0.1 定积分

0.1.1 建立积分递推

例题 0.1 计算 $\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) \mathrm{d}x, n \in \mathbb{N}$.

证明 利用分部积分和和差化积公式可得

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx) \mathrm{d}x \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)] \mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x [\sin(nx) \cos x + \cos(nx) \sin x] \mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(nx) \mathrm{d} \cos x \\
&= \frac{I_{n-1} + I_n}{2} - \frac{1}{2n} \int_0^{\frac{\pi}{2}} \cos(nx) \mathrm{d} \cos^n x \\
&= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) \mathrm{d}x \\
&= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
&= \frac{I_{n-1}}{2} + \frac{1}{2n}.
\end{aligned}$$

故 $I_n = \frac{I_{n-1}}{2} + \frac{1}{2n}$, 则两边同乘 2^n(强行裂项)

$$2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \cdots$$

又注意到 $I_0 = 0$, 从而

$$2^n I_n = 0 + \sum_{k=1}^{n} \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n} \sum_{k=1}^{n} \frac{2^{k-1}}{k}.$$

□

命题 0.1

证明:

(1) $\int_0^{\pi} \frac{\sin(nx)}{\sin x} \mathrm{d}x = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}$.

(2) $\int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} \mathrm{d}x = n\pi$.

(3) $\int_0^{\pi} \frac{\sin^2(nx)}{\sin x} \mathrm{d}x = \sum_{k=1}^{n} \frac{2}{2k-1}$.

笔记 提示:$\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y)$(证明见命题**??**).

证明

(1) 记 $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x} \mathrm{d}x$, 则

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin((n+2)x) - \sin(nx)}{\sin x} \mathrm{d}x = \int_0^{\pi} \frac{2\cos((n+1)x)\sin x}{\sin x} \mathrm{d}x = 2\int_0^{\pi} \cos((n+1)x) \mathrm{d}x = 0.$$

于是

$$\int_0^\pi \frac{\sin(nx)}{\sin x}\,\mathrm{d}x = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$$

(2) 记 $I_n = \int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x}\,\mathrm{d}x \\ &= \int_0^\pi \frac{\sin((2n+1)x)}{\sin x}\,\mathrm{d}x \xlongequal{\text{命题 0.1(1)}} \pi. \end{aligned} \tag{1}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记 $I_n = \int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin x}\,\mathrm{d}x \\ &= \int_0^\pi \sin((2n+1)x)\,\mathrm{d}x = \frac{1}{2n+1}\cos((2n+1)x)\Big|_\pi^0 = \frac{2}{2n+1}. \end{aligned} \tag{2}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1}\frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1}\frac{2}{2k+1} = \sum_{k=1}^{n}\frac{2}{2k-1}.$$

□

0.1.2 区间再现

定理 0.1 (区间再现恒等式)

当下述积分有意义时, 我们有

1.
$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\mathrm{d}x.$$

2.
$$\int_0^\infty f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^\infty f(x)\mathrm{d}x = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2}\right]\mathrm{d}x.$$

♡

笔记 注意: 倒代换具有将 $[0,1]$ 转化为 $[1,+\infty)$ 的功能.

证明 证明是显然的. □

命题 0.2

证明

1. $\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
2. $\int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x - \frac{\pi}{2}\ln 2.$
3. $\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln 2.$

♠

证明

1.

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln \sin x \mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln \cos x + \ln\left(\frac{\pi}{2} - x\right)\right] \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x \sin x) \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2} \mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x) \mathrm{d}x \\
&= -\frac{\pi}{4} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin x \mathrm{d}x = -\frac{\pi}{4} \ln 2 + \frac{1}{2} I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln \cos x \mathrm{d}x = -\frac{\pi}{2} \ln 2.
\end{aligned}
$$

2.

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln \cos x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln \sin x \mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln \cos x + \ln\left(\frac{\pi}{2} - x\right)\right] \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x \sin x) \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2} \mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x) \mathrm{d}x \\
&= -\frac{\pi}{4} \ln 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln \sin x \mathrm{d}x = -\frac{\pi}{4} \ln 2 + \frac{1}{2} I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln \cos x \mathrm{d}x = -\frac{\pi}{2} \ln 2.
\end{aligned}
$$

3.

$$
\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2} \mathrm{d}x &\xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2} \mathrm{d}\tan\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta \cdot \ln(1+\tan\theta)}{\sec^2\theta} \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta) \mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right)\right] \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right)\right] \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln \frac{2}{1+\tan\theta}\right] \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \ln 2 \mathrm{d}\theta = \frac{\pi}{8} \ln 2.
\end{aligned}
$$

