

0.1 微分学计算

0.1.1 单变量微分学计算

例题 0.1

- (1) 设 $f(x) = \prod_{k=0}^n (x-k)$. 对整数 $0 \leq j \leq n$, 求导数 $f'(j)$.
- (2) 设 $g(x) = \prod_{k=0}^n (e^x - k)$, 求 $g'(\ln j), j = 0, 1, 2, \dots, n$.

解

- (1) 解法一: 注意到 $f'(x) = \sum_{i=0}^n \prod_{\substack{k=0 \\ k \neq i}}^n (x-k)$, 故

$$\begin{aligned} f'(j) &= \sum_{i=0}^n \prod_{\substack{k=0 \\ k \neq i}}^n (j-k) = \prod_{\substack{k=0 \\ k \neq j}}^n (j-k) + \sum_{\substack{i=0 \\ i \neq j}}^n \prod_{\substack{k=0 \\ k \neq i}}^n (j-k) \\ &= (-1)^{n-j} j! (n-j)! + \sum_{\substack{i=0 \\ i \neq j}}^n (j-j) \prod_{\substack{k=0 \\ k \neq i, j}}^n (j-k) \\ &= (-1)^{n-j} j! (n-j)! \end{aligned}$$

解法二:

$$\begin{aligned} f'(j) &= \lim_{x \rightarrow j} \frac{f(x) - f(j)}{x - j} = \lim_{x \rightarrow j} \frac{\prod_{k=0}^n (x-k) - \prod_{k=0}^n (j-k)}{x - j} \\ &= \prod_{\substack{k=0 \\ k \neq j}}^n (j-k) + \lim_{x \rightarrow j} \frac{(j-j) \prod_{\substack{k=0 \\ k \neq j}}^n (j-k)}{x - j} \\ &= \prod_{\substack{k=0 \\ k \neq j}}^n (j-k) = (-1)^{n-j} j! (n-j)! \end{aligned}$$

- (2) 记 $f(x) = \prod_{i=0}^n (x-i)$, 则 $g(x) = f(e^x)$. 从而 $g'(x) = e^x f'(e^x)$, 于是由 (1) 可知

$$g'(\ln j) = j f'(j) = j \cdot (-1)^{n-j} j! (n-j)!$$

□

例题 0.2 对 $n \in \mathbb{N}$,

- (1) 设 $f(x) = \sin(ax), a \in \mathbb{R}$, 求 $f^{(n)}$.
- (2) 设 $f(x) = e^x \cos x$, 求 $f^{(n)}$.
- (3) 设 $f(x) = \frac{\ln x}{x}$, 求 $f^{(n)}$.
- (4) 设 $f(x) = \frac{1}{1-x^2}$, 求 $f^{(n)}$.
- (5) 设 $f(x) = \arctan x, x > 0$, 求 $f^{(n)}$.

解

- (1) 我们断言

$$f^{(n)}(x) = a^n \sin\left(ax + \frac{n}{2}\pi\right), \quad \forall n \in \mathbb{N}. \quad (1)$$

当 $n=0$ 时, 上式显然成立. 假设当 $n=k$ 时上式成立, 则

$$f^{(k+1)}(x) = a^{k+1} \cos\left(ax + \frac{k}{2}\pi\right) = a^{k+1} \sin\left(ax + \frac{k+1}{2}\pi\right).$$

故由数学归纳法可知 (1) 式成立.

(2) 由 Euler 公式可知, $\cos x = \operatorname{Re}(e^{ix})$, 从而 $f(x) = \operatorname{Re}[e^{(1+i)x}]$. 于是

$$f^{(n)}(x) = \operatorname{Re}[(1+i)^n e^{(1+i)x}], \quad \forall n \in \mathbb{N}.$$

注意到

$$1+i = \sqrt{2} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right) = \sqrt{2} e^{\frac{\pi}{4}i},$$

进而 $(1+i)^n = 2^{\frac{n}{2}} e^{\frac{n\pi}{4}i}$. 故

$$f^{(n)}(x) = \operatorname{Re} \left[2^{\frac{n}{2}} e^{\frac{n\pi}{4}i + (1+i)x} \right] = 2^{\frac{n}{2}} e^x \operatorname{Re} \left[e^{(x + \frac{n\pi}{4})i} \right] = 2^{\frac{n}{2}} e^x \cos \left(x + \frac{n\pi}{4} \right).$$

