

0.1 定积分

0.1.1 建立积分递推

例题 0.1 计算 $\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx, n \in \mathbb{N}$.

证明 利用分部积分和和差化积公式可得

$$\begin{aligned}
 I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx) dx \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)] dx \\
 &= \frac{I_{n-1}}{2} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x [\sin(nx) \cos x + \cos(nx) \sin x] dx \\
 &= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(nx) d \cos x \\
 &= \frac{I_{n-1} + I_n}{2} - \frac{1}{2n} \int_0^{\frac{\pi}{2}} \cos(nx) d \cos^n x \\
 &= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx \\
 &= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
 &= \frac{I_{n-1}}{2} + \frac{1}{2n}.
 \end{aligned}$$

故 $I_n = \frac{I_{n-1}}{2} + \frac{1}{2n}$, 则两边同乘 2^n (强行裂项)

$$2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \dots$$

又注意到 $I_0 = 0$, 从而

$$2^n I_n = 0 + \sum_{k=1}^n \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n} \sum_{k=1}^n \frac{2^{k-1}}{k}.$$

□

命题 0.1


证明:

$$(1) \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx = \begin{cases} 0, & n \text{ 为偶数} \\ \pi, & n \text{ 为奇数} \end{cases}.$$

$$(2) \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx = n\pi.$$

$$(3) \int_0^{\pi} \frac{\sin^2(nx)}{\sin x} dx = \sum_{k=1}^n \frac{2}{2k-1}.$$

♣

 **笔记** 提示: $\sin^2 x - \sin^2 y = \sin(x-y) \sin(x+y)$ (证明见命题??).

证明

(1) 记 $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx$, 则

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin((n+2)x) - \sin(nx)}{\sin x} dx = \int_0^{\pi} \frac{2 \cos((n+1)x) \sin x}{\sin x} dx = 2 \int_0^{\pi} \cos((n+1)x) dx = 0.$$

于是

$$\int_0^{\pi} \frac{\sin(nx)}{\sin x} dx = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n \text{ 为偶数} \\ I_1, & n \text{ 为奇数} \end{cases} = \begin{cases} 0, & n \text{ 为偶数} \\ \pi, & n \text{ 为奇数} \end{cases}.$$

(2) 记 $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^{\pi} \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x} dx = \int_0^{\pi} \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x} dx \\ &= \int_0^{\pi} \frac{\sin((2n+1)x)}{\sin x} dx \stackrel{\text{命题 0.1(1)}}{=} \pi. \end{aligned} \quad (1)$$

于是

$$\int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记 $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin x} dx$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^{\pi} \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x} dx = \int_0^{\pi} \frac{\sin x \cdot \sin((2n+1)x)}{\sin x} dx \\ &= \int_0^{\pi} \sin((2n+1)x) dx = \frac{1}{2n+1} \cos((2n+1)x) \Big|_{\pi}^0 = \frac{2}{2n+1}. \end{aligned} \quad (2)$$

于是

$$\int_0^{\pi} \frac{\sin^2(nx)}{\sin x} dx = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1} \frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1} \frac{2}{2k+1} = \sum_{k=1}^n \frac{2}{2k-1}.$$

□

0.1.2 区间再现

定理 0.1 (区间再现恒等式)

当下述积分有意义时, 我们有

1.

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx = \frac{1}{2} \int_a^b [f(x) + f(a+b-x)] dx = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] dx.$$

2.

$$\int_0^{\infty} f(x) dx = \int_0^1 f(x) dx + \int_1^{\infty} f(x) dx = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2} \right] dx.$$

♡



笔记 注意: 倒代换具有将 $[0, 1]$ 转化为 $[1, +\infty)$ 的功能.

证明 证明是显然的.

□

命题 0.2

证明

$$1. \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{2} \ln 2.$$

$$2. \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2.$$

$$3. \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2.$$

♠

证明

1.

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{2}} \ln \sin x dx = \int_0^{\frac{\pi}{4}} \left[\ln \cos x + \ln \left(\frac{\pi}{2} - x \right) \right] dx \\
 &= \int_0^{\frac{\pi}{4}} \ln (\cos x \sin x) dx = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2} dx + \int_0^{\frac{\pi}{4}} \ln (\sin 2x) dx \\
 &= -\frac{\pi}{4} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} I \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2.
 \end{aligned}$$

2.

