0.1 定积分

0.1.1 建立积分递推

例题 0.1 计算 $\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)dx, n \in \mathbb{N}.$

证明 利用分部积分和和差化积公式可得

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx)\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x[\sin(nx)\cos x + \cos(nx)\sin x]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x\cos(nx)\mathrm{d}\cos x \\
&= \frac{I_{n-1}+I_n}{2} - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos(nx)\mathrm{d}\cos^n x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^n x\sin(nx)\mathrm{d}x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
&= \frac{I_{n-1}}{2} + \frac{1}{2n}.
\end{aligned}$$

故 $I_n = \frac{I_{n-1}}{2} + \frac{1}{2n}$, 则两边同乘 2^n（强行裂项）

$$2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \cdots$$

又注意到 $I_0 = 0$, 从而

$$2^n I_n = 0 + \sum_{k=1}^{n} \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n}\sum_{k=1}^{n} \frac{2^{k-1}}{k}.$$

□

命题 0.1

证明:

(1) $\int_0^{\pi} \frac{\sin(nx)}{\sin x}\mathrm{d}x = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$

(2) $\int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x}\mathrm{d}x = n\pi.$

(3) $\int_0^{\pi} \frac{\sin^2(nx)}{\sin x}\mathrm{d}x = \sum_{k=1}^{n} \frac{2}{2k-1}.$

笔记 提示:$\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y)$(证明见命题**??**).

证明

(1) 记 $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x}\mathrm{d}x$, 则

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin((n+2)x) - \sin(nx)}{\sin x}\mathrm{d}x = \int_0^{\pi} \frac{2\cos((n+1)x)\sin x}{\sin x}\mathrm{d}x = 2\int_0^{\pi} \cos((n+1)x)\mathrm{d}x = 0.$$

于是

$$\int_0^\pi \frac{\sin(nx)}{\sin x}\mathrm{d}x = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$$

(2) 记 $I_n = \int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x}\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x}\mathrm{d}x \\ &= \int_0^\pi \frac{\sin((2n+1)x)}{\sin x}\mathrm{d}x \xlongequal{\text{命题 0.1(1)}} \pi. \end{aligned} \tag{1}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\mathrm{d}x = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记 $I_n = \int_0^\pi \frac{\sin^2(nx)}{\sin x}\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x}\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin x}\mathrm{d}x \\ &= \int_0^\pi \sin((2n+1)x)\,\mathrm{d}x = \frac{1}{2n+1}\cos((2n+1)x)\Big|_\pi^0 = \frac{2}{2n+1}. \end{aligned} \tag{2}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin x}\mathrm{d}x = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1}\frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1}\frac{2}{2k+1} = \sum_{k=1}^{n}\frac{2}{2k-1}.$$

□

0.1.2 区间再现

定理 0.1 (区间再现恒等式)

当下述积分有意义时, 我们有

1. $$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\mathrm{d}x.$$
2. $$\int_0^\infty f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^\infty f(x)\mathrm{d}x = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2}\right]\mathrm{d}x.$$

笔记 注意: 倒代换具有将 $[0,1]$ 转化为 $[1,+\infty)$ 的功能.

证明 证明是显然的. □

命题 0.2

证明

1. $\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
2. $\int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x - \frac{\pi}{2}\ln 2.$
3. $\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln 2.$

证明

1.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2} - x\right)\right] \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

2.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2} - x\right)\right] \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

3.

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &\xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2}\mathrm{d}\tan\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\cdot\ln(1+\tan\theta)}{\sec^2\theta}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right] \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\right] \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\frac{2}{1+\tan\theta}\right] \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \ln 2\mathrm{d}\theta = \frac{\pi}{8}\ln 2.
\end{aligned}$$

□

例题 0.2 计算

1. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+a^2}\mathrm{d}x, a>0.$
2. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+x+1}\mathrm{d}x.$
3. $\displaystyle\int_0^{1} \frac{\ln x}{\sqrt{x-x^2}}\mathrm{d}x.$

