

0.1 Cauchy-Riemann 方程

定义 0.1

设 $f(z) = u(x, y) + iv(x, y)$ 是定义在域 D 上的函数, $z_0 = x_0 + iy_0 \in D$. 我们说 f 在 z_0 处**实可微**, 是指 u 和 v 作为 x, y 的二元函数在 (x_0, y_0) 处可微.

命题 0.1

设 $f: D \rightarrow \mathbb{C}$ 是定义在域 D 上的函数, $z_0 \in D$, 那么 f 在 z_0 处实可微的充分必要条件是

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z} + o(|\Delta z|). \quad (1)$$

成立, 其中

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \end{aligned}$$

证明 设 f 在 z_0 处实可微, 由二元实值函数可微的定义, 有

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|), \quad (2)$$

$$v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|), \quad (3)$$

这里, $|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. 于是

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)) \\ &= \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|) + i \left(\frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|) \right) \\ &= \left(\frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \right) \Delta x + \left(\frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0) \right) \Delta y + o(|\Delta z|) \\ &= \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|). \end{aligned}$$

把 $\Delta x = \frac{1}{2}(\Delta z + \overline{\Delta z})$, $\Delta y = \frac{1}{2i}(\Delta z - \overline{\Delta z})$ 代入上式, 得

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= \frac{1}{2} \frac{\partial f}{\partial x}(x_0, y_0)(\Delta z + \overline{\Delta z}) - \frac{i}{2} \frac{\partial f}{\partial y}(x_0, y_0)(\Delta z - \overline{\Delta z}) + o(|\Delta z|) \\ &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f(x_0, y_0)\Delta z + \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(x_0, y_0)\overline{\Delta z} + o(|\Delta z|). \end{aligned}$$

引进算子

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \end{aligned} \quad (4)$$

则上式可写为

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z} + o(|\Delta z|). \quad (5)$$

容易看出, (5) 式和 (2), (3) 两式等价. □

注 为什么要像 (4) 式那样来定义算子 $\frac{\partial}{\partial z}$ 和 $\frac{\partial}{\partial \bar{z}}$ 呢? 这是因为如果把复变函数 $f(z)$ 写成

$$f(x, y) = f\left(\frac{z + \bar{z}}{2}, -i\frac{z - \bar{z}}{2}\right),$$

把 z, \bar{z} 看成独立变量, 分别对 z 和 \bar{z} 求偏导数, 则得

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \\ \frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).\end{aligned}$$

这就是表达式(4)的来源. 这说明在进行微分运算时, 可以把 z, \bar{z} 看成独立的变量.

现在很容易得到 f 在 z_0 处可微的条件了.

定理 0.1

设 f 是定义在域 D 上的函数, $z_0 \in D$, 那么 f 在 z_0 处可微的充要条件是 f 在 z_0 处实可微且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. 在可微的情况下, $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

证明 如果 f 在 z_0 处可微, 由(??)式得

$$f(z_0 + \Delta z) - f(z_0) = f'(z_0)\Delta z + o(|\Delta z|)$$

与(1)式比较就知道, f 在 z_0 处是实可微的, 而且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0, f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

反之, 若 f 在 z_0 处实可微, 且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$, 则由(1)式得

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + o(|\Delta z|)$$


由此即知

$$\lim_{\Delta z} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial f}{\partial z}(z_0).$$

故 f 在 z_0 处可微, 而且 $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$. □

定义 0.2 (Cauchy-Riemann 方程)

设 f 是定义在域 D 上的函数, $\frac{\partial f}{\partial \bar{z}} = 0$ 称为 **Cauchy - Riemann 方程**.

 **笔记** 从这个方程可以得到 f 的实部和虚部应满足的条件.

命题 0.2 (Cauchy-Riemann 方程的等价定义)

设 $z = x + iy, f(z) = u(x, y) + iv(x, y)$, 则 Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 等价于

(i)

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases} \quad (6)$$

(ii)

$$\frac{\partial \bar{f}}{\partial z} = 0.$$

(iii) 令 $x = r \cos \theta, y = r \sin \theta$, 进而 $z = r(\cos \theta + i \sin \theta), f(z) = u(r, \theta) + iv(r, \theta)$, 则 Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 等价于

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \end{cases}$$

证明 (i) 由(4)式得

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right).$$

因此, Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 就等价于

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$$

(ii) 又注意到

$$\frac{\partial \bar{f}}{\partial z} = \frac{\partial u}{\partial z} - i \frac{\partial v}{\partial z} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) - \frac{i}{2} \left(\frac{\partial v}{\partial x} - i \frac{\partial v}{\partial y} \right) = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) - \frac{i}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$

故 Cauchy-Riemann 方程 $\frac{\partial \bar{f}}{\partial z} = 0$ 也等价于 $\frac{\partial f}{\partial \bar{z}} = 0$.