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{2}} \ln \cos x dx = \int_0^{\frac{\pi}{2}} \ln \sin x dx = \int_0^{\frac{\pi}{4}} \left[\ln \cos x + \ln \left(\frac{\pi}{2} - x \right) \right] dx \\
 &= \int_0^{\frac{\pi}{4}} \ln (\cos x \sin x) dx = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2} dx + \int_0^{\frac{\pi}{4}} \ln (\sin 2x) dx \\
 &= -\frac{\pi}{4} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin x dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} I \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2.
 \end{aligned}$$

3.

$$\begin{aligned}
 \int_0^1 \frac{\ln(1+x)}{1+x^2} dx &\stackrel{x=\tan \theta}{=} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan \theta)}{1+\tan^2 \theta} d \tan \theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2 \theta \cdot \ln(1+\tan \theta)}{\sec^2 \theta} d\theta \\
 &= \int_0^{\frac{\pi}{4}} \ln(1+\tan \theta) d\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan \theta) + \ln \left(1 + \tan \left(\frac{\pi}{4} - \theta \right) \right) \right] d\theta \\
 &= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan \theta) + \ln \left(1 + \frac{1-\tan \theta}{1+\tan \theta} \right) \right] d\theta \\
 &= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan \theta) + \ln \frac{2}{1+\tan \theta} \right] d\theta \\
 &= \int_0^{\frac{\pi}{8}} \ln 2 d\theta = \frac{\pi}{8} \ln 2.
 \end{aligned}$$

□

例题 0.2 计算

$$1. \int_0^{\infty} \frac{\ln x}{x^2 + a^2} dx, a > 0.$$

$$2. \int_0^{\infty} \frac{\ln x}{x^2 + x + 1} dx.$$

$$3. \int_0^1 \frac{\ln x}{\sqrt{x-x^2}} dx.$$

解

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2 + a^2} dx \stackrel{x=at}{=} \frac{1}{a} \int_0^{+\infty} \frac{\ln(at)}{1+t^2} dt = \frac{1}{a} \int_0^{+\infty} \frac{\ln a}{1+t^2} dt + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} dt = \frac{\pi \ln a}{2a} + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} dt. \quad (3)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2} dt \stackrel{t=\frac{1}{x}}{=} \int_0^{+\infty} \frac{\ln \frac{1}{x}}{1+\frac{1}{x^2}} \frac{1}{x^2} dx = \int_0^{+\infty} \frac{-\ln x}{1+x^2} dx \Rightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2} dt = 0.$$

于是再结合(3)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2 + a^2} dx = \frac{\pi \ln a}{2a}.$$

2.

$$\int_0^{\infty} \frac{\ln x}{x^2 + x + 1} dx \stackrel{x=\frac{1}{t}}{=} \int_0^{\infty} \frac{-\ln t}{1 + \frac{1}{t} + \frac{1}{t^2}} d\frac{1}{t} = \int_0^{+\infty} \frac{-\ln t}{1 + t + t^2} dt \Rightarrow \int_0^{\infty} \frac{\ln x}{x^2 + x + 1} dx = 0.$$

3.

$$\begin{aligned} \int_0^1 \frac{\ln x}{\sqrt{x-x^2}} dx &\stackrel{x=\sin^2 y}{=} \int_0^{\frac{\pi}{2}} \frac{\ln \sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}} d\sin^2 y \\ &= 4 \int_0^{\frac{\pi}{2}} \ln \sin y dy \stackrel{\text{命题 0.2}}{=} 4 \cdot \left(-\frac{\pi}{2} \ln 2\right) = -2\pi \ln 2. \end{aligned}$$

□

例题 0.31. 对 $n \in \mathbb{N}$, 计算 $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx$.2. $\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} dx$.3. 对 $n \in \mathbb{N}$, 计算 $\int_0^{2\pi} \sin(\sin x + nx) dx$.

解

1.

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx &= \int_{-\pi}^0 \left[\frac{\sin(nx)}{(1+2^x)\sin x} + \frac{\sin(nx)}{(1+2^{-x})\sin x} \right] dx = \int_{-\pi}^0 \frac{\sin(nx)}{\sin x} \left(\frac{1}{1+2^x} + \frac{1}{1+2^{-x}} \right) dx \\ &= \int_{-\pi}^0 \frac{\sin(nx)}{\sin x} \cdot \frac{2+2^x+2^{-x}}{2+2^x+2^{-x}} dx = \int_{-\pi}^0 \frac{\sin(nx)}{\sin x} dx = \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx \stackrel{\text{例题 0.1}}{=} \begin{cases} 0, n \text{ 为偶数} \\ \pi, n \text{ 为奇数} \end{cases}. \end{aligned}$$

2.

