0.1 定积分

0.1.1 建立积分递推

例题 0.1 计算 $\int_0^{\frac{\pi}{2}} \cos^n x\sin(nx)\mathrm{d}x, n\in\mathbb{N}$.

证明 利用分部积分和和差化积公式可得

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x\sin(nx)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x\cdot\cos x\sin(nx)\mathrm{d}x\\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x\cdot[\sin((n+1)x)+\sin((n-1)x)]\mathrm{d}x\\
&= \frac{I_{n-1}}{2}+\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x[\sin(nx)\cos x+\cos(nx)\sin x]\mathrm{d}x\\
&= \frac{I_{n-1}}{2}+\frac{I_n}{2}-\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x\cos(nx)\mathrm{d}\cos x\\
&= \frac{I_{n-1}+I_n}{2}-\frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos(nx)\mathrm{d}\cos^n x\\
&= \frac{I_{n-1}+I_n}{2}+\frac{1}{2n}-\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^n x\sin(nx)\mathrm{d}x\\
&= \frac{I_{n-1}+I_n}{2}+\frac{1}{2n}-\frac{I_n}{2}\\
&= \frac{I_{n-1}}{2}+\frac{1}{2n}.
\end{aligned}$$

故 $I_n=\frac{I_{n-1}}{2}+\frac{1}{2n}$, 则两边同乘 2^n(强行裂项)

$$2^n I_n = 2^{n-1}I_{n-1}+\frac{2^{n-1}}{n}, n=1,2,\cdots$$

又注意到 $I_0=0$, 从而

$$2^n I_n = 0+\sum_{k=1}^n \frac{2^{k-1}}{k} \Rightarrow I_n=\frac{1}{2^n}\sum_{k=1}^n\frac{2^{k-1}}{k}.$$

□

命题 0.1

证明:

(1) $\int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x=\begin{cases}0, & n\text{为偶数}\\ \pi, & n\text{为奇数}\end{cases}.$

(2) $\int_0^{\pi}\frac{\sin^2(nx)}{\sin^2 x}\mathrm{d}x=n\pi.$

(3) $\int_0^{\pi}\frac{\sin^2(nx)}{\sin x}\mathrm{d}x=\sum_{k=1}^n\frac{2}{2k-1}.$

笔记 提示:$\sin^2 x-\sin^2 y=\sin(x-y)\sin(x+y)$(证明见命题**??**).

证明

(1) 记 $I_n=\int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x$, 则

$$I_{n+2}-I_n=\int_0^{\pi}\frac{\sin((n+2)x)-\sin(nx)}{\sin x}\mathrm{d}x=\int_0^{\pi}\frac{2\cos((n+1)x)\sin x}{\sin x}\mathrm{d}x=2\int_0^{\pi}\cos((n+1)x)\mathrm{d}x=0.$$

于是

$$\int_0^\pi \frac{\sin(nx)}{\sin x}\,\mathrm{d}x = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$$

(2) 记 $I_n = \displaystyle\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x}\,\mathrm{d}x \\ &= \int_0^\pi \frac{\sin((2n+1)x)}{\sin x}\,\mathrm{d}x \xlongequal{\text{命题 0.1(1)}} \pi. \end{aligned} \tag{1}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记 $I_n = \displaystyle\int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin x}\,\mathrm{d}x \\ &= \int_0^\pi \sin((2n+1)x)\,\mathrm{d}x = \frac{1}{2n+1}\cos((2n+1)x)\Big|_\pi^0 = \frac{2}{2n+1}. \end{aligned} \tag{2}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1}\frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1}\frac{2}{2k+1} = \sum_{k=1}^{n}\frac{2}{2k-1}.$$

□

例题 0.2 设 $a > 1$, 计算积分 $\displaystyle\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x$.

注 很多情况下不需求出被积函数的原函数, 只需充分利用换元、分部积分以及被积函数的性质, 即可求出积分的值. 见下述解法二.

解 解法一:设 $a_0 = a > 1$. 构造数列如下:

$$a_{n+1} = 2a_n^2 - 1 \quad (n = 0, 1, \cdots),$$

则由例题**??**可知, 存在 $x_0 > 0$ 使得

$$a_0 = \mathrm{ch}(x_0), \quad a_n = \mathrm{ch}(2^n x_0),$$

其中 $\mathrm{ch}(x) = \dfrac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$. 可以解得

$$x_0 = \ln\left(a_0 + \sqrt{a_0^2 - 1}\right). \tag{3}$$

故

$$a_n = \frac{\mathrm{e}^{2^n x_0} + \mathrm{e}^{-2^n x_0}}{2}.$$

设

$$I_n = \int_0^\pi \ln(a_n - \cos x)\,\mathrm{d}x,$$

则

$$\begin{aligned} I_0 &= \int_0^\pi \ln(a_0 - \cos x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^\pi \ln(a_0 - \cos x)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \ln(a_0 + \cos x)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0^2 - \cos^2 x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\left(a_0^2 - \frac{1 + \cos 2x}{2}\right)\,\mathrm{d}x \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{a_1 - \cos 2x}{2}\right) \mathrm{d}x = \frac{1}{2}\int_0^{\pi} \ln\left(\frac{a_1 - \cos x}{2}\right) \mathrm{d}x = \frac{1}{2}I_1 - \frac{\pi}{2}\ln 2.$$

同理, 有

$$I_n = \frac{1}{2}I_{n+1} - \frac{\pi}{2}\ln 2. \tag{4}$$

由此递推公式, 可得

$$I_0 = \frac{1}{2^n}I_n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)\frac{\pi}{2}\ln 2. \tag{5}$$

因为

$$\begin{aligned} I_n &= \int_0^{\pi} \ln(a_n - \cos x)\,\mathrm{d}x = \int_0^{\pi} \ln\left(\frac{\mathrm{e}^{2^n x_0} + \mathrm{e}^{-2^n x_0}}{2} - \cos x\right) \mathrm{d}x \\ &= 2^n x_0 \pi + \int_0^{\pi} \ln\left(\frac{1 + \mathrm{e}^{-2^{n+1}x_0}}{2} - \mathrm{e}^{-2^n x_0}\cos x\right) \mathrm{d}x, \end{aligned}$$

所以

$$\frac{1}{2^n}I_n \to x_0\pi \quad (n \to +\infty).$$

故从式 (5) 可得

$$I_0 = x_0\pi - \pi\ln 2 = \pi\ln\left(\frac{a_0 + \sqrt{a_0^2 - 1}}{2}\right),$$

即所求的积分为

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right).$$

解法二:我们有

$$F(a) = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x = \int_0^{\pi} \ln(a - \cos x)\,\mathrm{d}x.$$

由定理**??**, 关于 a 求导得到

$$F'(a) = \int_0^{\pi} \frac{1}{a - \cos x}\,\mathrm{d}x \xlongequal{\text{万能公式}} \int_0^{+\infty} \frac{2}{a(1+t^2) - (1-t^2)}\,\mathrm{d}t = \frac{\pi}{\sqrt{a^2 - 1}}, \quad a > 1.$$

因此

$$F(a) = \int_1^a F'(a)\,\mathrm{d}a = \pi\ln\left(a + \sqrt{a^2 - 1}\right) + C, \quad a > 1.$$

结合

$$F(1) = 2\int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x = -\pi\ln 2.$$

可得

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right), \quad a > 1.$$

□

0.1.2 区间再现

定理 0.1 (区间再现恒等式)

当下述积分有意义时, 我们有

1.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)]\mathrm{d}x.$$

2.
$$\int_0^{\infty} f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^{\infty} f(x)\mathrm{d}x = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2}\right]\mathrm{d}x.$$

笔记 注意: 倒代换具有将 $[0,1]$ 转化为 $[1,+\infty)$ 的功能.