□

例题 0.2 计算

1. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x, a > 0.$
2. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+x+1} \mathrm{d}x.$
3. $\displaystyle\int_0^{1} \frac{\ln x}{\sqrt{x-x^2}} \mathrm{d}x.$

解

1. 注意到

$$
\int_0^{+\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x \xlongequal{x=at} \frac{1}{a} \int_0^{+\infty} \frac{\ln(at)}{1+t^2} dt = \frac{1}{a} \int_0^{+\infty} \frac{\ln a}{1+t^2} dt + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} dt = \frac{\pi \ln a}{2a} + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} dt. \quad (3)
$$

又注意到

$$
\int_0^{+\infty} \frac{\ln t}{1+t^2} dt \xlongequal{t=\frac{1}{x}} \int_0^{+\infty} \frac{\ln \frac{1}{x}}{1+\frac{1}{x^2}} \frac{1}{x^2} \mathrm{d}x = \int_0^{+\infty} \frac{-\ln x}{1+x^2} \mathrm{d}x \Longrightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2} dt = 0.
$$

于是再结合(3)式可得

$$
\int_0^{+\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x = \frac{\pi \ln a}{2a}.
$$

2.

$$\int_0^{\infty}\frac{\ln x}{x^2+x+1}\mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_0^{\infty}\frac{-\ln t}{1+\frac{1}{t}+\frac{1}{t^2}}\mathrm{d}\frac{1}{t}=\int_0^{+\infty}\frac{-\ln t}{1+t+t^2}dt \Longrightarrow \int_0^{\infty}\frac{\ln x}{x^2+x+1}\mathrm{d}x=0.$$

3.

$$\int_0^1\frac{\ln x}{\sqrt{x-x^2}}\mathrm{d}x \xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}}\frac{\ln\sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}}\mathrm{d}\sin^2 y$$

$$=4\int_0^{\frac{\pi}{2}}\ln\sin y\mathrm{d}y \xlongequal{\text{命题 0.2}} 4\cdot\left(-\frac{\pi}{2}\ln 2\right)=-2\pi\ln 2.$$

□

例题 0.3

1. 对 $n\in\mathbb{N}$, 计算 $\displaystyle\int_{-\pi}^{\pi}\frac{\sin(nx)}{(1+2^x)\sin x}\mathrm{d}x$.
2. $\displaystyle\int_{-\pi}^{\pi}\frac{x\sin x\arctan e^x}{1+\cos^2 x}\mathrm{d}x$.
3. 对 $n\in\mathbb{N}$, 计算 $\displaystyle\int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x$.

解

1.

$$\int_{-\pi}^{\pi}\frac{\sin(nx)}{(1+2^x)\sin x}\mathrm{d}x=\int_{-\pi}^{0}\left[\frac{\sin(nx)}{(1+2^x)\sin x}+\frac{\sin(nx)}{(1+2^{-x})\sin x}\right]\mathrm{d}x=\int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\left(\frac{1}{1+2^x}+\frac{1}{1+2^{-x}}\right)\mathrm{d}x$$

$$=\int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\cdot\frac{2+2^x+2^{-x}}{2+2^x+2^{-x}}\mathrm{d}x=\int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\mathrm{d}x=\int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x \xlongequal{\text{例题 0.1}} \begin{cases}0, n\text{为偶数}\\ \pi, n\text{为奇数}\end{cases}.$$

2.

$$\int_{-\pi}^{\pi}\frac{x\sin x\arctan e^x}{1+\cos^2 x}\mathrm{d}x=\int_{-\pi}^{0}\left(\frac{x\sin x\arctan e^x}{1+\cos^2 x}+\frac{x\sin x\arctan e^{-x}}{1+\cos^2 x}\right)\mathrm{d}x=\int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}(\arctan e^x+\arctan e^{-x})\mathrm{d}x$$

$$\xlongequal{\text{命题??(1)}}\int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}\cdot\frac{\pi}{2}\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\frac{x\sin x}{1+\cos^2 x}+\frac{(\pi-x)\sin x}{1+\cos^2 x}\right)\mathrm{d}x=\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x$$

$$=\frac{\pi^2}{2}\arctan\cos x\Big|_{\frac{\pi}{2}}^{0}=\frac{\pi^2}{2}\cdot\frac{\pi}{4}=\frac{\pi^3}{8}.$$

3.

$$\int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x=\int_0^{2\pi}\sin[\sin(2\pi-x)+n(2\pi-x)]\mathrm{d}x$$

$$=\int_0^{2\pi}\sin(-\sin x-nx)\mathrm{d}x=-\int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x$$

$$\Longrightarrow\int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x=0.$$

□

0.1.3 化成多元累次积分 (换序)

命题 0.3

证明:

(1) $\displaystyle\int_0^{\infty}e^{-x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2}$.

