

## 0.1 定积分

### 0.1.1 建立积分递推

**例题 0.1** 计算  $\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx, n \in \mathbb{N}$ .

**证明** 利用分部积分和和差化积公式可得

$$\begin{aligned}
 I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx) dx \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)] dx \\
 &= \frac{I_{n-1}}{2} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x [\sin(nx) \cos x + \cos(nx) \sin x] dx \\
 &= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(nx) d \cos x \\
 &= \frac{I_{n-1} + I_n}{2} - \frac{1}{2n} \int_0^{\frac{\pi}{2}} \cos(nx) d \cos^n x \\
 &= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx \\
 &= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
 &= \frac{I_{n-1}}{2} + \frac{1}{2n}.
 \end{aligned}$$

故  $I_n = \frac{I_{n-1}}{2} + \frac{1}{2n}$ , 则两边同乘  $2^n$  (强行裂项)

$$2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \dots$$

又注意到  $I_0 = 0$ , 从而

$$2^n I_n = 0 + \sum_{k=1}^n \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n} \sum_{k=1}^n \frac{2^{k-1}}{k}.$$


□

#### 命题 0.1

**证明:**

- (1)  $\int_0^{\pi} \frac{\sin(nx)}{\sin x} dx = \begin{cases} 0, & n \text{ 为偶数} \\ \pi, & n \text{ 为奇数} \end{cases}.$
- (2)  $\int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx = n\pi.$
- (3)  $\int_0^{\pi} \frac{\sin^2(nx)}{\sin x} dx = \sum_{k=1}^n \frac{2}{2k-1}.$

♣

 **笔记** 提示:  $\sin^2 x - \sin^2 y = \sin(x-y) \sin(x+y)$  (证明见命题??).

**证明**

(1) 记  $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx$ , 则

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin((n+2)x) - \sin(nx)}{\sin x} dx = \int_0^{\pi} \frac{2 \cos((n+1)x) \sin x}{\sin x} dx = 2 \int_0^{\pi} \cos((n+1)x) dx = 0.$$

于是

$$\int_0^{\pi} \frac{\sin(nx)}{\sin x} dx = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n \text{ 为偶数} \\ I_1, & n \text{ 为奇数} \end{cases} = \begin{cases} 0, & n \text{ 为偶数} \\ \pi, & n \text{ 为奇数} \end{cases}.$$

(2) 记  $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx$ , 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^{\pi} \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x} dx = \int_0^{\pi} \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x} dx \\ &= \int_0^{\pi} \frac{\sin((2n+1)x)}{\sin x} dx \stackrel{\text{命题 0.1(1)}}{=} \pi. \end{aligned} \quad (1)$$

于是

$$\int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记  $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin x} dx$ , 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^{\pi} \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x} dx = \int_0^{\pi} \frac{\sin x \cdot \sin((2n+1)x)}{\sin x} dx \\ &= \int_0^{\pi} \sin((2n+1)x) dx = \frac{1}{2n+1} \cos((2n+1)x) \Big|_0^{\pi} = \frac{2}{2n+1}. \end{aligned} \quad (2)$$

于是

$$\int_0^{\pi} \frac{\sin^2(nx)}{\sin x} dx = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1} \frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1} \frac{2}{2k+1} = \sum_{k=1}^n \frac{2}{2k-1}.$$

□

**例题 0.2** 设  $a > 1$ , 计算积分  $\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x) dx$ .

**注** 很多情况下不需求出被积函数的原函数, 只需充分利用换元、分部积分以及被积函数的性质, 即可求出积分的值. 见下述解法二.

**解 解法一:** 设  $a_0 = a > 1$ . 构造数列如下:

$$a_{n+1} = 2a_n^2 - 1 \quad (n = 0, 1, \cdots),$$

则由例题??可知, 存在  $x_0 > 0$  使得

$$a_0 = \operatorname{ch}(x_0), \quad a_n = \operatorname{ch}(2^n x_0),$$

其中  $\operatorname{ch}(x) = \frac{1}{2}(e^x + e^{-x})$ . 可以解得

$$x_0 = \ln \left( a_0 + \sqrt{a_0^2 - 1} \right). \quad (3)$$

故

$$a_n = \frac{e^{2^n x_0} + e^{-2^n x_0}}{2}.$$

设

$$I_n = \int_0^{\pi} \ln(a_n - \cos x) dx,$$

则

$$\begin{aligned} I_0 &= \int_0^{\pi} \ln(a_0 - \cos x) dx = \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x) dx + \int_{\frac{\pi}{2}}^{\pi} \ln(a_0 - \cos x) dx \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x) dx + \int_0^{\frac{\pi}{2}} \ln(a_0 + \cos x) dx \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0^2 - \cos^2 x) dx = \int_0^{\frac{\pi}{2}} \ln \left( a_0^2 - \frac{1 + \cos 2x}{2} \right) dx \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \ln \left( \frac{a_1 - \cos 2x}{2} \right) dx = \frac{1}{2} \int_0^{\pi} \ln \left( \frac{a_1 - \cos x}{2} \right) dx = \frac{1}{2} I_1 - \frac{\pi}{2} \ln 2.$$

同理, 有

$$I_n = \frac{1}{2} I_{n+1} - \frac{\pi}{2} \ln 2. \quad (4)$$

由此递推公式, 可得

$$I_0 = \frac{1}{2^n} I_n - \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \right) \frac{\pi}{2} \ln 2. \quad (5)$$

因为

$$\begin{aligned} I_n &= \int_0^{\pi} \ln(a_n - \cos x) dx = \int_0^{\pi} \ln \left( \frac{e^{2^n x_0} + e^{-2^n x_0}}{2} - \cos x \right) dx \\ &= 2^n x_0 \pi + \int_0^{\pi} \ln \left( \frac{1 + e^{-2^{n+1} x_0}}{2} - e^{-2^n x_0} \cos x \right) dx, \end{aligned}$$

所以

$$\frac{1}{2^n} I_n \rightarrow x_0 \pi \quad (n \rightarrow +\infty).$$

故从式 (5) 可得

$$I_0 = x_0 \pi - \pi \ln 2 = \pi \ln \left( \frac{a_0 + \sqrt{a_0^2 - 1}}{2} \right),$$

即所求的积分为

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x) dx = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x) dx = \pi \ln \left( \frac{a + \sqrt{a^2 - 1}}{2} \right).$$

解法二: 我们有

$$F(a) = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x) dx = \int_0^{\pi} \ln(a - \cos x) dx.$$

由定理??, 关于  $a$  求导得到

$$F'(a) = \int_0^{\pi} \frac{1}{a - \cos x} dx \stackrel{\text{万能公式}}{=} \int_0^{+\infty} \frac{2}{a(1+t^2) - (1-t^2)} dt = \frac{\pi}{\sqrt{a^2 - 1}}, \quad a > 1.$$

因此

$$F(a) = \int_1^a F'(a) da = \pi \ln(a + \sqrt{a^2 - 1}) + C, \quad a > 1.$$

结合

$$F(1) = 2 \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\pi \ln 2.$$

可得

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x) dx = \pi \ln \left( \frac{a + \sqrt{a^2 - 1}}{2} \right), \quad a > 1.$$

□

## 0.1.2 区间再现

### 定理 0.1 (区间再现恒等式)

当下述积分有意义时, 我们有

1.

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] dx = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] dx.$$

2.

$$\int_0^{\infty} f(x) dx = \int_0^1 f(x) dx + \int_1^{\infty} f(x) dx = \int_0^1 \left[ f(x) + \frac{f(\frac{1}{x})}{x^2} \right] dx.$$



**笔记** 注意: 倒代换具有将  $[0, 1]$  转化为  $[1, +\infty)$  的功能.

**证明** 证明是显然的.(第 1 问中最后一个等号是由轴对称得到的)

□

### 命题 0.2

证明

$$1. \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{2} \ln 2.$$

$$2. \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2.$$

$$3. \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2.$$



**证明**

1.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \ln \sin x dx = \int_0^{\frac{\pi}{4}} \left[ \ln \cos x + \ln \left( \frac{\pi}{2} - x \right) \right] dx \\ &= \int_0^{\frac{\pi}{4}} \ln (\cos x \sin x) dx = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2} dx + \int_0^{\frac{\pi}{4}} \ln (\sin 2x) dx \\ &= -\frac{\pi}{4} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} I \\ \Rightarrow I &= \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2. \end{aligned}$$

