

0.1 积分性态分析

例题 0.1 已知 $f(x) \in C[a, b]$, 且

$$\int_a^b f(x) dx = \int_a^b x f(x) dx = 0.$$

证明: $f(x)$ 在 (a, b) 上至少 2 个零点.

证明 设 $F_1(x) = \int_a^x f(t) dt$, 则 $F_1(a) = F_1(b) = 0$. 再设 $F_2(x) = \int_a^x F_1(t) dt = \int_a^x \left[\int_a^t f(s) ds \right] dt$, 则 $F_2(a) = 0, F_2'(x) = F_1(x), F_2''(x) = F_1'(x) = f(x)$. 由条件可知


$$0 = \int_a^b x f(x) dx = \int_a^b x F_1'(x) dx = \int_a^b x dF_1(x) = x F_1(x) \Big|_a^b - \int_a^b F_1(x) dx = -F_2(b).$$

于是由 Rolle 中值定理可知, 存在 $\xi \in (a, b)$, 使得 $F_2'(\xi) = F_1(\xi) = 0$. 从而再由 Rolle 中值定理可知, 存在 $\eta_1 \in (a, \xi), \eta_2 \in (\xi, b)$, 使得 $F_1'(\eta_1) = F_1'(\eta_2) = 0$. 即 $f(\eta_1) = f(\eta_2) = 0$. \square

例题 0.2 已知 $f(x) \in C[a, b]$, 且

$$\int_a^b x^k f(x) dx = 0, k = 0, 1, 2, \dots, n.$$

证明: $f(x)$ 在 (a, b) 上至少 $n+1$ 个零点.

 **笔记** 利用分部积分转换导数的技巧.

证明 令 $F(x) = \int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_1) dx_1 \dots dx_{n-1}$. 则 $F(a) = F'(a) = \dots = F^{(n)}(a) = 0, F^{(n+1)}(x) = f(x)$. 由已知条件, 再反复分部积分, 可得当 $1 \leq k \leq n$ 且 $k \in \mathbb{N}$ 时, 有


$$\begin{aligned} 0 &= \int_a^b f(x) dx = \int_a^b F^{(n+1)}(x) dx = F^{(n)}(x) \Big|_a^b = F^{(n)}(b), \\ 0 &= \int_a^b x f(x) dx = \int_a^b x F^{(n+1)}(x) dx = \int_a^b x dF^{(n)}(x) = x F^{(n)}(x) \Big|_a^b - \int_a^b F^{(n)}(x) dx = -F^{(n-1)}(b), \\ &\dots\dots \\ 0 &= \int_a^b x^n f(x) dx = \int_a^b x^n F^{(n+1)}(x) dx = \int_a^b x^n dF^{(n)}(x) = x^n F^{(n)}(x) \Big|_a^b - n \int_a^b x^{n-1} F^{(n)}(x) dx \\ &= -n \int_a^b x^{n-1} F^{(n)}(x) dx = \dots = (-1)^n n! \int_a^b F'(x) dx = (-1)^n n! F(b). \end{aligned}$$

从而 $F(b) = F'(b) = \dots = F^{(n)}(b) = 0$. 于是由 Rolle 中值定理可知, 存在 $\xi_1^1 \in (a, b)$, 使得 $F'(\xi_1^1) = 0$. 再利用 Rolle 中值定理可知存在 $\xi_1^2, \xi_2^2 \in (a, b)$, 使得 $F''(\xi_1^2) = F''(\xi_2^2) = 0$. 反复利用 Rolle 中值定理可得, 存在 $\xi_1^{n+1}, \xi_2^{n+1}, \dots, \xi_{n+1}^{n+1} \in (a, b)$, 使得 $F^{(n+1)}(\xi_1^{n+1}) = F^{(n+1)}(\xi_2^{n+1}) = \dots = F^{(n+1)}(\xi_{n+1}^{n+1}) = 0$. 即 $f(\xi_1^{n+1}) = f(\xi_2^{n+1}) = \dots = f(\xi_{n+1}^{n+1}) = 0$. \square

例题 0.3 已知 $f(x) \in D^2[0, 1]$, 且

$$\int_0^1 f(x) dx = \frac{1}{6}, \int_0^1 x f(x) dx = 0, \int_0^1 x^2 f(x) dx = \frac{1}{60}.$$

证明: 存在 $\xi \in (0, 1)$, 使得 $f''(\xi) = 16$.