证明 证明是显然的.(第 1 问中最后一个等号是由轴对称得到的) □

命题 0.2

证明

1. $\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
2. $\int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
3. $\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln 2.$

证明

1.
$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

2.
$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

3. 解法一:
$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &\xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2}\mathrm{d}\tan\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\cdot\ln(1+\tan\theta)}{\sec^2\theta}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\frac{2}{1+\tan\theta}\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \ln 2\mathrm{d}\theta = \frac{\pi}{8}\ln 2.
\end{aligned}$$

解法二:考虑含参量积分
$$\varphi(\alpha) = \int_0^1 \frac{\ln(1+\alpha x)}{1+x^2}\mathrm{d}x,\quad \alpha\in[0,1].$$

显然 $\varphi(0)=0,\varphi(1)=I$, 且函数 $\dfrac{\ln(1+\alpha x)}{1+x^2}$ 在 $R=[0,1]\times[0,1]$ 上满足定理**??** 的条件, 于是

$$\varphi'(\alpha)=\int_0^1\frac{x}{(1+x^2)(1+\alpha x)}\,\mathrm{d}x.$$

因为

$$\frac{x}{(1+x^2)(1+\alpha x)}=\frac{1}{1+\alpha^2}\left(\frac{\alpha+x}{1+x^2}-\frac{\alpha}{1+\alpha x}\right),$$

所以

$$\begin{aligned}\varphi'(\alpha)&=\frac{1}{1+\alpha^2}\left(\int_0^1\frac{\alpha}{1+x^2}\,\mathrm{d}x+\int_0^1\frac{x}{1+x^2}\,\mathrm{d}x-\int_0^1\frac{\alpha}{1+\alpha x}\,\mathrm{d}x\right)\\&=\frac{1}{1+\alpha^2}\left[\alpha\arctan x\Big|_0^1+\frac12\ln\left(1+x^2\right)\Big|_0^1-\ln\left(1+\alpha x\right)\Big|_0^1\right]\\&=\frac{1}{1+\alpha^2}\left[\alpha\cdot\frac{\pi}{4}+\frac12\ln 2-\ln\left(1+\alpha\right)\right].\end{aligned}$$

因此

$$\begin{aligned}\int_0^1\varphi'\left(\alpha\right)\,\mathrm{d}\alpha&=\int_0^1\frac{1}{1+\alpha^2}\left[\frac{\pi}{4}\alpha+\frac12\ln 2-\ln\left(1+\alpha\right)\right]\,\mathrm{d}\alpha\\&=\frac{\pi}{8}\ln\left(1+\alpha^2\right)\Big|_0^1+\frac12\ln 2\arctan\alpha\Big|_0^1-\varphi\left(1\right)\\&=\frac{\pi}{8}\ln 2+\frac{\pi}{8}\ln 2-\varphi\left(1\right)\\&=\frac{\pi}{4}\ln 2-\varphi\left(1\right).\end{aligned}$$

另一方面,

$$\int_0^1\varphi'(\alpha)\,d\alpha=\varphi(1)-\varphi(0)=\varphi(1),$$

所以 $I=\varphi(1)=\dfrac{\pi}{8}\ln 2.$

□

例题 0.3 计算

1. $\displaystyle\int_0^{\infty}\frac{\ln x}{x^2+a^2}\mathrm{d}x,a>0.$
2. $\displaystyle\int_0^{\infty}\frac{\ln x}{x^2+x+1}\mathrm{d}x.$
3. $\displaystyle\int_0^{1}\frac{\ln x}{\sqrt{x-x^2}}\mathrm{d}x.$

解

1. 注意到

$$\int_0^{+\infty}\frac{\ln x}{x^2+a^2}\mathrm{d}x\xlongequal{x=at}\frac1a\int_0^{+\infty}\frac{\ln(at)}{1+t^2}\mathrm{d}t=\frac1a\int_0^{+\infty}\frac{\ln a}{1+t^2}\mathrm{d}t+\frac1a\int_0^{+\infty}\frac{\ln t}{1+t^2}\mathrm{d}t=\frac{\pi\ln a}{2a}+\frac1a\int_0^{+\infty}\frac{\ln t}{1+t^2}\mathrm{d}t.\tag{6}$$

又注意到

$$\int_0^{+\infty}\frac{\ln t}{1+t^2}\mathrm{d}t\xlongequal{t=\frac1x}\int_0^{+\infty}\frac{\ln\frac1x}{1+\frac{1}{x^2}}\frac{1}{x^2}\mathrm{d}x=\int_0^{+\infty}\frac{-\ln x}{1+x^2}\mathrm{d}x\Longrightarrow\int_0^{+\infty}\frac{\ln t}{1+t^2}\mathrm{d}t=0.$$

于是再结合(6)式可得

$$\int_0^{+\infty}\frac{\ln x}{x^2+a^2}\mathrm{d}x=\frac{\pi\ln a}{2a}.$$

2.
$$\int_0^{\infty}\frac{\ln x}{x^2+x+1}\mathrm{d}x\xlongequal{x=\frac1t}\int_0^{\infty}\frac{-\ln t}{1+\frac1t+\frac{1}{t^2}}\mathrm{d}\frac1t=\int_0^{+\infty}\frac{-\ln t}{1+t+t^2}\mathrm{d}t\Longrightarrow\int_0^{\infty}\frac{\ln x}{x^2+x+1}\mathrm{d}x=0.$$

3.

$$\int_0^1 \frac{\ln x}{\sqrt{x-x^2}}\mathrm{d}x \xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}} \frac{\ln \sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}}\mathrm{d}\sin^2 y$$

$$= 4\int_0^{\frac{\pi}{2}} \ln \sin y \mathrm{d}y \xlongequal{\text{命题 0.2}} 4\cdot\left(-\frac{\pi}{2}\ln 2\right) = -2\pi \ln 2.$$

□

例题 0.4

1. 对 $n\in\mathbb{N}$, 计算 $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\mathrm{d}x$.
2. $\int_{-\pi}^{\pi} \frac{x\sin x \arctan e^x}{1+\cos^2 x}\mathrm{d}x$.
3. 对 $n\in\mathbb{N}$, 计算 $\int_0^{2\pi} \sin(\sin x + nx)\mathrm{d}x$.

解

1.

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\mathrm{d}x = \int_{-\pi}^{0}\left[\frac{\sin(nx)}{(1+2^x)\sin x} + \frac{\sin(nx)}{(1+2^{-x})\sin x}\right]\mathrm{d}x = \int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\left(\frac{1}{1+2^x}+\frac{1}{1+2^{-x}}\right)\mathrm{d}x$$

$$= \int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\cdot\frac{2+2^x+2^{-x}}{2+2^x+2^{-x}}\mathrm{d}x = \int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\mathrm{d}x = \int_{0}^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x \xlongequal{\text{例题 0.1}} \begin{cases} 0, n\text{为偶数} \\ \pi, n\text{为奇数} \end{cases}.$$

2.

$$\int_{-\pi}^{\pi} \frac{x\sin x \arctan e^x}{1+\cos^2 x}\mathrm{d}x = \int_{-\pi}^{0}\left(\frac{x\sin x \arctan e^x}{1+\cos^2 x} + \frac{x\sin x\arctan e^{-x}}{1+\cos^2 x}\right)\mathrm{d}x = \int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}(\arctan e^x + \arctan e^{-x})\mathrm{d}x$$

$$\xlongequal{\text{命题??(1)}} \int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}\cdot\frac{\pi}{2}\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\frac{x\sin x}{1+\cos^2 x} + \frac{(\pi-x)\sin x}{1+\cos^2 x}\right)\mathrm{d}x = \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x$$

$$= \frac{\pi^2}{2}\arctan\cos x\Big|_{\frac{\pi}{2}}^{0} = \frac{\pi^2}{2}\cdot\frac{\pi}{4} = \frac{\pi^3}{8}.$$

3.

$$\int_0^{2\pi}\sin(\sin x + nx)\mathrm{d}x = \int_0^{2\pi}\sin[\sin(2\pi - x) + n(2\pi - x)]\mathrm{d}x$$

$$= \int_0^{2\pi}\sin(-\sin x - nx)\mathrm{d}x = -\int_0^{2\pi}\sin(\sin x + nx)\mathrm{d}x$$

$$\Longrightarrow \int_0^{2\pi}\sin(\sin x + nx)\mathrm{d}x = 0.$$

□

0.1.3 Frullani(傅汝兰尼) 积分

定理 0.2 (Frullani(傅汝兰尼) 积分)

设 $f\in C(0,+\infty)$.

