

Applied Partial Differential Equations

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1 Ordinary Differential Equation

1.1 Modelling

- Free Falling Law

$$F = ma, a = \ddot{x} = \frac{d^2x}{dt^2}, m\ddot{x} = -mg, \ddot{x} = -g, \Rightarrow x(t) = \frac{1}{2}gt^2 + C_0 + C_1t, x(0) = C_0, \dot{x}(0) = C_1.$$

- Population Model (Malthus)

$x(t)$ = # of population at t , $\frac{\dot{x}(t)}{x(t)}$ = relative growth rate = $\alpha > 0$. Solu: $x(t) = x(0)e^{\alpha t}$.

Modification: Logistic Model

Modelled $x(t)$, healthy $y(t)$, $x(t) + y(t) = N$, $\frac{\dot{x}(t)}{x(t)} = \alpha y = \alpha(N - x)$, denote $\bar{x} = \frac{x}{N}$, we have
 $\dot{x} = \beta\bar{x}(1 - \bar{x})$, $\bar{x}(t) = \frac{\bar{x}(0)e^{\beta t}}{1 + \bar{x}(0)(e^{\beta t} - 1)}$.

- Harmonic Oscillator

$F = ma, x(t)$ = displacement of t , $F = -kx$ (Hooke's Law), then $m\ddot{x} = -kx, \ddot{x} + \omega^2x = 0$.

Solu: $x(t) = e^{\pm i\omega t}$.

- Free Fall at Large Distance

$$m\ddot{x} = F = -\frac{GMm}{x^2}, \ddot{x} = -\frac{GM}{x^2}, x(0) = R, \dot{x}(0) = 0.$$

ANSATZ: $x(t) = at^b$, we have $ab(b-1)t^{b-2} = -GMa^{-2}t^{-2b}$, $b = \frac{2}{3}$, $a = (\frac{9GM}{2})^{\frac{1}{3}}$.

We find that initial conditions do not match, thus modifying solution as $a(t \pm c)^b$, etc.

1.2 Matrix Series

$$\begin{cases} \dot{x} = Ax \\ x|_{t=0} = x_0 \end{cases} \quad \text{Theorem: } \exists 1 \ x(t) = e^{At}x_0, \text{ where } e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Generally speaking, to compute e^{At} , there are 3 methods as follows.

- $PAP^{-1} = \Lambda$

- regulation. For example, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = -I$.

- putzer method.

$$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t)P_j := \Phi(t), \text{ where}$$

$$P_0 = I \quad P_j = (A - \lambda_j I)P_{j-1} \quad \begin{cases} \dot{r}_1(t) = \lambda_1 r_1(t) \\ r_1(0) = 1 \end{cases} \quad \begin{cases} \dot{r}_j(t) = \lambda_j r_j(t) + r_{j-1}(t) \\ r_j(0) = 0 \end{cases}$$

Proof: $\dot{\Phi}(t) = \sum_{j=1}^n \dot{r}_j(t)P_{j-1} = \sum_{j=1}^n (\lambda_j r_j(t) + r_{j-1}(t))P_{j-1} = \sum_{j=1}^n \lambda_j r_j(t)P_{j-1} + \sum_{j=0}^{n-1} r_j(t)(A - \lambda_j I)P_{j-1} = A\Phi(t)$. The last step: $P_n = 0$ (Hamilton-Cayley Theorem), thus $AP_{n-1} = \lambda_n P_{n-1}$.

1.3 Matrix Norm and Periodic Coefficient Matrix

Vector Norm: $\|x\|_p = (\sum |x_i|^p)^{\frac{1}{p}}$. Operator norm: $\|A\|_p = \max_{\|x\|_p \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p$.

For example, $\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max(|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|)$, i.e. $\|A\|_1 = \max$ of column ($\max_i \sum_j |A_{ji}|$), and $\|A\|_\infty = \max$ of row ($\max_i \sum_j |A_{ij}|$).

ORDINARY DIFFERENTIAL EQUATION

Prop: $\|AB\| \leq \|A\| \cdot \|B\|$, $\left\| \sum \frac{A^k}{k!} \right\| \leq e^{\|A\|} < \infty$, $AB = BA \Leftrightarrow e^{(A+B)t} = e^{At} \cdot e^{Bt} = e^{Bt} \cdot e^{At}, \forall t$

Proof: $\begin{cases} \dot{x}_t = (A + B)x \\ x|_{t=0} = x_0 \end{cases}$. Use the uniqueness of the solution, where $e^{(A+B)t}$ and $e^{At} \cdot e^{Bt}$ are both solutions.

Consider $\begin{cases} \dot{x}_t = A(t)x \\ x|_{t=0} = x_0 \end{cases}$. Denote $x(t) = \Phi(t)x_0$ (fundamental matrix), then $\begin{cases} \dot{\Phi}_t = A(t)\Phi(t) \\ \Phi(0) = I \end{cases}$.

Claim: $\exists 1$ solution.

Liouville's Th: $\det \Phi(t) = \det \Phi(0) \cdot e^{\int_0^t \text{tr}A(s)ds}$.

In fact, $\forall Z(t)$ is invertible, $\det Z(t) = \det Z(0) \cdot e^{\int_0^t \text{tr}(Z^{-1}(s)\dot{Z}(s))ds}$.

We note that even if $A(t+T) = A(t)$, $x(t)$ or $\Phi(t)$ can be not periodic. However, we have

Floquet/Bloch Thm: $\Phi(t+T) = \Phi(t)Q$ if $A(t+T) = A(t)$ where Q is a constant matrix.

Proof: $\frac{d}{dt}(\Phi^{-1}(t)\Phi(t+T)) = -\Phi^{-1}(t)\dot{\Phi}(t)\Phi^{-1}(t)\Phi(t+T) + \Phi^{-1}(t)\dot{\Phi}(t+T) = -\Phi^{-1}(t)A(t)\Phi(t+T) + \Phi^{-1}(t)A(t+T)\Phi(t+T) = 0$.

Prop: $\det Q \neq 0$, $Q = e^{TB}$ where $B \in C^{n \times n}$. And if denoting $\Psi(t) = \Phi(t)e^{-tB}$, we have $\Psi(t+T) = \Psi(t)$. To go further, by denoting $x(t) = \Psi(t)z(t)$ and calculating, we have $\dot{z} = Bz$.

Proof: $x(t) = \Psi(t)z(t)$, hence $\dot{x} = \dot{\Psi}(t)z(t) + \Psi(t)\dot{z}(t) = A(t)\Psi(t)z(t), \dot{\Phi}(t)e^{-tB}z(t) - \Phi(t)e^{-tB}Bz(t) + \Psi(t)\dot{z}(t) = A(t)\Phi(t)e^{-tB}z(t), \dot{\Phi}(t) = A(t)\Phi(t), \dot{z}(t) = Bz(t)$.

We must point out that this conclusion has only theoretical value, because it uses the equation $Q = e^{TB}$ where matrix B is very difficult to compute.

Example: $A(t) = \begin{pmatrix} 1 & 0 \\ \cos t & 0 \end{pmatrix}$. Obviously $A(t)$ is periodic and the fundamental matrix $\Phi(t)$ is $\begin{pmatrix} e^t & 0 \\ \int_0^t e^s \cos s ds & 1 \end{pmatrix}$. Here $Q = \Phi^{-1}(t)\Phi(t+2\pi) = \begin{pmatrix} e^{2\pi} & 0 \\ \int_0^{t+2\pi} e^s \cos s ds - e^{2\pi} \int_0^t e^s \cos s ds & 1 \end{pmatrix}$. It seems $Q = Q(t)$, but if you compute $\frac{dQ}{dt}$ you will find it a constant.

D'Alambert's reduction (not required): for differential equation $\dot{x} = A(t)x$ where $A \in R^{n \times n}$, if one solution $y(t)$ is known, we can reduce the dimension of solution space from n to $n-1$.

ANSATZ: $x(t) = \phi(t)y(t) + z(t)$. By computing we get the result: $\dot{\phi}(t) = \sum_{j=2}^n A_{1j}z_j/y_1(t), \dot{z}_i(t) = \sum_{j=2}^n (A_{ij} - A_{1j}\frac{y_i}{y_1})z_j$ where $i = 2, \dots, n$.

1.4 Duhammel's Principle (Variation of Constants)

Consider differential equations $\begin{cases} \dot{x} = Ax + f \\ x|_{t=0} = x_0 \end{cases}$ and $\begin{cases} \dot{x} = A(t)x + f(t) \\ x|_{t=0} = x_0 \end{cases}$.

ANSATZ: For LHS, $x(t) = e^{At}C(t), \dot{x} = Ax + e^{At}\dot{C}(t) = Ax + f(t), \dot{C}(t) = e^{-At}f(t)$, hence $x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds$. For RHS, we denote $\Phi(t)$ as fundamental matrix, then $x(t) = \Phi(t)C(t), \dot{x} = \dot{\Phi}(t)C(t) + \Phi(t)\dot{C}(t), \dot{C}(t) = \Phi^{-1}(t)f(t)$.

Example: $\begin{cases} \dot{Z} = AZ + ZB \\ Z(0) = Q \end{cases}$. Assume $Z(t) = e^{At}Y(t)$, hence $\dot{Z} = AZ + e^{At}Y(t)B$, so we have

$$\begin{cases} \dot{Y}(t) = Y(t)B \\ Y(0) = Q \end{cases}, Y(t) = Qe^{Bt}, Z(t) = e^{At}Qe^{Bt}.$$

ORDINARY DIFFERENTIAL EQUATION

High-order ODE \rightarrow 1st ODEs: $\begin{cases} \ddot{x} + \omega^2 x = f(t) \\ x(0) = x_0, \dot{x}(0) = x_1 \end{cases}$. By denoting $Z = (x, \dot{x}/\omega)^T$, we have $\dot{Z} = AZ + F$ where $A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, F = (0, f(t)/\omega)^T$. The solution is $Z(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} x_0 \\ \frac{x_1}{\omega} \end{pmatrix} + \int_0^t \begin{pmatrix} \cos[\omega(t-s)] & \sin[\omega(t-s)] \\ -\sin[\omega(t-s)] & \cos[\omega(t-s)] \end{pmatrix} \begin{pmatrix} 0 \\ \frac{f(s)}{\omega} \end{pmatrix} ds$. If $x_0 = x_1 = 0$ and $f(t) = e^{i\nu t}$, $x(t) = \frac{1}{\omega} \int_0^t \sin[\omega(t-s)] f(s) ds \sim \int_0^t e^{i\omega(t-s)} e^{i\nu s} ds = e^{i\omega t} \int_0^t e^{i(\nu-\omega)s} ds$. We find $\omega \rightarrow \nu$, resonance forms.

Consider $x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = 0(f(t))$. By denoting $Z = (x, \dot{x}, \ddot{x}, \dots, x^{(n-1)})^T$, we have $\dot{Z} = AZ + F$ where $F = (0, \dots, 0, f)^T$. BTW, if $a_i \rightarrow a_i(t)$, we have $\dot{Z} = A(t)Z(t) + F(t)$. Assume $x = e^{\lambda t}$, then $\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0$, e.g. $\det(\lambda I - A) = 0$. Hence the fundamental solutions are $\{e^{\lambda_i t}\}_{i=1}^n$. If $\lambda_1 = \lambda_2$, they become $\{e^{\lambda t}, \dots, t^{n-1} e^{\lambda t}\}$.

Euler equation: $a_i = t^i \frac{d^i x}{dt^i}$, e.g. $t^2 \ddot{x} + t \dot{x} + x = 0$. Via denoting $t = e^s$, we can get $x(t) = x(e^s) = y(s)$ and $t \frac{dx}{dt} = \frac{dy}{ds}$, thus initial question becomes an ode of constant coefficients. To go further, the solution has the form $y(s) = e^{\lambda s}$ while $t = e^s$, hence $x(t) = t^\lambda$. More specifically, considering the example, we have $\lambda^2 + 1 = 0, \lambda = \pm i$.

1.5 Existence and Uniqueness of Solution

$$\begin{cases} \dot{x} = f(t, x) \\ x|_{t=0} = x_0 \end{cases}, \text{ Lip: } \|f(t, x_1) - f(t, x_2)\| \leq L \|x_1 - x_2\|. \text{ A special example: } f(t, x) = A(t)x.$$

Granwall's inequality: $x(t) = x_0 + \int_0^t f(s, x(s)) ds, \|x_1(t) - x_2(t)\| \leq L \int_0^t \|x_1(s) - x_2(s)\| ds := LF(t)$, we claim $F(t) \leq F(0)e^{tL}$ because $\frac{\dot{F}(t)}{F(t)} \leq L$. To dig a little deeper, if $f(t) \leq A + \int_0^t b(s)f(s) ds$ with $f(t) \geq 0, A \geq 0, b(s) \geq 0$, we have $f(t) \leq Ae^{\int_0^t b(s) ds}$.

Proof: denoting $F(t) = A + \int_0^t b(s)f(s) ds$, we have $\frac{\dot{F}(t)}{F(t)} \leq b(t), F(t) \leq F(0)e^{\int_0^t b(s) ds}$.

Uniqueness: $z = x_1 - x_2$, then $\frac{1}{2} \frac{d}{dt} \|z\|^2 = (z, \dot{z}) = (z, f(t, x_1) - f(t, x_2)) \leq \|z\| \cdot \|f(t, x_1) - f(t, x_2)\| \leq \|z\| \cdot L \|x_1 - x_2\| = L \|z\|^2$, i.e. $\frac{\frac{d}{dt} \|z\|^2}{\|z\|^2} \leq 2L$. $\|z(t_0)\| = 0$ and $\|z\| \geq 0$, thus $\|z\| \equiv 0$.

$$\text{Existence: } \begin{cases} \dot{x} = f(t, x) \\ x|_{t=0} = x_0 \end{cases} \quad \{(t, x) | |t - t_0| \leq a, |x - x_0| \leq b\}. \text{ If 1) } |f(t, x)| \leq M, 2) |\frac{\partial f}{\partial x}| \leq L \Rightarrow$$

$f(t, x)$ is Lip w.r.t. x , then $\exists 1 x$ s.t. $\begin{cases} \dot{x} = f(t, x) \\ x|_{t=0} = x_0 \end{cases} \rightarrow$ classical solu. $x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds \rightarrow$

strong solu, $\int_I x(t) \phi(t) dt = x_0 \int_I \phi(t) dt + \int_I \int_{t_0}^t f(s, x(s)) ds \phi(t) dt \forall \phi(t) \rightarrow$ weak solu.

Lemma: classical solu, strong solu, weak solu are essentially equivalent.

Proof: Picard iteration. $x_n = x_0 + \int_{t_0}^t f(s, x_{n-1}(s)) ds$ for $n = 1, 2, \dots$. Then $|x_1 - x_0| \leq M(t - t_0), |x_2 - x_1| \leq \frac{LM}{2}(t - t_0)^2, \dots, \Rightarrow \sum |(x_n - x_{n-1})(t)| \leq \sum \frac{L^{n-1}M}{n!}(t - t_0)^n, x_n(t) \rightarrow x_\infty(t)$. Usually, $x_\infty(t)$ is weak solu.

Prop: global Lip will lead to global solu, while local Lip may lead to local solu. Examples: $\dot{x} = 1 + x^2, x = \tan(t); |f(t, x_1) - f(t, x_2)| \leq A|x_1 - x_2|^k$ with $k > 1$.

For the former, consider a new norm $\|x(t)\| = \max_{t \in I} |x(t)e^{-\alpha t}|$. Def $Tx = x_0 + \int_{t_0}^t f(t, x) dt$, $|Tx - Ty| \leq L \int_{t_0}^t |(x - y)(s)| ds = \int_{t_0}^t |(x - y)(s)e^{-\alpha s}| e^{\alpha s} ds \leq L \|x - y\| \int_{t_0}^t e^{\alpha s} ds \leq \frac{L}{\alpha} e^{\alpha t} \|x - y\| \Rightarrow$

$\|Tx - Ty\| \leq \frac{L}{\alpha} \|x - y\|$. Let $\alpha = 2L$ and get global Lip.

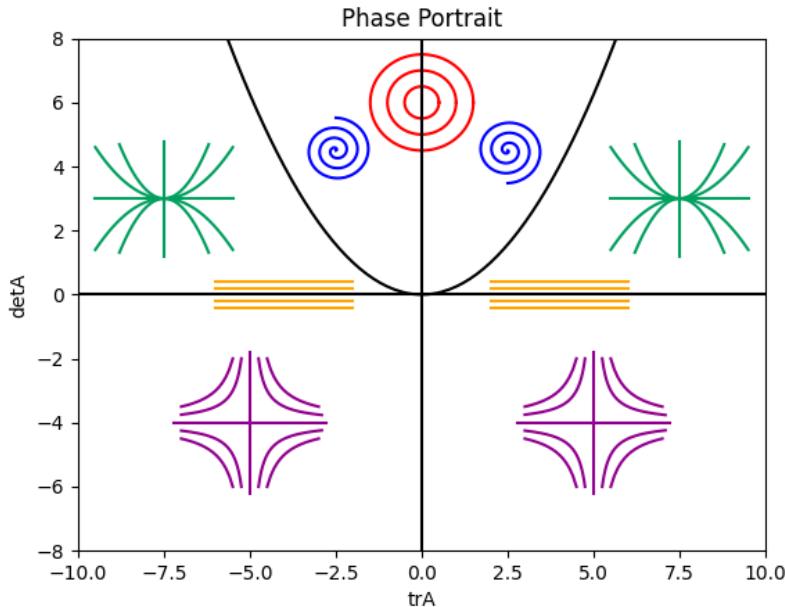
Consider $\begin{cases} \dot{x} = f(t, x) \\ x|_{t=0} = x_0 \end{cases}$ $\tilde{x}_0 \rightarrow x_0$, we have $\tilde{x} - x = \tilde{x}_0 - x_0 + \int_0^t f(s, \tilde{x}(s)) - f(s, x) ds$, $|\tilde{x} - x| \leq |\tilde{x}_0 - x_0| + L \int_0^t |\tilde{x}(s) - x(s)| ds$, $|\tilde{x} - x| \leq |\tilde{x}_0 - x_0| e^{tL}$.