2.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \ln \cos x dx = \int_0^{\frac{\pi}{2}} \ln \sin x dx = \int_0^{\frac{\pi}{4}} \left[ \ln \cos x + \ln \left( \frac{\pi}{2} - x \right) \right] dx \\ &= \int_0^{\frac{\pi}{4}} \ln (\cos x \sin x) dx = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2} dx + \int_0^{\frac{\pi}{4}} \ln (\sin 2x) dx \\ &= -\frac{\pi}{4} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} I \\ \Rightarrow I &= \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2. \end{aligned}$$

3.

$$\begin{aligned} \int_0^1 \frac{\ln(1+x)}{1+x^2} dx &\stackrel{x=\tan \theta}{=} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{1+\tan^2 \theta} d \tan \theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta \cdot \ln(1+\tan \theta)}{\sec^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta = \int_0^{\frac{\pi}{8}} \left[ \ln(1+\tan \theta) + \ln \left( 1 + \tan \left( \frac{\pi}{4} - \theta \right) \right) \right] d\theta \\ &= \int_0^{\frac{\pi}{8}} \left[ \ln(1+\tan \theta) + \ln \left( 1 + \frac{1-\tan \theta}{1+\tan \theta} \right) \right] d\theta \\ &= \int_0^{\frac{\pi}{8}} \left[ \ln(1+\tan \theta) + \ln \frac{2}{1+\tan \theta} \right] d\theta \\ &= \int_0^{\frac{\pi}{8}} \ln 2 d\theta = \frac{\pi}{8} \ln 2. \end{aligned}$$

□

**例题 0.3** 计算

1.  $\int_0^{\infty} \frac{\ln x}{x^2 + a^2} dx, a > 0.$
2.  $\int_0^{\infty} \frac{\ln x}{x^2 + x + 1} dx.$
3.  $\int_0^1 \frac{\ln x}{\sqrt{x - x^2}} dx.$

解

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2 + a^2} dx \stackrel{x=at}{=} \frac{1}{a} \int_0^{+\infty} \frac{\ln(at)}{1+t^2} dt = \frac{1}{a} \int_0^{+\infty} \frac{\ln a}{1+t^2} dt + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} dt = \frac{\pi \ln a}{2a} + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} dt. \quad (6)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2} dt \stackrel{t=\frac{1}{x}}{=} \int_0^{+\infty} \frac{\ln \frac{1}{x}}{1+\frac{1}{x^2}} \frac{1}{x^2} dx = \int_0^{+\infty} \frac{-\ln x}{1+x^2} dx \Rightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2} dt = 0.$$

于是再结合(6)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2 + a^2} dx = \frac{\pi \ln a}{2a}.$$

2.

$$\int_0^{\infty} \frac{\ln x}{x^2 + x + 1} dx \stackrel{x=\frac{1}{t}}{=} \int_0^{\infty} \frac{-\ln t}{1+\frac{1}{t}+\frac{1}{t^2}} d\frac{1}{t} = \int_0^{\infty} \frac{-\ln t}{1+t+t^2} dt \Rightarrow \int_0^{\infty} \frac{\ln x}{x^2 + x + 1} dx = 0.$$

3.

$$\begin{aligned} \int_0^1 \frac{\ln x}{\sqrt{x-x^2}} dx &\stackrel{x=\sin^2 y}{=} \int_0^{\frac{\pi}{2}} \frac{\ln \sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}} d\sin^2 y \\ &= 4 \int_0^{\frac{\pi}{2}} \ln \sin y dy \stackrel{\text{命题 0.2}}{=} 4 \cdot \left(-\frac{\pi}{2} \ln 2\right) = -2\pi \ln 2. \end{aligned}$$

□

## 例题 0.4

1. 对  $n \in \mathbb{N}$ , 计算  $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx.$
2.  $\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} dx.$
3. 对  $n \in \mathbb{N}$ , 计算  $\int_0^{2\pi} \sin(\sin x + nx) dx.$

解

1.

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx &= \int_{-\pi}^0 \left[ \frac{\sin(nx)}{(1+2^x)\sin x} + \frac{\sin(nx)}{(1+2^{-x})\sin x} \right] dx = \int_{-\pi}^0 \frac{\sin(nx)}{\sin x} \left( \frac{1}{1+2^x} + \frac{1}{1+2^{-x}} \right) dx \\ &= \int_{-\pi}^0 \frac{\sin(nx)}{\sin x} \cdot \frac{2+2^x+2^{-x}}{2+2^x+2^{-x}} dx = \int_{-\pi}^0 \frac{\sin(nx)}{\sin x} dx = \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx \stackrel{\text{例题 0.1}}{=} \begin{cases} 0, n \text{ 为偶数} \\ \pi, n \text{ 为奇数} \end{cases}. \end{aligned}$$

2.

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} dx &= \int_{-\pi}^0 \left( \frac{x \sin x \arctan e^x}{1+\cos^2 x} + \frac{x \sin x \arctan e^{-x}}{1+\cos^2 x} \right) dx = \int_{-\pi}^0 \frac{x \sin x}{1+\cos^2 x} (\arctan e^x + \arctan e^{-x}) dx \\ &\stackrel{\text{命题 ??(1)}}{=} \int_{-\pi}^0 \frac{x \sin x}{1+\cos^2 x} \cdot \frac{\pi}{2} dx = \frac{\pi}{2} \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \left( \frac{x \sin x}{1+\cos^2 x} + \frac{(\pi-x) \sin x}{1+\cos^2 x} \right) dx = \frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx \\ &= \frac{\pi^2}{2} \arctan \cos x \Big|_{\frac{\pi}{2}}^0 = \frac{\pi^2}{2} \cdot \frac{\pi}{4} = \frac{\pi^3}{8}. \end{aligned}$$

3.

$$\begin{aligned}
\int_0^{2\pi} \sin(\sin x + nx) dx &= \int_0^{2\pi} \sin[\sin(2\pi - x) + n(2\pi - x)] dx \\
&= \int_0^{2\pi} \sin(-\sin x - nx) dx = - \int_0^{2\pi} \sin(\sin x + nx) dx \\
&\Rightarrow \int_0^{2\pi} \sin(\sin x + nx) dx = 0.
\end{aligned}$$

□

**例题 0.5** 计算积分  $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$ .

**注** 此例中无法求出被积函数的原函数, 但通过积分的性质仍可算出积分的值.

**解 解法一:** 作变换  $x = \tan \varphi$ , 则  $d\varphi = \frac{1}{1+x^2} dx$ , 且当  $x = 0$  时,  $\varphi = 0$ ; 当  $x = 1$  时,  $\varphi = \frac{\pi}{4}$ . 于是

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos \varphi + \sin \varphi}{\cos \varphi}\right) d\varphi \\
&= \int_0^{\frac{\pi}{4}} \left\{ \ln\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos \varphi + \frac{1}{\sqrt{2}}\sin \varphi\right)\right) - \ln(\cos \varphi) \right\} d\varphi \\
&= \int_0^{\frac{\pi}{4}} \left\{ \ln \sqrt{2} + \ln\left(\sin\left(\varphi + \frac{\pi}{4}\right)\right) - \ln(\cos \varphi) \right\} d\varphi \\
&= \frac{\pi}{8} \ln 2 + \int_0^{\frac{\pi}{4}} \ln\left(\sin\left(\varphi + \frac{\pi}{4}\right)\right) d\varphi - \int_0^{\frac{\pi}{4}} \ln(\cos \varphi) d\varphi.
\end{aligned}$$

因为

$$\int_0^{\frac{\pi}{4}} \ln\left(\sin\left(\varphi + \frac{\pi}{4}\right)\right) d\varphi \stackrel{\varphi = \frac{\pi}{4} - t}{=} - \int_{\frac{\pi}{4}}^0 \ln\left(\sin\left(\frac{\pi}{2} - t\right)\right) dt = \int_0^{\frac{\pi}{4}} \ln(\cos t) dt,$$

所以  $I = \frac{\pi}{8} \ln 2$ .