(iii) 令 $F = x - r \cos \theta, G = y - r \sin \theta$, 则

$$\begin{pmatrix} F_x & F_y & F_r & F_\theta \\ G_x & G_y & G_r & G_\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\cos \theta & r \sin \theta \\ 0 & 1 & -\sin \theta & -r \cos \theta \end{pmatrix}.$$

直接计算 Jacobi 行列式得

$$J = \frac{\partial(F, G)}{\partial(r, \theta)} = \begin{vmatrix} -\cos \theta & r \sin \theta \\ -\sin \theta & -r \cos \theta \end{vmatrix} = r,$$

于是

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(x, \theta)} \right] + \frac{\partial u}{\partial \theta} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(r, x)} \right] = \frac{\partial u}{\partial r} \cdot \cos \theta - \frac{\partial u}{\partial \theta} \cdot \frac{\sin \theta}{r}; \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(y, \theta)} \right] + \frac{\partial u}{\partial \theta} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(r, y)} \right] = \frac{\partial u}{\partial r} \cdot \sin \theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta}{r}; \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(x, \theta)} \right] + \frac{\partial v}{\partial \theta} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(r, x)} \right] = \frac{\partial v}{\partial r} \cdot \cos \theta - \frac{\partial v}{\partial \theta} \cdot \frac{\sin \theta}{r}; \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(y, \theta)} \right] + \frac{\partial v}{\partial \theta} \cdot \left[-\frac{1}{J} \frac{\partial(F, G)}{\partial(r, y)} \right] = \frac{\partial v}{\partial r} \cdot \sin \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos \theta}{r}. \end{aligned}$$

从而

$$\begin{aligned} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &\iff \frac{\partial u}{\partial r} \cdot \cos \theta - \frac{\partial u}{\partial \theta} \cdot \frac{\sin \theta}{r} = \frac{\partial v}{\partial r} \cdot \sin \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos \theta}{r} \\ &\iff \frac{\partial u}{\partial r} \cdot r \cos \theta - \frac{\partial u}{\partial \theta} \cdot \sin \theta = \frac{\partial v}{\partial r} \cdot r \sin \theta + \frac{\partial v}{\partial \theta} \cdot \cos \theta, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &\iff \frac{\partial u}{\partial r} \cdot \sin \theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta}{r} = -\frac{\partial v}{\partial r} \cdot \cos \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\sin \theta}{r} \\ &\iff \frac{\partial u}{\partial r} \cdot r \sin \theta + \frac{\partial u}{\partial \theta} \cdot \cos \theta = -\frac{\partial v}{\partial r} \cdot r \cos \theta + \frac{\partial v}{\partial \theta} \cdot \sin \theta. \end{aligned}$$

由 (i) 可知 Cauchy-Riemann 方程等价于 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, 故此时 Cauchy-Riemann 方程等价于

$$\begin{cases} \frac{\partial u}{\partial r} \cdot r \cos \theta - \frac{\partial u}{\partial \theta} \cdot \sin \theta = \frac{\partial v}{\partial r} \cdot r \sin \theta + \frac{\partial v}{\partial \theta} \cdot \cos \theta, \\ \frac{\partial u}{\partial r} \cdot r \sin \theta + \frac{\partial u}{\partial \theta} \cdot \cos \theta = -\frac{\partial v}{\partial r} \cdot r \cos \theta + \frac{\partial v}{\partial \theta} \cdot \sin \theta. \end{cases}$$

化简可得

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \end{cases}$$

□

定理 0.2

设 $f = u + iv$ 是定义在域 D 上的函数, $z_0 = x_0 + iy_0 \in D$, 那么 f 在 z_0 处可微的充要条件是 $u(x, y), v(x, y)$ 在 (x_0, y_0) 处可微, 且在 (x_0, y_0) 处满足 Cauchy-Riemann 方程, 即

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{f}}{\partial z} = 0, \quad \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}.$$

在可微的情况下, 有

$$f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

这里的偏导数都在 (x_0, y_0) 处取值.

♡

证明 最后这个 $f'(z_0)$ 的表达式是从 **定理 0.1** 中的 $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ 和 **Cauchy-Riemann** 方程的等价定义得到的. □

定义 0.3

1. 设 D 是 \mathbb{C} 中的域, 我们用 $C(D)$ 记 D 上连续函数的全体, 用 $H(D)$ 记 D 上全纯函数的全体.
2. 设 $f = u + iv$, 记 $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$, $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体.
3. 用 $C^k(D)$ 记在 D 上有 k 阶连续偏导数的函数的全体, $C^\infty(D)$ 记在 D 上有任意阶连续偏导数的函数的全体.

♣

命题 0.3

- (1) $H(D) \subset C(D)$.
- (2) $C^1(D) \subset C(D)$.
- (3) 域 D 上的全纯函数在 D 上有任意阶的连续偏导数, 并且有如下的包含关系:

$$H(D) \subset C^\infty(D) \subset C^k(D) \subset C^1(D) \subset C(D).$$

这里, k 是大于 1 的自然数.