1. 若存在极限

$$\lim_{x\to 0^+} f(x),\ \lim_{x\to+\infty} f(x), \tag{7}$$

则对 $a,b>0$ 有

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x = \left[\lim_{x\to 0^+} f(x) - \lim_{x\to+\infty} f(x)\right]\ln\frac{b}{a}.$$

2. 若存在极限和积分

$$\lim_{x\to 0^+} f(x)=\alpha, \int_A^{\infty} \frac{f(x)}{x}\mathrm{d}x. \tag{8}$$

则对 $a,b>0$, 有

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x=\alpha \ln \frac{b}{a}.$$

3. 若存在极限和积分

$$\lim_{x\to +\infty} f(x)=\alpha, \int_0^{1} \frac{f(x)}{x}\mathrm{d}x. \tag{9}$$

则对 $a,b>0$, 有

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x=\alpha \ln \frac{a}{b}.$$

4. 若 f 是周期 $T>0$ 函数且 $\lim\limits_{x\to 0^+} f(x)$ 存在, 则对 $a,b>0$ 有

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to 0^+} f(x)-\frac{1}{T}\int_0^T f(x)\mathrm{d}x\right]\ln \frac{b}{a}.$$

5. 若 f 满足 $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to +\infty} \frac{1}{x}\int_0^x f(y)\mathrm{d}y$ 存在, 则对 $a,b>0$ 有

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to 0^+} f(x)-\lim_{x\to +\infty}\frac{1}{x}\int_0^x f(y)\mathrm{d}y\right]\ln \frac{b}{a}.$$

笔记 傅汝兰尼积分有诸多变种, 无需记忆具体表达式, 知道有大概这么一个东西即可.

证明 不妨设 $b>a$.

1. 给定 $A>\delta>0$, 考虑

$$\begin{aligned}\int_\delta^A \frac{f(ax)-f(bx)}{x}\mathrm{d}x&=\int_\delta^A \frac{f(ax)}{x}\mathrm{d}x-\int_\delta^A \frac{f(bx)}{x}\mathrm{d}x\\&=\int_{a\delta}^{aA} \frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{bA} \frac{f(x)}{x}\mathrm{d}x\\&=\int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x\\&\xlongequal{\text{积分中值定理}} f(\theta_1)\int_{bA}^{aA} \frac{1}{x}\mathrm{d}x-f(\theta_2)\int_{b\delta}^{a\delta} \frac{1}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta_1\in(aA,bA), \theta_2\in(a\delta,b\delta)$, 于是让 $A\to+\infty, \delta\to 0^+$, 由(7), 我们知

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to 0^+} f(x)-\lim_{x\to +\infty} f(x)\right]\ln \frac{b}{a}.$$

2. 给定 $A>\delta>0$, 考虑

$$\begin{aligned}\int_\delta^A \frac{f(ax)-f(bx)}{x}\mathrm{d}x&=\int_\delta^A \frac{f(ax)}{x}\mathrm{d}x-\int_\delta^A \frac{f(bx)}{x}\mathrm{d}x\\&=\int_{a\delta}^{aA} \frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{bA} \frac{f(x)}{x}\mathrm{d}x\\&=\int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x\\&\xlongequal{\text{积分中值定理}} \int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x-f(\theta)\int_{b\delta}^{a\delta} \frac{1}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta\in(a\delta,b\delta)$, 于是让 $A\to+\infty, \delta\to 0^+$, 由 (8), 我们知

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x=\alpha \ln \frac{b}{a}.$$

3. 给定 $A > \delta > 0$, 考虑

$$\begin{aligned}\int_{\delta}^{A} \frac{f(ax)-f(bx)}{x}\mathrm{d}x &= \int_{\delta}^{A} \frac{f(ax)}{x}\mathrm{d}x - \int_{\delta}^{A} \frac{f(bx)}{x}\mathrm{d}x \\ &= \int_{a\delta}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{bA} \frac{f(x)}{x}\mathrm{d}x \\ &= \int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x \\ &\xlongequal{\text{积分中值定理}} f(\theta)\int_{bA}^{aA} \frac{1}{x}\mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta \in (aA, bA)$, 于是让 $A \to +\infty, \delta \to 0^+$, 由(9), 我们知

$$\int_{0}^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x = \alpha \ln \frac{a}{b}.$$

4. 给定 $A > \delta > 0$, 考虑

$$\begin{aligned}\int_{\delta}^{A} \frac{f(ax)-f(bx)}{x}\mathrm{d}x &= \int_{\delta}^{A} \frac{f(ax)}{x}\mathrm{d}x - \int_{\delta}^{A} \frac{f(bx)}{x}\mathrm{d}x \\ &= \int_{a\delta}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{bA} \frac{f(x)}{x}\mathrm{d}x \\ &= \int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x \\ &\xlongequal{\text{积分中值定理}} \int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x - f(\theta)\int_{b\delta}^{a\delta} \frac{1}{x}\mathrm{d}x \\ &= \int_{b}^{a} \frac{f(Ax)}{x}\mathrm{d}x - f(\theta)\int_{b\delta}^{a\delta} \frac{1}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta \in (a\delta, b\delta)$. 现在

$$\lim_{\delta\to0^+}\left(-f(\theta)\int_{b\delta}^{a\delta}\frac{1}{x}\mathrm{d}x\right) = \lim_{x\to0^+} f(x)\ln\frac{b}{a}.$$

由 Riemann 引理, 我们有

$$\lim_{A\to+\infty}\int_{b}^{a}\frac{f(Ax)}{x}\mathrm{d}x = \int_{b}^{a}\frac{1}{x}\mathrm{d}x \cdot \frac{1}{T}\int_{0}^{T} f(x)\mathrm{d}x = -\frac{1}{T}\int_{0}^{T} f(x)\mathrm{d}x \cdot \ln\frac{b}{a},$$

这就证明了

$$\int_{0}^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x = \left[\lim_{x\to0^+} f(x) - \frac{1}{T}\int_{0}^{T} f(x)\mathrm{d}x\right]\ln\frac{b}{a}.$$

5. 上一问证明中把使用的 Riemann 引理用平均值极限版本的 Riemann 引理代替即可.

□

0.1.4 化成多元累次积分 (换序)

命题 0.3

证明:

(1) $\displaystyle\int_{0}^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

(2) $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$.

(3) $\displaystyle\int_{0}^{\infty} \sin x^2\mathrm{d}x, \int_{0}^{\infty} \cos x^2\mathrm{d}x = \sqrt{\frac{\pi}{8}}$.

笔记 本结果可以直接使用.

证明

(1) 注意到

$$\left(\int_0^{+\infty} e^{-x^2}\mathrm{d}x\right)^2=\left(\int_0^{+\infty} e^{-x^2}\mathrm{d}x\right)\left(\int_0^{+\infty} e^{-y^2}\mathrm{d}y\right)\xlongequal{\text{把}\int_0^{+\infty}e^{-y^2}\mathrm{d}y\text{看作常数}}\int_0^{+\infty} e^{-x^2}\left(\int_0^{+\infty} e^{-y^2}\mathrm{d}x\right)\mathrm{d}y$$

$$\xlongequal{\text{把}e^{-x^2}\text{看作常数}}\int_0^{+\infty}\left(\int_0^{+\infty} e^{-(x^2+y^2)}\mathrm{d}x\right)\mathrm{d}y\xlongequal{e^{-(x^2+y^2)}\text{连续}}\iint_{R^2} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

$$=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{+\infty} re^{-r^2}\mathrm{d}r=\frac{\pi}{2}\int_0^{+\infty} re^{-r^2}\mathrm{d}r$$

$$=\frac{\pi}{4}\int_0^{+\infty} e^{-r^2}\mathrm{d}r^2=\frac{\pi}{4}.$$

故 $\int_0^{\infty} e^{-x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2}$.

(2) 注意到

$$\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x=\operatorname{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\,\mathrm{d}x=\operatorname{Im}\int_0^{+\infty} e^{-(y-\mathrm{i})x}\,\mathrm{d}x=\operatorname{Im}\frac{1}{y-\mathrm{i}}=\operatorname{Im}\frac{y+\mathrm{i}}{y^2+1}=\frac{1}{y^2+1}.$$

因此就有

$$\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x=\int_0^{+\infty}\sin x\left(\int_0^{+\infty} e^{-yx}\,\mathrm{d}y\right)\mathrm{d}x=\int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x$$

$$=\int_0^{+\infty}\mathrm{d}y\left(\operatorname{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\right)\mathrm{d}x=\int_0^{+\infty}\frac{1}{y^2+1}\,\mathrm{d}y=\frac{\pi}{2}.$$

当然本题也可以直接利用分部积分计算 $\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x=\frac{1}{y^2+1}$.

