0.1 Cauchy-Riemann 方程

定义 0.1

设 $f(z)=u(x,y)+\mathrm{i}v(x,y)$ 是定义在域 D 上的函数,$z_0=x_0+\mathrm{i}y_0\in D$. 我们说 f 在 z_0 处**实可微**, 是指 u 和 v 作为 x,y 的二元函数在 (x_0,y_0) 处可微.

命题 0.1

设 $f:D\to\mathbb{C}$ 是定义在域 D 上的函数,$z_0\in D$, 那么 f 在 z_0 处实可微的充分必要条件是

$$f(z_0+\Delta z)-f(z_0)=\frac{\partial f}{\partial z}(z_0)\Delta z+\frac{\partial f}{\partial \bar z}(z_0)\overline{\Delta z}+o(|\Delta z|). \tag{1}$$

成立, 其中

$$\begin{aligned}\frac{\partial}{\partial z}&=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right),\\ \frac{\partial}{\partial \bar z}&=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right).\end{aligned}$$

证明 设 f 在 z_0 处实可微, 由二元实值函数可微的定义, 有

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)=\frac{\partial u}{\partial x}(x_0,y_0)\Delta x+\frac{\partial u}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|), \tag{2}$$

$$v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)=\frac{\partial v}{\partial x}(x_0,y_0)\Delta x+\frac{\partial v}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|), \tag{3}$$

这里,$|\Delta z|=\sqrt{(\Delta x)^2+(\Delta y)^2}$. 于是

$$\begin{aligned}f(z_0+\Delta z)-f(z_0)&=u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)+\mathrm{i}(v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0))\\ &=\frac{\partial u}{\partial x}(x_0,y_0)\Delta x+\frac{\partial u}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|)+\mathrm{i}\left(\frac{\partial v}{\partial x}(x_0,y_0)\Delta x+\frac{\partial v}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|)\right)\\ &=\left(\frac{\partial u}{\partial x}(x_0,y_0)+\mathrm{i}\frac{\partial v}{\partial x}(x_0,y_0)\right)\Delta x+\left(\frac{\partial u}{\partial y}(x_0,y_0)+\mathrm{i}\frac{\partial v}{\partial y}(x_0,y_0)\right)\Delta y+o(|\Delta z|)\\ &=\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|).\end{aligned}$$

把 $\Delta x=\frac{1}{2}(\Delta z+\overline{\Delta z})$,$\Delta y=\frac{1}{2\mathrm{i}}(\Delta z-\overline{\Delta z})$ 代入上式, 得

$$\begin{aligned}f(z_0+\Delta z)-f(z_0)&=\frac{1}{2}\frac{\partial f}{\partial x}(x_0,y_0)(\Delta z+\overline{\Delta z})-\frac{\mathrm{i}}{2}\frac{\partial f}{\partial y}(x_0,y_0)(\Delta z-\overline{\Delta z})+o(|\Delta z|)\\ &=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right)f(x_0,y_0)\Delta z+\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right)f(x_0,y_0)\overline{\Delta z}+o(|\Delta z|).\end{aligned}$$

引进算子

$$\begin{aligned}\frac{\partial}{\partial z}&=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right),\\ \frac{\partial}{\partial \bar z}&=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right),\end{aligned} \tag{4}$$

则上式可写为

$$f(z_0+\Delta z)-f(z_0)=\frac{\partial f}{\partial z}(z_0)\Delta z+\frac{\partial f}{\partial \bar z}(z_0)\overline{\Delta z}+o(|\Delta z|). \tag{5}$$

容易看出,(5)式和(2),(3)两式等价.

□

注 为什么要像(4)式那样来定义算子 $\frac{\partial}{\partial z}$ 和 $\frac{\partial}{\partial \bar z}$ 呢? 这是因为如果把复变函数 $f(z)$ 写成

$$f(x,y)=f\left(\frac{z+\bar z}{2},-\mathrm{i}\frac{z-\bar z}{2}\right),$$

把 $z, \bar{z}$ 看成独立变量, 分别对 z 和 $\bar{z}$ 求偏导数, 则得

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right).$$

这就是表达式(4)的来源. 这说明在进行微分运算时, 可以把 $z, \bar{z}$ 看成独立的变量.

现在很容易得到 f 在 z_0 处可微的条件了.

