0.1 定积分

0.1.1 建立积分递推

例题 0.1 计算 $\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x, n \in \mathbb{N}$.

证明 利用分部积分和和差化积公式可得

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx)\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x[\sin(nx)\cos x + \cos(nx)\sin x]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(nx)\mathrm{d}\cos x \\
&= \frac{I_{n-1}+I_n}{2} - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos(nx)\mathrm{d}\cos^n x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
&= \frac{I_{n-1}}{2} + \frac{1}{2n}.
\end{aligned}$$

故 $I_n = \frac{I_{n-1}}{2} + \frac{1}{2n}$, 则两边同乘 2^n(强行裂项)

$$2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \cdots$$

又注意到 $I_0 = 0$, 从而

$$2^n I_n = 0 + \sum_{k=1}^{n} \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n}\sum_{k=1}^{n} \frac{2^{k-1}}{k}.$$

□

命题 0.1

证明:

(1) $\int_0^{\pi} \frac{\sin(nx)}{\sin x}\mathrm{d}x = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$

(2) $\int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x}\mathrm{d}x = n\pi.$

(3) $\int_0^{\pi} \frac{\sin^2(nx)}{\sin x}\mathrm{d}x = \sum_{k=1}^{n} \frac{2}{2k-1}.$

笔记 提示:$\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y)$(证明见命题**??**).

证明

(1) 记 $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x}\mathrm{d}x$, 则

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin((n+2)x) - \sin(nx)}{\sin x}\mathrm{d}x = \int_0^{\pi} \frac{2\cos((n+1)x)\sin x}{\sin x}\mathrm{d}x = 2\int_0^{\pi} \cos((n+1)x)\mathrm{d}x = 0.$$

于是

$$\int_0^{\pi} \frac{\sin(nx)}{\sin x}\,\mathrm{d}x = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$$

(2) 记 $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^{\pi} \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = \int_0^{\pi} \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x}\,\mathrm{d}x \\ &= \int_0^{\pi} \frac{\sin((2n+1)x)}{\sin x}\,\mathrm{d}x \xlongequal{\text{命题 0.1(1)}} \pi. \end{aligned} \tag{1}$$

于是

$$\int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记 $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^{\pi} \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x}\,\mathrm{d}x = \int_0^{\pi} \frac{\sin x \cdot \sin((2n+1)x)}{\sin x}\,\mathrm{d}x \\ &= \int_0^{\pi} \sin((2n+1)x)\,\mathrm{d}x = \frac{1}{2n+1}\cos((2n+1)x)\Big|_{\pi}^{0} = \frac{2}{2n+1}. \end{aligned} \tag{2}$$

于是

$$\int_0^{\pi} \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1} \frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1} \frac{2}{2k+1} = \sum_{k=1}^{n} \frac{2}{2k-1}.$$

□

例题 0.2 设 $a > 1$, 计算积分 $\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,dx$.

注 很多情况下不需求出被积函数的原函数, 只需充分利用换元、分部积分以及被积函数的性质, 即可求出积分的值. 见下述解法二.

解 解法一:设 $a_0 = a > 1$. 构造数列如下:

$$a_{n+1} = 2a_n^2 - 1 \quad (n = 0, 1, \cdots),$$

则由例题**??**可知, 存在 $x_0 > 0$ 使得

$$a_0 = \mathrm{ch}(x_0), \quad a_n = \mathrm{ch}(2^n x_0),$$

其中 $\mathrm{ch}(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$. 可以解得

$$x_0 = \ln\left(a_0 + \sqrt{a_0^2 - 1}\right). \tag{3}$$

故

$$a_n = \frac{\mathrm{e}^{2^n x_0} + \mathrm{e}^{-2^n x_0}}{2}.$$

设

$$I_n = \int_0^{\pi} \ln(a_n - \cos x)\,dx,$$

则

$$\begin{aligned} I_0 = \int_0^{\pi} \ln(a_0 - \cos x)\,dx &= \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,dx + \int_{\frac{\pi}{2}}^{\pi} \ln(a_0 - \cos x)\,dx \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,dx + \int_0^{\frac{\pi}{2}} \ln(a_0 + \cos x)\,dx \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0^2 - \cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} \ln\left(a_0^2 - \frac{1 + \cos 2x}{2}\right)\,dx \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{a_1 - \cos 2x}{2}\right) dx = \frac{1}{2}\int_0^{\pi} \ln\left(\frac{a_1 - \cos x}{2}\right) dx = \frac{1}{2}I_1 - \frac{\pi}{2}\ln 2.$$

同理, 有

$$I_n = \frac{1}{2}I_{n+1} - \frac{\pi}{2}\ln 2. \tag{4}$$

由此递推公式, 可得

$$I_0 = \frac{1}{2^n}I_n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)\frac{\pi}{2}\ln 2. \tag{5}$$

因为

$$\begin{aligned} I_n &= \int_0^{\pi} \ln(a_n - \cos x)\, dx = \int_0^{\pi} \ln\left(\frac{\mathrm{e}^{2^n x_0} + \mathrm{e}^{-2^n x_0}}{2} - \cos x\right) dx \\ &= 2^n x_0 \pi + \int_0^{\pi} \ln\left(\frac{1 + \mathrm{e}^{-2^{n+1} x_0}}{2} - \mathrm{e}^{-2^n x_0}\cos x\right) dx, \end{aligned}$$

