

0.1 降阶法

定义 0.1 (组合数定义的扩充)

$$C_n^k \triangleq \begin{cases} 0 & , k < 0, \\ \frac{n!}{k!(n-k)!} & , 0 \leq k \leq n, \\ 0 & , k > n, \end{cases}$$

定理 0.1 (Vandermode 恒等式)

$$C_{p+q}^l = \sum_{k=0}^{p+q} C_p^k C_q^{l-k}, \quad l = 1, 2, \dots, p+q.$$

证明 注意到

$$(x+1)^p (x+1)^q = (x+1)^{p+q}.$$

由二项式定理可得

$$\sum_{r=0}^p C_p^r x^r \cdot \sum_{r=0}^q C_q^r x^r = \sum_{r=0}^{p+q} C_{p+q}^r x^r.$$

对 $\forall l = \{1, 2, \dots, p+q\}$, 考虑上式 x^l 的系数, 就有

$$C_{p+q}^l = C_p^0 C_q^l + C_p^1 C_q^{l-1} + \dots + C_p^l C_q^0.$$

□

例题 0.1 计算 n 阶行列式:

$$|A| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & C_2^1 & \cdots & C_n^1 \\ 1 & C_3^2 & \cdots & C_{n+1}^2 \\ \vdots & \vdots & & \vdots \\ 1 & C_n^{n-1} & \cdots & C_{2n-2}^{n-1} \end{vmatrix}.$$

笔记 解法一的关键就是组合数公式: $C_m^{k-1} + C_m^k = C_{m+1}^k$.

于是有

$$C_m^k = C_{m+1}^k - C_m^{k-1}$$

$$C_m^{k-1} = C_{m+1}^k - C_m^k$$

解法二的核心想法就是: 将 Vandermode 恒等式与矩阵乘法的定义联系起来.

解 解法一:

$$|A| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & C_2^1 & \cdots & C_n^1 \\ 1 & C_3^2 & \cdots & C_{n+1}^2 \\ \vdots & \vdots & & \vdots \\ 1 & C_n^{n-1} & \cdots & C_{2n-2}^{n-1} \end{vmatrix} \xrightarrow[i=n, \dots, 2]{(-1) \cdot r_{i-1} + r_i} \begin{vmatrix} C_0^0 & C_1^0 & \cdots & C_{n-1}^0 \\ 0 & C_2^1 - C_1^0 & \cdots & C_n^1 - C_{n-1}^0 \\ 0 & C_3^2 - C_2^1 & \cdots & C_{n+1}^2 - C_n^1 \\ \vdots & \vdots & & \vdots \\ 0 & C_n^{n-1} - C_{n-1}^{n-2} & \cdots & C_{2n-2}^{n-1} - C_{2n-3}^{n-2} \end{vmatrix}$$

$$\begin{aligned}
&= \begin{vmatrix} C_0^0 & C_1^0 & \cdots & C_{n-1}^0 \\ 0 & C_1^1 & \cdots & C_{n-1}^1 \\ 0 & C_2^2 & \cdots & C_n^2 \\ \vdots & \vdots & & \vdots \\ 0 & C_{n-1}^{n-1} & \cdots & C_{2n-3}^{n-1} \end{vmatrix} \xrightarrow{\text{按第一列展开}} \begin{vmatrix} C_1^1 & C_2^1 & \cdots & C_{n-1}^1 \\ C_2^2 & C_3^2 & \cdots & C_n^2 \\ \vdots & \vdots & & \vdots \\ C_{n-1}^{n-1} & C_n^{n-1} & \cdots & C_{2n-3}^{n-1} \end{vmatrix} \\
&\xrightarrow[i=n, \dots, 2]{(-1)^{j_{i-1}+j_i}} \begin{vmatrix} C_1^1 & C_2^1 - C_1^1 & \cdots & C_{n-1}^1 - C_{n-2}^1 \\ C_2^2 & C_3^2 - C_2^2 & \cdots & C_n^2 - C_{n-1}^2 \\ \vdots & \vdots & & \vdots \\ C_{n-1}^{n-1} & C_n^{n-1} - C_{n-1}^{n-1} & \cdots & C_{2n-3}^{n-1} - C_{2n-4}^{n-1} \end{vmatrix} = \begin{vmatrix} C_1^1 & C_1^0 & \cdots & C_{n-2}^0 \\ C_2^2 & C_2^1 & \cdots & C_{n-1}^1 \\ \vdots & \vdots & & \vdots \\ C_{n-1}^{n-1} & C_{n-1}^{n-2} & \cdots & C_{2n-4}^{n-2} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & C_2^1 & \cdots & C_{n-1}^1 \\ \vdots & \vdots & & \vdots \\ 1 & C_{n-1}^{n-2} & \cdots & C_{2n-4}^{n-2} \end{vmatrix}.
\end{aligned}$$

