## 0.1 递推法与数学归纳法

**命题 0.1 (三对角行列式)**

求下列行列式的递推关系式 (空白处均为 0):

$$D_n=\begin{vmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{n-1} & b_{n-1} \\ & & & & c_{n-1} & a_n \end{vmatrix}.$$

**笔记** 记忆三对角行列式的计算方法和结果: $\boldsymbol{D_n=a_nD_{n-1}-b_{n-1}c_{n-1}D_{n-2}(n\geq 2)}$, 即按最后一列 (或行) 展开得到递推公式.

**解** 显然 $D_0=1, D_1=a_1$. 当 $n\geq 2$ 时, 我们有

$$D_n=\begin{vmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{n-1} & b_{n-1} \\ & & & & c_{n-1} & a_n \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & & & & & \\ c_1 & a_2 & b_2 & & & & \\ & c_2 & a_3 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & a_{n-2} & b_{n-2} & \\ & & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & & c_{n-1} & a_n \end{vmatrix}$$

$$\xlongequal{\text{按最后一列展开}} a_n\begin{vmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{n-2} & b_{n-2} \\ & & & & c_{n-2} & a_{n-1} \end{vmatrix}-b_{n-1}\begin{vmatrix} a_1 & b_1 & & & & & \\ c_1 & a_2 & b_2 & & & & \\ & c_2 & a_3 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & a_{n-3} & b_{n-3} & \\ & & & & c_{n-3} & a_{n-2} & b_{n-2} \\ & & & & & 0 & c_{n-1} \end{vmatrix}$$

$$\xlongequal{\text{第二项按最后展开}} a_n\begin{vmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{n-2} & b_{n-2} \\ & & & & c_{n-2} & a_{n-1} \end{vmatrix}-b_{n-1}c_{n-1}\begin{vmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{n-3} & b_{n-3} \\ & & & & c_{n-3} & a_{n-2} \end{vmatrix}$$

$$=a_nD_{n-1}-b_{n-1}c_{n-1}D_{n-2}.$$

□

**推论 0.1**

计算 $n$ 阶行列式 $(bc \neq 0)$:

$$D_n = \begin{vmatrix} a & b & & & & \\ c & a & b & & & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \\ & & & & c & a \end{vmatrix}.$$

**笔记** 解递推式:$D_n = aD_{n-1} - bcD_{n-2}(n \geq 2)$ 对应的特征方程:$x^2 - ax + bc = 0$ 得到两根 $\alpha = \dfrac{a+\sqrt{a^2-4bc}}{2}, \beta = \dfrac{a-\sqrt{a^2-4bc}}{2}$, 由 Vieta 定理可知 $a = \alpha+\beta, bc = \alpha\beta$.

若 $a, b, c$ 均为复数, 则上述特征方程

**解** 由命题 0.1可知, 递推式为 $D_n = aD_{n-1} - bcD_{n-2}(n \geq 2)$. 又易知 $D_0 = 1, D_1 = a$. 令 $\alpha = \dfrac{a+\sqrt{a^2-4bc}}{2}, \beta = \dfrac{a-\sqrt{a^2-4bc}}{2}$, 则 $a = \alpha+\beta, bc = \alpha\beta$, 于是 $D_n = (\alpha+\beta)D_{n-1} - \alpha\beta D_{n-2}(n \geq 2)$. 从而

$$D_n - \alpha D_{n-1} = \beta\left(D_{n-1} - \alpha D_{n-2}\right), D_n - \beta D_{n-1} = \alpha\left(D_{n-1} - \beta D_{n-2}\right).$$

于是

$$D_n - \alpha D_{n-1} = \beta^{n-1}\left(D_1 - \alpha D_0\right) = \beta^{n-1}(a-\alpha) = \beta^n,$$
$$D_n - \beta D_{n-1} = \alpha^{n-1}\left(D_1 - \beta D_0\right) = \alpha^{n-1}(a-\beta) = \alpha^n.$$

因此, 若 $a^2 \neq 4bc$(即$\alpha \neq \beta$), 则联立上面两式, 解得

$$D_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta};$$

若 $a^2 = 4bc$(即$\alpha = \beta$), 则由 $a = \alpha+\beta$ 可知,$\alpha = \beta = \dfrac{a}{2}$. 又由 $D_n - \alpha D_{n-1} = \beta^n$ 可得

$$D_n = \left(\frac{a}{2}\right)^n + \frac{a}{2}D_{n-1} = \left(\frac{a}{2}\right)^n + \frac{a}{2}\left(\left(\frac{a}{2}\right)^{n-1} + \frac{a}{2}D_{n-2}\right) = 2\left(\frac{a}{2}\right)^n + \left(\frac{a}{2}\right)^2 D_{n-2} = \cdots = n\left(\frac{a}{2}\right)^n + \left(\frac{a}{2}\right)^n D_0 = (n+1)\left(\frac{a}{2}\right)^n.$$

综上, 我们有

$$D_n = \begin{cases} \dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, a^2 \neq 4bc, \\ (n+1)\left(\dfrac{\alpha}{2}\right)^n, a^2 = 4bc. \end{cases}$$

□

**练习 0.1** 求证:$n$ 阶行列式

$$|\boldsymbol{A}| = \begin{vmatrix} \cos x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2\cos x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2\cos x & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2\cos x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2\cos x \end{vmatrix} = \cos nx.$$

**解** 解法一:

设 $|\boldsymbol{A}| = D_n$, 其中 $n$ 表示 $|\boldsymbol{A}|$ 的阶数 $(n \geq 0)$. 易知 $D_0 = 1, D_1 = \cos x$.

从而 $|\boldsymbol{A}| = D_n \xlongequal[\text{命题0.1}]{\text{按最后一列展开}} 2\cos xD_{n-1} - D_{n-2}\ (n \geq 2)$.

其对应的特征方程为 $\lambda^2 = 2\cos x\lambda - 1$, 解得 $\lambda_1 = \cos x + i\sin x, \lambda_2 = \cos x - i\sin x$.

于是当 $n \geq 2$ 时, 我们有 $D_n = (\lambda_1 + \lambda_2) D_{n-1} + \lambda_1\lambda_2 D_{n-2}$.