所以

$$\frac{1}{2^n}I_n \to x_0\pi \quad (n \to +\infty).$$

故从式 (5) 可得

$$I_0 = x_0\pi - \pi\ln 2 = \pi\ln\left(\frac{a_0 + \sqrt{a_0^2 - 1}}{2}\right),$$

即所求的积分为

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\, dx = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right).$$

解法二:我们有

$$F(a) = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\, dx = \int_0^{\pi} \ln(a - \cos x)\, dx.$$

由定理**??**, 关于 a 求导得到

$$F'(a) = \int_0^{\pi} \frac{1}{a - \cos x}\, dx \xlongequal{\text{万能公式}} \int_0^{+\infty} \frac{2}{a(1+t^2) - (1-t^2)}\, dt = \frac{\pi}{\sqrt{a^2 - 1}}, \quad a > 1.$$

因此

$$F(a) = \int_1^a F'(a)\,\mathrm{d}a = \pi\ln\left(a + \sqrt{a^2 - 1}\right) + C, \quad a > 1.$$

结合

$$F(1) = 2\int_0^{\frac{\pi}{2}} \ln\sin x\, dx = -\pi\ln 2.$$

可得

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\, dx = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right), \quad a > 1.$$

□

0.1.2 区间再现

定理 0.1 (区间再现恒等式)

当下述积分有意义时, 我们有

1.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)]\mathrm{d}x.$$

2.
$$\int_0^{\infty} f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^{\infty} f(x)\mathrm{d}x = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2}\right]\mathrm{d}x.$$

笔记 注意: 倒代换具有将 $[0,1]$ 转化为 $[1,+\infty)$ 的功能.

证明 证明是显然的.(第 1 问中最后一个等号是由轴对称得到的) □

命题 0.2

证明

1. $\displaystyle\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
2. $\displaystyle\int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
3. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln 2.$

证明

1.
$$\begin{aligned}
I = \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2} - x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

2.
$$\begin{aligned}
I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2} - x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

3.
$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x \xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2}\mathrm{d}\tan\theta &= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\cdot\ln(1+\tan\theta)}{\sec^2\theta}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\tan\left(\frac{\pi}{4} - \theta\right)\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\frac{2}{1+\tan\theta}\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \ln 2\mathrm{d}\theta = \frac{\pi}{8}\ln 2.
\end{aligned}$$

□

例题 0.3 计算

1. $\int_0^{\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x, a>0$.
2. $\int_0^{\infty} \frac{\ln x}{x^2+x+1} \mathrm{d}x$.
3. $\int_0^{1} \frac{\ln x}{\sqrt{x-x^2}} \mathrm{d}x$.

解

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x \xlongequal{x=at} \frac{1}{a}\int_0^{+\infty} \frac{\ln(at)}{1+t^2} dt = \frac{1}{a}\int_0^{+\infty} \frac{\ln a}{1+t^2} dt + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2} dt = \frac{\pi \ln a}{2a} + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2} dt. \quad (6)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2} dt \xlongequal{t=\frac{1}{x}} \int_0^{+\infty} \frac{\ln \frac{1}{x}}{1+\frac{1}{x^2}} \frac{1}{x^2} \mathrm{d}x = \int_0^{+\infty} \frac{-\ln x}{1+x^2} \mathrm{d}x \Longrightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2} dt = 0.$$

于是再结合(6)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x = \frac{\pi \ln a}{2a}.$$

2.

$$\int_0^{\infty} \frac{\ln x}{x^2+x+1} \mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_0^{\infty} \frac{-\ln t}{1+\frac{1}{t}+\frac{1}{t^2}} \mathrm{d}\frac{1}{t} = \int_0^{+\infty} \frac{-\ln t}{1+t+t^2} dt \Longrightarrow \int_0^{\infty} \frac{\ln x}{x^2+x+1} \mathrm{d}x = 0.$$

3.

$$\begin{aligned}\int_0^1 \frac{\ln x}{\sqrt{x-x^2}} \mathrm{d}x &\xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}} \frac{\ln \sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}} \mathrm{d}\sin^2 y \\ &= 4\int_0^{\frac{\pi}{2}} \ln \sin y \mathrm{d}y \xlongequal{\text{命题 0.2}} 4\cdot\left(-\frac{\pi}{2}\ln 2\right) = -2\pi \ln 2.\end{aligned}$$

□

例题 0.4

1. 对 $n \in \mathbb{N}$, 计算 $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} \mathrm{d}x$.
2. $\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} \mathrm{d}x$.
3. 对 $n \in \mathbb{N}$, 计算 $\int_0^{2\pi} \sin(\sin x + nx) \mathrm{d}x$.

解

1.

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} \mathrm{d}x &= \int_{-\pi}^{0} \left[\frac{\sin(nx)}{(1+2^x)\sin x} + \frac{\sin(nx)}{(1+2^{-x})\sin x}\right] \mathrm{d}x = \int_{-\pi}^{0} \frac{\sin(nx)}{\sin x}\left(\frac{1}{1+2^x} + \frac{1}{1+2^{-x}}\right) \mathrm{d}x \\ &= \int_{-\pi}^{0} \frac{\sin(nx)}{\sin x} \cdot \frac{2+2^x+2^{-x}}{2+2^x+2^{-x}} \mathrm{d}x = \int_{-\pi}^{0} \frac{\sin(nx)}{\sin x} \mathrm{d}x = \int_0^{\pi} \frac{\sin(nx)}{\sin x} \mathrm{d}x \xlongequal{\text{例题 0.1}} \begin{cases} 0, n\text{为偶数} \\ \pi, n\text{为奇数} \end{cases}.\end{aligned}$$

2.

