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Approximate expressions for the ageing coefficient and the relaxation function in the viscoelastic analysis of concrete structures

G. Lacidogna[1], *M. Tarantino*[2]

(1) Department of Structural Engineering, Politecnico di Torino, viale Mattioli, 10125 Turin. Italy

(2) Institute of Structural Engineering, Ancona University, via Brecce Bianche, Monte D'Ago, 60131 Ancona. Italy

ABSTRACT

Creep analysis of concrete structures meets well-known computational difficulties when one needs to determine the relaxation function corresponding to a specified creep function through the inversion of Volterra's linear integral equation. For this reason, the recent proposals for the 1990 edition of the CEB Model Code introduce an approximate formulation of the relaxation function obtained from the expression of the ageing coefficient $\chi(t, t_0)$ of the algebraic age-adjusted-effective modulus (AAEM) method assumed as a function of age at loading, t_0, only.

In the first part of this paper, an approximate algebraic expression, $\tilde{\chi}$ for the ageing coefficient with reference to the MC 90 is presented. This formulation differs from previous proposals in that, in addition to the influence of t_0, it also takes into account the influence of other parameters, such as relative humidity, characteristic strength of concrete and effective thickness.

In the second part, this approximate algebraic expression of the ageing coefficient is used to obtain an approximate formulation for the relaxation function which is in very good agreement with the exact values of the relaxation function as obtained from the creep function through the solution of the basic integral equation.

Finally, an example of structural calculation is provided (by applying the AAEM method and comparing the values obtained with those yielded by the general numerical method): the example clearly shows the advantages offered by the new approximate formulation of the ageing coefficient, $\tilde{\chi}$, compared to the expression proposed in the CEB MC 90.

RÉSUMÉ

L'analyse du fluage des structures en béton a permis de relever des difficultés de calcul quand il s'agit de déterminer la fonction de relaxation relative à une fonction de fluage spécifique par l'inversion de l'équation intégrale de Volterra. Pour cette raison, les récentes propositions pour l'édition du manuel CEB Model Code 1990 introduisent une formule approchée de la fonction de relaxation obtenue à partir de l'expression du coefficient de vieillissement, $\chi(t, t_0)$*, de la méthode algébrique* « age-adjusted-effective modulus » *(AAEM) supposée comme étant uniquement fonction de l'âge au moment initial de charge,* t_0.

Dans la première partie de ce travail, une expression algébrique très proche de celle proposée dans le manuel CEB MC 90 est présentée pour le coefficient de vieillissement, $\tilde{\chi}$. *Cette formule diffère des propositions précédentes dans le sens où, en plus de l'influence de* t_0*, elle tient compte de l'influence d'autres paramètres, tels que l'humidité relative, la résistance caractéristique du béton ainsi que son épaisseur effective.*

Dans la seconde partie, cette expression algébrique approchée du coefficient de vieillissement est utilisée pour obtenir une formulation approchée de la fonction de relaxation, laquelle est en bon accord avec les valeurs exactes de la fonction de relaxation obtenues à partir de la fonction de fluage par l'intermédiaire de la solution de l'équation intégrale de base.

Pour terminer, un exemple de calcul des structures est fourni (par l'application de la méthode AAEM et la comparaison des valeurs obtenues à partir de la méthode numérique générale) : l'exemple montre clairement les avantages offerts par la nouvelle fonction approchée du coefficient de vieillissement, $\tilde{\chi}$*, par rapport à l'expression proposée dans le manuel CEB MC 90.*

Editorial note

Dr. Giuseppe Lacidogna is working at the Politecnico di Torino, which is a RILEM Titular Member.

1. INTRODUCTION

In the aim of providing simplified expressions for use in the structural design of members with a viscoelastic behaviour, some recent studies have worked out approximate expressions for the determination of the ageing coefficient adopted by the AAEM Method.

As is known, the AAEM (age-adjusted effective modulus) method, initially proposed by Trost [1] and subsequently perfected by Bažant [2, 3], is widely used in the solution of structural problems in that it permits a simplified mathematical formulation that yields reliable results [4].

The determination of the ageing coefficient on the basis of its definition:

$$\chi(t,t_0) = \frac{1}{1-\dfrac{R(t,t_0)}{E_c(t_0)}} - \frac{1}{E_c(t_0)J(t,t_0)-1} \qquad (1)$$

where $R(t,t_0)$, $J(t,t_0)$, and $E_c(t_0)$ stand for the relaxation function, the creep function and the elastic modulus of concrete, respectively, calls for the prior determination of the relaxation function which, as a rule, is not given in model codes and is difficult to obtain from tests. The function therefore has to be evaluated through the numerical solution of Volterra's integral equation – a complex and time-consuming process – from the creep function, which may be obtained more easily from tests and is frequently specified in codes.

As an alternative to equation (1) and with a view to simplifying the calculation of $\chi(t, t_0)$, this study proposes an approximate expression, $\tilde{\chi}$, for the assessment of the ageing coefficient based on the creep function set forth in the 1990 edition of the CEB-FIP Model Code (MC 90) [5].

In this context, Trost's proposal [1], suggesting a systematic adoption of constant value, 0.8, for the ageing coefficient, seems too simple, despite the fact that the latter value has been widely used in actual design practice.

An approximate expression of the ageing coefficient, which takes into account the age at loading t_0 only and neglects all other parameters, was first proposed by Trevino starting from the consideration, already presented in a paper of Bažant [2], that, as a rule, the values of χ are scarcely affected by the duration $(t - t_0)$ of the loading history, regardless of the creep model employed, whereas they are considerably affected by t_0. Later on, this expression was improved by Chiorino [7] and Chiorino-Lacidogna [8] with reference to the creep function proposed in MC 90 [5], by applying the statistical least squares method over a wide range of numerical values obtained by determining the ageing coefficient from its definition [9,12].

