UAGS Problem Set 2

Problem 1 If $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a Zariski-open cover of $\mathbb{A}^n$, show that it has a finite subcover $\{U_{\alpha_1} \dots U_{\alpha_n}\}$. (In the context of algebraic geometry, the property is often called *quasi-compactness*)

Problem 2 If $f \in k[\mathbb{A}^n]$, we let $D(f)$ denote the Zariski open set

$$D(f) = \{p \in \mathbb{A}^n \mid f(p) \neq 0\}.$$

Show that the collection of $D(f)$ for all $f \in k[\mathbb{A}^n]$ forms a base for the Zariski topology: given any point $p \in \mathbb{A}^n$ and any Zariski-open U containing p, there is some $D(f)$ such that $p \in D(f) \subseteq U$.

Problem 3 Let I be an ideal of some ring A. On Friday, we defined the radical of I as

$$\sqrt{I} = \{f \in A \mid f^n \in I \text{ for some } k\}.$$

Show that this set is actually an ideal of A. Show that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Show that

$$\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}.$$

Problem 4 Suppose $f, g \in k[\mathbb{A}^n]$ are two irreducible polynomials. Show that $V(f) = V(g)$ if and only if $f = \alpha g$ for some nonzero α in k.

Problem 5 Suppose $\phi : X \to Y$ is a continuous map of topological spaces. Suppose also that X is irreducible. Conclude that $\operatorname{Im} \phi$ is irreducible (with the subspace topology).

Problem 6 Prove that a map $\phi : \mathbb{A}^n \to \mathbb{A}^m$ of the form

$$\phi(x_1 \dots x_n) = (\phi_1(x_1 \dots x_n) \dots \phi_m(x_1 \dots x_n)),$$

where ϕ_j is a polynomial in x_i for all i, j, is continuous if $\mathbb{A}^n$ and $\mathbb{A}^m$ are given the Zariski topology. (These kinds of maps are called *regular*)

Problem 7 Let A be an $n \times n$ matrix. Prove that $\det(A)$ is an irreducible polynomial of the entries of A:

7.a Show that $V(\det(A))$ is the image of $\mathbb{A}^{2n^2} \cong \operatorname{Mat}_n(k) \times \operatorname{Mat}_n(k)$ under the regular map ϕ such that

$$\phi(S, T) = S \cdot \operatorname{diag}(0, 1 \dots 1) \cdot T.$$

Conclude that $V(\det(A))$ is irreducible.

7.b Since $k[\mathbb{A}^{n^2}]$ is a UFD, write $\det(A) = \prod_{i=1}^m p_i$, where p_i is irreducible. Use the fact that $V(\det(A))$ is irreducible to conclude that $p_i = \alpha_i p_1$ for all i and some $\alpha_i \in k^\times$, so $\det(A) = \alpha p^m$ for some irreducible p.

7.c Use the fact that $\det(A)$ has degree one in any particular entry of A to conclude that $m = 1$.