(3) 注意到

$$\int_0^{+\infty} e^{-ax^2}\,\mathrm{d}x\xlongequal{x=\frac{t}{\sqrt{a}}}\frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t=\frac{1}{2}\sqrt{\frac{\pi}{a}},\quad a>0.$$

并且 $-\mathrm{i}=e^{-\frac{\pi}{2}\mathrm{i}}$, 从而 $\sqrt{-\mathrm{i}}=e^{-\frac{\pi}{4}\mathrm{i}}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}$. 于是

$$\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x=\int_0^{+\infty} e^{-\mathrm{i}x^2}\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{\mathrm{i}}}=\frac{1}{2}\sqrt{-\mathrm{i}\pi}$$

$$=\frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}.$$

故

$$\int_0^{+\infty}\cos x^2\,\mathrm{d}x=\operatorname{Re}\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x=\operatorname{Re}\left(\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}=\sqrt{\frac{\pi}{8}},$$

$$\int_0^{+\infty}\sin x^2\,\mathrm{d}x=\operatorname{Im}\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x=\operatorname{Im}\left(\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}=\sqrt{\frac{\pi}{8}}.$$

□

例题 0.5 计算 $\int_0^1\sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x\ (b>a>0)$.

证明

$$\int_0^1\sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x=\int_0^1\sin\ln\frac{1}{x}\left(\int_a^b x^y\mathrm{d}y\right)\mathrm{d}x=\int_a^b\mathrm{d}y\int_0^1 x^y\sin\ln\frac{1}{x}\mathrm{d}x$$

$$\xlongequal{x=e^{-t}}\int_a^b\mathrm{d}y\int_{+\infty}^0 e^{-ty}\sin t\mathrm{d}e^{-t}=\int_a^b\mathrm{d}y\int_0^{+\infty} e^{-t(y+1)}\sin t\mathrm{d}t$$

$$\xlongequal{\text{命题 0.3(2) 的证明过程}}\int_a^b\frac{1}{1+(y+1)^2}\mathrm{d}y=\arctan(b+1)-\arctan(a+1).$$

□

0.1.5 化成含参积分 (求导)

例题 0.6 设 $a,b \geqslant 0$ 且不全为 0, 计算 $\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$.

注 实际上, 根据 $a>b$ 时得到的结果, 可以看出 $F(a,b)=\pi\ln\frac{a+b}{2}$ 对 a,b 有轮换对称性, 故这个结果对其他情况显然也成立.

证明 设 $F(a,b)=\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$, 当 $a>b$ 时, 则

$$\begin{aligned}
\frac{\partial}{\partial b}F(a,b) &= \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial b}\ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{2b\sin^2 x}{a^2\cos^2 x + b^2\sin^2 x}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{2b\tan^2 x}{a^2 + b^2\tan^2 x}\mathrm{d}x = \int_0^{+\infty} \frac{2bt^2}{(a^2+b^2t^2)(1+t^2)}\mathrm{d}t \\
&= \frac{1}{a^2-b^2}\int_0^{+\infty}\left(\frac{2a^2b}{a^2+b^2t^2} - \frac{2b}{1+t^2}\right)\mathrm{d}t \\
&= \frac{1}{a^2-b^2}\int_0^{+\infty}\frac{2a^2b}{a^2+b^2t^2}\mathrm{d}t - \frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+t^2}\mathrm{d}t \\
&= \frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+\left(\frac{b}{a}t\right)^2}\mathrm{d}t - \frac{b\pi}{a^2-b^2} \\
&= \frac{2b}{a^2-b^2}\cdot\frac{a}{b}\cdot\frac{\pi}{2} - \frac{b\pi}{a^2-b^2} = \frac{\pi}{a+b}.
\end{aligned}$$

于是

$$\begin{aligned}
F(a,b) &= F(a,0) + \int_0^b \frac{\partial}{\partial b'}F(a,b')\mathrm{d}b' = F(a,0) + \int_0^b \frac{\pi}{a+b'}\mathrm{d}b' \\
&= 2\int_0^{\frac{\pi}{2}} \ln(a\cos x)\mathrm{d}x + \pi\ln\frac{a+b}{a} \xlongequal{\text{例题 0.2}} \pi\ln\frac{a+b}{2}.
\end{aligned}$$

当 $a<b$ 时, 类似可得 $F(a,b)=\pi\ln\frac{a+b}{2}$. 当 $a=b$ 时,$F(a,b)=\int_0^{\frac{\pi}{2}}\ln a^2\mathrm{d}x=\pi\ln a=\pi\ln\frac{a+b}{2}$.

综上, 对 $\forall a,b\geqslant 0$, 都有 $F(a,b)=\pi\ln\frac{a+b}{2}$. □

0.1.6 级数展开方法

积分和求和换序 $\sum\limits_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum\limits_{n=1}^{\infty} f_n(x)\mathrm{d}x$, 等价于

$$\lim_{m\to\infty}\sum_{n=1}^{m}\int_a^b f_n(x)\mathrm{d}x = \int_a^b\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x.$$

又由于有限和随意交换, 因此上式等价于

$$\lim_{m\to\infty}\int_a^b\sum_{n=1}^{m}f_n(x)\mathrm{d}x = \int_a^b\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x,$$

于是

$$\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x \Longleftrightarrow \lim_{m\to\infty}\int_a^b\sum_{n=m+1}^{\infty}f_n(x)\mathrm{d}x = 0.$$

例题 0.7 计算 $\int_0^{\infty}\frac{x}{1+e^x}\mathrm{d}x$.

解 由于

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n,\quad 0<x<1.$$

并且 $0 < e^{-x} < 1$, 故

$$\begin{aligned}\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x &= \int_0^{+\infty}\frac{xe^{-x}}{1+e^{-x}}\mathrm{d}x = \int_0^{+\infty}x\sum_{n=0}^{\infty}(-1)^n e^{-(n+1)x}\mathrm{d}x \\ &\xlongequal{\text{换序}} \sum_{n=0}^{\infty}(-1)^n\int_0^{+\infty}xe^{-(n+1)x}\mathrm{d}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2} \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty}\frac{1}{(2n)^2}.\end{aligned}$$

又因为 $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, 所以

$$\sum_{n=0}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{24},$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=0}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{8}.$$

故

$$\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} - \sum_{n=0}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.$$

下面证明(**??**)式换序成立, 等价于证明 $\lim\limits_{m\to+\infty}\int_0^{+\infty}\sum_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\mathrm{d}x = 0$. 由交错级数不等式及 $xe^{-(n+1)x}$ 关于 n 非负递减, 对 $\forall m \in \mathbb{N}$, 都有

$$\int_0^{+\infty}\left|\sum_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\right|\mathrm{d}x \leqslant \int_0^{+\infty}xe^{-(m+1)x}\mathrm{d}x = -\frac{xe^{-(m+1)x}}{m+1}\bigg|_0^{+\infty} + \frac{1}{m+1}\int_0^{+\infty}e^{-(m+1)x}\mathrm{d}x = \frac{1}{(m+1)^2}.$$

令 $m \to +\infty$, 得 $\lim\limits_{m\to+\infty}\int_0^{+\infty}\sum_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\mathrm{d}x = 0$. 故(**??**)式换序成立. □

命题 0.4

证明:

(1) $\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n} = \arctan\frac{q\sin x}{1-q\cos x}, |q| \leqslant 1.$

(2) $\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n} = -\frac{1}{2}\ln(1+q^2-2q\cos x), |q| \leqslant 1.$

(3) $\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!} = e^{q\cos x}\cos(q\sin x) - 1, |q| \leqslant 1, x \in \mathbb{R}.$

(4) $\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!} = e^{q\cos x}\sin(q\sin x), |q| \leqslant 1, x \in \mathbb{R}.$

♠

笔记 在 $\mathbb{C}$ 上,

$$\mathrm{Ln}z = \ln|z| + i(\arg z + 2k\pi), k \in \mathbb{Z}.$$

我们定义主值支

$$\ln z = \ln|z| + i\arg z.$$

本部分内容无需记忆, 只需要大概有个可以算的感觉即可, 实际做题中可以围绕这种级数给出构造.