(3) 令 $y = f(x) = \frac{\ln x}{x}$, 则 $\ln x = xy$. 对 $\forall n \in \mathbb{N}$, 两边同时对 x 求 n 阶导, 得

$$(\ln x)^{(n)} = (xy)^{(n)} \iff \frac{(-1)^{n-1}(n-1)!}{x^n} = \sum_{k=0}^n x^{(k)} y^{(n-k)} = xy^{(n)} + ny^{(n-1)}.$$

从而对 $\forall n \in \mathbb{N}$, 都有

$$\begin{aligned} xy^{(n)} + ny^{(n-1)} &= \frac{(-1)^{n-1}(n-1)!}{x^n} \\ \iff (-1)^n x^{n+1} y^{(n)} - (-1)^{n-1} n x^n y^{(n-1)} &= -(n-1)! \\ \iff \frac{(-1)^n x^{n+1} y^{(n)}}{n!} - \frac{(-1)^{n-1} x^n y^{(n-1)}}{(n-1)!} &= -\frac{1}{n}. \end{aligned}$$

于是

$$\frac{(-1)^n x^{n+1} y^{(n)}}{n!} - xy = \sum_{k=1}^n \left(-\frac{1}{k} \right).$$

故

$$f^{(n)}(x) = y^{(n)} = \frac{(-1)^n n!}{x^{n+1}} \left(\sum_{k=1}^n \left(-\frac{1}{k} \right) - \ln x \right).$$


(4) 注意到 $f(x) = \frac{1}{2} \left(\frac{1}{1-x} + \frac{1}{1+x} \right)$, 则 $f^{(n)}(x) = \frac{n!}{2} \left(\frac{1}{(1-x)^{n+1}} + \frac{(-1)^n}{(1+x)^{n+1}} \right)$.

(5) 注意到 $f'(x) = \frac{1}{1+x^2} = \frac{1}{2i} \left(\frac{1}{x-i} - \frac{1}{x+i} \right)$, 故

$$\begin{aligned} f^{(n)}(x) &= \left(\frac{1}{1+x^2} \right)^{(n-1)} = \frac{(-1)^{n-1}(n-1)!}{2i} \left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n} \right] = \frac{(-1)^{n-1}(n-1)!}{2i(x^2+1)^n} [(x+i)^n - (x-i)^n] \\ &= \frac{(-1)^{n-1}(n-1)!}{2i(x^2+1)^{\frac{n}{2}}} \left[\left(\sqrt{1+x^2} e^{i \arctan \frac{1}{x}} \right)^n - \left(\sqrt{1+x^2} e^{-i \arctan \frac{1}{x}} \right)^n \right] \\ &= \frac{(-1)^{n-1}(n-1)!}{2i(x^2+1)^{\frac{n}{2}}} \left(e^{i n \arctan \frac{1}{x}} - e^{-i n \arctan \frac{1}{x}} \right) = \frac{(-1)^{n-1}(n-1)!}{2i(x^2+1)^{\frac{n}{2}}} \cdot 2i \cdot \sin \left(n \arctan \frac{1}{x} \right) \\ &= \frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{\frac{n}{2}}} \sin \left(n \arctan \frac{1}{x} \right). \end{aligned}$$

□

例题 0.3 设 $f(x) = x^2 \ln(x + \sqrt{1+x^2})$, 计算 $f^{(n)}(0), n \in \mathbb{N}$.

 **笔记** 此类问题都是通过背 Taylor 公式之后通过拼凑来得到 $f^{(n)}(0)$, 这是因为

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

解 注意到

$$\begin{aligned} \left[\ln \left(x + \sqrt{1+x^2} \right) \right]' &= [\operatorname{arcsinh} x]' = \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} \\ &\stackrel{\text{广义二项式定理}}{=} \sum_{n=0}^{\infty} C_{-\frac{1}{2}}^n x^{2n}, \end{aligned}$$