**解法二:** 考虑含参量积分

$$\varphi(\alpha) = \int_0^1 \frac{\ln(1+\alpha x)}{1+x^2} dx, \quad \alpha \in [0, 1].$$

显然  $\varphi(0) = 0, \varphi(1) = I$ , 且函数  $\frac{\ln(1+\alpha x)}{1+x^2}$  在  $R = [0, 1] \times [0, 1]$  上满足定理?? 的条件, 于是

$$\varphi'(\alpha) = \int_0^1 \frac{x}{(1+x^2)(1+\alpha x)} dx.$$

因为

$$\frac{x}{(1+x^2)(1+\alpha x)} = \frac{1}{1+\alpha^2} \left( \frac{\alpha+x}{1+x^2} - \frac{\alpha}{1+\alpha x} \right),$$

所以

$$\begin{aligned}
\varphi'(\alpha) &= \frac{1}{1+\alpha^2} \left( \int_0^1 \frac{\alpha}{1+x^2} dx + \int_0^1 \frac{x}{1+x^2} dx - \int_0^1 \frac{\alpha}{1+\alpha x} dx \right) \\
&= \frac{1}{1+\alpha^2} \left[ \alpha \arctan x \Big|_0^1 + \frac{1}{2} \ln(1+x^2) \Big|_0^1 - \ln(1+\alpha x) \Big|_0^1 \right] \\
&= \frac{1}{1+\alpha^2} \left[ \alpha \cdot \frac{\pi}{4} + \frac{1}{2} \ln 2 - \ln(1+\alpha) \right].
\end{aligned}$$

因此

$$\begin{aligned}
\int_0^1 \varphi'(\alpha) d\alpha &= \int_0^1 \frac{1}{1+\alpha^2} \left[ \frac{\pi}{4} \alpha + \frac{1}{2} \ln 2 - \ln(1+\alpha) \right] d\alpha \\
&= \frac{\pi}{8} \ln(1+\alpha^2) \Big|_0^1 + \frac{1}{2} \ln 2 \arctan \alpha \Big|_0^1 - \varphi(1) \\
&= \frac{\pi}{8} \ln 2 + \frac{\pi}{8} \ln 2 - \varphi(1)
\end{aligned}$$

$$= \frac{\pi}{4} \ln 2 - \varphi(1).$$

另一方面,

$$\int_0^1 \varphi'(\alpha) d\alpha = \varphi(1) - \varphi(0) = \varphi(1),$$

所以  $I = \varphi(1) = \frac{\pi}{8} \ln 2$ . □

### 0.1.3 Frullani(傅汝兰尼) 积分

#### 定理 0.2 (Frullani(傅汝兰尼) 积分)

设  $f \in C(0, +\infty)$ .

1. 若存在极限

$$\lim_{x \rightarrow 0^+} f(x), \lim_{x \rightarrow +\infty} f(x), \quad (7)$$

则对  $a, b > 0$  有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \left[ \lim_{x \rightarrow 0^+} f(x) - \lim_{x \rightarrow +\infty} f(x) \right] \ln \frac{b}{a}.$$

2. 若存在极限和积分

$$\lim_{x \rightarrow 0^+} f(x) = \alpha, \int_A^\infty \frac{f(x)}{x} dx. \quad (8)$$

则对  $a, b > 0$ , 有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \alpha \ln \frac{b}{a}.$$

3. 若存在极限和积分

$$\lim_{x \rightarrow +\infty} f(x) = \alpha, \int_0^1 \frac{f(x)}{x} dx. \quad (9)$$

则对  $a, b > 0$ , 有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \alpha \ln \frac{a}{b}.$$

4. 若  $f$  是周期  $T > 0$  函数且  $\lim_{x \rightarrow 0^+} f(x)$  存在, 则对  $a, b > 0$  有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \left[ \lim_{x \rightarrow 0^+} f(x) - \frac{1}{T} \int_0^T f(x) dx \right] \ln \frac{b}{a}.$$

5. 若  $f$  满足  $\lim_{x \rightarrow 0^+} f(x), \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(y) dy$  存在, 则对  $a, b > 0$  有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \left[ \lim_{x \rightarrow 0^+} f(x) - \lim_{x \rightarrow +\infty} \frac{1}{x} \int_0^x f(y) dy \right] \ln \frac{b}{a}.$$



**笔记** 傅汝兰尼积分有诸多变种, 无需记忆具体表达式, 知道有大概这么一个东西即可.

**证明** 不妨设  $b > a$ .

1. 给定  $A > \delta > 0$ , 考虑

$$\begin{aligned} \int_\delta^A \frac{f(ax) - f(bx)}{x} dx &= \int_\delta^A \frac{f(ax)}{x} dx - \int_\delta^A \frac{f(bx)}{x} dx \\ &= \int_{a\delta}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{bA} \frac{f(x)}{x} dx \\ &= \int_{bA}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{a\delta} \frac{f(x)}{x} dx \\ &\stackrel{\text{积分中值定理}}{=} f(\theta_1) \int_{bA}^{aA} \frac{1}{x} dx - f(\theta_2) \int_{b\delta}^{a\delta} \frac{1}{x} dx, \end{aligned}$$

这里  $\theta_1 \in (aA, bA), \theta_2 \in (a\delta, b\delta)$ , 于是让  $A \rightarrow +\infty, \delta \rightarrow 0^+$ , 由(??), 我们知

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \left[ \lim_{x \rightarrow 0^+} f(x) - \lim_{x \rightarrow +\infty} f(x) \right] \ln \frac{b}{a}.$$

2. 给定  $A > \delta > 0$ , 考虑

$$\begin{aligned} \int_\delta^A \frac{f(ax) - f(bx)}{x} dx &= \int_\delta^A \frac{f(ax)}{x} dx - \int_\delta^A \frac{f(bx)}{x} dx \\ &= \int_{a\delta}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{bA} \frac{f(x)}{x} dx \\ &= \int_{bA}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{a\delta} \frac{f(x)}{x} dx \\ &\stackrel{\text{积分中值定理}}{=} \int_{bA}^{aA} \frac{f(x)}{x} dx - f(\theta) \int_{b\delta}^{a\delta} \frac{1}{x} dx, \end{aligned}$$

这里  $\theta \in (a\delta, b\delta)$ , 于是让  $A \rightarrow +\infty, \delta \rightarrow 0^+$ , 由(??), 我们知

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \alpha \ln \frac{b}{a}.$$

3. 给定  $A > \delta > 0$ , 考虑

$$\begin{aligned} \int_\delta^A \frac{f(ax) - f(bx)}{x} dx &= \int_\delta^A \frac{f(ax)}{x} dx - \int_\delta^A \frac{f(bx)}{x} dx \\ &= \int_{a\delta}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{bA} \frac{f(x)}{x} dx \\ &= \int_{bA}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{a\delta} \frac{f(x)}{x} dx \\ &\stackrel{\text{积分中值定理}}{=} f(\theta) \int_{bA}^{aA} \frac{1}{x} dx - \int_{b\delta}^{a\delta} \frac{f(x)}{x} dx, \end{aligned}$$

这里  $\theta \in (aA, bA)$ , 于是让  $A \rightarrow +\infty, \delta \rightarrow 0^+$ , 由(??), 我们知

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \alpha \ln \frac{a}{b}.$$

4. 给定  $A > \delta > 0$ , 考虑

$$\begin{aligned} \int_\delta^A \frac{f(ax) - f(bx)}{x} dx &= \int_\delta^A \frac{f(ax)}{x} dx - \int_\delta^A \frac{f(bx)}{x} dx \\ &= \int_{a\delta}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{bA} \frac{f(x)}{x} dx \\ &= \int_{bA}^{aA} \frac{f(x)}{x} dx - \int_{b\delta}^{a\delta} \frac{f(x)}{x} dx \\ &\stackrel{\text{积分中值定理}}{=} \int_{bA}^{aA} \frac{f(x)}{x} dx - f(\theta) \int_{b\delta}^{a\delta} \frac{1}{x} dx \\ &= \int_b^a \frac{f(Ax)}{x} dx - f(\theta) \int_{b\delta}^{a\delta} \frac{1}{x} dx, \end{aligned}$$

这里  $\theta \in (a\delta, b\delta)$ . 现在

$$\lim_{\delta \rightarrow 0^+} \left( -f(\theta) \int_{b\delta}^{a\delta} \frac{1}{x} dx \right) = \lim_{x \rightarrow 0^+} f(x) \ln \frac{b}{a}.$$

由 Riemann 引理, 我们有

$$\lim_{A \rightarrow +\infty} \int_b^a \frac{f(Ax)}{x} dx = \int_b^a \frac{1}{x} dx \cdot \frac{1}{T} \int_0^T f(x) dx = -\frac{1}{T} \int_0^T f(x) dx \cdot \ln \frac{b}{a},$$

这就证明了

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = \left[ \lim_{x \rightarrow 0^+} f(x) - \frac{1}{T} \int_0^T f(x) dx \right] \ln \frac{b}{a}.$$

5. 上一问证明中把使用的 Riemann 引理用平均值极限版本的 Riemann 引理代替即可.