The expression proposed by Chiorino-Lacidogna is as follows:

$$\tilde{\chi} = \tilde{\chi}(3 \cdot 10^4, t_0) \approx \frac{t_0^{0.5}}{1+t_0^{0.5}} \qquad (2)$$

As can be seen, this relationship suggests adopting for $\tilde{\chi}$ the approximate values $\tilde{\chi}(3\cdot10^4,t_0)$ of the ageing coefficient at $3\cdot10^4$ days, i.e., once the creep phenomenon has been virtually exhausted, expressed as a function of parameter t_0 (age at loading) only. This function was obtained by interpolating the limit curves of the full range of numerical values [9]; therefore, equation(2) provides the mean values of $\tilde{\chi}(3\cdot10^4,t_0)$ as a function of t_0 by describing the development of the bundle of curves, as illustrated in Fig. 1.

Yet, a certain difference between actual values and the mean values supplied by equation (2) (which decreases with increasing t_0, since when t_0 increases, the bundle of curves is seen to converge asymptotically) is apparent (see curve 2 in Fig. 1). This difference should be ascribed to the influence of the additional parameters, such as effective thickness (h_0), relative humidity ($R.H.$) and concrete strength (f_c).

As already pointed out by Bažant [2], $\chi = 1$ values are

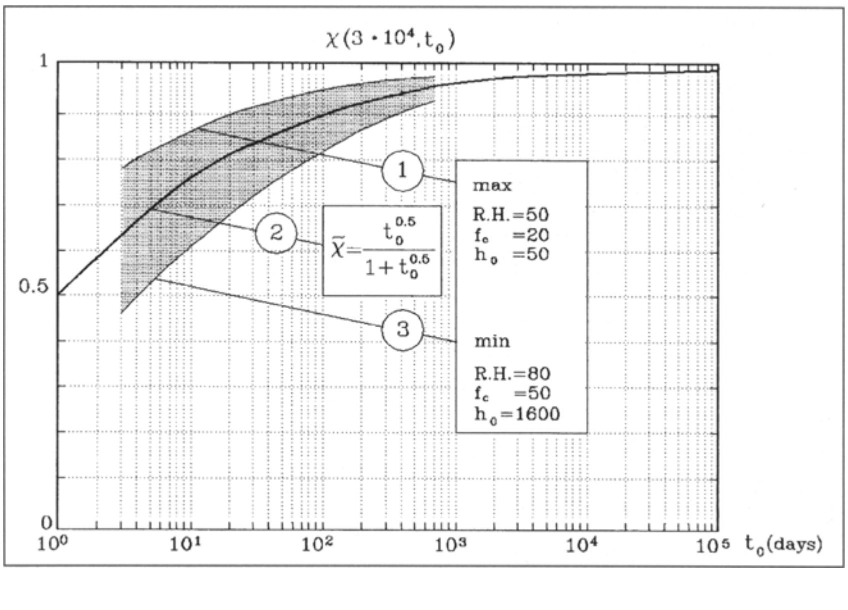

Fig. 1 - Long-term evolution of the ageing coefficient $\chi(3\cdot10^4,t_0)$, according to the CEB 1990 creep model, as a function of age at loading t_0. The boundaries of the band correspond to combinations of the parameters $R.H.$, f_c, h_0 designed to determine high creep deformability [curve 1, $\varphi_{CEB}(3\cdot10^4,28) = 3.644$] or low creep deformability [curve 3, $\varphi_{CEB}(3\cdot10^4,28) = 1.244$], respectively. Curve 2, corresponding to equation (2), represents with good approximation the average evolution of the bundle.

provided by the expressions of the simplified method of inversion of the integral relationship between J and R – the effective modulus (EM) method – which, as is known, yields accurate long-term results – i.e. for high $(t - t_0)$ values – in non-ageing materials. Thus, the fact that the value of $\tilde{\chi}(3{\cdot}10^4,t_0)$ approaches 1 with increasing age, t_0, shows that the adjustment introduced in the EM method by χ is essentially associated with the ageing of the material.

The objective therefore was to improve equation (2) by including the influence of parameters h_0, $R.H.$ and f_c.

2. APPROXIMATE EXPRESSION OF THE AGEING COEFFICIENT

The approximate expression of $\tilde{\chi}(3{\cdot}10^4,t_0)$ proposed below, always with reference to the CEB 1990 creep function, consists of four terms which independently describe the influence of parameters t_0, h_0, $R.H.$ and f_c. The first of these terms coincides with equation (2), which is taken to represent 'average' behaviour, while the other three terms are viewed as corrective coefficients.

It is important to emphasize that the mutual influence of parameters h_0, $R.H.$ and f_c is virtually negligible for any value of t_0. This is confirmed by Tables 1a and 1b where the values of $\tilde{\chi}(3{\cdot}10^4,28)$ are given as a function of h_0 = 50, 100, 200, 400, 800 and 1600 for f_c = 50 MPa and $R.H.$ = 50% – 80% (Table 1a) and for f_c = 20 – 50 MPa and R.H. = 80% (Table 1b).

The third row of each table gives the ratios between the first two: the virtually constant value of this ratio as a function of effective thickness shows the independence of the effects of the individual parameters on χ.

As a result, the influence of the three parameters (h_0, $R.H.$ and f_c) on χ can be introduced separately and hence we may provide an approximate expression as the product of four functions.

