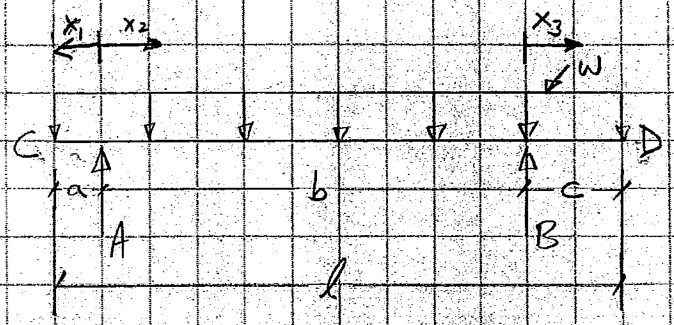
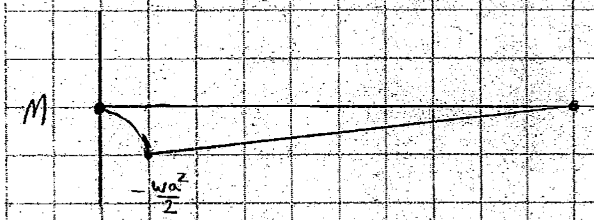


① Moments due to w over a only:



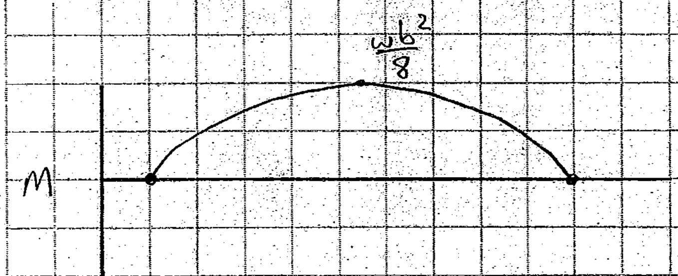
$\Sigma M_B = 0 \quad Ab = wa\left(b + \frac{a}{2}\right)$

$A = \frac{wa}{b}\left(b + \frac{a}{2}\right)$

$\Sigma M_A = 0 \quad Bb + wa\left(\frac{a}{2}\right) = 0$

$B = -\frac{wa}{b}\left(\frac{a}{2}\right)$

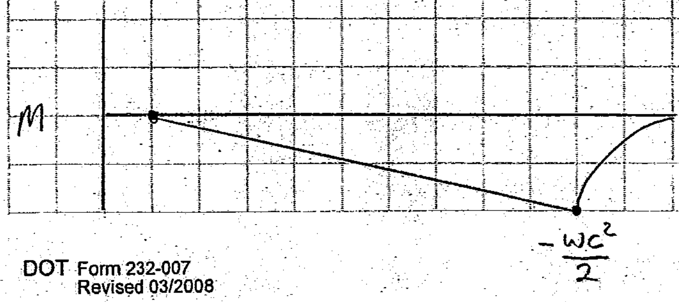
② Moments due to w over b only:



$\Sigma M_B = 0 \quad Ab = wb\left(\frac{b}{2}\right)$

$A = \frac{wb}{2}$ = B by symmetry

③ Moments due to w over c only:



$\Sigma M_A = 0 \quad Bb = wc\left(b + \frac{c}{2}\right)$

$B = \frac{wc}{b}\left(b + \frac{c}{2}\right)$

$\Sigma M_B = 0 \quad Ab + wc\left(\frac{c}{2}\right) = 0$

$A = -\frac{wc}{b}\left(\frac{c}{2}\right)$

By the Moment Area Theorems, the tangential deviation at B from the tangent line at A is equal to the moment about B of the area of the $^{M}/_{EI}$ diagram between the two points.

