theta)}{\psi(\theta)}$  is strictly increasing and  $a\theta - \log \psi(\theta)$  is concave. Since we have  $\frac{\psi'(0)}{\psi(0)} = \mu$ , this shows that for each  $a > \mu$  there is at most one  $\theta_a \geq 0$  that solves  $a = \frac{\psi'(\theta_a)}{\psi(\theta_a)}$ , and this value of  $\theta$  maximizes  $a\theta - \log \psi(\theta)$ . Let  $F^n$  be the c.d.f. of  $S_n = X_1 + \dots + X_n$  and  $F_\lambda^n$  be the c.d.f. of  $S_n^\lambda = X_1^\lambda + \dots + X_n^\lambda$  where  $X_i$  i.i.d.  $\sim F$  and  $X_i^\lambda$  i.i.d.  $\sim F_\lambda = \frac{1}{\psi(\lambda)} \int_{-\infty}^x e^{y\lambda} dF(y)$ . By induction,  $\frac{dF_\lambda^n}{dF_\lambda^n} = e^{-\lambda x} \psi(\lambda)^n$ . Then as  $n \rightarrow \infty$ ,  $n^{-1} \log P(S_n \geq na) \rightarrow -a\theta_a + \log \psi(\theta_a)$ .
- Some important information:  $\kappa(\theta) = \log \psi(\theta)$ ,  $\kappa'(\theta) = \frac{\psi'(\theta)}{\psi(\theta)}$ ,  $\theta_a$  solves  $\kappa'(\theta_a) = a$ ,  $\gamma(a) = \lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n \geq na) = -a\theta_a + \kappa(\theta_a)$ .
- Suppose  $x_o = \sup\{x : F(x) < 1\} = \infty$ ,  $\theta_+ < \infty$ , and  $\psi'(\theta)/\psi(\theta)$  increases to a finite limit  $a_0$  as  $\theta \uparrow \theta_+$ . If  $a_0 \leq a < \infty$ ,  $n^{-1} \log P(S_n \geq na) \rightarrow -a\theta_+ + \log \psi(\theta_+)$ , i.e.  $\gamma(a)$  is linear for  $a \geq a_0$ .
- Suppose  $x_o = \sup\{x : F(x) < 1\} < \infty$  and  $F$  has no mass at  $x_o$ . Then  $\psi(\theta) < \infty$  for all  $\theta > 0$  and  $\psi'(\theta)/\psi(\theta) \rightarrow x_o$  as  $\theta \rightarrow \infty$ .
- Now, we have shown the decaying asymptotic for all possible situations:
 
$$\left\{ \begin{array}{l} \text{If } x_o < \infty : \left\{ \begin{array}{l} a < x_o : \text{exponential, rate} = \theta_a \\ a = x_o : \text{exponential if } P(X_1 = x_o) > 0, 0 \text{ otherwise} \\ a > x_o : 0 \end{array} \right. \\ \text{If } x_o = \infty : \left\{ \begin{array}{l} \text{If } \theta_+ = \infty : \text{exponential, rate} = \theta_a \\ \text{If } \theta_+ < \infty : \left\{ \begin{array}{l} \text{If } \psi'(\theta)/\psi(\theta) \rightarrow \infty \text{ as } \theta \rightarrow \theta_+ : \text{exponential, rate} = \theta_a \\ \text{If } \psi'(\theta)/\psi(\theta) \rightarrow a_0 \text{ as } \theta \rightarrow \theta_+ : \left\{ \begin{array}{l} a < a_0 : \text{exponential, rate} = \theta_a \\ a \geq a_0 : \text{exponential, rate} = \theta_+ \end{array} \right. \end{array} \right. \end{array} \right.$$
- Cramér’s theorem: Let  $I(a)$  be the Legendre transform of  $\log \psi(\cdot)$ :  $I(a) := \sup_{\theta \in \mathbb{R}} (a\theta - \log \psi(\theta))$ . Then for any closed set  $F$ ,  $\limsup_{n \rightarrow \infty} n^{-1} \log P(\frac{S_n}{n} \in F) \leq -\inf_{x \in F} I(x)$ ; for any open set  $G$ ,  $\liminf_{n \rightarrow \infty} n^{-1} \log P(\frac{S_n}{n} \in G) \geq -\inf_{x \in G} I(x)$ .

- Intuition behind the tilting: Why do we want to introduce the measure  $F_\theta$ ? Intuitively, the new measure is like a “distorting mirror” – it “distorts” our view on how each event is likely to happen. So, when we want to estimate a rare event  $A$  under  $P$ , suppose (1) we can construct a new measure  $Q$  such that  $Q[A]$  is easily calculable, e.g.,  $Q[A] \approx 1$ ; (2) we have a uniform lower bound of the R-N derivative  $dP/dQ \geq c$  on  $A$ . Then we can conclude that  $P[A] = \int_A \frac{dP}{dQ} dQ \geq cQ[A]$ .
- Let  $\Sigma = \{a_1, \dots\}$  stand for a finite-size alphabet. Let  $M_1(\Sigma)$  be the space of all probability measures on  $\Sigma$ . The entropy of some  $\nu \in M_1(\Sigma)$  is  $H(\nu) := -\sum_{i=1}^{|\Sigma|} \nu(a_i) \log(\nu(a_i))$ . The relative entropy of  $\nu$  with respect to some other  $\mu \in M_1(\Sigma)$  is  $H(\nu|\mu) := \sum_{i=1}^{|\Sigma|} \nu(a_i) \log \frac{\nu(a_i)}{\mu(a_i)}$ .
- Let  $Y_i$  be i.i.d. r.v.'s,  $\mu \in M_1(\Sigma)$ . For  $n \geq 1$ , write  $Y = (Y_1, \dots, Y_n)$  and call  $L_n^Y \in M_1(\Sigma)$  be the empirical frequency of  $Y$ . Let  $T_n(\nu)$  be the set of  $y$  a sequence of  $n$  letters whose empirical measure is  $\nu$ .
- If  $y \in T_n(\nu)$ , then  $P_\mu(Y = y) = e^{-n(H(\nu) + H(\nu|\mu))}$ . In particular, if  $y \in T_n(\mu)$ , then  $P_\mu(Y = y) = e^{-nH(\mu)}$ .
- For every possible empirical measure  $\nu$  of  $n$  letters,  $(n+1)^{-|\Sigma|} e^{nH(\nu)} \leq |T_n(\nu)| \leq e^{nH(\nu)}$ .
- For every possible empirical measure  $\nu$  of  $n$  letters,  $(n+1)^{-|\Sigma|} e^{nH(\nu|\mu)} \leq P_\mu(L_n^T = \nu) \leq e^{nH(\nu|\mu)}$ .
- Sanov's theorem: For every set  $\Gamma \subset M_1(\Sigma)$ ,  $-\inf_{\nu \in \Gamma} H(\nu|\mu) \leq \liminf \frac{1}{n} \log P_\mu(L_n^Y \in \Gamma) \leq \limsup \frac{1}{n} \log P_\mu(L_n^Y \in \Gamma) \leq -\inf_{\nu \in \Gamma} H(\nu|\mu)$ .

