

Advanced Theory of Probability

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1 Review of Measure Theory

- Fatou's lemma: If $f_n \geq 0$ then $\liminf_{n \rightarrow \infty} \int f_n d\mu \geq \int \liminf_{n \rightarrow \infty} f_n d\mu$.
- Monotone convergence theorem: If $f_n \geq 0$ and $f_n \uparrow f$ then $\int f_n d\mu \uparrow \int f d\mu$.
- Dominated convergence theorem: If $f_n \rightarrow f$ a.e., $|f_n| \leq g$ for all n , and g is integrable, then $\int f_n d\mu \rightarrow \int f d\mu$.
- Suppose $X_n \rightarrow X$ a.s. Let $g(x), h(x)$ be continuous functions with (i) $g(x) \geq 0$ and $g(x) \rightarrow \infty$ as $|x| \rightarrow \infty$; (ii) $|h(x)|/g(x) \rightarrow 0$ as $|x| \rightarrow \infty$; (iii) $\mathbb{E}g(X_n) \leq K < \infty$ for all n . Then $\mathbb{E}h(X_n) \rightarrow \mathbb{E}h(X)$.
- Fubini's theorem: If $f \geq 0$ or $\int |f| d\mu < \infty$, $\int_X \int_Y f(x, y) \mu_2(dy) \mu_1(dx) = \int_{X \times Y} f d\mu = \int_Y \int_X f(x, y) \mu_1(dx) \mu_2(dy)$.

2 Laws of Large Numbers

2.1 Independence

- Two events A and B are independent if $P(A \cap B) = P(A)P(B)$. Two random variables X and Y are independent if for all $C, D \in \mathbb{R}$, $P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$. Two σ -fields \mathcal{F} and \mathcal{G} are independent if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$ the events A and B are independent.
- σ -fields $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent if whenever $A_i \in \mathcal{F}_i$ for $i = 1, \dots, n$, we have $P(\cap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i)$. Random variables X_1, \dots, X_n are independent if whenever $B_i \in \mathbb{R}$ for $i = 1, \dots, n$ we have $P(\cap_{i=1}^n \{X_i \in B_i\}) = \prod_{i=1}^n P(X_i \in B_i)$. Sets A_1, \dots, A_n are independent if whenever $I \subset \{1, \dots, n\}$ we have $P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$.
- A sequence of events A_1, \dots, A_n with $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$ is called pairwise independent.
- π - λ theorem: If \mathcal{P} is a π -system and \mathcal{L} is a λ -system that contains \mathcal{P} then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
- Suppose $\mathcal{A}_1, \dots, \mathcal{A}_n$ are independent and each \mathcal{A}_i is a π -system. Then $\sigma(\mathcal{A}_1), \dots, \sigma(\mathcal{A}_n)$ are independent.
- Suppose $\mathcal{F}_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m(i)$ are independent and let $\mathcal{G}_i = \sigma(\cup_j \mathcal{F}_{i,j})$. Then $\mathcal{G}_1, \dots, \mathcal{G}_n$ are independent.
- If for $1 \leq i \leq n, 1 \leq j \leq m(i)$, $X_{i,j}$ are independent and $f_i : \mathbb{R}^{m(i)} \rightarrow \mathbb{R}$ are measurable then $f_i(X_{i,1}, \dots, X_{i,m(i)})$ are independent.
- If X_1, \dots, X_n are independent and have (a) $X_i \geq 0$ for all i , or (b) $\mathbb{E}|X_i| < \infty$ for all i then $\mathbb{E}(\prod_{i=1}^n X_i) = \prod_{i=1}^n \mathbb{E}X_i$.
- If X and Y are independent, $F(x) = P(X \leq x)$, and $G(y) = P(Y \leq y)$, then $P(X + Y \geq z) = \int F(z - y) dG(y)$.

2.2 Weak Laws of Large Numbers

- L^2 weak law: Let X_1, X_2, \dots be uncorrelated random variables with $\mathbb{E}X_i = \mu$ and $\text{var}(X_i) \leq C < \infty$. If $S_n = X_1 + \dots + X_n$, then as $n \rightarrow \infty$, $S_n/n \rightarrow \mu$ in L^2 and in probability.
- Let $\mu_n = \mathbb{E}[S_n], \sigma_n^2 = \text{var}(S_n)$. If $\sigma_n^2/b_n^2 \rightarrow 0$ then $\frac{S_n - \mu_n}{b_n} \rightarrow 0$ in probability.
- Truncation: To truncate a random variable X at level M means to consider $\bar{X}_M = X1_{\{|X| \leq M\}}$.
- For each n , let $X_{n,k}, 1 \leq k \leq n$ be independent. Let $0 < b_n \rightarrow \infty$ and $\bar{X}_{n,k} = X_{n,k}1_{\{|X_{n,k}| \leq b_n\}}$. Suppose that as $n \rightarrow \infty$ (1) $\sum_{k=1}^n P(|X_{n,k}| > b_n) \rightarrow 0$; (2) $b_n^{-2} \sum_{k=1}^n \text{var}(\bar{X}_{n,k}) \rightarrow 0$. If we let $S_n = \sum_{k=1}^n X_{n,k}$ and $a_n = \sum_{k=1}^n \mathbb{E}[\bar{X}_{n,k}]$, then $\frac{S_n - a_n}{b_n} \rightarrow 0$ in probability.

- Let X_1, X_2, \dots be i.i.d. with $xP(|X_1| > x) \rightarrow 0$ as $x \rightarrow \infty$. Let $S_n = X_1 + \dots + X_n$ and let $\mu_n = \mathbb{E}[X_1 1_{\{|X_1| \leq n\}}]$. Then $S_n/n - \mu_n \rightarrow 0$ in probability.
- If $Y \geq 0$ and $p > 0$ then $\mathbb{E}[Y^p] = \int_0^\infty py^{p-1}P(Y > y)dy$.
- Let $\{X_i\}_{i=1}^\infty$ be i.i.d. with $\mathbb{E}[|X_i|] < \infty$. Let $S_n = X_1 + \dots + X_n$ and let $\mu = \mathbb{E}[X_1]$. Then $S_n/n \rightarrow \mu$ in probability.
- The distribution of X is infinitely divisible iff for any $n \in \mathbb{N}$, there exists i.i.d. Y_i 's such that $X = \sum_{i=1}^n Y_i$.
- The distribution of X is stable if for all $a, b > 0$, and X_1, X_2 i.i.d. copies of X , $aX_1 + bX_2 \stackrel{d}{=} cX + d$ for some $c > 0$.

2.3 Borel-Cantelli Lemmas

- If A_n is a sequence of subsets of Ω , then we write

$$\limsup A_n = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m = \{\omega : \omega \text{ in infinitely many } A_i\text{'s}\}$$

$$\liminf A_n = \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m = \{\omega : \omega \text{ in all but finitely many } A_i\text{'s}\}$$

- $P(\limsup A_n) \geq \limsup P(A_n)$, $P(\liminf A_n) \leq \liminf P(A_n)$.
- Borel-Cantelli lemma: If $\sum_i P(A_i) < \infty$, then $P(A_n \text{ i.o.}) = 0$.
- Let y_n be a sequence of elements of a topological space. If every subsequence $y_{n(m)}$ has a further subsubsequence $y_{n(m_k)}$ that converges to y , then $y_n \rightarrow y$.
- $X_n \rightarrow X$ in probability iff for every subsequence $X_{n(m)}$ there is a further subsubsequence $X_{n(m_k)}$ that converges a.s. to X .
- If f is continuous and $X_n \rightarrow X$ in probability then $f(X_n) \rightarrow f(X)$ in probability. If in addition f is bounded then $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X)]$.
- Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}[X_i] = \mu$ and $\mathbb{E}[X_i^4] < \infty$. Then $S_n/n \rightarrow \mu$ a.s.
- For events $A_n, n = 1, 2, \dots$, independent such that $\sum_{n=1}^\infty P(A_n) = \infty$, then $P(A_n \text{ i.o.}) = 1$.
- If X_1, X_2, \dots are i.i.d. r.v.'s with $\mathbb{E}[X_i] = \infty$, then $P(|X_n| \geq n \text{ i.o.}) = 1$. Let $C = \{\lim S_n/n \text{ exists \& is finite}\}$. Then $P(C) = 0$.
- If A_1, A_2, \dots are pairwise independent and $\sum_{n=1}^\infty P(A_n) = \infty$ then $\sum_{i=1}^n 1_{A_i} / \sum_{i=1}^n P(A_i) \rightarrow 1$ a.s. as $n \rightarrow \infty$.
- For a sequence of increasing events A_n , $P(A_n \text{ i.o.}) = 1$ iff $\sum_n P(A_n | A_{n-1}^c) = \infty$.