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} dx &= \int_{-\pi}^0 \left(\frac{x \sin x \arctan e^x}{1+\cos^2 x} + \frac{x \sin x \arctan e^{-x}}{1+\cos^2 x} \right) dx = \int_{-\pi}^0 \frac{x \sin x}{1+\cos^2 x} (\arctan e^x + \arctan e^{-x}) dx \\ &\stackrel{\text{命题 ??(1)}}{=} \int_{-\pi}^0 \frac{x \sin x}{1+\cos^2 x} \cdot \frac{\pi}{2} dx = \frac{\pi}{2} \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \left(\frac{x \sin x}{1+\cos^2 x} + \frac{(\pi-x) \sin x}{1+\cos^2 x} \right) dx = \frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx \\ &= \frac{\pi^2}{2} \arctan \cos x \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{2} \cdot \frac{\pi}{4} = \frac{\pi^3}{8}. \end{aligned}$$

3.

$$\begin{aligned} \int_0^{2\pi} \sin(\sin x + nx) dx &= \int_0^{2\pi} \sin[\sin(2\pi-x) + n(2\pi-x)] dx \\ &= \int_0^{2\pi} \sin(-\sin x - nx) dx = - \int_0^{2\pi} \sin(\sin x + nx) dx \\ &\Rightarrow \int_0^{2\pi} \sin(\sin x + nx) dx = 0. \end{aligned}$$

□

0.1.3 化成多元累次积分 (换序)**命题 0.3**

证明:

$$(1) \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

$$(2) \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\sqrt{\pi}}{2}.$$

$$(3) \int_0^{\infty} \sin x^2 dx, \int_0^{\infty} \cos x^2 dx = \sqrt{\frac{\pi}{8}}.$$



笔记 本结果可以直接使用.

证明

(1) 注意到

$$\begin{aligned} \left(\int_0^{+\infty} e^{-x^2} dx \right)^2 &= \left(\int_0^{+\infty} e^{-x^2} dx \right) \left(\int_0^{+\infty} e^{-y^2} dy \right) \xrightarrow{\text{把 } \int_0^{+\infty} e^{-y^2} dy \text{ 看作常数}} \int_0^{+\infty} e^{-x^2} \left(\int_0^{+\infty} e^{-y^2} dx \right) dy \\ &\xrightarrow{\text{把 } e^{-x^2} \text{ 看作常数}} \int_0^{+\infty} \left(\int_0^{+\infty} e^{-(x^2+y^2)} dx \right) dy \xrightarrow{e^{-(x^2+y^2)} \text{ 连续}} \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} r e^{-r^2} dr = \frac{\pi}{2} \int_0^{+\infty} r e^{-r^2} dr \\ &= \frac{\pi}{4} \int_0^{+\infty} e^{-r^2} dr^2 = \frac{\pi}{4}. \end{aligned}$$

$$\text{故 } \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(2) 注意到

$$\int_0^{+\infty} \sin x e^{-yx} dx = \operatorname{Im} \int_0^{+\infty} e^{ix-yx} dx = \operatorname{Im} \int_0^{+\infty} e^{-(y-i)x} dx = \operatorname{Im} \frac{1}{y-i} = \operatorname{Im} \frac{y+i}{y^2+1} = \frac{1}{y^2+1}.$$

因此就有

$$\begin{aligned} \int_0^{+\infty} \frac{\sin x}{x} dx &= \int_0^{+\infty} \sin x \left(\int_0^{+\infty} e^{-yx} dy \right) dx = \int_0^{+\infty} dy \int_0^{+\infty} \sin x e^{-yx} dx \\ &= \int_0^{+\infty} dy \left(\operatorname{Im} \int_0^{+\infty} e^{ix-yx} dx \right) = \int_0^{+\infty} \frac{1}{y^2+1} dy = \frac{\pi}{2}. \end{aligned}$$

当然本题也可以直接利用分部积分计算 $\int_0^{+\infty} \sin x e^{-yx} dx = \frac{1}{y^2+1}$.