此时得到的行列式恰好是原行列式的左上角部分, 并具有相同的规律. 不断这样做下去, 最后可得 $|A| = 1$

解法二: 设 $A = (a_{ij})_{n \times n}$, 则 $a_{ij} = C_{i+j-2}^{i-1}$, $i, j = 1, 2, \dots, n$. 从而由 **Vandermode 恒等式** 及组合数定义的扩充可得

$$\begin{aligned}
a_{ij} &= C_{i+j-2}^{i-1} = \sum_{k=0}^{i+j-2} C_{i-1}^{i-1-k} C_{j-1}^k = \sum_{k=1}^{i+j-1} C_{i-1}^{i-k} C_{j-1}^{k-1} \\
&= \sum_{k=1}^n C_{i-1}^{i-k} C_{j-1}^{k-1} = \sum_{k=1}^n C_{i-1}^{k-1} C_{j-1}^{i-k} = \sum_{k=1}^n l_{ik} l_{jk},
\end{aligned}$$

其中 $l_{ij} = \begin{cases} C_{i-1}^{j-1}, & 1 \leq j \leq i \leq n, \\ 0, & 1 \leq i < j \leq n, \end{cases}$. 记 $L = (l_{ij})_{n \times n}$, 则根据矩阵乘法的定义可知

$$A = LL^T \Rightarrow |A| = |L|^2.$$

因为当 $1 \leq i < j \leq n$ 时, $l_{ij} = 0$, 所以 L 是上三角矩阵. 于是

$$|L| = \prod_{i=1}^n l_{ii} = \prod_{i=1}^n C_{i-1}^{i-1} = 1.$$

故 $|A| = |L|^2 = 1$. □

例题 0.2 计算 n 阶行列式:

$$|A| = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & 3 & \cdots & n \\ -1 & -2 & 0 & \cdots & n \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix}$$

解

$$|A| = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & 3 & \cdots & n \\ -1 & -2 & 0 & \cdots & n \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix} \xrightarrow[i=2, \dots, n]{r_1 + r_i} \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 2 & * & \cdots & * \\ 0 & 0 & 3 & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n \end{vmatrix} = n!$$

□

例题 0.3 计算 n 阶行列式:

$$|A| = \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1b_n \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2b_n \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1b_n & a_2b_n & a_3b_n & \cdots & a_nb_n \end{vmatrix}.$$

解

$$\begin{aligned} |A| &= \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \cdots & a_1b_n \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2b_n \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1b_n & a_2b_n & a_3b_n & \cdots & a_nb_n \end{vmatrix} = a_1 \begin{vmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ a_1b_2 & a_2b_2 & a_2b_3 & \cdots & a_2b_n \\ a_1b_3 & a_2b_3 & a_3b_3 & \cdots & a_3b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1b_n & a_2b_n & a_3b_n & \cdots & a_nb_n \end{vmatrix} \\ &\stackrel{\substack{(-a_i)r_1+r_i \\ i=2,\cdots,n}}{=} a_1 \begin{vmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ a_1b_2 - a_2b_1 & 0 & 0 & \cdots & 0 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1b_n - a_nb_1 & a_2b_n - a_nb_2 & a_3b_n - a_nb_3 & \cdots & 0 \end{vmatrix} \\ &\stackrel{\text{按第 } n \text{ 列展开}}{=} (-1)^{n+1} a_1b_n \begin{vmatrix} a_1b_2 - a_2b_1 & 0 & \cdots & 0 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1b_n - a_nb_1 & a_2b_n - a_nb_2 & \cdots & a_{n-1}b_n - a_nb_{n-1} \end{vmatrix} \\ &= (-1)^{n-1} a_1b_n \prod_{i=1}^{n-1} (a_ib_{i+1} - a_{i+1}b_i) \\ &= a_1b_n \prod_{i=1}^{n-1} (a_{i+1}b_i - a_ib_{i+1}). \end{aligned}$$

□

命题 0.1 (“爪”型行列式)

证明 n 阶行列式:

$$|A| = \begin{vmatrix} a_1 & b_2 & \cdots & b_n \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{vmatrix} = a_1a_2 \cdots a_n - \sum_{i=2}^n a_2 \cdots \widehat{a_i} \cdots a_nb_ic_i.$$



笔记 记忆“爪”型行列式的计算方法和结论.