定理 0.1

设 f 是定义在域 D 上的函数,$z_0 \in D$, 那么 f 在 z_0 处可微的充要条件是 f 在 z_0 处实可微且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. 在可微的情况下,$f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

♡

证明 如果 f 在 z_0 处可微, 由(??)式得

$$f(z_0 + \Delta z) - f(z_0) = f'(z_0)\Delta z + o(|\Delta z|)$$

与(1)式比较就知道,f 在 z_0 处是实可微的, 而且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0, f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

反之, 若 f 在 z_0 处实可微, 且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$, 则由(1)式得

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + o(|\Delta z|)$$

由此即知

$$\lim_{\Delta z} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial f}{\partial z}(z_0).$$

故 f 在 z_0 处可微, 而且 $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

□

定义 0.2 (Cauchy-Riemann 方程)

设 f 是定义在域 D 上的函数,$\frac{\partial f}{\partial \bar{z}} = 0$ 称为 **Cauchy − Riemann 方程**.

♣

笔记 从这个方程可以得到 f 的实部和虚部应满足的条件.

命题 0.2 (Cauchy-Riemann 方程的等价定义)

设 $z = x + \mathrm{i}y, f(z) = u(x, y) + \mathrm{i}v(x, y)$, 则 Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 等价于

(i)

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases} \tag{6}$$

(ii)

$$\frac{\partial \bar{f}}{\partial z} = 0.$$

(iii) 令 $x = r\cos\theta, y = r\sin\theta$, 进而 $z = r(\cos\theta + \mathrm{i}\sin\theta), f(z) = u(r, \theta) + \mathrm{i}v(r, \theta)$, 则 Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 等价于

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \end{cases}$$

♠

证明 (i) 由(4)式得

$$\frac{\partial f}{\partial \bar z}=\frac{\partial u}{\partial \bar z}+\mathrm{i}\frac{\partial v}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial u}{\partial y}\right)+\frac{\mathrm{i}}{2}\left(\frac{\partial v}{\partial x}+\mathrm{i}\frac{\partial v}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

因此,Cauchy-Riemann 方程 $\dfrac{\partial f}{\partial \bar z}=0$ 就等价于

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}.\end{cases}$$

(ii) 又注意到

$$\frac{\partial \overline{f}}{\partial z}=\frac{\partial u}{\partial z}-\mathrm{i}\frac{\partial v}{\partial z}=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\mathrm{i}\frac{\partial u}{\partial y}\right)-\frac{\mathrm{i}}{2}\left(\frac{\partial v}{\partial x}-\mathrm{i}\frac{\partial v}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-\frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

故 Cauchy-Riemann 方程 $\dfrac{\partial f}{\partial \bar z}=0$ 也等价于 $\dfrac{\partial \overline{f}}{\partial z}=0$.

(iii) 令 $F=x-r\cos\theta, G=y-r\sin\theta$, 则

$$\begin{pmatrix}F_x & F_y & F_r & F_\theta\\ G_x & G_y & G_r & G_\theta\end{pmatrix}=\begin{pmatrix}1 & 0 & -\cos\theta & r\sin\theta\\ 0 & 1 & -\sin\theta & -r\cos\theta\end{pmatrix}.$$

直接计算 Jacobi 行列式得

$$J=\frac{\partial(F,G)}{\partial(r,\theta)}=\begin{vmatrix}-\cos\theta & r\sin\theta\\ -\sin\theta & -r\cos\theta\end{vmatrix}=r,$$

于是

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial u}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,\theta)}\right]+\frac{\partial u}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,x)}\right]=\frac{\partial u}{\partial r}\cdot\cos\theta-\frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{r};$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}=\frac{\partial u}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,\theta)}\right]+\frac{\partial u}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,y)}\right]=\frac{\partial u}{\partial r}\cdot\sin\theta+\frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r};$$

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial v}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,\theta)}\right]+\frac{\partial v}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,x)}\right]=\frac{\partial v}{\partial r}\cdot\cos\theta-\frac{\partial v}{\partial \theta}\cdot\frac{\sin\theta}{r};$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}=\frac{\partial v}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,\theta)}\right]+\frac{\partial v}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,y)}\right]=\frac{\partial v}{\partial r}\cdot\sin\theta+\frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r}.$$

从而

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\iff\frac{\partial u}{\partial r}\cdot\cos\theta-\frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{r}=\frac{\partial v}{\partial r}\cdot\sin\theta+\frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

$$\iff\frac{\partial u}{\partial r}\cdot r\cos\theta-\frac{\partial u}{\partial \theta}\cdot\sin\theta=\frac{\partial v}{\partial r}\cdot r\sin\theta+\frac{\partial v}{\partial \theta}\cdot\cos\theta,$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\iff\frac{\partial u}{\partial r}\cdot\sin\theta+\frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r}=-\frac{\partial v}{\partial r}\cdot\cos\theta+\frac{\partial v}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$\iff\frac{\partial u}{\partial r}\cdot r\sin\theta+\frac{\partial u}{\partial \theta}\cdot\cos\theta=-\frac{\partial v}{\partial r}\cdot r\cos\theta+\frac{\partial v}{\partial \theta}\cdot\sin\theta.$$