## 2.7 Percolation

- Fix  $p \in [0, 1]$  and consider the  $d$ -dimensional lattice  $\mathbb{Z}^d$ . Assign to each edge  $e \in \mathbb{E}$  an independent Bernoulli r.v.  $I(e)$  with parameter  $p$ . If  $I(e) = 1$ , we say that this edge is open, otherwise closed. Consider the connected components of open edges, then for any  $p \in [0, 1]$ ,  $P_p(A) = 0$  or  $1$  where  $A = \{\exists \text{ infinite open clusters}\}$ .
- If  $A$  is translation-invariant, then  $P(A) = 0$  or  $1$ .
- Actually we can go further and show that for any  $N = 0, 1, \dots, \infty$ ,  $P_p[A(N)] = 0$  or  $1$ , where  $A(N) = \{\exists N \text{ infinite open clusters}\}$ . Or even further: for  $N = 2, 3, \dots$  and  $N = \infty$ ,  $P_p[A(N)] = 0$ .
- Let  $p_c = p_c(d) = \sup\{p : P_p(A) = 0\}$ . Then one can show that  $1/3 \leq p_c(2) \leq 2/3$ . More generally,  $p_c(1) = 1$  and for  $d \geq 2$ ,  $1/(2d-1) \leq p_c(d) \leq p_c(2) (= 1/2)$ .
- By knowledge of Galton-Watson tree and the analogy between  $\mathbb{Z}^d$  and  $2d$ -regular tree in high dimensions, we can take an educated guess that  $p_c(d) \sim \frac{1}{2d}$  as  $d \rightarrow \infty$ .

## 3 Central Limit Theorems

### 3.1 The De Moivre-Laplace Theorem

- Central Limit Theorem: Let  $X_1, X_2, \dots$  be i.i.d. with mean  $\mu$  and variance  $\sigma^2 \in (0, \infty)$ . Write  $S_n = X_1 + \dots + X_n$ , then  $\frac{S_n - \mu n}{\sqrt{n}\sigma} \Rightarrow \mathcal{N}(0, 1)$ .
- Before discussing the central limit theorem in full generality, we first see a special example for Bernoulli random variables. Let  $X_1, X_2, \dots$  be i.i.d. random variables such that  $P(X_1 = 1) = P(X_1 = -1) = 1/2$  and write  $S_n = X_1 + \dots + X_n$ . For integers  $|k| \leq n$ ,  $P(S_{2n} = 2k) = C_{2n}^{n+k} 2^{-2n}$  since  $(S_{2n} + 2n)/2 \sim \text{Binomial}(2n, 1/2)$ .

- Local central limit theorem: If  $2k/\sqrt{2n} \rightarrow x$ , then  $\lim_{n \rightarrow \infty} (\pi n)^{1/2} e^{x^2/2} P(S_{2n} = 2k) = 1$ .
- The De Moivre-Laplace Theorem: For  $a < b$ ,  $P(a \leq S_n/\sqrt{n} \leq b) \rightarrow \int_a^b (2\pi)^{-1/2} e^{-x^2/2} dx$ .

### 3.2 Weak Convergence

- A sequence of distribution function  $F_n$  is said to converge weakly to a limit  $F$ , denoted by  $F_n \Rightarrow F$ , if  $F_n(y) \rightarrow F(y)$  at every point of continuity of  $F$ , i.e. every  $y \in \mathbb{R}$  such that  $F(\cdot)$  is continuous at  $y$ .
- A sequence of random variables  $X_n$  is said to converge weakly or converge in distribution / law to a limit  $X_\infty$  if their distribution functions  $F_n$  converges weakly.
- Skorokhod's representation theorem: If  $F_n \Rightarrow F$  then there are random variables  $Y_n, 1 \leq n < \infty$  and  $Y$  with living in the same probability space such that  $Y_n \sim F_n, Y \sim F$  and  $Y_n \rightarrow Y$  a.s.
- $X_n \Rightarrow X$  if and only if for every bounded continuous function  $g$  we have  $\mathbb{E}g(X_n) \rightarrow \mathbb{E}g(X)$ .
- Continuous mapping theorem: Let  $g$  be a measurable function and  $D_g = \{x : g \text{ is discontinuous at } x\}$ . If  $X_n \Rightarrow X$ , and  $P(X \in D_g) = 0$ , then  $g(X_n) \Rightarrow g(X)$ .
- Portmanteau theorem: The following statements are equivalent: (1)  $X_n \Rightarrow X$ ; (2)  $G$  open,  $\liminf_{n \rightarrow \infty} P(X_n \in G) \geq P(X \in G)$ ; (3)  $G$  closed,  $\limsup_{n \rightarrow \infty} P(X_n \in G) \leq P(X \in G)$ ; (4) If  $P(X \in \partial A) = 0$ , then  $\lim_{n \rightarrow \infty} P(X_n \in A) = P(X \in A)$ .
- Helly's selection theorem: For every sequence  $F_n$  of distribution functions, there is a subsequence  $F_{n(k)}$  and a right continuous nondecreasing function  $F$  so that at all points of continuity  $y$  of  $F$ ,  $\lim_{k \rightarrow \infty} F_{n(k)}(y) = F(y)$ .
- Every subsequential limit of the sequence  $F_n$  is the distribution function of a probability measure iff the sequence is tight, i.e., for all  $\epsilon > 0$ , there is an  $M_\epsilon$  so that  $\limsup_{n \rightarrow \infty} [1 - F_n(M_\epsilon) + F_n(-M_\epsilon)] \leq \epsilon$ .
- If there is a function  $\phi \geq 0$  so that  $\phi(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$  and  $C = \sup_n \int \phi(x) dF_n(x) < \infty$ , then  $F_n$  is tight.