(2) $\displaystyle\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x=\frac{\pi}{2}$.

(3) $\int_0^{\infty}\sin x^2\mathrm{d}x,\ \int_0^{\infty}\cos x^2\mathrm{d}x=\sqrt{\frac{\pi}{8}}.$

笔记 本结果可以直接使用.

证明

(1) 注意到

$$\begin{aligned}\left(\int_0^{+\infty}e^{-x^2}\mathrm{d}x\right)^2&=\left(\int_0^{+\infty}e^{-x^2}\mathrm{d}x\right)\left(\int_0^{+\infty}e^{-y^2}\mathrm{d}y\right)\xlongequal{\text{把}\int_0^{+\infty}e^{-y^2}\mathrm{d}y\text{看作常数}}\int_0^{+\infty}e^{-x^2}\left(\int_0^{+\infty}e^{-y^2}\mathrm{d}x\right)\mathrm{d}y\\&\xlongequal{\text{把}e^{-x^2}\text{看作常数}}\int_0^{+\infty}\left(\int_0^{+\infty}e^{-(x^2+y^2)}\mathrm{d}x\right)\mathrm{d}y\xlongequal{e^{-(x^2+y^2)}\text{连续}}\iint_{R^2}e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y\\&=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{+\infty}re^{-r^2}\mathrm{d}r=\frac{\pi}{2}\int_0^{+\infty}re^{-r^2}\mathrm{d}r\\&=\frac{\pi}{4}\int_0^{+\infty}e^{-r^2}\mathrm{d}r^2=\frac{\pi}{4}.\end{aligned}$$

故 $\int_0^{\infty}e^{-x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2}.$

(2) 注意到

$$\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x=\mathrm{Im}\int_0^{+\infty}e^{\mathrm{i}x-yx}\,\mathrm{d}x=\mathrm{Im}\int_0^{+\infty}e^{-(y-\mathrm{i})x}\,\mathrm{d}x=\mathrm{Im}\frac{1}{y-\mathrm{i}}=\mathrm{Im}\frac{y+\mathrm{i}}{y^2+1}=\frac{1}{y^2+1}.$$

因此就有

$$\begin{aligned}\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x&=\int_0^{+\infty}\sin x\left(\int_0^{+\infty}e^{-yx}\,\mathrm{d}y\right)\mathrm{d}x=\int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x\\&=\int_0^{+\infty}\mathrm{d}y\left(\mathrm{Im}\int_0^{+\infty}e^{\mathrm{i}x-yx}\right)\mathrm{d}x=\int_0^{+\infty}\frac{1}{y^2+1}\,\mathrm{d}y=\frac{\pi}{2}.\end{aligned}$$

当然本题也可以直接利用分部积分计算 $\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x=\frac{1}{y^2+1}.$

(3) 注意到

$$\int_0^{+\infty}e^{-ax^2}\,\mathrm{d}x\xlongequal{x=\frac{t}{\sqrt{a}}}\frac{1}{\sqrt{a}}\int_0^{+\infty}e^{-t^2}\,dt=\frac{1}{2}\sqrt{\frac{\pi}{a}},\quad a>0.$$

并且 $-\mathrm{i}=e^{-\frac{\pi}{2}\mathrm{i}}$, 从而 $\sqrt{-\mathrm{i}}=e^{-\frac{\pi}{4}\mathrm{i}}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}$. 于是

$$\begin{aligned}\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x&=\int_0^{+\infty}e^{-\mathrm{i}x^2}\,\mathrm{d}x=\frac{1}{2}\sqrt{\frac{\pi}{\mathrm{i}}}=\frac{1}{2}\sqrt{-\mathrm{i}\pi}\\&=\frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}.\end{aligned}$$

故

$$\begin{aligned}\int_0^{+\infty}\cos x^2\,\mathrm{d}x&=\mathrm{Re}\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x=\mathrm{Re}\left(\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}=\sqrt{\frac{\pi}{8}},\\\int_0^{+\infty}\sin x^2\,\mathrm{d}x&=\mathrm{Im}\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x=\mathrm{Im}\left(\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}=\sqrt{\frac{\pi}{8}}.\end{aligned}$$

□

例题 0.4 计算 $\int_0^1\sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x\ (b>a>0).$