2.4 Strong Law of Large Numbers

- Strong law of large numbers: Let X_1, X_2, \dots be pairwise independent identically distributed random variables with $\mathbb{E}[X_i] < \infty$. Let $\mathbb{E}X_i = \mu$ and $S_n = X_1 + \dots + X_n$. Then $S_n/n \rightarrow \mu$ a.s. as $n \rightarrow \infty$.
- Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}[X^+] = \infty$ and $\mathbb{E}[X^-] < \infty$, then $S_n/n \rightarrow \infty$ a.s.
- Let X_1, X_2, \dots be i.i.d. with $0 < X_i < \infty$, write $T_n = X_1 + \dots + X_n$ and let $N_t = \sup\{n : T_n \leq t\}$. If $\mathbb{E}[X_1] = \mu \leq \infty$, then as $t \rightarrow \infty$, $N_t/t \rightarrow 1/\mu$, a.s.

- If $X_n \rightarrow X_\infty$ a.s. and $N(n) \rightarrow \infty$ a.s. then $X_{N(n)} \rightarrow X_\infty$ a.s. But the analogous result for convergence in probability is false!
- Empirical distribution functions: Let X_1, X_2, \dots be i.i.d. with distribution F and let $F_n(x) = \frac{\sum_{i=1}^n 1_{X_i \leq x}}{n}$. As $n \rightarrow \infty$, $\sup_x |F_n(x) - F(x)| \rightarrow 0$ a.s.
- Uniform law of large numbers: Suppose $f(x, \theta)$ is continuous in $\theta \in \Theta$ for some compact Θ . Let X_1, X_2, \dots be a sequence of i.i.d. random variables. If f is continuous at θ for a.s. all $x \in \mathbb{R}$ and measurable of x at each θ and there exists some function $d(x)$ such that $\mathbb{E}[d(X_i)] < \infty$ and for all $\theta \in \Theta$, $|f(x, \theta)| \leq d(x)$. Then $\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n f(X_i, \theta) - \mathbb{E}[f(X_1, \theta)]| \xrightarrow{\text{a.s.}} 0$.

2.5 Convergence of Random Series

- Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of random variables. Define $\mathcal{F}'_n = \sigma(X_n, X_{n+1}, \dots)$ as the information of the future after time n . Let $\mathcal{I} = \cap_{n=1}^\infty \mathcal{F}'_n$ be the tail σ -field, i.e., the information in the remote future. Intuitively, $A \in \mathcal{I}$ if and only if changing a finite number of values does not affect the occurrence of the event.
- Kolmogorov's 0-1 law: If $X_1, X_2, \dots, X_n, \dots$ are independent and $A \in \mathcal{I}$, then $P(A) = 0$ or 1 .
- A finite permutation of \mathbb{N} is a map from \mathbb{N} onto \mathbb{N} such that there is a finite I with $\pi(i) = i$ for all $i \geq I$. For $S^\mathbb{N}$, associated with its natural product sigma field $\mathcal{F}^\mathbb{N}$, and any $\omega = (\omega_1, \omega_2, \dots)$, let $\pi(\omega) = (\omega_{\pi(1)}, \omega_{\pi(2)}, \dots)$. An event $A \in \mathcal{F}^\mathbb{N}$ is permutable if $\pi^{-1}(A) = A$ for any finite permutation π . All permutable events form the exchangeable σ -field, denoted by \mathcal{E} . All events in the tail σ -field \mathcal{I} are permutable.
- Hewitt-Savage 0-1 law: If X_1, X_2, \dots , are i.i.d. and $B \in \mathcal{E}(\mathbb{R}^\mathbb{N})$. Denote $X = (X_1, X_2, \dots)$. Then $P(X \in B) = 0$ or 1 .
- Kolmogorov's maximal inequality: Suppose X_1, X_2, \dots, X_n are independent with $\mathbb{E}[X_i] = 0$, $\text{var}(X_i) < \infty$. Let $S_n = X_1 + \dots + X_n$, then $P(\max_{k \leq n} |S_k| \geq x) \leq \frac{\text{var}(S_n)}{x^2}$.
- We call a sequence of r.v.'s S_1, S_2, \dots a martingale if (i) there is a sequence of σ -algebras $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ and $S_i \in \mathcal{F}_i$ for all i ; (ii) S_i 's are integrable; (iii) For each k , $\mathbb{E}[S_{k+1} | \mathcal{F}_k] = S_k$. If the "=" in (iii) is replaced by \geq (resp. \leq), then we say that this sequence is a submartingale (resp. supermartingale).
- Second-moment criterion: Suppose X_1, X_2, \dots are independent and centered (i.e., for all i , $\mathbb{E}[X_i] = 0$). If $\sum_{n=1}^\infty \text{var}(X_n) < \infty$, then $P(\sum_{n=1}^\infty X_n(\omega) \text{ converges}) = 1$.
- Kronecker's lemma: If $a_n \uparrow \infty$ and $\sum_{n=1}^\infty x_n/a_n$ converges, then $a_n^{-1} \sum_{m=1}^n x_m \rightarrow 0$.
- Let X_1, X_2, \dots be i.i.d. random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \sigma^2 < \infty$. Let $S_n = X_1 + \dots + X_n$. If $\epsilon > 0$, then $S_n/n^{1/2}(\log n)^{1/2+\epsilon} \rightarrow 0$ a.s.
- Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[|X_i|^p] < \infty$ where $1 < p < 2$. Write $S_n = X_1 + \dots + X_n$. Then $S_n/n^{1/p} \rightarrow 0$ a.s.
- Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}[X_1] = \infty$ and let $S_n = X_1 + \dots + X_n$. Let a_n be a sequence of positive numbers with a_n/n increasing. Then $\limsup_{n \rightarrow \infty} |S_n|/a_n = 0$ or ∞ according as $\sum_n P(|X_1| \geq a_n) < \infty$ or $= \infty$.
- Kolmogorov's three-series theorem: Let $X_1, X_2, \dots, X_n, \dots$ be independent random variables. Let $A > 0$ and $Y_i = X_i 1_{|X_i| \leq A}$. In order to show that $\sum X_i$ converges a.s., it is necessary and sufficient that (i) $\sum_{n=1}^\infty P(|X_n| > A) < \infty$; (ii) $\sum_{n=1}^\infty \mathbb{E}[Y_n]$ converges; (iii) $\sum_{n=1}^\infty \text{var}(Y_n) < \infty$.