证明 当 $a_i \neq 0 (\forall i \in [2, n] \cap \mathbb{N})$ 时, 我们有

$$|A| = \begin{vmatrix} a_1 & b_2 & \cdots & b_n \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{vmatrix} \stackrel{\substack{(-\frac{c_i}{a_i})j_i+j_1 \\ i=2,\cdots,n}}{=} \begin{vmatrix} a_1 - \sum_{i=2}^n \frac{b_ic_i}{a_i} & b_2 & \cdots & b_n \\ 0 & a_2 & & \\ \vdots & & \ddots & \\ 0 & & & a_n \end{vmatrix}$$

$$= \left(a_1 - \sum_{i=2}^n \frac{b_i c_i}{a_i} \right) \prod_{i=2}^n a_i = a_1 a_2 \cdots a_n - \sum_{i=2}^n a_2 \cdots \widehat{a_i} \cdots a_n b_i c_i.$$

当 $\exists i \in [2, n] \cap \mathbb{N}$ s.t. $a_i = 0$ 时, 则 $a_1 a_2 \cdots a_n - \sum_{i=2}^n a_2 \cdots \widehat{a_i} \cdots a_n b_i c_i = -a_2 \cdots \widehat{a_i} \cdots a_n b_i c_i$. 此时, 我们有

$$|A| = \begin{vmatrix} a_1 & b_2 & \cdots & b_{i-1} & b_i & b_{i+1} & \cdots & b_n \\ c_2 & a_2 & & & & & & \\ \vdots & & \ddots & & & & & \\ c_{i-1} & & & a_{i-1} & & & & \\ c_i & & & & 0 & & & \\ c_{i+1} & & & & & a_{i+1} & & \\ \vdots & & & & & & \ddots & \\ c_n & & & & & & & a_n \end{vmatrix} \xrightarrow[\text{(按 } c_i \text{ 所在行展开)}]{\text{按第 } i \text{ 行展开}} (-1)^{i+1} c_i \begin{vmatrix} b_2 & \cdots & b_{i-1} & b_i & b_{i+1} & \cdots & b_n \\ a_2 & & & & & & \\ & \ddots & & & & & \\ & & a_{i-1} & 0 & 0 & & \\ & & 0 & 0 & a_{i+1} & & \\ & & & & & \ddots & \\ & & & & & & a_n \end{vmatrix}$$


$$\xrightarrow[\text{(按 } b_i \text{ 所在列展开)}]{\text{按第 } i-1 \text{ 列展开}} (-1)^{i+1} (-1)^i b_i c_i \begin{vmatrix} a_2 & & & & \\ & \ddots & & & \\ & & a_{i-1} & & \\ & & & a_{i+1} & \\ & & & & \ddots & \\ & & & & & a_n \end{vmatrix} = -a_2 \cdots \widehat{a_i} \cdots a_n b_i c_i.$$

综上所述, 原命题得证. □

命题 0.2 (分块“爪”型行列式)

计算 n 阶行列式 ($a_{ii} \neq 0, i = k+1, k+2, \dots, n$):

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{k,k+1} & \cdots & a_{kn} \\ a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1,k+1} & & \\ \vdots & & \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nk} & & & a_{nn} \end{vmatrix}.$$

 **笔记** 记忆分块“爪”型行列式的计算方法即可, 计算方法和“爪”型行列式的计算方法类似.

解

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{k,k+1} & \cdots & a_{kn} \\ a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1,k+1} & & \\ \vdots & & \vdots & & \ddots & \\ a_{n1} & \cdots & a_{nk} & & & a_{nn} \end{vmatrix}$$

$$\begin{aligned}
& \frac{-\frac{a_{i1}}{a_{ii}}j_i+j_1, -\frac{a_{i2}}{a_{ii}}j_i+j_2, \dots, -\frac{a_{in}}{a_{ii}}j_i+j_n}{i=k+1, k+2, \dots, n} \begin{vmatrix} c_{11} & \cdots & c_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{k1} & \cdots & c_{kk} & a_{k,k+1} & \cdots & a_{kn} \\ 0 & \cdots & 0 & a_{k+1,k+1} & & \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & & & a_{nn} \end{vmatrix} \\
& = \begin{vmatrix} C & B \\ O & \Lambda \end{vmatrix} = |C| \cdot |\Lambda| = |C| \prod_{i=k+1}^n a_{ii}.
\end{aligned}$$