1.6 Qualitative Analysis

Examples: $\dot{x} = Ax$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$. Their phase portraits are parabolas, hyperbolas, circles and spirals.

$\dot{x} = f(x)$ where $f(x^*) = 0$. $\dot{x} = f(x^* + \delta x) = (\delta x \cdot \nabla)f(x^*) + o(\|\delta x\|) \sim A\delta x$.

Consider dimension = 2 where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2 \times 2}$. $\det(\lambda I - A) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$, $\Delta = (\text{tr}A)^2 - 4\det(A)$. See the following picture.



Def: L-Stable: $\forall \epsilon > 0, \exists \delta > 0$, s.t. $\forall x_0 \in O(x^*, \delta), |x(t) - x^*| < \epsilon$. A-Stable: $x(t) \rightarrow x^*$ with $t \rightarrow +\infty$.

Lyapunov 1st Thm: If $\text{Re}\lambda(A) < 0 \Rightarrow$ A-Stable.

Lyapunov 2nd Thm: $\text{Re}\lambda(A) < 0 \Leftrightarrow \exists G, G = G^*, G > 0$ s.t. $GA + A^*G < 0$.

Proof: $\Leftrightarrow ((GA + A^*G)h, h) = (GAh, h) + (A^*Gh, h) = \lambda(Gh, h) + (Gh, Ah) = (\lambda + \bar{\lambda})(Gh, h) = 2\text{Re}\lambda(Gh, h) < 0 \Rightarrow \text{Re}\lambda < 0$.

\Rightarrow : Let $G = \int_0^\infty e^{A^*t} e^{At} dt$. $(Gx, x) = \int_0^\infty \|e^{At}\|^2 dt > 0$, $GA + A^*G = \int_0^\infty \frac{d}{dt} e^{A^*t} e^{At} dt = -I < 0$.

Remark: 1) G is not unique; 2) If $\text{Re}\lambda(A) < 0$, $\int_0^\infty e^{A^*t} e^{At} dt \sim \sum_n e^{A^*n} e^{An}$ is well defined.

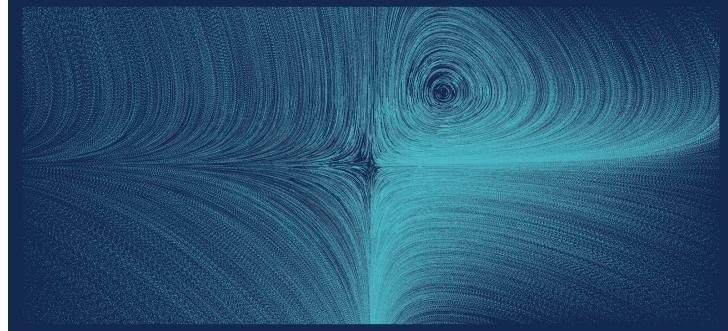
Consider $\dot{x} = Ax + F(t, x)$. If $\lim_{x \rightarrow 0} \frac{|F(t, x)|}{|x|} = 0$, L-1st Thm is still valid. (Hint: Duhammel)

Example: $\begin{cases} \dot{x} = x(ay - b) \\ \dot{y} = y(c - dx) \end{cases}, a, b, c, d, > 0$. $\nabla F = \begin{pmatrix} ay - b & ax \\ -dy & c - dx \end{pmatrix}$.

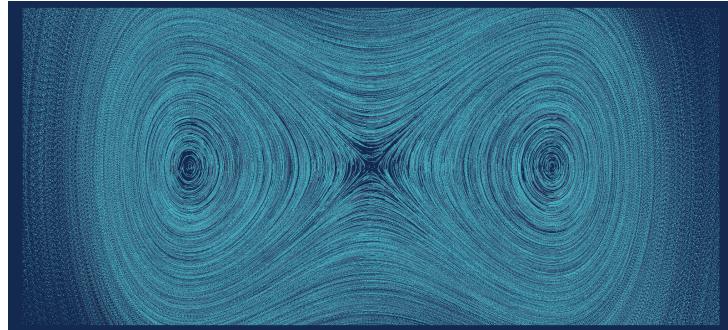
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$\nabla F|_{(0,0)} = \begin{pmatrix} -b & 0 \\ 0 & c \end{pmatrix} \rightarrow$ hyperbola. $\nabla F|_{(\frac{c}{a}, \frac{b}{a})} = \begin{pmatrix} 0 & \frac{ac}{d} \\ -\frac{db}{a} & 0 \end{pmatrix} \rightarrow$ a center for linearized system.

In fact we have $\frac{dx}{dy} = \frac{x(ay-b)}{y(c-dx)}$, thus $V(x, y) = ay + dx - \ln x - b \ln y$ where $\frac{d}{dt}V = 0$, $ay - b \ln y + dx - \ln y = ay_0 - b \ln y_0 + dx_0 - \ln x_0 \Rightarrow e^{d(x-x_0)} \left(\frac{x_0}{x}\right)^c = e^{-a(y-y_0)} \left(\frac{y_0}{y}\right)^b$.

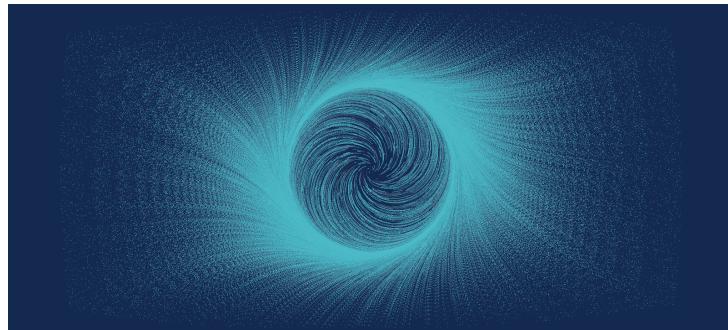


Example: $m\ddot{x} = -\frac{\partial V}{\partial x}$, $p = m\dot{x}$, $z = (x, p)$, $H = \frac{p^2}{2m} + V(x)$. Let $V = \frac{1}{4}(1-x^2)^2$, $m = 1$, then $F = (p, x - x^3)^T$, $\nabla F = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$. $\nabla F|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ saddle. $\nabla F|_{0,\pm 1} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$. In fact, $H = \frac{p^2}{2} + \frac{1}{4}(1-x^2)^2 \equiv C$.



Example (Bifurcation): $\frac{dx}{dt} = (a-1)x - x^3$. If $a < 1$, $\frac{df}{dx}|_{x=0} = a-1 < 0 \rightarrow$ stable. If $a > 1$, $\frac{df}{dx}|_{x=0} > 0 \rightarrow$ unstable, $\frac{df}{dx}|_{x=\pm\sqrt{a-1}} = -2(a-1) < 0 \rightarrow$ stable. If $a = 1$, $\frac{dx}{dt} = -x^3 \rightarrow x = 0$ is stable.

Example: $\begin{cases} \dot{x} = ax - y - x(x^2 + y^2) \\ \dot{y} = x + ay - y(x^2 + y^2) \end{cases}$. Let $x = r(t)\cos\theta(t)$, $y = r(t)\sin\theta(t)$, then $\begin{cases} \dot{r} = ar - r^3 \\ \dot{\theta} = 1 \end{cases}$, $r^* = \sqrt{a} \rightarrow$ limiting circle.



ASYMPTOTIC ANALYSIS

Lyapunov function method: $L(x, y) = L(x(t), y(t))$. 1) L is C^1 , $L(x^*, y^*) = 0$, $L(x, y) \neq 0$ if

$$(x, y) \neq (x^*, y^*). 2) \frac{d}{dt}L \begin{cases} < 0 \rightarrow \text{A-stable} \\ \leq 0 \rightarrow \text{stable} \\ > 0 \rightarrow \text{unstable} \end{cases} \cdot \frac{d}{dt}L = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = f\frac{\partial L}{\partial x} + g\frac{\partial L}{\partial y} = (F \cdot \nabla)L, F = (f, g).$$

$L^{-1}(c) \rightarrow$ closed curve (for enough small c).

Example: $\dot{x} = -\nabla V(x) \cdot L = V - V(x^*)$, $\dot{L} = \dot{V} = \nabla V \cdot \dot{x} = -|\nabla V|^2 \leq 0$.

$$\text{Example: } \begin{cases} \dot{x} = xf(y) \\ \dot{y} = yg(x) \end{cases} \Rightarrow \frac{dx}{dy} = \frac{xf(y)}{yg(x)} \Rightarrow \frac{f(y)}{y}dy = \frac{g(x)}{x}dx. \text{ Let } V = \int_0^y \frac{f(t)}{t}dt - \int_0^x \frac{g(s)}{s}ds, L = V - V(x^*, y^*) \Rightarrow \dot{L} = \dot{V} = 0.$$

$$\text{Example: } \begin{cases} \dot{x} = -2xy \\ \dot{y} = x^2 - 2y^3 \end{cases}, (x^*, y^*) = (0, 0), \nabla F|_{(0,0)} = 0. \text{ We guess } L = ax^{2m} + by^{2n}, \dot{L} = 2max^{2m-1}\dot{x} + 2nbgy^{2n-1}\dot{y} = -4max^{2m}y + 2nbx^2y^{2n-1} - 4nbgy^{2n+2}. \text{ Let } 4am = 2nb, 2m = 2, 2n - 1 = 1 \Rightarrow m = n = 1, b = 2a, \text{ thus } L = x^2 + 2y^2, \dot{L} = -8y^4 \leq 0.$$

Example: $\ddot{x} = -\nabla V, V \in C^2$. If $(x \cdot \nabla)V \leq -CV (C \geq 2)$, then $\partial_t^2(|x(t)|^2) = 2\partial_t(x, \dot{x}) = 2|\dot{x}|^2 + 2x\ddot{x} = 2|\dot{x}|^2 - 2(x \cdot \nabla)V(x) \geq 2|\dot{x}|^2 + 2CV = 4(\frac{1}{2}|\dot{x}|^2 + \frac{C}{2}V) \geq 4E(t) \geq 0 \Rightarrow |x(t)|^2$ is convex.

2 Asymptotic Analysis

2.1 Regular Expansion

$\mathcal{L}_\epsilon = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ where $\mathcal{L}_0, \mathcal{L}_1$ is differential operator.

ANSATZ: $y_\epsilon = y_0 + \epsilon y_1 + \dots \Rightarrow (\mathcal{L}_0 + \epsilon\mathcal{L}_1)(y_0 + \epsilon y_1 + \dots) = f$. $O(1) : \mathcal{L}_0 y_0 = f$; $O(\epsilon) : \mathcal{L}_0 y_1 + \mathcal{L}_1 y_0 = 0$; $O(\epsilon^2) : \mathcal{L}_0 y_2 + \mathcal{L}_1 y_1 = 0; \dots$. Assume $\mathcal{L}_0^{-1} \exists$, so $y_0 = \mathcal{L}_0^{-1}f$, $y_1 = -\mathcal{L}_0^{-1}\mathcal{L}_1 y_0 = -\mathcal{L}_0^{-1}\mathcal{L}_1\mathcal{L}_0^{-1}f$.

$$\text{Example: } \begin{cases} \ddot{x} + \sin x = 0 \\ x(0) = A, \dot{x}(0) = 0 \end{cases} \text{ where } A \text{ is small. Question: } T(A) = ?$$

$$\text{Assume } x = x_0 + Ax_1 + A^2x_2 + A^3x_3 + o(A^4), \sin x = \sin(x_0 + Ax_1 + A^2x_2 + A^3x_3 + o(A^4)) = \sin x_0 + \cos x_0(Ax_1 + A^2x_2 + A^3x_3 + o(A^4)) + \frac{1}{2}(-\sin x_0)(Ax_1 + A^2x_2 + A^3x_3 + o(A^4))^2 + \frac{1}{6}(-\cos x_0)(Ax_1 + A^2x_2 + A^3x_3 + o(A^4))^3 + o(A^4). O(1) : \begin{cases} \ddot{x}_0 + \sin x_0 = 0 \\ x_0(0) = 0, \dot{x}_0(0) = 0 \end{cases} \Rightarrow x_0 \equiv 0; O(A) : \begin{cases} \ddot{x}_1 + x_1 = 0 \\ x_1(0) = 1, \dot{x}_1(0) = 0 \end{cases} \Rightarrow$$

$$x_1(t) = \text{cost}; O(A^2) : \begin{cases} \ddot{x}_2 + x_2 = 0 \\ x_2(0) = 0, \dot{x}_2(0) = 0 \end{cases} \Rightarrow x_2 \equiv 0; O(A^3) : \begin{cases} \ddot{x}_3 + x_3 - \frac{1}{6}x_1^3 = 0 \\ x_3(0) = 0, \dot{x}_3(0) = 0 \end{cases} \Rightarrow$$

$$x_3(t) = \frac{1}{6} \int_0^t \sin(t-s) \cos^3 s ds = -\frac{1}{192}(\cos 3t - \text{cost}) + \frac{1}{16}ts \sin t. \text{ Thus } x(t) = A \text{cost} + A^3(-\frac{1}{192}(\cos 3t - \text{cost}) + \frac{1}{16}ts \sin t) + o(A^4). \dot{x}(T(A)) = 0, T(A) = 2\pi + B, B \ll 1, \sin B \sim B, \cos B \sim 1. \text{ By specific computing, we have } T(A) = 2\pi(1 + \frac{1}{16}A^2 + o(A^4)).$$

2.2 Singular Perturbation

We will begin with a special example and introduce multiple-scale expansion in sequence.

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Example: $\begin{cases} \ddot{x} + (1 + \epsilon)x = 0 \\ x(0) = 1, \dot{x}(0) = 0 \end{cases}$. $x_\epsilon(t) = \cos(\sqrt{1+\epsilon}t)$. $\epsilon \rightarrow 0$, $x_0(t) = \text{cost}$, $x_\epsilon(t) \rightarrow x_0(t)$. We have estimation $|x_\epsilon(t) - x_0(t)| \leq Ct\epsilon$. $x_\epsilon(t) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$, then $O(1) : \ddot{x}_0 + x_0 = 0$; $O(\epsilon) : \ddot{x}_1 + x_1 = -x_0; \dots$. $x_\epsilon(0) = 1, \dot{x}_\epsilon(0) = 0 \Rightarrow x_0(0) = 1, x_i(0) = 0 (i \geq 1), \dot{x}_j(0) = 0 (j \geq 0)$. $x_0(t) = \text{cost}$, $x_1(t) = -\frac{t}{2}\sin t \Rightarrow x_\epsilon(t) \sim \text{cost} - \frac{1}{2}\epsilon t \sin t + \dots$. We notice that if $t \sim O(\frac{1}{\epsilon})$, the conventional method does not work.

Denote $\tau = \epsilon t$, $x(t) \rightarrow Z(t, \tau)$, $\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$, $\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2}{\partial \tau^2}$. The PLK method has two steps: 1) Operator expansion; 2) $Z(t, \tau) = Z_0(t, \tau) + \epsilon Z_1(t, \tau) + O(\epsilon^2)$. When computing, we must remove the singular term (e.g. $e^{i\omega t}$) which may make the solution drift.

$(\frac{\partial^2}{\partial t^2} + (1 + \epsilon))Z = (\frac{\partial^2 Z}{\partial t^2} + Z) + \epsilon(2\frac{\partial^2 Z}{\partial t \partial \tau} + Z) + \epsilon^2 \frac{\partial^2 Z}{\partial \tau^2} = 0$. $O(1) : \frac{\partial^2 Z_0}{\partial t^2} + Z_0 = 0$; $O(\epsilon) : \frac{\partial^2 Z_1}{\partial t^2} + Z_1 = -Z_0 - 2\frac{\partial^2 Z_0}{\partial t \partial \tau}$. $Z_0 = A(\tau)e^{it} + A^*(\tau)e^{-it}$, $\frac{\partial^2 Z_1}{\partial t^2} + Z_1 = [-A(\tau) - 2iA'(\tau)]e^{it} + [-A^*(\tau) + 2i(A^*(\tau))']e^{-it}$. We hope $A(\tau) + 2iA'(\tau) \equiv 0 \Rightarrow A'(\tau) = -\frac{1}{2i}A(\tau) \Rightarrow A(\tau) = A(0)e^{\frac{1}{2}i\tau}$. Because IC $Z_0(0, 0) = 1, \frac{\partial Z_0}{\partial t}(0, 0) = 0 \Rightarrow Z_0(t, \tau) = \cos(1 + \frac{\epsilon}{2})t$. We have the estimation $|x_\epsilon - Z_0 - \epsilon Z_1| \leq c(t)\epsilon^2$ where $t \sim O(\frac{1}{\epsilon})$.

