UAGS Problem Set 1

September 7, 2019

Problem 1 If X is a subset of $k[\mathbb{A}^n]$, define $V(X)$ to be the subset of $\mathbb{A}^n$ such that $p \in V(X)$ if and only if $f(p) = 0$ for all $f \in X$. Prove that if I is the ideal generated by X, then $V(I) = V(X)$.

Problem 2 Let $n = 2$, and write $k[\mathbb{A}^2] = k[x, y]$. Compute and draw:

- $V(x - a, y - b)$
- $V(x^2 - y)$
- $V(x^2 - x)$
- $V(x^2)$

Problem 3 Let A be a commutative ring with a multiplicative unit. We say that A is *Noetherian* if every increasing chain of ideals in A stabilizes. In other words, for any set of ideals $\{I_k\}_{k=1}^{\infty}$ such that

$$I_1 \subseteq I_2 \subseteq I_3 \dots$$

we know that there is some N such that $I_k = I_N$ for all $k \geq N$. Prove that $\mathbb{Z}$ is Noetherian. [Hint: it is a Principal Ideal Domain]

Problem 4 Prove Hilbert's Basis Theorem: if A is Noetherian, then $A[x]$ is Noetherian.