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} \mathrm{d}x &= \int_{-\pi}^{0} \left(\frac{x \sin x \arctan e^x}{1+\cos^2 x} + \frac{x \sin x \arctan e^{-x}}{1+\cos^2 x}\right) \mathrm{d}x = \int_{-\pi}^{0} \frac{x \sin x}{1+\cos^2 x} (\arctan e^x + \arctan e^{-x}) \mathrm{d}x \\ &\xlongequal{\text{命题??(1)}} \int_{-\pi}^{0} \frac{x \sin x}{1+\cos^2 x} \cdot \frac{\pi}{2} \mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \mathrm{d}x \\ &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{x \sin x}{1+\cos^2 x} + \frac{(\pi - x)\sin x}{1+\cos^2 x}\right) \mathrm{d}x = \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} \mathrm{d}x \\ &= \frac{\pi^2}{2} \arctan \cos x \Big|_{\frac{\pi}{2}}^{0} = \frac{\pi^2}{2} \cdot \frac{\pi}{4} = \frac{\pi^3}{8}.\end{aligned}$$

3.

$$\begin{aligned}\int_0^{2\pi}\sin(\sin x+nx)\,\mathrm{d}x &= \int_0^{2\pi}\sin[\sin(2\pi-x)+n(2\pi-x)]\,\mathrm{d}x\\ &= \int_0^{2\pi}\sin(-\sin x-nx)\,\mathrm{d}x = -\int_0^{2\pi}\sin(\sin x+nx)\,\mathrm{d}x\\ &\Longrightarrow \int_0^{2\pi}\sin(\sin x+nx)\,\mathrm{d}x = 0.\end{aligned}$$

□

例题 0.5 计算积分 $I=\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$.

注 此例中无法求出被积函数的原函数, 但通过积分的性质仍可算出积分的值.

解 解法一: 作变换 $x=\tan\varphi$, 则 $\mathrm{d}\varphi=\frac{1}{1+x^2}\mathrm{d}x$, 且当 $x=0$ 时,$\varphi=0$; 当 $x=1$ 时,$\varphi=\frac{\pi}{4}$. 于是

$$\begin{aligned}I &= \int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos\varphi+\sin\varphi}{\cos\varphi}\right)\mathrm{d}\varphi\\ &= \int_0^{\frac{\pi}{4}}\left\{\ln\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\varphi+\frac{1}{\sqrt{2}}\sin\varphi\right)\right)-\ln(\cos\varphi)\right\}\mathrm{d}\varphi\\ &= \int_0^{\frac{\pi}{4}}\left\{\ln\sqrt{2}+\ln\left(\sin\left(\varphi+\frac{\pi}{4}\right)\right)-\ln(\cos\varphi)\right\}\mathrm{d}\varphi\\ &= \frac{\pi}{8}\ln 2+\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(\varphi+\frac{\pi}{4}\right)\right)\mathrm{d}\varphi-\int_0^{\frac{\pi}{4}}\ln(\cos\varphi)\,\mathrm{d}\varphi.\end{aligned}$$

因为

$$\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(\varphi+\frac{\pi}{4}\right)\right)\mathrm{d}\varphi \xlongequal{\varphi=\frac{\pi}{4}-t} -\int_{\frac{\pi}{4}}^{0}\ln\left(\sin\left(\frac{\pi}{2}-t\right)\right)\mathrm{d}t=\int_0^{\frac{\pi}{4}}\ln(\cos t)\,\mathrm{d}t,$$

所以 $I=\frac{\pi}{8}\ln 2$.

解法二:考虑含参量积分

$$\varphi(\alpha)=\int_0^1\frac{\ln(1+\alpha x)}{1+x^2}\,dx,\quad \alpha\in[0,1].$$

显然 $\varphi(0)=0,\varphi(1)=I$, 且函数 $\frac{\ln(1+\alpha x)}{1+x^2}$ 在 $R=[0,1]\times[0,1]$ 上满足定理**??** 的条件, 于是

$$\varphi'(\alpha)=\int_0^1\frac{x}{(1+x^2)(1+\alpha x)}\,dx.$$

因为

$$\frac{x}{(1+x^2)(1+\alpha x)}=\frac{1}{1+\alpha^2}\left(\frac{\alpha+x}{1+x^2}-\frac{\alpha}{1+\alpha x}\right),$$

所以

$$\begin{aligned}\varphi'(\alpha) &= \frac{1}{1+\alpha^2}\left(\int_0^1\frac{\alpha}{1+x^2}\mathrm{d}x+\int_0^1\frac{x}{1+x^2}\mathrm{d}x-\int_0^1\frac{\alpha}{1+\alpha x}\mathrm{d}x\right)\\ &= \frac{1}{1+\alpha^2}\left[\alpha\arctan x\Big|_0^1+\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1-\ln(1+\alpha x)\Big|_0^1\right]\\ &= \frac{1}{1+\alpha^2}\left[\alpha\cdot\frac{\pi}{4}+\frac{1}{2}\ln 2-\ln(1+\alpha)\right].\end{aligned}$$