Based on the foregoing considerations, the following approximate expression of the ageing coefficient is proposed:

$$\tilde{\chi} = \tilde{\chi}\left(3 \cdot 10^4, t_0\right) \approx \alpha_{t_0} \cdot \alpha_{h_0} \cdot \alpha_{f_c} \cdot R.H. \tag{3}$$

where

$$\alpha_{t_0} = \frac{t_0^{0.5}}{1 + t_0^{0.5}}$$

$$\alpha_{h_0} = 1 + \frac{\gamma_{h_0}}{t_0^{0.5}}, \quad \gamma_{h_0} = \frac{4.5}{\ln\left(h_0\right)}$$

$$\alpha_{f_c} = 1 + \frac{\gamma f_c}{t_0^{0.5}}, \quad \gamma f_c = 0.21 - 0.007 f_c$$

$$\alpha_{R.H.} = 1 + \frac{\gamma_{R.H.}}{t_0^{0.5}}, \quad \gamma_{R.H.} = 0.55\left(1 - R.H./80\right)$$

for 50 mm ≤ h_0 ≤ 1600 mm, 20 MPa ≤ f_c ≤ 50 MPa and 50% ≤ $R.H.$ ≤ 80%.

The expression of the α functions are suggested by Fig. 2, where it can be seen that the influence of effective thickness, characteristic strength and humidity decreases with increasing t_0.

Equation (3) shows that in order to provide a more accurate expression for the ageing coefficients some enhancements in its formulation with respect to equation (2) are needed. Nevertheless, in 'normal' ambient conditions, i.e. by assuming $R.H.$ = 80%, and for concrete strength f_c = 30 MPa, functions α_{f_c} and $\alpha_{R.H.}$ take on the value of 1 and equation (3) becomes:

$$\tilde{\chi} = \tilde{\chi}\left(3 \cdot 10^4, t_0\right) = \frac{\gamma_{h_0} + t_0^{0.5}}{1 + t_0^{0.5}} \tag{4}$$

Thus, equation (4) takes on a simple form similar to equation(2) which proves very practical for applications.

Table 1a – Ratios between the values of $\chi(3{\cdot}10^4,28)$ as determined for f_c = 30 MPa and R.H. = 50 – 80%

h_0 (mm)	50	100	200	400	800	1600
$R.H.$ = 50%	0.904	0.886	.861	.826	.789	.771
$R.H.$ = 80%	.871	.850	.820	.780	.754	.751
ratio	1.038	1.042	1.051	1.059	1.036	1.027

Table 1b – Ratios between the values of $\chi(3{\cdot}10^4,28)$ as determined for f_c = 20 – 50 MPa and R.H. = 80%

h_0 (mm)	50	100	200	400	800	1600
f_c = 20 MPa	0.882	0.862	.833	.795	.770	.766
f_c = 50 MPa	.896	.833	.801	.759	.732	.728
ratio	1.031	1.035	1.040	1.047	1.052	1.052

Table 2 – Values of $\chi(3 \cdot 10^4, t_0)$, $\tilde{\chi}(3 \cdot 10^4, t_0)$, percentage error for f_c = 30 MPa and R.H. = 50%

				R.H. = 50%, f_c = 30 MPa				
		t_0 = 3	t_0 = 7	t_0 = 14	t_0 = 28	t_0 = 90	t_0 = 365	t_0 = 730
χ	h_0 = 50	.763	.830	.872	.904	.944	.973	.982
$\tilde{\chi}$		.771	.827	.866	.899	.939	.968	.977
error %		- 1.05%	0.36%	0.69%	0.55%	0.53%	0.51%	0.50%
	h_0 = 100	.728	.803	.849	.886	.932	.967	.977
		.700	.776	.827	.870	.922	.960	.970
		3.84%	3.36%	2.59%	1.80%	1.07%	0.72%	0.72%
	h_0 = 200	.683	.765	.818	.861	.915	.957	.971
		.648	.738	.799	.849	.909	.953	.966
		5.12%	3.53%	2.32%	1.39%	0.66%	0.42%	0.51%
	h_0 = 400	.627	.717	.777	.826	.890	.942	.960
		.607	.709	.777	.833	.900	.948	.963
		3.19%	1.12%	0%	0.85%	- 1.12%	- 0.64%	- 0.31%
	h_0 = 800	.562	.660	.725	.781	.856	.921	.944
		.576	.686	.760	.820	.892	.944	.960
		- 2.49%	- 3.94%	- 4.83%	- 4.99%	- 4.20%	- 2.50%	- 1.69%
	h_0 = 1600	.544	.645	.713	.771	.849	.917	.941
		.550	.667	.746	.809	.886	.941	.958
		- 1.10%	- 3.41%	- 4.63%	- 4.93%	- 4.36%	- 2.62%	- 1.80%

Table 3 – Values of $\chi(3 \cdot 10^4, t_0)$, $\tilde{\chi}(3 \cdot 10^4, t_0)$, percentage error for f_c = 50 MPa and R.H. = 50%