$$\text{Area} = \left(-\frac{wa^2}{2}\right)\left(\frac{b}{2}\right)\left(\frac{1}{EI}\right)$$

$$\text{Area } 2 = \frac{1}{EI}\int_0^b \left(\frac{wb}{2}\right)x_2 - \left(\frac{w}{2}\right)x_2^2 \, dx_2 = \frac{1}{EI}\Big|_0^b \left(\frac{wb}{2}\right)\left(\frac{1}{2}\right)x_2^2 - \left(\frac{w}{2}\right)\left(\frac{1}{3}\right)x_2^3 = \frac{wb^3}{4EI} - \frac{wb^3}{6EI} = \frac{wb^3}{12EI}$$

$$\text{Area } 3 = \left(-\frac{wc^2}{2}\right)\left(\frac{b}{2}\right)\left(\frac{1}{EI}\right)$$

$$t_{BA} = \left(-\frac{wa^2b}{4EI}\right)\left(\frac{2b}{3}\right) + \left(\frac{wb^3}{12EI}\right)\left(\frac{b}{2}\right) + \left(-\frac{wbc^2}{4EI}\right)\left(\frac{b}{3}\right)$$

$$= -\frac{wa^2b^2}{6EI} + \frac{wb^4}{24EI} - \frac{wb^2c^2}{12EI}$$

$$\tan\theta_A = -\frac{t_{BA}}{b} \approx \theta_A \quad (\text{for small angles})$$

$$\theta_A = \frac{w}{24EI}\left(4a^2b - b^3 + 2bc^2\right)$$

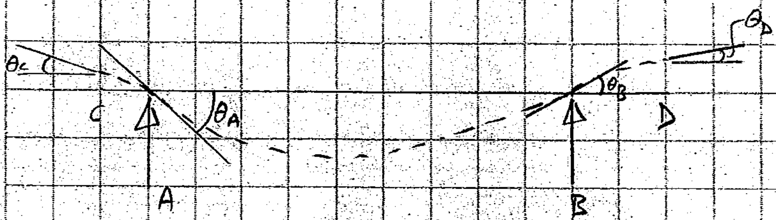
Similarly for θ at support B:

$$t_{AB} = \left(-\frac{wa^2b}{4EI}\right)\left(\frac{b}{3}\right) + \left(\frac{wb^3}{12EI}\right)\left(\frac{b}{2}\right) + \left(-\frac{wbc^2}{4EI}\right)\left(\frac{2b}{3}\right)$$

$$= -\frac{wa^2b^2}{12EI} + \frac{wb^4}{24EI} - \frac{wb^2c^2}{6EI}$$

$$\tan\theta_B = \frac{t_{AB}}{b} \approx \theta_B$$

$$\theta_B = \frac{w}{24EI}\left(-2a^2b + b^3 - 4bc^2\right)$$



Assumed Deflected Shape

By the Moment Area Theorems, the change in slope between any two points is equal to the area under the M/EI curve between these points.

$$\theta_C + Area_{CA} = \theta_A$$

$$\theta_C = \theta_A - Area_{CA}$$

$$Area_{CA} = \int_0^a -\frac{w}{2EI} x_1^2 \, dx_1 = \Big|_0^a \left(-\frac{w}{2EI}\right)\left(\frac{1}{3}\right) x_1^3 = -\frac{wa^3}{6EI}$$

$$\therefore \theta_C = \frac{w}{24EI}\left(4a^2 b - b^3 + 2bc^2\right) - \left(-\frac{wa^3}{6EI}\right)$$

$$= \frac{w}{24EI}\left(4a^3 + 4a^2 b - b^3 + 2bc^2\right) \quad \checkmark$$

Similarly for θ_D:

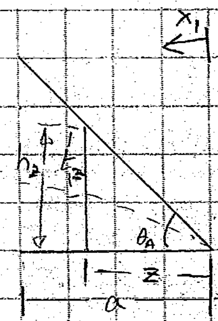
$$\theta_B + Area_{BD} = \theta_D$$

$$Area_{BD} = -\frac{wc^3}{6EI}$$

$$\theta_D = \frac{w}{24EI}\left(-2a^2 b + b^3 - 4bc^2\right) + \left(-\frac{wc^3}{6EI}\right)$$

$$= \frac{w}{24EI}\left(-2a^2 b + b^3 - 4bc^2 - 4c^3\right) \quad \checkmark$$

Write deflection equation for portion CA:



$$\frac{M}{EI} = \frac{w(a-x_1)^2}{2EI} = \frac{w}{2EI}\left(a^2 - 2ax_1 + x_1^2\right)$$

$$t_z = \int_0^z \frac{w}{2EI}\left(a^2 - 2ax_1 + x_1^2\right)(z - x_1)\,dx_1$$

$$= \int_0^z \frac{w}{2EI}\left(a^2 z - 2ax_1 z + x_1^2 z - a^2 x_1 + 2ax_1^2 - x_1^3\right)dx_1$$