Example: $\begin{cases} \ddot{x} + x + \epsilon x^3 = 0 \\ x(0) = 1, \dot{x}(0) = 0 \end{cases}$. $\tau = \epsilon t$, $\begin{cases} O(1) : \frac{d^2 Z_0}{dt^2} + Z_0 = 0 \\ O(\epsilon) : \frac{d^2 Z_1}{dt^2} + Z_1 = -2\frac{\partial^2 Z}{\partial t \partial \tau} - Z_0^3 \end{cases}$. $Z_0 = A(\tau)e^{it} + A^*(\tau)e^{-it}, -2\frac{\partial^2 Z_0}{\partial t \partial \tau} - Z_0^3 = [-3A^2(\tau)A^*(\tau) - 2iA'(\tau)]e^{it} + [-3A(\tau)(A^*(\tau))^2 + 2i(A^*(\tau))']e^{-it} - A^3(\tau)e^{3it} - (A^*(\tau))^3 e^{-3it} \Rightarrow A'(\tau) = \frac{3}{2}iA^2(\tau)A^*(\tau)$. Let $A(\tau) = R(\tau)e^{i\theta(\tau)} \Rightarrow \dot{R}(\tau) + iR(\tau)\dot{\theta}(\tau) = \frac{3}{2}iR^3(\tau) \Rightarrow \dot{R}(\tau) = 0, \dot{\theta}(\tau) = \frac{3}{2}R^2(\tau) \Rightarrow R(\tau) = R(0), \theta(\tau) = \theta(0) + \frac{3}{2}R^2(0)\tau, A(\tau) = R(0)e^{i(\theta(0) + \frac{3}{2}R^2(0)\tau)}, Z_0 = R(0)e^{i(t + \theta(0) + \frac{3}{2}R^2(0)\tau)} + R(0)e^{-i(t + \theta(0) + \frac{3}{2}R^2(0)\tau)}$. Beacause IC $Z_0(0, 0) = 0, \frac{\partial Z_0}{\partial t}(0, 0) = 0 \Rightarrow R(0) = \frac{1}{2}, \theta(0) = 0 \Rightarrow Z_0(t, \tau) = \cos(1 + \frac{3}{8}\epsilon)t$.

Remark: PLK method only works for $t \sim O(\epsilon^{-k})$.

3 Derivation of Partial Differential Equation

3.1 Conservation Law

Consider traffic flow(flux). $\rho(x, t) \rightarrow$ density of cars, $J(\rho) \rightarrow$ flow function. So $\frac{d}{dt} \int_a^b \rho(x, t) dx = J(a) - J(b) = J(\rho(a, t)) - J(\rho(b, t)) = - \int_a^b J'(\rho) dx \Rightarrow \frac{d}{dt} \int_a^b \partial_t \rho + \partial_x J = 0 \Rightarrow \partial_t \rho + \partial_x J = 0$. $J(\rho) = C\rho \Rightarrow \partial_t \rho + \partial_x(C\rho) = 0$. $J(\rho) = C\rho(M - \rho)$ and $\bar{\rho} = \rho/M \Rightarrow \partial_t \bar{\rho} + \partial_x[\beta \bar{\rho}(1 - \bar{\rho})] = 0, \beta = CM$. $J(\rho) = C\rho - \mu \rho_x \Rightarrow \partial_t \rho + \partial_x(C\rho) - \mu \rho_{xx} = 0$. $J(\rho) = \frac{\rho^2}{2} - \mu \rho_x \Rightarrow \partial_t \rho + \frac{1}{2}(\rho^2)_x - \mu \rho_{xx} = 0$.

For high-dimension cases, $\frac{d}{dt} \int_\Omega \rho(x, t) dx = - \int_{\partial\Omega} J \cdot \vec{n} dx = - \int_\Omega \nabla \cdot J dx \Rightarrow \partial_t \rho + \nabla \cdot J = 0$.

Example: String Vibration: 1) Homogeneous: $\rho(x) \equiv \rho$; 2)Elastic String (only stretching, no shearing, no bending). $u(x, t) =$ vertical distance at (x, t) , $|\partial_x u| \ll 1$. $T(x) =$ tension, $|\partial_x u| \ll 1 \Rightarrow T(x) \approx T(x + \Delta x) \Rightarrow T(x) \equiv T$. So $\int_x^{x+\Delta x} \rho \frac{\partial^2 u}{\partial t^2} dx = T \sin \alpha(x + \Delta x) - T \sin \alpha(x) = T \partial_x u(x + \Delta x, t) - T \partial_x u(x, t) = T \int_x^{x+\Delta x} \frac{\partial^2 u}{\partial x^2} dx \Rightarrow \rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}$. If $E =$ Young's modulus, then $\rho S(x) \frac{\partial^2 u}{\partial t^2} = E \frac{\partial}{\partial x}(S(x) \frac{\partial u}{\partial x})$.

For high-dimension cases, $\int_x^{x+\Delta x} \int_y^{y+\Delta y} \rho \frac{\partial^2 u}{\partial t^2} dy dx = T(\frac{\partial u}{\partial x}(x + \Delta x, y, t) - \frac{\partial u}{\partial x}(x, y, t)) \Delta y + T(\frac{\partial u}{\partial y}(x, y + \Delta y, t) - \frac{\partial u}{\partial y}(x, y, t)) \Delta x \Rightarrow \frac{1}{c^2} \partial_t^2 u = \Delta u$ where $c^2 = \frac{T}{\rho}$. For this class, we will impose some BC/IC.

BC: 1) Dirichlet BC: $u(0, t) = u(L, t) = 0$ (homo) or $u(0, t) = f(t), u(L, t) = g(t)$ (inhomo).

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2) Neumann BC: $u'_x(0, t) = u'_x(L, t) = 0$ or $-Tu'_x(0, t) = f(t), -Tu'_x(L, t) = g(t)$.

3) Robin BC: $\alpha u(0, t) + \beta u_x(0, t) = \alpha u(L, t) + \beta u_x(L, t) = 0$.

4) Newton's Law of Cooling: $-\frac{k\alpha}{\beta}(u(0, t) - g(t)) = -ku_x(0, t)$.

3.2 Calculus of Variation

Consider energy function $I[u] = \frac{1}{2} \int_0^1 |u'|^2 dx - \int_0^1 f(x)u(x)dx$, our goal is to $\min_{u \in A} I[u]$ where A is admissible set/space. We assume the minimizer u exists. $u_t = u + tv, I[u_t] \geq I[u]$. We denote $g(t) = I[u_t]$, then $g(t) \geq g(0), g'(0) = 0$. $g(t) = \frac{1}{2} \int_0^1 |u' + tv'|^2 dx - \int_0^1 f(u + tv)dx, g'(t) = \int_0^1 u'(x)v'(x)dx + t \int_0^1 (v')^2 dx - \int_0^1 f(x)v(x)dx, g'(0) = 0 \Rightarrow \int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx$ (Euler-Lagrange EQ) \rightarrow weak solu. $\int_0^1 u'(x)v'(x)dx = \int_0^1 u'(x)dv(x) = u'(1)v(1) - u'(0)v(0) - \int_0^1 u''(x)v(x)dx = \int_0^1 f(x)v(x)dx \Rightarrow u'(1)v(1) - u'(0)v(0) = \int_0^1 (u'' + f)vdx$.

1) Assume $A = \{u \in C^2(0, 1) | I[u] < \infty, u'(0) = u'(1) = 0\}$, we have $\begin{cases} -u'' = f \\ u'(0) = u'(1) = 0 \end{cases} \rightarrow$

Neumann BVP.

2) Assume $A = \{u \in C^2(0, 1) | I[u] < \infty, u(0) = A, u(1) = B\}$. In particular, $u, u_t \in A$, so we have $v(0) = v(1) = 0 \Rightarrow \begin{cases} -u'' = f \\ u(0) = A, u(1) = B \end{cases} \rightarrow$ Dirichlet BVP.

3) Assume $A = \{u \in C^2(0, 1) | I[u] < \infty, u'(0) + a(0)u(0) = u'(1) + a(1)u(1) = 0\}$ and consider $I[u] = \frac{1}{2} \int_0^1 |u'(x)|^2 dx - \int_0^1 f(x)u(x)dx + \frac{1}{2}[a(1)u^2(1) - a(0)u^2(0)]dx$. By computing $g(t) = I[u + tv], g'(0) = 0$, we have $\begin{cases} -u'' = f \\ u'(0) + a(0)u(0) = u'(1) + a(1)u(1) = 0 \end{cases} \rightarrow$ Robin BC.

Example: $I[u] = \frac{1}{2} \int_{\Omega} \mathcal{C}\epsilon(u):\epsilon(u)dx - \int_{\Omega} f(x)u(x)dx$. $\mathcal{C}A = 2\mu A + \lambda \text{tr}(A)I, A \in R^{2 \times 2}, \mu, \lambda = \text{Lamé constants} > 0, u = (u_1, u_2) = \text{displacement}, fu = f_1u_1 + f_2u_2, \epsilon(u) = \text{strain tensor} \in R^{2 \times 2} = \frac{1}{2}(\nabla u + \nabla u^T)$. Denote $g(t) = \frac{1}{2} \int_{\Omega} \mathcal{C}\epsilon(u + tv):\epsilon(u + tv)dx - \int_{\Omega} f(x)(u(x) + tv(x))dx, \epsilon(u + tv) = \epsilon(u) + t\epsilon(v)$, then $g'(0) = \frac{1}{2} \int_{\Omega} \mathcal{C}\epsilon(u):\epsilon(v)dx + \frac{1}{2}\mathcal{C}\epsilon(v):\epsilon(u)dx - \int_{\Omega} f(x)v(x)dx = 0 \Rightarrow \int_{\Omega} \mathcal{C}\epsilon(u):\epsilon(v)dx = \int_{\Omega} f(x)v(x)dx$. Define $\sigma := \mathcal{C}\epsilon(u) = \text{stress tensor}$, then $\sigma = \sigma^T, \int_{\Omega} \sigma:\epsilon(v)dx = \frac{1}{2} \int \sigma:(\nabla v + \nabla v^T)dx = \int \sigma:\nabla v dx = - \int_{\Omega} \nabla \cdot \sigma v dx + \int_{\partial\Omega} (\sigma \cdot n)v d\sigma(x) = \int_{\Omega} f(x)v(x) = 0 \Rightarrow -\nabla \cdot \sigma = f, \sigma \cdot n = 0$ (Navier EQ). Here, $A:B = \text{tr}(AB^T)$.

Least action principle: $A[u] = \int_{t_0}^{t_1} (K(t) - P(t))dt$. $A \rightarrow$ action, $K \rightarrow$ kinetic, $P \rightarrow$ potential. Motion path u always follows the extreme point of $A[u]$.

Example: $\min_{\theta} \int_{t_0}^{t_1} (\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta))dt$. By computing we get $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

Consider $L = \int_{t_0}^{t_1} \mathcal{L}(t, q, \dot{q})dt$ where $\mathcal{L} \rightarrow$ Lagrangian. By computing we get $\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}}$ (E-L EQ). $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \rightarrow$ Generalized Momentum, $\mathcal{H} = p \cdot \dot{q} - \mathcal{L} \rightarrow$ Hamiltonian. Using E-L EQ, we have $\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i, \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i \Rightarrow \frac{d\mathcal{H}}{dt} = \dot{p}_i \cdot \dot{q}_i - \frac{\partial \mathcal{L}}{\partial p_i} \dot{p}_i = 0$.

Example: $\begin{cases} \dot{x}(s) = f(x(s), \alpha(s)), t < s < T \\ x(t) = x \end{cases}$ where $\alpha \rightarrow$ control and $x \rightarrow$ response. Consider $u(x, t) = \inf_{\alpha} \int_t^T W(x(s), \alpha(s))ds + g(x(T))$ where $W \rightarrow$ running cost and $g \rightarrow$ final cost. Admissible solution set $A = \{\alpha | \alpha \text{ is measurable}\}$. $\forall h > 0, u(x, t) = \inf \int_t^{t+h} W(x(s), \alpha(s))ds + u(x(t+h), t+h)$. By Taylor expansion, we have $x(t+h) \approx x(t) + \dot{x}(t)h = x(t) + f(x(t), \alpha(t))h, u(x(t+h), t+h) \approx$

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$u(x(t) + f(x, \alpha)h, t+h) \approx u(x(t), t) + \partial_x u f(x, \alpha)h + \partial_t u h$. Then $u(x, t) \approx \inf_{\alpha} \int_t^{t+h} W(x(s), \alpha(s))ds + u(x, t) + \partial_x u f(x, \alpha)h + \partial_t u h \Rightarrow \partial_t u + \inf_{\alpha} (\partial_x u f + W(x, \alpha)) = 0$. Define $H(p) = \inf_{\alpha} (pf(x, \alpha) + W(x, \alpha))$, we get $\partial_t u + H(\partial_x u) = 0 \rightarrow$ Hamilton-Jacobi EQ. If $f(x, \alpha) = -\alpha, W(x, \alpha) = \frac{1}{2}\alpha^2$, we have $\partial_t u - \frac{1}{2}(\partial_x u)^2 = 0$.

Consider $A(a, \alpha), B(b, \beta)$, we want to find the shortest route connecting A with B . $I[u] = \int ds = \int_a^b \sqrt{1+u'(x)^2}dx$, admissible solution set $A = \{u \in C^1 | u(a) = \alpha, u(b) = \beta\}$. Denote $v_t = u + tv$ where u is minimizer and v is test function, $g(t) = \int_a^b \sqrt{1+(u'+tv')^2}dx, g'(0) = 0 \Rightarrow \int_a^b \frac{u'v'}{\sqrt{1+(u')^2}}dx = 0 = -\int_a^b \frac{d}{dx} \left(\frac{u'}{\sqrt{1+u'^2}} \right) v dx \Rightarrow -\frac{d}{dx} \left(\frac{u'}{\sqrt{1+(u')^2}} \right) = 0 \Rightarrow u'' = 0$.

Consider ray-propagation rules. $T = \int dt = \int \frac{ds}{c(x,y)} = \int_a^b \frac{\sqrt{1+u'^2}}{c(x,u(x))}dx = \int_a^b n(x, u(x))\sqrt{1+u'^2}dx$. $g(t) = \int_a^b n(x, u+tv)\sqrt{1+(u'+tv')^2}dx, g'(t) = \int_a^b \frac{\partial n}{\partial y} \sqrt{1+v_t^2}v + n \frac{(u'+tv')v'}{\sqrt{1+(u'+tv')^2}}dx. g'(0) = 0 \Rightarrow \int_a^b \frac{\partial n}{\partial y}(x, u)\sqrt{1+u'^2}v + n \frac{u'v'}{\sqrt{1+u'^2}}dx \Rightarrow -\frac{d}{dx} \left(\frac{u'}{\sqrt{1+u'^2}} \right) + \partial_y n \sqrt{1+u'^2} = 0$.

A special case is $n = \begin{cases} n_1, & a < x < x_0 \\ n_2, & x_0 < x < b \end{cases}$. Then $0 = g'(0) = \int_a^b \frac{n_1 u' v'}{\sqrt{1+u'^2}}dx = \int_a^{x_0} \frac{n_1 u' v'}{\sqrt{1+u'^2}}dx + \int_{x_0}^b \frac{n_2 u' v'}{\sqrt{1+u'^2}}dx \Rightarrow \frac{n_1 u'}{\sqrt{1+u'^2}}(x_0^-) - \frac{n_2 u'}{\sqrt{1+u'^2}}(x_0^+) = 0$. Explanation: $\frac{n_1 u'}{\sqrt{1+u'^2}} = \frac{n_1 u' dx}{\sqrt{1+u'^2} dx} = n_1 \frac{dy}{ds} = n_1 \sin\theta_1 = n_2 \sin\theta_2$, i.e. refraction law.

As for high-dimension cases, $I[u] = \int_{\Omega} \sqrt{1+|\nabla u|^2}dx$, E-L equation: $\int_{\Omega} \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}}dx = 0 \Rightarrow -\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0 (|\nabla u| \ll 1) \Rightarrow \Delta u = 0$.

Consider $\min_{u \in A} I[u] = \int_a^b \sqrt{1+u'^2}dx$ s.t. $\int_a^b u(x)dx = A$. $I_{\lambda}[u] = I[u] + \lambda(\int_a^b u(x)dx - A), g'(0) = 0 \Rightarrow \int_a^b \frac{u'v'}{\sqrt{1+u'^2}} + \lambda u dx = 0 \Rightarrow -\frac{d}{dx} \left(\frac{u'}{\sqrt{1+u'^2}} \right) + \lambda = 0 \Rightarrow \frac{u'}{\sqrt{1+u'^2}} = \lambda x + \mu \Rightarrow \frac{u'^2}{1+u'^2} = (\lambda x + \mu)^2 \Rightarrow u' = \frac{\lambda x + \mu}{\sqrt{1-(\lambda x + \mu)^2}} \Rightarrow u(x) - \alpha = \int_a^x \frac{\lambda y + \mu}{\sqrt{1-(\lambda y + \mu)^2}} dy = (\lambda^{-2} - (x + \frac{\mu}{\lambda})^2)^{\frac{1}{2}} - (\lambda^{-2} - (a + \frac{\mu}{\lambda})^2)^{\frac{1}{2}} \Rightarrow (u(x) - (\alpha - \alpha_0)^2) + (x + \frac{\mu}{\lambda})^2 = \lambda^{-2} \rightarrow$ circle.