解

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2}\mathrm{d}x \xlongequal{x=at} \frac{1}{a}\int_0^{+\infty} \frac{\ln(at)}{1+t^2}\mathrm{d}t = \frac{1}{a}\int_0^{+\infty} \frac{\ln a}{1+t^2}\mathrm{d}t + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t = \frac{\pi\ln a}{2a} + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t. \quad (3)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t \xlongequal{t=\frac{1}{x}} \int_0^{+\infty} \frac{\ln\frac{1}{x}}{1+\frac{1}{x^2}}\frac{1}{x^2}\mathrm{d}x = \int_0^{+\infty} \frac{-\ln x}{1+x^2}\mathrm{d}x \Longrightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t = 0.$$

于是再结合(3)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2}\mathrm{d}x = \frac{\pi\ln a}{2a}.$$

2.

$$\int_0^{\infty} \frac{\ln x}{x^2+x+1}\mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_0^{\infty} \frac{-\ln t}{1+\frac{1}{t}+\frac{1}{t^2}}\mathrm{d}\frac{1}{t} = \int_0^{+\infty} \frac{-\ln t}{1+t+t^2}\mathrm{d}t \Longrightarrow \int_0^{\infty} \frac{\ln x}{x^2+x+1}\mathrm{d}x = 0.$$

3.

$$\begin{aligned}\int_0^1 \frac{\ln x}{\sqrt{x-x^2}}\mathrm{d}x &\xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}} \frac{\ln \sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}}\mathrm{d}\sin^2 y\\ &= 4\int_0^{\frac{\pi}{2}} \ln \sin y \mathrm{d}y \xlongequal{\text{命题 0.2}} 4\cdot\left(-\frac{\pi}{2}\ln 2\right) = -2\pi\ln 2.\end{aligned}$$

□

例题 0.3

1. 对 $n\in\mathbb{N}$, 计算 $\displaystyle\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}dx$.
2. $\displaystyle\int_{-\pi}^{\pi} \frac{x\sin x\arctan e^x}{1+\cos^2 x}dx$.
3. 对 $n\in\mathbb{N}$, 计算 $\displaystyle\int_0^{2\pi} \sin(\sin x+nx)dx$.

解

1.

$$\begin{aligned}&\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\mathrm{d}x = \int_{-\pi}^{0}\left[\frac{\sin(nx)}{(1+2^x)\sin x}+\frac{\sin(nx)}{(1+2^{-x})\sin x}\right]\mathrm{d}x = \int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\left(\frac{1}{1+2^x}+\frac{1}{1+2^{-x}}\right)\mathrm{d}x\\ &= \int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\cdot\frac{2+2^x+2^{-x}}{2+2^x+2^{-x}}\mathrm{d}x = \int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\mathrm{d}x = \int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x \xlongequal{\text{例题 0.1}} \begin{cases}0, n\text{为偶数}\\ \pi, n\text{为奇数}\end{cases}.\end{aligned}$$

2.

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{x\sin x\arctan e^x}{1+\cos^2 x}\mathrm{d}x &= \int_{-\pi}^{0}\left(\frac{x\sin x\arctan e^x}{1+\cos^2 x}+\frac{x\sin x\arctan e^{-x}}{1+\cos^2 x}\right)\mathrm{d}x = \int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}(\arctan e^x+\arctan e^{-x})\mathrm{d}x\\ &\xlongequal{\text{命题??(1)}} \int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}\cdot\frac{\pi}{2}\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x\\ &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\frac{x\sin x}{1+\cos^2 x}+\frac{(\pi-x)\sin x}{1+\cos^2 x}\right)\mathrm{d}x = \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x\\ &= \frac{\pi^2}{2}\arctan\cos x\Big|_{\frac{\pi}{2}}^{0} = \frac{\pi^2}{2}\cdot\frac{\pi}{4} = \frac{\pi^3}{8}.\end{aligned}$$

3.

$$\begin{aligned}\int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x &= \int_0^{2\pi}\sin[\sin(2\pi-x)+n(2\pi-x)]\mathrm{d}x\\ &= \int_0^{2\pi}\sin(-\sin x-nx)\mathrm{d}x = -\int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x\\ &\Longrightarrow \int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x = 0.\end{aligned}$$

□

0.1.3 化成含参积分/多元累次积分 (换序)

命题 0.3

证明:

(1) $\displaystyle\int_0^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

(2) $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

(3) $\int_0^{\infty}\sin x^2 dx, \int_0^{\infty}\cos x^2 dx = \sqrt{\frac{\pi}{8}}.$

笔记 本结果可以直接使用.