进而

$$\begin{aligned} D_n - \lambda_1 D_{n-1} &= \lambda_2 (D_n - \lambda_1 D_{n-1}), \\ D_n - \lambda_2 D_{n-1} &= \lambda_1 (D_n - \lambda_2 D_{n-1}). \end{aligned} \tag{1}$$

由此可得

$$D_n - \lambda_1 D_{n-1} = {\lambda_2}^{n-1} (D_1 - \lambda_1 D_0) = -i \sin x \cdot {\lambda_2}^{n-1},$$
$$D_n - \lambda_2 D_{n-1} = {\lambda_1}^{n-1} (D_1 - \lambda_2 D_0) = i \sin x \cdot {\lambda_1}^{n-1}.$$

若 $x \neq k\pi(k \in \mathbb{Z})$, 则联立上面两式, 解得

$$\begin{aligned} D_n &= \frac{i\sin x \cdot {\lambda_1}^n + i\sin x \cdot {\lambda_2}^n}{\lambda_1 - \lambda_2} = \frac{i\sin x \cdot (\cos x + i\sin x)^n + i\sin x \cdot (\cos x - i\sin x)^n}{2i\sin x} \\ &\xlongequal[e^{ix}=\cos x + i\sin x, e^{-ix}=\cos x - i\sin x.]{Euler公式} \frac{i\sin x \cdot e^{nxi} + i\sin x \cdot e^{-nxi}}{2i\sin x} = \frac{i\sin x \cdot (\cos nx + i\sin nx) + i\sin x \cdot (\cos nx - i\sin nx)}{2i\sin x} \\ &= \frac{2i\sin x \cdot \cos nx}{2i\sin x} = \cos nx. \end{aligned}$$

若 $x = k\pi(k \in \mathbb{Z})$, 则 $\lambda_1 = \lambda_2 = \cos k\pi$. 从而由(1)式可得,$D_n - \cos k\pi D_{n-1} = -i\sin x \cdot (\cos k\pi) = 0$.

于是

$$D_n = \cos k\pi D_{n-1} = (\cos k\pi)^2 D_{n-2} = \cdots = (\cos k\pi)^n D_0 = (\cos k\pi)^n = (-1)^{kn} = \cos(nk\pi) = \cos nx.$$

解法二:仿照练习0.3中的数学归纳法证明. □

**练习 0.2** 求下列 $n$ 阶行列式的值:

$$D_n = \begin{vmatrix} 1-a_1 & a_2 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1-a_2 & a_3 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1-a_3 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1-a_n \end{vmatrix}.$$

笔记 观察原行列式我们可以得到,$D_n$ 的每列和有一定的规律, 即除了第一列和最后一列, 中间每列和均为 0. 并且 $D_n$ 是三对角行列式. 因此, 我们既可以直接应用三对角行列式的结论 (即命题0.1), 又可以使用求和法进行求解. 如果我们直接应用三对角行列式的结论 (即命题0.1), 按照对一般的三对角行列式展开的方法能得到相应递推式, 但是这样得到的递推式并不是相邻两项之间的递推, 后续求解通项并不简便. 又因为使用求和法计算行列式后续计算一般比较简便所以我们先采用求和法进行尝试.

解 解法一: 当 $n \geq 1$ 时, 我们有

$$\begin{aligned} D_n &= \begin{vmatrix} 1-a_1 & a_2 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1-a_2 & a_3 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1-a_3 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1-a_n \end{vmatrix} \xlongequal[i=2,\cdots,n]{r_i+r_1} \begin{vmatrix} -a_1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1-a_2 & a_3 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1-a_3 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1-a_n \end{vmatrix} \\ &\xlongequal{按第一行展开} -a_1 D_{n-1} + (-1)^{n+1} \begin{vmatrix} -1 & 1-a_2 & a_3 & 0 & \cdots & 0 \\ 0 & -1 & 1-a_3 & a_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{vmatrix} \\ &= -a_1 D_{n-1} + (-1)^{n+1} (-1)^{n-1} \\ &= 1 - a_1 D_{n-1}. \end{aligned}$$

其中 $D_{n-i}$ 表示 $D_{n-i+1}$ 去掉第一行和第一列得到的 $n-i$ 阶行列式,$i=1,2,\cdots,n-1$. (或者称 $D_{n-i}$ 表示以 $a_{i+1},\cdots,a_n$ 为未定元的 $n-i$ 阶行列式,$i=1,2,\cdots,n-1$)

由递推不难得到

$$D_n=1-a_1(1-a_2D_{n-2})=1-a_1+a_1a_2D_{n-2}=\cdots=1-a_1+a_1a_2-a_1a_2a_3+\cdots+(-1)^n a_1a_2\cdots a_n.$$

解法二:仿照练习0.3中的数学归纳法证明. □

**命题 0.2**

计算 $n$ 阶行列式:

$$D_n=\begin{vmatrix} x_1 & y & y & \cdots & y & y\\ z & x_2 & y & \cdots & y & y\\ z & z & x_3 & \cdots & y & y\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ z & z & z & \cdots & x_{n-1} & y\\ z & z & z & \cdots & z & x_n \end{vmatrix}.$$

笔记 解法二: $f(x)\triangleq\begin{vmatrix} x_1+x & y+x & \cdots & y+x\\ z+x & x_2+x & \cdots & y+x\\ \vdots & \vdots & & \vdots\\ z+x & z+x & \cdots & x_n+x \end{vmatrix}=\begin{vmatrix} x_1+x & y+x & \cdots & y+x\\ z-x_1 & x_2-y & \cdots & 0\\ \vdots & \vdots & & \vdots\\ z-x_1 & z-y & \cdots & x_n-y \end{vmatrix}$, 再按第一行展开可得 $f(x)$ 一定为关于 $x$ 的线性函数.

解 解法一 (小拆分法): 对第 $n$ 列进行拆分即可得到递推式: (对第 1 或 n 行 (或列) 拆分都可以得到相同结果)

$$D_n=\begin{vmatrix} x_1 & y & y & \cdots & y & y+0\\ z & x_2 & y & \cdots & y & y+0\\ z & z & x_3 & \cdots & y & y+0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ z & z & z & \cdots & x_{n-1} & y+0\\ z & z & z & \cdots & z & y+x_n-y \end{vmatrix}=\begin{vmatrix} x_1 & y & y & \cdots & y & y\\ z & x_2 & y & \cdots & y & y\\ z & z & x_3 & \cdots & y & y\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ z & z & z & \cdots & x_{n-1} & y\\ z & z & z & \cdots & z & y \end{vmatrix}+\begin{vmatrix} x_1 & y & y & \cdots & y & 0\\ z & x_2 & y & \cdots & y & 0\\ z & z & x_3 & \cdots & y & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ z & z & z & \cdots & x_{n-1} & 0\\ z & z & z & \cdots & z & x_n-y \end{vmatrix}$$

$$=\begin{vmatrix} x_1-z & 0 & 0 & \cdots & 0 & 0\\ 0 & x_2-z & 0 & \cdots & 0 & 0\\ 0 & 0 & x_3-z & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & x_{n-1}-z & 0\\ z & z & z & \cdots & z & y \end{vmatrix}+(x_n-y)D_{n-1}=y\prod_{i=1}^{n-1}(x_i-z)+(x_n-y)D_{n-1}. \tag{2}$$