证明

$$\begin{aligned}\int_0^1\sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x&=\int_0^1\sin\ln\frac{1}{x}\left(\int_a^b x^y\mathrm{d}y\right)\mathrm{d}x=\int_a^b\mathrm{d}y\int_0^1 x^y\sin\ln\frac{1}{x}\mathrm{d}x\\&\xlongequal{x=e^{-t}}\int_a^b\mathrm{d}y\int_{+\infty}^0 e^{-ty}\sin t\mathrm{d}e^{-t}=\int_a^b\mathrm{d}y\int_0^{+\infty}e^{-t(y+1)}\sin t\,dt\end{aligned}$$

$$\xlongequal{\text{命题 0.3(2) 的证明过程}} \int_a^b \frac{1}{1+(y+1)^2}\mathrm{d}y = \arctan(b+1) - \arctan(a+1).$$

□

0.1.4 化成含参积分 (求导)

例题 0.5 设 $a,b \geqslant 0$ 且不全为 0, 计算 $\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$.

注 实际上, 根据 $a>b$ 时得到的结果, 可以看出 $F(a,b)=\pi\ln\frac{a+b}{2}$ 对 a,b 有轮换对称性, 故这个结果对其他情况显然也成立.

证明 设 $F(a,b)=\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$, 当 $a>b$ 时, 则

$$\begin{aligned}
\frac{\partial}{\partial b}F(a,b) &= \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial b}\ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{2b\sin^2 x}{a^2\cos^2 x + b^2\sin^2 x}\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{2b\tan^2 x}{a^2+b^2\tan^2 x}\mathrm{d}x = \int_0^{+\infty} \frac{2bt^2}{(a^2+b^2t^2)(1+t^2)}dt \\
&= \frac{1}{a^2-b^2}\int_0^{+\infty}\left(\frac{2a^2b}{a^2+b^2t^2} - \frac{2b}{1+t^2}\right)dt \\
&= \frac{1}{a^2-b^2}\int_0^{+\infty}\frac{2a^2b}{a^2+b^2t^2}dt - \frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+t^2}dt \\
&= \frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+\left(\frac{b}{a}t\right)^2}dt - \frac{b\pi}{a^2-b^2} \\
&= \frac{2b}{a^2-b^2}\cdot\frac{a}{b}\cdot\frac{\pi}{2} - \frac{b\pi}{a^2-b^2} = \frac{\pi}{a+b}.
\end{aligned}$$

于是

$$\begin{aligned}
F(a,b) &= F(a,0) + \int_0^b \frac{\partial}{\partial b'}F(a,b')\mathrm{d}b' = F(a,0) + \int_0^b \frac{\pi}{a+b'}\mathrm{d}b' \\
&= 2\int_0^{\frac{\pi}{2}} \ln(a\cos x)\mathrm{d}x + \pi\ln\frac{a+b}{a} \xlongequal{\text{例题 0.2}} \pi\ln\frac{a+b}{2}.
\end{aligned}$$

当 $a<b$ 时, 类似可得 $F(a,b)=\pi\ln\frac{a+b}{2}$. 当 $a=b$ 时,$F(a,b)=\int_0^{\frac{\pi}{2}}\ln a^2\mathrm{d}x = \pi\ln a = \pi\ln\frac{a+b}{2}$.

综上, 对 $\forall a,b \geqslant 0$, 都有 $F(a,b)=\pi\ln\frac{a+b}{2}$. □

0.1.5 级数展开方法

积分和求和换序 $\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x$, 等价于

$$\lim_{m\to\infty}\sum_{n=1}^{m}\int_a^b f_n(x)\mathrm{d}x = \int_a^b\sum_{n=1}^{\infty} f_n(x)\mathrm{d}x.$$

又由于有限和随意交换, 因此上式等价于

$$\lim_{m\to\infty}\int_a^b\sum_{n=1}^{m} f_n(x)\mathrm{d}x = \int_a^b\sum_{n=1}^{\infty} f_n(x)\mathrm{d}x,$$

于是

$$\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b\sum_{n=1}^{\infty} f_n(x)\mathrm{d}x \Longleftrightarrow \lim_{m\to\infty}\int_a^b\sum_{n=m+1}^{\infty} f_n(x)\mathrm{d}x = 0.$$

例题 0.6 计算 $\int_0^{\infty}\frac{x}{1+e^x}\mathrm{d}x$.

