Notes on Differential Equations

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Abstract

I'm compiling these notes for a Directed Reading Program I'm currently doing. These are for personal use so buyer beware!

1 Banach Spaces

One of the basic ways that we'll prove the existence of solutions to differential equations is to construct a Cauchy sequence of "approximate solutions" which come closer and closer to solving the specified differential equation. For this sequence to converge, our underlying space of functions will need to be complete: the theory of Banach spaces is the natural setting to study complete function spaces.

Definition 1.1. A **Normed Vector Space** is a pair $(V, ||\cdot||)$, where V is a vector space over field K which is either $\mathbb{R}$ or $\mathbf{C}$ and $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ is a function satisfying the following three conditions:

1. $||v|| = 0$ if and only if $v = 0$.
2. $||\lambda v|| = |\lambda| \, ||v||$ for all $\lambda \in K$.
3. $||v + w|| \leq ||v|| + ||w||$.

In what follows, we will often abuse notation and use V to refer to both the NVS $(V, ||\cdot||)$ and its underlying vector space. We can naturally give a NVS $(V, ||\cdot||)$ the structure of a metric space by decreeing that the distance between any two vectors $v, w \in V$ is $||v - w||$ (it is left as an exercise to check that this distance function satisfies the appropriate axioms for a metric space).

todo: at some point I should probably mention/prove that the norm is continuous

Definition 1.2. A **Banach space** is a NVS V which is complete with respect to the metric associated to its norm.

Example 1.3. The vector space $\mathbb{R}^n$ equipped with the norm $||v|| = \sqrt{v_1^2 + \ldots + v_n^2}$ a Banach space (this should be familiar from undergraduate analysis). Surprisingly, it is the case that every *finite dimensional* Banach space is in some sense "basically the same" as this example. There are, however, many different *infinite dimensional* Banach spaces. We will meet many of them in these notes.

Definition 1.4. For a compact topological space K and a Banach space V, let $C(K, V)$ denote the vector space of continuous functions $K \to V$.

Theorem 1.5. *$C(K, V)$ becomes a Banach space when equipped with the norm $||\cdot||_\infty$, where*

$$||f||_\infty = \sup_{x \in K} ||f(x)||_V$$

Proof. For any $f \in C(K, V)$, because K is compact and f is continuous, we know that $||f(K)||_V$ is also compact and therefore a bounded subset of $\mathbb{R}$. This shows that $||f||_\infty$ is some finite number, so $||\cdot||_\infty$ is a well-defined function from $C(K, V)$ to $\mathbb{R}_{\geq 0}$. A straightforward application of the basic properties of sup shows that $(C(K, V), ||\cdot||_\infty)$ is a NVS (this is left as an exercise).

To see that $C(K, V)$ is complete with respect to $||\cdot||_\infty$, suppose that $\{f_n\}$ is a Cauchy sequence with respect to $||\cdot||_\infty$. In other words, $||f_i - f_j||_\infty$ goes to zero as i and j go to ∞. Then for every point $x \in K$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in V, since

$$\begin{aligned} ||f_i(x) - f_j(x)||_V &\leq \sup_{x \in K} ||f_i(x) - f_j(x)||_V \\ &= ||f_i - f_j||_\infty \end{aligned}$$

so $||f_i(x) - f_j(x)||_V$ also goes to zero as i, j go to ∞. Since V is complete, this implies that $f_n(x)$ converges to some value y_x as n goes to ∞. Define $f : K \to V$ to be the function such that $f(x) = y_x$ for all $x \in K$. To conclude, we want to show that f_n converges to f and that f is continuous.

To show that f_n converges to f, fix $\varepsilon > 0$ and let N be large enough that $||f_i - f_j||_\infty < \varepsilon$ for all $i, j \geq N$. Then for any $k \geq N$ and any $x \in K$ we know that

$$\begin{aligned} ||f(x) - f_k(x)||_V &= \lim_{i \to \infty} ||f_i(x) - f_k(x)||_V \\ &\leq \lim_{i \to \infty} \sup_{x \in K} ||f_i(x) - f_k(x)||_V \\ &= \lim_{i \to \infty} ||f_i - f_k||_\infty \\ &\leq \varepsilon \end{aligned}$$

so

$$||f - f_k||_\infty \leq \sup_{x \in K} ||f(x) - f_k(x)||_V \leq \varepsilon.$$

This shows that $||f - f_k||_\infty$ goes to 0 as k goes to ∞ as desired.