其中 $C = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{kk} \end{pmatrix}$, $B = \begin{pmatrix} a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k,k+1} & \cdots & a_{kn} \end{pmatrix}$, $\Lambda = \begin{pmatrix} a_{k+1} & & \\ & \ddots & \\ & & a_n \end{pmatrix}$. 并且 $c_{pq} = a_{pq} - \sum_{i=k+1}^n \frac{a_{iq}a_{pi}}{a_{ii}}$, $p, q = 1, 2, \dots, n$. □

推论 0.1 (“爪”型行列式的推广)

计算 n 阶行列式:

$$|A| = \begin{vmatrix} x_1 - a_1 & x_2 & x_3 & \cdots & x_n \\ x_1 & x_2 - a_2 & x_3 & \cdots & x_n \\ x_1 & x_2 & x_3 - a_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n - a_n \end{vmatrix}.$$



笔记 这是一个有用的模板 (即行列式除了主对角元素外, 每行都一样).

记忆该命题的计算方法即可. 即先化为“爪”型行列式, 再利用“爪”型行列式的计算结果.

解 当 $a_i \neq 0 (\forall i \in [2, n] \cap \mathbb{N})$ 时, 我们有

$$\begin{aligned}
|A| &= \begin{vmatrix} x_1 - a_1 & x_2 & x_3 & \cdots & x_n \\ x_1 & x_2 - a_2 & x_3 & \cdots & x_n \\ x_1 & x_2 & x_3 - a_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n - a_n \end{vmatrix} \xrightarrow[\substack{(-1)r_1+r_i \\ i=2, \dots, n}]{\substack{(-1)r_1+r_i \\ i=2, \dots, n}} \begin{vmatrix} x_1 - a_1 & x_2 & x_3 & \cdots & x_n \\ a_1 & -a_2 & 0 & \cdots & 0 \\ a_1 & 0 & -a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & 0 & 0 & \cdots & -a_n \end{vmatrix} \\
&\stackrel{\text{命题0.1}}{=} \left[(x_1 - a_1) + \sum_{i=2}^n \frac{a_1 x_i}{a_i} \right] \prod_{i=2}^n (-a_i) = (-1)^{n-1} \left[(x_1 - a_1) + \sum_{i=2}^n \frac{a_1 x_i}{a_i} \right] \prod_{i=2}^n a_i \\
&= (-1)^{n-1} \left[(x_1 - a_1) \prod_{i=2}^n a_i + \sum_{i=2}^n a_1 a_2 \cdots \widehat{a_i} \cdots a_n x_i \right].
\end{aligned}$$

当 $\exists i \in [2, n] \cap \mathbb{N}$ s.t. $a_i = 0$ 时, 我们有

$$|A| = \begin{vmatrix} x_1 - a_1 & x_2 & x_3 & \cdots & x_n \\ x_1 & x_2 - a_2 & x_3 & \cdots & x_n \\ x_1 & x_2 & x_3 - a_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n - a_n \end{vmatrix} \xrightarrow[i=2, \dots, n]{(-1)r_1+r_i} \begin{vmatrix} x_1 - a_1 & x_2 & x_3 & \cdots & x_n \\ a_1 & -a_2 & 0 & \cdots & 0 \\ a_1 & 0 & -a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \cdots & -a_n \end{vmatrix}$$

$$\stackrel{\text{命题0.1}}{=} (x_1 - a_1)(-a_2)(-a_3) \cdots (-a_n) - \sum_{i=2}^n (-a_2) \cdots \widehat{(-a_i)} \cdots (-a_n) a_1 x_i$$

$$= (-1)^{n-1} (x_1 - a_1) \prod_{i=2}^n a_i + (-1)^{n-1} \sum_{i=2}^n a_1 a_2 \cdots \widehat{a_i} \cdots a_n x_i$$

$$= (-1)^{n-1} \left[(x_1 - a_1) \prod_{i=2}^n a_i + \sum_{i=2}^n a_1 a_2 \cdots \widehat{a_i} \cdots a_n x_i \right].$$