(3) 注意到

$$\int_0^{+\infty} e^{-ax^2} dx \xrightarrow{x=\frac{t}{\sqrt{a}}} \frac{1}{\sqrt{a}} \int_0^{+\infty} e^{-t^2} dt = \frac{1}{2} \sqrt{\frac{\pi}{a}}, \quad a > 0.$$

并且 $-i = e^{-\frac{\pi}{2}i}$, 从而 $\sqrt{-i} = e^{-\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$. 于是

$$\begin{aligned} \int_0^{+\infty} (\cos x^2 - i \sin x^2) dx &= \int_0^{+\infty} e^{-ix^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{i}} = \frac{1}{2} \sqrt{-i\pi} \\ &= \frac{\sqrt{\pi}}{2} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i. \end{aligned}$$

故

$$\begin{aligned} \int_0^{+\infty} \cos x^2 dx &= \operatorname{Re} \int_0^{+\infty} (\cos x^2 - i \sin x^2) dx = \operatorname{Re} \left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i \right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}, \\ \int_0^{+\infty} \sin x^2 dx &= \operatorname{Im} \int_0^{+\infty} (\cos x^2 - i \sin x^2) dx = \operatorname{Im} \left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i \right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}. \end{aligned}$$

□

例题 0.4 计算 $\int_0^1 \sin \ln \frac{1}{x} \cdot \frac{x^b - x^a}{\ln x} dx$ ($b > a > 0$).

证明

$$\begin{aligned} \int_0^1 \sin \ln \frac{1}{x} \cdot \frac{x^b - x^a}{\ln x} dx &= \int_0^1 \sin \ln \frac{1}{x} \left(\int_a^b x^y dy \right) dx = \int_a^b dy \int_0^1 x^y \sin \ln \frac{1}{x} dx \\ &\xrightarrow{x=e^{-t}} \int_a^b dy \int_{+\infty}^0 e^{-ty} \sin t de^{-t} = \int_a^b dy \int_0^{+\infty} e^{-t(y+1)} \sin t dt \end{aligned}$$

$$\underline{\underline{\text{命题 0.3(2) 的证明过程}}} \int_a^b \frac{1}{1+(y+1)^2} dy = \arctan(b+1) - \arctan(a+1).$$

□

0.1.4 化成含参积分 (求导)

例题 0.5 设 $a, b \geq 0$ 且不全为 0, 计算 $\int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx$.

注 实际上, 根据 $a > b$ 时得到的结果, 可以看出 $F(a, b) = \pi \ln \frac{a+b}{2}$ 对 a, b 有轮换对称性, 故这个结果对其他情况显然也成立.

证明 设 $F(a, b) = \int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx$, 当 $a > b$ 时, 则

$$\begin{aligned} \frac{\partial}{\partial b} F(a, b) &= \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial b} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \int_0^{\frac{\pi}{2}} \frac{2b \sin^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{2b \tan^2 x}{a^2 + b^2 \tan^2 x} dx = \int_0^{+\infty} \frac{2bt^2}{(a^2 + b^2 t^2)(1+t^2)} dt \\ &= \frac{1}{a^2 - b^2} \int_0^{+\infty} \left(\frac{2a^2 b}{a^2 + b^2 t^2} - \frac{2b}{1+t^2} \right) dt \\ &= \frac{1}{a^2 - b^2} \int_0^{+\infty} \frac{2a^2 b}{a^2 + b^2 t^2} dt - \frac{2b}{a^2 - b^2} \int_0^{+\infty} \frac{1}{1+t^2} dt \\ &= \frac{2b}{a^2 - b^2} \int_0^{+\infty} \frac{1}{1 + (\frac{b}{a}t)^2} dt - \frac{b\pi}{a^2 - b^2} \\ &= \frac{2b}{a^2 - b^2} \cdot \frac{a}{b} \cdot \frac{\pi}{2} - \frac{b\pi}{a^2 - b^2} = \frac{\pi}{a+b}. \end{aligned}$$

于是

$$\begin{aligned} F(a, b) &= F(a, 0) + \int_0^b \frac{\partial}{\partial b'} F(a, b') db' = F(a, 0) + \int_0^b \frac{\pi}{a+b'} db' \\ &= 2 \int_0^{\frac{\pi}{2}} \ln(a \cos x) dx + \pi \ln \frac{a+b}{a} \stackrel{\text{例题 0.2}}{=} \pi \ln \frac{a+b}{2}. \end{aligned}$$

当 $a < b$ 时, 类似可得 $F(a, b) = \pi \ln \frac{a+b}{2}$. 当 $a = b$ 时, $F(a, b) = \int_0^{\frac{\pi}{2}} \ln a^2 dx = \pi \ln a = \pi \ln \frac{a+b}{2}$.