♠

证明

- (1) 命题??告诉我们, $H(D) \subset C(D)$.
- (2) 设 $f = u + iv$, 记 $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$, $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体. 进而 u, v 关于 x, y 的偏导在 D 上都连续, 由多元微积分的知识知道, u, v 在 D 上都可微. 于是对于任意 $f \in C^1(D)$, f 在 D 上实可微, 从(5)式知道

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z} + o(|\Delta z|).$$

令 $\Delta z \rightarrow 0$, 则 $\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial f}{\partial z}(z_0)$, 故 f 在 D 上连续, 因而 $C^1(D) \subset C(D)$.

(3)

□

例题 0.1 研究函数 $f(z) = z^n, n$ 是自然数.

解 显然, $\frac{\partial f}{\partial \bar{z}} = 0$, 且 f 在整个平面上是实可微的. 因而, f 是 \mathbb{C} 上的全纯函数, 而且

$$f'(z) = \frac{\partial f}{\partial z} = nz^{n-1}. \quad (7)$$

□

例题 0.2 研究函数 $f(z) = e^{-|z|^2}$.

解 把 f 写为 $f(z) = e^{-z\bar{z}}$, 于是 $\frac{\partial f}{\partial \bar{z}} = -e^{-z\bar{z}}z$, 它只有在 $z = 0$ 处才等于零. 因此, $e^{-|z|^2}$ 只有在 $z = 0$ 处可微, 它在任

何点处都不是全纯的. 但它对 x, y 有任意阶连续偏导数, 所以它是 $C^\infty(\mathbb{C})$ 中的函数. \square

命题 0.4

设 D 是 \mathbb{C} 中的域, $f \in H(D)$. 如果对每一个 $z \in D$, 都有 $f'(z) = 0$, 证明 f 是一常数. \clubsuit

证明 因为 $f'(z) = 0$, 所以由 **定理 0.2** 可知 $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0$, 并且 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. 于是 $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. 因此 u, v 都是常数, 故 f 是一常数. \square

定义 0.4 (调和函数)

设 u 是域 D 上的实值函数, 如果 $u \in C^2(D)$, 且对任意 $z \in D$, 有

$$\Delta u(z) = \frac{\partial^2 u(z)}{\partial x^2} + \frac{\partial^2 u(z)}{\partial y^2} = 0, \quad (8)$$

就称 u 是 D 中的**调和函数**. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ 称为 **Laplace 算子**. \clubsuit

命题 0.5

设 $u \in C^2(D)$, 那么 $\Delta u = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}$. \clubsuit

证明 由(4)式, 有

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \end{aligned}$$

所以

$$\begin{aligned} \frac{\partial^2 u}{\partial z \partial \bar{z}} &= \frac{\partial}{\partial \bar{z}} \frac{\partial u}{\partial z} = \frac{1}{4} \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) + i \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) \right] \\ &= \frac{1}{4} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{1}{4} \Delta u. \end{aligned}$$

\square

定理 0.3

设 $f = u + iv \in H(D)$, 那么 u 和 v 都是 D 上的调和函数. \heartsuit

证明 因为 $f \in H(D)$, 由 **定理 0.2**, 有

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{f}}{\partial z} = 0.$$

所以

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 \bar{f}}{\partial z \partial \bar{z}} = 0.$$

于是, 由 $u = \frac{1}{2}(f + \bar{f})$ 即得

$$\Delta u = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

同理可证 $\Delta v = 0$. \square

定义 0.5 (共轭调和函数)

设 u 和 v 是 D 上的一对调和函数, 如果它们还满足 Cauchy-Riemann 方程

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \end{cases} \quad (9)$$

就称 v 为 u 的**共轭调和函数**.

命题 0.6

全纯函数的实部和虚部就构成一对共轭调和函数.

证明 由定理 0.3 和定理 0.2 立得. □

定理 0.4

设 u 是单连通域 D 上的调和函数, 则必存在 u 的共轭调和函数 v , 使得 $u + iv$ 是 D 上的全纯函数. ♥

证明 因为 u 满足 Laplace 方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

若令 $P = -\frac{\partial u}{\partial y}, Q = \frac{\partial u}{\partial x}$, 则

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial y},$$

所以

$$Pdx + Qdy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

是一个全微分, 因而积分

$$\int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

与路径无关. 令

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy,$$

则

$$\begin{aligned} v(x, y) &= \int_{(x_0, y_0)}^{(x, y_0)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy + \int_{(x, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= \int_{x_0}^x -\frac{\partial u}{\partial y}dx + 0 + 0 + \int_{y_0}^y \frac{\partial u}{\partial x}dy \\ &= \int_{x_0}^x -\frac{\partial u}{\partial y}dx + \int_{y_0}^y \frac{\partial u}{\partial x}dy. \end{aligned}$$

那么

$$\begin{cases} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}. \end{cases}$$

所以, v 就是要求的 u 的共轭调和函数. □