于是

$$\begin{aligned} \ln \left(x + \sqrt{1+x^2} \right) &= \sum_{n=0}^{\infty} \frac{C_{-\frac{1}{2}}^n}{2n+1} x^{2n+1} = x + \sum_{n=0}^{\infty} \frac{C_{-\frac{1}{2}}^n}{2n+1} x^{2n+1} \\ &= x + \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right) \left(-\frac{1}{2}-1\right) \cdots \left(-\frac{1}{2}-n+1\right)}{(2n+1) \cdot n!} x^{2n+1} \\ &= x + \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n+1) \cdot n!} x^{2n+1}. \end{aligned}$$

从而 $f(x) = x^3 + \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n+1) \cdot n!} x^{2n+3}$, 因此

$$f^{(n)}(0) = \begin{cases} 6, & n=3 \\ \frac{(-1)^m (2m-1)!! (2m+3)!}{m! \cdot 2^m (2m+1)}, & n=2m+3, m=1, 2, \cdots \\ 0, & \text{其他} \end{cases}$$


□

命题 0.1

$$\arcsin^2 x = \sum_{n=1}^{\infty} \frac{2^{2n-1} ((n-1)!)^2}{(2n)!} x^{2n}, \quad x \in (-1, 1).$$

例题 0.4 生成级数或者建立递推法求解高阶导数值 对 $n \in \mathbb{N}_0$,

- (1) 设 $f(x) = \arcsin^2 x$, 求 $f^{(n)}(0)$.
- (2) 设 $f(x) = \arcsin x \cdot \arccos x$, 求 $f^{(n)}(0)$.
- (3) 设 $f(x) = (x + \sqrt{x^2+1})^m, m \in \mathbb{N}$, 求 $f^{(n)}(0)$.
- (4) 设 $f(x) = \arctan^2 x$, 求 $f^{(n)}(0)$.

 **笔记** 此类问一般是先建立函数满足的微分方程, 然后用乘积求导法则或者形式幂级数对比系数来得到导数的递推, 从而完成了证明.

解

- (1) **解法一:** 注意到

$$f'(x) = \frac{2 \arcsin x}{\sqrt{1-x^2}} \iff \sqrt{1-x^2} f' = 2 \arcsin x,$$

令 $y = f(x)$, 则对上式两边同时求导得

$$-\frac{x}{\sqrt{1-x^2}} f' + \sqrt{1-x^2} f'' = \frac{2}{\sqrt{1-x^2}} \iff -xy' + (1-x^2)y'' = 2.$$

再对上式两边同时对 x 求 $n (n \geq 2)$ 阶导, 得

$$\begin{aligned} &[-xy' + (1-x^2)y'']^{(n)} = 2^{(n)} \\ \iff &-[ny^{(n)} + xy^{(n+1)}] + \left[\binom{n}{2} \cdot (-2)y^{(n)} + \binom{n}{1} (-2x)y^{(n+1)} + (1-x^2)y^{(n+2)} \right] = 0 \end{aligned}$$

将 $x=0$ 代入上式得

$$f^{(n+2)}(0) = n^2 f^{(n)}(0), \quad \forall n \geq 2. \quad (2)$$

显然上式对 $n=1$ 也成立. 又注意到 $f''(0)=2$, 因此对 $\forall n \in \mathbb{N}_1$, 由(2)式可得

$$\frac{f^{(2n+2)}(0)}{f^{(2n)}(0)} = 4n^2 \Rightarrow \frac{f^{(2n+2)}(0)}{f^{(2)}(0)} = \prod_{i=1}^n 4i^2 \Rightarrow f^{(2n+2)}(0) = 2^{2n+1}(n!)^2.$$

显然上式对 $n=0$ 也成立. 故

$$f^{(2n+2)}(0) = 2^{2n+1}(n!)^2, \forall n \in \mathbb{N}_0.$$

又 $f'''(0)=0$, 故由(2)式可得

$$f^{(2n-1)}(0) = (2n-1)^2 f^{(2n-3)}(0) = \cdots = [(2n-1)!!]^2 f^{(3)}(0) = 0, \forall n \in \mathbb{N}_1.$$

解法二: 注意到

$$f'(x) = \frac{2 \arcsin x}{\sqrt{1-x^2}} \iff \sqrt{1-x^2} f' = 2 \arcsin x,$$

令 $y = f(x)$, 则对上式两边同时求导得

$$-\frac{x}{\sqrt{1-x^2}} f' + \sqrt{1-x^2} f'' = \frac{2}{\sqrt{1-x^2}} \iff -xy' + (1-x^2)y'' = 2. \quad (3)$$