### 3.3 Characteristic Functions

- If  $X$  is a r.v., we define its Characteristic function (ch.f.) by  $\phi(t) := \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)]$ .
- All characteristic functions have the following properties: (i)  $\phi(0) = 1$ ; (ii)  $\phi(-t) = \overline{\phi(t)}$ ; (iii)  $|\phi(t)| = |\mathbb{E}e^{itX}| \leq \mathbb{E}|e^{itX}| = 1$ ; (iv)  $|\phi(t+h) - \phi(t)| \leq \mathbb{E}|e^{ihX} - 1|$ , so  $\phi(t)$  is uniformly continuous on  $\mathbb{R}$ ; (v)  $\mathbb{E}e^{it(aX+b)} = e^{itb}\phi(at)$ .
- If  $X_1$  and  $X_2$  are independent and have ch.f.'s  $\phi_1$  and  $\phi_2$ . Then  $X_1 + X_2$  has ch.f.  $\phi_1 \cdot \phi_2$ .
- Stein's Lemma: If  $X, Y$  are jointly Gaussian, then for differentiable  $g : \mathbb{R} \rightarrow \mathbb{R}$ , as long as the expectations are well-defined,  $\text{cov}(g(X), Y) = \text{cov}(X, Y)\mathbb{E}[g'(X)]$ .
- If  $F_1, \dots, F_n$  have ch.f.  $\phi_1, \dots, \phi_n$  and  $\lambda_i \geq 0, 1 \leq i \leq n$  have  $\lambda_1 + \dots + \lambda_n = 1$ . Then  $\sum \lambda_i F_i$  has ch.f.  $\sum \lambda_i \phi_i$ .
- The inversion formula: If  $a < b$ , then  $\frac{1}{2\pi} \lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt = \mu(a, b) + \frac{1}{2}\mu(\{a, b\})$ .
- If  $\int |\phi(t)| dt < \infty$ , then  $\mu$  has bounded continuous density  $f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \phi(t) dt$ .
- Continuity theorem: Let  $\mu_n, 1 \leq n \leq \infty$  be probability measures with ch.f.  $\phi_n$ . (i) If  $\mu_n \Rightarrow \mu_\infty$  then  $\phi_n(t) \rightarrow \phi_\infty(t)$  for all  $t$ . (ii) If  $\phi_n(t) \rightarrow \phi(t)$  for all  $t$ , and  $\phi(t)$  is continuous at 0. Then  $\{\mu_n\}_{n=1}^\infty$  is tight and has a weak limit with ch.f.  $\phi$ .

## CENTRAL LIMIT THEOREMS

- Let  $\mu$  be a probability measure and  $\phi$  be its ch.f. Then  $\mu(\{x : |x| \geq 2u^{-1}\}) \leq u^{-1} \int_{-u}^u [1 - \phi(t)] dt$ .
- If  $\int |x|^n \mu(dx) < \infty$ , then its ch.f.  $\phi$  has a continuous derivative of order  $n$  given by  $\phi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx)$ . In particular,  $\phi^{(n)}(0) = \mathbb{E}[(iX)^n]$ .
- However, if a characteristic function  $\phi_X$  has a  $k$ -th derivative at zero, then the random variable  $X$  has all moments up to  $k$  if  $k$  is even, but only up to  $(k-1)$  if  $k$  is odd.
- $|e^{ix} - \sum_{m=0}^n \frac{(ix)^m}{m!}| \leq \min(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!})$ .
- If  $\mathbb{E}|X|^2 < \infty$ , then  $\phi(t) = 1 + it\mathbb{E}X - t^2\mathbb{E}|X|^2/2 + o(t^2)$ .
- If  $\limsup_{h \downarrow 0} \frac{\phi(h) - 2\phi(0) + \phi(-h)}{h^2} > -\infty$ , then  $\mathbb{E}[X^2] < \infty$ .
- Given  $\phi$  and  $x_1, \dots, x_n \in \mathbb{R}$ , we can consider the matrix with  $(i, j)$  entry given by  $\phi(x_i - x_j)$ . Call  $\phi$  positive definite if this matrix is always positive semi-definite Hermitian.
- Bochner's theorem: A function from  $\mathbb{R}$  to  $\mathbb{C}$  which is continuous at origin with  $\phi(0) = 1$  is a ch.f. of some probability measure on  $\mathbb{R}$  if and only if it is positive definite.
- Pólya's theorem: If  $\phi$  is real-valued, even and continuous such that (i)  $\phi(0) = 1$ ; (ii)  $\phi$  is convex for  $t > 0$ ; (iii)  $\phi(\infty) = 0$ ; then  $\phi(t)$  is the ch.f. of a distribution symmetric about 0.

### 3.4 Central Limit Theorems

- Central Limit Theorem: Let  $X_1, X_2, \dots$  be i.i.d. with  $\mathbb{E}[X_1] = \mu, \text{var}(X_1) = \sigma^2 \in (0, \infty)$ . If  $S_n = X_1 + X_2 + \dots + X_n$ ,  $\frac{S_n - n\mu}{n^{1/2}\sigma} \Rightarrow \mathcal{N}(0, 1)$ .
- The Lindeberg-Feller theorem: For each  $n$ , let  $X_{n,m}, 1 \leq m \leq n$ , be independent random variables for each  $n$  with  $\mathbb{E}[X_{n,m}] = 0$ . Suppose (i)  $\sum_{m=1}^n \mathbb{E}[X_{n,m}^2] \rightarrow \sigma^2 > 0$ ; (ii) For all  $\epsilon > 0$ ,  $\lim_{n \rightarrow \infty} \sum_{m=1}^n \mathbb{E}[X_{n,m}^2 1_{|X_{n,m}| > \epsilon}] = 0$ . Then  $S_n = X_{n,1} + \dots + X_{n,n} \Rightarrow \mathcal{N}(0, \sigma^2)$  as  $n \rightarrow \infty$ .
- Converging together lemma: If  $X_n \Rightarrow X$  and  $Y_n \Rightarrow c$ ,  $X_n + Y_n \Rightarrow X + c$ . A useful consequence of this result is that if  $X_n \Rightarrow X$  and  $Z_n - X_n \Rightarrow 0$  then  $Z_n \Rightarrow X$ .
- Lévy's condition for CLT: Let  $X_1, X_2, \dots$  be i.i.d. and  $S_n = X_1 + \dots + X_n$ . In order that there exist constants  $a_n$  and  $b_n > 0$  so that  $(S_n - a_n)/b_n \Rightarrow \mathcal{N}(0, 1)$ , it is necessary and sufficient that  $\frac{y^2 P(|X_1| > y)}{\mathbb{E}[X_1^2 1_{|X_1| \leq y}]} \rightarrow 0$ .
- Chernoff bound: Let  $X_i$  be independent Bernoulli r.v.'s. Write  $S_n = X_1 + \dots + X_n$  and let  $\mu = \mathbb{E}[S_n]$ . Then for  $\delta > 0$ ,  $P(S_n > (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2 + \delta}}, P(S_n < (1 - \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$ .
- Hoeffding's inequality for bounded r.v. Let  $X_i$  be independent r.v.'s such that  $X_i \in [a_i, b_i]$  a.s. Write  $S_n = X_1 + \dots + X_n$  and let  $\mu = \mathbb{E}[S_n]$ . Then for  $\delta > 0$ ,  $P(|S_n - \mu| \geq \delta) \leq 2 \exp(-\frac{2n^2 \delta^2}{\sum_{i=1}^n (b_i - a_i)^2})$ .
- A random variable is sub-Gaussian, if and only if for some  $C < \infty$  and  $c > 0$ ,  $P(|X| \geq t) \leq C e^{-ct^2}$ .
- Hoeffding's inequality for sub-Gaussian r.v.'s: Let  $X_i$  be independent zero-mean sub-Gaussian r.v.'s. Write  $S_n = X_1 + \dots + X_n$ . Then there exists some  $c > 0$  such that for any  $\delta > 0$ ,  $P(|S_n| \geq \delta) \leq 2 \exp(-c\delta^2 / \sum_{i=1}^n \|X_i\|_{\psi_2})$ , where  $\|X\|_{\psi_2} = \inf\{c \geq 0 : \mathbb{E}[e^{X^2/c^2}] \leq 2\}$ .
- Let  $X_1, X_2, \dots$  be i.i.d. with  $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^2] = \sigma^2$ , and  $\mathbb{E}[|X_i|^3] = \rho < \infty$ . Let  $\mathcal{N}(x)$  is the distribution of the standard normal distribution, then for all  $n \geq 1$  and  $x \in \mathbb{R}$ ,  $|F_n(x) - \mathcal{N}(x)| \leq 3\rho/(\sigma^3 \sqrt{n})$ .