将原行列式转置后, 同理可得

$$D_n=D_n^T=\begin{vmatrix} x_1 & z & z & \cdots & z & z+0\\ y & x_2 & z & \cdots & z & z+0\\ y & y & x_3 & \cdots & z & z+0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ y & y & y & \cdots & x_{n-1} & z+0\\ y & y & y & \cdots & y & z+x_n-z \end{vmatrix}=\begin{vmatrix} x_1 & z & z & \cdots & z & z\\ y & x_2 & z & \cdots & z & z\\ y & y & x_3 & \cdots & z & z\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ y & y & y & \cdots & x_{n-1} & z\\ y & y & y & \cdots & y & z \end{vmatrix}+\begin{vmatrix} x_1 & z & z & \cdots & z & 0\\ y & x_2 & z & \cdots & z & 0\\ y & y & x_3 & \cdots & z & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ y & y & y & \cdots & x_{n-1} & 0\\ y & y & y & \cdots & y & x_n-z \end{vmatrix}$$

$$= \begin{vmatrix} x_1-y & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_2-y & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_3-y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-1}-y & 0 \\ y & y & y & \cdots & y & z \end{vmatrix} + (x_n-z)\,D_{n-1}^T = z\prod_{i=1}^{n-1}(x_i-y) + (x_n-z)\,D_{n-1}. \tag{3}$$

若 $z\neq y$, 则联立(2)(3)式, 解得

$$D_n = \frac{1}{z-y}\left[z\prod_{i=1}^{n}(x_i-y) - y\prod_{i=1}^{n}(x_i-z)\right];$$

若 $z=y$, 则由(2)式递推可得

$$\begin{aligned} D_n &= y\prod_{i=1}^{n-1}(x_i-y) + (x_n-y)\,D_{n-1} \\ &= y\prod_{i=1}^{n-1}(x_i-y) + (x_n-y)\left(y\prod_{i=1}^{n-2}(x_i-y) + (x_{n-1}-y)\,D_{n-2}\right) \\ &= y\prod_{j\neq n}(x_i-y) + y\prod_{j\neq n-1}(x_i-y) + (x_n-y)\,(x_{n-1}-y)\,D_{n-2} \\ &= \cdots = y\sum_{i=1}^{n}\prod_{j\neq i}(x_j-y) + \prod_{i=1}^{n}(x_i-y)D_0 \\ &= y\sum_{i=1}^{n}\prod_{j\neq i}(x_j-y) + \prod_{i=1}^{n}(x_i-y). \end{aligned}$$

解法二 (大拆分法):令 $f(x) \triangleq \begin{vmatrix} x_1+x & y+x & \cdots & y+x \\ z+x & x_2+x & \cdots & y+x \\ \vdots & \vdots & & \vdots \\ z+x & z+x & \cdots & x_n+x \end{vmatrix}$, 则 $f(x)$ 一定是线性函数, 从而设 $f(x)=ax+b$. 注意到

$$f(-z) = \begin{vmatrix} x_1-z & y-z & \cdots & y-z \\ 0 & x_2-z & \cdots & y-z \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_n-z \end{vmatrix} = \prod_{i=1}^{n}(x_i-z), \quad f(-y) = \begin{vmatrix} x_1-y & 0 & \cdots & 0 \\ z-y & x_2-y & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ z-y & z-y & \cdots & x_n-y \end{vmatrix} = \prod_{i=1}^{n}(x_i-y).$$

当 $y\neq z$ 时, 将上式代入 $f(x)=ax+b$(即线性函数 $f(x)$ 过两点 $(-y,f(-y))$,$(-z,f(-z))$, 再利用两点式) 解得

$$f(x) = \frac{f(-z)-f(-y)}{-z-(-y)}(x+y) + f(-y) = \frac{\prod\limits_{i=1}^{n}(x_i-z) - \prod\limits_{i=1}^{n}(x_i-y)}{y-z}(x+y) + \prod_{i=1}^{n}(x_i-y).$$

从而此时就有

$$D_n = f(0) = \frac{y\prod\limits_{i=1}^{n}(x_i-z) - z\prod\limits_{i=1}^{n}(x_i-y)}{y-z}. \tag{4}$$

当 $y=z$ 时, 将 $D_n$ 看作关于 $y$ 的连续函数, 记为 $g(y)=D_n$, 则此时由 $g$ 的连续性及(4)式和 L'Hospital 法则可得

$$D_n = g(z) = \lim_{y\to z} g(y) = \lim_{y\to z}\frac{y\prod\limits_{i=1}^{n}(x_i-z) - z\prod\limits_{i=1}^{n}(x_i-y)}{y-z}$$

$$= \lim_{y\to z} \frac{\prod_{i=1}^{n}(x_i - z) + y\sum_{i=1}^{n}\prod_{j\neq i}(x_j - y)}{1} = \prod_{i=1}^{n}(x_i - z) + z\sum_{i=1}^{n}\prod_{j\neq i}(x_j - z).$$

□

**例题 0.1**

(1) 计算

$$|B| = \begin{vmatrix} 2a_1 & a_1+a_2 & \cdots & a_1+a_n \\ a_2+a_1 & 2a_2 & \cdots & a_2+a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+a_1 & a_n+a_2 & \cdots & 2a_n \end{vmatrix}.$$

(2) 求下列 $n$ 阶行列式的值:

$$|\boldsymbol{A}| = \begin{vmatrix} 0 & a_1+a_2 & \cdots & a_1+a_{n-1} & a_1+a_n \\ a_2+a_1 & 0 & \cdots & a_2+a_{n-1} & a_2+a_n \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1}+a_1 & a_{n-1}+a_2 & \cdots & 0 & a_{n-1}+a_n \\ a_n+a_1 & a_n+a_2 & \cdots & a_n+a_{n-1} & 0 \end{vmatrix}.$$

笔记 第 (2) 问解法一中不仅使用了升阶法还使用了分块” 爪” 型行列式的计算方法. 观察到各行各列有不同的公共项, 因此可以利用升阶法将各行各列的公共项消去.

注 因为第 (2) 问中, 当 $a_i \neq 0(i = 1,2,\cdots,n)$ 时, 最后的结果不含 $a_i$ 的分式结构, 所以当存在 $a_i = 0$, 其中 $i \in 1,2,\cdot,ns$ 时, 根据行列式 (可以看作多元多项式函数) 的连续性可知, 此时最后的结果就是将 $a_i$ 中相应为零的值代入当 $a_i \neq 0(i = 1,2,\cdots,n)$ 时的结果中. 因此我吗们可以直接不妨设 $a_i \neq 0(i = 1,2,\cdots,n)$, 只需考虑这一种情况即可.

**解**

(1) 注意到 $B = \begin{pmatrix} 2a_1 & a_1+a_2 & \cdots & a_1+a_n \\ a_2+a_1 & 2a_2 & \cdots & a_2+a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+a_1 & a_n+a_2 & \cdots & 2a_n \end{pmatrix} = \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$. 由 Cauchy-Binet 公式可知,$|B| = \begin{cases} 0, & n \geq 3, \\ -(a_1 - a_2)^2, & n = 2, \\ 2a_1, & n = 1. \end{cases}$.

