UAGS Lecture Notes 1

1 Introduction

What is algebraic geometry? At a basic level, it studies shapes that are defined as the solutions to some system of polynomial equations. Many shapes we care about – for example circles, hyperbolas, parabolas, cones, and spheres – arise in this way. Even though a general geometric object, like the graph of the Weierstrass everywhere continuous, nowhere differentiable function, might be hard to understand, shapes defined through polynomials will presumably be nicer. Because of this, one of the themes of algebraic geometry will be that **algebra tells us about geometry**. The converse of this statement is maybe less clear, but still true: **geometry tells us about algebra**. To get an idea for how this might work, consider the following theorem:

Theorem 1.1 (Cayley-Hamilton)**.** *Let V be a finite dimensional vector space and let $T : V \to V$ be a linear transformation. If p_T is the characteristic polynomial of T, then $p_T(T) = 0$.*

Proof. We will prove this statement only over the complex numbers. First, suppose that T is diagonalizable. Then there is some basis where the matrix representing T is $\operatorname{diag}(\lambda_1 \dots \lambda_n)$ (in other words T has diagonal entries $T_{ii} = \lambda_i$ and 0 elsewhere). Then

$$p_T(x) = \prod_{i=1}^{n}(x - \lambda_i),$$

so

$$p_T(T) = \operatorname{diag}(0, \lambda_2 - \lambda_1 \dots \lambda_n - \lambda_1) \cdot \operatorname{diag}(\lambda_1 - \lambda_2, 0 \dots \lambda_n - \lambda_2) \cdot \dots \cdot \operatorname{diag}(\lambda_1 - \lambda_n \dots 0) = 0.$$

Now choose a basis for V, so that we can identify $\operatorname{End}(V) \cong \operatorname{Mat}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. Note that the coefficients of the characteristic polynomial of a matrix are polynomials in the entries of the matrix. Moreover, note that the expression

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

gives the entries of the product of two matrices as a polynomial in terms of the entries of the matrices being multiplied. In a similar, the entries of the sum of two matrices are given by polynomials in terms of the entries of the matrices being summed. As a result, the expression $p_T(T) = 0$ can be written out explicitly as a system of n^2 polynomial equations in terms of the n^2 entries of T. As a result, if we can prove that diagonalizable transformations are dense inside of $\operatorname{End}(V) \cong \operatorname{Mat}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, then we'll be done: because polynomials are continuous, if they are 0 on a dense subset of our space, they'll be dense everywhere.

To see that diagonalizable transformations are dense in $\operatorname{End}(V)$, suppose that T is any linear transformation $V \to V$. Then choose a basis such that the matrix M representing T in this basis is in Jordan canonical form. By perturbing the diagonal entries of M by as small of an amount as we want, we can obtain a matrix M' where the diagonal entries are all distinct. But since M' is upper-triangular,

$$p_{M'}(x) = \prod_{i=1}^{n}(x - M'_{ii}).$$

Since the roots of $p_{M'}$ are all distinct, this shows that M' is diagonalizable, so we're done. □

We see that by using a "geometric argument" (about the density of some subset) we can prove a theorem with a purely algebraic statement. We were, however, only able to prove the Cayley-Hamilton theorem over $\mathbb{C}$, even though it's true over an arbitrary field. This is because we currently only understand the geometry of $\mathbb{C}$, and not of other fields. Can we find a way to think geometrically about an arbitrary field?

2 The Zariski Topology

Let k be any field. Let $\mathbb{A}^n$, called "affine n-space" be the collection of points in k^n. The ring of polynomial functions on $\mathbb{A}^n$, which we'll write $k[\mathbb{A}^n]$, is just $k[x_1 \ldots x_n]$. Let I be an ideal of $k[\mathbb{A}^n]$. Then define $V(I)$, the "vanishing locus" of I, to be

$$V(I) = \{p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } f \in I\}.$$

A set of the form $V(I)$ for some I will be called an "affine algebraic variety". The Zariski topology on $\mathbb{A}^n$ will be the topology such that the closed subsets are exactly the affine algebraic varieties.

Theorem 2.1. *The Zariski topology is really a topology.*

Proof. We must show three things.

1. $\mathbb{A}^n$ and $\emptyset$ are affine algebraic varieties.
2. $\bigcap_\alpha V(I_\alpha)$ is an affine algebraic variety.
3. $V(I_1) \cup V(I_2)$ is an affine algebraic variety.

For 1, note that $\mathbb{A}^n = V(0)$ and $\emptyset = V(1)$. For 2, consider the ideal $J = \sum_\alpha I_\alpha$. If $p \in V(J)$, then for any α and any $f \in I_\alpha$, we know that $f \in J$, so $f(p) = 0$. As a result, $p \in V(I_\alpha)$ for all α, so $p \in \bigcap_\alpha V(I_\alpha)$. Conversely, suppose that $p \in \bigcap_\alpha V(I_\alpha)$. For any $f \in J$, we know by definition that

$$f = \sum_{i=1}^{n} f_i,$$

where $f_i \in I_{\alpha_i}$ for some α_i. As a result,

$$f(p) = \sum_{i=1}^{n} f_i(p) = 0,$$

so $p \in V(J)$. We conclude that $\bigcap_\alpha V(I_\alpha)$ is an affine algebraic variety, since it equals $V(J)$. Finally, for 3, consider the ideal $J = I_1 I_2$. If $p \in V(J)$ and $p \notin V(I_1)$, then there is some $g \in I_1$ such that $g(p) \neq 0$. Since for any $h \in I_2$ we know that $f = gh \in J$, we see that

$$0 = f(p) = g(p)h(p),$$

and since $g(p) \neq 0$, we can conclude that $h(p) = 0$. Since this is true for all $h \in I_2$, we conclude that $p \in V(I_2)$. Since any $p \in V(J)$ is in $V(I_1)$ or, if not, $V(I_2)$, we see that $V(J) \subseteq V(I_1) \cup V(I_2)$. Conversely, suppose that $p \in V(I_1)$. Then for any $f \in J$, we know that

$$f = \sum_{i=1}^{n} g_i h_i,$$

with $g_i \in I_1$ and $h_i \in I_2$ for all i. As a result,

$$f(p) = \sum_{i=1}^{n} g_i(p) h_i(p) = \sum_{i=1}^{n} 0 \cdot h_i(p) = 0,$$

so $p \in V(J)$. By a similar argument we see that if $p \in V(I_2)$ then $p \in V(J)$, so we conclude that $V(I_1) \cup V(I_2) = V(J)$, which shows 3. $\square$