To see that f is continuous, fix $x_0 \in K$ and $\varepsilon > 0$. By our previous argument there is some k such that $||f - f_k||_\infty < \varepsilon/3$. Since f_k is continuous, there is some open neighborhood U of x_0 such that $||f(x) - f(x_0)||_V < \varepsilon/3$ for any $x \in U$. We therefore know that

$$\begin{aligned}||f(x) - f(x_0)||_V &\leq ||f(x) - f_k(x)||_V + ||f_k(x) - f_k(x_0)||_V + ||f_k(x_0) - f(x_0)||_V \\ &< ||f - f_k||_\infty + \varepsilon/3 + ||f_k - f||_\infty \\ &< \varepsilon.\end{aligned}$$

Since we can find such a U for any $\varepsilon > 0$, this implies that f is continuous at x_0, and since this is true for every x_0 in K, this implies that f is continuous. We have therefore shown that every Cauchy sequence f_n in $C(K, V)$ converges to an element f of $C(K, V)$, completing our proof that $C(K, V)$ is a Banach space. □

We will now prove the first of several important theorems that will allow us to prove the existence of solutions to differential equations in the context of Banach spaces.

Definition 1.6. If (X, d) is a metric space and $T : X \to X$ is a map such that there is a constant $0 \leq \alpha < 1$ so that $d(Tx, Ty) \leq \alpha d(x, y)$ for every $x, y \in X$ then we say that T is a **contraction mapping**.

Note that every contraction mapping is automatically continuous (this is an exercise).

Theorem 1.7 (Banach Fixed Point Theorem)**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping. Then T has a unique fixed point.*

Proof. First, to see that the fixed point is unique, suppose that x and y are both fixed points of T, so $Tx = x$ and $Ty = y$. Then

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y).$$

Since $\alpha < 1$, this implies that $d(x, y)$ must be 0, so $x = y$. Now to show the existence of a fixed point we must use the fact that X is complete. Pick any point $x \in X$ and consider the sequence $x_n = T^n x$. We will show that this sequence is Cauchy and it converges to the fixed point of T. Let δ denote $d(x, Tx)$. We want to bound $d(T^i x, T^j x)$ as i and j grow large. If $i = j$ then $d(T^i x, T^i x) = 0$, so suppose without loss of generality that $i > j$. Then

$$\begin{aligned}d(T^i x, T^j x) &\leq d(T^i x, T^{i-1} x) + d(T^{i-1} x, T^{i-2} x) + \ldots + d(T^{j+1} x, T^j x) \\ &\leq \alpha^{i-1}\delta + \alpha^{i-2}\delta + \ldots \alpha^j \delta \\ &= \alpha^j \delta \sum_{k=0}^{i-j-1} \alpha^k \\ &\leq \frac{\alpha^j \delta}{1 - \alpha}.\end{aligned}$$

For any $\varepsilon > 0$, since $\alpha < 1$ there is some large N such that for all $j \geq N$ we know that

$$\alpha^j < (1-\alpha)\varepsilon/\delta$$

so for all $i, j \geq N$ we see that $d(T^i x, T^j x) < \varepsilon$. Since this is true for all $\varepsilon > 0$ we see that the sequence $x_n = T^n x$ is Cauchy. Because we have assumed that (X, d) is a complete metric space, this implies that x_n converges to some value y. To see that y is a fixed point of T, note that since T is continuous,

$$\begin{aligned} Ty &= \lim_{n\to\infty} Tx_n \\ &= \lim_{n\to\infty} T^{n+1}x \\ &= \lim_{n\to\infty} x_{n+1} \\ &= y. \end{aligned}$$

We have therefore proven the existence and uniqueness of a fixed point of T in X, as desired. □

Exercises

Problem 1 Given a NVS $(V, ||\cdot||)$, prove that the function $d(v, w) = ||v - w||$ satisfies the axioms of a distance function on a metric space.

Problem 2 Prove that $(C(K), ||\cdot||_\infty)$ is a NVS for any compact topological space K.

Problem 3 Prove that every contraction mapping is continuous.