### 3.5 Local Limit Theorems

- A random variable  $X$  has a lattice distribution if  $\exists b, h > 0$  so that  $P(X \in b + h\mathbb{Z}) = 1$ . The largest  $h$  for which the last statement holds is called the span of the distribution.
- Trichotomy of a random variable: Let  $\phi(t)$  be the ch.f. of a random variable  $X$ . There are only three possibilities: (1)  $|\phi(t)| < 1$  for all  $t \neq 0$ ; (2) There is a  $\lambda > 0$  so that  $|\phi(\lambda)| = 1$  and  $|\phi(t)| < 1$  for  $0 < t < \lambda$ . In this case,  $X$  has a lattice distribution with span  $2\pi/\lambda$ ; (3)  $|\phi(t)| = 1$  for all  $t$ . In this case,  $X$  is deterministic.
- Let  $X_i$  be i.i.d. r.v.'s with  $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^2] = \sigma^2 \in (0, \infty)$ . Suppose in addition  $P(X_i \in b + h\mathbb{Z}) = 1$ , i.e.  $X_i$  are lattice with span  $h$ . Let  $p_n(x) = P(S_n/\sqrt{n} = x)$  for  $x \in \mathcal{L}_n = \{(nb + h\mathbb{Z})/\sqrt{n}\}$ , and  $n(x)$  be the density of  $\mathcal{N}(0, \sigma^2)$ . Then  $\lim_{n \rightarrow \infty} \sup_{x \in \mathcal{L}_n} |\frac{\sqrt{n}}{h} p_n(x) - n(x)| = 0$ .
- Let  $X_i$  be i.i.d. nonlattice r.v.'s with  $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = \sigma^2$ . If  $x_n/\sqrt{n} \rightarrow x$  and  $a < b$ ,  $\sqrt{n}P(S_n \in (x_n + a, x_n + b)) \rightarrow (b - a)n(x)$ .
- Let  $p_n^{(d)}(\cdot)$  stand for the  $n$ -step transition probability for  $d$ -dimensional simple random walk. Then  $p_{2n}^{(d)}(0) \rightarrow$  monotone decreasing in  $d$ .

### 3.6 Poisson Convergence

- For each  $n$  let  $X_{n,m}, 1 \leq m \leq n$  be independent random variables with  $P(X_{n,m} = 1) = p_{n,m}, P(X_{n,m} = 0) = 1 - p_{n,m}$ . Suppose (i)  $\lim_{n \rightarrow \infty} \sum_{m=1}^n p_{n,m} = \lambda$ ; (ii)  $\lim_{n \rightarrow \infty} \max_{m \leq n} p_{n,m} = 0$ . Let  $S_n := X_{n,1} + \dots + X_{n,n}$ , then  $S_n \Rightarrow \text{Poisson}(\lambda)$ .
- $d(\mu, \nu) = \|\mu - \nu\|_{\text{TV}}$  defines a metric on the set of probability measures on  $\mathbb{Z}$ .  $\|\mu_n - \mu\| \rightarrow 0$  if and only if  $\mu_n \Rightarrow \mu$ .
- The  $p$ -th Wasserstein distance between two probability measures  $\mu$  and  $\nu$  on  $M$  with  $p$ -th moment is defined as  $W_p(\mu, \nu) = (\inf_{\gamma \in \Gamma(\mu, \nu)} \int_{M \times M} d(x, y)^p d\gamma(x, y))^{1/p}$  where  $\Gamma(\mu, \nu)$  is the set of all couplings of  $\mu$  and  $\nu$ . One can show that  $W_p$  defines a metric and convergence under  $W_p$ -metric is equivalent to weak convergence plus convergence of the first  $p$ -th moment.
- Suppose that  $r$  balls are placed at random into  $n$  boxes. Then suppose  $r/n \rightarrow c$ , the number of balls in each box is approximately  $\text{Poisson}(c)$ . Let  $X_n$  be the number of empty boxes. Then if  $ne^{-r/n} \rightarrow \lambda$ ,  $X_n \rightarrow \text{Poisson}(\lambda)$ .
- Let  $X_{n,m}, 1 \leq m \leq n$  be independent random variables with  $P(X_{n,m} = 1) = p_{n,m}, P(X_{n,m} \geq 2) = \epsilon_{n,m}$ . Suppose  $\lim_{n \rightarrow \infty} \sum_{m=1}^n p_{n,m} = \lambda, \lim_{n \rightarrow \infty} \max_{m \leq n} p_{n,m} = 0, \lim_{n \rightarrow \infty} \sum_{m=1}^n \epsilon_{n,m} = 0$ . Let  $S_n = X_{n,1} + \dots + X_{n,n}$ , then  $S_n \Rightarrow \text{Poisson}(\lambda)$ .

### 3.7 Poisson Process

- Let  $N(s, t)$  be the number of students arriving at a certain dinning hall in the time interval  $(s, t]$ . Suppose the number of arrivals in intervals that are disjoint are independent, the distribution of  $N(s, t)$  only depends on  $t - s$ ,  $P(N(0, h) = 1) = \lambda h + o(h)$ ,  $P(N(0, h) \geq 2) = o(h)$ . Then  $N(0, t)$  has a Poisson distribution with mean  $\lambda t$ .
- A family of random variables  $N_t, t \geq 0$  is called a Poisson process with rate  $\lambda$ , if (i) for  $0 \leq t < s$ ,  $N(s) - N(t) \sim \text{Poisson}(\lambda(s - t))$ ; (ii) if  $0 < t_0 < t_1 < \dots < t_n, N(t_k) - N(t_{k-1}), 1 \leq k \leq n$  are independent.