由 (i) 可知 Cauchy-Riemann 方程等价于 $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$, 故此时 Cauchy-Riemann 方程等价于

$$\begin{cases}\dfrac{\partial u}{\partial r}\cdot r\cos\theta-\dfrac{\partial u}{\partial \theta}\cdot\sin\theta=\dfrac{\partial v}{\partial r}\cdot r\sin\theta+\dfrac{\partial v}{\partial \theta}\cdot\cos\theta,\\ \dfrac{\partial u}{\partial r}\cdot r\sin\theta+\dfrac{\partial u}{\partial \theta}\cdot\cos\theta=-\dfrac{\partial v}{\partial r}\cdot r\cos\theta+\dfrac{\partial v}{\partial \theta}\cdot\sin\theta.\end{cases}$$

化简可得

$$\begin{cases}\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial \theta},\\ \dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}.\end{cases}$$

□

定理 0.2

设 $f=u+\mathrm{i}v$ 是定义在域 D 上的函数,$z_0=x_0+\mathrm{i}y_0\in D$, 那么 f 在 z_0 处可微的充要条件是 $u(x,y)$,$v(x,y)$ 在 (x_0,y_0) 处可微, 且在 (x_0,y_0) 处满足 Cauchy-Riemann 方程, 即

$$\frac{\partial f}{\partial \bar{z}}=0,\qquad \frac{\partial \overline{f}}{\partial z}=0,\qquad \begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\end{cases}.$$

在可微的情况下, 有

$$f'(z_0)=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-\mathrm{i}\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}-\mathrm{i}\frac{\partial u}{\partial y}.$$

这里的偏导数都在 (x_0,y_0) 处取值.

♡

证明 最后这个 $f'(z_0)$ 的表达式是从 定理 0.1中的 $f'(z_0)=\dfrac{\partial f}{\partial z}(z_0)$ 和 Cauchy-Riemann 方程的等价定义得到的.

□

定义 0.3

1. 设 D 是 $\mathbb{C}$ 中的域, 我们用 $C(D)$ 记 D 上连续函数的全体, 用 $H(D)$ 记 D 上全纯函数的全体.
2. 设 $f=u+\mathrm{i}v$, 记 $\dfrac{\partial f}{\partial x}=\dfrac{\partial u}{\partial x}+\mathrm{i}\dfrac{\partial v}{\partial x}$,$\dfrac{\partial f}{\partial y}=\dfrac{\partial u}{\partial y}+\mathrm{i}\dfrac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\dfrac{\partial f}{\partial x}$,$\dfrac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体.
3. 用 $C^k(D)$ 记在 D 上有 k 阶连续偏导数的函数的全体,$C^\infty(D)$ 记在 D 上有任意阶连续偏导数的函数的全体.

♣

命题 0.3

(1) $H(D)\subset C(D)$.
(2) $C^1(D)\subset C(D)$.
(3) 域 D 上的全纯函数在 D 上有任意阶的连续偏导数, 并且有如下的包含关系:

$$H(D)\subset C^\infty(D)\subset C^k(D)\subset C^1(D)\subset C(D).$$

这里,k 是大于 1 的自然数.

♠

证明

(1) 命题**??**告诉我们,$H(D)\subset C(D)$.
(2) 设 $f=u+\mathrm{i}v$, 记 $\dfrac{\partial f}{\partial x}=\dfrac{\partial u}{\partial x}+\mathrm{i}\dfrac{\partial v}{\partial x}$,$\dfrac{\partial f}{\partial y}=\dfrac{\partial u}{\partial y}+\mathrm{i}\dfrac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\dfrac{\partial f}{\partial x}$,$\dfrac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体. 进而 u,v 关于 x,y 的偏导在 D 上都连续, 由多元微积分的知识知道,u,v 在 D 上都可微. 于是对于任意 $f\in C^1(D)$,f 在 D 上实可微, 从(5)式知道

$$f(z_0+\Delta z)-f(z_0)=\frac{\partial f}{\partial z}(z_0)\Delta z+\frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z}+o(|\Delta z|).$$

令 $\Delta z\to 0$, 则 $\lim\limits_{\Delta z\to 0}f(z_0+\Delta z)=f(z_0)$, 故 f 在 D 上连续, 因而 $C^1(D)\subset C(D)$.
(3)

□

例题 0.1 研究函数 $f(z)=z^n$,n 是自然数.

