0.1 Stirling 公式

对于阶乘问题,最好用的估计工具就是 Stirling 公式. 与组合数相关的极限问题,都可以尝试将其全部转化为阶乘然后估计大小.

定理 0.1 (Stirling 公式)

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, n \to \infty.$$

笔记 提示: 用欧拉麦克劳林公式估计 $\sum\limits_{k=1}^{n}\ln k, n \to \infty$ 的渐近展开式, 以此结合 Wallis 公式: $\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{\pi n}, n \to \infty$ 证明.

证明 由 E-M 公式可知, 对 $\forall n \in \mathbb{N}$, 都有

$$\sum_{k=1}^{n}\ln k = \frac{\ln n}{2} + \int_1^n \ln x\mathrm{d}x + \int_1^n \left(x-[x]-\frac{1}{2}\right)\frac{1}{x}\mathrm{d}x = \frac{\ln n}{2} + n\ln n - n + 1 + \int_1^n \left(x-[x]-\frac{1}{2}\right)\frac{1}{x}\mathrm{d}x. \tag{1}$$

由 Dirichlet 判别法可知, $\int_1^{+\infty}\left(x-[x]-\frac{1}{2}\right)\frac{1}{x}\mathrm{d}x$ 收敛. 则可设 $\lim\limits_{n\to\infty}\int_1^n \left(x-[x]-\frac{1}{2}\right)\frac{1}{x}\mathrm{d}x = \int_1^{+\infty}\left(x-[x]-\frac{1}{2}\right)\frac{1}{x}\mathrm{d}x \triangleq C_0 < \infty$. 记 $b_1(x) = x-[x]-\frac{1}{2}$, 再令 $b_2(x) = \frac{1}{2}(x-[x])^2 - \frac{1}{2}(x-[x]) + \frac{1}{12}, x \in \mathbb{R}$. 则不难发现 $b_2(x)$ 在 $\mathbb{R}$ 上连续且周期为 1, 并且

$$b_2(x) = \int_0^x b_1(y)\mathrm{d}y, \quad |b_2(x)| \leqslant \frac{1}{12}, \forall x \in \mathbb{R}.$$

从而对(1)式使用分部积分可得

$$\begin{aligned}
\sum_{k=1}^{n}\ln k &= \frac{\ln n}{2} + n\ln n - n + 1 + \int_1^n \frac{b_1(x)}{x}\mathrm{d}x = \frac{\ln n}{2} + n\ln n - n + 1 + \int_1^{+\infty}\frac{b_1(x)}{x}\mathrm{d}x - \int_n^{+\infty}\frac{b_1(x)}{x}\mathrm{d}x \\
&= \frac{\ln n}{2} + n\ln n - n + 1 + C_0 - \int_n^{+\infty}\frac{1}{x}db_2(x) = \frac{\ln n}{2} + n\ln n - n + 1 + C_0 - \frac{b_2(x)}{x}\Big|_n^{+\infty} - \int_n^{+\infty}\frac{b_2(x)}{x^2}\mathrm{d}x \\
&= \left(n+\frac{1}{2}\right)\ln n - n + 1 + C_0 + \frac{b_2(n)}{n} - \int_n^{+\infty}\frac{b_2(x)}{x^2}\mathrm{d}x, \forall n \in \mathbb{N}.
\end{aligned}$$

又因为 $|b_2(x)| \leqslant \frac{1}{12}, \forall x \in \mathbb{R}$. 所以对 $\forall n \in \mathbb{N}$, 我们有

$$\left|\frac{b_2(n)}{n} - \int_n^{+\infty}\frac{b_2(x)}{x^2}\mathrm{d}x\right| \leqslant \frac{1}{12}\left(\frac{1}{n} + \int_n^{+\infty}\frac{1}{x^2}\mathrm{d}x\right) = \frac{1}{6n}.$$

故 $\frac{b_2(n)}{n} - \int_n^{+\infty}\frac{b_2(x)}{x^2}\mathrm{d}x = O\left(\frac{1}{n}\right), \forall n \in \mathbb{N}$. 于是再记 $C = 1 + C_0$, 则

$$\sum_{k=1}^{n}\ln k = \left(n+\frac{1}{2}\right)\ln n - n + C + O\left(\frac{1}{n}\right), \forall n \in \mathbb{N}. \tag{2}$$