证明

(1) 注意到

$$\begin{aligned}
\left(\int_0^{+\infty} e^{-x^2} dx\right)^2 &= \left(\int_0^{+\infty} e^{-x^2} dx\right)\left(\int_0^{+\infty} e^{-y^2} dy\right) \xlongequal{\text{把}\int_0^{+\infty} e^{-y^2}dy\text{看作常数}} \int_0^{+\infty} e^{-x^2}\left(\int_0^{+\infty} e^{-y^2} dx\right) dy \\
&\xlongequal{\text{把}e^{-x^2}\text{看作常数}} \int_0^{+\infty}\left(\int_0^{+\infty} e^{-(x^2+y^2)} dx\right) dy \xlongequal{e^{-(x^2+y^2)}\text{连续}} \iint_{R^2} e^{-(x^2+y^2)} dxdy \\
&= \int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} re^{-r^2} dr = \frac{\pi}{2}\int_0^{+\infty} re^{-r^2} dr \\
&= \frac{\pi}{4}\int_0^{+\infty} e^{-r^2} dr^2 = \frac{\pi}{4}.
\end{aligned}$$

故 $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$

(2) 注意到

$$\int_0^{+\infty} \sin x e^{-yx}\, dx = \operatorname{Im}\int_0^{+\infty} e^{ix-yx}\, dx = \operatorname{Im}\int_0^{+\infty} e^{-(y-i)x}\, dx = \operatorname{Im}\frac{1}{y-i} = \operatorname{Im}\frac{y+i}{y^2+1} = \frac{1}{y^2+1}.$$

因此就有

$$\begin{aligned}
\int_0^{+\infty} \frac{\sin x}{x}\, dx &= \int_0^{+\infty} \sin x\left(\int_0^{+\infty} e^{-yx}\, dy\right) dx = \int_0^{+\infty} dy \int_0^{+\infty} \sin x e^{-yx}\, dx \\
&= \int_0^{+\infty} dy\left(\operatorname{Im}\int_0^{+\infty} e^{ix-yx}\right) dx = \int_0^{+\infty} \frac{1}{y^2+1}\, dy = \frac{\pi}{2}.
\end{aligned}$$

当然本题也可以直接利用分部积分计算 $\int_0^{+\infty} \sin x e^{-yx}\, dx = \frac{1}{y^2+1}.$

(3) 注意到

$$\int_0^{+\infty} e^{-ax^2}\, dx \xlongequal{x=\frac{t}{\sqrt{a}}} \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-t^2}\, dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a > 0.$$

并且 $-i = e^{-\frac{\pi}{2}i}$, 从而 $\sqrt{-i} = e^{-\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$. 于是

$$\begin{aligned}
\int_0^{+\infty} (\cos x^2 - i\sin x^2)\, dx &= \int_0^{+\infty} e^{-ix^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{i}} = \frac{1}{2}\sqrt{-i\pi} \\
&= \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i.
\end{aligned}$$

故

$$\begin{aligned}
\int_0^{+\infty} \cos x^2\, dx &= \operatorname{Re}\int_0^{+\infty} (\cos x^2 - i\sin x^2)\, dx = \operatorname{Re}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}, \\
\int_0^{+\infty} \sin x^2\, dx &= \operatorname{Im}\int_0^{+\infty} (\cos x^2 - i\sin x^2)\, dx = \operatorname{Im}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}.
\end{aligned}$$

□

例题 0.4 计算 $\int_0^1 \sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}dx\ (b > a > 0).$

证明

$$\begin{aligned}
\int_0^1 \sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}dx &= \int_0^1 \sin\ln\frac{1}{x}\left(\int_a^b x^y dy\right) dx = \int_a^b dy\int_0^1 x^y \sin\ln\frac{1}{x}dx \\
&\xlongequal{x=e^{-t}} \int_a^b dy\int_{+\infty}^0 e^{-ty}\sin t de^{-t} = \int_a^b dy\int_0^{+\infty} e^{-t(y+1)}\sin t dt
\end{aligned}$$

$$\xlongequal{\text{命题 0.3(2) 的证明过程}} \int_a^b \frac{1}{1+(y+1)^2} \mathrm{d}y = \arctan(b+1) - \arctan(a+1).$$

□