## 0.1.4 化成多元累次积分 (换序)


## 命题 0.3

证明:

$$(1) \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

$$(2) \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

$$(3) \int_0^{\infty} \sin x^2 dx, \int_0^{\infty} \cos x^2 dx = \sqrt{\frac{\pi}{8}}.$$

 **笔记** 本结果可以直接使用.

**证明**

(1) 注意到

$$\begin{aligned} \left( \int_0^{+\infty} e^{-x^2} dx \right)^2 &= \left( \int_0^{+\infty} e^{-x^2} dx \right) \left( \int_0^{+\infty} e^{-y^2} dy \right) \xrightarrow{\text{把 } \int_0^{+\infty} e^{-y^2} dy \text{ 看作常数}} \int_0^{+\infty} e^{-x^2} \left( \int_0^{+\infty} e^{-y^2} dx \right) dy \\ &\xrightarrow{\text{把 } e^{-x^2} \text{ 看作常数}} \int_0^{+\infty} \left( \int_0^{+\infty} e^{-(x^2+y^2)} dx \right) dy \xrightarrow{e^{-(x^2+y^2)} \text{ 连续}} \iint_{R^2} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} r e^{-r^2} dr = \frac{\pi}{2} \int_0^{+\infty} r e^{-r^2} dr \\ &= \frac{\pi}{4} \int_0^{+\infty} e^{-r^2} dr^2 = \frac{\pi}{4}. \end{aligned}$$

$$\text{故 } \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(2) 注意到

$$\int_0^{+\infty} \sin x e^{-yx} dx = \operatorname{Im} \int_0^{+\infty} e^{ix-yx} dx = \operatorname{Im} \int_0^{+\infty} e^{-(y-i)x} dx = \operatorname{Im} \frac{1}{y-i} = \operatorname{Im} \frac{y+i}{y^2+1} = \frac{1}{y^2+1}.$$

因此就有

$$\begin{aligned} \int_0^{+\infty} \frac{\sin x}{x} dx &= \int_0^{+\infty} \sin x \left( \int_0^{+\infty} e^{-yx} dy \right) dx = \int_0^{+\infty} dy \int_0^{+\infty} \sin x e^{-yx} dx \\ &= \int_0^{+\infty} dy \left( \operatorname{Im} \int_0^{+\infty} e^{ix-yx} dx \right) = \int_0^{+\infty} \frac{1}{y^2+1} dy = \frac{\pi}{2}. \end{aligned}$$

当然本题也可以直接利用分部积分计算  $\int_0^{+\infty} \sin x e^{-yx} dx = \frac{1}{y^2+1}.$

(3) 注意到

$$\int_0^{+\infty} e^{-ax^2} dx \xrightarrow{x=\frac{t}{\sqrt{a}}} \frac{1}{\sqrt{a}} \int_0^{+\infty} e^{-t^2} dt = \frac{1}{2} \sqrt{\frac{\pi}{a}}, \quad a > 0.$$

并且  $-i = e^{-\frac{\pi}{2}i}$ , 从而  $\sqrt{-i} = e^{-\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ . 于是

$$\begin{aligned} \int_0^{+\infty} (\cos x^2 - i \sin x^2) dx &= \int_0^{+\infty} e^{-ix^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{i}} = \frac{1}{2} \sqrt{-i\pi} \\ &= \frac{\sqrt{\pi}}{2} \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i. \end{aligned}$$

故

$$\int_0^{+\infty} \cos x^2 dx = \operatorname{Re} \int_0^{+\infty} (\cos x^2 - i \sin x^2) dx = \operatorname{Re} \left( \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i \right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}},$$

$$\int_0^{+\infty} \sin x^2 dx = \operatorname{Im} \int_0^{+\infty} (\cos x^2 - i \sin x^2) dx = \operatorname{Im} \left( \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4} i \right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}.$$

□

**例题 0.6** 计算  $\int_0^1 \sin \ln \frac{1}{x} \cdot \frac{x^b - x^a}{\ln x} dx$  ( $b > a > 0$ ).

**证明**

$$\begin{aligned} \int_0^1 \sin \ln \frac{1}{x} \cdot \frac{x^b - x^a}{\ln x} dx &= \int_0^1 \sin \ln \frac{1}{x} \left( \int_a^b x^y dy \right) dx = \int_a^b dy \int_0^1 x^y \sin \ln \frac{1}{x} dx \\ &\stackrel{x=e^{-t}}{=} \int_a^b dy \int_{+\infty}^0 e^{-ty} \sin t de^{-t} = \int_a^b dy \int_0^{+\infty} e^{-t(y+1)} \sin t dt \\ &\stackrel{\text{命题 0.3(2) 的证明过程}}{=} \int_a^b \frac{1}{1+(y+1)^2} dy = \arctan(b+1) - \arctan(a+1). \end{aligned}$$

□

### 0.1.5 化成含参积分 (求导)

**例题 0.7** 设  $a, b \geq 0$  且不全为 0, 计算  $\int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx$ .

**注** 实际上, 根据  $a > b$  时得到的结果, 可以看出  $F(a, b) = \pi \ln \frac{a+b}{2}$  对  $a, b$  有轮换对称性, 故这个结果对其他情况显然也成立.

**证明** 设  $F(a, b) = \int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx$ , 当  $a > b$  时, 则

$$\begin{aligned} \frac{\partial}{\partial b} F(a, b) &= \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial b} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \int_0^{\frac{\pi}{2}} \frac{2b \sin^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{2b \tan^2 x}{a^2 + b^2 \tan^2 x} dx = \int_0^{+\infty} \frac{2bt^2}{(a^2 + b^2 t^2)(1+t^2)} dt \\ &= \frac{1}{a^2 - b^2} \int_0^{+\infty} \left( \frac{2a^2 b}{a^2 + b^2 t^2} - \frac{2b}{1+t^2} \right) dt \\ &= \frac{1}{a^2 - b^2} \int_0^{+\infty} \frac{2a^2 b}{a^2 + b^2 t^2} dt - \frac{2b}{a^2 - b^2} \int_0^{+\infty} \frac{1}{1+t^2} dt \\ &= \frac{2b}{a^2 - b^2} \int_0^{+\infty} \frac{1}{1 + \left(\frac{b}{a}t\right)^2} dt - \frac{b\pi}{a^2 - b^2} \\ &= \frac{2b}{a^2 - b^2} \cdot \frac{a}{b} \cdot \frac{\pi}{2} - \frac{b\pi}{a^2 - b^2} = \frac{\pi}{a+b}. \end{aligned}$$

于是

$$\begin{aligned} F(a, b) &= F(a, 0) + \int_0^b \frac{\partial}{\partial b'} F(a, b') db' = F(a, 0) + \int_0^b \frac{\pi}{a+b'} db' \\ &= 2 \int_0^{\frac{\pi}{2}} \ln(a \cos x) dx + \pi \ln \frac{a+b}{a} \stackrel{\text{例题 0.2}}{=} \pi \ln \frac{a+b}{2}. \end{aligned}$$

当  $a < b$  时, 类似可得  $F(a, b) = \pi \ln \frac{a+b}{2}$ . 当  $a = b$  时,  $F(a, b) = \int_0^{\frac{\pi}{2}} \ln a^2 dx = \pi \ln a = \pi \ln \frac{a+b}{2}$ .

综上, 对  $\forall a, b \geq 0$ , 都有  $F(a, b) = \pi \ln \frac{a+b}{2}$ .

□

## 0.1.6 级数展开方法

积分和求和换序  $\sum_{n=1}^{\infty} \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx$ , 等价于

$$\lim_{m \rightarrow \infty} \sum_{n=1}^m \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx.$$

又由于有限和随意交换, 因此上式等价于

$$\lim_{m \rightarrow \infty} \int_a^b \sum_{n=1}^m f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx,$$

于是

$$\sum_{n=1}^{\infty} \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx \iff \lim_{m \rightarrow \infty} \int_a^b \sum_{n=m+1}^{\infty} f_n(x) dx = 0.$$

**例题 0.8** 计算  $\int_0^{\infty} \frac{x}{1+e^x} dx$ .

**解** 由于

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad 0 < x < 1.$$

并且  $0 < e^{-x} < 1$ , 故

$$\begin{aligned} \int_0^{+\infty} \frac{x}{1+e^x} dx &= \int_0^{+\infty} \frac{x e^{-x}}{1+e^{-x}} dx = \int_0^{+\infty} x \sum_{n=0}^{\infty} (-1)^n e^{-(n+1)x} dx \\ &\stackrel{\text{换序}}{=} \sum_{n=0}^{\infty} (-1)^n \int_0^{+\infty} x e^{-(n+1)x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2}. \end{aligned}$$

又因为  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ , 所以

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(2n)^2} &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}, \\ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8}. \end{aligned}$$

故

$$\int_0^{+\infty} \frac{x}{1+e^x} dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.$$