2.6 Large Deviations

- Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}[X_1] = \mu$ and let $S_n = X_1 + X_2 + \dots + X_n$. According to CLT, the typical value of $S_n - n\mu$ is $O(\sqrt{n})$. What about atypical deviations of $S_n - n\mu$? According to WLLN, we know that for any $a > \mu$, $P(S_n > na) \rightarrow 0$. We want to discuss the existence and value of the limit: $\lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n > na)$.
- Let $\pi_n = P(S_n \geq na)$. Then $\pi_{n+m} \geq P(S_n \geq na, S_{n+m} - S_n \geq ma) = \pi_n \pi_m$. Let $\gamma_n = \log \pi_n$, $\gamma_{n+m} \geq \gamma_n + \gamma_m$. As $n \rightarrow \infty$ the limit of γ_n exists and $\lim_{n \rightarrow \infty} \frac{\gamma_n}{n} = \sup_n \frac{\gamma_n}{n}$. We define $\gamma(a) = \lim_{n \rightarrow \infty} \gamma_n/n \leq 0$. Then for any distribution and any n and a , $P(S_n \geq na) \leq e^{n\gamma(a)}$. We want to show $\gamma(a) < 0$ if $a > \mu$.
- If the moment generating function $\psi(\theta) = \mathbb{E}[\exp(\theta X_1)] < \infty$ for some $\theta > 0$, then $P(S_n \geq na) \leq \exp[n(\log \psi(\theta) - \theta a)]$. Let $\kappa(\theta) = \log \psi(\theta)$. If $a > \mu$, then $a\theta - \kappa(\theta) > 0$ for all sufficiently small θ .
- We will further strengthen our upper bounds by finding the maximum of $\lambda(\theta) = a\theta - \kappa(\theta)$. Let $\theta_+ = \sup\{\theta : \psi(\theta) < \infty\}$ and $\theta_- = \inf\{\theta : \psi(\theta) < \infty\}$. Now since that $\psi(\theta) \in C^\infty$ within (θ_-, θ_+) , we have $\lambda'(\theta) = a - \frac{\psi'(\theta)}{\psi(\theta)}$. So the maximal point of λ must satisfy $\psi'(\theta)/\psi(\theta) = a$. For the existence and uniqueness of such point(s), we introduce a new distribution, and use a trick named “tilting”.
- We now introduce the distribution F_θ by “reweighting F ”: $F_\theta(x) = \frac{1}{\psi(\theta)} \int_{-\infty}^x e^{y\theta} dF(y)$. By simple calculus, $\int x dF_\theta(x) = \frac{\psi'(\theta)}{\psi(\theta)}$, $\psi''(\theta) = \int x^2 e^{\theta x} dF(x)$, $\frac{d}{d\theta} \frac{\psi'(\theta)}{\psi(\theta)} = \int x^2 dF_\theta(x) - (\int x dF_\theta(x))^2 \geq 0$. If we assume the distribution F is not a point mass at μ , then $\frac{\psi'(\theta)}{\psi(\theta)}$ is strictly increasing and $a\theta - \log \psi(\theta)$ is concave. Since we have $\frac{\psi'(0)}{\psi(0)} = \mu$, this shows that for each $a > \mu$ there is at most one $\theta_a \geq 0$ that solves $a = \frac{\psi'(\theta_a)}{\psi(\theta_a)}$, and this value of θ maximizes $a\theta - \log \psi(\theta)$. Let F^n be the c.d.f. of $S_n = X_1 + \dots + X_n$ and F_λ^n be the c.d.f. of $S_n^\lambda = X_1^\lambda + \dots + X_n^\lambda$ where X_i i.i.d. $\sim F$ and X_i^λ i.i.d. $\sim F_\lambda = \frac{1}{\psi(\lambda)} \int_{-\infty}^x e^{y\lambda} dF(y)$. By induction, $\frac{dF_\lambda^n}{dF_\lambda^n} = e^{-\lambda x} \psi(\lambda)^n$. Then as $n \rightarrow \infty$, $n^{-1} \log P(S_n \geq na) \rightarrow -a\theta_a + \log \psi(\theta_a)$.
- Some important information: $\kappa(\theta) = \log \psi(\theta)$, $\kappa'(\theta) = \frac{\psi'(\theta)}{\psi(\theta)}$, θ_a solves $\kappa'(\theta_a) = a$, $\gamma(a) = \lim_{n \rightarrow \infty} \frac{1}{n} \log P(S_n \geq na) = -a\theta_a + \kappa(\theta_a)$.
- Suppose $x_o = \sup\{x : F(x) < 1\} = \infty$, $\theta_+ < \infty$, and $\psi'(\theta)/\psi(\theta)$ increases to a finite limit a_0 as $\theta \uparrow \theta_+$. If $a_0 \leq a < \infty$, $n^{-1} \log P(S_n \geq na) \rightarrow -a\theta_+ + \log \psi(\theta_+)$, i.e. $\gamma(a)$ is linear for $a \geq a_0$.
- Suppose $x_o = \sup\{x : F(x) < 1\} < \infty$ and F has no mass at x_o . Then $\psi(\theta) < \infty$ for all $\theta > 0$ and $\psi'(\theta)/\psi(\theta) \rightarrow x_o$ as $\theta \rightarrow \infty$.
- Now, we have shown the decaying asymptotic for all possible situations:

$$\left\{ \begin{array}{l} \text{If } x_o < \infty : \left\{ \begin{array}{l} a < x_o : \text{exponential, rate} = \theta_a \\ a = x_o : \text{exponential if } P(X_1 = x_o) > 0, 0 \text{ otherwise} \\ a > x_o : 0 \end{array} \right. \\ \text{If } x_o = \infty : \left\{ \begin{array}{l} \text{If } \theta_+ = \infty : \text{exponential, rate} = \theta_a \\ \text{If } \theta_+ < \infty : \left\{ \begin{array}{l} \text{If } \psi'(\theta)/\psi(\theta) \rightarrow \infty \text{ as } \theta \rightarrow \theta_+ : \text{exponential, rate} = \theta_a \\ \text{If } \psi'(\theta)/\psi(\theta) \rightarrow a_0 \text{ as } \theta \rightarrow \theta_+ : \left\{ \begin{array}{l} a < a_0 : \text{exponential, rate} = \theta_a \\ a \geq a_0 : \text{exponential, rate} = \theta_+ \end{array} \right. \end{array} \right. \end{array} \right.$$
- Cramér’s theorem: Let $I(a)$ be the Legendre transform of $\log \psi(\cdot)$: $I(a) := \sup_{\theta \in \mathbb{R}} (\theta a - \log \psi(\theta))$. Then for any closed set F , $\limsup_{n \rightarrow \infty} n^{-1} \log P(\frac{S_n}{n} \in F) \leq -\inf_{x \in F} I(x)$; for any open set G , $\liminf_{n \rightarrow \infty} n^{-1} \log P(\frac{S_n}{n} \in G) \geq -\inf_{x \in G} I(x)$.

- Intuition behind the tilting: Why do we want to introduce the measure F_θ ? Intuitively, the new measure is like a “distorting mirror” – it “distorts” our view on how each event is likely to happen. So, when we want to estimate a rare event A under P , suppose (1) we can construct a new measure Q such that $Q[A]$ is easily calculable, e.g., $Q[A] \approx 1$; (2) we have a uniform lower bound of the R-N derivative $dP/dQ \geq c$ on A . Then we can conclude that $P[A] = \int_A \frac{dP}{dQ} dQ \geq cQ[A]$.
- Let $\Sigma = \{a_1, \dots\}$ stand for a finite-size alphabet. Let $M_1(\Sigma)$ be the space of all probability measures on Σ . The entropy of some $\nu \in M_1(\Sigma)$ is $H(\nu) := -\sum_{i=1}^{|\Sigma|} \nu(a_i) \log(\nu(a_i))$. The relative entropy of ν with respect to some other $\mu \in M_1(\Sigma)$ is $H(\nu|\mu) := \sum_{i=1}^{|\Sigma|} \nu(a_i) \log \frac{\nu(a_i)}{\mu(a_i)}$.
- Let Y_i be i.i.d. r.v.'s, $\mu \in M_1(\Sigma)$. For $n \geq 1$, write $Y = (Y_1, \dots, Y_n)$ and call $L_n^Y \in M_1(\Sigma)$ be the empirical frequency of Y . Let $T_n(\nu)$ be the set of y a sequence of n letters whose empirical measure is ν .
- If $y \in T_n(\nu)$, then $P_\mu(Y = y) = e^{-n(H(\nu) + H(\nu|\mu))}$. In particular, if $y \in T_n(\mu)$, then $P_\mu(Y = y) = e^{-nH(\mu)}$.
- For every possible empirical measure ν of n letters, $(n+1)^{-|\Sigma|} e^{nH(\nu)} \leq |T_n(\nu)| \leq e^{nH(\nu)}$.
- For every possible empirical measure ν of n letters, $(n+1)^{-|\Sigma|} e^{nH(\nu|\mu)} \leq P_\mu(L_n^T = \nu) \leq e^{nH(\nu|\mu)}$.
- Sanov's theorem: For every set $\Gamma \subset M_1(\Sigma)$, $-\inf_{\nu \in \Gamma} H(\nu|\mu) \leq \liminf \frac{1}{n} \log P_\mu(L_n^Y \in \Gamma) \leq \limsup \frac{1}{n} \log P_\mu(L_n^Y \in \Gamma) \leq -\inf_{\nu \in \Gamma} H(\nu|\mu)$.

2.7 Percolation

- Fix $p \in [0, 1]$ and consider the d -dimensional lattice \mathbb{Z}^d . Assign to each edge $e \in \mathbb{E}$ an independent Bernoulli r.v. $I(e)$ with parameter p . If $I(e) = 1$, we say that this edge is open, otherwise closed. Consider the connected components of open edges, then for any $p \in [0, 1]$, $P_p(A) = 0$ or 1 where $A = \{\exists \text{ infinite open clusters}\}$.
- If A is translation-invariant, then $P(A) = 0$ or 1.
- Actually we can go further and show that for any $N = 0, 1, \dots, \infty$, $P_p[A(N)] = 0$ or 1, where $A(N) = \{\exists N \text{ infinite open clusters}\}$. Or even further: for $N = 2, 3, \dots$ and $N = \infty$, $P_p[A(N)] = 0$.
- Let $p_c = p_c(d) = \sup\{p : P_p(A) = 0\}$. Then one can show that $1/3 \leq p_c(2) \leq 2/3$. More generally, $p_c(1) = 1$ and for $d \geq 2$, $1/(2d-1) \leq p_c(d) \leq p_c(2) (= 1/2)$.
- By knowledge of Galton-Watson tree and the analogy between \mathbb{Z}^d and $2d$ -regular tree in high dimensions, we can take an educated guess that $p_c(d) \sim \frac{1}{2d}$ as $d \rightarrow \infty$.