(2) (i) 当 $a_i \neq 0(1 \leq i \leq n)$ 时, 解法一 (升阶法):

$$|\boldsymbol{A}| \xlongequal{\text{升阶}} \begin{vmatrix} 1 & -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 0 & 0 & a_1+a_2 & \cdots & a_1+a_{n-1} & a_1+a_n \\ 0 & a_2+a_1 & 0 & \cdots & a_2+a_{n-1} & a_2+a_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1}+a_1 & a_{n-1}+a_2 & \cdots & 0 & a_{n-1}+a_n \\ 0 & a_n+a_1 & a_n+a_2 & \cdots & a_n+a_{n-1} & 0 \end{vmatrix}$$

$$\xlongequal[i=1,2,\cdots,n+1]{r_1+r_i}\begin{vmatrix}1 & -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n\\ 1 & -a_1 & a_1 & \cdots & a_1 & a_1\\ 1 & a_2 & -a_2 & \cdots & a_2 & a_2\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 1 & a_{n-1} & a_{n-1} & \cdots & -a_{n-1} & a_{n-1}\\ 1 & a_n & a_n & \cdots & a_n & -a_n\end{vmatrix}\xlongequal{\text{升阶}}\begin{vmatrix}1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n\\ -a_1 & 1 & -a_1 & a_1 & \cdots & a_1 & a_1\\ -a_2 & 1 & a_2 & -a_2 & \cdots & a_2 & a_2\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ -a_{n-1} & 1 & a_{n-1} & a_{n-1} & \cdots & -a_{n-1} & a_{n-1}\\ -a_n & 1 & a_n & a_n & \cdots & a_n & -a_n\end{vmatrix}$$

$$\xlongequal[i=1,3,4\cdots,n+2]{j_1+j_i}\begin{vmatrix}1 & 0 & 1 & 1 & \cdots & 1 & 1\\ 0 & 1 & -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n\\ -a_1 & 1 & -2a_1 & 0 & \cdots & 0 & 0\\ -a_2 & 1 & 0 & -2a_2 & \cdots & 0 & 0_2\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ -a_{n-1} & 1 & 0 & 0 & \cdots & -2a_{n-1} & 0\\ -a_n & 1 & 0 & 0 & \cdots & 0 & -2a_n\end{vmatrix}$$

$$\xlongequal[i=3,4\cdots,n+2]{\substack{-\frac{1}{2}j_i+j_1\\ \frac{1}{2a_{i-2}}j_i+j_2}}\begin{vmatrix}1-\frac{n}{2} & \frac{S}{2} & 1 & 1 & \cdots & 1 & 1\\ \frac{T}{2} & 1-\frac{n}{2} & -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n\\ 0 & 0 & -2a_1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & -2a_2 & \cdots & 0 & 0_2\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -2a_{n-1} & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & -2a_n\end{vmatrix}.$$

其中 $S=a_1+a_2+\cdots+a_n,T=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$. 注意到上述行列式是分块上三角行列式, 从而可得

$$\begin{aligned}|\boldsymbol{A}|&=(-2)^n\prod_{i=1}^{n}a_i\cdot\frac{(n-2)^2-ST}{4}=(-2)^{n-2}\prod_{i=1}^{n}a_i[(n-2)^2-(\sum_{i=1}^{n}a_i)(\sum_{i=1}^{n}\frac{1}{a_i})]\\&=(-2)^{n-2}\prod_{i=1}^{n}a_i\,(n-2)^2-(-2)^{n-2}\sum_{i=1}^{n}a_i\sum_{j=1}^{n}\prod_{k\neq j}a_k.\end{aligned}$$

解法二 (直接计算两个矩阵和的行列式)(不推荐使用!):

设 $\boldsymbol{B}=\begin{pmatrix}2a_1 & a_1+a_2 & \cdots & a_1+a_n\\ a_2+a_1 & 2a_2 & \cdots & a_2+a_n\\ \vdots & \vdots & & \vdots\\ a_n+a_1 & a_n+a_2 & \cdots & 2a_n\end{pmatrix},\boldsymbol{C}=\begin{pmatrix}-2a_1 & & & \\ & -2a_2 & & \\ & & \ddots & \\ & & & -2a_n\end{pmatrix}$, 则 $|\boldsymbol{A}|=|\boldsymbol{B}+\boldsymbol{C}|$.

从而利用直接计算两个矩阵和的行列式的结论得到

$$|\boldsymbol{A}|=|\boldsymbol{B}|+|\boldsymbol{C}|+\sum_{1\leqslant k\leqslant n-1}\left(\sum_{\substack{1\leqslant i_1<i_2<\cdots<i_k\leqslant n\\ 1\leqslant j_1<j_2<\cdots<j_k\leqslant n}}\boldsymbol{B}\begin{pmatrix}i_1 & i_2 & \cdots & i_k\\ j_1 & j_2 & \cdots & j_k\end{pmatrix}\widehat{\boldsymbol{C}}\begin{pmatrix}i_1 & i_2 & \cdots & i_k\\ j_1 & j_2 & \cdots & j_k\end{pmatrix}\right)\quad(5)$$

其中 $\widehat{\boldsymbol{C}}\begin{pmatrix}i_1 & i_2 & \cdots & i_k\\ j_1 & j_2 & \cdots & j_k\end{pmatrix}$ 是 $\boldsymbol{C}\begin{pmatrix}i_1 & i_2 & \cdots & i_k\\ j_1 & j_2 & \cdots & j_k\end{pmatrix}$ 的代数余子式.

我们先来计算 $\boldsymbol{B}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$,$k=1,2,\cdots,n$. 拆分 $\boldsymbol{B}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ 的第一列得到

$$\boldsymbol{B}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}=\begin{vmatrix} a_{i_1}+a_{j_1} & a_{i_1}+a_{j_2} & \cdots & a_{i_1}+a_{j_k} \\ a_{i_2}+a_{j_1} & a_{i_2}+a_{j_2} & \cdots & a_{i_2}+a_{j_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k}+a_{j_1} & a_{i_k}+a_{j_2} & \cdots & a_{i_k}+a_{j_k} \end{vmatrix}$$

$$=\begin{vmatrix} a_{i_1} & a_{i_1}+a_{j_2} & \cdots & a_{i_1}+a_{j_k} \\ a_{i_2} & a_{i_2}+a_{j_2} & \cdots & a_{i_2}+a_{j_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k} & a_{i_k}+a_{j_2} & \cdots & a_{i_k}+a_{j_k} \end{vmatrix}+\begin{vmatrix} a_{j_1} & a_{i_1}+a_{j_2} & \cdots & a_{i_1}+a_{j_k} \\ a_{j_1} & a_{i_2}+a_{j_2} & \cdots & a_{i_2}+a_{j_k} \\ \vdots & \vdots & & \vdots \\ a_{j_1} & a_{i_k}+a_{j_2} & \cdots & a_{i_k}+a_{j_k} \end{vmatrix}$$

$$=\begin{vmatrix} a_{i_1} & a_{j_2} & \cdots & a_{j_k} \\ a_{i_2} & a_{j_2} & \cdots & a_{j_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k} & a_{j_2} & \cdots & a_{j_k} \end{vmatrix}+\begin{vmatrix} a_{j_1} & a_{i_1} & \cdots & a_{i_1} \\ a_{j_1} & a_{i_2} & \cdots & a_{i_2} \\ \vdots & \vdots & & \vdots \\ a_{j_1} & a_{i_k} & \cdots & a_{i_k} \end{vmatrix}$$

