## 0.1 循环行列式

循环行列式关于单位根的计算公式见命题??.

**例题 0.1** 设 $a, n$ 是给定互素正整数，按 Build 除法，存在唯一确定的整数对 $(s, t)$ 使得 $a = sn + t, 0 \le t \le n-1$。令

$$u_i = \begin{cases} s+1, & 0 \le i < t \\ s, & t \le i \le n-1 \end{cases}$$

若 $t$ 与 $n$ 互素，计算

$$D_n = \begin{vmatrix} u_0 & u_1 & \cdots & u_{n-1} \\ u_{n-1} & u_0 & \cdots & u_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_0 \end{vmatrix}$$

**证明** 记 $f(x) \triangleq \sum_{i=1}^{n} u_i x^i, w_j \triangleq e^{\frac{2\pi j i}{n}}, j = 0, 1, 2, \cdots, n-1$, 则由命题??可知

$$D_n = \prod_{k=0}^{n} f(w_k) = f(1) \prod_{k=0}^{n-1} f(w_k).$$

由条件可知

$$f(1) = \sum_{i=0}^{t-1}(s+1) + \sum_{i=t}^{n-1} s = (s+1)t + (n-t)s = ns + t = a.$$

从而

$$\begin{aligned} D_n &= a \prod_{i=0}^{n-1} f(w_k) = a \prod_{i=0}^{n-1} \left[ \sum_{i=0}^{t-1} (s+1) w_k^i + \sum_{i=t}^{n-1} s w_k^i \right] \\ &= a \prod_{i=0}^{n-1} \left[ \sum_{i=0}^{t-1} w_k^i + s \sum_{i=0}^{n-1} w_k^i \right] = a \prod_{i=0}^{n-1} \left( \frac{1-w_k^t}{1-w_k} + s \frac{1-w_k^n}{1-w_k} \right). \end{aligned}$$

由 $w_k^n = 1, w_k = w_1^k$ 可知

$$D_n = a \prod_{i=0}^{n-1} \frac{1-w_1^{kt}}{1-w_1^k}.$$

由群论可知 $\{1, w_1, \cdots, w_1^{n-1}\}$ 是一个循环群且 $w_1$ 的阶为 $n$, 再根据群论的 Lagrange 定理及 $(t, n) = 1$ 可知, $w_1^t$ 的阶为 $\frac{n}{(n,t)} = n$。因此 $w_1^k = w_1$, 故 $\{w_1, w_1^2, \cdots, w_1^{n-1}\} = \{w_1^t, w_2^{2t}, \cdots, w_1^{(n-1)t}\}$。于是 $w_1^k = w_1^{tk}$, 故

$$D_n = a \prod_{i=0}^{n-1} \frac{1-w_1^{kt}}{1-w_1^k} = a.$$

□

**例题 0.2** 计算:

$$D_n = \begin{vmatrix} a & a+d & \cdots & a+(n-1)d \\ a+(n-1)d & a & \cdots & a+(n-2)d \\ \vdots & \vdots & & \vdots \\ a+d & a+2d & \cdots & a \end{vmatrix}.$$

**证明** 记 $f(x)=\sum\limits_{j=0}^{n-1}(a+jd)w_k^j$, 其中 $w_k=e^{\frac{2\pi k\mathrm{i}}{n}}, k=0,1,2,\cdots,n-1$. 由命题**??**可知

$$
\begin{aligned}
D_n &= f(w_0)f(w_1)\cdots f(w_{n-1})=\prod_{k=0}^{n-1}\sum_{j=0}^{n-1}(a+jd)w_k^j=\frac{2an+n(n-1)d}{2}\prod_{k=1}^{n-1}\sum_{j=0}^{n-1}(a+jd)w_k^j\\
&\xlongequal{\text{错位相减}}\frac{2an+n(n-1)d}{2}\prod_{k=1}^{n-1}\frac{aw_k^{n+1}-aw_k^n-aw_k+a-dw_k^{n+1}-dnw_k^n+dw_k}{(w_k-1)^2}\\
&\xlongequal{w_k^n=1}\frac{2an+n(n-1)d}{2}\prod_{k=1}^{n-1}\frac{dn}{w_k-1}=\frac{2an+n(n-1)d}{2}\cdot(dn)^{n-1}\prod_{k=1}^{n-1}\frac{1}{w_k-1}.
\end{aligned}
$$

注意到 $w_k-1, k=1,2,\cdots,n-1$ 是 $(x+1)^n-1=0$ 的 $n-1$ 个复根, 这些根和 $w_0-1=0$ 一起就是 $(x+1)^n-1=0$ 的全部复根. 从而由 Vieta 定理可得, $(x+1)^n-1$ 的一次项系数乘 $(-1)^{n-1}$ 为

$$
(-1)^{n-1}n=\sum_{0\leqslant i_1<i_2<\cdots<i_{n-1}\leqslant n-1}(w_{i_1}-1)(w_{i_2}-1)\cdots(w_{i_{n-1}}-1)=\prod_{k=1}^{n-1}(w_k-1).
$$

故

$$
D_n=\frac{2an+n(n-1)d}{2}\cdot(dn)^{n-1}\prod_{k=1}^{n-1}\frac{1}{w_k-1}=(-1)^{n-1}\cdot\frac{2a+(n-1)d}{2}\cdot(nd)^{n-1}.
$$

□