因此

$$\begin{aligned}\int_0^1\varphi'(\alpha)\,\mathrm{d}\alpha &= \int_0^1\frac{1}{1+\alpha^2}\left[\frac{\pi}{4}\alpha+\frac{1}{2}\ln 2-\ln(1+\alpha)\right]\mathrm{d}\alpha\\ &= \frac{\pi}{8}\ln\left(1+\alpha^2\right)\Big|_0^1+\frac{1}{2}\ln 2\arctan\alpha\Big|_0^1-\varphi(1)\\ &= \frac{\pi}{8}\ln 2+\frac{\pi}{8}\ln 2-\varphi(1)\end{aligned}$$

$$= \frac{\pi}{4}\ln 2 - \varphi(1).$$

另一方面,

$$\int_0^1 \varphi'(\alpha)\,d\alpha = \varphi(1) - \varphi(0) = \varphi(1),$$

所以 $I = \varphi(1) = \frac{\pi}{8}\ln 2.$ □

0.1.3 化成多元累次积分 (换序)

命题 0.3

证明:

(1) $\int_0^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$

(2) $\int_0^{\infty} \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}.$

(3) $\int_0^{\infty} \sin x^2\mathrm{d}x, \int_0^{\infty} \cos x^2\mathrm{d}x = \sqrt{\frac{\pi}{8}}.$

笔记 本结果可以直接使用.

证明

(1) 注意到

$$\begin{aligned}\left(\int_0^{+\infty} e^{-x^2}\mathrm{d}x\right)^2 &= \left(\int_0^{+\infty} e^{-x^2}\mathrm{d}x\right)\left(\int_0^{+\infty} e^{-y^2}\mathrm{d}y\right) \xlongequal{\text{把 }\int_0^{+\infty} e^{-y^2}\mathrm{d}y\text{看作常数}} \int_0^{+\infty} e^{-x^2}\left(\int_0^{+\infty} e^{-y^2}\mathrm{d}x\right)\mathrm{d}y \\ &\xlongequal{\text{把}e^{-x^2}\text{看作常数}} \int_0^{+\infty}\left(\int_0^{+\infty} e^{-(x^2+y^2)}\mathrm{d}x\right)\mathrm{d}y \xlongequal{e^{-(x^2+y^2)}\text{连续}} \iint_{R^2} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{+\infty} re^{-r^2}\mathrm{d}r = \frac{\pi}{2}\int_0^{+\infty} re^{-r^2}\mathrm{d}r \\ &= \frac{\pi}{4}\int_0^{+\infty} e^{-r^2}\mathrm{d}r^2 = \frac{\pi}{4}.\end{aligned}$$

故 $\int_0^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$

(2) 注意到

$$\int_0^{+\infty} \sin x e^{-yx}\,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{-(y-\mathrm{i})x}\,\mathrm{d}x = \mathrm{Im}\frac{1}{y-\mathrm{i}} = \mathrm{Im}\frac{y+\mathrm{i}}{y^2+1} = \frac{1}{y^2+1}.$$

因此就有

$$\begin{aligned}\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x &= \int_0^{+\infty}\sin x\left(\int_0^{+\infty} e^{-yx}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}\sin x e^{-yx}\,\mathrm{d}x \\ &= \int_0^{+\infty}\mathrm{d}y\left(\mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\right)\mathrm{d}x = \int_0^{+\infty}\frac{1}{y^2+1}\,\mathrm{d}y = \frac{\pi}{2}.\end{aligned}$$

当然本题也可以直接利用分部积分计算 $\int_0^{+\infty}\sin x e^{-yx}\,\mathrm{d}x = \frac{1}{y^2+1}.$

(3) 注意到

$$\int_0^{+\infty} e^{-ax^2}\,\mathrm{d}x \xlongequal{x=\frac{t}{\sqrt{a}}} \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-t^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a>0.$$

并且 $-\mathrm{i} = e^{-\frac{\pi}{2}\mathrm{i}}$, 从而 $\sqrt{-\mathrm{i}} = e^{-\frac{\pi}{4}\mathrm{i}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}$. 于是

$$\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2)\,\mathrm{d}x = \int_0^{+\infty} e^{-\mathrm{i}x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\mathrm{i}}} = \frac{1}{2}\sqrt{-\mathrm{i}\pi}$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}.$$

故

$$\int_0^{+\infty} \cos x^2 \,\mathrm{d}x = \mathrm{Re}\int_0^{+\infty} (\cos x^2 - \mathrm{i}\sin x^2)\,\mathrm{d}x = \mathrm{Re}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}},$$
$$\int_0^{+\infty} \sin x^2 \,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty} (\cos x^2 - \mathrm{i}\sin x^2)\,\mathrm{d}x = \mathrm{Im}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}.$$

□

例题 0.6 计算 $\int_0^1 \sin\ln\frac{1}{x}\cdot\frac{x^b - x^a}{\ln x}\mathrm{d}x\ (b > a > 0)$.