				R.H. = 50%, f_c = 50 MPa				
		t_0 = 3	t_0 = 7	t_0 = 14	t_0 = 28	t_0 = 90	t_0 = 365	t_0 = 730
χ	h_0 = 50	.732	.808	.854	.891	.936	.969	.979
$\tilde{\chi}$		.709	.783	.834	.875	.925	.960	.972
error %		3.14%	3.09%	2.34%	1.79%	1.17%	0.93%	0.71%
	h_0 = 100	.694	.777	.830	.872	.923	.962	.975
		.644	.734	.796	.847	.908	.952	.966
		7.20%	5.53%	4.09%	2.87%	1.62%	1.04%	0.92%
	h_0 = 200	.646	.738	.797	.844	.905	.952	.967
		.595	.699	.770	.826	.896	.946	.961
		7.89%	5.28%	3.39%	2.13%	0.99%	0.63%	0.62%
	h_0 = 400	.587	.687	.753	.807	.878	.936	.956
		.559	.671	.748	.811	.886	.941	.958
		4.77%	2.33%	0.66%	- 0.49%	- 0.91%	- 0.53%	- 0.21%
	h_0 = 800	.520	.627	.699	.759	.842	.913	.938
		.530	.659	.731	.798	.880	.937	.955
		- 1.92%	- 5.10%	- 4.58%	- 5.14%	- 4.51%	- 2.63%	- 1.81%
	h_0 = 1600	.501	.612	.686	.749	.835	.909	.935
		.505	.632	.718	.788	.873	.934	.953
		- 0.79%	- 3.27%	- 4.67%	- 5.20%	- 4.55%	- 2.75%	- 1.92%

Table 4 - Values of $\chi(3\cdot10^4,t_0)$, $\tilde{\chi}(3\cdot10^4,t_0)$, percentage error for f_c = 30 MPa and R.H. = 80%

		R.H. = 80%, f_c = 30 MPa						
		t_0 = 3	t_0 = 7	t_0 = 14	t_0 = 28	t_0 = 90	t_0 = 365	t_0 = 730
χ	h_0 = 50	.689	.775	.829	.871	.924	.963	.975
$\tilde{\chi}$		.689	.767	.821	.865	.919	.957	.969
error %		0%	1.03%	0.96%	0.69%	0.54%	0.62%	0.62%
	h_0 = 100	.652	.744	.803	.850	.909	.955	.970
		.626	.720	.784	.837	.903	.949	.963
		3.98%	3.22%	2.36%	1.53%	0.66%	0.63%	0.72%
	h_0 = 200	.604	.703	.767	.820	.888	.942	.960
		.579	.685	.757	.817	.890	.942	.959
		4.13%	2.56%	1.30%	0.36%	- 0.22%	0%	0.10%
	h_0 = 400	.547	.652	.722	.780	.858	.924	.947
		.543	.658	.728	.801	.881	.938	.955
		0.73%	- 0.92%	- 0.83%	- 2.69%	- 2.68%	- 1.51%	- 0.84%
	h_0 = 800	.511	.620	.693	.754	.838	.911	.937
		.514	.636	.720	.789	.873	.934	.952
		- 0.58%	- 2.58%	- 3.89%	- 4.64%	- 4.17%	- 2.52%	- 1.60%
	h_0 = 1600	.504	.614	.688	.751	.836	.910	.936
		.491	.619	.707	.779	.867	.931	.950
		2.57%	- 0.80%	- 2.76%	- 3.72%	- 3.70%	- 2.30%	- 1.49%

3. NUMERICAL CHECKS

Numerical checks of equation (3) are summarized in Tables 2 through 4, which give the exact value of $\chi(3\cdot10^4,t_0)$, the approximate value of $\tilde{\chi}(3\cdot10^4,t_0)$ as given by equation (3), and the percentage error for each value of the parameters being considered (t_0, h_0, $R.H.$ and f_c):

$$e\% = \frac{\chi-\tilde{\chi}}{\chi}\cdot 100$$

Maximum percentage error turns out to be 8% for $R.H.$ = 50% , f_c = 50 MPa, h_0 = 200 mm and t_0 = 3 days. In the conditions most widely used in design practice, i.e. $t_0 \geq 7$ days, $100 \leq h_0 \leq 400$, the percentage error does not exceed 3 – 4% as a rule (being as high as 5% in one instance only), and it becomes virtually negligible for $t_0 \geq 14$ days.

4. APPROXIMATE EXPRESSION OF THE RELAXATION FUNCTION

Provided that an approximate expression for the determination of the ageing coefficient is available from the definition of recent developments (equation (1)), it is possible to work out an approximate expression for the relaxation function as well:

$$\tilde{R}(t,t_0) = E_c(t_0)\left\{1-\frac{E_c(t_0)J(t,t_0)-1}{1+\tilde{\chi}\left[E_c(t_0)J(t,t_0)-1\right]}\right\} \qquad (5)$$

by applying the approximate expression of $\tilde{\chi}$ [10].

In Figs. 2 through 5, the evolution of the approximate relaxation function, $\tilde{R}(t-t_0)$, as obtained by introducing into equation (5) the expression (3) for $\tilde{\chi}$ (always with reference to the MC 90 creep function), is compared to its theoretically correct evolution, $R(t, t_0)$, as determined through the numerical inversion of the integral relationship with the creep function [9]. The diagrams refer to the non-dimensional relaxation function, $R(t, t_0)/E_c$, with E_c being the value of the $E_c(t_0)$ modulus for t_0 = 28 days [5, 9, 10].

The comparison is made for different values of relative humidity, $R.H.$, and effective thickness, h_0, for a concrete strength f_c = 40. The diagrams illustrate the relaxation functions up to $t = 10^5$ days.