Consider $\min_{u \in A} Q[u] = \int \frac{|\nabla u|^2 dx}{u^2 dx}$ where $A = H_0^1(\Omega) = \{u \in L^2, \nabla u \in L^2\}$. $Q[u+tv] = \frac{\int_{\Omega} |\nabla(u+tv)|^2 dx}{\int_{\Omega} u^2 dx} = \frac{\int_{\Omega} |\nabla u|^2 dx + 2t \int_{\Omega} \nabla u \cdot \nabla v dx + t^2 \int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} u^2 dx + 2t \int_{\Omega} uv dx + t^2 \int_{\Omega} v^2 dx}$. $g'(0) = 0 \Rightarrow \int_{\Omega} \nabla u \cdot \nabla v dx = Q[u] \int_{\Omega} u \cdot v dx$. Denote $Q[u] = \lambda$, we get $-\Delta u = \lambda u, x \in \Omega, u = 0, x \in \partial\Omega$. If $\Omega = (0, \pi)$, then $\min \lambda = 1, Q[u] \geq 1, \int_0^{\pi} u^2 dx \leq \int_0^{\pi} (u')^2 dx$ (Poincare inequality). If $\Omega = (0, L)$, then $\lambda_k = (\frac{k\pi}{L})^2, \int_0^L u^2 dx \leq (\frac{L}{\pi})^2 \int_0^L u'^2 dx$.

3.3 Second-order Variation

For general case, $I[u] = \int_a^b L(u, u')dx$. $g(t) = \int_a^b L(u+tv, u'+tv')dx$. Let $p = u'$, then $g'(t) = \int_a^b \frac{\partial L}{\partial u} v + \frac{\partial L}{\partial p} v dx, g''(t) = \int_a^b \left(\frac{\partial^2 L}{\partial u^2} v^2 + 2 \frac{\partial^2 L}{\partial u \partial p} v v' + \frac{\partial^2 L}{\partial p^2} v'^2 \right) dx$. $g'(0) = \int_a^b Av^2 + 2Bvv' + Cv'^2 dx$ where $A = \frac{\partial^2 L}{\partial u^2}(u, u'), B = \frac{\partial^2 L}{\partial u \partial p}(u, u'), C = \frac{\partial^2 L}{\partial p^2}(u, u')$. It is difficult to strictly prove $g''(0) \geq 0$, but when $L = (1+p^2)^{\frac{1}{2}}, A = B = 0, C = \frac{1}{(1+p^2)^{\frac{3}{2}}} > 0$, thus $g''(0) \geq 0$.

Thm: Let $Q[u] = \int_a^b Au^2 + 2Buu' + C(u')^2 dx$ where $A, B, C \in C^1(a, b)$, then $Q[u] \geq 0$ for all u satisfying $u(a) = u(b) = 0$ if 1) $C \geq 0$ (Legendre condition); 2) $\forall c \in (a, b)$, the equation $-(Cz')' + (A - B')z = 0, z(a) = z(c) = 0$ only has a trivial solution, i.e. there exists no conjugate point $c \in (a, b)$.

Proof: $g''(t) = \int_a^b Au^2 + B(u^2)' + C(u')^2 dx = \int_a^b C(u')^2 + (A - B')u^2 dx = \int_a^b pu'^2 + qu^2 dx (u(a) = u(b) = 0) = \int_a^b (pu'^2 + qu^2) dx + \int_a^b (wu^2)' dx = \int_a^b pu'^2 + qu^2 + w'u^2 + 2wuu' dx = \int_a^b p(u'^2 + \frac{2w}{p}uu') dx$

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$\frac{w'+q}{p}u')dx = \int_a^b p(u' + \frac{w}{p}u)^2 dx \Rightarrow \frac{w'+q}{p} = (\frac{w}{p})^2 \Leftrightarrow p(q+w) = w^2$ (Riccati EQ). Let $w = -\frac{z'}{z}p, w' = -\frac{z''z-(z')^2}{z^2}p - \frac{z'}{z}p'$, then $pq + pw' = pq - (\frac{p}{q})^2 + (\frac{pz'}{z})^2 - \frac{z'}{z}p'p = (\frac{z'}{z}p)^2 \Rightarrow -pz'' - p'z' + qz = 0 \Rightarrow -\frac{d}{dx}(pz') + qz = 0$. This equation with BC $z(a) = z(c) = 0$ has only zero equation, thus for all the BCs, it has a unique solution. In this way we can find a proper z and w and get the above-mentioned theorem.

Example: $Q[u] = \int_0^1 u'^2 - u^2 dx, u(0) = u(1) = 0$. $-u'' - u = 0, u(0) = u(1) = 0$ only has zero solution, so $Q[u] \geq 0$. But when modifying 1 with 4, there exists a conjugate point $\pi \in (0, 4)$.

3.4 Well-posedness of PDEs

- | | | | | |
|----|--|---|--------|------------|
| 1) | existence: classical, weak, strong; domain Ω , bound $\partial\Omega$ | } | A(x) { | measurable |
| 2) | uniqueness: $\begin{cases} -\Delta u + u = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$ or $\begin{cases} -\nabla \cdot (A(x)\nabla u) = f \text{ in } \Omega \\ \partial_n u = g \text{ on } \partial\Omega \end{cases}$ | | | |
| 3) | smoothness: how solution changes when t increases or decreases | | | |
| 4) | stability: how solution changes when data is given a small perturbation | | | |

Example: $AX = b$. The solution $A^{-1}b$ might be singular when $A = \text{diag}(1, \dots, 1, \epsilon)$.

Example: $\Delta u = 0, u(x, 0) = 0$ and solutions are $u_n(x, y) = \frac{1}{n}e^{-n}\sin(nx)\sinh(ny)$.

4 Method of Characteristic

4.1 Simple First Order PDE

A PDE strained in the characteristic line might perform a behavior which can be described as an ODE. So we can transform PDE to ODE.

Consider conversation law (1st PDE): $\partial_t\rho + \partial_x f(\rho) = 0$. A special case is $f(\rho) = \rho$ and $\rho|_{t=0} = \rho_0(x)$. Define $x = Z(t)$ and $p(t) = \rho(Z(t), t)$, then $\dot{p}(t) = \partial_x\rho\dot{Z} + \partial_t\rho = \partial_x\rho\dot{Z} - \partial_x\rho = \partial_x\rho(\dot{Z} - 1)$. Characteristic line is $\dot{Z} = 1, \dot{p}(t) = 0 \Rightarrow Z(t) = Z(0) + t, p(t) = p(0) = \rho(Z(0), 0) = \rho_0(Z(0)) = \rho_0(Z(t) - t) \Rightarrow \rho(x, t) = \rho_0(x - t)$.

Example: $\partial_t\rho - \partial_x\rho = 0, \rho|_{t=0} = \rho_0(x)$. $p(t) = \rho(Z(t), t), \dot{p} = \partial_x\rho\dot{Z} + \partial_t\rho = \partial_x\rho(\dot{Z} + 1)$, The characteristic line is $\dot{Z} = -1, \dot{p}(t) = 0 \Rightarrow Z(t) = Z(0) - t, p(t) = p(0) = \rho(Z(0), 0) = \rho_0(Z(t) + t) \Rightarrow \rho(x, t) = \rho_0(x + t)$.

Example: $\partial_t\rho + a(x)\partial_x\rho = 0$ and $a(x) = x$. $\dot{p}(t) = \partial_x\rho\dot{Z} + \partial_t\rho = \partial_x\rho(\dot{Z} - a(Z)) = \partial_x\rho(\dot{Z} - Z)$. The characteristic line is $\dot{Z} = Z, \dot{p} = 0 \Rightarrow Z(t) = Z(0)e^t, p(t) = p(0) = \rho_0(Z(0)) = \rho_0(Z(t)e^{-t}) \Rightarrow \rho(x, t) = \rho_0(xe^{-t})$.

Example: $\partial_t u = A\partial_x u$ where $A \in R^{2 \times 2}$ is constant matrix and $P^{-1}AP = \Lambda = \text{diag}(\lambda_1, \lambda_2)$. Then $\partial_t u = P\Lambda P^{-1}\partial_x u, P^{-1}\partial_t u = \Lambda P^{-1}\partial_x u \Rightarrow \partial_t W = \Lambda\partial_x W$ where $W = P^{-1}u$. The following process is familiar.

Another case is $\partial_t u_1 = \lambda\partial_x u_1 + \partial_x u_2, \partial_t u_2 = \lambda\partial_x u_2$ (i.e. A can not be diagonalized). $u_2(x, t) = \rho_2^2(x + \lambda t)$, thus $\partial_t u_1 = \lambda\partial_x u_1 + (\rho_2^2(x + \lambda t))'$. Consider the common case: $\partial_t\rho + \partial_x\rho = f(x, t)$, we have $\dot{p}(t) = \partial_x\rho\dot{Z} + \partial_t\rho + \partial_x\rho\dot{Z} + f(Z(t), t) - \partial_x\rho = \partial_x\rho(\dot{Z} - 1) + f(Z(t), t)$. Then $\dot{Z} = 1, \dot{p}(t) =$

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$f(Z(t), t) \Rightarrow Z(t) = Z(0) + t, p(t) = p(0) + \int_0^t f(Z(\tau), \tau) d\tau = \rho_0(Z(0)) + \int_0^t f(Z(\tau), \tau) d\tau = \rho_0(Z(t) - t) + \int_0^t f(Z(\tau), \tau) d\tau = \rho_0(Z(t) - t) + \int_0^t f(Z(0) + \tau, \tau) d\tau$. So $\rho(x, t) = \rho_0(x - t) + \int_0^t f(x - t + \tau, \tau) d\tau$.

Example: $\partial_t^2 u = \partial_x^2 u, u|_{t=0} = g(x), \partial_t u|_{t=0} = h(x)$. Denote $\vec{W} = (\partial_t u, \partial_x u)^T$, we have $\partial_t W = (\partial_{tt} u, \partial_{xt} u)^T = (\partial_{xx} u, \partial_{xt} u)^T = \partial_x (\partial_x u, \partial_t u)^T \Rightarrow \partial_t W_1 = \partial_x W_2, \partial_t W_2 = \partial_x W_1 \Rightarrow \partial_t (W_1 + W_2) = \partial_x (W_1 + W_2), \partial_t (W_1 - W_2) = -\partial_x (W_1 - W_2)$. The solutions are $W_1 + W_2 = (W_1^0 + W_2^0)(x + t), W_1 - W_2 = (W_1^0 - W_2^0)(x - t)$ where $W_1^0 = \partial_t u(x, 0) = h(x), W_2^0 = \partial_x u(x, 0) = g'(x)$. Thus $\partial_t u = \frac{1}{2}(h(x + t) + h(x - t)) + \frac{1}{2}(g'(x + t) - g'(x - t)), u(x, t) = u(x, 0) + \int_0^t \partial_t u d\tau = g(x) + \frac{1}{2} \int_0^t (h(x + \tau) + h(x - \tau)) d\tau + \frac{1}{2} \int_0^t g'(x + \tau) - g'(x - \tau) d\tau = \frac{1}{2} \int_{x-t}^{x+t} h(s) ds + \frac{1}{2}(g(x + t) + g(x - t))$ (D'Alembert formula).

Example: $\partial_t \rho + \rho \rho_x = 0, p(t) = \rho(Z(t), t), \dot{p}(t) = \partial_x \rho(\dot{Z}(t) - p(t)) = 0 \Rightarrow \dot{Z}(t) = p(t), \dot{p}(t) = 0 \Rightarrow p(t) = p(0) = \rho_0(Z(0)), Z(t) = \rho_0(Z(0))t + Z(0)$. If $\rho_0(x) = x$, then $Z(t) = Z(0)t + Z(0), Z(0) = \frac{Z(t)}{1+t}, p(t) = \rho_0(Z(0)) = Z(0) = \frac{Z(t)}{1+t} \Rightarrow \rho(x, t) = \frac{x}{1+t}$. If $\rho_0(x) = -x$, then $Z(t) = -Z(0)t + Z(0), Z(0) = \frac{Z(t)}{1-t}, p(t) = -Z(0) = -\frac{Z(t)}{1-t} \Rightarrow \rho(x, t) = \frac{x}{t-1}$. We find when $t = 1$, shock happens since characteristic lines intersect.

Shock solution: recall conservation law $\int_a^b \partial_t \rho + \partial_x J(\rho) dx = 0 \Leftrightarrow \frac{d}{dt} \int_a^b \rho(x, t) dx = J(\rho(a, t)) - J(\rho(b, t))$. Denote shock solution as $S(x)$, $J(\rho(a, t)) - J(\rho(b, t)) = \frac{d}{dt} (\int_a^{S(t)} \rho(x, t) dx + \int_{S(t)}^b \rho(x, t) dx) = \int_a^{S(t)} \partial_t \rho dx + \rho(S(t)^-, t) \dot{S}(t) + \int_{S(t)}^b \partial_t \rho dx - \rho(S(t)^+, t) \dot{S}(t) = J(\rho(a, t)) - J(\rho(S(t)^-, t)) + J(\rho(S(t)^+, t)) - J(\rho(b, t)) + (\rho(S(t)^-, t) - \rho(S(t)^+, t)) \dot{S}(t)$. Thus $J(\rho(S(t)^-, t)) - J(\rho(S(t)^+, t)) = (\rho(S(t)^-, t) - \rho(S(t)^+, t)) \dot{S}(t) \Rightarrow \dot{S}(t) = \frac{J(\rho(S(t)^-, t)) - J(\rho(S(t)^+, t))}{\rho(S(t)^-, t) - \rho(S(t)^+, t)}$.

4.2 Common Cases (Lagrange's Method)

Consider $a(x, y) \partial_x u + b(x, y) \partial_y u = c(x, y) \Rightarrow (a, b, c) \cdot (\partial_x u, \partial_y u, -1) = 0$. Denote curved surface $S = \{z = u(x, y) | x, y \in R\}$ in R^3 , then (a, b, c) is the tangent direction of S . So denote characteristic curve as $(x(s), y(s), z(s))$, we have $\dot{x}(s) = a(x(s), y(s)), \dot{y}(s) = b(x(s), y(s)), \dot{z}(s) = c(x(s), y(s))$. A special case is $\partial_t \rho + \partial_x \rho = 0 \Rightarrow \dot{x}(s) = 1, \dot{t}(s) = 1, \dot{z}(s) = 0 \Rightarrow x(s) = x(0) + s, t(s) = t(0) + s, z(s) = z(0) \Rightarrow z(s) = \rho(x(s), t(s)) = \rho(x(0), t(0)) = \rho_0(x(0)) = \rho_0(x(s) - s) = \rho_0(x(s) - t(s))$.

How to impose IC condition? WLOG, let IC be $u(x, y)|_\Gamma = f, \Gamma = (\gamma_1(r), \gamma_2(r))$. Since we can't impose IC in the characteristic line which is in the tangent direction, so $(a, b) \cdot (-\gamma'_2(r), \gamma'_1(r)) \neq 0$ (non-characteristic condition). Then $\frac{d}{ds} x(r, s) = a(x, y), \frac{d}{ds} y(r, s) = b(x, y), \frac{d}{ds} z(r, s) = c(x, y)$ and we will get $z(r, s)$ and transform it into $z(H(x, y)) = u(x, y)$. Recall our characteristic line starts from $(r_0, 0, f(r_0))$. Back to original process, $J|_\Gamma = \frac{\partial(x, y)}{\partial(r, s)} = \det \begin{pmatrix} x_r & x_s \\ y_r & y_s \end{pmatrix} = \det \begin{pmatrix} \gamma'_1(r) & a \\ \gamma'_2(r) & b \end{pmatrix} \neq 0$, so $z(r, s) = z(H(x, y))$ is feasible (local existence).