下面证明(??)式换序成立, 等价于证明  $\lim_{m \rightarrow +\infty} \int_0^{+\infty} \sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x} dx = 0$ . 由交错级数不等式及  $x e^{-(n+1)x}$  关于  $n$  非负递减, 对  $\forall m \in \mathbb{N}$ , 都有

$$\int_0^{+\infty} \left| \sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x} \right| dx \leq \int_0^{+\infty} x e^{-(m+1)x} dx = -\frac{x e^{-(m+1)x}}{m+1} \Big|_0^{+\infty} + \frac{1}{m+1} \int_0^{+\infty} e^{-(m+1)x} dx = \frac{1}{(m+1)^2}.$$

令  $m \rightarrow +\infty$ , 得  $\lim_{m \rightarrow +\infty} \int_0^{+\infty} \sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x} dx = 0$ . 故(??)式换序成立. □

## 命题 0.4

证明:

$$(1) \sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n} = \arctan \frac{q \sin x}{1 - q \cos x}, |q| \leq 1.$$

$$(2) \sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n} = -\frac{1}{2} \ln(1 + q^2 - 2q \cos x), |q| \leq 1.$$

$$(3) \sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n!} = e^{q \cos x} \cos(q \sin x) - 1, |q| \leq 1, x \in \mathbb{R}.$$

$$(4) \sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n!} = e^{q \cos x} \sin(q \sin x), |q| \leq 1, x \in \mathbb{R}.$$



 笔记 在  $\mathbb{C}$  上,

$$\operatorname{Ln} z = \ln |z| + i(\arg z + 2k\pi), k \in \mathbb{Z}.$$

我们定义主值支

$$\ln z = \ln |z| + i \arg z.$$

本部分内容无需记忆,只需要大概有个可以算的感觉即可,实际做题中可以围绕这种级数给出构造.

**证明**  $\Im$  表示取虚部,  $\Re$  表示取实部.

(1) 利用欧拉公式有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n} &= \Im \left( \sum_{n=1}^{\infty} \frac{q^n e^{inx}}{n} \right) = \Im \left( \sum_{n=1}^{\infty} \frac{(qe^{ix})^n}{n} \right) = \Im(-\ln(1 - qe^{ix})) \\ &= -\Im \left( \ln |1 - qe^{ix}| + i \frac{-q \sin x}{1 - q \cos x} \right) = \arctan \frac{q \sin x}{1 - q \cos x}. \end{aligned}$$

(2) 利用欧拉公式有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n} &= -\Re \left( \ln |1 - qe^{ix}| + i \frac{-q \sin x}{1 - q \cos x} \right) = -\frac{1}{2} \ln [(1 - q \cos x)^2 + q^2 \sin^2 x] \\ &= -\frac{1}{2} \ln(1 + q^2 - 2q \cos x). \end{aligned}$$

(3) 利用欧拉公式有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n!} &= \Re \left( \sum_{n=1}^{\infty} \frac{(qe^{ix})^n}{n!} \right) = \Re(e^{qe^{ix}} - 1) = \Re(e^{q \cos x + iq \sin x} - 1) \\ &= \Re(e^{q \cos x} \cos(q \sin x) - 1 + ie^{q \cos x} \sin(q \sin x)) \\ &= e^{q \cos x} \cos(q \sin x) - 1. \end{aligned}$$

(4) 利用 (3) 有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n!} &= \Im(e^{q \cos x} \cos(q \sin x) - 1 + ie^{q \cos x} \sin(q \sin x)) \\ &= e^{q \cos x} \sin(q \sin x). \end{aligned}$$

□

**例题 0.9** 计算

- $\int_0^{2\pi} e^{\cos x} \cos(\sin x) dx.$
- $\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx, a \in (0, +\infty) \setminus \{1\}.$

注 由 1 的证明可得

$$e^{\cos x} \cos(\sin x) = \operatorname{Re} \left( \sum_{n=0}^{\infty} \frac{(e^{ix})^n}{n!} \right) = \operatorname{Re} \left( \sum_{n=0}^{\infty} \frac{e^{inx}}{n!} \right) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}.$$

实际上, 上式就是命题 0.4(3) 的结论.

注 第 2 问也可以用含参积分求导的方法进行计算 (这个方法更容易想到).

证明

1.

$$\begin{aligned} \int_0^{2\pi} e^{\cos x} \cos(\sin x) dx &= \operatorname{Re} \left( \int_0^{2\pi} e^{\cos x} e^{i \sin x} dx \right) = \operatorname{Re} \left( \int_0^{2\pi} e^{\cos x + i \sin x} dx \right) \\ &= \operatorname{Re} \left( \int_0^{2\pi} e^{e^{ix}} dx \right) = \operatorname{Re} \left[ \int_0^{2\pi} \sum_{n=0}^{+\infty} \frac{(e^{ix})^n}{n!} dx \right] = \operatorname{Re} \left[ \sum_{n=0}^{+\infty} \int_0^{2\pi} \frac{(e^{ix})^n}{n!} dx \right] \\ &= \operatorname{Re} \left( \sum_{n=0}^{+\infty} \int_0^{2\pi} \frac{e^{inx}}{n!} dx \right) = \operatorname{Re} \left( \int_0^{2\pi} \frac{e^{i \cdot 0 \cdot x}}{0!} dx + \sum_{n=1}^{+\infty} \frac{e^{2\pi i x} - 1}{in \cdot n!} \right) \\ &= \operatorname{Re} \left( \int_0^{2\pi} 1 dx + 0 \right) = 2\pi. \end{aligned}$$

2. 注意到当  $a \in (0, 1)$  时, 有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a^n \cos(nx)}{n} &= \operatorname{Re} \left[ \sum_{n=1}^{\infty} \frac{(ae^{ix})^n}{n} \right] = -\operatorname{Re} [\ln(1 - ae^{ix})] \\ &= -\operatorname{Re} [\ln |1 - ae^{ix}| + i \arg(1 - ae^{ix})] = -\ln |1 - ae^{ix}| \\ &= -\ln |(1 - a \cos x) + ai \sin x| = -\frac{1}{2} \ln(1 + a^2 - 2a \cos x). \end{aligned}$$

于是当  $a \in (0, 1)$  时, 就有

$$\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = -\frac{1}{2} \int_0^{\pi} \sum_{n=1}^{\infty} \frac{a^n \cos(nx)}{n} dx = 0.$$

若  $a > 1$ , 则  $\frac{1}{a} \in (0, 1)$ , 从而此时我们有

$$\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = \pi \ln a^2 + \int_0^{\pi} \ln \left( \frac{1}{a^2} - \frac{2}{a} \cos x + 1 \right) dx = \pi \ln a^2 = 2\pi \ln a.$$

又由  $\ln(1 - 2a \cos x + a^2)$  关于  $a$  的偏导存在可知  $\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx$  关于  $a$  连续. 于是由

$$\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = 2\pi \ln a, \quad \forall a > 1.$$

可知当  $a = 1$  时, 我们有

$$\int_0^{\pi} \ln(2 - 2 \cos x) dx = \lim_{a \rightarrow 1^+} \int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = \lim_{a \rightarrow 1^+} (2\pi \ln a) = 0.$$

□

### 定义 0.1 (多重对数函数-Li<sub>2</sub> 函数)

定义

$$\operatorname{Li}_2(x) \triangleq \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad x \in [-1, 1].$$

♣

## 命题 0.5

$$(1) \operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \cdot \ln(1-x), x \in (0, 1).$$

$$(2) \operatorname{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \operatorname{Li}_2(0) = 0, \quad \operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}.$$

证明

(1) 记  $f(x) \triangleq \operatorname{Li}_2(x)$ ,  $F(x) \triangleq f(x) + f(1-x) + \ln x \ln(1-x)$ . 则

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\frac{1}{x} \ln(1-x).$$

于是

$$F'(x) = -\frac{1}{x} \ln(1-x) + \frac{\ln x}{1-x} - \frac{\ln x}{1-x} + \frac{\ln(1-x)}{x} = 0.$$

$$\text{故 } F(x) \equiv F(1) = f(0) + f(1) = 0 + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(2) 显然  $\operatorname{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ ,  $\operatorname{Li}_2(0) = 0$ . 由 (1) 可得

$$\operatorname{Li}_2\left(\frac{1}{2}\right) + \operatorname{Li}_2\left(\frac{1}{2}\right) = 2\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - \ln^2 \frac{1}{2} \implies \operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}.$$

□

例题 0.10 计算  $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x} dx$ .