综上, 对 $\forall a, b \geq 0$, 都有 $F(a, b) = \pi \ln \frac{a+b}{2}$.

□

0.1.5 级数展开方法

积分和求和换序 $\sum_{n=1}^{\infty} \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx$, 等价于

$$\lim_{m \rightarrow \infty} \sum_{n=1}^m \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx.$$

又由于有限和随意交换, 因此上式等价于

$$\lim_{m \rightarrow \infty} \int_a^b \sum_{n=1}^m f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx,$$

于是

$$\sum_{n=1}^{\infty} \int_a^b f_n(x) dx = \int_a^b \sum_{n=1}^{\infty} f_n(x) dx \iff \lim_{m \rightarrow \infty} \int_a^b \sum_{n=m+1}^{\infty} f_n(x) dx = 0.$$

例题 0.6 计算 $\int_0^{\infty} \frac{x}{1+e^x} dx$.

解 由于

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad 0 < x < 1.$$

并且 $0 < e^{-x} < 1$, 故

$$\begin{aligned} \int_0^{+\infty} \frac{x}{1+e^x} dx &= \int_0^{+\infty} \frac{x e^{-x}}{1+e^{-x}} dx = \int_0^{+\infty} x \sum_{n=0}^{\infty} (-1)^n e^{-(n+1)x} dx \\ &\stackrel{\text{换序}}{=} \sum_{n=0}^{\infty} (-1)^n \int_0^{+\infty} x e^{-(n+1)x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2}. \end{aligned}$$

又因为 $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, 所以

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(2n)^2} &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}, \\ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8}. \end{aligned}$$

故

$$\int_0^{+\infty} \frac{x}{1+e^x} dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.$$

下面证明(??)式换序成立, 等价于证明 $\lim_{m \rightarrow +\infty} \int_0^{+\infty} \sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x} dx = 0$. 由交错级数不等式及 $x e^{-(n+1)x}$ 关于 n 非负递减, 对 $\forall m \in \mathbb{N}$, 都有

$$\int_0^{+\infty} \left| \sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x} \right| dx \leq \int_0^{+\infty} x e^{-(m+1)x} dx = -\frac{x e^{-(m+1)x}}{m+1} \Big|_0^{+\infty} + \frac{1}{m+1} \int_0^{+\infty} e^{-(m+1)x} dx = \frac{1}{(m+1)^2}.$$

令 $m \rightarrow +\infty$, 得 $\lim_{m \rightarrow +\infty} \int_0^{+\infty} \sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x} dx = 0$. 故(??)式换序成立. \square

命题 0.4

证明:

- (1) $\sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n} = \arctan \frac{q \sin x}{1 - q \cos x}, |q| \leq 1.$
- (2) $\sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n} = -\frac{1}{2} \ln(1 + q^2 - 2q \cos x), |q| \leq 1.$
- (3) $\sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n!} = e^{q \cos x} \cos(q \sin x) - 1, |q| \leq 1, x \in \mathbb{R}.$
- (4) $\sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n!} = e^{q \cos x} \sin(q \sin x), |q| \leq 1, x \in \mathbb{R}.$

 笔记 在 \mathbb{C} 上,

$$\operatorname{Ln} z = \ln |z| + i(\arg z + 2k\pi), k \in \mathbb{Z}.$$

我们定义主值支

$$\ln z = \ln |z| + i \arg z.$$

本部分内容无需记忆,只需要大概有个可以算的感觉即可,实际做题中可以围绕这种级数给出构造.

证明 \Im 表示取虚部, \Re 表示取实部.

(1) 利用欧拉公式有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n} &= \Im \left(\sum_{n=1}^{\infty} \frac{q^n e^{inx}}{n} \right) = \Im \left(\sum_{n=1}^{\infty} \frac{(qe^{ix})^n}{n} \right) = \Im(-\ln(1 - qe^{ix})) \\ &= -\Im \left(\ln |1 - qe^{ix}| + i \frac{-q \sin x}{1 - q \cos x} \right) = \arctan \frac{q \sin x}{1 - q \cos x}. \end{aligned}$$