Consider common case $F(x, y, u, \partial_x u, \partial_y u) = 0$. Denote $x = x(r, s), y = y(r, s), z = z(r, s) = u(x(r, s), y(r, s)), p(r, s) = \partial_x u|_{(x(r, s), y(r, s))}, q = \partial_y u|_{(x(r, s), y(r, s))}$ where $x(r, 0) = \gamma_1(r), y(r, 0) = \gamma_2(r)$. Then $0 = \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial u}{\partial x} + \frac{\partial F}{\partial p} \frac{\partial^2 u}{\partial x^2} + \frac{\partial F}{\partial q} \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p} p_x + \frac{\partial F}{\partial q} p_y, 0 = \frac{dF}{dy} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p} q_x + \frac{\partial F}{\partial q} q_y, \frac{\partial p}{\partial s} = \frac{\partial}{\partial s} (\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial x^2} \dot{x} + \frac{\partial^2 u}{\partial x \partial y} \dot{y} = p_x \dot{x} + p_y \dot{y} = -\frac{\partial F}{\partial x} - \frac{\partial F}{\partial z} p, \frac{\partial q}{\partial s} = \frac{\partial}{\partial s} (\frac{\partial u}{\partial y}) = q_x \dot{x} + q_y \dot{y} = -\frac{\partial F}{\partial y} - \frac{\partial F}{\partial z} q$ (the last two steps are because of the following definitions: $\frac{d}{ds} x(r, s) := \frac{\partial F}{\partial p}, \frac{d}{ds} y(r, s) := \frac{\partial F}{\partial q}$). Principle: 2nd order derivative should not appear in 1st order PDE. Then $\frac{\partial z}{\partial s} = \frac{\partial u}{\partial x} \dot{x} + \frac{\partial u}{\partial y} \dot{y} = p \dot{x} + q \dot{y} = p \frac{\partial F}{\partial p} + q \frac{\partial F}{\partial q}$. This is called Lagrange's method. Next, search for

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initial value. $x(r, 0) = \gamma_1(r), y(r, 0) = \gamma_2(r), z(r, 0) = f(r)$. Denote $p(r, 0) = \psi_1(r), q(r, 0) = \psi_2(r)$, so $F(\gamma_1(r), \gamma_2(r), f(r), \psi_1(r), \psi_2(r)) = 0$ and $\frac{\partial u}{\partial x} \frac{d}{dr} x(r, 0) + \frac{\partial u}{\partial y} \frac{d}{dr} y(r, 0)|_{\Gamma} = f'(r) \Leftrightarrow \psi_1(r)\gamma'_1(r) + \psi_2(r)\gamma'_2(r) = f'(r)$. Then we can get $\psi_1(r), \psi_2(r)$.

Example: $|\nabla u| = 1$ when $x^2 + y^2 < 1$ and $u = 0$ when $x^2 + y^2 = 1$ (Eikonal EQ). We have $|\nabla u|^2 = 1, F = p^2 + q^2 - 1, \dot{x}_s = 2p, \dot{y}_s = 2q, \dot{p}_s = 0, \dot{q}_s = 0, \dot{z}_s = 2p^2 + 2q^2$. IC: $x(r, 0) = \cos r, y(r, 0) = \sin r, \psi_1^2 + \psi_2^2 = 1, -\psi_1 \sin r + \psi_2 \cos r = 0$. We can get $\psi_1 = \pm \cos r, \psi_2 = \pm \sin r$, so $x(r, s) = \pm(2s+1)\cos r, y(r, s) = \pm(2s+1)\sin r, z = 2s \Rightarrow x^2 + y^2 = (2s+1)^2, u(x, y) = -1 + \sqrt{x^2 + y^2}$ or $1 - \sqrt{x^2 + y^2}$.

Example: $\partial_t u = \partial_x^2 u$. Denote $v = \partial_x u$, then $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$, namely $B\partial_t \vec{u} + A\partial_x \vec{u} = C\vec{u} = b(\vec{u})$. If there exists γ s.t. $\gamma^T B\partial_t \vec{u} + \gamma^T A\partial_x \vec{u} = \gamma^T b \Leftrightarrow m^T(\alpha\partial_t \vec{u} + \beta\partial_x \vec{u}) = \gamma^T b$, then characteristic lines satisfy $\dot{t}(s) = \alpha, \dot{x}(s) = \beta$ and $m^T \dot{z}(s) = \gamma^T b(t(s), x(s)) = f(s)$. Backwards, $\gamma^T B = \alpha m^T, \gamma^T A = \beta m^T \Rightarrow \gamma^T(\beta B - \alpha A) = 0 \Rightarrow \det(\beta B - \alpha A) = 0$. Here we find $\alpha = 0$, which leads $\frac{dx}{dt} = \infty$, so method of characteristic line is invalid.

Remark: \forall PDE, movement constrained in characteristic lines satisfy Newton's law of motion. For example, Burgers Equation $\partial_t u + u\partial_x u = 0$, if $\dot{\phi}(t) = u(x(t), t)$, then $\ddot{\phi}(t) = 0$.

5 Fourier Transform

5.1 Fourier Series

Fourier Trans: Given $f(x) \rightarrow \hat{f}(\xi)$, where $\hat{f}(\xi) = \int_{R^d} f(x)e^{-i2\pi x \cdot \xi} dx$. $f(x) \in$ real space, $\xi \in$ phase space. $\widehat{f'(x)}(\xi) = \int_{R^d} f'(x)e^{-i2\pi x \cdot \xi} dx = \int_{R^d} e^{-i2\pi x \cdot \xi} df(x) = i2\pi\xi \int_{R^d} f(x)e^{-i2\pi x \cdot \xi} dx = i2\pi\xi \hat{f}$.

Partial Fourier Trans: $\hat{u}(\xi, t) = \int_{R^d} u(x, t)e^{-i2\pi x \cdot \xi} dx$. Given $\partial_t u + \partial_x u = 0$, then $\widehat{\partial_t u} + \widehat{\partial_x u} = 0 \Rightarrow \partial_t \hat{u} + i2\pi\xi \hat{u} = 0$, which lets PDE \rightarrow Algebraic EQ \rightarrow ODEs.

Fourier inversion: $u(x, t) = \int_{R^d} \hat{u}(\xi, t)e^{i2\pi x \cdot \xi} d\xi$.

$f(x) \in R([-\pi, \pi])$, $f(x) \sim \sum_{n \in Z} C_n(f) e^{-inx}$ where $C_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{iny} dy$. Then $f(x) \sim C_0(f) + \sum_{n=1}^{\infty} C_n(f) e^{-inx} + \sum_{n=-1}^{-\infty} C_n(f) e^{-inx} = C_0(f) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(y) (e^{in(y-x)} + e^{-in(y-x)}) dy = C_0(f) + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(y) (\cos ny \cos nx + \sin ny \sin nx) dy = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$. Thus we have $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ky dy, b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ky dy$.

1) Parseval identity: $\|f\|_{L^2} = \|\{C_n(f)\}\|_{l^2}$. In fact, we have $\|f - S_N f\|_{L^2} \rightarrow 0$ with $N \rightarrow +\infty$ and $S_N f(x) \rightarrow f(x)$ a.e. What's more, if $f \rightarrow f^{(k)} \in L_1$, $C_n(f) \sim \frac{1}{n^k}$. If $f(x)$ is Lip, $C_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt = \frac{1}{2\pi} \int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} f(t) e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s + \frac{\pi}{n}) e^{-in(s+\frac{\pi}{n})} ds = -\frac{1}{2\pi} f(s + \frac{\pi}{n}) e^{-ins} ds \Rightarrow 2C_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t) - f(t + \frac{\pi}{n})) e^{-int} dt \Rightarrow 2|C_n(f)| \leq \frac{1}{2\pi} \text{Lip}(f) \frac{\pi}{|n|} \cdot 2\pi \Rightarrow |C_n(f)| \leq \frac{\pi}{2|n|} \text{Lip}(f)$.

2) R-L lemma: $\forall f \in L_1, C_n(f) \rightarrow 0$.

Gibbs phenomenon: Consider $f(x) = \frac{\pi-x}{2}, 0 < x < 2\pi; f(x) = 0, x = 0$. $S_N(f) = \sum_{n=1}^N \frac{\sin nx}{n}$, $\max_{0 < x < \frac{\pi}{N}} S_N(f)(x) = \max_{0 < x < \frac{\pi}{N}} \sum_{n=1}^N \frac{\sin nx}{n} x \rightarrow \int_0^{\pi} \frac{\sin t}{t} dt \approx 0.59\pi$, which increases by about 0.09π compared to the original function.

Example: $\Delta u = 0$ in $B(0, 1)$, $u = g$ on $\partial B(0, 1)$. Denote $u(1, \theta) = g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$ and $u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} r^n \int_{-\pi}^{\pi} g(\phi) \cos[n(\phi - \theta)] d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) d\phi + \frac{1}{\pi} \int_{-\pi}^{\pi} g(\phi) \sum_{n=1}^{\infty} r^n \cos[n(\phi - \theta)] d\phi \Rightarrow u(r, \theta) = \frac{1-r^2}{2\pi} \int_{-\pi}^{\pi} \frac{g(\phi)}{1-2r\cos(\phi-\theta)+r^2} d\phi$.

5.2 Wave Solution

1) Standing Wave: $u(x, t) = Z(x)T(t)$. 2) Travelling wave: $u(x, t) = Z(x - ct)$.

Consider $\square u = 0$, $(x, t) \in (0, \pi) \times (0, \infty)$, $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = f(x)$, $\partial_x u(x, t) = 0$ where $\square = \partial_t^2 - \partial_x^2$. ANSATZ $u(x, t) = Z(x)T(t)$, then $Z(x)\ddot{T}(t) = \ddot{Z}(x)T(t) \Leftrightarrow \frac{\dot{T}(t)}{T(t)} = \frac{\dot{Z}(x)}{Z(x)} = \lambda = \text{Const.}$ BC: $Z(0)T(t) = Z(\pi)T(t) = 0$. So $\ddot{Z}(x) = \lambda Z(x)$, $Z(0) = Z(\pi) = 0 \Rightarrow Z(x) = e^{\mu x}$ where $\mu^2 = \lambda$. 1) $\lambda = 0 \Rightarrow \mu = 0 \Rightarrow Z(x) = \text{Const.} \equiv 0$. 2) $\lambda > 0 \Rightarrow \mu = \pm\sqrt{\lambda} \Rightarrow Z(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$. Consider BC, $c_1 + c_2 = 0$, $c_1 e^{\sqrt{\lambda}\pi} + c_2 e^{-\sqrt{\lambda}\pi} = 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow Z(x) = \text{Const.} \equiv 0$. 3) $\lambda < 0 \Rightarrow \mu = \pm\sqrt{|\lambda|}i \Rightarrow Z(x) = c_1 e^{\sqrt{|\lambda|}ix} + c_2 e^{-\sqrt{|\lambda|}ix}$. Consider BC, $c_1 + c_2 = 0$, $c_1 e^{\sqrt{|\lambda|}i\pi} + c_2 e^{-\sqrt{|\lambda|}i\pi} = 0 \Rightarrow 2c_1 \sin(\sqrt{|\lambda|}\pi) = 0$, so $\lambda = -k^2$, $k = 1, 2, \dots$, $Z_k(x) = \sin(kx)$. Then $\frac{\dot{T}(t)}{T(t)} = \lambda = -k^2 \Rightarrow T(t) = c_1 \cos kt + c_2 \sin kt$. Thus $u_k(x, t) = (c_1 \cos kt + c_2 \sin kt) \sin kx$, $u(x, t) = \sum_{k=1}^{\infty} (c_k \cos kt + d_k \sin kt) \sin kx$. Consider IC, $\sum_{k=1}^{\infty} c_k \sin kx = f(x)$, $\sum_{k=1}^{\infty} k d_k \sin kx = 0 \Rightarrow c_l = \frac{2}{\pi} \int_0^{\pi} f(x) \sin l x dx$, $d_l = 0$, $\forall l \geq 1$. So $u(x, t) = \sum_{k=1}^{\infty} c_k \cos kt \sin kx$.

Denote $f(x) = \sum_{k=1}^{\infty} c_k \sin kx$, then $u(x, t) = \frac{1}{2} \sum_{k=1}^{\infty} c_k (\sin(k(x+t)) + \sin(k(x-t))) = \frac{1}{2}(f(x+t) + f(x-t))$ which is superposition of two waves. This usually leads weak solu.

5.3 Fourier Transform

Recall $\hat{f}(\xi) = \int_{R^d} \widehat{f(x)} e^{-i2\pi x \cdot \xi} dx$, $f(x) = \int_{R^d} \widehat{f}(\xi) e^{i2\pi x \cdot \xi} d\xi$.

Gauss function: $e^{-\pi|x|^2} = e^{-\pi|\xi|^2}$. $\widehat{f}(\xi) = \int_{R^d} e^{-\pi|x|^2} e^{-i2\pi x \cdot \xi} dx = \prod_{j=1}^d \int_R e^{-\pi x_j^2} e^{-i2\pi x_j \xi_j} dx_j = \prod_{j=1}^d e^{-\pi x_j^2}$. Denote $F(\xi) = \int_R e^{-\pi x^2} e^{-i2\pi x \xi} dx$, then $F'(\xi) = -2\pi \xi F(\xi) \Rightarrow F(\xi) = C e^{-\pi \xi^2}$ where $C = F(0) = \int_R e^{-\pi x^2} dx = 1$. So we get the conclusion.

Translation: $x \rightarrow x + h$. $\widehat{f(x+h)}(\xi) = e^{i2\pi h \cdot \xi} \widehat{f}(\xi)$.

Stretching: $x \rightarrow \lambda x$. $\widehat{f(\lambda x)}(\xi) = \lambda^{-d} \widehat{f}(\xi/\lambda)$.

Derivative: $\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i2\pi \xi_j \widehat{f}(\xi)$. To go further, $p(D) = \sum_k a_k D^k$, then $\widehat{p(D)f} = p(2\pi i \xi) \widehat{f}$.

$f(x)$ is radial function $\Leftrightarrow \widehat{f}$ is radial. We call a function radial iff $f(x) = f_0(|x|)$ or $f(Qx) = f(x)$ where Q is an arbitrary rotation.

$f(x)$ is a homogeneous function of order α , then \widehat{f} is a homo-func of order $-(d+\alpha)$.

Convolution: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.

δ function: $\widehat{\delta} = 1$ where $\delta(0) = 1$, $\delta(x) = 0 \forall x \neq 0$ and $(\delta, f) = f(0)$. $1^v = \delta(x)$.

Example: $(xf(x))^\wedge = \int_R x f(x) e^{-i2\pi x \cdot \xi} dx = \int_R \frac{1}{-i2\pi} f(x) \frac{\partial}{\partial \xi} (e^{-i2\pi x \cdot \xi}) dx = \frac{i}{2\pi} \frac{d}{d\xi} \int_R f(x) e^{-i2\pi x \cdot \xi} dx = \frac{i}{2\pi} (\widehat{f}(\xi))'$. So $(\frac{x^2}{(1+x^2)^2})^\wedge = (xf(x))^\wedge = \frac{i}{2\pi} (\widehat{f}(\xi))'$ where $\widehat{f}(x) = (\frac{x}{(1+x^2)^2})^\wedge = -\frac{1}{2} [((1+x^2)^{-1})']^\wedge = -\frac{1}{2} i2\pi \xi [(1+x^2)^{-1}]^\wedge = -\frac{1}{2} i2\pi \xi \pi e^{-2\pi|\xi|} = -i\pi^2 \xi e^{-2\pi|\xi|}$.

Example: $\int_{R^d} \widehat{f}(\xi) e^{i2\pi x \cdot \xi} d\xi = \int_{R^d} (\int_{R^d} f(y) e^{-i2\pi y \cdot \xi} dy) e^{i2\pi x \cdot \xi} d\xi = \int_{R^d} f(y) \int_{R^d} e^{i2\pi(x-y) \cdot \xi} d\xi dy = \int_{R^d} f(y) \delta(x-y) dy = f(x)$.

Example: $\partial_t u + \partial_x u = 0$, $u|_{t=0} = u_0(x)$. $\widehat{\partial_t u} + \widehat{\partial_x u} = 0 \Rightarrow \partial_t \hat{u} + i2\pi \xi \hat{u} = 0$, $\hat{u}|_{t=0} = \hat{u}_0(\xi) \Rightarrow \hat{u}(\xi, t) = e^{-i2\pi \xi t} \hat{u}_0(\xi)$. Then $u(x, t) = \int_{R^d} \hat{u}_0(\xi) e^{-i2\pi \xi t} e^{i2\pi \xi x} d\xi = u_0(x-t)$.

Example: $-\Delta G = \delta$, $x \in R^3$. $\widehat{-\Delta G} = \widehat{\delta} \Rightarrow 4\pi^2 |\xi|^2 \widehat{G} = 1 \Rightarrow \widehat{G} = \frac{1}{4\pi^2 |\xi|^2}$. Notice \widehat{G} is radial $\rightarrow G$ is radial; G is a homo-func of order $-(\lambda+d) = -3 - (-2) = -1$. So $G(x) = G(\frac{x}{|x|} |x|) = |x|^{-1} G(\frac{x}{|x|}) = |x|^{-1} G_0(|\frac{x}{|x|}|) = |x|^{-1} G_0(1) = C|x|^{-1}$. On the other hand, $\int_{B(0, \epsilon)} -\Delta G dx = \int_{B(0, \epsilon)} \delta(x) dx = 1 \Rightarrow -\int_{\partial B(0, \epsilon)} \frac{\partial G}{\partial n} d\sigma = 1 = -\int_{\partial B(0, \epsilon)} G'_0(\epsilon) d\sigma = -4\pi \epsilon^2 G'_0(\epsilon) \Rightarrow C = \frac{1}{4\pi}$. So $G(x) = \frac{1}{4\pi |x|}$.

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Recall Parsaval idenety: $\|f\|_{L^2} = \|C_n(f)\|_{l_2}$.