Even though in equation (5) the value of $\tilde{\chi}$ is the ultimate one ($t = 3\cdot10^4$ days), the approximate expression of $\tilde{R}$ obtained in this manner provides a consistent description of the evolution over time of the theoretically correct relaxation function, R, itself obtained

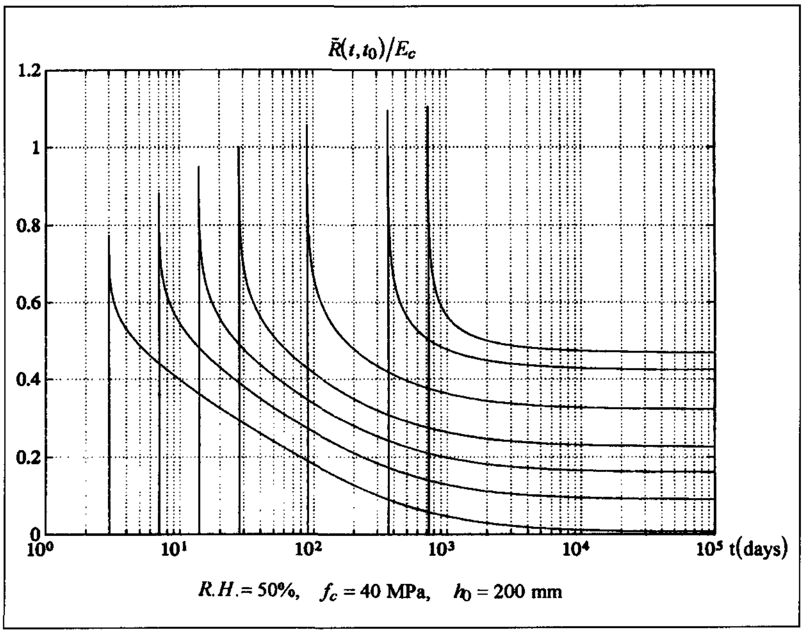

Fig. 2 - Approximate values of the non-dimensional function, $\tilde{R}(t,t_0)/E_c$. Parameters: *R.H.* = 50%, f_c = 40 MPa, h_0 = 200 mm.

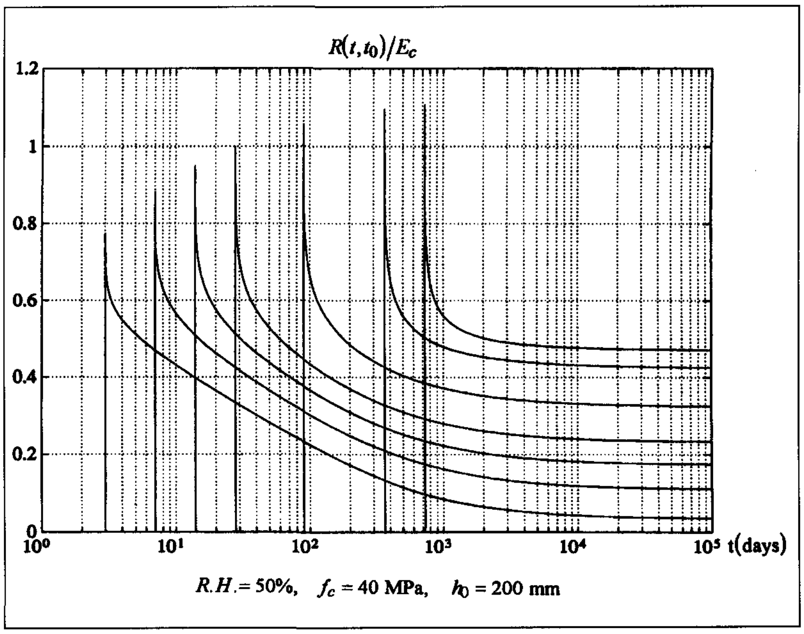

Fig. 3 - Numerically accurate values of the non-dimensional relaxation function, $R(t,t_0)/E_c$. Parameters: *R.H.* = 50%, f_c = 40 MPa, h_0 = 200 mm.

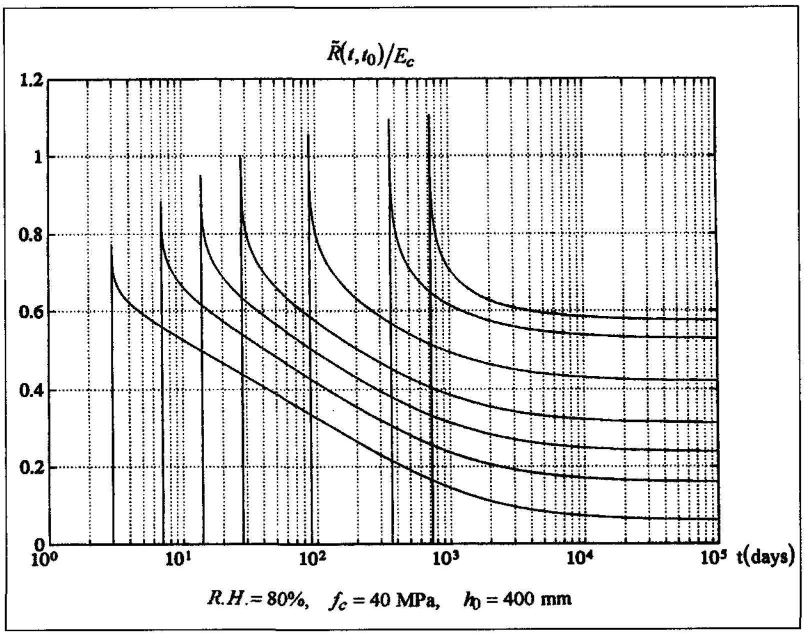

Fig. 4 - Approximate values of the non-dimensional function, $\tilde{R}(t,t_0)/E_c$. Parameters: *R.H.* = 80%, f_c = 40 MPa, h_0 = 400 mm.