注意到

$$(2n)!! = 2^n n!, n = 0, 1, 2, \cdots. \tag{3}$$

于是由 Wallis 公式: $\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{\pi n}, n \to \infty$. 再结合(2)(3)可得

$$\begin{aligned}
\sqrt{\pi} &= \lim_{n\to\infty}\frac{(2n)!!}{(2n-1)!!\sqrt{n}} = \lim_{n\to\infty}\frac{[(2n)!!]^2}{(2n)!\sqrt{n}} = \lim_{n\to\infty}\frac{(2^n n!)^2}{(2n)!\sqrt{n}} = \lim_{n\to\infty}\frac{4^n n!\cdot n!}{(2n)!\sqrt{n}} \\
&= \lim_{n\to\infty}\frac{4^n n!\prod\limits_{k=1}^{n}k}{\sqrt{n}\prod\limits_{k=n+1}^{2n}k} = \lim_{n\to\infty}\frac{4^n n! e^{\sum\limits_{k=1}^{n}\ln k}}{\sqrt{n}e^{\sum\limits_{k=1}^{2n}\ln k}} = \lim_{n\to\infty}\frac{4^n n! e^{\left(n+\frac{1}{2}\right)\ln n - n + C + O\left(\frac{1}{n}\right)}}{\sqrt{n}e^{\left(2n+\frac{1}{2}\right)\ln 2n - 2n + C + O\left(\frac{1}{n}\right)}}
\end{aligned}$$

$$= \lim_{n\to\infty}\frac{4^n n! e^{(n+\frac{1}{2})\ln n - n + C + O(\frac{1}{n}) - [(2n+\frac{1}{2})\ln 2n - 2n + C + O(\frac{1}{n})]}}{\sqrt{n}} = \lim_{n\to\infty}\frac{4^n n! e^{-n\ln n + n - (2n+\frac{1}{2})\ln 2 + O(\frac{1}{n})}}{\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{4^n n! 2^{-2n-\frac{1}{2}} e^n}{n^n\sqrt{n}} e^{O(\frac{1}{n})} = \lim_{n\to\infty}\frac{n! e^n}{n^n\sqrt{2n}} e^{O(\frac{1}{n})}.$$

从而 $\lim\limits_{n\to\infty}\frac{n!e^n}{n^n\sqrt{2n}} = \frac{\sqrt{\pi}}{\lim\limits_{n\to\infty} e^{O(\frac{1}{n})}} = \sqrt{\pi}$. 因此 $\lim\limits_{n\to\infty}\frac{n!}{\sqrt{n}\left(\frac{n}{e}\right)^n} = \lim\limits_{n\to\infty}\frac{n!e^n}{n^n\sqrt{n}} = \sqrt{2\pi}$. 故 $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, n\to\infty$.

□

例题 0.1 设 n, v 为正整数且 $1 < v < n$, 满足 $\lim\limits_{n\to\infty}\frac{v-\frac{n}{2}}{\sqrt{n}} = \lambda > 0$, 证明: $\lim\limits_{n\to\infty}\frac{\sqrt{n}}{2^n}C_n^v = \sqrt{\frac{2}{\pi}}e^{-2\lambda^2}$.

证明 根据条件, 显然在 $n\to\infty$ 时 v 也会趋于无穷, 设 $v = \frac{n}{2} + w\sqrt{n}$, 则 $w = \frac{v-\frac{n}{2}}{\sqrt{n}}$, 从而 $\lim\limits_{n\to\infty} w = \lambda > 0$, 则有

$$\frac{\sqrt{n}}{2^n}C_n^v = \frac{\sqrt{n}}{2^n}\frac{n!}{v!(n-v)!}, \quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, n\to\infty.$$

从而

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}e^{-2\lambda^2} &= \lim_{n\to\infty}\frac{\sqrt{n}}{2^n}C_n^v = \lim_{n\to\infty}\frac{\sqrt{n}}{2^n}\frac{n!}{v!\,(n-v)!} \\
&= \lim_{n\to\infty}\frac{\sqrt{n}}{2^n}\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi v}\left(\frac{v}{e}\right)^v\sqrt{2\pi(n-v)}\left(\frac{n-v}{e}\right)^{n-v}} = \frac{1}{\sqrt{2\pi}}\lim_{n\to\infty}\frac{n^n}{2^n v^v (n-v)^{n-v}}\frac{n}{\sqrt{v(n-v)}} \\
\Longleftrightarrow \lim_{n\to\infty}&\frac{n^n}{2^n\left(\frac{n}{2}+w\sqrt{n}\right)^v\left(\frac{n}{2}-w\sqrt{n}\right)^{n-v}}\frac{n}{2\sqrt{v(n-v)}} = e^{-2\lambda^2}.
\end{aligned}$$