综上所述, $|A| = (-1)^{n-1} \left[(x_1 - a_1) \prod_{i=2}^n a_i + \sum_{i=2}^n a_1 a_2 \cdots \widehat{a_i} \cdots a_n x_i \right].$ □

例题 0.4 计算 n 阶行列式:

$$|A| = \begin{vmatrix} a & 0 & \cdots & 0 & 1 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 1 & 0 & \cdots & 0 & a \end{vmatrix}.$$

解

$$|A| = \begin{vmatrix} a & 0 & \cdots & 0 & 1 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 1 & 0 & \cdots & 0 & a \end{vmatrix} \xrightarrow{\text{按第一列展开}} a^n + (-1)^{n+1} \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ a & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \end{vmatrix} = a^n + (-1)^{n+1+n} a^{n-2} = a^n - a^{n-2}.$$

□

注 本题也可由命题0.1直接得到, $|A| = a^n - a^{n-2}$.

命题 0.3


设 $|A| = |a_i|$ 是一个 n 阶行列式, A_{ij} 是它的第 (i, j) 元素的代数余子式, $X = (x_1, x_2, \dots, x_n)^T$, $Y = (y_1, y_2, \dots, y_n)^T$, z 是任意常数, 求证:

$$\begin{vmatrix} A & X \\ Y^T & z \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & x_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & x_n \\ y_1 & y_2 & \cdots & y_n & z \end{vmatrix} = z|A| - \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i y_j.$$

进而得到

$$\begin{vmatrix} A & X \\ Y^T & 0 \end{vmatrix} = -Y^T A^* X.$$

◆

 **笔记** 根据这个命题可以得到一个关于行列式 $|A|$ 的所有代数余子式求和的构造:

$$-\sum_{i,j=1}^n A_{ij} = \begin{vmatrix} \mathbf{A} & \mathbf{1} \\ \mathbf{1}' & 0 \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \mathbf{1} \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix} = \begin{vmatrix} \beta_1 & 1 \\ \beta_2 & 1 \\ \vdots & \vdots \\ \beta_n & 1 \\ \mathbf{1}' & 0 \end{vmatrix}.$$

其中 $|\mathbf{A}|$ 的列向量依次为 $\alpha_1, \alpha_2, \dots, \alpha_n$, $|\mathbf{A}|$ 的行向量依次为 $\beta_1, \beta_2, \dots, \beta_n$. 并且 $\mathbf{1}$ 表示元素均为 1 的列向量, $\mathbf{1}'$ 表示 $\mathbf{1}$ 的转置. (令上述命题中的 $z=0, x_i=y_i=1, i=1, 2, \dots, n$ 即可得到.)

注 如果需要证明的是矩阵的代数余子式的相关命题, 我们可以考虑一下这种构造, 即令上述命题中的 $z=0$ 并且待定/任取 x_i, y_i .

证明 证法一: 将上述行列式先按最后一列展开, 展开式的第一项为

$$(-1)^{n+2} x_1 \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ y_1 & y_2 & \cdots & y_n \end{vmatrix}.$$

再将上式按最后一行展开得到

$$\begin{aligned} & (-1)^{n+2} x_1 [(-1)^{n+1} (-1)^{1+1} y_1 A_{11} + (-1)^{n+2} (-1)^{1+2} y_2 A_{12} + \cdots + (-1)^{n+n} (-1)^{1+n} y_n A_{1n}] \\ &= (-1)^{n+2} x_1 (-1)^{n+1} [(-1)^2 y_1 A_{11} + (-1)^4 y_2 A_{12} + \cdots + (-1)^{2n} y_n A_{1n}] \\ &= -x_1 (y_1 A_{11} + y_2 A_{12} + \cdots + y_n A_{1n}) \\ &= -x_1 \sum_{j=1}^n y_j A_{1j}. \end{aligned}$$

同理可得原行列式展开式的第 i ($i=1, 2, \dots, n-1$) 项为

$$(-1)^{n+1+i} x_i \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ y_1 & y_2 & \cdots & y_n \end{vmatrix}.$$

将上式按最后一行展开得到 $z|\mathbf{A}|$.