- Suppose that between 12:00 and 1:00 cars arrive at the East Gate of PKU according to a Poisson process  $N_t$  with rate  $\lambda$ . Let  $Y_i$  be the number of people in the  $i$ -th vehicle which we assume to be i.i.d. and independent to  $N_t$ . Then consider  $M(t)$  be the total number of visitors within those vehicles by time  $t$ , i.e.  $M(t) = \sum_{i=1}^{N_t} Y_i$  with the convention that  $M(t) = 0$  if  $N_t = 0$ .
- Let  $Y_1, Y_2, \dots$  be i.i.d. r.v.'s;  $N$  and independent non-negative interger-valued r.v.;  $S = Y_1 + \dots + Y_N$  with  $S = 0$  when  $N = 0$ . (1) If  $\mathbb{E}[Y_i], \mathbb{E}[N] < \infty$ , then  $\mathbb{E}[S] = \mathbb{E}[N] \cdot \mathbb{E}[Y_i]$ ; (2) If  $\mathbb{E}[Y_i^2], \mathbb{E}[N^2] < \infty$ , then  $\text{var}(S) = \mathbb{E}[N]\text{Var}(Y_i) + \text{var}(N)(\mathbb{E}[Y_i])^2$ ; (iii) If  $N \sim \text{Poisson}(\lambda)$ , then  $\text{var}(S) = \lambda \mathbb{E}[Y_i^2]$ .
- Recall the problem of counting the number of cars arriving at the East Gate of PKU. Noting that  $Y_i$  now stands for the number of people in each vehicel,  $Y_i$  has to take positive integer values. Let  $N_t^j$  be the number of cars with exactly  $j$  passengers. For  $Y_i$  taking value on  $1, 2, \dots, m < \infty$ ,  $N_t^j$  are independent rate  $\lambda P(Y_i = j)$  Poisson processes.
- Suppose that in a Poisson process with rate  $\lambda$ , for a point that lands at time  $s$ , we keep it with probability  $p(s)$ . Then the result is an inhomogenous Poisson process with rate  $\lambda p(s)$ .
- Inhomogenous Poisson process as time change of Poisson process: For  $p(t)$ , and the standard Poisson process  $N_t$  with rate  $\lambda$ , we call  $\hat{N}(t) = N(\int_0^t \lambda p(s) ds)$  be the inhomogenous Poisson process with transition rate function  $\lambda(t) = \lambda p(t)$ .
- Suppose  $\lambda$  is  $\sigma$ -finite, we say a random measure  $\mu$  is a Poisson Point Process/Poisson random measure with intensity measure  $\lambda$  if (1) for all  $B \in \mathcal{S}$ ,  $\mu(B) \sim \text{Poisson}(\lambda(B))$ ; (2) If  $B_1, \dots, B_n$  be disjoint sets in  $\mathcal{S}$ , then the random variables  $\mu(B_1), \dots, \mu(B_n)$  are also independent.
- Let  $T_n$  be the time of the  $n$ -th arrival of a Poisson process with rate  $\lambda$ . Let  $U_1, U_2, \dots, U_n$  be independent uniform on  $(0, t)$  and let  $(V_k^n)_{k=1,2,\dots,n}$  be the order statistics of  $\{U_1, \dots, U_n\}$ , i.e.  $V_k^n$  is the  $k$ -th smallest number from  $(U_1, \dots, U_n)$ . Then, conditioning on  $N(t) = n$ , the vectors  $V = (V_1^n, \dots, V_n^n)$  and  $T = (T_1, \dots, T_n)$  have the same distribution.
- If  $0 < s < t$ , then  $P(N_s = m | N_t = n) = C_n^m (s/t)^m (1 - s/t)^{n-m}$ .

### 3.8 Limit Theorems in $\mathbb{R}^d$

- We say  $X_n \Rightarrow X_\infty$  if  $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$  for all bounded and continuous  $f$ .
- General Portmantean Theorem: The following statements are equivalent: (1)  $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$  for all bounded and continuous  $f$ ; (2)  $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$  for all bounded and Lipschitz-continuous  $f$ ; (3) For all closed sets  $K$ ,  $\limsup_{n \rightarrow \infty} P(X_n \in K) \leq P(X_\infty \in K)$ ; (4) For all open sets  $G$ ,  $\liminf_{n \rightarrow \infty} P(X_n \in G) \geq P(X_\infty \in G)$ ; (5) For all sets  $A$  with  $P(X_\infty \in \partial A) = 0$ ,  $\lim_{n \rightarrow \infty} P(X_n \in A) = P(X_\infty \in A)$ ; (6) Let  $D_f$  be the set of discontinuous of  $f$ . For all bounded functions  $f$  with  $P(X_\infty \in D_f) = 0$ , we have  $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$ .
- For distribution  $F_n$  and  $F$  on  $\mathbb{R}^d$ , we say that  $F_n$  converges weakly to  $F$ , and write  $F_n \Rightarrow F$ , if  $F_n(x) \rightarrow F(x)$  at all continuity points of  $F$ .
- Distribution function in  $\mathbb{R}^d$ : (i) Nondecreasing:  $x \leq y \Rightarrow F(x) \leq F(y)$ . (ii)  $\lim_{x \rightarrow \infty} F(x) = 1, \lim_{x_i \rightarrow -\infty} F(x) = 0$ . (iii)  $F$  is right continuous:  $\lim_{y \uparrow x} F(y) = F(x)$ . (iv)  $\triangle_A F \geq 0$  for all rectangles  $A$ .
- Equivalence of two definitions: On  $\mathbb{R}^d$  weak convergence defined in terms of convergence of distribution  $F_n \Rightarrow F_\infty$  is equivalent to notion of weak convergence defined for a general metric space.

- Tightness in  $\mathbb{R}^d$ : A sequence of probability measures  $\mu_n$  is said to be tight if for any  $\epsilon > 0$ , there is an  $M < \infty$  such that  $\liminf_{n \rightarrow \infty} \mu_n([-M, M]^d) \geq 1 - \epsilon$ .
- If  $\mu_n$  is tight, there is a weakly convergent subsequence.
- The characteristic function of  $\vec{X} = (X_1, \dots, X_d)$  is  $\phi(\vec{t}) = \mathbb{E}[\exp(i\vec{t} \cdot \vec{X})]$ . If  $A = [a_1, b_1] \times \dots \times [a_d, b_d]$  with  $\mu(\partial A) = 0$ , then  $\mu(A) = \lim_{T \rightarrow \infty} (2\pi)^{-d} \int_{[-T, T]^d} \left( \prod_{j=1}^d \psi_j(t_j) \right) \phi(\vec{t}) dt$ , where  $\psi_j(s) = \frac{\exp(-isa_j) - \exp(-isb_j)}{is}$ .
- Convergence theorem: Let  $X_n, 1 \leq n \leq \infty$  be random vectors with ch.f.  $\phi_n$ . A necessary and sufficient condition for  $F_n$  to converge weakly to a probability distribution  $F_\infty$  is that  $\phi_n(\vec{t}) \rightarrow \phi_\infty(\vec{t})$ , which is continuous at 0.
- Cramer-Wold device: A sufficient condition for  $X_n \Rightarrow X_\infty$  is that  $\vec{\theta} \cdot X_n \Rightarrow \vec{\theta} \cdot X_\infty$  for all  $\vec{\theta} \in \mathbb{R}^d$ .
- The central limit theorem in  $\mathbb{R}^d$ : Let  $X_1, X_2, \dots$  be i.i.d. random vectors with  $\mathbb{E}X_n = \mu$ , and finite covariances  $(\Gamma_{i,j})_{m \times m}$ . Then  $(S_n - n\mu)/n^{1/2} \Rightarrow \chi$ , where  $\chi$  is a multivariate normal with mean 0 and covariances  $(\Gamma_{i,j})_{m \times m}$ .