因此当 $k\geqslant 3$ 时,$\boldsymbol{B}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}=0$; 当 $k=2$ 时,$\boldsymbol{B}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}=\boldsymbol{B}\begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix}=\begin{vmatrix} a_{i_1} & a_{j_2} \\ a_{i_2} & a_{j_2} \end{vmatrix}+\begin{vmatrix} a_{j_1} & a_{i_1} \\ a_{j_1} & a_{i_2} \end{vmatrix}=(a_{i_1}a_{j_2}-a_{i_2}a_{j_2})(a_{i_2}a_{j_1}-a_{i_1}a_{j_1})$; 当 $k=1$ 时,$\boldsymbol{B}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}=\boldsymbol{B}\begin{pmatrix} i_1 \\ j_1 \end{pmatrix}=a_{i_1}+a_{j_1}$.

又注意到 $|\boldsymbol{C}|$ 只有主子式非零,而其主子式 $\boldsymbol{C}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}=(-2)^k a_{i_1}a_{i_2}\cdots a_{i_k}$. 于是当 $\exists m\in\{1,2,\cdots,k\}$,使得 $i_m\neq j_m$ 时,$\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}=0$; 当 $i_m\neq j_m,m=1,2,\cdots,k$ 时,$\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}=\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}=(-2)^{n-k}a_1\cdots\hat{a}_{i_1}\cdots\hat{a}_{i_2}\cdots\hat{a}_{i_k}\cdots a_n$.

故当 $n\geqslant 3$ 时,(5)式可化为

$$|\boldsymbol{A}|=|\boldsymbol{B}|+|\boldsymbol{C}|+\sum_{1\leqslant k\leqslant n-1}\left(\sum_{\substack{1\leqslant i_1<i_2<\cdots<i_k\leqslant n\\ 1\leqslant j_1<j_2<\cdots<j_k\leqslant n}}\boldsymbol{B}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}\right)$$

$$=|\boldsymbol{C}|+\sum_{\substack{1\leqslant i_1\leqslant n\\ 1\leqslant j_1\leqslant n}}\boldsymbol{B}\left(\begin{smallmatrix} i_1 \\ j_1 \end{smallmatrix}\right)\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 \\ j_1 \end{pmatrix}+\sum_{\substack{1\leqslant i_1<i_2\leqslant n\\ 1\leqslant j_1<j_2\leqslant n}}\boldsymbol{B}\begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix}\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix}$$

$$=|\boldsymbol{C}|+\sum_{1\leqslant i_1\leqslant n}\boldsymbol{B}\begin{pmatrix} i_1 \\ i_1 \end{pmatrix}\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 \\ i_1 \end{pmatrix}+\sum_{1\leqslant i_1<i_2\leqslant n}\boldsymbol{B}\begin{pmatrix} i_1 & i_2 \\ i_1 & i_2 \end{pmatrix}\widehat{\boldsymbol{C}}\begin{pmatrix} i_1 & i_2 \\ i_1 & i_2 \end{pmatrix}=|\boldsymbol{C}|+\sum_{1\leqslant i\leqslant n}\boldsymbol{B}\begin{pmatrix} i \\ i \end{pmatrix}\widehat{\boldsymbol{C}}\begin{pmatrix} i \\ i \end{pmatrix}+\sum_{1\leqslant i<j\leqslant n}\boldsymbol{B}\begin{pmatrix} i & j \\ i & j \end{pmatrix}\widehat{\boldsymbol{C}}\Big($$

$$=(-2)^n a_1a_2\cdots a_n+\sum_{1\leqslant i\leqslant n}2a_i(-2)^{n-1}a_1\cdots\hat{a}_i\cdots a_n+\sum_{1\leqslant i<j\leqslant n}[(a_ia_j-a_j^2)(a_ia_j-a_i^2)(-2)^{n-2}a_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_n]$$

$$=(-2)^n a_1a_2\cdots a_n-(-2)^n\sum_{1\leqslant i\leqslant n}a_1a_2\cdots\cdots a_n+(-2)^{n-2}\sum_{1\leqslant i<j\leqslant n}[-(a_i-a_j)^2a_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_n]$$

$$=(-2)^n a_1a_2\cdots a_n-(-2)^n na_1a_2\cdots\cdots a_n-(-2)^{n-2}\sum_{1\leqslant i<j\leqslant n}[(a_i-a_j)^2a_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_n]$$

$$=(-2)^n\prod_{i=1}^n a_i(1-n)-(-2)^{n-2}\prod_{i=1}^n a_i\sum_{1\leqslant i<j\leqslant n}\frac{(a_i-a_j)^2}{a_ia_j}$$

$$=(-2)^{n-2}\prod_{i=1}^n a_i[(n-2)^2-(\sum_{i=1}^n a_i)(\sum_{i=1}^n\frac{1}{a_i})]$$

$$
\begin{aligned}
&= (-2)^n \prod_{i=1}^{n} a_i (1-n) - (-2)^{n-2} \prod_{i=1}^{n} a_i \sum_{1\leqslant i<j\leqslant n} \frac{(a_i-a_j)^2}{a_i a_j} \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \left[ 4 - 4n - \sum_{1\leqslant i<j\leqslant n} \frac{(a_i-a_j)^2}{a_i a_j} \right] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \left[ 4 - 4n - \sum_{1\leqslant i<j\leqslant n} \left( \frac{a_j}{a_i} + \frac{a_i}{a_j} - 2 \right) \right] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \left[ 4 - 4n - \sum_{\substack{1\leqslant i,j\leqslant n \\ i\neq j}} \frac{a_i}{a_j} + \sum_{1\leqslant i<j\leqslant n} 2 \right] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \left[ 4 - 4n - \left( \sum_{1\leqslant i,j\leqslant n} \frac{a_i}{a_j} - \sum_{i=1}^{n} \frac{a_i}{a_i} \right) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2 \right] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \left[ 4 - 4n - \left( \sum_{1\leqslant i,j\leqslant n} \frac{a_i}{a_j} - n \right) + 2 \sum_{i=1}^{n-1} (n-i) \right] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \left[ 4 - 4n + n + n(n-1) - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i}{a_j} \right] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \left[ n^2 - 4n + 4 - \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \frac{1}{a_j} \right] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i [(n-2)^2 - (\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n} \frac{1}{a_i})] \\
&= (-2)^{n-2} \prod_{i=1}^{n} a_i \, (n-2)^2 - (-2)^{n-2} \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \prod_{k\neq j} a_k .
\end{aligned}
$$