$$\begin{aligned} & (-1)^{n+1+i} x_i [(-1)^{n+1} (-1)^{i+1} y_1 A_{i1} + (-1)^{n+2} (-1)^{i+2} y_2 A_{i2} + \cdots + (-1)^{n+n} (-1)^{i+n} y_n A_{in}] \\ &= (-1)^{n+1+i} x_i (-1)^{n+1} [(-1)^{i+1} y_1 A_{i1} + (-1)^{i+2+1} y_2 A_{i2} + \cdots + (-1)^{i+n+n-1} y_n A_{in}] \\ &= (-1)^{2i+1} y_1 A_{i1} + (-1)^{2i+3} y_2 A_{i2} + \cdots + (-1)^{2i+2n-1} y_n A_{in} \\ &= -x_i (y_1 A_{i1} + y_2 A_{i2} + \cdots + y_n A_{in}) \\ &= -x_i \sum_{j=1}^n y_j A_{ij}. \end{aligned}$$

而展开式的最后一项为 $z|\mathbf{A}|$.

因此, 原行列式的值为

$$z|\mathbf{A}| - \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i y_j.$$

证法二: 设 $\mathbf{x} = (x_1, x_2, \dots, x_n)'$, $\mathbf{y} = (y_1, y_2, \dots, y_n)'$. 若 A 是非异阵, 则由降阶公式可得

$$\begin{vmatrix} A & \mathbf{x} \\ \mathbf{y}' & z \end{vmatrix} = |A|(z - \mathbf{y}'A^{-1}\mathbf{x}) = z|A| - \mathbf{y}'A^*\mathbf{x}.$$

对于一般的方阵 A , 可取到一列有理数 $t_k \rightarrow 0$, 使得 $t_k I_n + A$ 为非异阵. 由非异阵情形的证明可得

$$\begin{vmatrix} t_k I_n + A & \mathbf{x} \\ \mathbf{y}' & z \end{vmatrix} = z|t_k I_n + A| - \mathbf{y}'(t_k I_n + A)^*\mathbf{x}.$$

注意到上式两边都是关于 t_k 的多项式, 从而关于 t_k 连续. 上式两边同时取极限, 令 $t_k \rightarrow 0$, 即有

$$\begin{vmatrix} A & \mathbf{x} \\ \mathbf{y}' & z \end{vmatrix} = z|A| - \mathbf{y}'A^*\mathbf{x} = z|A| - \sum_{i=1}^n \sum_{j=1}^n A_{ij}x_i y_j.$$

□

例题 0.5 设 n 阶行列式 $|A| = |a_{ij}|$, A_{ij} 是元素 a_{ij} 的代数余子式, 求证:

$$|B| = \begin{vmatrix} a_{11} - a_{12} & a_{12} - a_{13} & \cdots & a_{1,n-1} - a_{1n} & 1 \\ a_{21} - a_{22} & a_{22} - a_{23} & \cdots & a_{2,n-1} - a_{2n} & 1 \\ a_{31} - a_{32} & a_{32} - a_{33} & \cdots & a_{3,n-1} - a_{3n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} - a_{n2} & a_{n2} - a_{n3} & \cdots & a_{n,n-1} - a_{nn} & 1 \end{vmatrix} = \sum_{i,j=1}^n A_{ij}.$$

证明 证法一: 设 $|A|$ 的列向量依次为 $\alpha_1, \alpha_2, \dots, \alpha_n$, 并且 $\mathbf{1}$ 表示元素均为 1 的列向量. 则

$$|B| = |\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \dots, \alpha_{n-1} - \alpha_n, \mathbf{1}| \xrightarrow[i=n-1, n-2, \dots, 2]{ji+j_{i-1}} |\alpha_1 - \alpha_n, \alpha_2 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n, \mathbf{1}|.$$

将最后一列写成 $(\alpha_n + \mathbf{1}) - \alpha_n$, 进行拆分可得

$$\begin{aligned} |B| &= |\alpha_1 - \alpha_n, \alpha_2 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n, (\alpha_n + \mathbf{1}) - \alpha_n| \\ &= |\alpha_1 - \alpha_n, \alpha_2 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n, \alpha_n + \mathbf{1}| - |\alpha_1 - \alpha_n, \alpha_2 - \alpha_n, \dots, \alpha_{n-1} - \alpha_n, \alpha_n| \\ &= |\alpha_1 + \mathbf{1}, \alpha_2 + \mathbf{1}, \dots, \alpha_{n-1} + \mathbf{1}, \alpha_n + \mathbf{1}| - |\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n|. \end{aligned}$$