Plancherel's Thm: $\int |f|^2 dx = \int |\hat{f}|^2 d\xi$, i.e. $\|f\|_{\mathcal{L}_2} = \|\hat{f}\|_{\mathcal{L}_2}$.

Proof: $\int f \bar{f} dx = \int_{R^d} \int_{R^d} \hat{f}(\xi) e^{i2\pi x \cdot \xi} d\xi \int_{R^d} \hat{f}(y) e^{i2\pi x \cdot y} dy dx = \int_{R^d} \hat{f}(\xi) \bar{\hat{f}(y)} \int_{R^d} e^{i2\pi x \cdot (\xi-y)} dx d\xi dy$
 $= \int_{R^d} \hat{f}(\xi) \hat{f}(\xi) d\xi = \int_{R^d} |\hat{f}(\xi)|^2 d\xi$.

Corollary: $\int f(x) \overline{g(x)} dx = \sum_n C_n(f) \overline{C_n(g)}$, $\int f(x) \overline{g(x)} dx = \int \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$.

Example: $\square u = 0, u|_{t=0} = f(x), \partial_t u|_{t=0} = g(x)$. By fourier transform, we get $\partial_t^2 \hat{u} + 4\pi^2 \xi^2 \hat{u} = 0, \hat{u}|_{t=0} = \hat{f}(\xi), \partial_t \hat{u}|_{t=0} = \hat{g}(\xi) \Rightarrow \hat{u}(\xi, t) = \hat{f}(\xi) \cos(2\pi \xi t) + \hat{g}(\xi) \frac{\sin(2\pi \xi t)}{2\pi \xi}$. By fourier inversion, we get $(\hat{f}(\xi) \cos(2\pi \xi t))^v = \frac{1}{2}(f(x+t) + f(x-t)), (\hat{g}(\xi) \frac{\sin(2\pi \xi t)}{2\pi \xi})^v = \int_{R^d} \hat{g}(\xi) \frac{e^{i2\pi \xi(x+t)} - e^{i2\pi \xi(x-t)}}{4i\pi \xi} d\xi = \frac{1}{2} \int_{R^d} \hat{g}(\xi) \int_{x-t}^{x+t} e^{i2\pi \xi s} ds d\xi = \frac{1}{2} \int_{x-t}^{x+t} \int_{R^d} \hat{g}(\xi) e^{i2\pi \xi s} d\xi ds = \frac{1}{2} \int_{x-t}^{x+t} g(s) ds$.

Denote $K(t) = \frac{1}{2} \int |\partial_t u|^2 dx, P(t) = \frac{1}{2} \int |\partial_x u|^2 dx$, then $K(t) + P(t) = \frac{1}{2} \int (|\partial_t u|^2 + |\partial_x u|^2) dx = \frac{1}{2} \int (|\widehat{\partial_t u}|^2 + |\widehat{\partial_x u}|^2) dx = \frac{1}{2} \int |\partial_t \hat{u}|^2 + 4\pi^2 \xi^2 |\hat{u}|^2 dx$. $\partial_t \hat{u} = -2\pi \xi \hat{f}(\xi) \sin(2\pi \xi t) + \hat{g}(\xi) \cos(2\pi \xi t), 2\pi \xi \hat{u} = 2\pi \xi \hat{f}(\xi) \cos(2\pi \xi t) + \hat{g}(\xi) \sin(2\pi \xi t)$. Thus $K(t) + P(t) = \frac{1}{2} \int (4\pi^2 \xi^2 |\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2) d\xi = \frac{1}{2} \int (|\partial_x f(x)|^2 + |g(x)|^2) dx = K(0) + P(0) \rightarrow$ conservation of energy.

$\lim_{t \rightarrow \infty} K(t) = \lim_{t \rightarrow \infty} \frac{1}{2} \int_R 4\pi^2 \xi^2 |\hat{f}(\xi)|^2 \sin^2(2\pi \xi t) - 2\pi \xi \hat{f}(\xi) \hat{g}(\xi) \sin(4\pi \xi t) + |\hat{g}(\xi)|^2 \cos^2(2\pi \xi t) d\xi = \frac{1}{4} \int (4\pi^2 \xi^2 |\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2) d\xi = \frac{1}{2} (K(0) + P(0))$. And the same holds true for $\lim_{t \rightarrow \infty} P(t)$.

Heat Equation: $\partial_t H = \Delta H, H|_{t=0} = \delta(x)$. By fourier trans, we get $\partial_t \hat{H} = -4\pi^2 |\xi|^2 \hat{H}, \hat{H}|_{t=0} = 1 \Rightarrow \hat{H} = e^{-4\pi^2 |\xi|^2 t} \Rightarrow H = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$ (Heat Kernel).

Bessel potential: $-\Delta B + B = \delta$. By fourier trans, $\hat{B} = \frac{1}{1+4\pi^2 |\xi|^2} \Rightarrow B = \int_{R^d} \frac{1}{1+4\pi^2 |\xi|^2} e^{i2\pi x \cdot \xi} d\xi = \int_{R^d} \left(\int_0^\infty e^{-(1+4\pi^2 |\xi|^2)s} ds \right) e^{i2\pi x \cdot \xi} d\xi = \int_0^\infty e^{-s} \int_{R^d} e^{-4\pi^2 |\xi|^2 s} e^{i2\pi x \cdot \xi} d\xi ds = \int_0^\infty e^{-s} (4\pi s)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4s}} ds$.

Laplace Equation: $\Delta u = 0, u(x, 0) = f(x), (x, y) \in R \times R_+$. $\hat{u}(\xi, y) = \int_R u(x, y) e^{-i2\pi x \cdot \xi} dx, \partial_y^2 \hat{u} - 4\pi^2 \xi^2 \hat{u} = 0, \hat{u}|_{(\xi, 0)} = \hat{f}, \hat{u} = c_1 e^{-2\pi |\xi| y} \hat{f}(\xi) + c_2 e^{2\pi |\xi| y} \hat{f}(\xi)$. We ignore the singular term for $y > 0$, and $(e^{-2\pi |\xi| y})^v = \frac{y}{\pi(x^2+y^2)} \Rightarrow u(x, y) = \int \frac{f(x-t)y}{\pi(t^2+y^2)} dt$.

We find $(e^{-2\pi |\xi|})^v = \frac{1}{\pi(x^2+1)}$, thus $\widehat{\frac{1}{1+x^2}} = \pi e^{-2\pi |\xi|}, B(x) = (\frac{1}{1+4\pi^2 |\xi|^2})^v = \int_R \frac{1}{1+4\pi^2 |\xi|^2} e^{i2\pi x \cdot \xi} d\xi = (\eta = 2\pi \xi) = \frac{1}{2\pi} \int_R \frac{1}{1+\eta^2} e^{ix\eta} d\eta = \frac{1}{2} e^{-|x|}$. (dim = 1)

Newton Potential: $N(x) = (\frac{1}{4\pi^2 |\xi|^2})^v = \frac{1}{(d-2)\omega_{d-1}} |x|^{2-d}$. On the other hand, $\int_0^\infty H(x, t) dt (t = \frac{|x|^2}{4s}) = \int_0^\infty (\frac{\pi|x|^2}{s})^{-\frac{d}{2}} e^{-s \frac{|x|^2}{4}} s^{-2} ds = \frac{1}{4} \pi^{-\frac{d}{2}} |x|^{2-d} \int_0^\infty s^{\frac{d}{2}-2} e^{-ds} ds = \frac{1}{4} \pi^{-\frac{d}{2}} |x|^{2-d} \Gamma(\frac{d}{2} - 1) = N(x)$.

Assume $f(x)$ is radial and consider polar coordinates $\begin{cases} x_1 = r \sin \theta \cos \phi \\ x_2 = r \sin \theta \sin \phi \\ x_3 = r \cos \theta \end{cases}$ or $x = r\gamma$. Then

$\hat{f}(\xi) = \int_{R^3} f(x) e^{-i2\pi x \cdot \xi} dx = \int_0^\infty r^2 \int_{S^2} f(r\gamma) e^{-i2\pi r\gamma \cdot \xi} d\sigma(\gamma) dr = \int_0^\infty r^2 f_0(r) \int_{S^2} e^{-i2\pi r\gamma \cdot \xi} d\sigma(\gamma) dr$ and $d\sigma(\gamma) = \sin \theta d\theta d\phi$. $I(\xi) = \frac{1}{4\pi} \int_{S^2} e^{-i2\pi r\gamma \cdot \xi} d\sigma(\gamma), I(\xi) = I(Q\xi)$ (Q is an arbitrary rotation) $\Rightarrow I(\xi) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi e^{-i2\pi r|\xi| \cos \theta} \sin \theta d\theta = \frac{1}{2} \int_0^\pi e^{-i2\pi r|\xi| \cos \theta} \sin \theta d\theta (s = 2\pi r|\xi| \cos \theta) = \frac{1}{2} \int_{-2\pi r|\xi|}^{2\pi r|\xi|} \frac{\cos s}{2\pi r|\xi|} ds \Rightarrow \hat{f}(\xi) = \int_0^\infty r^2 f_0(r) \int_{S^2} e^{-i2\pi r\gamma \cdot \xi} d\sigma(\gamma) dr = 4\pi \int_0^\infty r^2 f_0(r) \frac{\sin 2\pi r|\xi|}{2\pi r|\xi|} dr = \frac{2}{|\xi|} \int_0^\infty r f_0(r) \sin 2\pi r|\xi| dr$. If dim = 2, $\hat{f}(\xi) = \int_{R^2} f(x) e^{-i2\pi x \cdot \xi} dx = \int_0^\infty r \int_{S^1} f(r\gamma) e^{-i2\pi r\gamma \cdot \xi} d\sigma(\gamma) dr = \int_0^\infty r f_0(r) \int_{S^1} e^{-i2\pi r\gamma \cdot \xi} d\sigma(\gamma) dr = [\xi = (0, -|\xi|)] = \int_0^\infty 2\pi r f_0(r) \times J_0(2\pi r|\xi|) dr$ where $J_0(2\pi r|\xi|) = \int_0^{2\pi} e^{i2\pi r|\xi| \sin \theta} d\theta$.

Yukawa potential: $-\Delta B + \mu^2 B = \delta$. By fourier trans, $\hat{B} = \frac{1}{\mu^2 + 4\pi^2 |\xi|^2}, B = \int_{R^3} \frac{1}{\mu^2 + 4\pi^2 |\xi|^2} e^{i2\pi x \cdot \xi} d\xi = \int_0^\infty \frac{4\pi r^2}{\mu^2 + 4\pi^2 r^2} \frac{\sin(2\pi|x|r)}{2\pi|x|r} dr = \frac{2}{|x|} \int_0^\infty \frac{r \sin(2\pi|x|r)}{\mu^2 + 4\pi^2 r^2} dr$ (transfer to complex variable) $= \frac{e^{-\mu|x|}}{4\pi|x|}$. (dim = 3)

Denote $f(\xi) = e^{-2\pi |\xi|}$. $(e^{-2\pi |\xi| y})^v = \frac{y}{\pi x^2 + y^2}, e^{-2\pi |\xi|} = \frac{1}{\pi} \int \frac{1}{x^2 + 1} e^{i2\pi \xi \cdot x} dx = \frac{1}{\pi} \int (\int_0^\infty e^{-(x^2+1)s} ds)$
 $e^{i2\pi \xi \cdot x} dx = \frac{1}{\pi} \int_0^\infty e^{-s} (e^{-sx^2})^v ds = \frac{1}{\pi} \int_0^\infty e^{-s} (\frac{s}{\pi})^{-\frac{1}{2}} e^{-\frac{\pi^2 |\xi|^2}{s}} ds$. $(e^{-2\pi |\xi|})^v = \pi^{-\frac{1}{2}} \{ \int_0^\infty s^{-\frac{1}{2}} e^{-(s+\frac{\pi^2 |\xi|^2}{s})} ds \}$

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$$= \pi^{-\frac{1}{2}} \int_0^\infty s^{-\frac{1}{2}} e^{-s} (e^{-\frac{\pi^2 |\xi|^2}{s}}) ds = \pi^{-\frac{1}{2}} \int_0^\infty s^{-\frac{1}{2}} e^{-s} (\frac{\pi}{s})^{-\frac{d}{2}} e^{-s|x|^2} ds = \pi^{-\frac{d+1}{2}} \int_0^\infty s^{\frac{d-1}{2}} e^{-s(1+|x|^2)} ds (t = s(1+|x|^2)) = \pi^{-\frac{d+1}{2}} \int_0^\infty (1+|x|^2)^{-\frac{d-1}{2}} t^{\frac{d-1}{2}} e^{-t} (1+|x|^2)^{-1} dt = \pi^{-\frac{d+1}{2}} (1+|x|^2)^{-\frac{d+1}{2}} \int_0^\infty t^{\frac{d-1}{2}} e^{-t} dt = \pi^{-\frac{d+1}{2}} (1+|x|^2)^{-\frac{d+1}{2}} \Gamma(\frac{d+1}{2}) \rightarrow \text{Gauss-Weistrass kernel.}$$

Consider $\square u = 0, u|_{t=0} = f(x), \partial_t u|_{t=0} = g(x)$. $\partial_t^2 \hat{u} + 4\pi^2 |\xi|^2 \hat{u} = 0, \hat{u}(\xi, t) = \hat{f}(\xi) \cos(2\pi|\xi|t) + \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$. Def $M_t(tg) = \int_{|y-x|=t} tg(y) d\sigma(\gamma)$, then $\widehat{M_t(tg)} = \int_{R^3} f_{S^2} tg(x+t\gamma) d\sigma(\gamma) e^{-i2\pi x \cdot \xi} dx = \int_{R^3} f_{S^2} tg(y) e^{-i2\pi(y-t\gamma) \cdot \xi} d\sigma(\gamma) dy = \int_{R^3} tg(y) e^{-i2\pi y \cdot \xi} dy \int_{S^2} e^{i2\pi t \gamma \cdot \xi} d\sigma(\gamma) = \hat{g}(\xi) \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$. Thus when $\dim = 3, u(x, t) = \frac{\partial}{\partial t}(M_t(tf)) + M_t(tg)$. Also, $M_t(f) = \frac{1}{4\pi} \int_{S^2} f(x-t\gamma) d\sigma(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi f(x_1 - t \sin \theta \cos \phi, x_2 - t \sin \theta \sin \phi) \sin \theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} f(x_1 - t \sin \theta \cos \phi, x_2 - t \sin \theta \sin \phi) \sin \theta d\theta (r = \sin \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^1 f(x_1 - t r \cos \phi, x_2 - t r \sin \phi) \frac{r}{\sqrt{1-r^2}} dr = \frac{1}{2\pi} \int_{|y| \leq 1} f(x-ty) \frac{1}{\sqrt{1-|y|^2}} dy := \widetilde{M}_t(f)$. When $\dim = 2, u(x, t) = \frac{\partial}{\partial t}(t \widetilde{M}_t(f)) + t \widetilde{M}_t(g)$. (Method of descent)

Regard fourier transform as an operator, i.e. $Tf = \hat{f}$, then $T : \mathcal{L}_1 \rightarrow \mathcal{L}_\infty, \|T\|_{\mathcal{L}_1 \rightarrow \mathcal{L}_\infty} \leq 1; T : \mathcal{L}_2 \rightarrow \mathcal{L}_2, \|T\|_{\mathcal{L}_2 \rightarrow \mathcal{L}_2} = 1$.

Riesz-Thorin Interpolation Thm: $T : \mathcal{L}_{p_1} \rightarrow \mathcal{L}_{q_1}, T : \mathcal{L}_{p_2} \rightarrow \mathcal{L}_{q_2}, \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, then $T : \mathcal{L}_p \rightarrow \mathcal{L}_q, \|T\|_{\mathcal{L}_p \rightarrow \mathcal{L}_q} \leq \|T\|_{\mathcal{L}_{p_1} \rightarrow \mathcal{L}_{q_1}}^\theta + \|T\|_{\mathcal{L}_{p_2} \rightarrow \mathcal{L}_{q_2}}^{1-\theta}$.

Hausdorff-Young's inequality: $\|\hat{f}\|_{\mathcal{L}^p} \leq \|f\|_{\mathcal{L}^q}, \frac{1}{p} + \frac{1}{q} = 1$.

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \text{ where } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Example: $\square u = f, u|_{t=0} = 0, \partial_t u|_{t=0} = 0$. $\partial_t \hat{u} + 4\pi^2 |\xi|^2 \hat{u} = \hat{f}, \hat{u}|_{t=0} = 0, \partial_t \hat{u}|_{t=0} = 0, \hat{u}(\xi, t) = \int_0^t \frac{\sin 2\pi \xi (t-s)}{2\pi \xi} \hat{f}(\xi, s) ds$. $u(x, t) = \int_0^t \int_R \frac{\sin 2\pi \xi (t-s)}{2\pi \xi} \hat{f}(\xi, s) e^{i2\pi x \cdot \xi} d\xi ds = \frac{1}{2} \int_0^t \int_R \frac{e^{i2\pi \xi (t-s)} - e^{-i2\pi \xi (t-s)}}{i2\pi \xi} e^{i2\pi x \cdot \xi} \hat{f}(\xi, s) d\xi ds = \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} f(x, s) dx ds$.