$$= \Big|_0^z \frac{w}{2EI}\left(a^2 z x_1 - azx_1^2 + \frac{zx_1^3}{3} - \frac{a^2 x_1^2}{2} + \frac{2ax_1^3}{3} - \frac{x_1^4}{4}\right)$$

$$= \frac{w}{2EI}\left(a^2z^2 - az^3 + \frac{z^4}{3} - \frac{a^2z^2}{2} + \frac{2az^3}{3} - \frac{z^4}{4}\right)$$

$$= \frac{w}{2EI}\left(\frac{a^2z^2}{2} - \frac{az^3}{3} + \frac{z^4}{12}\right) = \frac{w}{24EI}\left(6a^2z^2 - 4az^3 + z^4\right)$$

$\tan -\theta_A = \frac{h_z}{z} \approx -\theta_A$

$h_z = -\theta_A z$

$y = h_z - t_z$

(use negative value of θ_A because $-\theta_A$ corresponds with a positive h_z)

$$y = -\theta_A z - t$$

$$= -\frac{wz}{24EI}\left(4a^2b - b^3 + 2bc^2\right) - \frac{w}{24EI}\left(6a^2z^2 - 4az^3 + z^4\right)$$

$$= -\frac{wz}{24EI}\left(4a^2b - b^3 + 2bc^2 + 6a^2z - 4az^2 + z^3\right) \checkmark$$

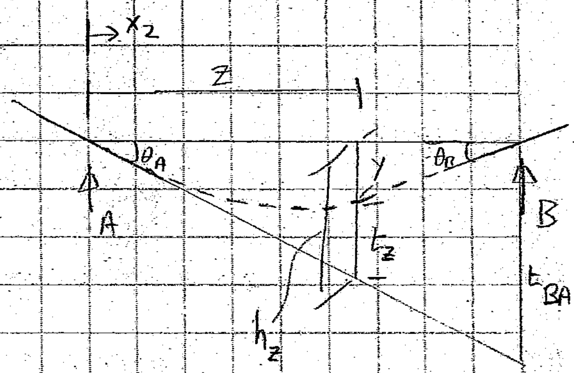
Write deflection equation for portion AB:

$$\frac{M}{EI}\ (CA) = \left[A x_2 - wa\left(x_2 + \frac{a}{2}\right)\right]\frac{1}{EI}$$

$$= \left[\frac{wa x_2}{b}\left(b + \frac{a}{2}\right) - wa\left(x_2 + \frac{a}{2}\right)\right]\frac{1}{EI}$$

$$= \left[\cancel{wa x_2} + \frac{wa^2 x_2}{2b} - \cancel{wa x_2} - \frac{wa^2}{2}\right]\frac{1}{EI}$$

$$= \left[\frac{wa^2}{2b}x_2 - \frac{wa^2}{2}\right]\frac{1}{EI}$$



$$\frac{M}{EI}\ (AB) = \left[A x_2 - w x_2\left(\frac{x_2}{2}\right)\right]\frac{1}{EI}$$

$$= \left[\frac{wb}{2}x_2 - \frac{w}{2}x_2^2\right]\frac{1}{EI}$$

$$\frac{M}{EI}\ (BD) = \left[A x_2\right]\frac{1}{EI} = -\frac{wc^2}{2b}x_2\left[\frac{1}{EI}\right]$$

$$\tan\theta_A \approx \theta_A = h_z/z$$

$$h_z = \theta_A z$$

$$y = h_z + t_z$$

$$t_z = \frac{1}{EI}\int_0^z\left[\left(\frac{wa^2}{2b}x_2 - \frac{wa^2}{2}\right)(z - x_2) + \left(\frac{wb}{2}x_2 - \frac{w}{2}x_2^2\right)(z - x_2) - \frac{wc^2}{2b}x_2(z - x_2)\right]dx_2$$