解

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{\ln x}{1-x} dx &= \int_{\frac{1}{2}}^1 \frac{\ln(1-x)}{x} dx = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{2}}^1 x^{n-1} dx \\ &= -\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{2^n n^2} = -\frac{\pi^2}{6} + \operatorname{Li}_2\left(\frac{1}{2}\right) \\ &\stackrel{\text{命题 0.5}}{=} -\frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}. \end{aligned}$$

□

## 0.1.7 重积分计算

## 定理 0.3 (二重积分换序)

证明:

$$\int_a^b dx \int_a^x f(x, y) dy = \int_a^b dy \int_y^b f(x, y) dx, \quad (10)$$

其中  $f(x, y)$  是在由直线  $y = a$ ,  $x = b$ ,  $y = x$  所围成的三角形 ( $\Delta$ ) 上连续的任意函数.

♡

证明

□

## 命题 0.6

设  $f(x)$  在  $[a, b]$  上连续, 试证: 对任意  $x \in (a, b)$ , 有

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_n} f(x_{n+1}) dx_{n+1} = \frac{1}{n!} \int_a^x (x-y)^n f(y) dy, \quad n = 1, 2, \dots$$

♣

**证明** 当  $n=1$  时, 由二重积分换序可知

$$\int_a^x dx_1 \int_a^{x_1} f(x_2) dx_2 = \int_a^x dx_2 \int_{x_2}^x f(x_2) dx_1 = \int_a^x (x-x_2)f(x_2) dx_2 = \int_a^x (x-y)f(y) dy.$$

设原结论对  $n=k-1$  的情形成立, 考虑  $n=k$  的情形, 由归纳假设可知

$$\int_a^{x_1} dx_1 \cdots \int_a^{x_k} f(x_{k+1}) dx_{k+1} = \frac{1}{(k-1)!} \int_a^{x_1} (x_1-y)^{k-1} f(y) dy.$$

于是再利用二重积分换序得

$$\begin{aligned} \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_k} f(x_{k+1}) dx_{k+1} &= \frac{1}{(k-1)!} \int_a^x dx_1 \int_a^{x_1} (x_1-y)^{k-1} f(y) dy \\ &= \frac{1}{(k-1)!} \int_a^x dy \int_y^x (x_1-y)^{k-1} f(y) dx_1 \\ &= \frac{1}{k!} \int_a^x (x-y)^k f(y) dy. \end{aligned}$$

故由数学归纳法知原结论成立.  $\square$

**例题 0.11** 求定义在星形区域  $D = \{(x, y) \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} \leq 1\}$  上满足  $f(1, 0) = 1$  的正值连续函数  $f$  使得  $\iint_D \frac{f(x, y)}{f(y, x)} dx dy$

达到最小, 并求出这个最小值.

**解** 对积分  $I = \iint_D \frac{f(x, y)}{f(y, x)} dx dy$  作变换  $x \rightarrow y, y \rightarrow x$ , 由  $D$  的对称性, 知  $I = \iint_D \frac{f(y, x)}{f(x, y)} dx dy$ . 因而由均值不等式可得

$$I = \frac{1}{2} \iint_D \left( \frac{f(x, y)}{f(y, x)} + \frac{f(y, x)}{f(x, y)} \right) dx dy \geq \iint_D 1 dx dy = \sigma(D),$$

这里  $\sigma(D)$  是  $D$  的面积.

$$I - \sigma(D) = \frac{1}{2} \iint_D \left( \sqrt{\frac{f(x, y)}{f(y, x)}} - \sqrt{\frac{f(y, x)}{f(x, y)}} \right)^2 dx dy \geq 0.$$

$I = \sigma(D)$  当且仅当  $f(x, y) = f(y, x)$ . 故所求函数为所有满足  $f(x, y) = f(y, x)$  及  $f(1, 0) = 1$  的连续正值函数.

$D$  的边界的参数方程为

$$x = \cos^3 \varphi, \quad y = \sin^3 \varphi \quad (0 \leq \varphi \leq 2\pi),$$

故  $I$  的最小值为

$$\begin{aligned} \sigma(D) &= \iint_D 1 dx dy = 4 \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{2}}} 3r \sin^2 \varphi \cos^2 \varphi dr d\varphi \\ &= 6 \int_0^{\frac{\pi}{2}} \sin^2 \varphi \cos^2 \varphi d\varphi = \frac{3}{8} \pi. \end{aligned}$$

所以所求最小值是  $\frac{3}{8}\pi$ , 且当  $f(x, y) = f(y, x)$  并满足  $f(1, 0) = 1$  时, 取到该最小值.  $\square$

**例题 0.12** 求证:  $\iint_{[0,1]^2} (xy)^{xy} dx dy = \int_0^1 t^t dt$ .

**证明** 首先化为累次积分

$$\iint_{[0,1]^2} (xy)^{xy} dx dy = \int_0^1 dx \int_0^1 (xy)^{xy} dy = \int_0^1 dx \int_0^x \frac{t^t}{x} dt = \int_0^1 \frac{f(x)}{x} dx,$$

其中  $f(x) = \int_0^x t^t dt$ . 由分部积分,

$$\int_0^1 \frac{f(x)}{x} dx = f(x) \ln x \Big|_0^1 - \int_0^1 x^x \ln x dx = - \int_0^1 x^x \ln x dx.$$

因为  $(x^x)' = x^x \ln x + x^x$ , 所以

$$\int_0^1 x^x \ln x \, dx = \int_0^1 ((x^x)' - x^x) \, dx = - \int_0^1 x^x \, dx.$$

于是

$$\iint_{[0,1]^2} (xy)^{xy} \, dx dy = \int_0^1 t^t \, dt.$$

□

**例题 0.13** 计算二重积分  $I = \iint_D \operatorname{sgn}(x^2 - y^2 + 2) \, dx dy$ , 其中  $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$ .

**解** 设  $D$  在第一象限部分为  $D_1$ , 则由对称性

$$I = 4 \iint_{D_1} \operatorname{sgn}(x^2 - y^2 + 2) \, dx dy.$$

设  $D_2$  是  $D_1$  中使得  $x^2 - y^2 + 2 < 0$  的部分,  $D_3$  是  $D_1$  中使得  $x^2 - y^2 + 2 \geq 0$  的部分, 则  $D_1 = D_2 \cup D_3$ . 因此

$$\begin{aligned} I &= 4 \left[ \iint_{D_3} dx dy - \iint_{D_2} dx dy \right] = 4[\sigma(D_3) - \sigma(D_2)] \\ &= 4 \left[ \frac{1}{4} \cdot \pi \cdot 2^2 - 2\sigma(D_2) \right] = 4\pi - 8\sigma(D_2), \end{aligned}$$

其中  $\sigma(D_2), \sigma(D_3)$  分别表示  $D_2$  和  $D_3$  的面积. 在极坐标  $x = r \cos \varphi, y = r \sin \varphi$  之下,  $D_2$  为

$$\left\{ (r, \varphi) \mid \frac{\pi}{3} \leq \varphi \leq \frac{\pi}{2}, \sqrt{-\frac{2}{\cos 2\varphi}} \leq r \leq 2 \right\}.$$

因而

$$\begin{aligned} \sigma(D_2) &= \iint_{D_2} dx dy = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_{\sqrt{-\frac{2}{\cos 2\varphi}}}^2 r \, dr \\ &= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left( 4 + \frac{2}{\cos 2\varphi} \right) d\varphi = \frac{\pi}{3} + \frac{1}{2} \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{\cos \varphi} d\varphi \\ &= \frac{\pi}{3} - \frac{1}{2} \ln(2 + \sqrt{3}), \end{aligned}$$

故

$$I = \frac{4\pi}{3} + 4 \ln(2 + \sqrt{3}).$$

□

**例题 0.14** 设  $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ . 求  $I = \iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy$ .

