

0.1 重积分方法

定理 0.1 (Chebeshev 不等式积分形式)

设 $p \in R[a, b]$ 且非负, f, g 在 $[a, b]$ 上是单调函数, 则

$$\left(\int_a^b p(x)f(x) dx \right) \left(\int_a^b p(x)g(x) dx \right) \leq \left(\int_a^b p(x) dx \right) \left(\int_a^b p(x)f(x)g(x) dx \right), f, g \text{ 单调性相同}$$

$$\left(\int_a^b p(x)f(x) dx \right) \left(\int_a^b p(x)g(x) dx \right) \geq \left(\int_a^b p(x) dx \right) \left(\int_a^b p(x)f(x)g(x) dx \right), f, g \text{ 单调性相反}$$



笔记 本不等式要牢记于心, 它是很多不等式的基本模型, 其特征就是出现单调性.

注 证法二中的 $d\mu$ 应该看作测度.

证明 证法一:

$$\begin{aligned} & \left(\int_a^b p(x)f(x) dx \right) \left(\int_a^b p(x)g(x) dx \right) - \left(\int_a^b p(x) dx \right) \left(\int_a^b p(x)f(x)g(x) dx \right) \\ &= \left(\int_a^b p(x)f(x) dx \right) \left(\int_a^b p(y)g(y) dy \right) - \left(\int_a^b p(x) dx \right) \left(\int_a^b p(y)f(y)g(y) dy \right) \\ &= \iint_{[a,b]^2} p(x)p(y)g(y)[f(x) - f(y)] dx dy \\ &\stackrel{\text{对称性}}{=} \iint_{[a,b]^2} p(y)p(x)g(x)[f(y) - f(x)] dx dy \\ &= \frac{1}{2} \iint_{[a,b]^2} p(x)p(y)[g(y) - g(x)][f(x) - f(y)] dx dy, \end{aligned}$$

故结论得证.

证法二: 令 $\frac{p(x)}{\int_a^b p(x) dx} dx = d\mu$, 则 $\int_a^b d\mu = \int_a^b \frac{p(x)}{\int_a^b p(x) dx} dx = 1$. 于是原不等式等价于

$$\begin{aligned} & \int_a^b f(x) d\mu \int_a^b g(x) d\mu - \int_a^b f(x)g(x) d\mu \\ &= \int_a^b f(x) d\mu \int_a^b g(y) d\mu - \int_a^b \int_a^b f(y)g(y) d\mu(y) d\mu(x) \\ &= \int_a^b \int_a^b [f(x) - f(y)]g(y) d\mu(y) d\mu(x) \\ &= \int_a^b \int_a^b [f(y) - f(x)]g(x) d\mu(y) d\mu(x) \\ &= \frac{1}{2} \int_a^b \int_a^b [f(x) - f(y)][g(y) - g(x)] \end{aligned}$$

故结论得证. □

例题 0.1 设 $f \in C[0, 1]$ 递减恒正, 证明

$$\frac{\int_0^1 f^2(x) dx}{\int_0^1 f(x) dx} \geq \frac{\int_0^1 x f^2(x) dx}{\int_0^1 x f(x) dx}.$$

证明

$$\frac{\int_0^1 f^2(x) dx}{\int_0^1 f(x) dx} \geq \frac{\int_0^1 x f^2(x) dx}{\int_0^1 x f(x) dx}.$$

原不等式等价于

$$\left(\int_0^1 f^2(x)dx\right)\left(\int_0^1 xf(x)dx\right) \geq \left(\int_0^1 xf^2(x)dx\right)\left(\int_0^1 f(x)dx\right).$$

令 $\frac{f(x)}{\int_0^1 f(x)dx}d\mu = d\mu$, 则上式等价于

$$\int_0^1 f(x)d\mu \int_0^1 xd\mu \geq \int_0^1 xf(x)d\mu.$$

上式由 **Chebyshev 不等式积分形式** 可直接得到. □

命题 0.1 (反向切比雪夫不等式)

设 $f, g \in R[a, b]$ 且 $m_1 \leq f(x) \leq M_1, m_2 \leq g(x) \leq M_2$, 证明

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{(M_2 - m_2)(M_1 - m_1)}{4}.$$


注 不妨设 $a = 0, b = 1$ 的原因: 假设当 $a = 0, b = 1$ 时,

$$\left| \int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx \right| \leq \frac{(M_2 - m_2)(M_1 - m_1)}{4}$$

成立. 则对一般的 $[a, b]$, 原不等式等价于

$$\left| \int_0^1 f(a + (b-a)x)g(a + (b-a)x)dx - \int_0^1 f(a + (b-a)x)dx \int_0^1 g(a + (b-a)x)dx \right| \leq \frac{(M_2 - m_2)(M_1 - m_1)}{4}. \quad (1)$$

又注意到 $f(a + (b-a)x), g(a + (b-a)x) \in R[0, 1]$, 且 $f(x) \in [m_1, M_1], g(x) \in [m_2, M_2]$. 故由假设可知(1)式成立. 因此不妨设也成立.