证明 $\Im$ 表示取虚部,$\Re$ 表示取实部.

(1) 利用欧拉公式有

$$\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n}=\Im\left(\sum_{n=1}^{\infty}\frac{q^ne^{inx}}{n}\right)=\Im\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n}\right)=\Im(-\ln(1-qe^{ix}))$$
$$=-\Im\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=\arctan\frac{q\sin x}{1-q\cos x}.$$

(2) 利用欧拉公式有

$$\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n}=-\Re\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=-\frac{1}{2}\ln\left[(1-q\cos x)^2+q^2\sin^2 x\right]$$
$$=-\frac{1}{2}\ln(1+q^2-2q\cos x).$$

(3) 利用欧拉公式有

$$\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!}=\Re\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n!}\right)=\Re\left(e^{qe^{ix}}-1\right)=\Re\left(e^{q\cos x+iq\sin x}-1\right)$$
$$=\Re\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)$$
$$=e^{q\cos x}\cos(q\sin x)-1.$$

(4) 利用 (3) 有

$$\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!}=\Im\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)$$
$$=e^{q\cos x}\sin(q\sin x).$$

□

例题 0.8 计算

1. $\int_0^{2\pi}e^{\cos x}\cos(\sin x)\mathrm{d}x$.
2. $\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x, a\in(0,+\infty)\setminus\{1\}$.

注 由 1 的证明可得

$$e^{\cos x}\cos(\sin x)=\mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{(e^{\mathrm{i}x})^n}{n!}\right)=\mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{e^{\mathrm{i}nx}}{n!}\right)=\sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}.$$

实际上, 上式就是命题 0.4(3)的结论.

注 第 2 问也可以用含参积分求导的方法进行计算 (这个方法更容易想到).

证明

1.

$$\int_0^{2\pi}e^{\cos x}\cos(\sin x)\,\mathrm{d}x=\mathrm{Re}\left(\int_0^{2\pi}e^{\cos x}e^{\mathrm{i}\sin x}\mathrm{d}x\right)=\mathrm{Re}\left(\int_0^{2\pi}e^{\cos x+\mathrm{i}\sin x}\mathrm{d}x\right)$$
$$=\mathrm{Re}\left(\int_0^{2\pi}e^{e^{\mathrm{i}x}}\mathrm{d}x\right)=\mathrm{Re}\left[\int_0^{2\pi}\sum_{n=0}^{+\infty}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right]=\mathrm{Re}\left[\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right]$$
$$=\mathrm{Re}\left(\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{e^{\mathrm{i}nx}}{n!}\mathrm{d}x\right)=\mathrm{Re}\left(\int_0^{2\pi}\frac{e^{\mathrm{i}\cdot 0\cdot x}}{n!}\mathrm{d}x+\sum_{n=1}^{+\infty}\frac{e^{2\pi\mathrm{i}x}-1}{\mathrm{i}n\cdot n!}\right)$$
$$=\mathrm{Re}\left(\int_0^{2\pi}1\mathrm{d}x+0\right)=2\pi.$$

2. 注意到当 $a\in(0,1)$ 时, 有

$$\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n}=\mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{(ae^{\mathrm{i}x})^n}{n}\right]=-\mathrm{Re}\left[\ln(1-ae^{\mathrm{i}x})\right]$$

$$= -\mathrm{Re}\left[\ln|1-ae^{\mathrm{i}x}| + \mathrm{i}\arg(1-ae^{\mathrm{i}x})\right] = -\ln|1-ae^{\mathrm{i}x}|$$
$$= -\ln|(1-a\cos x)+ai\sin x| = -\frac{1}{2}\ln(1+a^2-2a\cos x).$$

于是当 $a\in(0,1)$ 时，就有

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = -\frac{1}{2}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n}\mathrm{d}x = 0.$$

若 $a>1$，则 $\frac{1}{a}\in(0,1)$，从而此时我们有

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = \pi\ln a^2 + \int_0^{\pi}\ln\left(\frac{1}{a^2}-\frac{2}{a}\cos x+1\right)\mathrm{d}x = \pi\ln a^2 = 2\pi\ln a.$$

又由 $\ln(1-2a\cos x+a^2)$ 关于 a 的偏导存在可知 $\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x$ 关于 a 连续. 于是由

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = 2\pi\ln a,\quad \forall a>1.$$

可知当 $a=1$ 时，我们有

$$\int_0^{\pi}\ln(2-2\cos x)\mathrm{d}x = \lim_{a\to1^+}\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = \lim_{a\to1^+}(2\pi\ln a) = 0.$$

□

定义 0.1 (多重对数函数-Li_2 函数)

定义

$$\mathrm{Li}_2(x) \triangleq \sum_{n=1}^{\infty}\frac{x^n}{n^2},\qquad x\in[-1,1].$$

♣

命题 0.5

(1) $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x\cdot\ln(1-x), x\in(0,1)$.

(2) $\mathrm{Li}_2(1) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},\quad \mathrm{Li}_2(0)=0,\quad \mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.$

♠

证明

(1) 记 $f(x)\triangleq \mathrm{Li}_2(x), F(x)\triangleq f(x)+f(1-x)+\ln x\ln(1-x)$. 则

$$f'(x) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n} = -\frac{1}{x}\ln(1-x).$$

于是

$$F'(x) = -\frac{1}{x}\ln(1-x) + \frac{\ln x}{1-x} - \frac{\ln x}{1-x} + \frac{\ln(1-x)}{x} = 0.$$

故 $F(x)\equiv F(1) = f(0)+f(1) = 0+\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

(2) 显然 $\mathrm{Li}_2(1) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \mathrm{Li}_2(0)=0$. 由 (1) 可得

$$\mathrm{Li}_2\left(\frac{1}{2}\right)+\mathrm{Li}_2\left(\frac{1}{2}\right) = 2\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6}-\ln^2\frac{1}{2} \Longrightarrow \mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.$$

□

例题 0.9 计算 $\int_0^{\frac{1}{2}}\frac{\ln x}{1-x}\mathrm{d}x$.

解

$$\int_0^{\frac{1}{2}} \frac{\ln x}{1-x}\,\mathrm{d}x = \int_{\frac{1}{2}}^{1} \frac{\ln(1-x)}{x}\,\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{1}{2}}^{1} x^{n-1}\,\mathrm{d}x$$
$$= -\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{2^n n^2} = -\frac{\pi^2}{6} + \mathrm{Li}_2\left(\frac{1}{2}\right)$$
$$\xlongequal{\text{命题 0.5}} -\frac{\pi^2}{12} - \frac{1}{2}\ln^2\frac{1}{2}.$$

□

0.1.7 重积分计算

定理 0.3 (二重积分换序)

证明:

$$\int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y)\mathrm{d}x, \tag{10}$$

其中 $f(x,y)$ 是在由直线 $y=a, x=b, y=x$ 所围成的三角形 (Δ) 上连续的任意函数.

证明

□

命题 0.6

设 $f(x)$ 在 $[a,b]$ 上连续, 试证: 对任意 $x\in(a,b)$, 有

$$\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_n} f(x_{n+1})\mathrm{d}x_{n+1} = \frac{1}{n!}\int_a^x (x-y)^n f(y)\mathrm{d}y, \quad n=1,2,\cdots.$$

证明 当 $n=1$ 时, 由二重积分换序可知

$$\int_a^x \mathrm{d}x_1 \int_a^{x_1} f(x_2)\mathrm{d}x_2 = \int_a^x \mathrm{d}x_2 \int_{x_2}^x f(x_2)\mathrm{d}x_1 = \int_a^x (x-x_2)f(x_2)\mathrm{d}x_2 = \int_a^x (x-y)f(y)\mathrm{d}y.$$

设原结论对 $n=k-1$ 的情形成立, 考虑 $n=k$ 的情形, 由归纳假设可知

$$\int_a^{x_1} \mathrm{d}x_1 \cdots \int_a^{x_k} f(x_{k+1})\mathrm{d}x_{k+1} = \frac{1}{(k-1)!}\int_a^{x_1} (x_1-y)^{k-1} f(y)\mathrm{d}y.$$

于是再利用二重积分换序得

$$\begin{aligned}\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_k} f(x_{k+1})\mathrm{d}x_{k+1} &= \frac{1}{(k-1)!}\int_a^x \mathrm{d}x_1 \int_a^{x_1} (x_1-y)^{k-1} f(y)\mathrm{d}y \\ &= \frac{1}{(k-1)!}\int_a^x \mathrm{d}y \int_y^x (x_1-y)^{k-1} f(y)\mathrm{d}x_1 \\ &= \frac{1}{k!}\int_a^x (x-y)^k f(y)\mathrm{d}y.\end{aligned}$$

故由数学归纳法知原结论成立. □

例题 0.10 求定义在星形区域 $D=\{(x,y)\mid x^{\frac{2}{3}}+y^{\frac{2}{3}}\leqslant 1\}$ 上满足 $f(1,0)=1$ 的正值连续函数 f 使得 $\iint\limits_D \frac{f(x,y)}{f(y,x)}\mathrm{d}x\mathrm{d}y$ 达到最小, 并求出这个最小值.