证明

$$\int_0^1 \sin\ln\frac{1}{x}\cdot\frac{x^b - x^a}{\ln x}\mathrm{d}x = \int_0^1 \sin\ln\frac{1}{x}\left(\int_a^b x^y \mathrm{d}y\right)\mathrm{d}x = \int_a^b \mathrm{d}y\int_0^1 x^y \sin\ln\frac{1}{x}\mathrm{d}x$$
$$\xlongequal{x=e^{-t}} \int_a^b \mathrm{d}y\int_{+\infty}^0 e^{-ty}\sin t\mathrm{d}e^{-t} = \int_a^b \mathrm{d}y\int_0^{+\infty} e^{-t(y+1)}\sin t dt$$
$$\xlongequal{\text{命题 0.3(2) 的证明过程}} \int_a^b \frac{1}{1+(y+1)^2}\mathrm{d}y = \arctan(b+1) - \arctan(a+1).$$

□

0.1.4 化成含参积分 (求导)

例题 0.7 设 $a, b \geqslant 0$ 且不全为 0, 计算 $\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$.

注 实际上, 根据 $a > b$ 时得到的结果, 可以看出 $F(a,b) = \pi\ln\frac{a+b}{2}$ 对 a, b 有轮换对称性, 故这个结果对其他情况显然也成立.

证明 设 $F(a,b) = \int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$, 当 $a > b$ 时, 则

$$\frac{\partial}{\partial b}F(a,b) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial b}\ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{2b\sin^2 x}{a^2\cos^2 x + b^2\sin^2 x}\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{2}} \frac{2b\tan^2 x}{a^2 + b^2\tan^2 x}\mathrm{d}x = \int_0^{+\infty} \frac{2bt^2}{(a^2+b^2t^2)(1+t^2)}dt$$
$$= \frac{1}{a^2-b^2}\int_0^{+\infty}\left(\frac{2a^2b}{a^2+b^2t^2} - \frac{2b}{1+t^2}\right)dt$$
$$= \frac{1}{a^2-b^2}\int_0^{+\infty}\frac{2a^2b}{a^2+b^2t^2}dt - \frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+t^2}dt$$
$$= \frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+\left(\frac{b}{a}t\right)^2}dt - \frac{b\pi}{a^2-b^2}$$
$$= \frac{2b}{a^2-b^2}\cdot\frac{a}{b}\cdot\frac{\pi}{2} - \frac{b\pi}{a^2-b^2} = \frac{\pi}{a+b}.$$

于是

$$F(a,b) = F(a,0) + \int_0^b \frac{\partial}{\partial b'}F(a,b')\mathrm{d}b' = F(a,0) + \int_0^b \frac{\pi}{a+b'}\mathrm{d}b'$$
$$= 2\int_0^{\frac{\pi}{2}} \ln(a\cos x)\mathrm{d}x + \pi\ln\frac{a+b}{a} \xlongequal{\text{例题 0.2}} \pi\ln\frac{a+b}{2}.$$

当 $a < b$ 时, 类似可得 $F(a,b) = \pi\ln\frac{a+b}{2}$. 当 $a = b$ 时,$F(a,b) = \int_0^{\frac{\pi}{2}} \ln a^2\mathrm{d}x = \pi\ln a = \pi\ln\frac{a+b}{2}$.

综上, 对 $\forall a, b \geqslant 0$, 都有 $F(a,b) = \pi\ln\frac{a+b}{2}$.

□

0.1.5 级数展开方法

积分和求和换序 $\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x$, 等价于

$$\lim_{m\to\infty}\sum_{n=1}^{m}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x.$$

又由于有限和随意交换, 因此上式等价于

$$\lim_{m\to\infty}\int_a^b \sum_{n=1}^{m} f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x,$$

于是

$$\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x \Longleftrightarrow \lim_{m\to\infty}\int_a^b \sum_{n=m+1}^{\infty} f_n(x)\mathrm{d}x = 0.$$

例题 0.8 计算 $\int_0^{\infty}\frac{x}{1+e^x}\mathrm{d}x$.

解 由于

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n,\quad 0 < x < 1.$$

并且 $0 < e^{-x} < 1$, 故

$$\begin{aligned}\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x &= \int_0^{+\infty}\frac{xe^{-x}}{1+e^{-x}}\mathrm{d}x = \int_0^{+\infty} x\sum_{n=0}^{\infty}(-1)^n e^{-(n+1)x}\mathrm{d}x \\ &\xlongequal{\text{换序}} \sum_{n=0}^{\infty}(-1)^n\int_0^{+\infty} xe^{-(n+1)x}\mathrm{d}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty}\frac{1}{(2n)^2}.\end{aligned}$$

又因为 $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, 所以

$$\sum_{n=0}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{24},$$
$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=0}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{8}.$$

故

$$\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.$$

下面证明(??)式换序成立, 等价于证明 $\lim_{m\to+\infty}\int_0^{+\infty}\sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x}\mathrm{d}x = 0$. 由交错级数不等式及 $xe^{-(n+1)x}$ 关于 n 非负递减, 对 $\forall m\in\mathbb{N}$, 都有

$$\int_0^{+\infty}\left|\sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x}\right|\mathrm{d}x \leqslant \int_0^{+\infty} xe^{-(m+1)x}\mathrm{d}x = -\frac{xe^{-(m+1)x}}{m+1}\bigg|_0^{+\infty} + \frac{1}{m+1}\int_0^{+\infty} e^{-(m+1)x}\mathrm{d}x = \frac{1}{(m+1)^2}.$$

令 $m\to+\infty$, 得 $\lim_{m\to+\infty}\int_0^{+\infty}\sum_{n=m}^{\infty} x(-1)^n e^{-(n+1)x}\mathrm{d}x = 0$. 故(??)式换序成立. □

命题 0.4

证明:

(1) $\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n}=\arctan\frac{q\sin x}{1-q\cos x},|q|\leqslant 1.$

(2) $\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n}=-\frac{1}{2}\ln(1+q^2-2q\cos x),|q|\leqslant 1.$

(3) $\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!}=e^{q\cos x}\cos(q\sin x)-1,|q|\leqslant 1,x\in\mathbb{R}.$

(4) $\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!}=e^{q\cos x}\sin(q\sin x),|q|\leqslant 1,x\in\mathbb{R}.$

笔记 在 $\mathbb{C}$ 上,

$$\mathrm{Ln}z=\ln|z|+i(\arg z+2k\pi),k\in\mathbb{Z}.$$

我们定义主值支

$$\ln z=\ln|z|+i\arg z.$$

本部分内容无需记忆,只需要大概有个可以算的感觉即可,实际做题中可以围绕这种级数给出构造.