解法三 (降价公式)(推荐使用!): 令 $\Lambda = \begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2a_1 & & & \\ & -2a_2 & & \\ & & \ddots & \\ & & & -2a_n \end{pmatrix}$, 则

$$
A = \begin{pmatrix} -2a_1 & & & \\ & -2a_2 & & \\ & & \ddots & \\ & & & -2a_n \end{pmatrix} + \begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix} I_2^{-1} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} = B + \Lambda I_2^{-1} \Lambda' .
$$

于是由降价公式 (打洞原理) 我们有

$$
|A| = |I| \left| B + \Lambda I_2^{-1} \Lambda' \right| = \begin{vmatrix} I_2 & \Lambda' \\ \Lambda & B \end{vmatrix} = |B| \left| I_2 - \Lambda' B^{-1} \Lambda \right|
$$

$$
= \begin{vmatrix} -2a_1 & & & \\ & -2a_2 & & \\ & & \ddots & \\ & & & -2a_n \end{vmatrix} \cdot \left| I_2 - \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} -\frac{1}{2a_1} & & & \\ & -\frac{1}{2a_2} & & \\ & & \ddots & \\ & & & -\frac{1}{2a_n} \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix} \right|
$$

$$= (-2)^n \prod_{i=1}^n a_i \left| I_2 - \begin{pmatrix} -\dfrac{1}{2a_1} & -\dfrac{1}{2a_2} & \cdots & -\dfrac{1}{2a_n} \\ -\dfrac{1}{2} & -\dfrac{1}{2} & \cdots & -\dfrac{1}{2} \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix} \right|$$

$$= (-2)^n \prod_{i=1}^n a_i \left| I_2 - \begin{pmatrix} -\dfrac{n}{2} & -\dfrac{1}{2}\sum\limits_{i=1}^n \dfrac{1}{a_i} \\ -\dfrac{1}{2}\sum\limits_{i=1}^n a_i & -\dfrac{n}{2} \end{pmatrix} \right| = (-2)^n \prod_{i=1}^n a_i \begin{vmatrix} \dfrac{n+2}{2} & \dfrac{1}{2}\sum\limits_{i=1}^n \dfrac{1}{a_i} \\ \dfrac{1}{2}\sum\limits_{i=1}^n a_i & \dfrac{n+2}{2} \end{vmatrix}$$

$$= (-2)^{n-2} \prod_{i=1}^n a_i \left[ (n+2)^2 - \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n \frac{1}{a_i}\right) \right]$$

$$= (-2)^{n-2} \prod_{i=1}^n a_i \, (n-2)^2 - (-2)^{n-2} \sum_{i=1}^n a_i \sum_{j=1}^n \prod_{k \neq j} a_k.$$

(ii) 当存在 $a_i = 0$, 其中 $i \in 1, 2, \cdot, ns$ 时, 不妨设只有 $a_{i_1}, a_{i_2}, \cdots, a_{i_m} = 0, i_1, i_2, \cdots, i_m \in 1, 2, \cdots, n$, 则可将 $|A|$ 看作关于 $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$ 连续的多元多项式函数 $g(a_{i_1}, a_{i_2}, \cdots, a_{i_m})$, 于是由 $g$ 的连续性可得

$$g(0, 0, \cdots, 0) = \lim_{(a_{i_1}, a_{i_2}, \cdots, a_{i_m}) \to (0, 0, \cdots, 0)} g(a_{i_1}, a_{i_2}, \cdots, a_{i_m})$$

$$= \lim_{(a_{i_1}, a_{i_2}, \cdots, a_{i_m}) \to (0, 0, \cdots, 0)} \left[ (-2)^{n-2} \prod_{i=1}^n a_i \, (n-2)^2 - (-2)^{n-2} \sum_{i=1}^n a_i \sum_{j=1}^n \prod_{k \neq j} a_k \right] = 0.$$

即由行列式的连续性可知

$$|A| = (-2)^{n-2} \prod_{i=1}^n a_i \, (n-2)^2 - (-2)^{n-2} \sum_{i=1}^n a_i \sum_{j=1}^n \prod_{k \neq j} a_k.$$

对某些 $a_i$ 为 0 时也成立.

□

**结论 对角矩阵行列式的子式和余子式:**

设 $|\boldsymbol{A}| = \begin{vmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix}$, 则其 $k$ 阶子式 $\boldsymbol{A}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ 除 $k$ 阶主子式 $\boldsymbol{A}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$ 外都为零, 其中 $k = 1, 2, \cdots, n$.

记 $\widehat{\boldsymbol{A}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ 为 $\boldsymbol{A}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ 的代数余子式 ($n-k$ 阶). 于是 $\widehat{\boldsymbol{A}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix}$ 除 $\widehat{\boldsymbol{A}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$ 外也都为零, 其中 $k = 1, 2, \cdots, n$.

并且

$$\boldsymbol{A}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} = a_{i_1} a_{i_2} \cdots a_{i_k},$$

$$\widehat{\boldsymbol{A}}\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ i_1 & i_2 & \cdots & i_k \end{pmatrix} = a_1 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_2} \cdots \hat{a}_{i_k} \cdots a_n$$

其中 $k = 1, 2, \cdots, n$.

**命题 0.3 (Cauchy 行列式)**

证明:

$$|\boldsymbol{A}| = \begin{vmatrix} (a_1+b_1)^{-1} & (a_1+b_2)^{-1} & \cdots & (a_1+b_n)^{-1} \\ (a_2+b_1)^{-1} & (a_2+b_2)^{-1} & \cdots & (a_2+b_n)^{-1} \\ \vdots & \vdots & & \vdots \\ (a_n+b_1)^{-1} & (a_n+b_2)^{-1} & \cdots & (a_n+b_n)^{-1} \end{vmatrix} = \frac{\prod\limits_{1\leq i<j\leq n} (a_j-a_i)(b_j-b_i)}{\prod\limits_{1\leqslant i<j\leqslant m} (a_i+b_j)}.$$

**笔记** 需要记忆 Cauchy 行列式的计算方法.

1. 分式分母有公共部分可以作差, 得到的分子会变得相对简便.

2. 行列式内行列做加减一般都是加减同一行 (或列). 但是在循环行列式中, 我们一般采取相邻两行 (或列) 相加减的方法.