**解** 由极坐标变换  $x = r \cos \varphi, y = r \sin \varphi, 0 \leq r \leq 1, 0 \leq \varphi \leq 2\pi$ , 有

$$\begin{aligned} I &= \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \varphi \leq 2\pi}} \left| \frac{\cos \varphi + \sin \varphi}{\sqrt{2}} - r \right| r^2 dr d\varphi \\ &= \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \varphi \leq 2\pi}} \left| \sin \left( \varphi + \frac{\pi}{4} \right) - r \right| r^2 dr d\varphi = \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \varphi \leq 2\pi}} |\sin \varphi - r| r^2 dr d\varphi \\ &= \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \varphi \leq \pi}} |\sin \varphi - r| r^2 dr d\varphi + \iint_{\substack{0 \leq r \leq 1 \\ \pi \leq \varphi \leq 2\pi}} |\sin \varphi - r| r^2 dr d\varphi \\ &= \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \varphi \leq \pi}} |\sin \varphi - r| r^2 dr d\varphi + \iint_{\substack{0 \leq r \leq 1 \\ 0 \leq \varphi \leq \pi}} (\sin \varphi + r) r^2 dr d\varphi. \end{aligned}$$



因此, 有

$$\begin{aligned}
 I &= \int_0^\pi d\varphi \int_0^{\sin \varphi} (\sin \varphi - r)r^2 dr + \int_0^\pi d\varphi \int_{\sin \varphi}^1 (r - \sin \varphi)r^2 dr \\
 &\quad + \int_0^\pi d\varphi \int_0^{\sin \varphi} (\sin \varphi + r)r^2 dr + \int_0^\pi d\varphi \int_{\sin \varphi}^1 (\sin \varphi + r)r^2 dr \\
 &= \int_0^\pi d\varphi \int_0^{\sin \varphi} 2 \sin \varphi \cdot r^2 dr + \int_0^\pi d\varphi \int_{\sin \varphi}^1 2r \cdot r^2 dr \\
 &= \int_0^\pi \frac{2}{3} \sin^4 \varphi d\varphi + \int_0^\pi \frac{1}{2} (1 - \sin^4 \varphi) d\varphi \\
 &= \frac{1}{6} \int_0^\pi \sin^4 \varphi d\varphi + \frac{\pi}{2} = \frac{1}{6} \cdot \frac{3\pi}{8} + \frac{\pi}{2} = \frac{9}{16} \pi.
 \end{aligned}$$

□

**例题 0.15** 设  $f$  是定义在正方形  $S = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$  上的四阶连续可微函数, 在  $S$  的边界上为零, 并且

$$\left| \frac{\partial^4 f}{\partial x^2 \partial y^2} \right| \leq M.$$

求证:

$$\left| \iint_S f(x, y) dx dy \right| \leq \frac{1}{144} M.$$

**证明** 考虑函数  $g(x, y) = x(1-x)y(1-y)$ . 易知

$$\frac{\partial^4 g}{\partial x^2 \partial y^2} = 4, \quad \iint_S g(x, y) dx dy = \frac{1}{36}.$$

因为  $f$  在  $S$  的边界上为零, 所以  $\frac{\partial^2 f}{\partial y^2}$  在  $x=0$  和  $x=1$  时为零. 于是

$$\begin{aligned}
 \iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g dx dy &= \int_0^1 dy \int_0^1 \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g dx \\
 &= \int_0^1 dy \left( \left. \frac{\partial^3 f}{\partial x \partial y^2} \cdot g \right|_{x=0}^1 - \int_0^1 \frac{\partial^3 f}{\partial x \partial y^2} \cdot \frac{\partial g}{\partial x} dx \right) \\
 &= - \int_0^1 dy \int_0^1 \frac{\partial^3 f}{\partial x \partial y^2} \cdot \frac{\partial g}{\partial x} dx \\
 &= - \int_0^1 dy \left( \left. \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial g}{\partial x} \right|_{x=0}^1 - \int_0^1 \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2} dx \right) \\
 &= \int_0^1 dy \int_0^1 \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2} dx \\
 &= \iint_S \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2} dx dy.
 \end{aligned}$$

同理, 由于  $\frac{\partial^2 g}{\partial x^2}$  在  $y=0$  和  $y=1$  时为零, 作与上面类似的推导, 可得

$$\iint_S \frac{\partial^4 g}{\partial x^2 \partial y^2} \cdot f dx dy = \iint_S \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2} dx dy.$$

因此

$$\iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g dx dy = \iint_S \frac{\partial^4 g}{\partial x^2 \partial y^2} \cdot f dx dy.$$

从而

$$\begin{aligned}
 \left| \iint_S f dx dy \right| &= \frac{1}{4} \left| \iint_S 4f dx dy \right| = \frac{1}{4} \left| \iint_S \frac{\partial^4 g}{\partial x^2 \partial y^2} f dx dy \right| \\
 &= \frac{1}{4} \left| \iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g dx dy \right| \leq \frac{M}{4} \iint_S g dx dy = \frac{M}{144}.
 \end{aligned}$$

□

**定理 0.4 (Poincaré(庞加莱) 不等式)**

设  $\varphi, \psi$  是  $[a, b]$  上的连续函数,  $f$  在区域  $D = \{(x, y) \mid a \leq x \leq b, \varphi(x) \leq y \leq \psi(x)\}$  上连续可微, 且有  $f(x, \varphi(x)) = 0$  ( $x \in [a, b]$ ). 则存在  $M > 0$ , 使得

$$\iint_D f^2(x, y) \, dx dy \leq M \iint_D (f'_y(x, y))^2 \, dx dy.$$

♡

**证明** 由 Newton-Leibniz 公式和 Cauchy 不等式可得

$$\begin{aligned} f^2(x, y) &= [f(x, y) - f(x, \varphi(x))]^2 = \left( \int_{\varphi(x)}^y \frac{\partial f}{\partial t}(x, t) \, dt \right)^2 \\ &\leq (y - \varphi(x)) \int_{\varphi(x)}^y \left( \frac{\partial f}{\partial t}(x, t) \right)^2 \, dt, \end{aligned}$$

因此

$$\begin{aligned} \iint_D f^2(x, y) \, dx dy &= \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f^2(x, y) \, dy \\ &\leq \int_a^b dx \int_{\varphi(x)}^{\psi(x)} (y - \varphi(x)) \, dy \int_{\varphi(x)}^y \left( \frac{\partial f}{\partial t}(x, t) \right)^2 \, dt \\ &= \int_a^b dx \int_{\varphi(x)}^{\psi(x)} \left( \frac{\partial f}{\partial t}(x, t) \right)^2 \, dt \int_t^{\psi(x)} (y - \varphi(x)) \, dy \\ &\leq \int_a^b dx \int_{\varphi(x)}^{\psi(x)} \left( \frac{\partial f}{\partial t}(x, t) \right)^2 \, dt \int_{\varphi(x)}^{\psi(x)} (y - \varphi(x)) \, dy \\ &= \int_a^b dx \int_{\varphi(x)}^{\psi(x)} \frac{1}{2} (\psi(x) - \varphi(x))^2 \left( \frac{\partial f}{\partial t}(x, t) \right)^2 \, dt \\ &\leq M \int_a^b \left( \int_{\varphi(x)}^{\psi(x)} \left( \frac{\partial f}{\partial t}(x, t) \right)^2 \, dt \right) dx \\ &= M \iint_D \left( \frac{\partial f}{\partial y}(x, y) \right)^2 \, dx dy, \end{aligned}$$

这里  $M$  是满足  $M > \max_{a \leq x \leq b} \frac{1}{2} (\psi(x) - \varphi(x))^2$  的常数.

□

**例题 0.16** 设  $a > 0$ ,  $\Omega_n(a) : x_1 + x_2 + \cdots + x_n \leq a, x_i \geq 0$  ( $i = 1, 2, \cdots, n$ ). 求积分

$$I_n(a) = \int \cdots \int_{\Omega_n(a)} x_1 x_2 \cdots x_n \, dx_1 dx_2 \cdots dx_n.$$

**解** 作变换  $x_i = at_i, i = 1, 2, \cdots, n$ , 则

$$I_n(a) = a^{2n} \int \cdots \int_{\Omega_n(1)} t_1 t_2 \cdots t_n \, dt_1 dt_2 \cdots dt_n = a^{2n} I_n(1).$$

再用累次积分, 可得

$$\begin{aligned} I_n(1) &= \int \cdots \int_{\Omega_n(1)} t_1 t_2 \cdots t_n \, dt_1 dt_2 \cdots dt_n \\ &= \int_0^1 t_n \, dt_n \int \cdots \int_{t_1 + t_2 + \cdots + t_{n-1} \leq 1 - t_n} t_1 \cdots t_{n-1} \, dt_1 \cdots dt_{n-1} \\ &= \int_0^1 t_n I_{n-1}(1 - t_n) \, dt_n = \int_0^1 t_n (1 - t_n)^{2(n-1)} I_{n-1}(1) \, dt_n. \end{aligned}$$

因此,

$$I_n(1) = \frac{1}{2n(2n-1)} I_{n-1}(1).$$

注意到  $I_1(1) = \int_0^1 t \, dt = \frac{1}{2}$ . 由上面的递推公式, 可得  $I_n(1) = \frac{1}{(2n)!}$ . 故  $I_n(a) = \frac{a^{2n}}{(2n)!}$ .

□

## 0.1.8 其他

**例题 0.17** 证明积分  $\int_0^{\infty} e^{-ax^2 - \frac{b}{x^2}} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}, a, b > 0$ .