证明 $\mathfrak{I}$ 表示取虚部,$\mathfrak{R}$ 表示取实部.

(1) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n}&=\mathfrak{I}\left(\sum_{n=1}^{\infty}\frac{q^ne^{inx}}{n}\right)=\mathfrak{I}\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n}\right)=\mathfrak{I}(-\ln(1-qe^{ix}))\\&=-\mathfrak{I}\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=\arctan\frac{q\sin x}{1-q\cos x}.\end{aligned}$$

(2) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n}&=-\mathfrak{R}\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=-\frac{1}{2}\ln\left[(1-q\cos x)^2+q^2\sin^2 x\right]\\&=-\frac{1}{2}\ln(1+q^2-2q\cos x).\end{aligned}$$

(3) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!}&=\mathfrak{R}\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n!}\right)=\mathfrak{R}\left(e^{qe^{ix}}-1\right)=\mathfrak{R}\left(e^{q\cos x+iq\sin x}-1\right)\\&=\mathfrak{R}\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)\\&=e^{q\cos x}\cos(q\sin x)-1.\end{aligned}$$

(4) 利用 (3) 有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!}&=\mathfrak{I}\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)\\&=e^{q\cos x}\sin(q\sin x).\end{aligned}$$

□

例题 0.9 计算

1. $\int_0^{2\pi}e^{\cos x}\cos(\sin x)\mathrm{d}x.$
2. $\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x,a\in(0,+\infty)\setminus\{1\}.$

注 由 1 的证明可得

$$e^{\cos x}\cos(\sin x) = \mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{(e^{\mathrm{i}x})^n}{n!}\right) = \mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{e^{\mathrm{i}nx}}{n!}\right) = \sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}.$$

实际上, 上式就是命题 0.4(3)的结论.

注 第 2 问也可以用含参积分求导的方法进行计算 (这个方法更容易想到).

证明

1.

$$\begin{aligned}
\int_0^{2\pi} e^{\cos x}\cos(\sin x)\,\mathrm{d}x &= \mathrm{Re}\left(\int_0^{2\pi} e^{\cos x}e^{\mathrm{i}\sin x}\mathrm{d}x\right) = \mathrm{Re}\left(\int_0^{2\pi} e^{\cos x+\mathrm{i}\sin x}\mathrm{d}x\right)\\
&= \mathrm{Re}\left(\int_0^{2\pi} e^{e^{\mathrm{i}x}}\mathrm{d}x\right) = \mathrm{Re}\left[\int_0^{2\pi}\sum_{n=0}^{+\infty}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right] = \mathrm{Re}\left[\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right]\\
&= \mathrm{Re}\left(\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{e^{\mathrm{i}nx}}{n!}\mathrm{d}x\right) = \mathrm{Re}\left(\int_0^{2\pi}\frac{e^{\mathrm{i}\cdot 0\cdot x}}{n!}\mathrm{d}x + \sum_{n=1}^{+\infty}\frac{e^{2\pi\mathrm{i}x}-1}{\mathrm{i}n\cdot n!}\right)\\
&= \mathrm{Re}\left(\int_0^{2\pi} 1\mathrm{d}x + 0\right) = 2\pi.
\end{aligned}$$

2. 注意到当 $a\in(0,1)$ 时, 有

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n} &= \mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{(ae^{\mathrm{i}x})^n}{n}\right] = -\mathrm{Re}\left[\ln(1-ae^{\mathrm{i}x})\right]\\
&= -\mathrm{Re}\left[\ln|1-ae^{\mathrm{i}x}| + \mathrm{i}\arg(1-ae^{\mathrm{i}x})\right] = -\ln|1-ae^{\mathrm{i}x}|\\
&= -\ln|(1-a\cos x)+a\mathrm{i}\sin x| = -\frac{1}{2}\ln(1+a^2-2a\cos x).
\end{aligned}$$

于是当 $a\in(0,1)$ 时, 就有

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = -\frac{1}{2}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n}\mathrm{d}x = 0.$$

若 $a>1$, 则 $\frac{1}{a}\in(0,1)$, 从而此时我们有

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = \pi\ln a^2 + \int_0^{\pi}\ln\left(\frac{1}{a^2}-\frac{2}{a}\cos x+1\right)\mathrm{d}x = \pi\ln a^2 = 2\pi\ln a.$$

又由 $\ln(1-2a\cos x+a^2)$ 关于 a 的偏导存在可知 $\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x$ 关于 a 连续. 于是由

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = 2\pi\ln a,\quad \forall a>1.$$

可知当 $a=1$ 时, 我们有

$$\int_0^{\pi}\ln(2-2\cos x)\mathrm{d}x = \lim_{a\to1^+}\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = \lim_{a\to1^+}(2\pi\ln a) = 0.$$

□

定义 0.1 (多重对数函数-Li_2 函数)

定义

$$\mathrm{Li}_2(x) \triangleq \sum_{n=1}^{\infty}\frac{x^n}{n^2},\quad x\in[-1,1].$$

♣

命题 0.5

(1) $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\dfrac{\pi^2}{6}-\ln x\cdot\ln(1-x),x\in(0,1)$.