Example: $u_{tt} = u_{xx} + u, u|_{t=0} = g, \partial_t u|_{t=0} = h$. Denote $v(x, y, t) = u(x, t)e^y$, then $\partial_t^2 v = \partial_t^2 ue^y, \partial_x^2 v = \partial_x^2 ue^y, \partial_y^2 v = ue^y \Rightarrow v_{tt} = \partial_x^2 v + \partial_y^2 v, v(x, y, 0) = ge^y, \partial_t v(x, y, 0) = he^y$.

Example: (Duhammel's Principle) $U_t + AU = F$. The solution is $U(x, t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds$. Then for $u_{tt} = \Delta u + f, \frac{\partial}{\partial t} \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \Rightarrow U_t = AU + F$. Therefore, $U = \begin{pmatrix} u \\ u_t \end{pmatrix} = S(t) \begin{pmatrix} g \\ h \end{pmatrix} + \int_0^t S(t-s) \begin{pmatrix} 0 \\ f \end{pmatrix} ds$.

Kirchhoff formula($\dim = 3$): $\square_a u = \partial_t^2 u - a^2 \partial_x^2 u = f, u|_{t=0} = \phi(x), \partial_t u|_{t=0} = \psi(x)$. The solution is $u(x, t) = \frac{1}{4\pi a^2 t^2} \int_{\partial B(x, at)} [\phi(y) + D\phi(y) \cdot (y-x) + t\psi(y)] ds(Y) + \frac{1}{4\pi a^2} \int_{B(x, at)} \frac{f(y, t-|y-x|/a)}{|y-x|} dy$.

For $\dim = 2, u(x, t) = \frac{1}{2\pi a t} \int_{B(x, at)} \frac{[\phi(y) + D\phi(y) \cdot (y-x) + t\psi(y)]}{\sqrt{(at)^2 - |y-x|^2}} dy + \frac{1}{2\pi a} \iint_{C(x, t)} \frac{f(y, \tau)}{\sqrt{a^2(t-\tau)^2 - |y-x|^2}} dy d\tau$, where $C(x, t) = \{(y, \tau) \in R^3 : 0 \leq \tau \leq t, |y-x| \leq a(t-\tau)\}$.

Example: $\partial_t u = \partial_x^2 u + a \partial_x u + bu. \partial_t \hat{u} = -4\pi^2 \xi^2 \hat{u} + ai2\pi \xi \hat{u} + b\hat{u}, \hat{u}(\xi, t) = \hat{u}(\xi, 0) e^{-4\pi^2 \xi^2 t + i2\pi a \xi t + bt}$. $u(t) = \int_R e^{-4\pi^2 \xi^2 t + i2\pi a \xi t} e^{i2\pi x \cdot \xi} d\xi e^{bt} = \int_R e^{-4\pi^2 \xi^2 t} e^{i2\pi \xi(x+at)} d\xi e^{bt} = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x+at|^2}{4t} + bt}$.

Example: $\square u = c^2 u, u|_{t=0} = g(x), \partial_t u|_{t=0} = h(x)$. Denote $v(x, y, t) = u(x, t)e^{cy}$, we get $\partial_t^2 v - \Delta v = \square v = 0$.

Example: $\partial_t u = x^2 \frac{\partial^2 u}{\partial x^2} + ax \frac{\partial u}{\partial x}$. Denote $U(y, t) = u(e^{-y}, t)$ ($x = e^{-y}$), so $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial y^2} + (1-a) \frac{\partial U}{\partial y}$. Then use fourier transform.

Example: $\square u = 0, u|_{t=0} = x^3 + y^2 z, \partial_t u|_{t=0} = 0$. Consider $\square u_1 = 0, u_1|_{t=0} = x^3, \partial_t u_1|_{t=0} = 0$ and $\square u_2 = 0, u_2|_{t=0} = y^2 z, \partial_t u_2|_{t=0} = 0$. Assume $u_1 = \phi(x, t), u_2 = \psi(y, t)z$, we get $u_1 = \frac{1}{2}((x+t)^3 + (x-t)^3), u_2 = \frac{1}{2}((y+t)^2 + (y-t)^2)z$.

6 Fundamental Solution and Green's Function

6.1 Fundamental Solution

Consider $-\Delta u = f$, we hope $u = N * f$ where N is fundamental solution and satisfies $-\Delta N = \delta$, then $N(x) = \frac{1}{(d-2)w_d} |x|^{2-d}$ ($d > 2$). When $d = 2$, $N(x) = -\frac{1}{2\pi} \ln|x|$; $d = 1$, $N(x) = -\frac{1}{2}|x|$.

By fourier transform, we have already known that the solution to $-\Delta N = \delta$ is a radial function. So denote $N(x) = N_0(r)$, $r = |x|$, then $\frac{\partial N}{\partial x_i} = N'_0(r) \frac{\partial r}{\partial x_i} = N'_0(r) \frac{x_i}{r}$, $\frac{\partial^2 N}{\partial x_i^2} = N''_0(r) (\frac{x_i}{r})^2 + N'_0(r) (\frac{1}{r} - \frac{x_i^2}{r^3})$. $\Delta N = \Delta N_0 \Rightarrow rN''_0(r) + N'_0(r)(d-1) = 0 \Rightarrow (r^{d-1}N'_0(r))' = 0$, $N'_0(r) = \frac{c_1}{r^{d-1}}$, $N_0(r) = c_2 r^{2-d}$ ($d \neq 2$) or $c_2 \ln r$ ($d = 2$).

Yukawa potential: $-\Delta Y_\mu + \mu^2 Y_\mu = \delta$. $Y_\mu(x) = Y_0(|x|)$, $|x| = r$, so $-rY''_0(r) - 2Y'_0(r) + \mu^2 r Y_0(r) = 0$. Denote $W(r) = rY_0(r)$, then $-W''(r) + \mu^2 W(r) = 0$. Here we choose $W(r) = e^{-\mu r}$, $Y_0(r) = \frac{c}{r} e^{-\mu r}$. To get c , we consider the integral $\int_{B(0,\epsilon)} -\Delta Y_\mu dx + \mu^2 \int_{B(0,\epsilon)} Y_\mu dx = 1 \Rightarrow -\int_{\partial B(0,\epsilon)} \frac{\partial Y_\mu}{\partial n} d\sigma(x) + \mu^2 \int_{B(0,\epsilon)} Y_\mu dx = 1 \Rightarrow \frac{c}{\epsilon^2} (1 + \mu\epsilon) e^{-\mu\epsilon} 4\pi\epsilon^2 + \mu^2 \int_{B(0,\epsilon)} \frac{c}{r} e^{-\mu r} dx = 1$ ($\epsilon \rightarrow 0$) $\Rightarrow c = \frac{1}{4\pi}$.

Wave equation: $\square W = 0$, $W|_{t=0} = 0$, $\partial_t W|_{t=0} = \delta$. $\hat{W} = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$, $W = \int_{R^3} \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} e^{i2\pi x \cdot \xi} d\xi = \int_0^\infty \frac{r \sin(2\pi rt)}{2\pi} \int_{S^2} e^{i2\pi x \cdot r\gamma} d\sigma(\gamma) dr = \frac{1}{2\pi|x|} \int_0^\infty \sin(2\pi rt) \sin(2\pi r|x|) dr = \frac{-1}{8\pi|x|} \int_R (e^{i2\pi r(t+|x|)} - e^{-i2\pi r(t-|x|)} - e^{-i2\pi r(|x|-t)} - e^{-i2\pi r(t+|x|)}) dr = \frac{1}{4\pi|x|} \delta(t-|x|)$. So $W * h = \int_{R^3} \frac{\delta(t-|y|)}{4\pi|y|} h(x-y) dy = \frac{1}{4\pi} \int_{S^2} \int_0^\infty r \delta(t-r) h(x-r\gamma) dr d\sigma(\gamma) = \frac{1}{4\pi} \int_{S^2} t h(x-t\gamma) d\sigma(\gamma) = t M_t(h)$. The solution of common cases is given by $\partial_t(W * u_0) + W_t * u_1$. (All the discussions here are in the condition of $d = 3$)

Heat equation: $\partial_t H = \partial_x^2 H$, $H|_{t=0} = \delta(x)$. We notice that $\lambda^d H(\lambda x, \lambda^2 t)$ is a fundamental solution ($\delta(\lambda x) = \lambda^{-d} \delta(x)$), so $\lambda^d H(\lambda x, \lambda^2 t) = H(x, t)$ ($\lambda = t^{-\frac{1}{2}}$) $\Rightarrow H(x, t) = t^{-d/2} h(x/t^{1/2})$ (radial) $= t^{-d/2} w(|x|/t^{1/2})$. Taking it to the previous equation, we can get $\frac{1}{2}(r^d w(r))' + (r^{d-1} w'(r))' = 0 \Rightarrow \frac{r^d}{2} w(r) + r^{d-1} w'(r) = 0 \Rightarrow W(r) = C e^{-\frac{1}{4}r^2}$.

Example: $\partial_t u + \Delta(u^\gamma) = 0$, $x \in R^d$, $\gamma > 1$, $u \geq 0$. We assume the solution has the form $t^{-\alpha} v(\frac{x}{t^\beta})$ (self-similar), then $t^{-\alpha-1}(u + (y \cdot \nabla)v)(y) = t^{-(\alpha\gamma+2\beta)} \Delta v^\gamma$ where $y = \frac{x}{t^\beta} \Rightarrow \alpha+1 = \alpha\gamma+2\beta$. To find fundamental solution, we assume $v(\frac{x}{t^\beta}) = w(\frac{|x|}{t^\beta})$ and denote $r = \frac{|x|}{t^\beta}$. So $\Delta u^\gamma = t^{-\alpha\gamma} \Delta w^\gamma = t^{-\alpha\gamma-2\beta} (w^\gamma + \frac{d-1}{r} (w^\gamma)')$, $\partial_t u = t^{-\alpha-1} (\alpha w + \beta r w'(r)) \Rightarrow \alpha r w + \beta r^2 w'(r) + r(w^\gamma)'' + (d-1)(w^\gamma)' = 0 \Rightarrow \alpha r^{d-1} w + \beta r^{d-1} w'(r) + (r^{d-1}(w^\gamma))' = 0$. Assume $\alpha = d\beta$, then $(\beta r^d w + r^{d-1}(w^\gamma))' = 0 \Rightarrow (w^\gamma)' = -\beta r w \Rightarrow \frac{\gamma}{\gamma-1} (w^{\gamma-1})' = -\beta r \Rightarrow w = (C - \frac{\beta(\gamma-1)}{2\gamma} r^2)^{\frac{1}{\gamma-1}} = (C - \frac{\beta(\gamma-1)}{2\gamma} \frac{|x|^2}{t^{2\beta}})^{\frac{1}{\gamma-1}}$.

6.2 Green's Function

Consider $\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases}$ (Dirichlet BVP). We hope we can represent the solution in terms of Green's function. $0 = \int_\Omega (\Delta u + f) v dx = -\int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma(x) + \int_\Omega f \cdot v dx \Rightarrow \int_\Omega \Delta u \cdot v dx = -\int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v d\sigma(x)$, $\int_\Omega \Delta v \cdot u dx = -\int_\Omega \nabla v \nabla u dx + \int_{\partial\Omega} \frac{\partial v}{\partial n} u d\sigma(x) \Rightarrow \int_\Omega (\Delta u \cdot v - \Delta v \cdot u) dx = \int_{\partial\Omega} (\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u) d\sigma(x)$. Define Green function as $\begin{cases} -\Delta G_y(x) = \delta(x-y), x \in \Omega \\ G_y(x) = 0, x \in \partial\Omega \end{cases}$, so the

above can be rewritten as $-\int_\Omega f(x) G_y(x) dx + u(y) = -\int_{\partial\Omega} \frac{\partial G_y}{\partial n} g(x) d\sigma(x)$ where $G_y = v$ and then $u(y) = \int_\Omega f(x) G_y(x) dx - \int_{\partial\Omega} \frac{\partial G_y}{\partial n} g(x) d\sigma(x)$.

Example: $-u''(x) = f(x)$, $u(0) = u(1) = 0$. Green function $-G''_y(x) = \delta(x-y)$, $G_y(0) =$

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$G_y(1) = 0$. For $x \neq y, -G''_y(x) = 0 \Rightarrow G_y(x) = \begin{cases} A(y)x, & 0 < x < y \\ C(y)(x-1), & y < x < 1 \end{cases}$. $\int_{y-\epsilon}^{y+\epsilon} -G''_y(x)dx = \int_{y-\epsilon}^{y+\epsilon} \delta(x-y)dx = 1 \Rightarrow -G'_y(y+\epsilon) + G'_y(y-\epsilon) = 1, -C(y) + A(y) = 1$. $\int_{y-\epsilon}^{y+\epsilon} \int_0^x G''_y(x')dx'dx = \int_{y-\epsilon}^{y+\epsilon} \int_0^x \delta(x'-y)dx'dx = \epsilon(\epsilon \rightarrow 0) \Rightarrow G_y(x) \in C[0, 1] \Rightarrow A(y)y = C(y)(y-1)$. Thus $A(y) = 1-y, C(y) = -y$ and $u(y) = \int_0^1 G_y(x)f(x)dx = \int_0^1 (\min(x, y) - xy)f(x)dx$.

Consider $\begin{cases} -\Delta u = f \text{ in } \Omega \\ \partial_n u = g \text{ on } \partial\Omega \end{cases}$. Green function $\begin{cases} -\Delta G_y(x) = \delta(x-y) - \frac{1}{|\Omega|}, & x \in \Omega \\ \frac{\partial G_y}{\partial n} = 0, & x \in \partial\Omega \end{cases}$. $u(y) = \int_{\Omega} f(x)G_y(x)dx + \int_{\Omega} u(x)G_y(x)dx + \int_{\Omega} g(x)G_y(x)d\sigma(x)$. The solution is not unique. ($-\int_{\Omega} f dx = \int_{\partial\Omega} g d\sigma$).

Example: $\begin{cases} -\nabla \cdot (A\nabla u) + (b \cdot \nabla)u = f \\ u = g \end{cases}$. $\int_{\Omega} \mathcal{L}[u]vdx = \int_{\Omega} (-\nabla \cdot (A^T \nabla v) + \nabla \cdot (bv))udx + \int_{\partial\Omega} (b \cdot n)uvd\sigma(x) + \int_{\partial\Omega} (n \cdot (A^T \nabla v))u - n \cdot (A\nabla u)v d\sigma(x)$. Define $\mathcal{L}^*[u] = -\nabla(A^T \nabla u) + \nabla \cdot (bu)$.

Green function $\begin{cases} \mathcal{L}^*G_y(x) = \delta(x-y), & x \in \Omega \\ G_y(x) = 0, & x \in \partial\Omega \end{cases}$

Consider $\begin{cases} -\Delta G_y(x) = \delta(x-y), & x \in \Omega \\ G_y(x) = 0, & x \in \partial\Omega \end{cases}$. Assume $G_y(x) = N_y(x) + H(x)$ where $N_y(x)$ is

Newton potential, then $\begin{cases} -\Delta H_y(x) = 0, & x \in \Omega \\ H_y(x) = -N_y(x), & x \in \partial\Omega \end{cases}$

Example: $\begin{cases} -\Delta u = 0, & x \in R \times R_+ \\ u = f(x), & x \in \partial R \times R_+ \end{cases}$. Method of image: $N_y(x) = -\frac{1}{2\pi} \ln|x-y|$, and $G_y(x) = N_y(x) - N_{y^*}(x)$ where $y_2 + y_2^* = 0$. Then $\frac{\partial G_y}{\partial n} = -\frac{1}{\pi} \frac{y_2}{|x-y|^2}, u(y) = \frac{1}{\pi} \int_R \frac{f(x_1)y_2}{(x_1-y_1)^2+y_2^2} dx_1$.

Example: $\begin{cases} -\Delta u = 0, & x \in R_+ \times R_+ \\ u = f(x), & x \in \partial R_+ \times R_+ \end{cases}$. $G_y(x) = N_y(x) - N_{y_1}(x) - N_{y_2}(x) + N_{y_3}(x)$ where $y_1 = (1, -1)y, y_2 = (-1, 1)y, y_3 = (-1, -1)y$ (entry-wise).