解 对积分 $I=\iint\limits_D \frac{f(x,y)}{f(y,x)}\mathrm{d}x\mathrm{d}y$ 作变换 $x\to y,\ y\to x$, 由 D 的对称性, 知 $I=\iint\limits_D \frac{f(y,x)}{f(x,y)}\mathrm{d}x\mathrm{d}y$. 因而由均值不等式可得

$$I=\frac{1}{2}\iint\limits_D \left(\frac{f(x,y)}{f(y,x)}+\frac{f(y,x)}{f(x,y)}\right)\mathrm{d}x\mathrm{d}y \geqslant \iint\limits_D 1\,\mathrm{d}x\mathrm{d}y = \sigma(D),$$

这里 $\sigma(D)$ 是 D 的面积.

$$I-\sigma(D)=\frac{1}{2}\iint\limits_D\left(\sqrt{\frac{f(x,y)}{f(y,x)}}-\sqrt{\frac{f(y,x)}{f(x,y)}}\right)^2\mathrm{d}x\mathrm{d}y\geqslant 0.$$

$I=\sigma(D)$ 当且仅当 $f(x,y)=f(y,x)$. 故所求函数为所有满足 $f(x,y)=f(y,x)$ 及 $f(1,0)=1$ 的连续正值函数.

D 的边界的参数方程为

$$x=\cos^3\varphi,\quad y=\sin^3\varphi\ (0\leqslant\varphi\leqslant 2\pi),$$

故 I 的最小值为

$$\begin{aligned}\sigma(D)&=\iint\limits_D 1\,\mathrm{d}x\mathrm{d}y=4\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant\frac{\pi}{2}}}3r\sin^2\varphi\cos^2\varphi\,\mathrm{d}r\mathrm{d}\varphi\\&=6\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos^2\varphi\,\mathrm{d}\varphi=\frac{3}{8}\pi.\end{aligned}$$

所以所求最小值是 $\frac{3}{8}\pi$, 且当 $f(x,y)=f(y,x)$ 并满足 $f(1,0)=1$ 时, 取到该最小值. □

例题 0.11 求证: $\iint\limits_{[0,1]^2}(xy)^{xy}\,\mathrm{d}x\mathrm{d}y=\int_0^1 t^t\,\mathrm{d}t$.

证明 首先化为累次积分

$$\iint\limits_{[0,1]^2}(xy)^{xy}\,\mathrm{d}x\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^1(xy)^{xy}\,\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^x\frac{t^t}{x}\,\mathrm{d}t=\int_0^1\frac{f(x)}{x}\,\mathrm{d}x,$$

其中 $f(x)=\int_0^x t^t\,\mathrm{d}t$. 由分部积分,

$$\int_0^1\frac{f(x)}{x}\,\mathrm{d}x=f(x)\ln x\Big|_0^1-\int_0^1 x^x\ln x\,\mathrm{d}x=-\int_0^1 x^x\ln x\,\mathrm{d}x.$$

因为 $(x^x)'=x^x\ln x+x^x$, 所以

$$\int_0^1 x^x\ln x\,\mathrm{d}x=\int_0^1\left((x^x)'-x^x\right)\mathrm{d}x=-\int_0^1 x^x\,\mathrm{d}x.$$

于是

$$\iint\limits_{[0,1]^2}(xy)^{xy}\,\mathrm{d}x\mathrm{d}y=\int_0^1 t^t\,\mathrm{d}t.$$

□

例题 0.12 计算二重积分 $I=\iint\limits_D\operatorname{sgn}(x^2-y^2+2)\,\mathrm{d}x\mathrm{d}y$, 其中 $D=\{(x,y)\mid x^2+y^2\leqslant 4\}$.

解 设 D 在第一象限部分为 D_1, 则由对称性

$$I=4\iint\limits_{D_1}\operatorname{sgn}(x^2-y^2+2)\,\mathrm{d}x\mathrm{d}y.$$

设 D_2 是 D_1 中使得 $x^2-y^2+2<0$ 的部分, D_3 是 D_1 中使得 $x^2-y^2+2\geqslant 0$ 的部分, 则 $D_1=D_2\cup D_3$. 因此

$$\begin{aligned}I&=4\left[\iint\limits_{D_3}\mathrm{d}x\mathrm{d}y-\iint\limits_{D_2}\mathrm{d}x\mathrm{d}y\right]=4[\sigma(D_3)-\sigma(D_2)]\\&=4\left[\frac{1}{4}\cdot\pi\cdot 2^2-2\sigma(D_2)\right]=4\pi-8\sigma(D_2),\end{aligned}$$

其中 $\sigma(D_2),\sigma(D_3)$ 分别表示 D_2 和 D_3 的面积. 在极坐标 $x=r\cos\varphi, y=r\sin\varphi$ 之下, D_2 为

$$\left\{(r,\varphi)\mid\frac{\pi}{3}\leqslant\varphi\leqslant\frac{\pi}{2},\sqrt{-\frac{2}{\cos 2\varphi}}\leqslant r\leqslant 2\right\}.$$

因而

$$\sigma(D_2)=\iint_{D_2}\mathrm{d}x\mathrm{d}y=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\mathrm{d}\varphi\int_{\sqrt{-\frac{2}{\cos 2\varphi}}}^{2}r\,\mathrm{d}r$$
$$=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left(4+\frac{2}{\cos 2\varphi}\right)\mathrm{d}\varphi=\frac{\pi}{3}+\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}\frac{1}{\cos\varphi}\mathrm{d}\varphi$$
$$=\frac{\pi}{3}-\frac{1}{2}\ln(2+\sqrt{3}),$$

故

$$I=\frac{4\pi}{3}+4\ln(2+\sqrt{3}).$$

□

例题 0.13 设 $D=\{(x,y)\mid x^2+y^2\leqslant 1\}$. 求 $I=\iint_D\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|\mathrm{d}x\mathrm{d}y$.

解 由极坐标变换 $x=r\cos\varphi,\ y=r\sin\varphi,\ 0\leqslant r\leqslant 1,\ 0\leqslant\varphi\leqslant 2\pi$, 有

$$I=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}\left|\frac{\cos\varphi+\sin\varphi}{\sqrt{2}}-r\right|r^2\mathrm{d}r\mathrm{d}\varphi$$
$$=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}\left|\sin\left(\varphi+\frac{\pi}{4}\right)-r\right|r^2\mathrm{d}r\mathrm{d}\varphi=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant 2\pi}}|\sin\varphi-r|\,r^2\mathrm{d}r\mathrm{d}\varphi$$
$$=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant \pi}}|\sin\varphi-r|\,r^2\mathrm{d}r\mathrm{d}\varphi+\iint\limits_{\substack{0\leqslant r\leqslant 1\\\pi\leqslant\varphi\leqslant 2\pi}}|\sin\varphi-r|\,r^2\mathrm{d}r\mathrm{d}\varphi$$
$$=\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant \pi}}|\sin\varphi-r|\,r^2\mathrm{d}r\mathrm{d}\varphi+\iint\limits_{\substack{0\leqslant r\leqslant 1\\0\leqslant\varphi\leqslant \pi}}(\sin\varphi+r)r^2\mathrm{d}r\mathrm{d}\varphi.$$