**证明**

$$|\boldsymbol{A}| = \begin{vmatrix} \dfrac{1}{a_1+b_1} & \dfrac{1}{a_1+b_2} & \cdots & \dfrac{1}{a_1+b_n} \\ \dfrac{1}{a_2+b_1} & \dfrac{1}{a_2+b_2} & \cdots & \dfrac{1}{a_2+b_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{1}{a_n+b_1} & \dfrac{1}{a_n+b_2} & \cdots & \dfrac{1}{a_n+b_n} \end{vmatrix}$$

$$\xlongequal[i=n-1,\cdots,1]{-j_n+j_i} \begin{vmatrix} \dfrac{b_n-b_1}{(a_1+b_1)(a_1+b_n)} & \dfrac{b_n-b_2}{(a_1+b_2)(a_1+b_n)} & \cdots & \dfrac{b_n-b_{n-1}}{(a_1+b_{n-1})(a_1+b_n)} & \dfrac{1}{a_1+b_n} \\ \dfrac{b_n-b_1}{(a_2+b_1)(a_2+b_n)} & \dfrac{b_n-b_2}{(a_2+b_2)(a_2+b_n)} & \cdots & \dfrac{b_n-b_{n-1}}{(a_1+b_{n-1})(a_2+b_n)} & \dfrac{1}{a_2+b_n} \\ \vdots & \vdots & & \vdots & \vdots \\ \dfrac{b_n-b_1}{(a_n+b_1)(a_n+b_n)} & \dfrac{b_n-b_2}{(a_n+b_2)(a_n+b_n)} & \cdots & \dfrac{b_n-b_{n-1}}{(a_1+b_{n-1})(a_n+b_n)} & \dfrac{1}{a_n+b_n} \end{vmatrix}$$

$$= \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)} \begin{vmatrix} \dfrac{1}{a_1+b_1} & \dfrac{1}{a_1+b_2} & \cdots & \dfrac{1}{a_1+b_{n-1}} & 1 \\ \dfrac{1}{a_2+b_1} & \dfrac{1}{a_2+b_2} & \cdots & \dfrac{1}{a_2+b_{n-1}} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \dfrac{1}{a_n+b_1} & \dfrac{1}{a_n+b_2} & \cdots & \dfrac{1}{a_n+b_{n-1}} & 1 \end{vmatrix}$$

$$\xlongequal[i=n-1,\cdots,1]{-r_n+r_i} \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)} \begin{vmatrix} \dfrac{a_n-a_1}{(a_1+b_1)(a_n+b_1)} & \dfrac{a_n-a_1}{(a_1+b_2)(a_n+b_2)} & \cdots & \dfrac{a_n-a_1}{(a_1+b_{n-1})(a_n+b_{n-1})} & 0 \\ \dfrac{a_n-a_2}{(a_2+b_1)(a_n+b_1)} & \dfrac{a_n-a_2}{(a_1+b_2)(a_n+b_2)} & \cdots & \dfrac{a_n-a_2}{(a_1+b_{n-1})(a_n+b_{n-1})} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \dfrac{a_n-a_{n-1}}{(a_{n-1}+b_1)(a_n+b_1)} & \dfrac{a_n-a_{n-1}}{(a_{n-1}+b_2)(a_n+b_2)} & \cdots & \dfrac{a_n-a_{n-1}}{(a_{n-1}+b_{n-1})(a_n+b_{n-1})} & 0 \\ \dfrac{1}{a_n+b_1} & \dfrac{1}{a_n+b_2} & \cdots & \dfrac{1}{a_n+b_{n-1}} & 1 \end{vmatrix}$$

$$= \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)} \cdot \frac{\prod\limits_{i=1}^{n-1}(a_n-a_i)}{\prod\limits_{k=1}^{n-1}(a_n+b_k)} \begin{vmatrix} \dfrac{1}{a_1+b_1} & \dfrac{1}{a_1+b_2} & \cdots & \dfrac{1}{a_1+b_{n-1}} & 0 \\ \dfrac{1}{a_2+b_1} & \dfrac{1}{a_2+b_2} & \cdots & \dfrac{1}{a_2+b_{n-1}} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \dfrac{1}{a_{n-1}+b_1} & \dfrac{1}{a_{n-1}+b_2} & \cdots & \dfrac{1}{a_{n-1}+b_{n-1}} & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

$$\xlongequal{\text{按最后一列展开}} \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)(a_n-a_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)\prod\limits_{k=1}^{n-1}(a_n+b_k)} \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_{n-1}} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_{n-1}} \\ \vdots & \vdots & & \vdots \\ \frac{1}{a_{n-1}+b_1} & \frac{1}{a_{n-1}+b_2} & \cdots & \frac{1}{a_{n-1}+b_{n-1}} \end{vmatrix}$$

$$= \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)(a_n-a_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)\prod\limits_{k=1}^{n-1}(a_n+b_k)} \cdot D_{n-1}.$$

不断递推下去即得

$$D_n = \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)(a_n-a_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)\prod\limits_{k=1}^{n-1}(a_n+b_k)} \cdot D_{n-1} = \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)(a_n-a_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)\prod\limits_{k=1}^{n-1}(a_n+b_k)} \cdot \frac{\prod\limits_{i=1}^{n-2}(b_{n-1}-b_i)(a_{n-1}-a_i)}{\prod\limits_{j=1}^{n-1}\left(a_j+b_{n-1}\right)\prod\limits_{k=1}^{n-2}(a_{n-1}+b_k)} \cdot D_{n-2}$$

$$= \cdots = \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)(a_n-a_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)\prod\limits_{k=1}^{n-1}(a_n+b_k)} \cdot \frac{\prod\limits_{i=1}^{n-2}(b_{n-1}-b_i)(a_{n-1}-a_i)}{\prod\limits_{j=1}^{n-1}\left(a_j+b_{n-1}\right)\prod\limits_{k=1}^{n-2}(a_{n-1}+b_k)} \cdot \cdots \cdot \frac{\prod\limits_{i=1}^{2}(b_3-b_i)(a_3-a_i)}{\prod\limits_{j=1}^{3}\left(a_j+b_3\right)\prod\limits_{k=1}^{2}(a_3+b_k)} \cdot D_2$$

$$= \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)(a_n-a_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)\prod\limits_{k=1}^{n-1}(a_n+b_k)} \cdot \frac{\prod\limits_{i=1}^{n-2}(b_{n-1}-b_i)(a_{n-1}-a_i)}{\prod\limits_{j=1}^{n-1}\left(a_j+b_{n-1}\right)\prod\limits_{k=1}^{n-2}(a_{n-1}+b_k)} \cdots \frac{\prod\limits_{i=1}^{2}(b_3-b_i)(a_3-a_i)}{\prod\limits_{j=1}^{3}\left(a_j+b_3\right)\prod\limits_{k=1}^{2}(a_3+b_k)} \cdot \frac{(b_2-b_1)(a_2-a_1)}{\prod\limits_{j=1}^{2}\left(a_j+b_2\right)(a_2+b_1)} \cdot D_1$$