## 4 Martingales

### 4.1 Conditional Expectation

- Existence and uniqueness of conditional expectation: Let  $(\Omega, \mathcal{H}, P)$  be a probability space,  $X$  be a random variable such that  $\mathbb{E}[|X|] < \infty$ ,  $\mathcal{G} \subset \mathcal{H}$  be a sub  $\sigma$ -algebra of  $\mathcal{H}$ . Then (1) Existence:  $\exists$  r.v.  $Y$  such that  $Y \in \mathcal{G}, \mathbb{E}[|Y|] < \infty$  and  $\forall G \in \mathcal{G}, \mathbb{E}[Y; G] = \mathbb{E}[X; G]$ . We call such  $Y$  a version of  $\mathbb{E}[X|\mathcal{G}]$ . (2) Uniqueness: If  $Y, \tilde{Y}$  are versions of  $\mathbb{E}[X|\mathcal{G}]$ , then  $Y = \tilde{Y}$  a.s.
- Orthogonal projection in  $L^2$ : If  $\mathbb{E}[X^2] < \infty$ , then  $Y = \mathbb{E}[X|\mathcal{G}]$  is a version of the orthogonal projection of  $X$  from  $L^2(\Omega, \mathcal{H}, P)$  to  $L^2(\Omega, \mathcal{G}, P)$ , i.e.  $Y$  is the best  $G$ -measurable predictor of  $X$ , which minimizes  $\mathbb{E}[(Y - X)^2]$ .
- Properties of conditional expectation: (1)  $Y = \mathbb{E}[X|\mathcal{G}] \Rightarrow \mathbb{E}[Y] = \mathbb{E}[X]$ . (2)  $X \in \mathcal{G} \Rightarrow \mathbb{E}[X|\mathcal{G}] = X$  a.s. (3) Linearity:  $\mathbb{E}[aX_1 + bX_2|\mathcal{G}] = a\mathbb{E}[X_1|\mathcal{G}] + b\mathbb{E}[X_2|\mathcal{G}]$  a.s. (4) Positivity:  $X \geq 0 \Rightarrow \mathbb{E}[X|\mathcal{G}] \geq 0$  a.s. (5) Monotone convergence theorem:  $0 \leq X_n \uparrow X \Rightarrow \mathbb{E}[X_n|\mathcal{G}] \uparrow \mathbb{E}[X|\mathcal{G}]$  a.s. (6) Fatou's lemma:  $X_n \geq 0 \Rightarrow \mathbb{E}[\liminf_{n \rightarrow \infty} X_n|\mathcal{G}] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n|\mathcal{G}]$  a.s. (7) Dominated convergence theorem:  $|X_n(\omega)| \leq V(\omega)$  a.s.  $\forall n, \mathbb{E}[V] < \infty, X_n \rightarrow X$  a.s., then  $\mathbb{E}[X_n|\mathcal{G}] \rightarrow \mathbb{E}[X|\mathcal{G}]$  a.s. (8) If  $c(x)$  is convex,  $\mathbb{E}[|c(x)|] < \infty$ , then  $\mathbb{E}[c(x)|\mathcal{G}] \geq c(\mathbb{E}[X|\mathcal{G}])$  a.s. (9) Tower property: If  $\mathcal{H} \subset \mathcal{G}$ , then  $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathcal{H}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]\mathcal{G}] = \mathbb{E}[X|\mathcal{H}]$ . (10) If  $Z \in \mathcal{G}$  then  $\mathbb{E}[ZX|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}]$ . (11) If  $\mathcal{H} \perp \sigma(X, \mathcal{G})$  then  $\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}]$  a.s. In particular, if  $X \perp \mathcal{H}$ , then  $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$  a.s.

### 4.2 Martingales, Almost Sure Convergence

- Filtered spaces:  $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^\infty, P)$  satisfies  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$  (i.e.  $\{\mathcal{F}_n\}_{n=1}^\infty$  is a filtration) and  $\sigma(\cup_{i=0}^\infty \mathcal{F}_i) := \mathcal{F}_\infty \subset \mathcal{F}$  (but not necessarily  $\mathcal{F}_\infty = \mathcal{F}$ ). Given a filtration  $\{\mathcal{F}_n\}$ , if a sequence of r.v.'s  $\{X_n\}$  satisfies  $X_n \in \mathcal{F}_n$ , we say  $\{X_n\}$  is adapted to  $\{\mathcal{F}_n\}$ .
- Martingale:  $X = \{X_n\}$  discrete time stochastic process is a martingale if: (1)  $\{X_n\}$  is adapted to some filtration  $\{\mathcal{F}_n\}$ ; (2)  $\forall n, \mathbb{E}[|X_n|] < \infty$  (but not necessarily  $\mathbb{E}[|X_n|] < M < \infty$ ); (3)  $\forall n, \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ . If “=” in (3) is replaced by “ $\geq$ ” or “ $\leq$ ”, then we say  $X$  is a submartingale/supermartingale.
- $m < n, \{X_n\}$  is martingale/submartingale/supermartingale,  $\mathbb{E}[X_n|\mathcal{F}_m] = / \geq / \leq X_m$ .
- If  $X_n$  is a martingale w.r.t.  $\mathcal{F}_n$  and  $\phi$  is a convex function with  $\mathbb{E}[\phi(X_n)] < \infty$  for all  $n$ , then  $\phi(X_n)$  is a submartingale w.r.t.  $\mathcal{F}_n$ .