因此, 有

$$I=\int_0^{\pi}\mathrm{d}\varphi\int_0^{\sin\varphi}(\sin\varphi-r)r^2\mathrm{d}r+\int_0^{\pi}\mathrm{d}\varphi\int_{\sin\varphi}^{1}(r-\sin\varphi)r^2\mathrm{d}r$$
$$+\int_0^{\pi}\mathrm{d}\varphi\int_0^{\sin\varphi}(\sin\varphi+r)r^2\mathrm{d}r+\int_0^{\pi}\mathrm{d}\varphi\int_{\sin\varphi}^{1}(\sin\varphi+r)r^2\mathrm{d}r$$
$$=\int_0^{\pi}\mathrm{d}\varphi\int_0^{\sin\varphi}2\sin\varphi\cdot r^2\mathrm{d}r+\int_0^{\pi}\mathrm{d}\varphi\int_{\sin\varphi}^{1}2r\cdot r^2\mathrm{d}r$$
$$=\int_0^{\pi}\frac{2}{3}\sin^4\varphi\mathrm{d}\varphi+\int_0^{\pi}\frac{1}{2}(1-\sin^4\varphi)\mathrm{d}\varphi$$
$$=\frac{1}{6}\int_0^{\pi}\sin^4\varphi\mathrm{d}\varphi+\frac{\pi}{2}=\frac{1}{6}\cdot\frac{3\pi}{8}+\frac{\pi}{2}=\frac{9}{16}\pi.$$

□

例题 0.14 设 f 是定义在正方形 $S=\{(x,y)\mid 0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1\}$ 上的四阶连续可微函数, 在 S 的边界上为零, 并且

$$\left|\frac{\partial^4 f}{\partial x^2\partial y^2}\right|\leqslant M.$$

求证:

$$\left|\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y\right|\leqslant\frac{1}{144}M.$$

证明 考虑函数 $g(x,y)=x(1-x)y(1-y)$. 易知

$$\frac{\partial^4 g}{\partial x^2\partial y^2}=4,\qquad\iint_S g(x,y)\,\mathrm{d}x\mathrm{d}y=\frac{1}{36}.$$

因为 f 在 S 的边界上为零, 所以 $\dfrac{\partial^2 f}{\partial y^2}$ 在 $x=0$ 和 $x=1$ 时为零. 于是

$$\begin{aligned}
&\iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2}\cdot g\,\mathrm{d}x\mathrm{d}y=\int_0^1 \mathrm{d}y\int_0^1 \frac{\partial^4 f}{\partial x^2\partial y^2}\cdot g\,\mathrm{d}x\\
&=\int_0^1 \mathrm{d}y\left(\frac{\partial^3 f}{\partial x\partial y^2}\cdot g\Big|_{x=0}^1-\int_0^1 \frac{\partial^3 f}{\partial x\partial y^2}\cdot\frac{\partial g}{\partial x}\,\mathrm{d}x\right)\\
&=-\int_0^1 \mathrm{d}y\int_0^1 \frac{\partial^3 f}{\partial x\partial y^2}\cdot\frac{\partial g}{\partial x}\,\mathrm{d}x\\
&=-\int_0^1 \mathrm{d}y\left(\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial g}{\partial x}\Big|_{x=0}^1-\int_0^1 \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\right)\\
&=\int_0^1 \mathrm{d}y\int_0^1 \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\\
&=\iint_S \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

同理, 由于 $\dfrac{\partial^2 g}{\partial x^2}$ 在 $y=0$ 和 $y=1$ 时为零, 作与上面类似的推导, 可得

$$\iint_S \frac{\partial^4 g}{\partial x^2\partial y^2}\cdot f\,\mathrm{d}x\mathrm{d}y=\iint_S \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\mathrm{d}y.$$

因此

$$\iint_S \frac{\partial^4 f}{\partial x^2\partial y^2}\cdot g\,\mathrm{d}x\mathrm{d}y=\iint_S \frac{\partial^4 g}{\partial x^2\partial y^2}\cdot f\,\mathrm{d}x\mathrm{d}y.$$

从而

$$\begin{aligned}
&\left|\iint_S f\,\mathrm{d}x\mathrm{d}y\right|=\frac{1}{4}\left|\iint_S 4f\mathrm{d}x\mathrm{d}y\right|=\frac{1}{4}\left|\iint_S \frac{\partial^4 g}{\partial x^2\partial y^2}f\mathrm{d}x\mathrm{d}y\right|\\
&=\frac{1}{4}\left|\iint_S \frac{\partial^4 f}{\partial x^2\partial y^2}\cdot g\,\mathrm{d}x\mathrm{d}y\right|\leqslant\frac{M}{4}\iint_S g\,\mathrm{d}x\mathrm{d}y=\frac{M}{144}.
\end{aligned}$$

□

定理 0.4 (Poincaré(庞加莱) 不等式)

设 φ,ψ 是 $[a,b]$ 上的连续函数, f 在区域 $D=\{(x,y)\mid a\leqslant x\leqslant b,\varphi(x)\leqslant y\leqslant\psi(x)\}$ 上连续可微, 且有 $f(x,\varphi(x))=0\ (x\in[a,b])$. 则存在 $M>0$, 使得

$$\iint_D f^2(x,y)\,\mathrm{d}x\mathrm{d}y\leqslant M\iint_D (f_y'(x,y))^2\,\mathrm{d}x\mathrm{d}y.$$

♡

证明 由 Newton-Leibniz 公式和 Cauchy 不等式可得

$$\begin{aligned}
f^2(x,y)=[f(x,y)-f(x,\varphi(x))]^2&=\left(\int_{\varphi(x)}^y \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}t\right)^2\\
&\leqslant(y-\varphi(x))\int_{\varphi(x)}^y\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t,
\end{aligned}$$

因此

$$\begin{aligned}
&\iint_D f^2(x,y)\,\mathrm{d}x\mathrm{d}y=\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)} f^2(x,y)\,\mathrm{d}y\\
&\leqslant\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\int_{\varphi(x)}^y\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\\
&=\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\int_t^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\\
&\leqslant\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\int_{\varphi(x)}^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \frac{1}{2}(\psi(x)-\varphi(x))^2 \left(\frac{\partial f}{\partial t}(x,t)\right)^2 \mathrm{d}t \\
&\leqslant M \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \left(\frac{\partial f}{\partial t}(x,t)\right)^2 \mathrm{d}t\right) \mathrm{d}x \\
&= M \iint_D \left(\frac{\partial f}{\partial y}(x,y)\right)^2 \mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

这里 M 是满足 $M > \max\limits_{a\leqslant x\leqslant b} \frac{1}{2}(\psi(x)-\varphi(x))^2$ 的常数. □

例题 0.15 设 $a>0, \Omega_n(a): x_1+x_2+\cdots+x_n \leqslant a, x_i \geqslant 0\ (i=1,2,\cdots,n)$. 求积分

$$I_n(a) = \int\cdots\int_{\Omega_n(a)} x_1x_2\cdots x_n \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

解 作变换 $x_i = at_i, i=1,2,\cdots,n$, 则

$$I_n(a) = a^{2n}\int\cdots\int_{\Omega_n(1)} t_1t_2\cdots t_n \mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n = a^{2n}I_n(1).$$

再用累次积分, 可得

$$
\begin{aligned}
I_n(1) &= \int\cdots\int_{\Omega_n(1)} t_1t_2\cdots t_n \mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n \\
&= \int_0^1 t_n\mathrm{d}t_n \int\cdots\int_{t_1+t_2+\cdots+t_{n-1}\leqslant 1-t_n} t_1\cdots t_{n-1}\mathrm{d}t_1\cdots\mathrm{d}t_{n-1} \\
&= \int_0^1 t_nI_{n-1}(1-t_n)\mathrm{d}t_n = \int_0^1 t_n(1-t_n)^{2(n-1)}I_{n-1}(1)\mathrm{d}t_n.
\end{aligned}
$$

因此,

$$I_n(1) = \frac{1}{2n(2n-1)}I_{n-1}(1).$$

注意到 $I_1(1) = \int_0^1 t\mathrm{d}t = \frac{1}{2}$. 由上面的递推公式, 可得 $I_n(1) = \frac{1}{(2n)!}$. 故 $I_n(a) = \frac{a^{2n}}{(2n)!}$. □