$$= \frac{1}{EI}\int_0^z\left[\frac{wa^2 z}{2b}x_2 - \frac{wa^2 z}{2} - \frac{wa^2}{2b}x_2^2 + \frac{wa^2}{2}x_2 + \frac{wbz}{2}x_2 - \frac{wz}{2}x_2^2 - \frac{wb}{2}x_2^2 + \frac{w}{2}x_2^3 - \frac{wc^2 z}{2b}x_2 + \frac{wc^2}{2b}x_2^2\right]dx_2$$

$$= \frac{1}{EI}\Big|_0^z\ \frac{wa^2 z}{4b}x_2^2 - \frac{wa^2 z}{2}x_2 - \frac{wa^2}{6b}x_2^3 + \frac{wa^2}{4}x_2^2 + \frac{wbz}{4}x_2^2 - \frac{wz}{6}x_2^3 - \frac{wb}{6}x_2^3 + \frac{w}{8}x_2^4 - \frac{wc^2 z}{4b}x_2^2 + \frac{wc^2}{6b}x_2^3$$

$$= \frac{1}{EI}\left[\frac{wa^2z^3}{4b} - \frac{wa^2z^2}{2} - \frac{wa^2z^3}{6b} + \frac{wa^2z^2}{4} + \frac{wbz^3}{4} - \frac{wz^4}{6} - \frac{wbz^3}{6} + \frac{wz^4}{8} - \frac{wc^2z^3}{4b} + \frac{wc^2z^3}{6b}\right]$$

$$L_z = \frac{1}{24EI}\left[\frac{2wa^2z^3}{b} - 6wa^2z^2 + 2wbz^3 - wz^4 - \frac{2wc^2z^3}{b}\right]$$

$$y = h_z + L_z = \theta_A z + L_z$$

$$= \frac{wz}{24EI}\left(4a^2b - b^3 + 2bc^2\right) + \frac{wz}{24EI}\left[\frac{2a^2z^2}{b} - 6a^2z + 2bz^2 - z^3 - \frac{2c^2z^2}{b}\right]$$

$$= \frac{wz}{24EI}\left[4a^2b - b^3 + 2bc^2 + \frac{2a^2z^2}{b} - 6a^2z + 2bz^2 - z^3 - \frac{2c^2z^2}{b}\right]$$

Check Against Equation in PCI Handbook:

$a^2b - a^2z$

$$-\frac{wz(b-z)}{24EI}\left[z(b-z) + b^2 - 2a^2 - 2c^2 - \frac{z}{b}\left[c^2z + a^2(b-z)\right]\right]$$

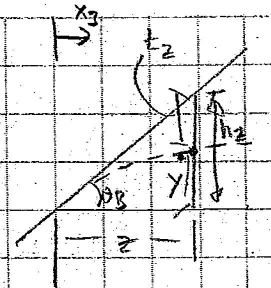
$$\frac{wz(z-b)}{24EI}\left[bz - z^2 + b^2 - 2a^2 - 2c^2 - \frac{2c^2z}{b} - 2a^2 + \frac{2a^2z}{b}\right]$$

$$\frac{wz}{24EI}\left[\cancel{bz^2} - z^3 + \cancel{b^2z} - \cancel{2a^2z} - \cancel{2c^2z} - \frac{2c^2z^2}{b} - 2a^2z + \frac{2a^2z^2}{b} - \cancel{b^2z} + \cancel{bz^2} - b^3 + 2a^2b\right.$$

$$\left. + 2bc^2 + \cancel{2c^2z} + 2a^2b - \cancel{2a^2z}\right]$$

$$= \frac{wz}{24EI}\left[2bz^2 - z^3 - 6a^2z - \frac{2c^2z^2}{b} + \frac{2a^2z^2}{b} - b^3 + 4a^2b + 2bc^2\right]$$ ✓ OK

Write deflection equation for portion BD:



$$\frac{M}{EI} = \left[A(b+x_3) + Bx_3 - \frac{w}{2}x_3^2\right]\frac{1}{EI}$$

$$= \left[-\frac{wc^2}{2b}(b+x_3) + \frac{wc}{b}\left(b+\frac{c}{2}\right)x_3 - \frac{w}{2}x_3^2\right]\frac{1}{EI}$$

$$= \left[-\frac{wc^2}{2} - \cancel{\frac{wc^2}{2b}x_3} + wcx_3 + \cancel{\frac{wc^2}{2b}x_3} - \frac{w}{2}x_3^2\right]\frac{1}{EI}$$