- A process is predictable if  $C_n \in \mathcal{F}_{n-1}$ .
- You can't beat the system: Let  $Y_n = \sum_{k=1}^n C_k(X_k - X_{k-1})$ ,  $C$  is a predictable process. (1) If  $C$  is non-negative,  $|C_n(\omega)| \leq K, \forall n, \forall \omega$ , and  $X$  is martingale/supermartingale, then  $Y$  is martingale/supermartingale. (2) If  $C$  is a bounded predictable process and  $X$  is a martingale, then  $Y$  is a martingale. (3) In (1) and (2), the boundness condition on  $C$  may be replaced by the condition  $C_n \in L^2, \forall n$ , provided we also insist that  $X_n \in L^2, \forall n$ .
- Stopping time:  $T : \Omega \rightarrow \mathbb{Z}_+$ , if  $\{T \leq n\} \in \mathcal{F}_n, \forall n \leq \infty$ .
- If  $X$  is a martingale/supermartingale and  $T$  is a stopping time, then the stopped process  $(X_{T \wedge n})_n$  is a martingale/supermartingale,  $\mathbb{E}[X_{T \wedge n}] = / \leq \mathbb{E}[X_0]$ .
- Doob's optional stopping theorem: Let  $T$  be a stopping time and  $X$  be a martingale/supermartingale. Then  $X_T$  is integrable and  $\mathbb{E}[X_T] = / \leq \mathbb{E}[X_0]$  in each of the following situations: (1)  $T$  is bounded; (2)  $X$  is bounded and  $T$  is a.s. finite; (3)  $\mathbb{E}[T] < \infty$ , and, for some  $K \in \mathbb{R}_+$ ,  $|X_n(\omega) - X_{n-1}(\omega)| \leq K$ .
- Define  $C_1 := I_{\{X_0 < a\}}$  and, for  $n \geq 2$ ,  $C_n := I_{\{C_{n-1}=1\}}I_{\{X_{n-1} \leq b\}} + I_{\{C_{n-1}=0\}}I_{\{X_{n-1} < a\}}$ .  $Y_n = \sum_{k=1}^n C_k(X_k - X_{k-1})$ . The number  $U_N[a, b](\omega)$  of upcrossings of  $[a, b]$  made by  $n \mapsto X_n(\omega)$  by time  $N$  is defined to be the largest  $k$  in  $\mathbb{Z}_+$  such that we can find  $0 \leq s_1 < t_1 < s_2 < t_2 < \dots < s_k < t_k \leq N$  with  $X_{s_i}(\omega) < a, X_{t_i}(\omega) > b, 1 \leq i \leq k$ .
- The fundamental inequality (recall that  $Y_0(\omega) = 0$ ) is obvious:  $Y_N(\omega) \geq (b - a)U_N[a, b](\omega) - [X_N(\omega) - a]^+$ .
- Doob's upcrossing lemma: Let  $X$  be a supermartingale. Let  $U_N[a, b]$  be the number of upcrossings of  $[a, b]$  by time  $N$ . Then  $(b - a)\mathbb{E}U_N[a, b] \leq \mathbb{E}[(X_N - a)^-]$ .
- Let  $X$  be a supermartingale bounded in  $L^1$  in that  $\sup_n \mathbb{E}|X_n| < \infty$ . Let  $a, b \in \mathbb{R}$  with  $a < b$ . Then, with  $U_\infty([a, b]) := \lim_N U_N[a, b]$ ,  $(b - a)\mathbb{E}U_\infty[a, b] \leq |a| + \sup_n \mathbb{E}|X_n| < \infty$  so that  $P(U_\infty[a, b] = \infty) = 0$ .
- Doob's forward convergence theorem: Let  $X$  be a supermartingale bounded in  $L_1$ :  $\sup_n \mathbb{E}|X_n| < \infty$ . Then, almost surely,  $X_\infty := \lim_n X_n$  exists and is finite. For definiteness, we define  $X_\infty(\omega) := \limsup_n X_n(\omega), \forall \omega$ , so that  $X_\infty$  is  $\mathcal{F}_\infty$  measurable and  $X_\infty = \lim_n X_n$ , a.s.
- Martingale convergence theorem: If  $X_n$  is a submartingale with  $\sup \mathbb{E}X_n^+ < \infty$ , then as  $n \rightarrow \infty$ ,  $X_n$  converges a.s. to a limit  $X$  with  $\mathbb{E}|X| < \infty$ .
- If  $X_n \geq 0$  is a supermartingale, then as  $n \rightarrow \infty$ ,  $X_n \rightarrow X$  a.s. and  $\mathbb{E}X \leq \mathbb{E}X_0$ .

### 4.3 Examples

- Doob's decomposition: Any submartingale  $X_n, n \geq 0$ , can be written in a unique way as  $X_n = M_n + A_n$ , where  $M_n$  is a martingale and  $A_n$  is a predictable increasing sequence with  $A_0 = 0$ .
- Let  $X_1, X_2, \dots$  be a martingale with  $|X_{n+1} - X_n| \leq M < \infty$ . Let  $C = \{\lim_n X_n \text{ exists and is finite}\}, D = \{\limsup_n X_n = +\infty \text{ and } \liminf_n X_n = -\infty\}$ . Then  $P(C \cup D) = 1$ .
- Second Borel-Cantelli lemma: Let  $\mathcal{F}_n, n \geq 0$  be a filtration with  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  and  $B_n, n \geq 1$  a sequence of events with  $B_n \in \mathcal{F}_n$ . Then  $\{B_n \text{ i.o.}\} = \{\sum_{n=1}^\infty P(B_n | \mathcal{F}_{n-1}) = \infty\}$ .
- Let  $\mu, \nu$  be two probability measures on  $(\Omega, \mathcal{F})$ . Let  $\mathcal{F}_n \uparrow \mathcal{F}$  be  $\sigma$ -fields. Let  $\mu_n$  and  $\nu_n$  be the restrictions of  $\mu$  and  $\nu$  to  $\mathcal{F}_n$ . Suppose  $\mu_n \ll \nu_n$  for all  $n$ . Let  $X_n = d\mu_n/d\nu_n$  and let  $X = \limsup_n X_n$ . Then  $\mu(A) = \int_A X d\nu + \mu(A \cap \{X = \infty\}) := \mu_r(A) + \mu_s(A)$ , which gives the Lebesgue decomposition of  $\mu$ , i.e.,  $\mu_r \ll \nu, \mu_s \perp \nu$ .

- Kakutani dichotomy for infinite product measures: Let  $\mu, \nu$  be two probability measures on sequence space  $(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}})$  that make the coordinates  $\xi_n(\omega) = \omega_n$  independent. Let  $F_n(x) = \mu(\xi_n \leq x), G_n(x) = \nu(\xi_n \leq x)$ . Suppose  $F_n \ll G_n$  and let  $q_n = dF_n/dG_n > 0, G_n$ -a.s. Let  $\mathcal{F}_n = \sigma(\xi_m : m \leq n)$ , let  $\mu_n, \nu_n$  be the restrictions of  $\mu$  and  $\nu$  to  $\mathcal{F}_n$ , and let  $X_n = \frac{d\mu_n}{d\nu_n} = \prod_{m=1}^n q_m$ . Then  $X_n \rightarrow X, \nu$ -a.s.  $\sum_{m=1}^{\infty} \log(q_m) > \infty$  is a tail event, so the Kolmogorov 0-1 law implies  $\nu(X = 0) \in \{0, 1\}$  and it follows that either  $\mu \ll \nu$  or  $\mu \perp \nu$ , according as  $\prod_{m=1}^{\infty} \int \sqrt{q_m} dG_m > 0$  or  $= 0$ .

#### 4.4 Doob's Inequality, Convergence in $L^p, p > 1$

- If  $X_n$  is a submartingale and  $N$  is a stopping time with  $P(N \leq k) = 1$ , then  $\mathbb{E}X_0 \leq \mathbb{E}X_N \leq \mathbb{E}X_k$ .
- Doob's inequality: Let  $X_m$  be a submartingale,  $\bar{X}_n = \max_{0 \leq m \leq n} X_m^+, \lambda > 0$  and  $A = \{\bar{X}_n \geq \lambda\}$ . Then  $\lambda P(A) \leq \mathbb{E}X_n 1_A \leq \mathbb{E}X_n^+$ .
- $L^p$  maximum inequality: If  $X_n$  is a submartingale, then for  $1 < p < \infty, \mathbb{E}(\bar{X}_n^p) \leq (\frac{p}{p-1})^p \mathbb{E}(X_n^+)^p$ . Consequently, if  $Y_n$  is a martingale and  $Y_n^* = \max_{0 \leq m \leq n} |Y_m|, \mathbb{E}|Y_n^*|^p \leq (\frac{p}{p-1})^p \mathbb{E}(|Y_n|^p)$ .
- $L^p$  convergence theorem: If  $X_n$  is a martingale with  $\sup \mathbb{E}|X_n|^p < \infty$  where  $p > 1$ , then  $X_n \rightarrow X$  a.s. and in  $L^p$ .