(2) $\mathrm{Li}_2(1)=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\quad \mathrm{Li}_2(0)=0,\quad \mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}$.

证明

(1) 记 $f(x)\triangleq\mathrm{Li}_2(x),F(x)\triangleq f(x)+f(1-x)+\ln x\ln(1-x)$. 则

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}=-\frac{1}{x}\ln(1-x).$$

于是

$$F'(x)=-\frac{1}{x}\ln(1-x)+\frac{\ln x}{1-x}-\frac{\ln x}{1-x}+\frac{\ln(1-x)}{x}=0.$$

故 $F(x)\equiv F(1)=f(0)+f(1)=0+\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

(2) 显然 $\mathrm{Li}_2(1)=\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\mathrm{Li}_2(0)=0$. 由 (1) 可得

$$\mathrm{Li}_2\left(\frac{1}{2}\right)+\mathrm{Li}_2\left(\frac{1}{2}\right)=2\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{6}-\ln^2\frac{1}{2}\Longrightarrow \mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.$$

□

例题 0.10 计算 $\displaystyle\int_0^{\frac{1}{2}}\frac{\ln x}{1-x}\,\mathrm{d}x$.

解

$$\begin{aligned}\int_0^{\frac{1}{2}}\frac{\ln x}{1-x}\,\mathrm{d}x&=\int_{\frac{1}{2}}^{1}\frac{\ln(1-x)}{x}\,\mathrm{d}x=-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{1}{2}}^{1}x^{n-1}\,\mathrm{d}x\\&=-\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{2^n n^2}=-\frac{\pi^2}{6}+\mathrm{Li}_2\left(\frac{1}{2}\right)\\&\xlongequal{\text{命题 0.5}}-\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.\end{aligned}$$

□

0.1.6 其他

例题 0.11 证明积分 $\displaystyle\int_0^{\infty}e^{-ax^2-\frac{b}{x^2}}\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}},a,b>0$.

证明 当 $a=1$ 时, 就有

$$\begin{aligned}\int_0^{+\infty}e^{-x^2-\frac{b}{x^2}}\,\mathrm{d}x&=e^{-2\sqrt{b}}\int_0^{+\infty}e^{-\left(x-\frac{\sqrt{b}}{x}\right)^2}\,\mathrm{d}x\xlongequal{y=\frac{\sqrt{b}}{x}}e^{-2\sqrt{b}}\int_0^{+\infty}\frac{\sqrt{b}}{y^2}e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2}\,\mathrm{d}y\\&=\frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty}\left(1+\frac{\sqrt{b}}{y^2}\right)e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\,\mathrm{d}y=\frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty}e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\,\mathrm{d}\left(y-\frac{\sqrt{b}}{y}\right)\\&=\frac{e^{-2\sqrt{b}}}{2}\int_{-\infty}^{+\infty}e^{-t^2}\,dt=\frac{\sqrt{\pi}}{2}e^{-2\sqrt{b}}.\end{aligned}$$

于是对 $\forall a>0$, 就有

$$\int_0^{+\infty}e^{-ax^2-\frac{b}{x^2}}\,\mathrm{d}x=\frac{1}{\sqrt{a}}\int_0^{+\infty}e^{-x^2-\frac{ab}{x^2}}\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}.$$

□

例题 0.12 计算 $\displaystyle\int_0^{\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x,a\in\mathbb{R}$.

注 本题可以用复变函数的方法 (留数定理) 来计算. 但是我们这里用基本的高等数学的方法来计算. $\int_0^{\infty}\frac{\sin(ax)}{1+x^2}\mathrm{d}x$ 这个积分没办法算出具体的初等数值.