$$= \left[-\frac{wc^2}{2} + wcx_3 - \frac{w}{2}x_3^2\right]\frac{1}{EI}$$

$$\tan\theta_B = \frac{h_z}{z} \approx \theta_B$$

$$h_z = \theta_B z$$

$$y = h_z + t_z$$

$$t_z = \frac{1}{EI}\int_0^z \left(-\frac{wc^2}{2} + wcx_3 - \frac{w}{2}x_3^2\right)(z - x_3)\,dx_3$$

$$= \frac{1}{EI}\int_0^z \left(-\frac{wc^2z}{2} + wczx_3 - \frac{wz}{2}x_3^2 + \frac{wc^2}{2}x_3 - wcx_3^2 + \frac{w}{2}x_3^3\right)dx_3$$

$$= \frac{1}{EI}\Bigg|_0^z -\frac{wc^2z}{2}x_3 + \frac{wcz}{2}x_3^2 - \frac{wz}{6}x_3^3 + \frac{wc^2}{4}x_3^2 - \frac{wc}{3}x_3^3 + \frac{w}{8}x_3^4$$

$$= \frac{w}{EI}\left(-\frac{c^2z^2}{2} + \frac{cz^3}{2} - \frac{z^4}{6} + \frac{c^2z^2}{4} - \frac{cz^3}{3} + \frac{z^4}{8}\right)$$

$$= \frac{w}{24EI}\left(-12c^2z^2 + 12cz^3 - 4z^4 + 6c^2z^2 - 8cz^3 + 3z^4\right)$$

$$= \frac{wz}{24EI}\left(-6c^2z + 4cz^2 - z^3\right)$$

$$y = \theta_B z + t_z$$

$$= \frac{wz}{24EI}\left(-2a^2b + b^3 - 4bc^2\right) + \frac{wz}{24EI}\left(-6c^2z + 4cz^2 - z^3\right)$$

$$= \frac{wz}{24EI}\left(-2a^2b + b^3 - 4bc^2 - 6c^2z + 4cz^2 - z^3\right)$$

From calculus, the average value of f over $[a,b]$ is

$$\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

For segment CA:

$$\bar{y}_{CA} = \frac{1}{a-0}\int_0^a -\frac{w}{24EI}\left(4a^2bz - b^3z + 2bc^2z + 6a^2z^2 - 4az^3 + z^4\right)dz$$

$$= -\frac{w}{24EIa}\Big|_0^a 2a^2bz^2 - \frac{b^3z^2}{2} + bc^2z^2 + 2a^2z^3 - az^4 + \frac{z^5}{5}$$

$$= -\frac{w}{24EIa}\left[2a^4b - \frac{b^3a^2}{2} + a^2bc^2 + 2a^5 - a^5 + \frac{a^5}{5}\right]$$

$$= -\frac{w}{24EIa}\left[\frac{6a^5}{5} + 2a^4b - \frac{a^2b^3}{2} + a^2bc^2\right]$$

$$= \frac{w}{a24EI}\left[-\frac{6a^5}{5} - 2a^4b + \frac{a^2b^3}{2} - a^2bc^2\right]$$

For segment AB:

$$\bar{y}_{AB} = \frac{1}{b-0}\int_0^b \frac{w}{24EI}\left[4a^2bz - b^3z + 2bc^2z + \frac{2a^2z^3}{b} - 6a^2z^2 + 2bz^3 - z^4 - \frac{2c^2z^3}{b}\right]dz$$

$$= \frac{w}{24EIb}\Big|_0^b 2a^2bz^2 - \frac{b^3z^2}{2} + bc^2z^2 + \frac{a^2z^4}{2b} - 2a^2z^3 + \frac{bz^4}{2} - \frac{z^5}{5} - \frac{c^2z^4}{2b}$$

$$= \frac{w}{24EIb}\left[\cancel{2a^2b^3} - \cancel{\frac{b^5}{2}} + b^3c^2 + \frac{a^2b^3}{2} - \cancel{2a^2b^3} + \cancel{\frac{b^5}{2}} - \frac{b^5}{5} - \frac{c^2b^3}{2}\right]$$

$$= \frac{w}{24EIb}\left[\frac{a^2b^3}{2} + \frac{b^3c^2}{2} - \frac{b^5}{5}\right]$$