#### 4.5 Square Integrable Martingales

- In this subsection, we will suppose  $X_n$  is a martingale with  $X_0 = 0$  and  $\mathbb{E}X_n^2 < \infty$  for all  $n$ .
- Let  $X_n^2 = M_n + A_n$  be the Doob decomposition of  $X_n^2$ . Then  $X_n$  is  $L^2$ -bounded iff  $\mathbb{E}A_{\infty} = \sum_{n=1}^{\infty} \mathbb{E}(X_n - X_{n-1})^2 < \infty$ .
- $\mathbb{E}(\sup_m |X_m|^2) \leq 4\mathbb{E}A_{\infty}$ .
- $\lim_{n \rightarrow \infty} X_n$  exists and is finite a.s. on  $\{A_{\infty} < \infty\}$ .
- Let  $f \geq 1$  be increasing with  $\int_0^{\infty} f(t)^{-2} dt < \infty$ . Then  $X_n/f(A_n) \rightarrow 0$  a.s. on  $\{A_{\infty} = \infty\}$ .
- Second Borel-Cantelli Lemma: Suppose  $B_n$  is adapted to  $\mathcal{F}_n$  and  $p_n = P(B_n | \mathcal{F}_{n-1})$ .  $\sum_{m=1}^n 1_{B(m)} / \sum_{m=1}^n p_m \rightarrow 1$  a.s. on  $\{\sum_{m=1}^{\infty} p_m = \infty\}$ .
- $\mathbb{E}(\sup_n |X_n|) \leq 3\mathbb{E}A_{\infty}^{1/2}$ .

#### 4.6 Uniform Integrability, Convergence in $L^1$

- $\{X_i\}_{i \in I}$  is uniformly integrable if  $\lim_{M \rightarrow \infty} (\sup_{i \in I} \mathbb{E}(|X_i|; |X_i| > M)) = 0$ .
- Given a probability space  $(\Omega, \mathcal{F}_0, P)$  and an  $X \in L^1$ , then  $\{\mathbb{E}(X | \mathcal{F}) : \mathcal{F} \text{ is a } \sigma\text{-field } \subset \mathcal{F}_0\}$  is uniformly integrable.
- Let  $\phi \geq 0$  be any function with  $\phi(x)/x \rightarrow \infty$  as  $x \rightarrow \infty$ . If  $\mathbb{E}\phi(|X_i|) \leq C$  for all  $i \in I$ , then  $\{X_i, i \in I\}$  is uniformly integrable.
- Suppose that  $\mathbb{E}|X_n| < \infty$  for all  $n$ . If  $X_n \rightarrow X$  in probability, then the following are equivalent: (i)  $\{X_n : n \geq 0\}$  is uniformly integrable. (ii)  $X_n \rightarrow X$  in  $L^1$ . (iii)  $\mathbb{E}|X_n| \rightarrow \mathbb{E}|X| < \infty$ .
- For a submartingale, the following are equivalent: (i) It is uniformly integrable. (ii) It converges a.s. and in  $L^1$ . (iii) It converges in  $L^1$ .

- If a martingale  $X_n \rightarrow X$  in  $L^1$ , then  $X_n = \mathbb{E}(X|\mathcal{F}_n)$ .
- For a martingale, the following are equivalent: (i) It is uniformly integrable. (ii) It converges a.s. and in  $L^1$ . (iii) It converges in  $L^1$ . (iv) There is an integrable random variable  $X$  so that  $X_n = \mathbb{E}(X|\mathcal{F}_n)$ .
- Suppose  $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ , i.e.,  $\mathcal{F}_n$  is an increasing sequence of  $\sigma$ -fields and  $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$ . As  $n \rightarrow \infty$ ,  $\mathbb{E}(X|\mathcal{F}_n) \rightarrow \mathbb{E}(X|\mathcal{F}_\infty)$  a.s. and in  $L^1$ .
- Lévy's 0-1 law: If  $\mathcal{F}_n \uparrow \mathcal{F}_\infty$  and  $A \in \mathcal{F}_\infty$ , then  $\mathbb{E}(1_A|\mathcal{F}_n) \rightarrow 1_A$  a.s.

#### 4.7 Backwards Martingales

- A backwards martingale is a martingale indexed by the negative integers, i.e.,  $X_n, n \leq 0$ , adapted to an increasing sequence of  $\sigma$ -fields  $\mathcal{F}_n$  with  $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$  for  $n \leq -1$ .
- $X_{-\infty} = \lim_{n \rightarrow -\infty} X_n$  exists a.s. and in  $L^1$ .
- If  $X_{-\infty} = \lim_{n \rightarrow -\infty} X_n$  and  $\mathcal{F}_{-\infty} = \cap_n \mathcal{F}_n$ , then  $X_{-\infty} = \mathbb{E}(X_0|\mathcal{F}_{-\infty})$ .
- A sequence  $X_1, X_2, \dots$  is said to be exchangeable if for each  $n$  and permutation  $\pi$  of  $\{1, \dots, n\}$ ,  $(X_1, \dots, X_n)$  and  $(X_{\pi(1)}, \dots, X_{\pi(n)})$  have the same distribution. If  $X_1, X_2, \dots$  are exchangeable then conditional on  $\mathcal{E}$  (exchangeable  $\sigma$ -field),  $X_1, X_2, \dots$  are independent and identically distributed.
- If  $X_1, X_2, \dots$  are exchangeable and take values in  $\{0, 1\}$ , then there is a probability distribution on  $[0, 1]$  so that  $P(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0) = \int_0^1 \theta^k (1 - \theta)^{n-k} dF(\theta)$ .

#### 4.8 Optional Stopping Theorems

- If  $X_n$  is a uniformly integrable submartingale, then for any stopping time  $N$ ,  $X_{N \wedge n}$  is uniformly integrable.
- If  $\mathbb{E}|X_N| < \infty$  and  $X_n 1_{(N > n)}$  is uniformly integrable, then  $X_{N \wedge n}$  is uniformly integrable and hence  $\mathbb{E}X_0 \leq \mathbb{E}X_N$ .
- If  $X_n$  is a uniformly integrable submartingale, then for any stopping time  $N \leq \infty$ , we have  $\mathbb{E}X_0 \leq \mathbb{E}X_N \leq \mathbb{E}X_\infty$ , where  $X_\infty = \lim_n X_n$ .
- If  $X_n$  is a nonnegative supermartingale and  $N \leq \infty$  is a stopping time, then  $\mathbb{E}X_0 \leq \mathbb{E}X_N$ , where  $X_\infty = \lim_n X_n$ .