又

$$\lim_{n\to\infty}\frac{n}{2\sqrt{v(n-v)}} = \lim_{n\to\infty}\frac{n}{2\sqrt{\left(\frac{n}{2}+w\sqrt{n}\right)\left(\frac{n}{2}-w\sqrt{n}\right)}} = \lim_{n\to\infty}\frac{1}{\sqrt{1-\frac{4w^2}{\sqrt{n}}}} = 1,$$

故

$$\begin{aligned}
&\lim_{n\to\infty}\frac{n^n}{2^n\left(\frac{n}{2}+w\sqrt{n}\right)^v\left(\frac{n}{2}-w\sqrt{n}\right)^{n-v}}\frac{n}{2\sqrt{v(n-v)}} = e^{-2\lambda^2} \\
\Longleftrightarrow &\lim_{n\to\infty}\frac{n^{\left(\frac{n}{2}+w\sqrt{n}\right)+\left(\frac{n}{2}-w\sqrt{n}\right)}}{2^{\left(\frac{n}{2}+w\sqrt{n}\right)+\left(\frac{n}{2}-w\sqrt{n}\right)}\left(\frac{n}{2}+w\sqrt{n}\right)^{\frac{n}{2}+w\sqrt{n}}\left(\frac{n}{2}-w\sqrt{n}\right)^{\frac{n}{2}-w\sqrt{n}}} = e^{-2\lambda^2} \\
\Longleftrightarrow &\lim_{n\to\infty}\frac{n^{\left(\frac{n}{2}+w\sqrt{n}\right)+\left(\frac{n}{2}-w\sqrt{n}\right)}}{(n+2w\sqrt{n})^{\frac{n}{2}+w\sqrt{n}}(n-2w\sqrt{n})^{\frac{n}{2}-w\sqrt{n}}} = e^{-2\lambda^2} \\
\Longleftrightarrow &\lim_{n\to\infty}\frac{1}{\left(1+\frac{2w}{\sqrt{n}}\right)^{\frac{n}{2}+w\sqrt{n}}\left(1-\frac{2w}{\sqrt{n}}\right)^{\frac{n}{2}-w\sqrt{n}}} = e^{-2\lambda^2} \\
\Longleftrightarrow &\lim_{n\to\infty}\left[\left(\frac{n}{2}+w\sqrt{n}\right)\ln\left(1+\frac{2w}{\sqrt{n}}\right)+\left(\frac{n}{2}-w\sqrt{n}\right)\ln\left(1-\frac{2w}{\sqrt{n}}\right)\right] = 2\lambda^2.
\end{aligned} \tag{4}$$

又由 Taylor 公式可得

$$\begin{aligned}
&\left(\frac{n}{2}+w\sqrt{n}\right)\ln\left(1+\frac{2w}{\sqrt{n}}\right)+\left(\frac{n}{2}-w\sqrt{n}\right)\ln\left(1-\frac{2w}{\sqrt{n}}\right) \\
&= \left(\frac{n}{2}+w\sqrt{n}\right)\left(\frac{2w}{\sqrt{n}}-\frac{2w^2}{n}+O\left(\frac{1}{n\sqrt{n}}\right)\right)+\left(\frac{n}{2}-w\sqrt{n}\right)\left(-\frac{2w}{\sqrt{n}}-\frac{2w^2}{n}+O\left(\frac{1}{n\sqrt{n}}\right)\right) \\
&= w\sqrt{n}+w^2+O\left(\frac{1}{\sqrt{n}}\right)-w\sqrt{n}+w^2+O\left(\frac{1}{\sqrt{n}}\right) = 2w^2+O\left(\frac{1}{\sqrt{n}}\right), n\to\infty.
\end{aligned}$$

再结合 $\lim\limits_{n\to\infty} w = \lambda$ 可知(4)式成立, 因此结论得证.

□