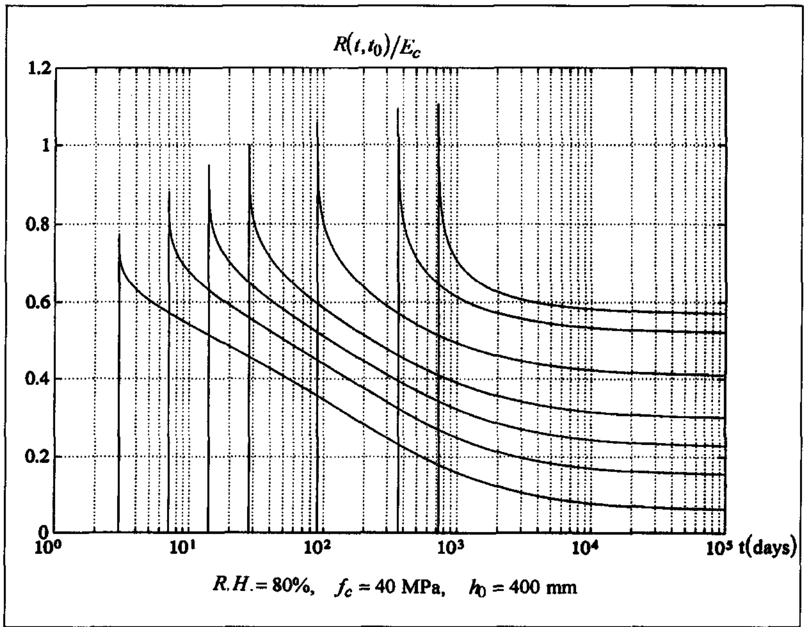

Fig. 5 - Numerically accurate values of the non-dimensional relaxation function, $R(t,t_0)/E_c$. Parameters: R.H. = 80%, f_c = 40 MPa, h_0 = 400 mm.

through the direct method. The diagrams of the two functions are, in fact, seen to agree very closely, and their final values show very limited differences.

On account of the greater accuracy of equation (3) compared to equation (2), results obtained are more accurate, for any combination of rheological parameters, than those obtained by introducing equation (2) into equation (5) [10].

Apart from the numerical reliability of the results, the determination of the relaxation function from equation (5) turns out to be simplified in comparison with other methods proposed in the literature (see, for instance, the expression of $\tilde{R}$ proposed by Bažant and Kim [11]).

5. EXAMPLE: HOMOGENEOUS STRUCTURES WITH A SINGLE ELASTIC RESTRAINT

The analysis of homogeneous structures with a single elastic restraint makes it possible to evaluate the behaviour of visco-elastic systems as a function of the ratio between the deformability of the concrete portion and that of the structure with the elastic restraint taken as a whole [4,13].

This ratio can be expressed through the relationship $\delta = a_{c,11}/(a_{c,11} + a_{s,11})$, where $a_{c,11}$ is the deformability of the concrete portion and $a_{s,11}$ is the deformability of the elastic restraint.

This structural model has already been discussed in some length [4] for comparing the results provided by algebraic methods with those obtained through the general numerical method.

The example proposed makes it possible to compare the results obtained through expression (3) of the approximate ageing coefficient, $\tilde{\chi}$, with those determined through the earlier formulation, function (2), proposed in the CEB MC 90. Furthermore, in order to assess the degree of accuracy offered by expressions (2) and (3) of the AAEM method, the design values are compared with those yielded by the general method.

Let us consider the structure depicted in Fig. 6 where, as in [4], the influence of mass forces and shrinkage is assumed to be negligible. In order to determine the value of the reaction, $X_1(t)$, taking place in the elastic restraint at time t due to the initial imposed deformation $y_1(t_0)$, assumed to be constant in time and positive because of its downward direction, the solution is obtained from the equation of compatibility at the restraint:

$$\begin{aligned} y_1(t_0) &= a_{c,11} \cdot E_c(t_0) \cdot X_1(t_0) J(t,t_0) \\ &+ a_{c,11} \cdot E_c(t_0) \cdot \left[X_1(t) - X_1(t_0)\right] \cdot \\ &\cdot \left[\frac{1}{E_c(t_0)} + \chi(t,t_0)\frac{\varphi(t,t_0)}{E_c}\right] + a_{s,11} \cdot X_1(t) \end{aligned} \tag{6}$$

where $\varphi(t,t_0)$ is the concrete creep coefficient [5], and $\chi(t,t_0)$ is given by equation (1).

By dividing equation (6) by $(a_{c,11} + a_{s,11})$ and recognizing that the elastic solution is $X_1(t_0) = y_1(t_0)/(a_{c,11} + a_{s,11})$ we get the evolution over time of the non-dimensional relationship $X_1(t)/X_1(t_0)$ [4]:

$$\frac{X_1(t)}{X_1(t_0)} = \frac{1 - \delta \cdot E_c(t_0) \cdot \dfrac{\varphi(t,t_0)}{E_c}\left[1 - \tilde{\chi}\right]}{1 + \delta \cdot E_c(t_0) \cdot \dfrac{\varphi(t,t_0)}{E_c} \cdot \tilde{\chi}} \tag{7}$$

which appears as a very simple relationship wherein we have introduced the approximate value of $\chi(t,t_0)$ worked out from expressions (2) and (3).