Problem 4 Prove that the mapping $Tx = x/2 + 1/x$ is a contraction mapping on $[1, \infty)$. Show that $\sqrt{2}$ is a fixed point of T. Use the Banach fixed point theorem to describe an iterative algorithm for computing $\sqrt{2}$.

2 The Picard-Lindelöf Theorem

(todo) Give example of $x^2 \partial_x f + f = 0$

Theorem 2.1 (Picard-Lindelöf)**.** *Let U be an open subset of $\mathbb{R}^n$ and let $\xi : U \to \mathbb{R}^n$ be a vector field which is locally Lipschitz on U. Then for any $t_0 \in \mathbb{R}$ and any $p \in U$ there is an $\varepsilon > 0$ and a function $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ which satisfies the initial value problem $\gamma(t_0) = p$ and $\gamma'(t) = \xi(\gamma(t))$.*

Proof. Because ξ is locally Lipschitz, there is a constant L and a closed ball $B = \overline{B_R(p)}$ of radius $R > 0$ around p such that for all $x, y \in B$ we know

$$||\xi(x) - \xi(y)|| \leq L\,||x - y||\,.$$

Also, since ξ is continuous and B is compact, we know that ξ is bounded on B: there is some M such that $||\xi(x)|| \leq M$ for all $x \in B$. Let $a > 0$ be such that $a \leq R/M$ and $a < 1/L$. We wish to show that there is a function $\gamma : [t_0 - a, t_0 + a] \to U$ which satisfies the integral equation

$$\gamma(t) = p + \int_{t_0}^{t} \xi(\gamma(t))\ dt. \tag{2.2}$$

Then by the fundamental theorem of calculus, for any $\varepsilon \leq a$ we know that the restriction of γ to $(t_0 - \varepsilon, t_0 + \varepsilon)$ will be a solution to our initial value problem, so if we can find such a γ and show it is unique we will be finished. Our strategy for finding a γ satisfying the desired integral equation will be to use the Banach fixed point theorem.

Let I be the closed interval $[t_0 - a, t_0 + a]$ and let $C(I, \mathbb{R}^n)$ be as in Definition 1.4. Let $\gamma_0 : I \to \mathbb{R}^n$ be the constant function at p, so $\gamma_0(t) = p$ for all t. Let X be the closed ball of radius R around γ_0; by the definition of the norm on $C(I, \mathbb{R}^n)$ the functions in X are exactly the functions whose image lies in B. Also, because X is a closed subspace of a Banach space, X is itself complete. Let $T : X \to X$ be the operator which maps a function $f \in X$ to the function Tf such that

$$(Tf)(t) = p + \int_{t_0}^{t} \xi(f(t))\ dt.$$

First, we claim that T does actually map X to itself. To see this, note that

$$\begin{aligned}
||Tf - \gamma_0||_\infty &= \sup_{t \in I} \left|\left| \int_{t_0}^{t} \xi(f(t))\ dt \right|\right| \\
&\leq \sup_{t \in I} \int_{t_0}^{t} ||\xi(f(t))||\ \ dt \\
&\leq \sup_{t \in I} \int_{t_0}^{t} M\ dt \\
&\leq Ma \\
&\leq R
\end{aligned}$$

where the third-to-last inequality follows because $f \in X$ so we know that $f(t) \in B$ for all $t \in I$ so $||\xi(f(t))|| \leq M$ by our definition of M. Now we wish to show that T is a contraction mapping. To see this, let f and g be in X. Then

$$\begin{aligned}
||Tf - Tg||_\infty &= \sup_{t\in I} \left|\left| \int_{t_0}^{t} \xi(f(t)) - \xi(g(t))\ dt \right|\right| \\
&\leq \sup_{t\in I} \int_{t_0}^{t} ||\xi(f(t)) - \xi(g(t))||\ dt \\
&\leq \sup_{t\in I} \int_{t_0}^{t} L\,||f(t) - g(t)||\ dt \\
&\leq \sup_{t\in I} \int_{t_0}^{t} L\,||f - g||_\infty\ dt \\
&\leq aL\,||f - g||_\infty
\end{aligned}$$

which shows that T is a contraction mapping since $aL < 1$ by our choice of a. The Banach fixed point theorem, Theorem 1.7, then tells us that $T : X \to X$ has a unique fixed point γ. Since