3 Central Limit Theorems

3.1 The De Moivre-Laplace Theorem

- Central Limit Theorem: Let X_1, X_2, \dots be i.i.d. with mean μ and variance $\sigma^2 \in (0, \infty)$. Write $S_n = X_1 + \dots + X_n$, then $\frac{S_n - \mu n}{\sqrt{n}\sigma} \Rightarrow \mathcal{N}(0, 1)$.
- Before discussing the central limit theorem in full generality, we first see a special example for Bernoulli random variables. Let X_1, X_2, \dots be i.i.d. random variables such that $P(X_1 = 1) = P(X_1 = -1) = 1/2$ and write $S_n = X_1 + \dots + X_n$. For integers $|k| \leq n$, $P(S_{2n} = 2k) = C_{2n}^{n+k} 2^{-2n}$ since $(S_{2n} + 2n)/2 \sim \text{Binomial}(2n, 1/2)$.

- Local central limit theorem: If $2k/\sqrt{2n} \rightarrow x$, then $\lim_{n \rightarrow \infty} (\pi n)^{1/2} e^{x^2/2} P(S_{2n} = 2k) = 1$.
- The De Moivre-Laplace Theorem: For $a < b$, $P(a \leq S_n/\sqrt{n} \leq b) \rightarrow \int_a^b (2\pi)^{-1/2} e^{-x^2/2} dx$.

3.2 Weak Convergence

- A sequence of distribution function F_n is said to converge weakly to a limit F , denoted by $F_n \Rightarrow F$, if $F_n(y) \rightarrow F(y)$ at every point of continuity of F , i.e. every $y \in \mathbb{R}$ such that $F(\cdot)$ is continuous at y .
- A sequence of random variables X_n is said to converge weakly or converge in distribution / law to a limit X_∞ if their distribution functions F_n converges weakly.
- Skorokhod's representation theorem: If $F_n \Rightarrow F$ then there are random variables $Y_n, 1 \leq n < \infty$ and Y with living in the same probability space such that $Y_n \sim F_n, Y \sim F$ and $Y_n \rightarrow Y$ a.s.
- $X_n \Rightarrow X$ if and only if for every bounded continuous function g we have $\mathbb{E}g(X_n) \rightarrow \mathbb{E}g(X)$.
- Continuous mapping theorem: Let g be a measurable function and $D_g = \{x : g \text{ is discontinuous at } x\}$. If $X_n \Rightarrow X$, and $P(X \in D_g) = 0$, then $g(X_n) \Rightarrow g(X)$.
- Portmanteau theorem: The following statements are equivalent: (1) $X_n \Rightarrow X$; (2) G open, $\liminf_{n \rightarrow \infty} P(X_n \in G) \geq P(X \in G)$; (3) G closed, $\limsup_{n \rightarrow \infty} P(X_n \in G) \leq P(X \in G)$; (4) If $P(X \in \partial A) = 0$, then $\lim_{n \rightarrow \infty} P(X_n \in A) = P(X \in A)$.
- Helly's selection theorem: For every sequence F_n of distribution functions, there is a subsequence $F_{n(k)}$ and a right continuous nondecreasing function F so that at all points of continuity y of F , $\lim_{k \rightarrow \infty} F_{n(k)}(y) = F(y)$.
- Every subsequential limit of the sequence F_n is the distribution function of a probability measure iff the sequence is tight, i.e., for all $\epsilon > 0$, there is an M_ϵ so that $\limsup_{n \rightarrow \infty} [1 - F_n(M_\epsilon) + F_n(-M_\epsilon)] \leq \epsilon$.
- If there is a function $\phi \geq 0$ so that $\phi(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ and $C = \sup_n \int \phi(x) dF_n(x) < \infty$, then F_n is tight.

3.3 Characteristic Functions

- If X is a r.v., we define its Characteristic function (ch.f.) by $\phi(t) := \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)]$.
- All characteristic functions have the following properties: (i) $\phi(0) = 1$; (ii) $\phi(-t) = \overline{\phi(t)}$; (iii) $|\phi(t)| = |\mathbb{E}e^{itX}| \leq \mathbb{E}|e^{itX}| = 1$; (iv) $|\phi(t+h) - \phi(t)| \leq \mathbb{E}|e^{ihX} - 1|$, so $\phi(t)$ is uniformly continuous on \mathbb{R} ; (v) $\mathbb{E}e^{it(aX+b)} = e^{itb}\phi(at)$.
- If X_1 and X_2 are independent and have ch.f.'s ϕ_1 and ϕ_2 . Then $X_1 + X_2$ has ch.f. $\phi_1 \cdot \phi_2$.
- Stein's Lemma: If X, Y are jointly Gaussian, then for differentiable $g : \mathbb{R} \rightarrow \mathbb{R}$, as long as the expectations are well-defined, $\text{cov}(g(X), Y) = \text{cov}(X, Y)\mathbb{E}[g'(X)]$.
- If F_1, \dots, F_n have ch.f. ϕ_1, \dots, ϕ_n and $\lambda_i \geq 0, 1 \leq i \leq n$ have $\lambda_1 + \dots + \lambda_n = 1$. Then $\sum \lambda_i F_i$ has ch.f. $\sum \lambda_i \phi_i$.
- The inversion formula: If $a < b$, then $\frac{1}{2\pi} \lim_{T \rightarrow \infty} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt = \mu(a, b) + \frac{1}{2}\mu(\{a, b\})$.
- If $\int |\phi(t)| dt < \infty$, then μ has bounded continuous density $f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity} \phi(t) dt$.
- Continuity theorem: Let $\mu_n, 1 \leq n \leq \infty$ be probability measures with ch.f. ϕ_n . (i) If $\mu_n \Rightarrow \mu_\infty$ then $\phi_n(t) \rightarrow \phi_\infty(t)$ for all t . (ii) If $\phi_n(t) \rightarrow \phi(t)$ for all t , and $\phi(t)$ is continuous at 0. Then $\{\mu_n\}_{n=1}^\infty$ is tight and has a weak limit with ch.f. ϕ .

CENTRAL LIMIT THEOREMS

- Let μ be a probability measure and ϕ be its ch.f. Then $\mu(\{x : |x| \geq 2u^{-1}\}) \leq u^{-1} \int_{-u}^u [1 - \phi(t)] dt$.
- If $\int |x|^n \mu(dx) < \infty$, then its ch.f. ϕ has a continuous derivative of order n given by $\phi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx)$. In particular, $\phi^{(n)}(0) = \mathbb{E}[(iX)^n]$.
- However, if a characteristic function ϕ_X has a k -th derivative at zero, then the random variable X has all moments up to k if k is even, but only up to $(k-1)$ if k is odd.
- $|e^{ix} - \sum_{m=0}^n \frac{(ix)^m}{m!}| \leq \min(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!})$.
- If $\mathbb{E}|X|^2 < \infty$, then $\phi(t) = 1 + it\mathbb{E}X - t^2\mathbb{E}|X|^2/2 + o(t^2)$.
- If $\limsup_{h \downarrow 0} \frac{\phi(h) - 2\phi(0) + \phi(-h)}{h^2} > -\infty$, then $\mathbb{E}[X^2] < \infty$.
- Given ϕ and $x_1, \dots, x_n \in \mathbb{R}$, we can consider the matrix with (i, j) entry given by $\phi(x_i - x_j)$. Call ϕ positive definite if this matrix is always positive semi-definite Hermitian.
- Bochner's theorem: A function from \mathbb{R} to \mathbb{C} which is continuous at origin with $\phi(0) = 1$ is a ch.f. of some probability measure on \mathbb{R} if and only if it is positive definite.
- Pólya's theorem: If ϕ is real-valued, even and continuous such that (i) $\phi(0) = 1$; (ii) ϕ is convex for $t > 0$; (iii) $\phi(\infty) = 0$; then $\phi(t)$ is the ch.f. of a distribution symmetric about 0.