For Segment BD:

$$\bar{y}_{BD} = \frac{1}{c-0}\int_0^c \frac{w}{24EI}\left[-2a^2bz + b^3z - 4bc^2z - 6c^2z^2 + 4cz^3 - z^4\right]dz$$

$$= \frac{w}{24EIc}\Big|_0^c -a^2bz^2 + \frac{b^3z^2}{2} - 2bc^2z^2 - 2c^2z^3 + cz^4 - \frac{z^5}{5}$$

$$= \frac{w}{24EIc}\left[-a^2bc^2 + \frac{b^3c^2}{2} - 2bc^4 - 2c^5 + c^5 - \frac{c^5}{5}\right]$$

$$= \frac{w}{24EIc}\left[-a^2bc^2 + \frac{b^3c^2}{2} - 2bc^4 - \frac{6c^5}{5}\right]$$

To find centroid of all 3 sections together, use weighted average:

$$\bar{z}_o = \bar{y}_{CA}\left(\frac{a}{\ell}\right) + \bar{y}_{AB}\left(\frac{b}{\ell}\right) + \bar{y}_{BD}\left(\frac{c}{\ell}\right)$$

$$\bar{z}_o = \frac{\bar{y}_{CA}\,a + \bar{y}_{AB}\,b + \bar{y}_{BD}\,c}{\ell}$$

$$\bar{z}_o = \left(\frac{w}{24EI\ell}\right)\left[-\frac{6a^5}{5} - 2a^4b + \frac{a^2b^3}{2} - a^2bc^2 + \frac{a^2b^3}{2} + \frac{b^3c^2}{2} - \frac{b^5}{5} - a^2bc^2 + \frac{b^3c^2}{2} - 2bc^4 - \frac{6c^5}{5}\right]$$

$$\bar{z}_o = \frac{w}{24EI\ell}\left[-\frac{6a^5}{5} - 2a^4b + a^2b^3 - 2a^2bc^2 + b^3c^2 - \frac{b^5}{5} - 2bc^4 - \frac{6c^5}{5}\right]$$

$a = L_L \quad b = L_S \quad c = L_T$

Check for case where $a = c$

$$\bar{z}_o = \frac{w}{24EI\ell}\left[-\frac{6a^5}{5} - 2a^4b + a^2b^3 - 2a^4b + b^3a^2 - \frac{b^5}{5} - 2ba^4 - \frac{6a^5}{5}\right]$$

$$= \frac{w}{24EI\ell}\left[-\frac{12a^5}{5} - 6a^4b + 2a^2b^3 - \frac{b^5}{5}\right]$$

$$= \frac{w}{12EI\ell}\left[-\frac{6a^5}{5} - 3a^4b + a^2b^3 - \frac{b^5}{10}\right]$$ (sign is opposite) - ok

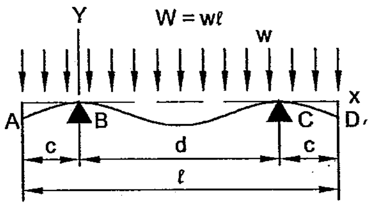
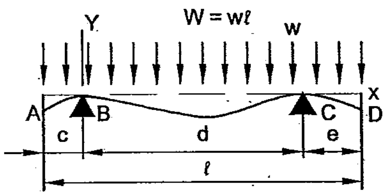
Check for case where $a = c = 0$

$$\bar{z}_o = \frac{w}{24EI\ell}\left[-\frac{b^5}{5}\right] = -\frac{wb^5}{24EI\ell} \quad \text{since } b = \ell \quad = -\frac{w\ell^4}{120EI}$$ (sign is opposite) - ok

∴ Add negative sign to $\bar{z}_o$ to show correct direction of displacement consistent with derivations of equations in references.