解 显然,$\dfrac{\partial f}{\partial \bar{z}}=0$, 且 f 在整个平面上是实可微的. 因而,f 是 $\mathbb{C}$ 上的全纯函数, 而且

$$f'(z)=\frac{\partial f}{\partial z}=nz^{n-1}.\tag{7}$$

□

例题 0.2 研究函数 $f(z)=\mathrm{e}^{-|z|^2}$.

解 把 f 写为 $f(z)=\mathrm{e}^{-z\bar{z}}$, 于是 $\dfrac{\partial f}{\partial \bar{z}}=-\mathrm{e}^{-z\bar{z}}z$, 它只有在 $z=0$ 处才等于零. 因此,$\mathrm{e}^{-|z|^2}$ 只有在 $z=0$ 处可微, 它在任

何点处都不是全纯的. 但它对 x, y 有任意阶连续偏导数, 所以它是 $C^\infty(\mathbb{C})$ 中的函数.

□

命题 0.4

设 D 是 $\mathbb{C}$ 中的域,$f \in H(D)$. 如果对每一个 $z \in D$, 都有 $f'(z) = 0$, 证明 f 是一常数.

证明 因为 $f'(z) = 0$, 所以由定理 0.2可知 $f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = 0$, 并且 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. 于是 $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$. 因此 u, v 都是常数, 故 f 是一常数.

□

定义 0.4 (调和函数)

设 u 是域 D 上的实值函数, 如果 $u \in C^2(D)$, 且对任意 $z \in D$, 有

$$\Delta u(z) = \frac{\partial^2 u(z)}{\partial x^2} + \frac{\partial^2 u(z)}{\partial y^2} = 0, \tag{8}$$

就称 u 是 D 中的**调和函数**. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ 称为 **Laplace 算子**.

命题 0.5

设 $u \in C^2(D)$, 那么 $\Delta u = 4\frac{\partial^2 u}{\partial z \partial \bar{z}}$.

证明 由(4)式, 有

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

所以

$$\begin{aligned}\frac{\partial^2 u}{\partial z \partial \bar{z}} &= \frac{\partial}{\partial \bar{z}}\frac{\partial u}{\partial z} = \frac{1}{4}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}\right) + \mathrm{i}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}\right)\right] \\ &= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{1}{4}\Delta u.\end{aligned}$$

□

定理 0.3

设 $f = u + \mathrm{i}v \in H(D)$, 那么 u 和 v 都是 D 上的调和函数.

证明 因为 $f \in H(D)$, 由定理 0.2, 有

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{f}}{\partial z} = 0.$$

所以

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 \bar{f}}{\partial z \partial \bar{z}} = 0.$$

于是, 由 $u = \frac{1}{2}(f + \bar{f})$ 即得

$$\Delta u = 4\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

同理可证 $\Delta v = 0$.

□

定义 0.5 (共轭调和函数)

设 u 和 v 是 D 上的一对调和函数, 如果它们还满足 Cauchy-Riemann 方程

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \end{cases} \tag{9}$$

就称 v 为 u 的**共轭调和函数**.

命题 0.6

全纯函数的实部和虚部就构成一对共轭调和函数.

证明 由定理 0.3和定理 0.2立得.

□

定理 0.4

设 u 是单连通域 D 上的调和函数, 则必存在 u 的共轭调和函数 v, 使得 $u+\mathrm{i}v$ 是 D 上的全纯函数.

证明 因为 u 满足 Laplace 方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

若令 $P = -\dfrac{\partial u}{\partial y}, Q = \dfrac{\partial u}{\partial x}$, 则

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial y},$$

所以

$$P\mathrm{d}x + Q\mathrm{d}y = -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$$

是一个全微分, 因而积分

$$\int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$$

与路径无关. 令

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y,$$

则

$$\begin{aligned} v(x,y) &= \int_{(x_0,y_0)}^{(x,y_0)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y + \int_{(x,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y \\ &= \int_{x_0}^{x} -\frac{\partial u}{\partial y}\mathrm{d}x + 0 + 0 + \int_{y_0}^{y} \frac{\partial u}{\partial x}\mathrm{d}y \\ &= \int_{x_0}^{x} -\frac{\partial u}{\partial y}\mathrm{d}x + \int_{y_0}^{y} \frac{\partial u}{\partial x}\mathrm{d}y. \end{aligned}$$

那么

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}, \\ \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}. \end{cases}$$

所以,v 就是要求的 u 的共轭调和函数.

□