解 由于

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n,\quad 0<x<1.$$

并且 $0<e^{-x}<1$, 故

$$\begin{aligned}\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x&=\int_0^{+\infty}\frac{xe^{-x}}{1+e^{-x}}\mathrm{d}x=\int_0^{+\infty}x\sum_{n=0}^{\infty}(-1)^n e^{-(n+1)x}\mathrm{d}x\\&\xlongequal{\text{换序}}\sum_{n=0}^{\infty}(-1)^n\int_0^{+\infty}xe^{-(n+1)x}\mathrm{d}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}\\&=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(2n)^2}.\end{aligned}$$

又因为 $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, 所以

$$\sum_{n=0}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{24},$$
$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=0}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{8}.$$

故

$$\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{8}-\frac{\pi^2}{24}=\frac{\pi^2}{12}.$$

下面证明(??)式换序成立, 等价于证明 $\lim\limits_{m\to+\infty}\int_0^{+\infty}\sum\limits_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\mathrm{d}x=0$. 由交错级数不等式及 $xe^{-(n+1)x}$ 关于 n 非负递减, 对 $\forall m\in\mathbb{N}$, 都有

$$\int_0^{+\infty}\left|\sum_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\right|\mathrm{d}x\leqslant\int_0^{+\infty}xe^{-(m+1)x}\mathrm{d}x=-\frac{xe^{-(m+1)x}}{m+1}\Big|_0^{+\infty}+\frac{1}{m+1}\int_0^{+\infty}e^{-(m+1)x}\mathrm{d}x=\frac{1}{(m+1)^2}.$$

令 $m\to+\infty$, 得 $\lim\limits_{m\to+\infty}\int_0^{+\infty}\sum\limits_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\mathrm{d}x=0$. 故(??)式换序成立. □

命题 0.4

证明:

(1) $\sum\limits_{n=1}^{\infty}\frac{q^n\sin(nx)}{n}=\arctan\frac{q\sin x}{1-q\cos x},|q|\leqslant 1.$

(2) $\sum\limits_{n=1}^{\infty}\frac{q^n\cos(nx)}{n}=-\frac{1}{2}\ln(1+q^2-2q\cos x),|q|\leqslant 1.$

(3) $\sum\limits_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!}=e^{q\cos x}\cos(q\sin x)-1,|q|\leqslant 1,x\in\mathbb{R}.$

(4) $\sum\limits_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!}=e^{q\cos x}\sin(q\sin x),|q|\leqslant 1,x\in\mathbb{R}.$

笔记 在 $\mathbb{C}$ 上,

$$\mathrm{Ln}z=\ln|z|+i(\arg z+2k\pi),k\in\mathbb{Z}.$$

我们定义主值支

$$\ln z=\ln|z|+i\arg z.$$

本部分内容无需记忆, 只需要大概有个可以算的感觉即可, 实际做题中可以围绕这种级数给出构造.

证明 $\Im$ 表示取虚部,$\Re$ 表示取实部.

(1) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n}&=\Im\left(\sum_{n=1}^{\infty}\frac{q^ne^{inx}}{n}\right)=\Im\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n}\right)=\Im(-\ln(1-qe^{ix}))\\&=-\Im\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=\arctan\frac{q\sin x}{1-q\cos x}.\end{aligned}$$

(2) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n}&=-\Re\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=-\frac{1}{2}\ln\left[(1-q\cos x)^2+q^2\sin^2x\right]\\&=-\frac{1}{2}\ln(1+q^2-2q\cos x).\end{aligned}$$

(3) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!}&=\Re\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n!}\right)=\Re\left(e^{qe^{ix}}-1\right)=\Re\left(e^{q\cos x+iq\sin x}-1\right)\\&=\Re\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)\\&=e^{q\cos x}\cos(q\sin x)-1.\end{aligned}$$

(4) 利用 (3) 有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!}&=\Im\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)\\&=e^{q\cos x}\sin(q\sin x).\end{aligned}$$

□

例题 0.7 计算

1. $\displaystyle\int_0^{2\pi}e^{\cos x}\cos(\sin x)\mathrm{d}x.$
2. $\displaystyle\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x,\ a\in(0,+\infty)\setminus\{1\}.$

注 由 1 的证明可得

$$e^{\cos x}\cos(\sin x)=\mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{(e^{ix})^n}{n!}\right)=\mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{e^{inx}}{n!}\right)=\sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}.$$

实际上, 上式就是命题 0.4(3)的结论.

注 第 2 问也可以用含参积分求导的方法进行计算 (这个方法更容易想到).