DESIGN INFORMATION

Design Aid 11.1.8 Moments, Shears, and Deflections in Beams with Overhangs

LOADING AND SUPPORT	REACTIONS AND VERTICAL SHEAR	BENDING MOMENT M AND MAXIMUM BENDING MOMENT	DEFLECTION y, MAXIMUM DEFLECTION, AND END SLOPE θ
EQUAL OVERHANGS, UNIFORM LOAD  	$R_B = R_C = \frac{W}{2}$ (A to B) $V = \frac{W(c-x)}{\ell}$ (B to C) $V = W\left(\frac{1}{2} - \frac{x+c}{\ell}\right)$ (C to D) $V = \frac{W(c+d-x)}{\ell}$	(A to B) $M = -\frac{W}{2\ell}(c-x)^2$ (B to C) $M = -\frac{W}{2\ell}[(c^2 - x(d-x)]$ $M = -\frac{Wc^2}{2\ell}$ at B and C $M = -\frac{W}{2\ell}\left(c^2 - \frac{d^2}{4}\right)$ at $x = \frac{d}{2}$ if $d > 2c$, $M = 0$ at $x = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - c^2}$ if $c = 0.207\ell$, $M = -\frac{W\ell}{46.62}$ at $x = 0 = d$ and $M = \frac{W\ell}{46.62}$ ar $x = \frac{d}{2}$ x is considered positive on both sides of the origin.	(A to B) $y = -\frac{Wx}{24EI\ell}[6c^2(d+x) - x^2(4c-x) - d^3]$ (B to C) $y = -\frac{Wx(d-x)}{24EI\ell}[x(d-x) + d^2 - 6c^2]$ $y = -\frac{Wc}{24EI\ell}[3c^2(c+2d) - d^3]$ at A and D $y = -\frac{Wd^2}{384EI\ell}(5d^2 - 24c^2)$ AT $x = \frac{d}{2}$ if $2c < d < 2.449c$, the maximum deflection between supports is: $y = \frac{W}{96EI\ell}(6c^2 - d^2)^2$ AT $x = \frac{d}{2} \pm \sqrt{3\left(\frac{d^2}{4} - c^2\right)}$ $\theta = \frac{W}{24EI\ell}(6c^2d + 4c^3 - d^3)$ AT A $\theta = -\frac{W}{24EI\ell}(6c^2d + 4c^3 - d^3)$ AT D
UNEQUAL OVERHANGS, UNIFORM LOAD x ←|→ x  	$R_B = \frac{W}{2d}(c+d-e)$ $R_C = \frac{W}{2d}(d+e-c)$ (A to B) $V = -\frac{W}{\ell}(c-x)$ (B to C) $V = R_B - \frac{W}{\ell}(c+x)$ (C to D) $V = -\frac{W}{\ell}(d+e-x)$	(A to B) $M = -\frac{W}{2\ell}(c-x)^2$ (B to C) $M = -\frac{W}{2\ell}(c-x)^2 + R_B x$ (C to D) $M = -\frac{W}{2\ell}(e+d-x)^2$ $M = -\frac{Wc^2}{2\ell}$ at B $M = -\frac{We^2}{2\ell}$ at C M_{max} between supports $= \frac{W}{2\ell}(c^2 - x_1^2)$ at $x = x_1$ $= \frac{c^2 + d^2 - e^2}{2d}$ if $x_1 > c$, $M = 0$ AT $x = x_1 \pm \sqrt{x_1^2 - c^2}$ x is considered positive on both sides of the origin.	(A to B) $y = -\frac{Wx}{24EI\ell}$ $[2d(e^2 + 2c^2) + 6c^2x - x^2(4c-x) - d^3]$ (B to C) $y = -\frac{Wx(d-x)}{24EI\ell}$ $\left\{x(d-x) + d^2 - 2(c^2 + e^2) - \frac{2}{d}[e^2x + c^2(d-x)]\right\}$ (C to D) $y = -\frac{W(x-d)}{24EI\ell}[2d(c^2 + 2e^2) + 6e^2(x-d)$ $-(x-d)^2(4e+d-x) - d^3]$ $y = -\frac{Wc}{24EI\ell}[2d(e^2 + 2c^2) + 3c^3 - d^3]$ AT A $y = -\frac{We}{24EI\ell}[2d(c^2 + 2e^2) + 3e^3 - d^3]$ AT D This case is too complicated to obtain a general expression for critical deflections between the supports. $\theta = \frac{W}{24EI\ell}(4c^3 + 4c^2d - d^3 + 2de^2)$ AT A $\theta = -\frac{W}{24EI\ell}(2c^2d + 4de^2 - d^3 + 4e^3)$ AT D