By applying the general method [4,13], the same problem is defined by means of the following equation:

$$y_1(t_0) = a_{c,11} \cdot E_c(t_0) \cdot X_1(t_0) \cdot J(t,t_0) + \delta \cdot E_c(t_0) \int_{t_0^+}^{t} J(t,\tau) dX_1(\tau) + a_{s,11} \cdot X_1(t) \quad (8)$$

from which, by introducing δ and dividing by $X_1(t_0)$, we get:

$$1 = \delta \cdot E_c(t_0) \cdot J(t,t_0) + \delta \frac{E_c(t_0)}{X_1(t_0)} E_c(t_0) \int_{t_0^+}^{t} J(t,\tau) dX_1(\tau) + (1-\delta) \frac{X_1(t)}{X_1(t_0)} \quad (9)$$

which calls for the time-consuming numerical procedure involving the inversion of Volterra's integral with the trapezoidal rule [3,4,13].

For an infinitely elastic restraint ($a_{s,11} \rightarrow \infty$; $\delta = 0$), equations (7) and (9) yield $X_1(t)/X_1(t_0) = 1$, corresponding to a creep problem; conversely, when dealing with an infinitely-rigid restraint ($a_{s,11} = 0$; $\delta = 1$), being substituted into equation (7), the expression (1) of $\chi(t,t_0)$ is transformed as follows:

$$\chi(t,t_0) = \frac{E_c(t_0)}{E_c(t_0) - R(t,t_0)} - \frac{E_c}{E_c(t_0) \cdot \varphi(t,t_0)} \quad (10)$$

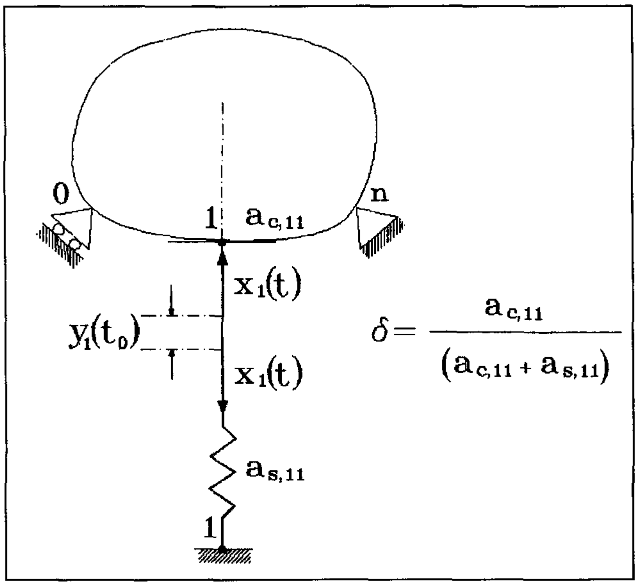

Fig. 6 - Homogeneous concrete structure with a single additional elastic restraint.

We then obtain:

$$\frac{X_1(t)}{X_1(t_0)} = \frac{R(t,t_0)}{R(t_0,t_0)} = \frac{R(t,t_0)}{E_c(t_0)} \quad (11)$$

corresponding to a relaxation problem.

Furthermore, since in these two extreme cases the AAEM yields exact solutions, we may expect that, by varying δ, the closer the approximation of expressions (2) and (3) of $\tilde{\chi}$ to $\chi(t, t_0)$, the greater the accuracy of the solutions provided by equation (7) compared to those yielded by (9).

In particular, for $\delta \rightarrow 1$, by comparing the results, we may assess the reliability of the approximate expression of the proposed ageing coefficient (3) in the solution of a relaxation problem.

6. DISCUSSION OF THE RESULTS OF THE EXAMPLE

The results of the example are summarised in the diagrams of Fig. 7, where, for different values of parameter δ (between 0 and 1), we find the evolution of the final values at 10^5 days of the $X_1(t)/X_1(t_0)$ ratio obtained by introducing into equation (7) the values of $\tilde{\chi}$ from expressions (2) and (3) and then solving equation (9).

Fig. 7 has been produced by using the rheological parameters resulting in maximum creep deformability of concrete, $R.H.$ = 50%, f_c = 20 MPa, h_0 = 50 mm, and an age at loading t_0 = 3 days, when the deformation $y_1(t_0)$ is imposed.

An analysis of Fig. 7 shows that the values obtained from expression (3) of $\tilde{\chi}$ are in good agreement with those yielded by the general method for each value of δ between 0 and 1; on the other hand, the values obtained from expression (2) of $\tilde{\chi}$ are seen to progressively depart from the exact values as δ approaches 1, i.e., when the deformability of the elastic restraint is such ($a_{s,11} \rightarrow 0$) that it corresponds to a relaxation problem.

In particular, expression (2) of $\tilde{\chi}$ yields negative values of $X_1(10^5)/X_1(t_0)$ in the range: $0.6 < \delta \leq 1$, $R.H.$ = 50%, $f_c \leq 50$ MPa, $h_0 \leq 800$ mm and $t_0 \leq 7$. This solution, which is clearly unacceptable from the physical standpoint, entails a limitation of the field of application of the MC 90's expression (2) of $\tilde{\chi}$ and underscores the improvements achieved by adopting expression (3).

Nevertheless, in normal ambient conditions for rheological parameters commonly encountered in concrete structures ($R.H.$ = 80%, f_c = 30 MPa, h_0 = 200 mm), both expressions lead to satisfactory results.

Finally, in Fig. 8, the evolution over time of $X_1(t)/X_1(t_0)$ obtained from equation (7), by substituting expression (3) of $\tilde{\chi}$, is compared with the evolution determined on the basis of equation (9) of the general method.