0.1.8 其他

例题 0.16 证明积分 $\int_0^{\infty} e^{-ax^2-\frac{b}{x^2}}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}, a,b>0.$

证明 当 $a=1$ 时, 就有

$$
\begin{aligned}
\int_0^{+\infty} e^{-x^2-\frac{b}{x^2}}\mathrm{d}x &= e^{-2\sqrt{b}}\int_0^{+\infty} e^{-\left(x-\frac{\sqrt{b}}{x}\right)^2}\mathrm{d}x \xlongequal{y=\frac{\sqrt{b}}{x}} e^{-2\sqrt{b}}\int_0^{+\infty}\frac{\sqrt{b}}{y^2}e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2}\mathrm{d}y \\
&= \frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty}\left(1+\frac{\sqrt{b}}{y^2}\right)e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\mathrm{d}y = \frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty}e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\mathrm{d}\left(y-\frac{\sqrt{b}}{y}\right) \\
&= \frac{e^{-2\sqrt{b}}}{2}\int_{-\infty}^{+\infty}e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}e^{-2\sqrt{b}}.
\end{aligned}
$$

于是对 $\forall a>0$, 就有

$$\int_0^{+\infty}e^{-ax^2-\frac{b}{x^2}}\mathrm{d}x = \frac{1}{\sqrt{a}}\int_0^{+\infty}e^{-x^2-\frac{ab}{x^2}}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}.$$

□

例题 0.17 计算 $\int_0^{\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x, a\in\mathbb{R}.$

注 本题可以用复变函数的方法 (留数定理) 来计算. 但是我们这里用基本的高等数学的方法来计算. $\int_0^{\infty}\frac{\sin(ax)}{1+x^2}\mathrm{d}x$ 这个积分没办法算出具体的初等数值.

证明

$$\int_0^{+\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\cos(ax)\left(\int_0^{+\infty}e^{-(1+x^2)y}\mathrm{d}y\right)\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_{0}^{+\infty} e^{-(1+x^2)y}\cos(ax)\,\mathrm{d}y\right)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{0}^{+\infty} e^{-(1+x^2)y}\cos(ax)\,\mathrm{d}y\mathrm{d}x\\
&= \frac{1}{2}\int_{0}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-(1+x^2)y}\cos(ax)\,\mathrm{d}x\right)\mathrm{d}y = \frac{1}{2}\int_{0}^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-x^2y}\cos(ax)\,\mathrm{d}x\right)\mathrm{d}y\\
&= \frac{1}{2}\mathrm{Re}\left(\int_{0}^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-x^2y+\mathrm{i}ax}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_{0}^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x^2-\frac{a\mathrm{i}x}{y}+\left(\frac{a\mathrm{i}}{2y}\right)^2\right)-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\
&= \frac{1}{2}\mathrm{Re}\left(\int_{0}^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x-\frac{a\mathrm{i}}{2y}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_{0}^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2\mathrm{i}y}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\
&= \frac{1}{2}\mathrm{Re}\left(\int_{0}^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2\mathrm{i}y}\right)^2-\frac{a^2}{4y}}\mathrm{d}\left(x+\frac{a}{2\mathrm{i}y}\right)\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_{0}^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-yx^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\
&= \frac{1}{2}\int_{0}^{+\infty} e^{-y-\frac{a^2}{4y}}\left(\int_{-\infty}^{+\infty} e^{-yx^2}\mathrm{d}x\right)\mathrm{d}y = \frac{\sqrt{\pi}}{2}\int_{0}^{+\infty}\frac{1}{\sqrt{y}}e^{-y-\frac{a^2}{4y}}\mathrm{d}y\\
&\xlongequal{y=t^2} \sqrt{\pi}\int_{0}^{+\infty} e^{-t^2-\frac{a^2}{4t^2}}\mathrm{d}t \xlongequal{\text{例题 0.17}} \sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2}e^{-|a|} = \frac{\pi}{2}e^{-|a|}.
\end{aligned}
$$

□

例题 0.18 计算 $\int_{0}^{\infty}\frac{1}{(1+x^8)^2}\mathrm{d}x$.

注 由命题??可知对 $\forall s>0$, 都有

$$
\frac{1}{t^s} = \frac{1}{\Gamma(s)}\int_{0}^{+\infty} y^{s-1}e^{-ty}\mathrm{d}y, \forall t\in\mathbb{R}.
$$

本题的核心想法就是利用上式将 $\frac{z}{1+x^8}$ 转化成积分形式.

证明 注意到

$$
\int_{0}^{+\infty} ye^{-(1+x^8)y}\mathrm{d}y \xlongequal{y=\frac{z}{1+x^8}} \frac{1}{(1+x^8)^2}\int_{0}^{+\infty} ze^{-z}\mathrm{d}z = \frac{1}{(1+x^8)^2},
$$

因此

$$
\begin{aligned}
\int_{0}^{+\infty}\frac{1}{(1+x^8)^2}\mathrm{d}x &= \int_{0}^{+\infty}\left(\int_{0}^{+\infty} ye^{-(1+x^8)y}\mathrm{d}y\right)\mathrm{d}x = \int_{0}^{+\infty}\left(\int_{0}^{+\infty} ye^{-(1+x^8)y}\mathrm{d}x\right)\mathrm{d}y\\
&= \int_{0}^{+\infty} ye^{-y}\left(\int_{0}^{+\infty} e^{-x^8y}\mathrm{d}x\right)\mathrm{d}y \xlongequal{x=y^{-\frac{1}{8}}z^{\frac{1}{8}}} \int_{0}^{+\infty} ye^{-y}\left(\int_{0}^{+\infty} y^{-\frac{1}{8}}e^{-z}\mathrm{d}z^{\frac{1}{8}}\right)\mathrm{d}y\\
&= \frac{1}{8}\int_{0}^{+\infty} y^{\frac{7}{8}}e^{-y}\left(\int_{0}^{+\infty} z^{-\frac{7}{8}}e^{-z}\mathrm{d}z\right)\mathrm{d}y = \frac{1}{8}\int_{0}^{+\infty} y^{\frac{7}{8}}e^{-y}\Gamma\left(\frac{1}{8}\right)\mathrm{d}y\\
&= \frac{1}{8}\Gamma\left(\frac{15}{8}\right)\Gamma\left(\frac{1}{8}\right) = \frac{1}{8}\cdot\frac{7}{8}\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{1}{8}\right)\\
&\xlongequal{??} \frac{7\pi}{64\sin\left(\frac{\pi}{8}\right)} = \frac{7\pi}{32\sqrt{2-\sqrt{2}}}.
\end{aligned}
$$

□

例题 0.19 计算积分 $I=\int_{-1}^{2}\frac{1+x^2}{1+x^4}\mathrm{d}x$.

注 在此例中 $I\neq F(2)-F(-1)$. 这是因为 F 并不是 f 在区间 $[-1,2]$ 上的原函数.

解 在不包含 0 的区间上作变换 $t=x-\frac{1}{x}$ 得

$$
\begin{aligned}
\int\frac{1+x^2}{1+x^4}\mathrm{d}x &= \int\frac{x-\frac{1}{x}}{2+\left(x-\frac{1}{x}\right)^2}\mathrm{d}x = \int\frac{\mathrm{d}t}{2+t^2}\\
&= \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C = \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}+C.
\end{aligned}
$$

这说明在区间 $[-1,0)$ 和 $(0,2]$ 上, 函数 $f(x)=\dfrac{1+x^2}{1+x^4}$ 的一个原函数是

$$F(x)=\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}.$$

因此

$$\int_{-1}^{0} f(x)\,\mathrm{d}x=F(0^-)-F(-1)=\frac{\pi}{2\sqrt{2}}-0=\frac{\pi}{2\sqrt{2}},$$
$$\int_{0}^{2} f(x)\,\mathrm{d}x=F(2)-F(0^+)=\frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}}+\frac{\pi}{2\sqrt{2}}.$$

故

$$I=\int_{-1}^{0} f(x)\,\mathrm{d}x+\int_{0}^{2} f(x)\,\mathrm{d}x=\frac{\pi}{\sqrt{2}}+\frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}}.$$

□