(2) 利用欧拉公式有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n} &= -\Re \left(\ln |1 - qe^{ix}| + i \frac{-q \sin x}{1 - q \cos x} \right) = -\frac{1}{2} \ln [(1 - q \cos x)^2 + q^2 \sin^2 x] \\ &= -\frac{1}{2} \ln(1 + q^2 - 2q \cos x). \end{aligned}$$

(3) 利用欧拉公式有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n!} &= \Re \left(\sum_{n=1}^{\infty} \frac{(qe^{ix})^n}{n!} \right) = \Re(e^{qe^{ix}} - 1) = \Re(e^{q \cos x + iq \sin x} - 1) \\ &= \Re(e^{q \cos x} \cos(q \sin x) - 1 + ie^{q \cos x} \sin(q \sin x)) \\ &= e^{q \cos x} \cos(q \sin x) - 1. \end{aligned}$$

(4) 利用 (3) 有

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n!} &= \Im(e^{q \cos x} \cos(q \sin x) - 1 + ie^{q \cos x} \sin(q \sin x)) \\ &= e^{q \cos x} \sin(q \sin x). \end{aligned}$$

□

例题 0.7 计算

- $\int_0^{2\pi} e^{\cos x} \cos(\sin x) dx.$
- $\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx, a \in (0, +\infty) \setminus \{1\}.$

注 由 1 的证明可得

$$e^{\cos x} \cos(\sin x) = \operatorname{Re} \left(\sum_{n=0}^{\infty} \frac{(e^{ix})^n}{n!} \right) = \operatorname{Re} \left(\sum_{n=0}^{\infty} \frac{e^{inx}}{n!} \right) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}.$$

实际上, 上式就是命题 0.4(3) 的结论.

注 第 2 问也可以用含参积分求导的方法进行计算 (这个方法更容易想到).

证明

1.

$$\begin{aligned} \int_0^{2\pi} e^{\cos x} \cos(\sin x) dx &= \operatorname{Re} \left(\int_0^{2\pi} e^{\cos x} e^{i \sin x} dx \right) = \operatorname{Re} \left(\int_0^{2\pi} e^{\cos x + i \sin x} dx \right) \\ &= \operatorname{Re} \left(\int_0^{2\pi} e^{e^{ix}} dx \right) = \operatorname{Re} \left[\int_0^{2\pi} \sum_{n=0}^{+\infty} \frac{(e^{ix})^n}{n!} dx \right] = \operatorname{Re} \left[\sum_{n=0}^{+\infty} \int_0^{2\pi} \frac{(e^{ix})^n}{n!} dx \right] \\ &= \operatorname{Re} \left(\sum_{n=0}^{+\infty} \int_0^{2\pi} \frac{e^{inx}}{n!} dx \right) = \operatorname{Re} \left(\int_0^{2\pi} \frac{e^{i \cdot 0 \cdot x}}{0!} dx + \sum_{n=1}^{+\infty} \frac{e^{2\pi i x} - 1}{in \cdot n!} \right) \\ &= \operatorname{Re} \left(\int_0^{2\pi} 1 dx + 0 \right) = 2\pi. \end{aligned}$$

2. 注意到当 $a \in (0, 1)$ 时, 有

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a^n \cos(nx)}{n} &= \operatorname{Re} \left[\sum_{n=1}^{\infty} \frac{(ae^{ix})^n}{n} \right] = -\operatorname{Re} [\ln(1 - ae^{ix})] \\ &= -\operatorname{Re} [\ln|1 - ae^{ix}| + i \arg(1 - ae^{ix})] = -\ln|1 - ae^{ix}| \\ &= -\ln|(1 - a \cos x) + ai \sin x| = -\frac{1}{2} \ln(1 + a^2 - 2a \cos x).\end{aligned}$$

于是当 $a \in (0, 1)$ 时, 就有

$$\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = -\frac{1}{2} \int_0^{\pi} \sum_{n=1}^{\infty} \frac{a^n \cos(nx)}{n} dx = 0.$$

若 $a > 1$, 则 $\frac{1}{a} \in (0, 1)$, 从而此时我们有

$$\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = \pi \ln a^2 + \int_0^{\pi} \ln \left(\frac{1}{a^2} - \frac{2}{a} \cos x + 1 \right) dx = \pi \ln a^2 = 2\pi \ln a.$$

又由 $\ln(1 - 2a \cos x + a^2)$ 关于 a 的偏导存在可知 $\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx$ 关于 a 连续. 于是由

$$\int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = 2\pi \ln a, \quad \forall a > 1.$$