证明

1.

$$\begin{aligned}\int_0^{2\pi}e^{\cos x}\cos(\sin x)\,\mathrm{d}x&=\mathrm{Re}\left(\int_0^{2\pi}e^{\cos x}e^{\mathrm{i}\sin x}\mathrm{d}x\right)=\mathrm{Re}\left(\int_0^{2\pi}e^{\cos x+\mathrm{i}\sin x}\mathrm{d}x\right)\\&=\mathrm{Re}\left(\int_0^{2\pi}e^{e^{\mathrm{i}x}}\mathrm{d}x\right)=\mathrm{Re}\left[\int_0^{2\pi}\sum_{n=0}^{+\infty}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right]=\mathrm{Re}\left[\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right]\\&=\mathrm{Re}\left(\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{e^{inx}}{n!}\mathrm{d}x\right)=\mathrm{Re}\left(\int_0^{2\pi}\frac{e^{\mathrm{i}\cdot0\cdot x}}{n!}\mathrm{d}x+\sum_{n=1}^{+\infty}\frac{e^{2\pi\mathrm{i}x}-1}{\mathrm{i}n\cdot n!}\right)\\&=\mathrm{Re}\left(\int_0^{2\pi}1\mathrm{d}x+0\right)=2\pi.\end{aligned}$$

2. 注意到当 $a\in(0,1)$ 时, 有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n}&=\mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{(ae^{\mathrm{i}x})^n}{n}\right]=-\mathrm{Re}\left[\ln(1-ae^{\mathrm{i}x})\right]\\&=-\mathrm{Re}\left[\ln|1-ae^{\mathrm{i}x}|+\mathrm{i}\arg(1-ae^{\mathrm{i}x})\right]=-\ln|1-ae^{\mathrm{i}x}|\\&=-\ln|(1-a\cos x)+a\mathrm{i}\sin x|=-\frac{1}{2}\ln(1+a^2-2a\cos x).\end{aligned}$$

于是当 $a\in(0,1)$ 时, 就有

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x=-\frac{1}{2}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n}\mathrm{d}x=0.$$

若 $a>1$, 则 $\frac{1}{a}\in(0,1)$, 从而此时我们有

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x=\pi\ln a^2+\int_0^{\pi}\ln\left(\frac{1}{a^2}-\frac{2}{a}\cos x+1\right)\mathrm{d}x=\pi\ln a^2=2\pi\ln a.$$

又由 $\ln(1-2a\cos x+a^2)$ 关于 a 的偏导存在可知 $\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x$ 关于 a 连续. 于是由

$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x=2\pi\ln a,\quad\forall a>1.$$

可知当 $a=1$ 时, 我们有

$$\int_0^{\pi}\ln(2-2\cos x)\mathrm{d}x=\lim_{a\to1^+}\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x=\lim_{a\to1^+}(2\pi\ln a)=0.$$

□

定义 0.1 (多重对数函数-Li_2 函数)

定义

$$\mathrm{Li}_2(x)\triangleq\sum_{n=1}^{\infty}\frac{x^n}{n^2},\quad x\in[-1,1].$$

♣

命题 0.5

(1) $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\frac{\pi^2}{6}-\ln x\cdot\ln(1-x),x\in(0,1)$.

(2) $\mathrm{Li}_2(1)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\quad\mathrm{Li}_2(0)=0,\quad\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}$.

♠

证明

(1) 记 $f(x)\triangleq\mathrm{Li}_2(x),F(x)\triangleq f(x)+f(1-x)+\ln x\ln(1-x)$. 则

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}=-\frac{1}{x}\ln(1-x).$$

于是

$$F'(x)=-\frac{1}{x}\ln(1-x)+\frac{\ln x}{1-x}-\frac{\ln x}{1-x}+\frac{\ln(1-x)}{x}=0.$$

故 $F(x)\equiv F(1)=f(0)+f(1)=0+\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

(2) 显然 $\mathrm{Li}_2(1)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\mathrm{Li}_2(0)=0$. 由 (1) 可得

$$\mathrm{Li}_2\left(\frac{1}{2}\right)+\mathrm{Li}_2\left(\frac{1}{2}\right)=2\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{6}-\ln^2\frac{1}{2}\implies\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.$$

□

例题 0.8 计算 $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x} \mathrm{d}x$.