$$\gamma(t) = T\gamma(t) = p + \int_{t_0}^{t} \gamma(t)\ dt$$

we see that this fixed point satisfies Equation (2.2). As we already noted, this implies that for any $\varepsilon \leq a$ we know that γ satisfies the initial value problem when restricted to $(t_0 - \varepsilon, t_0 + \varepsilon)$. To see that γ is the unique such function, suppose that there were some function $\sigma : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}^n$ which also satisfied the initial value problem but which didn't equal to γ. Then there would be some $s \in (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\sigma(s) \neq \gamma(s)$. But since

$$|s - t_0| < \varepsilon \leq a$$

we see that we can apply the above argument with a replaced by $|s - t_0|$. Then the uniqueness part of the contraction mapping theorem would force $\sigma(s) = \gamma(s)$, contradicting the existence of a $\sigma \neq \gamma$. this ending is sort of messy. Maybe separate equivalence of integral and differential formulations into different lemma? □

(todo)Prove continuous dependence of solution on initial conditions.

3 L^p Spaces

todo: include motivation for PDEs.

todo: include proof that these are NVS.

Theorem 3.1. *$L^p(\Omega)$ is a Banach space for any measure space Ω and any $1 \leq p < \infty$.*

Proof. Let f_n be a Cauchy sequence in $L^p(\Omega)$. Using the Cauchy property (todo: expand?), choose a subsequence f_{n_k} such that

$$\left|\left|f_{n_{k+1}} - f_{n_k}\right|\right|_p \leq 2^{-k}.$$

Now let g_i be the nonnegative function on Ω such that

$$g_i = |f_{n_1}| + \sum_{k=1}^{i-1} |f_{n_{k+1}} - f_{n_k}|$$

and let $g = \lim_{i\to\infty} g_i$. Note that

$$\begin{aligned} ||g_i||_p &\leq ||f_{n_1}||_p + \sum_{k=1}^{i-1} \left|\left|f_{n_{k+1}} - f_{n_k}\right|\right|_p \\ &= ||f_{n_1}||_p + \sum_{k=1}^{i-1} 2^{-k}. \end{aligned}$$

Moreover, g_i converges to g monotonically from below, so since $x \mapsto x^p$ is a monotone continuous function on $\mathbb{R}_{\geq 0}$ it's also the case that g_i^p converges to g^p monotonically from below. As a result, the Monotone Convergence Theorem shows that

$$\begin{aligned} \int_\Omega g^p &= \lim_{i\to\infty} \int_\Omega g_i^p \\ &= \lim_{i\to\infty} ||g_i||_p^p \\ &\leq \lim_{i\to\infty} \left(||f_{n_1}||_p + \sum_{k=1}^{i-1} 2^{-k} \right)^p \\ &= \left(1 + ||f_{n_1}||_p\right)^p \\ &< \infty. \end{aligned}$$

Since g^p is in $L^1(\Omega)$, it must be finite almost everywhere, so it must also be the case that g is finite almost everywhere. Because of this, for almost every $x \in \Omega$ we know that

$$\sum_{i=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$$

converges absolutely, since

$$\sum_{i=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| = g(x) - |f_{n_1}(x)| < \infty.$$

As a result, we can define a function f on Ω almost everywhere as

$$f = f_{n_1} + \sum_{i=0}^{\infty} \left(f_{n_{k+1}} - f_{n_k} \right).$$

Since $|f| \leq g$, we know $|f|^p \leq g^p$, so since $g \in L^1(\Omega)$ we know that $|f|^p \in L^1(\Omega)$, which implies that $f \in L^p(\Omega)$. We now want to show that our original sequence converges to f. Note that for any k,

$$\begin{aligned} ||f - f_{n_k}||_p &= \left|\left| \sum_{i=k}^{\infty} f_{n_{i+1}} - f_{n_i} \right|\right|_p \\ &\leq \sum_{i=k}^{\infty} 2^{-i} \\ &= 2^{1-k}. \end{aligned}$$

so $||f - f_{n_k}||_p \to 0$ as $k \to \infty$. Since our original sequence f_n was Cauchy and since a subsequence f_{n_k} converges to f, we can conclude that f_n also converges to f (todo: expand?). □

4 Hilbert Spaces

todo: define inner product space, prove Cauchy-Schwartz

todo: define Hilbert space, projection operators

todo: Riesz representation theorem. Lax-Milgram later?

5 Sobolev Spaces