Example: $\begin{cases} -\Delta u = f, & u \in B_R(0) \\ u = g, & u \in \partial B_R(0) \end{cases}$. Kelvin transform: $y^* = \frac{R^2}{|y|^2}y$. Thus $|x - y^*|^2 = |x|^2 - 2(x, y)\frac{R^2}{|y|^2} + \frac{R^4}{|y|^2}(x \in \partial B_R(0)) = \frac{R^2}{y^2}(|y|^2 - 2(x, y) + |x|^2) \Rightarrow |x - y^*| = \frac{R}{|y|}|x - y|$. Then $G_y(x) = \frac{1}{(d-2)w_{d-1}}|x-y|^{2-d} - \frac{1}{(d-2)w_{d-1}}\left(\frac{|y|}{R}|x-y^*|\right)^{2-d}, d \geq 3$ and $G_y(x) = -\frac{1}{2\pi} \ln|x-y| + \frac{1}{2\pi} \ln\frac{|y|}{R}|x-y^*|, d = 2$. $u(y) = \int_{B_R(0)} = \int_{\Omega} f(x)G_y(x)dx - \int_{\partial B_R(0)} g(x)\frac{\partial G_y}{\partial n}d\sigma(x)$. If $d \geq 3$, $\frac{\partial G_y}{\partial n} = \frac{x_i}{R}\frac{\partial G_y}{\partial x_i} = \frac{x_i}{R}(-\frac{1}{w_{d-1}}|x-y|^{1-d}\frac{x_i-y_i}{|x-y|} + \frac{1}{w_{d-1}}|x-y^*|^{1-d}\frac{x_i^*-y_i^*}{|x-y^*|}(\frac{|y|}{R})^{2-d}) = \frac{x_i}{R}\frac{1}{w_{d-1}}(-\frac{x_i-y_i}{|x-y|^d} + \frac{x_i-y_i^*}{|x-y^*|^d}(\frac{|y|}{R})^{d-2}) = \frac{-1}{w_{d-1}R|x-y|^d}(|y|^2 - R^2)$. Assume $f(x) = 0, u(y) = \frac{R^2-y^2}{w_{d-1}R} \int_{\partial B_R(0)} \frac{g(y)}{|x-y|^d} d\sigma(x)$, and $u(0) = \int_{\partial B_R(0)} u(y)d\sigma(y)$. By taking integral w.r.t. r from 0 to R , $u(0) = \int_{B_R(0)} u(y)dy$.

Symmetric: $\begin{cases} -\Delta G_a(x) = \delta(x-a), & x \in \Omega \\ G_a(x) = 0, & x \in \partial\Omega \end{cases}$, $\begin{cases} -\Delta G_b(x) = \delta(x-b), & x \in \Omega \\ G_b(x) = 0, & x \in \partial\Omega \end{cases}$. $G_b(a) - G_a(b) = \int_{\Omega} (\Delta G_a)G_b - (\Delta G_b)G_a dx = \int_{\partial\Omega} \frac{\partial G_a}{\partial n}G_b - \frac{\partial G_b}{\partial n}G_a d\sigma = 0$.

Example: $\begin{cases} -\Delta u = f, & x \in B(0, R) \\ \frac{\partial u}{\partial n} = g, & x \in \partial B(0, R) \end{cases}$ ($d = 2$). Assume $G_y(x) = N_y(x) + N_{y^*}(x) = -\frac{1}{2\pi} \ln|x-y|$.

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$y| - \frac{1}{2\pi} \ln \frac{|y|}{R} |x - y^*| \cdot \frac{\partial G_y}{\partial n} = -\frac{1}{2\pi R}$. Transfer the former question to $\begin{cases} -\Delta H(x) = -\frac{1}{\pi R^2} \\ \frac{\partial H}{\partial n} = \frac{1}{2\pi R} \end{cases}$. Assume $H(x) = H_0(r)$, then $H_0''(r) + \frac{1}{r} H_0'(r) = \frac{1}{\pi R^2}$, $H_0'(R) = \frac{1}{2\pi R} \Rightarrow H_0(r) = \frac{r^2}{4\pi R^2} + C$.

7 Fourier Method

7.1 Eigenvalue Problem and Separation of Variables

Consider $\begin{cases} -u''(x) = \lambda u \\ u(0) = u(\pi) = 0 \end{cases}, \lambda_k = k^2, u_k = \sin kx$. $\begin{cases} -u''(x) = \lambda u \\ u'(0) = u'(\pi) = 0 \end{cases}, \lambda_k = k^2, u_k = \cos kx$. $\begin{cases} -u''(x) = \lambda u \\ u(0) = 0, u'(\pi) = 0 \end{cases}, \lambda_k = (k + \frac{1}{2})^2, u_k = \sin(k + \frac{1}{2})x$. $\begin{cases} -u''(x) = \lambda u \\ u'(0) = 0, u(\pi) = 0 \end{cases}, \lambda_k = (k + \frac{1}{2})^2, u_k = \cos(k + \frac{1}{2})x$.

Example: $\begin{cases} -u''(x) = f(x) \\ u(0) = u(\pi) = 0 \end{cases} . \tilde{u}_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, u(x) = \sum a_k \tilde{u}_k(x), -u''(x) = -\sum a_k \tilde{u}_k''(x) = \sum \lambda_k a_k \tilde{u}_k(x)$. $f(x) = \sum f_k \tilde{u}_k(x) \Rightarrow \lambda_k a_k = f_k, u(x) = \sum \frac{f_k}{\lambda_k} \tilde{u}_k(x) = \sum_{k=1}^{\infty} \lambda_k^{-1} \int_0^1 f(y) \tilde{u}_k(y) dy \tilde{u}_k(x) = \int_0^1 f(y) \left(\sum_{k=1}^{\infty} \frac{\tilde{u}_k(x) \tilde{u}_k(y)}{\lambda_k} \right) dy = \int_0^1 f(y) G_x(y) dy$ where $G_x(y) = \sum_{k=1}^{\infty} \frac{2 \sin kx \sin ky}{\pi k^2} = \min(x, y) - \frac{xy}{\pi}$.

Example: $\begin{cases} \partial_t u = \Delta u \\ u(0) = u(\pi) = 0 \\ u|_{t=0} = u_0(x) \end{cases} . \text{Assume } u(x, t) = T(t)Z(x), Z_k(x) = \tilde{u}_k(x), \text{ then } \dot{T}(t) = -\lambda_k T(t), T(t) = e^{-\lambda_k t} T(0), u(x, t) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k^2 t} \tilde{u}_k(x), u(x, 0) = \sum_{k=1}^{\infty} a_k \tilde{u}_k(x) = u_0(x), \text{ thus } a_k = \int_0^{\pi} u_0(x) \tilde{u}_k(x) dx$.

Example: $\begin{cases} \square u = 0 \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = g \\ u(0, t) = u(\pi, t) = 0 \end{cases} . \text{Assume } u = \sum_k T_k(t)Z_k(x) \text{ where } Z_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \text{ we get } u = \sum_{k=1}^{\infty} (c_k \cos kt + d_k \sin kt) Z_k(x) \text{ where } c_k = \int_0^{\pi} f(x) Z_k(x) dx, d_k = \frac{1}{k} \int_0^{\pi} g(x) Z_k(x) dx$. Denote $u_n = \sum_{k=1}^n (c_k \cos kt + d_k \sin kt) Z_k(x)$, then $\square u_n = 0, 0 = \int_0^{\infty} dt \int_0^{\pi} \square u_n \phi(x, t) dx = \int_0^{\infty} dt \int_0^{\pi} (\partial_t^2 u_n - \partial_x^2 u_n) \phi(x, t) dx$. $\int_0^{\pi} dx \int_0^{\infty} \partial_t^2 u_n \phi dt = \int_0^{\pi} \int_0^{\infty} \phi d\partial_t u_n = \int_0^{\pi} dx (-\phi(x, 0) \partial_t u_n(x, 0)) - \int_0^{\infty} \partial_t u_n \partial_t \phi dx = -\int_0^{\pi} \phi(x, 0) g_n dx - \int_0^{\pi} \int_0^{\infty} \partial_t \phi du_n = -\int_0^{\pi} \phi(x, 0) g_n dx + \int_0^{\pi} \partial_t \phi(x, 0) f_n dx, \int_0^{\infty} dt \int_0^{\pi} -\partial_x^2 u_n \phi(x, t) dx = \int_0^{\pi} \int_0^{\infty} u_n \square \phi dx dt$. Let $n \rightarrow \infty$, we get $-\int_0^{\pi} \phi(x, 0) g(x) dx + \int_0^{\pi} \partial_t \phi(x, 0) f(x) dx + \int_0^{\pi} \int_0^{\infty} u \square \phi dx dt \rightarrow$ weak solu.

If we replace BC with inhomogeneous condition, i.e. $u(0, t) = A(t), u(\pi, t) = B(t)$. Denote $w(x, t) = \frac{B-A}{\pi} x + A(t)$ and $v = u - w$, we get $\square v = -\partial_t^2 w := f(x, t), v(x, 0) = f(x) - w(x, 0), \partial_t v(x, 0) = g(x) - \partial_t w(x, 0) \rightarrow \square v = f(x, t), v(x, 0) = f_1, \partial_t v(x, 0) = g_1, v(0, t) = v(\pi, t) = 0$. Then use Duhammel's principle to acquire a special solution.

Example: $-\Delta u = 0, \Omega = (0, a) \times (0, b), u(x, 0) = f(x), u(x, b) = u(0, y) = u(a, y) = 0$. Assume $u(x, y) = Z(x)Y(y), \ddot{Z}(x)Y(y) + Z(x)\ddot{Y}(y) = 0, \frac{\ddot{Z}(x)}{Z(x)} = -\frac{\ddot{Y}(y)}{Y(y)}$. So $-\ddot{Z}(x) = \lambda Z(x), Z(0) = Z(a) = 0$.

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Eignevalue $\lambda_k = (\frac{k\pi}{a})^2$, $Z_k(x) = \sqrt{\frac{2}{a}} \sin \frac{k\pi x}{a}$. $\ddot{Y}(y) = \lambda_k Y(y)$, $Y(b) = 0$, $Y_k(y) = \sinh \frac{k\pi}{a} (b - y)$. Thus $u(x, y) = \sum_{k=1}^{\infty} a_k Z_k(x) Y_k(y)$, $a_k = \frac{(f, Z_k)}{Y_k(0)} \Rightarrow u(x, y) = \sum_k f_k \frac{Y_k(y)}{Y_k(0)} Z_k(x)$. $\frac{Y_k(y)}{Y_k(0)} = \frac{\sinh \frac{k\pi}{a} (b - y)}{\sinh \frac{k\pi}{a} b} = \frac{e^{-\frac{k\pi}{a} y} - e^{-\frac{k\pi}{a} (2b-y)}}{1 - e^{-\frac{2k\pi}{a} b}}$. $\lim_{y \rightarrow 0} u(x, y) = f(x)$.

Example: $-\Delta u = f$, $\Omega = B_R(0)$, $u(R, \theta) = g(\theta)$. Assume $u(r, \theta) = R(r)\Theta(\theta)$, then $\ddot{R}(r)\Theta(0) + \frac{1}{r}\dot{R}(r)\Theta(0) + \frac{R(r)}{r^2}\ddot{\Theta}(0) = 0 \Rightarrow r^2 \frac{\ddot{R}(r)}{R(r)} + r \frac{\dot{R}(r)}{R(r)} = -\frac{\ddot{\Theta}(0)}{\Theta(0)}$. Periodic BC: $\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$. Eigenvalue $\lambda_k = k^2$, $r^2 \ddot{R}(r) + r \dot{R}(r) = -k^2 R(r)$ (Euler Equation), $R(r) = r^\alpha$, $\alpha^2 r^\alpha = k^2 r^\alpha$, $\alpha = \pm k \Rightarrow R(r) = r^k$, r^{-k} ($k \geq 1$) and $R(r) = 1$, $\ln r$ ($k = 0$). Thus $u = \alpha + \beta \ln r + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta) + \sum_{k=1}^{\infty} r^{-k} (c_k \cos k\theta + d_k \sin k\theta)$.

Example: $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$. Assume $u(x, y) = Z(x)Y(y)$, $(\dot{Z}(x)Y(y))^2 \ddot{Z}(x)Y(y) + 2\dot{Z}(x)Y(y)Z(x)\dot{Y}(y)\ddot{Y}(y) + (Z(x)\dot{Y}(y))^2 Z(x)\ddot{Y}(y) = 0$. Let $\dot{Z}(x) = \lambda_1 Z(x)$, $\dot{Y}(y) = \lambda_2 Y(y)$, then $\lambda_1^4 Z^3 Y^3 + 2\lambda_1^2 \lambda_2^2 Z^3 Y^3 + \lambda_2^4 Z^3 Y^3 = 0 \Rightarrow \lambda_1^2 + \lambda_2^2 = 0$. So $u(x, y) = e^{\lambda_1 x} e^{\lambda_2 y} = e^{\lambda_2(y \pm ix)}$ ($\lambda_2 < 0$). In anthoer way, assume $u(x, y) = Z(x) + Y(y)$, $(\dot{Z})^2 \ddot{Z} + (\dot{Y}(y))^2 \ddot{Y}(y) = 0$, $\ddot{Z}(x)(\dot{Z})^2 = \lambda$, $\ddot{Y}(y)(\dot{Y}(y))^2 = -\lambda \Rightarrow (\dot{Z})^3 = 3\lambda x$, $\dot{Z} = (3\lambda x)^{1/3}$, $Z = C + \frac{4}{3}(3\lambda x)^{4/3}$.

Example: $\partial_t u = \Delta(u^\gamma)$. Assume $u = T(t)Z(x)$, $\dot{T}(t)Z(x) = T^\gamma(t)\Delta Z^\gamma$, $\frac{\dot{T}(t)}{T^\gamma(t)} = \frac{\Delta Z^\gamma}{Z(x)} = \lambda$. $\frac{1}{1-\gamma} T^{1-\gamma}(t) = C_1 + \lambda t$, $T(t) = ((1-\gamma)C_1 + (1-\gamma)\lambda t)^{\frac{1}{1-\gamma}}$. Also, $\Delta Z^\gamma = \lambda Z$, $Z(x) = |x|^\alpha$, $Z^\gamma = |x|^{\alpha\gamma}$, $\alpha\gamma(\alpha\gamma-1)|x|^{\alpha\gamma-2} + (d-1)\alpha\gamma|x|^{\alpha\gamma-2} = \lambda|x|^\alpha \Rightarrow \alpha\gamma-2 = \alpha$, $\alpha = \frac{2}{\gamma-1}$, $\lambda = \alpha\gamma(\alpha\gamma+d-2)$.

Consider $\begin{cases} \mathcal{L}u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$, $\begin{cases} \mathcal{L}u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$. Green function $\begin{cases} \mathcal{L}G_y(x) = \delta(x-y) \text{ in } \Omega \\ G_y(x) = 0 \text{ on } \partial\Omega \end{cases}$, $G_y(x) = \sum_{k=1}^{\infty} a_k(y)u_k(x)$, $\mathcal{L}G_y(x) = \sum_{k=1}^{\infty} a_k(y)\mathcal{L}u_k(x) = \sum_{k=1}^{\infty} a_k(y)\lambda_k u_k(x) = \delta(x-y)$. So $(\delta(x-y), u_k(x)) = a_k(y)\lambda_k(u_k, u_k) \Rightarrow u_k(y) = \lambda_k a_k(y)$, $a_k(y) = \lambda_k^{-1} u_k(y)$, $G_y(x) = \sum_{k=1}^{\infty} \lambda_k^{-1} u_k(x)u_k(y)$.

7.2 Variation of Constant for PDE

Recall $\dot{u} = Au + f$, $u(0) = u_0$. The solu is $u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(s)ds$.

Consider $u_t = \Delta u + f$, $u|_{t=0} = u_0$. Regard $A = \Delta$, $e^{At}u_0 = \int_{R^d} (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy$, then $u(t) = \int_{R^d} (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy + \int_0^t \int_{R^d} (4\pi(t-s))^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) dy ds$.

Consider $\dot{u} = Au + f$, $u(0) = 0$. Denote $v(t, s)$ satisfies $v_t(t, s) = Av(t, s)$, $v(t, t) = f(t)$, thus $v(t, s) = e^{A(t-s)}f(s)$ and $u(t) = \int_0^t v(t, s)ds$.

Consider $\partial_t u = \Delta u + f$, $u|_{t=0} = 0$. Denote $v(x, t, s)$ satisfies $v_t(x, t, s) = \Delta v(x, t, s)$, $v(x, t, s)|_{s=t} = f(x, t)$. The solution is given by $u(x, t) = \int_0^t v(x, t, s)ds$.

8 Energy Method

Consider wave equation $\square u = 0$, $u = g$, $u|_{t=0} = u_0$, $\partial_t u|_{t=0} = u_1$, $\Omega = \{(x, t) \in R^3 \times R : |x-x_0| \leq |t-t_0|\}$, and energy density $e(t) = \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2$. $\dot{e}(t) = u_t u_{tt} + \nabla u \nabla u_t = u_t \Delta u + \nabla u \nabla u_t = \nabla \cdot (u_t \nabla u)$. Denote $m(t) = u_t \nabla u$, then $e_t - \nabla \cdot m(t) = 0 = \int_F (e_t - \nabla \cdot m(t)) dx dt$ where F is a frustum (i.e. a piece of solid light cone). By Gauss's formula, we obtain $\int_{\partial F} (n_t e(t) - n \cdot m(t)) dx = 0$. Denoting T, B, K as the top, the bottom and the side of F , we have $\int_T e(t) dx - \int_B e(t) dx + \int_K (n_t e(t) - u_t \frac{\partial u}{\partial n}) dx = 0$. The third term is equal to $\frac{1}{\sqrt{2}} \int_K (e(t) - u_t \frac{\partial u}{\partial n}) dx \geq \frac{1}{\sqrt{2}} \int_K (e(t) - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2) dx = 0$, so $\int_T e(t) dx \leq \int_B e(t) dx$.