**证明** 当  $a = 1$  时, 就有

$$\begin{aligned} \int_0^{\infty} e^{-x^2 - \frac{b}{x^2}} dx &= e^{-2\sqrt{b}} \int_0^{\infty} e^{-\left(x - \frac{\sqrt{b}}{x}\right)^2} dx \stackrel{y = \frac{\sqrt{b}}{x}}{=} e^{-2\sqrt{b}} \int_0^{\infty} \frac{\sqrt{b}}{y^2} e^{-\left(\frac{\sqrt{b}}{y} - y\right)^2} dy \\ &= \frac{e^{-2\sqrt{b}}}{2} \int_0^{\infty} \left(1 + \frac{\sqrt{b}}{y^2}\right) e^{-\left(y - \frac{\sqrt{b}}{y}\right)^2} dy = \frac{e^{-2\sqrt{b}}}{2} \int_0^{\infty} e^{-\left(y - \frac{\sqrt{b}}{y}\right)^2} d\left(y - \frac{\sqrt{b}}{y}\right) \\ &= \frac{e^{-2\sqrt{b}}}{2} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} e^{-2\sqrt{b}}. \end{aligned}$$

于是对  $\forall a > 0$ , 就有

$$\int_0^{\infty} e^{-ax^2 - \frac{b}{x^2}} dx = \frac{1}{\sqrt{a}} \int_0^{\infty} e^{-x^2 - \frac{ab}{x^2}} dx = \frac{1}{\sqrt{a}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$$

□

**例题 0.18** 计算  $\int_0^{\infty} \frac{\cos(ax)}{1+x^2} dx, a \in \mathbb{R}$ .

**注** 本题可以用复变函数的方法 (留数定理) 来计算. 但是我们这里用基本的高等数学的方法来计算.

$\int_0^{\infty} \frac{\sin(ax)}{1+x^2} dx$  这个积分没办法算出具体的初等数值.

**证明**

$$\begin{aligned} \int_0^{\infty} \frac{\cos(ax)}{1+x^2} dx &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\cos(ax)}{1+x^2} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \cos(ax) \left( \int_0^{\infty} e^{-(1+x^2)y} dy \right) dx \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \left( \int_0^{\infty} e^{-(1+x^2)y} \cos(ax) dy \right) dx = \frac{1}{2} \int_{-\infty}^{+\infty} \int_0^{\infty} e^{-(1+x^2)y} \cos(ax) dy dx \\ &= \frac{1}{2} \int_0^{\infty} \left( \int_{-\infty}^{+\infty} e^{-(1+x^2)y} \cos(ax) dx \right) dy = \frac{1}{2} \int_0^{\infty} e^{-y} \left( \int_{-\infty}^{+\infty} e^{-x^2 y} \cos(ax) dx \right) dy \\ &= \frac{1}{2} \operatorname{Re} \left( \int_0^{\infty} e^{-y} \left( \int_{-\infty}^{+\infty} e^{-x^2 y + iax} dx \right) dy \right) = \frac{1}{2} \operatorname{Re} \left( \int_0^{\infty} e^{-y} \left( \int_{-\infty}^{+\infty} e^{-y \left( x^2 - \frac{aix}{y} + \left(\frac{ai}{2y}\right)^2 \right) - \frac{a^2}{4y}} dx \right) dy \right) \\ &= \frac{1}{2} \operatorname{Re} \left( \int_0^{\infty} e^{-y} \left( \int_{-\infty}^{+\infty} e^{-y \left( x - \frac{ai}{2y} \right)^2 - \frac{a^2}{4y}} dx \right) dy \right) = \frac{1}{2} \operatorname{Re} \left( \int_0^{\infty} e^{-y} \left( \int_{-\infty}^{+\infty} e^{-y \left( x + \frac{a}{2iy} \right)^2 - \frac{a^2}{4y}} dx \right) dy \right) \\ &= \frac{1}{2} \operatorname{Re} \left( \int_0^{\infty} e^{-y} \left( \int_{-\infty}^{+\infty} e^{-y \left( x + \frac{a}{2iy} \right)^2 - \frac{a^2}{4y}} d \left( x + \frac{a}{2iy} \right) \right) dy \right) = \frac{1}{2} \operatorname{Re} \left( \int_0^{\infty} e^{-y} \left( \int_{-\infty}^{+\infty} e^{-yx^2 - \frac{a^2}{4y}} dx \right) dy \right) \\ &= \frac{1}{2} \int_0^{\infty} e^{-y - \frac{a^2}{4y}} \left( \int_{-\infty}^{+\infty} e^{-yx^2} dx \right) dy = \frac{\sqrt{\pi}}{2} \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-y - \frac{a^2}{4y}} dy \\ &\stackrel{y=t^2}{=} \sqrt{\pi} \int_0^{\infty} e^{-t^2 - \frac{a^2}{4t^2}} dt \stackrel{\text{例题 0.17}}{=} \sqrt{\pi} \cdot \frac{\sqrt{\pi}}{2} e^{-|a|} = \frac{\pi}{2} e^{-|a|}. \end{aligned}$$

□

**例题 0.19** 计算  $\int_0^{\infty} \frac{1}{(1+x^8)^2} dx$ .

**注** 由命题??可知对  $\forall s > 0$ , 都有

$$\frac{1}{t^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} y^{s-1} e^{-ty} dy, \forall t \in \mathbb{R}.$$

本题的核心想法就是利用上式将  $\frac{1}{1+x^8}$  转化成积分形式.

**证明** 注意到

$$\int_0^{\infty} y e^{-(1+x^8)y} dy \stackrel{y = \frac{z}{1+x^8}}{=} \frac{1}{(1+x^8)^2} \int_0^{\infty} z e^{-z} dz = \frac{1}{(1+x^8)^2},$$

因此

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{(1+x^8)^2} dx &= \int_0^{+\infty} \left( \int_0^{+\infty} y e^{-(1+x^8)y} dy \right) dx = \int_0^{+\infty} \left( \int_0^{+\infty} y e^{-(1+x^8)y} dx \right) dy \\
&= \int_0^{+\infty} y e^{-y} \left( \int_0^{+\infty} e^{-x^8 y} dx \right) dy \stackrel{x=y^{-\frac{1}{8}} z^{\frac{1}{8}}}{=} \int_0^{+\infty} y e^{-y} \left( \int_0^{+\infty} y^{-\frac{1}{8}} e^{-z} dz^{\frac{1}{8}} \right) dy \\
&= \frac{1}{8} \int_0^{+\infty} y^{\frac{7}{8}} e^{-y} \left( \int_0^{+\infty} z^{-\frac{7}{8}} e^{-z} dz \right) dy = \frac{1}{8} \int_0^{+\infty} y^{\frac{7}{8}} e^{-y} \Gamma\left(\frac{1}{8}\right) dy \\
&= \frac{1}{8} \Gamma\left(\frac{15}{8}\right) \Gamma\left(\frac{1}{8}\right) = \frac{1}{8} \cdot \frac{7}{8} \Gamma\left(\frac{7}{8}\right) \Gamma\left(\frac{1}{8}\right) \\
&\stackrel{??}{=} \frac{7\pi}{64 \sin\left(\frac{\pi}{8}\right)} = \frac{7\pi}{32\sqrt{2-\sqrt{2}}}.
\end{aligned}$$

□

**例题 0.20** 计算积分  $I = \int_{-1}^2 \frac{1+x^2}{1+x^4} dx$ .

**注** 在此例中  $I \neq F(2) - F(-1)$ . 这是因为  $F$  并不是  $f$  在区间  $[-1, 2]$  上的原函数.

**解** 在不包含 0 的区间上作变换  $t = x - \frac{1}{x}$  得

$$\begin{aligned}
\int \frac{1+x^2}{1+x^4} dx &= \int \frac{x - \frac{1}{x}}{2 + \left(x - \frac{1}{x}\right)^2} dx = \int \frac{dt}{2+t^2} \\
&= \frac{1}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \arctan \frac{x^2 - 1}{\sqrt{2}x} + C.
\end{aligned}$$

这说明在区间  $[-1, 0)$  和  $(0, 2]$  上, 函数  $f(x) = \frac{1+x^2}{1+x^4}$  的一个原函数是

$$F(x) = \frac{1}{\sqrt{2}} \arctan \frac{x^2 - 1}{\sqrt{2}x}.$$

因此

$$\begin{aligned}
\int_{-1}^0 f(x) dx &= F(0^-) - F(-1) = \frac{\pi}{2\sqrt{2}} - 0 = \frac{\pi}{2\sqrt{2}}, \\
\int_0^2 f(x) dx &= F(2) - F(0^+) = \frac{1}{\sqrt{2}} \arctan \frac{3}{2\sqrt{2}} + \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

故

$$I = \int_{-1}^0 f(x) dx + \int_0^2 f(x) dx = \frac{\pi}{\sqrt{2}} + \frac{1}{\sqrt{2}} \arctan \frac{3}{2\sqrt{2}}.$$

□