3.4 Central Limit Theorems

- Central Limit Theorem: Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}[X_1] = \mu, \text{var}(X_1) = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + X_2 + \dots + X_n$, $\frac{S_n - n\mu}{n^{1/2}\sigma} \Rightarrow \mathcal{N}(0, 1)$.
- The Lindeberg-Feller theorem: For each n , let $X_{n,m}, 1 \leq m \leq n$, be independent random variables for each n with $\mathbb{E}[X_{n,m}] = 0$. Suppose (i) $\sum_{m=1}^n \mathbb{E}[X_{n,m}^2] \rightarrow \sigma^2 > 0$; (ii) For all $\epsilon > 0$, $\lim_{n \rightarrow \infty} \sum_{m=1}^n \mathbb{E}[X_{n,m}^2 1_{|X_{n,m}| > \epsilon}] = 0$. Then $S_n = X_{n,1} + \dots + X_{n,n} \Rightarrow \mathcal{N}(0, \sigma^2)$ as $n \rightarrow \infty$.
- Converging together lemma: If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, $X_n + Y_n \Rightarrow X + c$. A useful consequence of this result is that if $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$ then $Z_n \Rightarrow X$.
- Lévy's condition for CLT: Let X_1, X_2, \dots be i.i.d. and $S_n = X_1 + \dots + X_n$. In order that there exist constants a_n and $b_n > 0$ so that $(S_n - a_n)/b_n \Rightarrow \mathcal{N}(0, 1)$, it is necessary and sufficient that $\frac{y^2 P(|X_1| > y)}{\mathbb{E}[X_1^2 1_{|X_1| \leq y}]} \rightarrow 0$.
- Chernoff bound: Let X_i be independent Bernoulli r.v.'s. Write $S_n = X_1 + \dots + X_n$ and let $\mu = \mathbb{E}[S_n]$. Then for $\delta > 0$, $P(S_n > (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2 + \delta}}, P(S_n < (1 - \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$.
- Hoeffding's inequality for bounded r.v. Let X_i be independent r.v.'s such that $X_i \in [a_i, b_i]$ a.s. Write $S_n = X_1 + \dots + X_n$ and let $\mu = \mathbb{E}[S_n]$. Then for $\delta > 0$, $P(|S_n - \mu| \geq \delta) \leq 2 \exp(-\frac{2n^2 \delta^2}{\sum_{i=1}^n (b_i - a_i)^2})$.
- A random variable is sub-Gaussian, if and only if for some $C < \infty$ and $c > 0$, $P(|X| \geq t) \leq C e^{-ct^2}$.
- Hoeffding's inequality for sub-Gaussian r.v.'s: Let X_i be independent zero-mean sub-Gaussian r.v.'s. Write $S_n = X_1 + \dots + X_n$. Then there exists some $c > 0$ such that for any $\delta > 0$, $P(|S_n| \geq \delta) \leq 2 \exp(-c\delta^2 / \sum_{i=1}^n \|X_i\|_{\psi_2})$, where $\|X\|_{\psi_2} = \inf\{c \geq 0 : \mathbb{E}[e^{X^2/c^2}] \leq 2\}$.
- Let X_1, X_2, \dots be i.i.d. with $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^2] = \sigma^2$, and $\mathbb{E}[|X_i|^3] = \rho < \infty$. Let $\mathcal{N}(x)$ is the distribution of the standard normal distribution, then for all $n \geq 1$ and $x \in \mathbb{R}$, $|F_n(x) - \mathcal{N}(x)| \leq 3\rho/(\sigma^3 \sqrt{n})$.

3.5 Local Limit Theorems

- A random variable X has a lattice distribution if $\exists b, h > 0$ so that $P(X \in b + h\mathbb{Z}) = 1$. The largest h for which the last statement holds is called the span of the distribution.
- Trichotomy of a random variable: Let $\phi(t)$ be the ch.f. of a random variable X . There are only three possibilities: (1) $|\phi(t)| < 1$ for all $t \neq 0$; (2) There is a $\lambda > 0$ so that $|\phi(\lambda)| = 1$ and $|\phi(t)| < 1$ for $0 < t < \lambda$. In this case, X has a lattice distribution with span $2\pi/\lambda$; (3) $|\phi(t)| = 1$ for all t . In this case, X is deterministic.
- Let X_i be i.i.d. r.v.'s with $\mathbb{E}[X_i] = 0, \mathbb{E}[X_i^2] = \sigma^2 \in (0, \infty)$. Suppose in addition $P(X_i \in b + h\mathbb{Z}) = 1$, i.e. X_i are lattice with span h . Let $p_n(x) = P(S_n/\sqrt{n} = x)$ for $x \in \mathcal{L}_n = \{(nb + h\mathbb{Z})/\sqrt{n}\}$, and $n(x)$ be the density of $\mathcal{N}(0, \sigma^2)$. Then $\lim_{n \rightarrow \infty} \sup_{x \in \mathcal{L}_n} |\frac{\sqrt{n}}{h} p_n(x) - n(x)| = 0$.
- Let X_i be i.i.d. nonlattice r.v.'s with $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = \sigma^2$. If $x_n/\sqrt{n} \rightarrow x$ and $a < b$, $\sqrt{n}P(S_n \in (x_n + a, x_n + b)) \rightarrow (b - a)n(x)$.
- Let $p_n^{(d)}(\cdot)$ stand for the n -step transition probability for d -dimensional simple random walk. Then $p_{2n}^{(d)}(0) \rightarrow$ monotone decreasing in d .

3.6 Poisson Convergence

- For each n let $X_{n,m}, 1 \leq m \leq n$ be independent random variables with $P(X_{n,m} = 1) = p_{n,m}, P(X_{n,m} = 0) = 1 - p_{n,m}$. Suppose (i) $\lim_{n \rightarrow \infty} \sum_{m=1}^n p_{n,m} = \lambda$; (ii) $\lim_{n \rightarrow \infty} \max_{m \leq n} p_{n,m} = 0$. Let $S_n := X_{n,1} + \dots + X_{n,n}$, then $S_n \Rightarrow \text{Poisson}(\lambda)$.
- $d(\mu, \nu) = \|\mu - \nu\|_{\text{TV}}$ defines a metric on the set of probability measures on \mathbb{Z} . $\|\mu_n - \mu\| \rightarrow 0$ if and only if $\mu_n \Rightarrow \mu$.
- The p -th Wasserstein distance between two probability measures μ and ν on M with p -th moment is defined as $W_p(\mu, \nu) = (\inf_{\gamma \in \Gamma(\mu, \nu)} \int_{M \times M} d(x, y)^p d\gamma(x, y))^{1/p}$ where $\Gamma(\mu, \nu)$ is the set of all couplings of μ and ν . One can show that W_p defines a metric and convergence under W_p -metric is equivalent to weak convergence plus convergence of the first p -th moment.
- Suppose that r balls are placed at random into n boxes. Then suppose $r/n \rightarrow c$, the number of balls in each box is approximately $\text{Poisson}(c)$. Let X_n be the number of empty boxes. Then if $ne^{-r/n} \rightarrow \lambda$, $X_n \rightarrow \text{Poisson}(\lambda)$.
- Let $X_{n,m}, 1 \leq m \leq n$ be independent random variables with $P(X_{n,m} = 1) = p_{n,m}, P(X_{n,m} \geq 2) = \epsilon_{n,m}$. Suppose $\lim_{n \rightarrow \infty} \sum_{m=1}^n p_{n,m} = \lambda, \lim_{n \rightarrow \infty} \max_{m \leq n} p_{n,m} = 0, \lim_{n \rightarrow \infty} \sum_{m=1}^n \epsilon_{n,m} = 0$. Let $S_n = X_{n,1} + \dots + X_{n,n}$, then $S_n \Rightarrow \text{Poisson}(\lambda)$.

3.7 Poisson Process

- Let $N(s, t)$ be the number of students arriving at a certain dinning hall in the time interval $(s, t]$. Suppose the number of arrivals in intervals that are disjoint are independent, the distribution of $N(s, t)$ only depends on $t - s$, $P(N(0, h) = 1) = \lambda h + o(h)$, $P(N(0, h) \geq 2) = o(h)$. Then $N(0, t)$ has a Poisson distribution with mean λt .
- A family of random variables $N_t, t \geq 0$ is called a Poisson process with rate λ , if (i) for $0 \leq t < s$, $N(s) - N(t) \sim \text{Poisson}(\lambda(s - t))$; (ii) if $0 < t_0 < t_1 < \dots < t_n, N(t_k) - N(t_{k-1}), 1 \leq k \leq n$ are independent.