可知当 $a = 1$ 时, 我们有

$$\int_0^{\pi} \ln(2 - 2 \cos x) dx = \lim_{a \rightarrow 1^+} \int_0^{\pi} \ln(1 - 2a \cos x + a^2) dx = \lim_{a \rightarrow 1^+} (2\pi \ln a) = 0.$$

□

定义 0.1 (多重对数函数-Li₂ 函数)

定义

$$\operatorname{Li}_2(x) \triangleq \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad x \in [-1, 1].$$

♣

命题 0.5

$$(1) \operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \cdot \ln(1-x), x \in (0, 1).$$

$$(2) \operatorname{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \operatorname{Li}_2(0) = 0, \quad \operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}.$$

♣

证明

(1) 记 $f(x) \triangleq \operatorname{Li}_2(x)$, $F(x) \triangleq f(x) + f(1-x) + \ln x \ln(1-x)$. 则

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\frac{1}{x} \ln(1-x).$$

于是

$$F'(x) = -\frac{1}{x} \ln(1-x) + \frac{\ln x}{1-x} - \frac{\ln x}{1-x} + \frac{\ln(1-x)}{x} = 0.$$

$$\text{故 } F(x) \equiv F(1) = f(0) + f(1) = 0 + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(2) 显然 $\operatorname{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\operatorname{Li}_2(0) = 0$. 由 (1) 可得

$$\operatorname{Li}_2\left(\frac{1}{2}\right) + \operatorname{Li}_2\left(\frac{1}{2}\right) = 2\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - \ln^2 \frac{1}{2} \implies \operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}.$$

例题 0.8 计算 $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x} dx$.

解

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{\ln x}{1-x} dx &= \int_{\frac{1}{2}}^1 \frac{\ln(1-x)}{x} dx = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{2}}^1 x^{n-1} dx \\&= -\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{2^n n^2} = -\frac{\pi^2}{6} + \text{Li}_2\left(\frac{1}{2}\right) \\&\stackrel{\text{命题 0.5}}{=} -\frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}.\end{aligned}$$

0.1.6 其他

例题 0.9 证明积分 $\int_0^{\infty} e^{-ax^2 - \frac{b}{x^2}} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}, a, b > 0$.

证明 当 $a = 1$ 时, 就有

$$\begin{aligned}\int_0^{+\infty} e^{-x^2 - \frac{b}{x^2}} dx &= e^{-2\sqrt{b}} \int_0^{+\infty} e^{-\left(x - \frac{\sqrt{b}}{x}\right)^2} dx \stackrel{y = \frac{\sqrt{b}}{x}}{=} e^{-2\sqrt{b}} \int_0^{+\infty} \frac{\sqrt{b}}{y^2} e^{-\left(\frac{\sqrt{b}}{y} - y\right)^2} dy \\&= \frac{e^{-2\sqrt{b}}}{2} \int_0^{+\infty} \left(1 + \frac{\sqrt{b}}{y^2}\right) e^{-\left(y - \frac{\sqrt{b}}{y}\right)^2} dy = \frac{e^{-2\sqrt{b}}}{2} \int_0^{+\infty} e^{-\left(y - \frac{\sqrt{b}}{y}\right)^2} d\left(y - \frac{\sqrt{b}}{y}\right) \\&= \frac{e^{-2\sqrt{b}}}{2} \int_{-\infty}^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} e^{-2\sqrt{b}}.\end{aligned}$$

于是对 $\forall a > 0$, 就有

$$\int_0^{+\infty} e^{-ax^2 - \frac{b}{x^2}} dx = \frac{1}{\sqrt{a}} \int_0^{+\infty} e^{-x^2 - \frac{ab}{x^2}} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$$

例题 0.10 计算 $\int_0^{\infty} \frac{\cos(ax)}{1+x^2} dx, a \in \mathbb{R}$.

注 本题可以用复变函数的方法 (留数定理) 来计算. 但是我们这里用基本的高等数学的方法来计算.

$\int_0^{\infty} \frac{\sin(ax)}{1+x^2} dx$ 这个积分没办法算出具体的初等数值.

