

0.1 三角函数相关

0.1.1 三角函数

命题 0.1 (三角平方差公式)

$$\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y) = \cos(y-x)\cos(y+x) = \cos^2 y - \cos^2 x.$$

证明 首先, 我们有

$$\cos^2 x - \cos^2 y = 1 - \sin^2 x - (1 - \sin^2 y) = \sin^2 y - \sin^2 x.$$

接着, 我们有

$$\begin{aligned}\sin(x-y)\sin(x+y) &= (\sin x \cos y - \cos x \sin y)(\sin x \cos y + \cos x \sin y) \\ &= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y \\ &= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y \\ &= \sin^2 x - \sin^2 y;\end{aligned}$$

$$\begin{aligned}\cos(y-x)\cos(y+x) &= (\cos x \cos y + \sin x \sin y)(\cos x \cos y - \sin x \sin y) \\ &= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y \\ &= \cos^2 x \cos^2 y - (1 - \cos^2 x)(1 - \cos^2 y) \\ &= \cos^2 x - \cos^2 y.\end{aligned}$$

故结论得证. □

0.1.2 反三角函数

命题 0.2

$$\begin{aligned}(1) \quad \arctan x + \arctan \frac{1}{x} &= \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases} \\ (2) \quad \arctan x - \arctan y &= \arctan \frac{x-y}{1+xy}, \forall x, y \in \mathbf{R}.\end{aligned}$$

证明

1. 令 $f(x) = \arctan x + \arctan \frac{1}{x}$, 则

$$f'(x) = \frac{1}{x^2+1} + \frac{1}{(\frac{1}{x})^2+1}(-\frac{1}{x^2}) = \frac{1}{x^2+1} - \frac{1}{x^2+1} = 0$$

故 $f(x)$ 为常函数, 于是就有 $f(x) = f(1) = \frac{\pi}{2}, \forall x > 0; f(x) = f(-1) = -\frac{\pi}{2}, \forall x < 0$.

2. □

0.1.3 双曲三角函数

命题 0.3

$$(1) \cosh x = \frac{e^x + e^{-x}}{2} \geq 1,$$

$$(2) \sinh x = \frac{e^x - e^{-x}}{2} \geq x.$$



证明 可以分别利用均值不等式和求导进行证明.