- Suppose that between 12:00 and 1:00 cars arrive at the East Gate of PKU according to a Poisson process N_t with rate λ . Let Y_i be the number of people in the i -th vehicle which we assume to be i.i.d. and independent to N_t . Then consider $M(t)$ be the total number of visitors within those vehicles by time t , i.e. $M(t) = \sum_{i=1}^{N_t} Y_i$ with the convention that $M(t) = 0$ if $N_t = 0$.
- Let Y_1, Y_2, \dots be i.i.d. r.v.'s; N and independent non-negative interger-valued r.v.; $S = Y_1 + \dots + Y_N$ with $S = 0$ when $N = 0$. (1) If $\mathbb{E}[Y_i], \mathbb{E}[N] < \infty$, then $\mathbb{E}[S] = \mathbb{E}[N] \cdot \mathbb{E}[Y_i]$; (2) If $\mathbb{E}[Y_i^2], \mathbb{E}[N^2] < \infty$, then $\text{var}(S) = \mathbb{E}[N]\text{Var}(Y_i) + \text{var}(N)(\mathbb{E}[Y_i])^2$; (iii) If $N \sim \text{Poisson}(\lambda)$, then $\text{var}(S) = \lambda\mathbb{E}[Y_i^2]$.
- Recall the problem of counting the number of cars arriving at the East Gate of PKU. Noting that Y_i now stands for the number of people in each vehicel, Y_i has to take positive integer values. Let N_t^j be the number of cars with exactly j passengers. For Y_i taking value on $1, 2, \dots, m < \infty$, N_t^j are independent rate $\lambda P(Y_i = j)$ Poisson processes.
- Suppose that in a Poisson process with rate λ , for a point that lands at time s , we keep it with probability $p(s)$. Then the result is an inhomogenous Poisson process with rate $\lambda p(s)$.
- Inhomogenous Poisson process as time change of Poisson process: For $p(t)$, and the standard Poisson process N_t with rate λ , we call $\hat{N}(t) = N(\int_0^t \lambda p(s) ds)$ be the inhomogenous Poisson process with transition rate function $\lambda(t) = \lambda p(t)$.
- Suppose λ is σ -finite, we say a random measure μ is a Poisson Point Process/Poisson random measure with intensity measure λ if (1) for all $B \in \mathcal{S}$, $\mu(B) \sim \text{Poisson}(\lambda(B))$; (2) If B_1, \dots, B_n be disjoint sets in \mathcal{S} , then the random variables $\mu(B_1), \dots, \mu(B_n)$ are also independent.
- Let T_n be the time of the n -th arrival of a Poisson process with rate λ . Let U_1, U_2, \dots, U_n be independent uniform on $(0, t)$ and let $(V_k^n)_{k=1,2,\dots,n}$ be the order statistics of $\{U_1, \dots, U_n\}$, i.e. V_k^n is the k -th smallest number from (U_1, \dots, U_n) . Then, conditioning on $N(t) = n$, the vectors $V = (V_1^n, \dots, V_n^n)$ and $T = (T_1, \dots, T_n)$ have the same distribution.
- If $0 < s < t$, then $P(N_s = m | N_t = n) = C_n^m (s/t)^m (1 - s/t)^{n-m}$.

3.8 Limit Theorems in \mathbb{R}^d

- We say $X_n \Rightarrow X_\infty$ if $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$ for all bounded and continuous f .
- General Portmantean Theorem: The following statements are equivalent: (1) $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$ for all bounded and continuous f ; (2) $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$ for all bounded and Lipschitz-continuous f ; (3) For all closed sets K , $\limsup_{n \rightarrow \infty} P(X_n \in K) \leq P(X_\infty \in K)$; (4) For all open sets G , $\liminf_{n \rightarrow \infty} P(X_n \in G) \geq P(X_\infty \in G)$; (5) For all sets A with $P(X_\infty \in \partial A) = 0$, $\lim_{n \rightarrow \infty} P(X_n \in A) = P(X_\infty \in A)$; (6) Let D_f be the set of discontinuous of f . For all bounded functions f with $P(X_\infty \in D_f) = 0$, we have $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X_\infty)]$.
- For distribution F_n and F on \mathbb{R}^d , we say that F_n converges weakly to F , and write $F_n \Rightarrow F$, if $F_n(x) \rightarrow F(x)$ at all continuity points of F .
- Distribution function in \mathbb{R}^d : (i) Nondecreasing: $x \leq y \Rightarrow F(x) \leq F(y)$. (ii) $\lim_{x \rightarrow \infty} F(x) = 1, \lim_{x_i \rightarrow -\infty} F(x) = 0$. (iii) F is right continuous: $\lim_{y \uparrow x} F(y) = F(x)$. (iv) $\triangle_A F \geq 0$ for all rectangles A .
- Equivalence of two definitions: On \mathbb{R}^d weak convergence defined in terms of convergence of distribution $F_n \Rightarrow F_\infty$ is equivalent to notion of weak convergence defined for a general metric space.

- Tightness in \mathbb{R}^d : A sequence of probability measures μ_n is said to be tight if for any $\epsilon > 0$, there is an $M < \infty$ such that $\liminf_{n \rightarrow \infty} \mu_n([-M, M]^d) \geq 1 - \epsilon$.
- If μ_n is tight, there is a weakly convergent subsequence.
- The characteristic function of $\vec{X} = (X_1, \dots, X_d)$ is $\phi(\vec{t}) = \mathbb{E}[\exp(i\vec{t} \cdot \vec{X})]$. If $A = [a_1, b_1] \times \dots \times [a_d, b_d]$ with $\mu(\partial A) = 0$, then $\mu(A) = \lim_{T \rightarrow \infty} (2\pi)^{-d} \int_{[-T, T]^d} \left(\prod_{j=1}^d \psi_j(t_j) \right) \phi(\vec{t}) dt$, where $\psi_j(s) = \frac{\exp(-isa_j) - \exp(-isb_j)}{is}$.
- Convergence theorem: Let $X_n, 1 \leq n \leq \infty$ be random vectors with ch.f. ϕ_n . A necessary and sufficient condition for F_n to converge weakly to a probability distribution F_∞ is that $\phi_n(\vec{t}) \rightarrow \phi_\infty(\vec{t})$, which is continuous at 0.
- Cramer-Wold device: A sufficient condition for $X_n \Rightarrow X_\infty$ is that $\vec{\theta} \cdot X_n \Rightarrow \vec{\theta} \cdot X_\infty$ for all $\vec{\theta} \in \mathbb{R}^d$.
- The central limit theorem in \mathbb{R}^d : Let X_1, X_2, \dots be i.i.d. random vectors with $\mathbb{E}X_n = \mu$, and finite covariances $(\Gamma_{i,j})_{m \times m}$. Then $(S_n - n\mu)/n^{1/2} \Rightarrow \chi$, where χ is a multivariate normal with mean 0 and covariances $(\Gamma_{i,j})_{m \times m}$.

4 Martingales

4.1 Conditional Expectation

- Existence and uniqueness of conditional expectation: Let (Ω, \mathcal{H}, P) be a probability space, X be a random variable such that $\mathbb{E}[|X|] < \infty$, $\mathcal{G} \subset \mathcal{H}$ be a sub σ -algebra of \mathcal{H} . Then (1) Existence: \exists r.v. Y such that $Y \in \mathcal{G}, \mathbb{E}[|Y|] < \infty$ and $\forall G \in \mathcal{G}, \mathbb{E}[Y; G] = \mathbb{E}[X; G]$. We call such Y a version of $\mathbb{E}[X|\mathcal{G}]$. (2) Uniqueness: If Y, \tilde{Y} are versions of $\mathbb{E}[X|\mathcal{G}]$, then $Y = \tilde{Y}$ a.s.
- Orthogonal projection in L^2 : If $\mathbb{E}[X^2] < \infty$, then $Y = \mathbb{E}[X|\mathcal{G}]$ is a version of the orthogonal projection of X from $L^2(\Omega, \mathcal{H}, P)$ to $L^2(\Omega, \mathcal{G}, P)$, i.e. Y is the best G -measurable predictor of X , which minimizes $\mathbb{E}[(Y - X)^2]$.
- Properties of conditional expectation: (1) $Y = \mathbb{E}[X|\mathcal{G}] \Rightarrow \mathbb{E}[Y] = \mathbb{E}[X]$. (2) $X \in \mathcal{G} \Rightarrow \mathbb{E}[X|\mathcal{G}] = X$ a.s. (3) Linearity: $\mathbb{E}[aX_1 + bX_2|\mathcal{G}] = a\mathbb{E}[X_1|\mathcal{G}] + b\mathbb{E}[X_2|\mathcal{G}]$ a.s. (4) Positivity: $X \geq 0 \Rightarrow \mathbb{E}[X|\mathcal{G}] \geq 0$ a.s. (5) Monotone convergence theorem: $0 \leq X_n \uparrow X \Rightarrow \mathbb{E}[X_n|\mathcal{G}] \uparrow \mathbb{E}[X|\mathcal{G}]$ a.s. (6) Fatou's lemma: $X_n \geq 0 \Rightarrow \mathbb{E}[\liminf_{n \rightarrow \infty} X_n|\mathcal{G}] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[X_n|\mathcal{G}]$ a.s. (7) Dominated convergence theorem: $|X_n(\omega)| \leq V(\omega)$ a.s. $\forall n, \mathbb{E}[V] < \infty, X_n \rightarrow X$ a.s., then $\mathbb{E}[X_n|\mathcal{G}] \rightarrow \mathbb{E}[X|\mathcal{G}]$ a.s. (8) If $c(x)$ is convex, $\mathbb{E}[|c(x)|] < \infty$, then $\mathbb{E}[c(x)|\mathcal{G}] \geq c(\mathbb{E}[X|\mathcal{G}])$ a.s. (9) Tower property: If $\mathcal{H} \subset \mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathcal{H}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]\mathcal{G}] = \mathbb{E}[X|\mathcal{H}]$. (10) If $Z \in \mathcal{G}$ then $\mathbb{E}[ZX|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}]$. (11) If $\mathcal{H} \perp \sigma(X, \mathcal{G})$ then $\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}]$ a.s. In particular, if $X \perp \mathcal{H}$, then $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$ a.s.