因为 $f \in C^\infty(\mathbb{R})$, 所以由 Taylor 公式可知

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

其中 $a_n = \frac{f^{(n)}(0)}{n!}$, $n \in \mathbb{N}_0$. 再将上式代入(3)式可得

$$\begin{aligned} 2 &= -\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n \\ &= -\sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n \\ &= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n - n(n-1) a_n] x^n. \end{aligned}$$

比较上式两边系数, 得对 $\forall n \in \mathbb{N}_1$, 都有

$$\begin{aligned} (n+2)(n+1) a_{n+2} - n a_n - n(n-1) a_n &= 0 \\ \iff (n+2)(n+1) \cdot \frac{f^{(n+2)}(0)}{(n+2)!} - n \cdot \frac{f^{(n)}(0)}{n!} - n(n-1) \cdot \frac{f^{(n)}(0)}{n!} &= 0 \\ \iff f^{(n+2)}(0) &= n^2 f^{(n)}(0). \end{aligned} \quad (4)$$

又 $f''(0)=2$, 因此对 $\forall n \in \mathbb{N}_1$, 由(4)式可得

$$\frac{f^{(2n+2)}(0)}{f^{(2n)}(0)} = 4n^2 \Rightarrow \frac{f^{(2n+2)}(0)}{f^{(2)}(0)} = \prod_{i=1}^n 4i^2 \Rightarrow f^{(2n+2)}(0) = 2^{2n+1}(n!)^2.$$

显然上式对 $n=0$ 也成立. 故

$$f^{(2n+2)}(0) = 2^{2n+1}(n!)^2, \forall n \in \mathbb{N}_0.$$

又 $f'''(0)=0$, 故由(4)式可得

$$f^{(2n-1)}(0) = (2n-1)^2 f^{(2n-3)}(0) = \cdots = [(2n-1)!!]^2 f^{(3)}(0) = 0, \forall n \in \mathbb{N}_1.$$

(2)

(3)

(4)

□

命题 0.2

设 f 在 a 处 $n+1$ 阶连续可导的, 证明:

$$\lim_{x \rightarrow a} \frac{d^n}{dx^n} \left[\frac{f(x) - f(a)}{x - a} \right] = \frac{f^{(n+1)}(a)}{n+1}.$$

▲

注 不妨设 $a=0, f(a)=0$ 的原因: 先证不妨设 $f(a)=0$ 成立. 假设 $f(a) \neq 0$ 时, 令 $g(x) = f(x) - f(a)$, 则 $g(a)=0$, 从而由假设可知

$$\lim_{x \rightarrow a} \frac{d^n}{dx^n} \left[\frac{f(x) - f(a)}{x - a} \right] = \lim_{x \rightarrow a} \frac{d^n}{dx^n} \left[\frac{g(x)}{x - a} \right] = \frac{g^{(n+1)}(a)}{n+1} = \frac{f^{(n+1)}(a)}{n+1}.$$

故可以不妨设 $f(a)=0$.

再证不妨设 $a=0$ 成立. 假设 $a \neq 0$ 时, 令 $g(x) = f(x+a)$, 则由假设可知

$$\lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left[\frac{g(x)}{x} \right] = \frac{g^{(n+1)}(0)}{n+1}.$$

从而

$$\begin{aligned} \lim_{x \rightarrow a} \frac{d^n}{dx^n} \left[\frac{f(x)}{x - a} \right] &= \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left[\frac{f(x+a)}{x} \right] = \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left[\frac{g(x)}{x} \right] \\ &= \frac{g^{(n+1)}(0)}{n+1} = \frac{f^{(n+1)}(a)}{n+1}. \end{aligned}$$

故可以不妨设 $a=0$.