解

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{\ln x}{1-x} \mathrm{d}x &= \int_{\frac{1}{2}}^1 \frac{\ln(1-x)}{x} \mathrm{d}x = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{2}}^1 x^{n-1} \mathrm{d}x \\
&= -\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{2^n n^2} = -\frac{\pi^2}{6} + \mathrm{Li}_2\left(\frac{1}{2}\right) \\
&\xlongequal{\text{命题 0.5}} -\frac{\pi^2}{12} - \frac{1}{2}\ln^2 \frac{1}{2}.
\end{aligned}$$

□

0.1.6 其他

例题 0.9 证明积分 $\int_0^{\infty} e^{-ax^2-\frac{b}{x^2}} \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}, a, b > 0.$

证明 当 $a=1$ 时, 就有

$$\begin{aligned}
\int_0^{+\infty} e^{-x^2-\frac{b}{x^2}} \mathrm{d}x &= e^{-2\sqrt{b}} \int_0^{+\infty} e^{-\left(x-\frac{\sqrt{b}}{x}\right)^2} \mathrm{d}x \xlongequal{y=\frac{\sqrt{b}}{x}} e^{-2\sqrt{b}} \int_0^{+\infty} \frac{\sqrt{b}}{y^2} e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2} \mathrm{d}y \\
&= \frac{e^{-2\sqrt{b}}}{2} \int_0^{+\infty} \left(1+\frac{\sqrt{b}}{y^2}\right) e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2} \mathrm{d}y = \frac{e^{-2\sqrt{b}}}{2} \int_0^{+\infty} e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2} \mathrm{d}\left(y-\frac{\sqrt{b}}{y}\right) \\
&= \frac{e^{-2\sqrt{b}}}{2} \int_{-\infty}^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} e^{-2\sqrt{b}}.
\end{aligned}$$

于是对 $\forall a > 0$, 就有

$$\int_0^{+\infty} e^{-ax^2-\frac{b}{x^2}} \mathrm{d}x = \frac{1}{\sqrt{a}} \int_0^{+\infty} e^{-x^2-\frac{ab}{x^2}} \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$$

□

例题 0.10 计算 $\int_0^{\infty} \frac{\cos(ax)}{1+x^2} \mathrm{d}x, a \in \mathbb{R}.$

注 本题可以用复变函数的方法 (留数定理) 来计算. 但是我们这里用基本的高等数学的方法来计算. $\int_0^{\infty} \frac{\sin(ax)}{1+x^2} \mathrm{d}x$ 这个积分没办法算出具体的初等数值.

证明

$$\begin{aligned}
\int_0^{+\infty} \frac{\cos(ax)}{1+x^2} \mathrm{d}x &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\cos(ax)}{1+x^2} \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{+\infty} \cos(ax) \left(\int_0^{+\infty} e^{-(1+x^2)y} \mathrm{d}y\right) \mathrm{d}x \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} e^{-(1+x^2)y} \cos(ax) \mathrm{d}y\right) \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-(1+x^2)y} \cos(ax) \mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2} \int_0^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-(1+x^2)y} \cos(ax) \mathrm{d}x\right) \mathrm{d}y = \frac{1}{2} \int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-x^2 y} \cos(ax) \mathrm{d}x\right) \mathrm{d}y \\
&= \frac{1}{2} \mathrm{Re}\left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-x^2 y + \mathrm{i}ax} \mathrm{d}x\right) \mathrm{d}y\right) = \frac{1}{2} \mathrm{Re}\left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y\left(x^2 - \frac{aix}{y} + \left(\frac{ai}{2y}\right)^2\right) - \frac{a^2}{4y}} \mathrm{d}x\right) \mathrm{d}y\right) \\
&= \frac{1}{2} \mathrm{Re}\left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y\left(x-\frac{ai}{2y}\right)^2 - \frac{a^2}{4y}} \mathrm{d}x\right) \mathrm{d}y\right) = \frac{1}{2} \mathrm{Re}\left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2\mathrm{i}y}\right)^2 - \frac{a^2}{4y}} \mathrm{d}x\right) \mathrm{d}y\right) \\
&= \frac{1}{2} \mathrm{Re}\left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2\mathrm{i}y}\right)^2 - \frac{a^2}{4y}} \mathrm{d}\left(x+\frac{a}{2\mathrm{i}y}\right)\right) \mathrm{d}y\right) = \frac{1}{2} \mathrm{Re}\left(\int_0^{+\infty} e^{-y} \left(\int_{-\infty}^{+\infty} e^{-yx^2 - \frac{a^2}{4y}} \mathrm{d}x\right) \mathrm{d}y\right) \\
&= \frac{1}{2} \int_0^{+\infty} e^{-y-\frac{a^2}{4y}} \left(\int_{-\infty}^{+\infty} e^{-yx^2} \mathrm{d}x\right) \mathrm{d}y = \frac{\sqrt{\pi}}{2} \int_0^{+\infty} \frac{1}{\sqrt{y}} e^{-y-\frac{a^2}{4y}} \mathrm{d}y \\
&\xlongequal{y=t^2} \sqrt{\pi} \int_0^{+\infty} e^{-t^2-\frac{a^2}{4t^2}} dt \xlongequal{\text{例题 0.9}} \sqrt{\pi} \cdot \frac{\sqrt{\pi}}{2} e^{-|a|} = \frac{\pi}{2} e^{-|a|}.
\end{aligned}$$

□

例题 0.11 计算 $\int_0^\infty \frac{1}{(1+x^8)^2}\mathrm{d}x$.