证明

$$\begin{aligned}\int_0^{+\infty} \frac{\cos(ax)}{1+x^2} dx &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\cos(ax)}{1+x^2} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \cos(ax) \left(\int_0^{+\infty} e^{-(1+x^2)y} dy \right) dx \\&= \frac{1}{2} \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} e^{-(1+x^2)y} \cos(ax) dy \right) dx = \frac{1}{2} \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-(1+x^2)y} \cos(ax) dy dx \\&= \frac{1}{2} \int_0^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-(1+x^2)y} \cos(ax) dx \right) dy = \frac{1}{2} \int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-x^2 y} \cos(ax) dx \right) dy \\&= \frac{1}{2} \text{Re} \left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-x^2 y + iax} dx \right) dy \right) = \frac{1}{2} \text{Re} \left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y \left(x^2 - \frac{aix}{y} + \left(\frac{ai}{2y} \right)^2 \right) - \frac{a^2}{4y}} dx \right) dy \right) \\&= \frac{1}{2} \text{Re} \left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y \left(x - \frac{ai}{2y} \right)^2 - \frac{a^2}{4y}} dx \right) dy \right) = \frac{1}{2} \text{Re} \left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y \left(x + \frac{a}{2iy} \right)^2 - \frac{a^2}{4y}} dx \right) dy \right) \\&= \frac{1}{2} \text{Re} \left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y \left(x + \frac{a}{2iy} \right)^2 - \frac{a^2}{4y}} d \left(x + \frac{a}{2iy} \right) \right) dy \right) = \frac{1}{2} \text{Re} \left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-yx^2 - \frac{a^2}{4y}} dx \right) dy \right) \\&= \frac{1}{2} \int_0^{+\infty} e^{-y - \frac{a^2}{4y}} \left(\int_{-\infty}^{+\infty} e^{-yx^2} dx \right) dy = \frac{\sqrt{\pi}}{2} \int_0^{+\infty} \frac{1}{\sqrt{y}} e^{-y - \frac{a^2}{4y}} dy \\&\stackrel{y=t^2}{=} \sqrt{\pi} \int_0^{+\infty} e^{-t^2 - \frac{a^2}{4t^2}} dt \stackrel{\text{例题 0.9}}{=} \sqrt{\pi} \cdot \frac{\sqrt{\pi}}{2} e^{-|a|} = \frac{\pi}{2} e^{-|a|}.\end{aligned}$$

□

例题 0.11 计算 $\int_0^{\infty} \frac{1}{(1+x^8)^2} dx$.

注 由命题??可知对 $\forall s > 0$, 都有

$$\frac{1}{t^s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} y^{s-1} e^{-ty} dy, \forall t \in \mathbb{R}.$$

本题的核心想法就是利用上式将 $\frac{z}{1+x^8}$ 转化成积分形式.

证明 注意到

$$\int_0^{+\infty} y e^{-(1+x^8)y} dy \stackrel{y=\frac{z}{1+x^8}}{=} \frac{1}{(1+x^8)^2} \int_0^{+\infty} z e^{-z} dz = \frac{1}{(1+x^8)^2},$$

因此

$$\begin{aligned} \int_0^{+\infty} \frac{1}{(1+x^8)^2} dx &= \int_0^{+\infty} \left(\int_0^{+\infty} y e^{-(1+x^8)y} dy \right) dx = \int_0^{+\infty} \left(\int_0^{+\infty} y e^{-(1+x^8)y} dx \right) dy \\ &= \int_0^{+\infty} y e^{-y} \left(\int_0^{+\infty} e^{-x^8 y} dx \right) dy \stackrel{x=y^{-\frac{1}{8}} z^{\frac{1}{8}}}{=} \int_0^{+\infty} y e^{-y} \left(\int_0^{+\infty} y^{-\frac{1}{8}} e^{-z} dz^{\frac{1}{8}} \right) dy \\ &= \frac{1}{8} \int_0^{+\infty} y^{\frac{7}{8}} e^{-y} \left(\int_0^{+\infty} z^{-\frac{7}{8}} e^{-z} dz \right) dy = \frac{1}{8} \int_0^{+\infty} y^{\frac{7}{8}} e^{-y} \Gamma\left(\frac{1}{8}\right) dy \\ &= \frac{1}{8} \Gamma\left(\frac{15}{8}\right) \Gamma\left(\frac{1}{8}\right) = \frac{1}{8} \cdot \frac{7}{8} \Gamma\left(\frac{7}{8}\right) \Gamma\left(\frac{1}{8}\right) \\ &\stackrel{??}{=} \frac{7\pi}{64 \sin\left(\frac{\pi}{8}\right)} = \frac{7\pi}{32\sqrt{2-\sqrt{2}}}. \end{aligned}$$

□