证明

$$
\begin{aligned}
\int_0^{+\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\cos(ax)\left(\int_0^{+\infty}e^{-(1+x^2)y}\mathrm{d}y\right)\mathrm{d}x\\
&= \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}e^{-(1+x^2)y}\cos(ax)\,\mathrm{d}y\right)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^{+\infty}e^{-(1+x^2)y}\cos(ax)\,\mathrm{d}y\mathrm{d}x\\
&= \frac{1}{2}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}e^{-(1+x^2)y}\cos(ax)\,\mathrm{d}x\right)\mathrm{d}y = \frac{1}{2}\int_0^{+\infty}e^{-y}\left(\int_{-\infty}^{+\infty}e^{-x^2y}\cos(ax)\,\mathrm{d}x\right)\mathrm{d}y\\
&= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty}e^{-y}\left(\int_{-\infty}^{+\infty}e^{-x^2y+\mathrm{i}ax}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty}e^{-y}\left(\int_{-\infty}^{+\infty}e^{-y\left(x^2-\frac{a\mathrm{i}x}{y}+\left(\frac{a\mathrm{i}}{2y}\right)^2\right)-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\
&= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty}e^{-y}\left(\int_{-\infty}^{+\infty}e^{-y\left(x-\frac{a\mathrm{i}}{2y}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty}e^{-y}\left(\int_{-\infty}^{+\infty}e^{-y\left(x+\frac{a}{2\mathrm{i}y}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\
&= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty}e^{-y}\left(\int_{-\infty}^{+\infty}e^{-y\left(x+\frac{a}{2\mathrm{i}y}\right)^2-\frac{a^2}{4y}}\mathrm{d}\left(x+\frac{a}{2\mathrm{i}y}\right)\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty}e^{-y}\left(\int_{-\infty}^{+\infty}e^{-yx^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\
&= \frac{1}{2}\int_0^{+\infty}e^{-y-\frac{a^2}{4y}}\left(\int_{-\infty}^{+\infty}e^{-yx^2}\mathrm{d}x\right)\mathrm{d}y = \frac{\sqrt{\pi}}{2}\int_0^{+\infty}\frac{1}{\sqrt{y}}e^{-y-\frac{a^2}{4y}}\mathrm{d}y\\
&\xlongequal{y=t^2}\sqrt{\pi}\int_0^{+\infty}e^{-t^2-\frac{a^2}{4t^2}}dt \xlongequal{\text{例题 0.11}}\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2}e^{-|a|} = \frac{\pi}{2}e^{-|a|}.
\end{aligned}
$$

□

例题 0.13 计算 $\int_0^{\infty}\frac{1}{(1+x^8)^2}\mathrm{d}x$.

注 由命题**??**可知对 $\forall s>0$, 都有

$$
\frac{1}{t^s} = \frac{1}{\Gamma(s)}\int_0^{+\infty}y^{s-1}e^{-ty}\mathrm{d}y, \forall t\in\mathbb{R}.
$$

本题的核心想法就是利用上式将 $\frac{z}{1+x^8}$ 转化成积分形式.

证明 注意到

$$
\int_0^{+\infty}ye^{-(1+x^8)y}\mathrm{d}y \xlongequal{y=\frac{z}{1+x^8}}\frac{1}{(1+x^8)^2}\int_0^{+\infty}ze^{-z}\mathrm{d}z = \frac{1}{(1+x^8)^2},
$$

因此

$$
\begin{aligned}
\int_0^{+\infty}\frac{1}{(1+x^8)^2}\mathrm{d}x &= \int_0^{+\infty}\left(\int_0^{+\infty}ye^{-(1+x^8)y}\mathrm{d}y\right)\mathrm{d}x = \int_0^{+\infty}\left(\int_0^{+\infty}ye^{-(1+x^8)y}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_0^{+\infty}ye^{-y}\left(\int_0^{+\infty}e^{-x^8y}\mathrm{d}x\right)\mathrm{d}y \xlongequal{x=y^{-\frac{1}{8}}z^{\frac{1}{8}}}\int_0^{+\infty}ye^{-y}\left(\int_0^{+\infty}y^{-\frac{1}{8}}e^{-z}\mathrm{d}z^{\frac{1}{8}}\right)\mathrm{d}y\\
&= \frac{1}{8}\int_0^{+\infty}y^{\frac{7}{8}}e^{-y}\left(\int_0^{+\infty}z^{-\frac{7}{8}}e^{-z}\mathrm{d}z\right)\mathrm{d}y = \frac{1}{8}\int_0^{+\infty}y^{\frac{7}{8}}e^{-y}\Gamma\left(\frac{1}{8}\right)\mathrm{d}y\\
&= \frac{1}{8}\Gamma\left(\frac{15}{8}\right)\Gamma\left(\frac{1}{8}\right) = \frac{1}{8}\cdot\frac{7}{8}\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{1}{8}\right)\\
&\xlongequal{??}\frac{7\pi}{64\sin\left(\frac{\pi}{8}\right)} = \frac{7\pi}{32\sqrt{2-\sqrt{2}}}.
\end{aligned}
$$

□

例题 0.14 计算积分 $I=\int_{-1}^{2}\frac{1+x^2}{1+x^4}\mathrm{d}x$.

注 在此例中 $I\neq F(2)-F(-1)$. 这是因为 F 并不是 f 在区间 $[-1,2]$ 上的原函数.

解 在不包含 0 的区间上作变换 $t=x-\frac{1}{x}$ 得

$$
\int\frac{1+x^2}{1+x^4}\mathrm{d}x = \int\frac{x-\frac{1}{x}}{2+\left(x-\frac{1}{x}\right)^2}\mathrm{d}x = \int\frac{\mathrm{d}t}{2+t^2}
$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x} + C.$$

这说明在区间 $[-1,0)$ 和 $(0,2]$ 上, 函数 $f(x) = \dfrac{1+x^2}{1+x^4}$ 的一个原函数是

$$F(x) = \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}.$$

因此

$$\int_{-1}^{0} f(x)\,\mathrm{d}x = F(0^-) - F(-1) = \frac{\pi}{2\sqrt{2}} - 0 = \frac{\pi}{2\sqrt{2}},$$

$$\int_{0}^{2} f(x)\,\mathrm{d}x = F(2) - F(0^+) = \frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}} + \frac{\pi}{2\sqrt{2}}.$$

故

$$I = \int_{-1}^{0} f(x)\,\mathrm{d}x + \int_{0}^{2} f(x)\,\mathrm{d}x = \frac{\pi}{\sqrt{2}} + \frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}}.$$

$\square$