UAGS Problem Set 1

Problem 1 If X is a subset of $k[\mathbb{A}^n]$, define $V(X)$ to be the subset of $\mathbb{A}^n$ such that $p \in V(X)$ if and only if $f(p) = 0$ for all $f \in X$. Prove that if I is the ideal generated by X, then $V(I) = V(X)$.

Problem 2 Let $n = 2$, and write $k[\mathbb{A}^2] = k[x, y]$. Compute and draw:

- $V(x - a, y - b)$
- $V(x^2 - y)$
- $V(x^2 - x)$
- $V(x^2)$

Problem 3 Let A be a commutative ring with a multiplicative unit. We say that A is *Noetherian* if every increasing chain of ideals in A stabilizes. In other words, for any set of ideals $\{I_k\}_{k=1}^{\infty}$ such that

$$I_1 \subseteq I_2 \subseteq I_3 \ldots$$

we know that there is some N such that $I_k = I_N$ for all $k \geq N$. Prove that fields are Noetherian. Prove that $\mathbb{Z}$ is Noetherian. [Hint: $\mathbb{Z}$ is a Principal Ideal Domain]

Problem 4 Prove that the following are equivalent:

- A is Noetherian.
- Every collection $\mathcal{C}$ of ideals of A has a maximal element.
- Every ideal of A is finitely generated.

Problem 5 By filling in the following outline, prove Hilbert's Basis Theorem: if A is Noetherian, then $A[x]$ is Noetherian. Conclude that every variety is cut out by finitely many polynomial equations.

5.a For any ideal I in $A[x]$, show that the set of leading coefficients of polynomials in I forms an ideal I' in A.

5.b Since A is Noetherian, I' is finitely generated, say by $\{a_1 \ldots a_n\}$. Let $\{p_1 \ldots p_n\}$ be polynomials in I such that the leading coefficient of p_i is a_i. Let N be the largest degree of any p_i. Show that for any $f \in I$, we can write

$$f = \tilde{f} + \sum_{i=1}^{n} g_i p_i,$$

where $g_i \in A[x]$ and $\tilde{f}$ has degree less than N.

5.c Show that for a fixed k, the set of leading coefficients of polynomials in I of degree k forms an ideal I_k in A.

5.d Let $\{b_{k,1} \ldots b_{k,m_k}\}$ generate I_k, and let $q_{k,i}$ be a polynomial in I with leading coefficient $b_{k,i}$. Show that the finite collection of $p_1 \ldots p_n$ and $q_{k,1} \ldots q_{k,m_k}$ for all $k < N$ generates I.

Problem 6 (ZG) Consider the function $R : \mathbb{Z}^{>1} \to \mathbb{N}$ defined by taking $R(m)$ to be the number of zeros of $x^2 - 1$ in $\mathbb{Z}/m\mathbb{Z}$. Can you give a general formula for computing $R(m)$? [Hint: Chinese Remainder Theorem] More generally, given $f \in \mathbb{Z}[x]$, define $R_f : \mathbb{Z}^{>1} \to \mathbb{N}$ by $R_f(m) := \#\{a \in \mathbb{Z}/m\mathbb{Z} : f(a) = 0\}$. Can you give a general formula for computing $R_f(m)$? What does geometry have to do with all of this?