The diagrams were plotted for δ = 0.5 by keeping relative humidity and concrete strength constant (f_c = 30

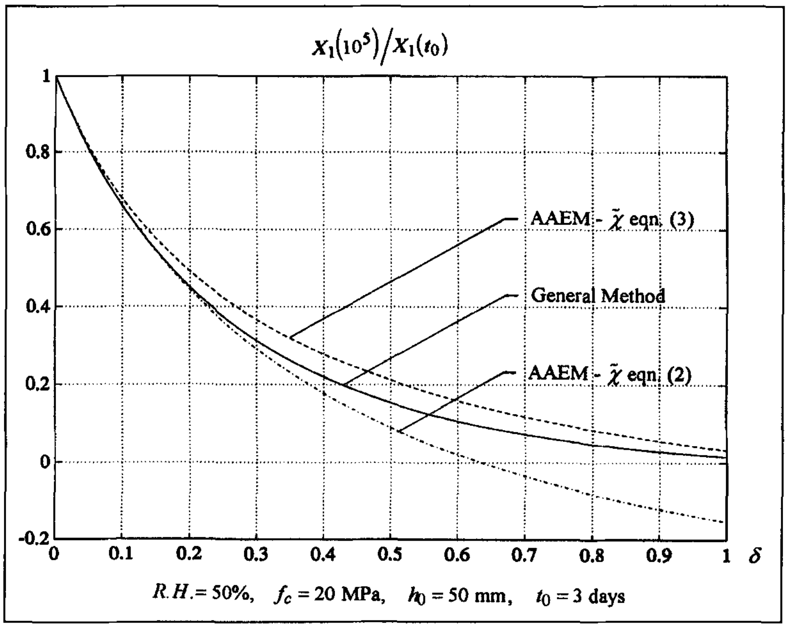

Fig. 7 - Evolution of the final values of the $X_1(t)/X_1(t_0)$ ratio, at 10^5 days, as a function of δ: comparison between the results yielded by the AAEM method and the general method.
Parameters:
R.H. = 50%, f_c = 20 MPa, h_0 = 50 mm, t_0 = 3 days.

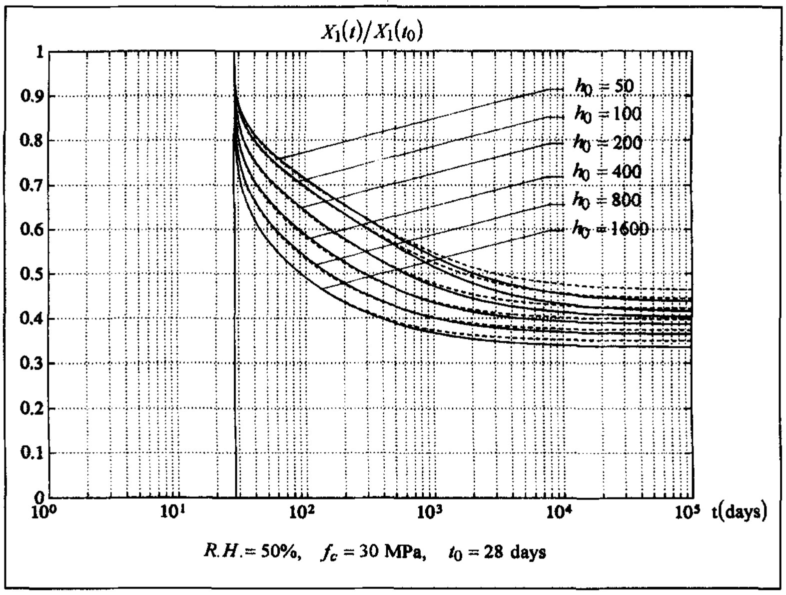

Fig. 8 - Structure with a single elastic restraint. Evolution in time of the $X_1(t)/X_1(t_0)$ ratio for δ = 0,5: comparison between the results yielded by the AAEM method (dashed line) and the general method (solid line).
Parameters:
R.H. = 50%, f_c = 30 MPa, h_0 = variable, t_0 = 28 days.

MPa, *R.H.* = 50%), for t_0 = 28 days and for different values of h_0. In this manner, it proves possible to assess the accuracy of the proposed function over a wide variety of cases in relation to an intermediate value of the deformability ratio, δ, between the concrete structure and the elastic restraint.

Thus, on the basis of the example provided, it can be stated that expression (3) of $\tilde{\chi}$ can be applied with very good results over the entire field of variation of the rheological parameters (as defined in paragraph 2) and of the deformability ratio between concrete structure-elastic restraint, in keeping with the objectives of the AAEM method for structural design in the linear visco-elastic domain.

7. CONCLUSIONS

With reference to the creep function recently suggested by CEB Model Code 1990, the approximate algebraic expression (3) of the ageing coefficient proposed in this work improves equation (2) by reducing the error involved. It amounts to the product of four terms, the first of which, coinciding with equation (2), is taken to represent average behaviour, while the three remaining terms independently describe the influence of other parameters, such as effective thickness, relative humidity and concrete characteristic strength.

The numerical checks carried out on equation (3) show that the percentage errors with respect to the exact values of $\tilde{\chi}$ supplied by equation (1) are extremely modest. In the range of values of h_0, $R.H.$ and f_c most widely used for practical design purposes, errors generally do not exceed 3-4% for $t_0 \geq 7$ days and become virtually negligible for $t_0 \geq 14$ days.

Moreover, the new expression of the ageing coefficient has made it possible to work out an approximate expression $\tilde{R}(t, t_0)$ (equation(5)) of the relaxation function which turns out to be sufficiently reliable as its diagrams are virtually identical, with minimal deviations, to the numerically-exact diagrams obtained from the direct method that requires the inversion of Volterra's integral equation.

Finally, an example of a structural calculation is provided to prove the applicability of the approximate expression (3) of the ageing coefficient, in compliance with the objectives of the AAEM method: the example shows that the results obtained in the design of structures in the linear visco-elastic domain are extremely good.

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