4.2 Martingales, Almost Sure Convergence

- Filtered spaces: $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^\infty, P)$ satisfies $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$ (i.e. $\{\mathcal{F}_n\}_{n=1}^\infty$ is a filtration) and $\sigma(\cup_{i=0}^\infty \mathcal{F}_i) := \mathcal{F}_\infty \subset \mathcal{F}$ (but not necessarily $\mathcal{F}_\infty = \mathcal{F}$). Given a filtration $\{\mathcal{F}_n\}$, if a sequence of r.v.'s $\{X_n\}$ satisfies $X_n \in \mathcal{F}_n$, we say $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$.
- Martingale: $X = \{X_n\}$ discrete time stochastic process is a martingale if: (1) $\{X_n\}$ is adapted to some filtration $\{\mathcal{F}_n\}$; (2) $\forall n, \mathbb{E}[|X_n|] < \infty$ (but not necessarily $\mathbb{E}[|X_n|] < M < \infty$); (3) $\forall n, \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$. If “=” in (3) is replaced by “ \geq ” or “ \leq ”, then we say X is a submartingale/supermartingale.
- $m < n, \{X_n\}$ is martingale/submartingale/supermartingale, $\mathbb{E}[X_n|\mathcal{F}_m] = / \geq / \leq X_m$.
- If X_n is a martingale w.r.t. \mathcal{F}_n and ϕ is a convex function with $\mathbb{E}[\phi(X_n)] < \infty$ for all n , then $\phi(X_n)$ is a submartingale w.r.t. \mathcal{F}_n .

- A process is predictable if $C_n \in \mathcal{F}_{n-1}$.
- You can't beat the system: Let $Y_n = \sum_{k=1}^n C_k(X_k - X_{k-1})$, C is a predictable process. (1) If C is non-negative, $|C_n(\omega)| \leq K, \forall n, \forall \omega$, and X is martingale/supermartingale, then Y is martingale/supermartingale. (2) If C is a bounded predictable process and X is a martingale, then Y is a martingale. (3) In (1) and (2), the boundness condition on C may be replaced by the condition $C_n \in L^2, \forall n$, provided we also insist that $X_n \in L^2, \forall n$.
- Stopping time: $T : \Omega \rightarrow \mathbb{Z}_+$, if $\{T \leq n\} \in \mathcal{F}_n, \forall n \leq \infty$.
- If X is a martingale/supermartingale and T is a stopping time, then the stopped process $(X_{T \wedge n})_n$ is a martingale/supermartingale, $\mathbb{E}[X_{T \wedge n}] = / \leq \mathbb{E}[X_0]$.
- Doob's optional stopping theorem: Let T be a stopping time and X be a martingale/supermartingale. Then X_T is integrable and $\mathbb{E}[X_T] = / \leq \mathbb{E}[X_0]$ in each of the following situations: (1) T is bounded; (2) X is bounded and T is a.s. finite; (3) $\mathbb{E}[T] < \infty$, and, for some $K \in \mathbb{R}_+$, $|X_n(\omega) - X_{n-1}(\omega)| \leq K$.
- Define $C_1 := I_{\{X_0 < a\}}$ and, for $n \geq 2$, $C_n := I_{\{C_{n-1}=1\}}I_{\{X_{n-1} \leq b\}} + I_{\{C_{n-1}=0\}}I_{\{X_{n-1} < a\}}$. $Y_n = \sum_{k=1}^n C_k(X_k - X_{k-1})$. The number $U_N[a, b](\omega)$ of upcrossings of $[a, b]$ made by $n \mapsto X_n(\omega)$ by time N is defined to be the largest k in \mathbb{Z}_+ such that we can find $0 \leq s_1 < t_1 < s_2 < t_2 < \dots < s_k < t_k \leq N$ with $X_{s_i}(\omega) < a, X_{t_i}(\omega) > b, 1 \leq i \leq k$.
- The fundamental inequality (recall that $Y_0(\omega) = 0$) is obvious: $Y_N(\omega) \geq (b - a)U_N[a, b](\omega) - [X_N(\omega) - a]^-$.
- Doob's upcrossing lemma: Let X be a supermartingale. Let $U_N[a, b]$ be the number of upcrossings of $[a, b]$ by time N . Then $(b - a)\mathbb{E}U_N[a, b] \leq \mathbb{E}[(X_N - a)^-]$.
- Let X be a supermartingale bounded in L^1 in that $\sup_n \mathbb{E}|X_n| < \infty$. Let $a, b \in \mathbb{R}$ with $a < b$. Then, with $U_\infty([a, b]) := \lim_N U_N[a, b]$, $(b - a)\mathbb{E}U_\infty[a, b] \leq |a| + \sup_n \mathbb{E}|X_n| < \infty$ so that $P(U_\infty[a, b] = \infty) = 0$.
- Doob's forward convergence theorem: Let X be a supermartingale bounded in L_1 : $\sup_n \mathbb{E}|X_n| < \infty$. Then, almost surely, $X_\infty := \lim_n X_n$ exists and is finite. For definiteness, we define $X_\infty(\omega) := \limsup_n X_n(\omega), \forall \omega$, so that X_∞ is \mathcal{F}_∞ measurable and $X_\infty = \lim_n X_n$, a.s.
- Martingale convergence theorem: If X_n is a submartingale with $\sup \mathbb{E}X_n^+ < \infty$, then as $n \rightarrow \infty$, X_n converges a.s. to a limit X with $\mathbb{E}|X| < \infty$.
- If $X_n \geq 0$ is a supermartingale, then as $n \rightarrow \infty$, $X_n \rightarrow X$ a.s. and $\mathbb{E}X \leq \mathbb{E}X_0$.

4.3 Examples

- Doob's decomposition: Any submartingale $X_n, n \geq 0$, can be written in a unique way as $X_n = M_n + A_n$, where M_n is a martingale and A_n is a predictable increasing sequence with $A_0 = 0$.
- Let X_1, X_2, \dots be a martingale with $|X_{n+1} - X_n| \leq M < \infty$. Let $C = \{\lim_n X_n \text{ exists and is finite}\}, D = \{\limsup_n X_n = +\infty \text{ and } \liminf_n X_n = -\infty\}$. Then $P(C \cup D) = 1$.
- Second Borel-Cantelli lemma: Let $\mathcal{F}_n, n \geq 0$ be a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $B_n, n \geq 1$ a sequence of events with $B_n \in \mathcal{F}_n$. Then $\{B_n \text{ i.o.}\} = \{\sum_{n=1}^\infty P(B_n | \mathcal{F}_{n-1}) = \infty\}$.
- Let μ, ν be two probability measures on (Ω, \mathcal{F}) . Let $\mathcal{F}_n \uparrow \mathcal{F}$ be σ -fields. Let μ_n and ν_n be the restrictions of μ and ν to \mathcal{F}_n . Suppose $\mu_n \ll \nu_n$ for all n . Let $X_n = d\mu_n/d\nu_n$ and let $X = \limsup_n X_n$. Then $\mu(A) = \int_A X d\nu + \mu(A \cap \{X = \infty\}) := \mu_r(A) + \mu_s(A)$, which gives the Lebesgue decomposition of μ , i.e., $\mu_r \ll \nu, \mu_s \perp \nu$.