 **笔记** 构造 $g(x) = f(x) - (8x^2 - 9x + 2)$ 的原因: 受到上一题的启发, 我们希望找到一个 $g(x) = f(x) - p(x)$, 使得

$$\int_0^1 x^k g(x) dx = \int_0^1 x^k [f(x) - p(x)] dx = 0, \quad k = 0, 1, 2.$$

成立. 即

$$\int_0^1 x^k f(x) dx = \int_0^1 x^k p(x) dx, \quad k = 0, 1, 2.$$

待定 $p(x) = ax^2 + bx + c$, 则代入上述公式, 再结合已知条件可得

$$\frac{1}{6} = \int_0^1 p(x) dx = \int_0^1 (ax^2 + bx + c) dx = \frac{a}{3} + \frac{b}{2} + c,$$

$$0 = \int_0^1 xp(x)dx = \int_0^1 (ax^3 + bx^2 + cx) dx = \frac{a}{4} + \frac{b}{3} + \frac{c}{2},$$

$$\frac{1}{60} = \int_0^1 x^2 p(x)dx = \int_0^1 (ax^4 + bx^3 + cx^2) dx = \frac{a}{5} + \frac{b}{4} + \frac{c}{3}.$$

解得: $a = 8, b = -9, c = 2$. 于是就得到 $g(x) = f(x) - (8x^2 - 9x + 2)$.

证明 令 $g(x) = f(x) - (8x^2 - 9x + 2)$, 则由条件可得

$$\int_0^1 x^k g(x)dx = 0, \quad k = 0, 1, 2.$$

再令 $G(x) = \int_0^x \left[\int_0^t \left(\int_0^s g(y)dy \right) ds \right] dt$, 则 $G(0) = G'(0) = G''(0) = 0, G'''(x) = g(x)$. 利用分部积分可得

$$0 = \int_0^1 g(x) dx = \int_0^1 G'''(x) dx = G''(1),$$

$$0 = \int_0^1 xg(x) dx = \int_0^1 xG'''(x) dx = \int_0^1 x dG''(x) = xG''(x) \Big|_0^1 - \int_0^1 G''(x) dx = -G'(1),$$

$$0 = \int_0^1 x^2 g(x) dx = \int_0^1 x^2 G'''(x) dx = \int_0^1 x^2 dG''(x) = x^2 G''(x) \Big|_0^1 - 2 \int_0^1 xG''(x) dx$$

$$= -2 \int_0^1 x dG'(x) = 2 \int_0^1 G'(x) dx - 2xG'(x) \Big|_0^1 = 2G(1).$$

从而 $G(1) = G'(1) = G''(1) = 0$. 于是由 *Rolle* 中值定理可知, 存在 $\xi_1^1 \in (0, 1)$, 使得 $G'(\xi_1^1) = 0$. 再利用 *Rolle* 中值定理可知, 存在 $\xi_1^2, \xi_2^2 \in (0, 1)$, 使得 $G''(\xi_1^2) = G''(\xi_2^2) = 0$. 反复利用 *Rolle* 中值定理可得, 存在 $\xi_1^3, \xi_2^3, \xi_3^3 \in (0, 1)$, 使得 $G'''(\xi_1^3) = G'''(\xi_2^3) = G'''(\xi_3^3) = 0$. 即 $g(\xi_1^3) = g(\xi_2^3) = g(\xi_3^3) = 0$. 再反复利用 *Rolle* 中值定理可得, 存在 $\xi \in (0, 1)$, 使得 $g''(\xi) = 0$. 即 $f''(\xi) = 16$. □