注 由命题??可知对 $\forall s>0$, 都有

$$\frac{1}{t^s}=\frac{1}{\Gamma(s)}\int_0^{+\infty} y^{s-1}e^{-ty}\mathrm{d}y, \forall t\in\mathbb{R}.$$

本题的核心想法就是利用上式将 $\frac{z}{1+x^8}$ 转化成积分形式.

证明 注意到

$$\int_0^{+\infty} ye^{-(1+x^8)y}\mathrm{d}y \xlongequal{y=\frac{z}{1+x^8}} \frac{1}{(1+x^8)^2}\int_0^{+\infty} ze^{-z}\mathrm{d}z=\frac{1}{(1+x^8)^2},$$

因此

$$\begin{aligned}
\int_0^{+\infty}\frac{1}{(1+x^8)^2}\mathrm{d}x &= \int_0^{+\infty}\left(\int_0^{+\infty} ye^{-(1+x^8)y}\mathrm{d}y\right)\mathrm{d}x=\int_0^{+\infty}\left(\int_0^{+\infty} ye^{-(1+x^8)y}\mathrm{d}x\right)\mathrm{d}y\\
&=\int_0^{+\infty} ye^{-y}\left(\int_0^{+\infty} e^{-x^8y}\mathrm{d}x\right)\mathrm{d}y \xlongequal{x=y^{-\frac{1}{8}}z^{\frac{1}{8}}} \int_0^{+\infty} ye^{-y}\left(\int_0^{+\infty} y^{-\frac{1}{8}}e^{-z}\mathrm{d}z^{\frac{1}{8}}\right)\mathrm{d}y\\
&=\frac{1}{8}\int_0^{+\infty} y^{\frac{7}{8}}e^{-y}\left(\int_0^{+\infty} z^{-\frac{7}{8}}e^{-z}\mathrm{d}z\right)\mathrm{d}y=\frac{1}{8}\int_0^{+\infty} y^{\frac{7}{8}}e^{-y}\Gamma\left(\frac{1}{8}\right)\mathrm{d}y\\
&=\frac{1}{8}\Gamma\left(\frac{15}{8}\right)\Gamma\left(\frac{1}{8}\right)=\frac{1}{8}\cdot\frac{7}{8}\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{1}{8}\right)\\
&\xlongequal{??}\frac{7\pi}{64\sin\left(\frac{\pi}{8}\right)}=\frac{7\pi}{32\sqrt{2-\sqrt{2}}}.
\end{aligned}$$

□

例题 0.12 计算积分 $I=\int_{-1}^2\frac{1+x^2}{1+x^4}\mathrm{d}x$.

注 在此例中 $I\neq F(2)-F(-1)$. 这是因为 F 并不是 f 在区间 $[-1,2]$ 上的原函数.

解 在不包含 0 的区间上作变换 $t=x-\frac{1}{x}$ 得

$$\begin{aligned}
\int\frac{1+x^2}{1+x^4}\mathrm{d}x &= \int\frac{x-\frac{1}{x}}{2+\left(x-\frac{1}{x}\right)^2}\mathrm{d}x=\int\frac{\mathrm{d}t}{2+t^2}\\
&=\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C=\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}+C.
\end{aligned}$$

这说明在区间 $[-1,0)$ 和 $(0,2]$ 上, 函数 $f(x)=\frac{1+x^2}{1+x^4}$ 的一个原函数是

$$F(x)=\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}.$$

因此

$$\begin{aligned}
\int_{-1}^0 f(x)\,\mathrm{d}x &= F(0^-)-F(-1)=\frac{\pi}{2\sqrt{2}}-0=\frac{\pi}{2\sqrt{2}},\\
\int_0^2 f(x)\,\mathrm{d}x &= F(2)-F(0^+)=\frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}}+\frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

故

$$I=\int_{-1}^0 f(x)\,\mathrm{d}x+\int_0^2 f(x)\,\mathrm{d}x=\frac{\pi}{\sqrt{2}}+\frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}}.$$

□