- Kakutani dichotomy for infinite product measures: Let μ, ν be two probability measures on sequence space $(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}})$ that make the coordinates $\xi_n(\omega) = \omega_n$ independent. Let $F_n(x) = \mu(\xi_n \leq x), G_n(x) = \nu(\xi_n \leq x)$. Suppose $F_n \ll G_n$ and let $q_n = dF_n/dG_n > 0, G_n$ -a.s. Let $\mathcal{F}_n = \sigma(\xi_m : m \leq n)$, let μ_n, ν_n be the restrictions of μ and ν to \mathcal{F}_n , and let $X_n = \frac{d\mu_n}{d\nu_n} = \prod_{m=1}^n q_m$. Then $X_n \rightarrow X, \nu$ -a.s. $\sum_{m=1}^{\infty} \log(q_m) > \infty$ is a tail event, so the Kolmogorov 0-1 law implies $\nu(X = 0) \in \{0, 1\}$ and it follows that either $\mu \ll \nu$ or $\mu \perp \nu$, according as $\prod_{m=1}^{\infty} \int \sqrt{q_m} dG_m > 0$ or $= 0$.

4.4 Doob's Inequality, Convergence in $L^p, p > 1$

- If X_n is a submartingale and N is a stopping time with $P(N \leq k) = 1$, then $\mathbb{E}X_0 \leq \mathbb{E}X_N \leq \mathbb{E}X_k$.
- Doob's inequality: Let X_m be a submartingale, $\bar{X}_n = \max_{0 \leq m \leq n} X_m^+, \lambda > 0$ and $A = \{\bar{X}_n \geq \lambda\}$. Then $\lambda P(A) \leq \mathbb{E}X_n 1_A \leq \mathbb{E}X_n^+$.
- L^p maximum inequality: If X_n is a submartingale, then for $1 < p < \infty, \mathbb{E}(\bar{X}_n^p) \leq (\frac{p}{p-1})^p \mathbb{E}(X_n^+)^p$. Consequently, if Y_n is a martingale and $Y_n^* = \max_{0 \leq m \leq n} |Y_m|, \mathbb{E}|Y_n^*|^p \leq (\frac{p}{p-1})^p \mathbb{E}(|Y_n|^p)$.
- L^p convergence theorem: If X_n is a martingale with $\sup \mathbb{E}|X_n|^p < \infty$ where $p > 1$, then $X_n \rightarrow X$ a.s. and in L^p .

4.5 Square Integrable Martingales

- In this subsection, we will suppose X_n is a martingale with $X_0 = 0$ and $\mathbb{E}X_n^2 < \infty$ for all n .
- Let $X_n^2 = M_n + A_n$ be the Doob decomposition of X_n^2 . Then X_n is L^2 -bounded iff $\mathbb{E}A_{\infty} = \sum_{n=1}^{\infty} \mathbb{E}(X_n - X_{n-1})^2 < \infty$.
- $\mathbb{E}(\sup_m |X_m|^2) \leq 4\mathbb{E}A_{\infty}$.
- $\lim_{n \rightarrow \infty} X_n$ exists and is finite a.s. on $\{A_{\infty} < \infty\}$.
- Let $f \geq 1$ be increasing with $\int_0^{\infty} f(t)^{-2} dt < \infty$. Then $X_n/f(A_n) \rightarrow 0$ a.s. on $\{A_{\infty} = \infty\}$.
- Second Borel-Cantelli Lemma: Suppose B_n is adapted to \mathcal{F}_n and $p_n = P(B_n | \mathcal{F}_{n-1})$. $\sum_{m=1}^n 1_{B(m)} / \sum_{m=1}^n p_m \rightarrow 1$ a.s. on $\{\sum_{m=1}^{\infty} p_m = \infty\}$.
- $\mathbb{E}(\sup_n |X_n|) \leq 3\mathbb{E}A_{\infty}^{1/2}$.

4.6 Uniform Integrability, Convergence in L^1

- $\{X_i\}_{i \in I}$ is uniformly integrable if $\lim_{M \rightarrow \infty} (\sup_{i \in I} \mathbb{E}(|X_i|; |X_i| > M)) = 0$.
- Given a probability space $(\Omega, \mathcal{F}_0, P)$ and an $X \in L^1$, then $\{\mathbb{E}(X | \mathcal{F}) : \mathcal{F} \text{ is a } \sigma\text{-field } \subset \mathcal{F}_0\}$ is uniformly integrable.
- Let $\phi \geq 0$ be any function with $\phi(x)/x \rightarrow \infty$ as $x \rightarrow \infty$. If $\mathbb{E}\phi(|X_i|) \leq C$ for all $i \in I$, then $\{X_i, i \in I\}$ is uniformly integrable.
- Suppose that $\mathbb{E}|X_n| < \infty$ for all n . If $X_n \rightarrow X$ in probability, then the following are equivalent: (i) $\{X_n : n \geq 0\}$ is uniformly integrable. (ii) $X_n \rightarrow X$ in L^1 . (iii) $\mathbb{E}|X_n| \rightarrow \mathbb{E}|X| < \infty$.
- For a submartingale, the following are equivalent: (i) It is uniformly integrable. (ii) It converges a.s. and in L^1 . (iii) It converges in L^1 .

- If a martingale $X_n \rightarrow X$ in L^1 , then $X_n = \mathbb{E}(X|\mathcal{F}_n)$.
- For a martingale, the following are equivalent: (i) It is uniformly integrable. (ii) It converges a.s. and in L^1 . (iii) It converges in L^1 . (iv) There is an integrable random variable X so that $X_n = \mathbb{E}(X|\mathcal{F}_n)$.
- Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, i.e., \mathcal{F}_n is an increasing sequence of σ -fields and $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$. As $n \rightarrow \infty$, $\mathbb{E}(X|\mathcal{F}_n) \rightarrow \mathbb{E}(X|\mathcal{F}_\infty)$ a.s. and in L^1 .
- Lévy's 0-1 law: If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $A \in \mathcal{F}_\infty$, then $\mathbb{E}(1_A|\mathcal{F}_n) \rightarrow 1_A$ a.s.

4.7 Backwards Martingales

- A backwards martingale is a martingale indexed by the negative integers, i.e., $X_n, n \leq 0$, adapted to an increasing sequence of σ -fields \mathcal{F}_n with $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ for $n \leq -1$.
- $X_{-\infty} = \lim_{n \rightarrow -\infty} X_n$ exists a.s. and in L^1 .
- If $X_{-\infty} = \lim_{n \rightarrow -\infty} X_n$ and $\mathcal{F}_{-\infty} = \cap_n \mathcal{F}_n$, then $X_{-\infty} = \mathbb{E}(X_0|\mathcal{F}_{-\infty})$.
- A sequence X_1, X_2, \dots is said to be exchangeable if for each n and permutation π of $\{1, \dots, n\}$, (X_1, \dots, X_n) and $(X_{\pi(1)}, \dots, X_{\pi(n)})$ have the same distribution. If X_1, X_2, \dots are exchangeable then conditional on \mathcal{E} (exchangeable σ -field), X_1, X_2, \dots are independent and identically distributed.
- If X_1, X_2, \dots are exchangeable and take values in $\{0, 1\}$, then there is a probability distribution on $[0, 1]$ so that $P(X_1 = 1, \dots, X_k = 1, X_{k+1} = 0, \dots, X_n = 0) = \int_0^1 \theta^k (1 - \theta)^{n-k} dF(\theta)$.

4.8 Optional Stopping Theorems

- If X_n is a uniformly integrable submartingale, then for any stopping time N , $X_{N \wedge n}$ is uniformly integrable.
- If $\mathbb{E}|X_N| < \infty$ and $X_n 1_{(N > n)}$ is uniformly integrable, then $X_{N \wedge n}$ is uniformly integrable and hence $\mathbb{E}X_0 \leq \mathbb{E}X_N$.
- If X_n is a uniformly integrable submartingale, then for any stopping time $N \leq \infty$, we have $\mathbb{E}X_0 \leq \mathbb{E}X_N \leq \mathbb{E}X_\infty$, where $X_\infty = \lim_n X_n$.
- If X_n is a nonnegative supermartingale and $N \leq \infty$ is a stopping time, then $\mathbb{E}X_0 \leq \mathbb{E}X_N$, where $X_\infty = \lim_n X_n$.