证明 不妨设 $a=0, f(a)=0$, 从而

$$\frac{d^n}{dx^n} \left[\frac{f(x)}{x} \right] = \sum_{k=0}^n C_n^k f^{(k)}(x) \frac{(-1)^{n-k}(n-k)!}{x^{n-k+1}} = \frac{n!(-1)^n}{x^{n+1}} \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x)(-x)^k.$$

于是由 L'Hospital 法则可得


$$\begin{aligned} \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left[\frac{f(x)}{x} \right] &= n!(-1)^n \lim_{x \rightarrow 0} \frac{\sum_{k=0}^n \frac{1}{k!} f^{(k)}(x)(-x)^k}{x^{n+1}} \\ &\stackrel{\text{L'Hospital 法则}}{=} n!(-1)^n \lim_{x \rightarrow 0} \frac{\sum_{k=0}^n \frac{1}{k!} f^{(k+1)}(x)(-x)^k - \sum_{k=1}^n \frac{1}{(k-1)!} f^{(k)}(x)(-x)^{k-1}}{(n+1)x^n} \\ &= n!(-1)^n \lim_{x \rightarrow 0} \frac{\sum_{k=0}^n \frac{1}{k!} f^{(k+1)}(x)(-x)^k - \sum_{k=0}^{n-1} \frac{1}{(k)!} f^{(k+1)}(x)(-x)^k}{(n+1)x^n} \\ &= n!(-1)^n \lim_{x \rightarrow 0} \frac{\frac{1}{n!} f^{(n+1)}(x)(-x)^n}{(n+1)x^n} \\ &= \frac{f^{(n+1)}(0)}{n+1}. \end{aligned}$$

□

例题 0.5 设 $f \in C^\infty(\mathbb{R}), n \in \mathbb{N}$ 满足

$$f^{(j)}(0) = 0, j = 0, 1, 2, \dots, n-1, f^{(n)}(0) \neq 0.$$

证明: $g(x) = \begin{cases} \frac{f(x)}{x^n}, & x \neq 0 \\ \frac{f^{(n)}(0)}{n!}, & x = 0 \end{cases}$ 在 \mathbb{R} 上无穷次可微.

 **笔记** 本题不能对 Taylor 公式的 peano 余项求导来说明 g 可微分性, 这是不严格的.

证明 当 $n=0$ 时, $g=f \in C^\infty(\mathbb{R})$ 显然成立. 假设命题对 $n \in \mathbb{N}$ 成立, 考虑 $n+1$ 的情形. 令 $h(x) = \frac{f(x)}{x}$, 则

$$\frac{f(x)}{x^{n+1}} = \frac{\frac{f(x)}{x}}{x^n} = \frac{h(x)}{x^n}. \quad (100.2412424234-2-4-:::-24..119)$$

对 $\forall k \in \mathbb{N}$, 由命题 0.2 可知

$$\lim_{x \rightarrow 0} h^{(k)}(x) = \lim_{x \rightarrow 0} \left[\frac{f(x)}{x} \right]^{(k)} = \frac{f^{(k+1)}(0)}{k+1}.$$

于是由导数极限定理可知 $h^{(k)}(0) = \frac{f^{(k+1)}(0)}{k+1}$, $\forall k \in \mathbb{N}$, 故 h 在 $x = 0$ 处无穷次可微. 又由 $f \in C^\infty(\mathbb{R})$, 从而 $h \in C^\infty(\mathbb{R} \setminus \{0\})$, 故 $h \in C^\infty(\mathbb{R})$. 于是

$$\begin{aligned} h^{(j)}(0) &= \lim_{x \rightarrow 0} h^{(j)}(x) = \frac{f^{(j+1)}(0)}{j+1} = 0, \quad 0 \leq j \leq n-1, \\ h^{(n)}(0) &= \lim_{x \rightarrow 0} h^{(n)}(x) = \frac{f^{(n+1)}(0)}{n+1} \neq 0. \end{aligned} \quad (5)$$

因此 $h(x)$ 满足归纳假设条件, 进而由归纳假设及 (4)(5) 式可知

$$g(x) = \begin{cases} \frac{f(x)}{x^{n+1}}, & x \neq 0 \\ \frac{f^{(n+1)}(0)}{(n+1)!}, & x = 0 \end{cases} = \begin{cases} \frac{h(x)}{x^n}, & x \neq 0 \\ \frac{h^{(n)}(0)}{n!}, & x = 0 \end{cases} \in C^\infty(\mathbb{R}).$$

因此由数学归纳法可知, 结论成立. □