根据行列式的性质将 $|\alpha_1 + \mathbf{1}, \alpha_2 + \mathbf{1}, \dots, \alpha_{n-1} + \mathbf{1}, \alpha_n + \mathbf{1}|$ 每一列都拆分成两列, 然后按 $\mathbf{1}$ 所在的列展开得到

$$\begin{aligned} |B| &= |\alpha_1 + \mathbf{1}, \alpha_2 + \mathbf{1}, \dots, \alpha_{n-1} + \mathbf{1}, \alpha_n + \mathbf{1}| - |\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n| \\ &= |\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n| + \sum_{i,j=1}^n A_{ij} - |\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n| = \sum_{i,j=1}^n A_{ij}. \end{aligned}$$

证法二: 设 $|A|$ 的列向量依次为 $\alpha_1, \alpha_2, \dots, \alpha_n$, 并且 $\mathbf{1}$ 表示元素均为 1 的列向量. 注意到

$$-\sum_{i,j=1}^n A_{ij} = \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \mathbf{1} \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix}.$$

依次将第 i 列乘以 -1 加到第 $i-1$ 列上去 ($i=2, 3, \dots, n$), 再按第 $n+1$ 行展开可得

$$\begin{aligned} -\sum_{i,j=1}^n A_{ij} &= \begin{vmatrix} \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \cdots & \alpha_{n-1} - \alpha_n & \alpha_n & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{vmatrix} \\ &= -|\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \dots, \alpha_{n-1} - \alpha_n, \mathbf{1}| = -|B|. \end{aligned}$$

结论得证.

□

例题 0.6 设 n 阶矩阵 A 的每一行、每一列的元素之和都为零, 证明: A 的每个元素的代数余子式都相等.

证明 证法一: 设 $A = (a_{ij})$, $\mathbf{x} = (x_1, x_2, \dots, x_n)'$, $\mathbf{y} = (y_1, y_2, \dots, y_n)'$, 不妨设 $x_i y_j$ 均不相同, $i, j = 1, 2, \dots, n$. 考虑如下 $n+1$ 阶矩阵的行列式求值:

$$B = \begin{pmatrix} A & \mathbf{x} \\ \mathbf{y}' & 0 \end{pmatrix}$$

一方面,由命题 0.3 可得 $|B| = -\sum_{i=1}^n \sum_{j=1}^n A_{ij}x_iy_j$. 另一方面,先把行列式 $|B|$ 的第二行, ..., 第 n 行全部加到第一行上;再将第二列, ..., 第 n 列全部加到第一列上,可得

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & x_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & x_n \\ y_1 & y_2 & \cdots & y_n & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & \cdots & 0 & \sum_{i=1}^n x_i \\ a_{21} & a_{22} & \cdots & a_{2n} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & x_n \\ y_1 & y_2 & \cdots & y_n & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & \cdots & 0 & \sum_{i=1}^n x_i \\ 0 & a_{22} & \cdots & a_{2n} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} & x_n \\ \sum_{j=1}^n y_j & y_2 & \cdots & y_n & 0 \end{vmatrix}$$

依次按照第一行和第一列进行展开,可得 $|B| = -A_{11} \sum_{i=1}^n \sum_{j=1}^n x_iy_j$. 比较上述两个结果,又由于 x_iy_j 均不相同,因此可得 A 的所有代数余子式都相等.

证法二:由假设可知 $|A| = 0$ (每行元素全部加到第一行即得),从而 A 是奇异矩阵. 若 A 的秩小于 $n-1$,则 A 的任意一个代数余子式 A_{ij} 都等于零,结论显然成立. 若 A 的秩等于 $n-1$,则线性方程组 $Ax = 0$ 的基础解系只含一个向量. 又因为 A 的每一行元素之和都等于零,所以由命题??可知,我们可以选取 $\alpha = (1, 1, \dots, 1)'$ 作为 $Ax = 0$ 的基础解系. 由命题??的证明可知 A^* 的每一列都是 $Ax = 0$ 的解,从而 A^* 的每一列与 α 成比例,特别地, A^* 的每一行都相等. 对 A' 重复上面的讨论,可得 $(A')^*$ 的每一行都相等. 注意到 $(A')^* = (A^*)'$, 从而 A^* 的每一列都相等,于是 A 的所有代数余子式 A_{ij} 都相等. \square