 **笔记** 积累本题的想法.

证明 不妨设 $a = 0, b = 1$, 则记 $A = \int_0^1 f(x)dx, B = \int_0^1 g(x)dx$. 于是

$$\begin{aligned} & \left| \int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx \right|^2 = \left| \int_0^1 (f(x) - A)(g(x) - B)dx \right|^2 \\ & \stackrel{\text{Cauchy不等式}}{\leq} \int_0^1 |f(x) - A|^2 dx \cdot \int_0^1 |g(x) - B|^2 dx \\ & = \left(\int_0^1 |f(x)|^2 dx - \left(\int_0^1 f(x)dx \right)^2 \right) \cdot \left(\int_0^1 |g(x)|^2 dx - \left(\int_0^1 g(x)dx \right)^2 \right). \end{aligned}$$

注意到

$$\int_0^1 (M_1 - f)(f - m_1)dx = M_1 A + m_1 A - M_1 m_1 - \int_0^1 |f(x)|^2 dx,$$

于是我们有

$$\begin{aligned} \int_0^1 |f(x)|^2 dx - \left(\int_0^1 f(x)dx \right)^2 &= \int_0^1 |f(x)|^2 dx - A^2 \\ &= (M_1 - A)(A - m_1) - \int_0^1 (M_1 - f)(f - m_1)dx \\ &\leq (M_1 - A)(A - m_1) \leq \frac{(M_1 - m_1)^2}{4}. \end{aligned}$$

最后一个不等号可由均值不等式或看出二次函数取最值得到. 类似的有

$$\int_0^1 |g(x)|^2 dx - \left(\int_0^1 g(x)dx \right)^2 \leq \frac{(M_2 - m_2)^2}{4},$$

这就证明了

$$\left| \int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \int_0^1 g(x)dx \right|^2 \leq \frac{(M_1 - m_1)^2}{4} \frac{(M_2 - m_2)^2}{4},$$

即原不等式成立. □

例题 0.2 设 $f \in C[a, b]$ 且

$$0 \leq f(x) \leq M, \forall x \in [a, b].$$

证明

$$\left(\int_a^b f(x) \cos x dx \right)^2 + \left(\int_a^b f(x) \sin x dx \right)^2 + \frac{M^2(b-a)^4}{12} \geq \left(\int_a^b f(x) dx \right)^2. \quad (2)$$

注 由 Taylor 公式可得 inequality:

$$\cos x \geq 1 - \frac{x^2}{2}, \forall x \in \mathbb{R}. \quad (3)$$

$\sin x < x$ 两边同时在 $[0, 1]$ 上积分也可得 $1 - \cos x \leq \frac{x^2}{2}$.

证明 一方面

$$\begin{aligned} \left(\int_a^b f(x) \cos x dx \right)^2 + \left(\int_a^b f(x) \sin x dx \right)^2 &= \int_a^b f(x) \cos x dx \int_a^b f(y) \cos y dy + \int_a^b f(x) \sin x dx \int_a^b f(y) \sin y dy \\ &= \iint_{[a,b]^2} f(x)f(y)[\cos x \cos y + \sin x \sin y] dx dy = \iint_{[a,b]^2} f(x)f(y) \cos(x-y) dx dy. \end{aligned}$$

另外一方面

$$\left(\int_a^b f(x) dx \right)^2 = \int_a^b f(x) \cos x dx \int_a^b f(y) \cos y dy = \iint_{[a,b]^2} f(x)f(y) dx dy.$$

于是不等式(2)变为

$$\iint_{[a,b]^2} f(x)f(y)[1 - \cos(x-y)] dx dy \leq \frac{M^2(b-a)^4}{12}. \quad (4)$$

事实上

$$\iint_{[a,b]^2} f(x)f(y)[1 - \cos(x-y)] dx dy \stackrel{(3)}{\leq} M^2 \iint_{[a,b]^2} \frac{(x-y)^2}{2} dx dy = \frac{M^2(b-a)^4}{12},$$

这就得到了不等式(4). □