$$= \frac{\prod\limits_{i=1}^{n-1}(b_n-b_i)(a_n-a_i)}{\prod\limits_{j=1}^{n}\left(a_j+b_n\right)\prod\limits_{k=1}^{n-1}(a_n+b_k)} \cdot \frac{\prod\limits_{i=1}^{n-2}(b_{n-1}-b_i)(a_{n-1}-a_i)}{\prod\limits_{j=1}^{n-1}\left(a_j+b_{n-1}\right)\prod\limits_{k=1}^{n-2}(a_{n-1}+b_k)} \cdots \frac{\prod\limits_{i=1}^{2}(b_3-b_i)(a_3-a_i)}{\prod\limits_{j=1}^{3}\left(a_j+b_3\right)\prod\limits_{k=1}^{2}(a_3+b_k)} \cdot \frac{(b_2-b_1)(a_2-a_1)}{\prod\limits_{j=1}^{2}\left(a_j+b_2\right)(a_2+b_1)} \cdot \frac{1}{a_1+b_1}$$

$$= \frac{\prod\limits_{1\le i<j\le n}(a_j-a_i)(b_j-b_i)}{\prod\limits_{1\le i\le j\le n}(a_i+b_j)\prod\limits_{1\le j<i\le n}(a_i+b_j)} = \frac{\prod\limits_{1\le i<j\le n}(a_j-a_i)(b_j-b_i)}{\prod\limits_{1\leqslant i<j\leqslant m}(a_i+b_j)}.$$

□

**例题 0.2** 证明:

$$A = \left(\frac{1}{i+j}\right)_{1\leqslant i,j\leqslant n} \in \mathbb{R}^{n\times n}$$

是正定矩阵.

**证明** 由Cauchy 行列式可知, 对 $A$ 的所有 $m$ 阶顺序主子式, 我们都有

$$\begin{vmatrix} (1+1)^{-1} & (1+2)^{-1} & \cdots & (1+m)^{-1} \\ (2+1)^{-1} & (2+2)^{-1} & \cdots & (2+m)^{-1} \\ \vdots & \vdots & & \vdots \\ (m+1)^{-1} & (m+2)^{-1} & \cdots & (m+m)^{-1} \end{vmatrix} = \frac{\prod\limits_{1\leqslant i<j\leqslant m}(j-i)^2}{\prod\limits_{1\leqslant i<j\leqslant m}(i+j)} > 0.$$

故 $A$ 是正定矩阵. □

**例题 0.3** 设 $n$ 阶行列式

$$A_n=\begin{vmatrix} a_0+a_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_1+a_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & a_2+a_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-1}+a_n \end{vmatrix},$$

求证:

$$A_n=a_0a_1\cdots a_n\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right).$$

**笔记** 用**数学归纳法**证明与行列式有关的结论.

练习0.1和练习0.2都可同理使用用数学归纳法证明 (对阶数 $n$ 进行归纳即可).

**证明** (数学归纳法) 对阶数 $n$ 进行归纳. 当 $n=1,2$ 时, 结论显然成立. 假设阶数小于 $n$ 结论成立.

现证明 $n$ 阶的情形. 注意到

$$A_n=\begin{vmatrix} a_0+a_1 & a_1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_1+a_2 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & a_2+a_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-1}+a_n \end{vmatrix}=(a_{n-1}+a_n)A_{n-1}-a_{n-1}^2A_{n-2}.$$

将归纳假设代入上面的式子中得

$$\begin{aligned} A_n&=(a_{n-1}+a_n)A_{n-1}-a_{n-1}^2A_{n-2}\\ &=(a_{n-1}+a_n)a_0a_1\cdots a_{n-1}\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_{n-1}}\right)-a_{n-1}^2a_0a_1\cdots a_{n-2}\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_{n-2}}\right)\\ &=a_0a_1\cdots a_n\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_{n-1}}\right)+a_0a_1\cdots a_{n-2}a_{n-1}^2\frac{1}{a_{n-1}}\\ &=a_0a_1\cdots a_{n-1}\left[a_n\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_{n-1}}\right)+1\right]\\ &=a_0a_1\cdots a_{n-1}a_n\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_{n-1}}+\frac{1}{a_n}\right). \end{aligned}$$

故由数学归纳法可知, 结论对任意正整数 $n$ 都成立. □

**例题 0.4** 设 $n(n>2)$ 阶行列式 $|A|$ 的所有元素或为 1 或为 $-1$, 求证:$|A|$ 的绝对值小于等于 $\frac{2}{3}n!$.

**解** 对阶数 $n$ 进行归纳. 当 $n=3$ 时, 将 $|A|$ 的第一列元素为-1 的行都乘以-1, 再将 $|A|$ 的第一行元素为 1 的列都乘以-1,$|A|$ 的绝对值不改变.

因此不妨设 $|A|=\begin{vmatrix} 1 & -1 & -1 \\ 1 & a_0 & b_0 \\ 1 & c_0 & d_0 \end{vmatrix}$, 其中$a_0,b_0,c_0,d_0=1$或$-1$.

从而

$$|A|=\begin{vmatrix} 1 & -1 & -1 \\ 1 & a_0 & b_0 \\ 1 & c_0 & d_0 \end{vmatrix}\xlongequal[i=2,3]{j_1+j_i}\begin{vmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{vmatrix}, \text{其中} a,b,c,d=0\text{或}2.$$

于是

$$abs\left(|A|\right)=abs\left(\begin{vmatrix} 1 & 0 & 0 \\ 1 & a & b \\ 1 & c & d \end{vmatrix}\right)=abs\left(ad-bc\right)\leqslant 4=\frac{2}{3}\cdot 3!$$

假设 n-1 阶时结论成立, 现证 $n$ 阶的情形. 将 $|A|$ 按第一行展开得

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}, \text{其中} a_{1i} = 1 \text{或} -1\,(i = 1, 2\cdots, n).$$

从而由归纳假设可得

$$\begin{aligned} abs\,(|A|) &= abs\,(a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}) \leqslant abs\,(A_{11}) + abs\,(A_{12}) + \cdots + abs\,(A_{1n}) \\ &\leqslant \frac{2}{3}(n-1)! + \frac{2}{3}(n-1)! + \cdots + \frac{2}{3}(n-1)! \\ &= n \cdot \frac{2}{3}(n-1)! = \frac{2}{3}n!. \end{aligned}$$

故由数学归纳法可知结论对任意正整数都成立. □

**命题 0.4 (行列式的求导运算)**

设 $f_{ij}(t)$ 是可微函数,

$$F(t) = \begin{vmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2n}(t) \\ \vdots & \vdots & & \vdots \\ f_{n1}(t) & f_{n2}(t) & \cdots & f_{nn}(t) \end{vmatrix}$$

求证:$\dfrac{d}{dt}F(t) = \sum\limits_{j=1}^{n} F_j(t)$, 其中

$$F_j(t) = \begin{vmatrix} f_{11}(t) & f_{12}(t) & \cdots & \frac{d}{dt}f_{1j}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & \frac{d}{dt}f_{2j}(t) & \cdots & f_{2n}(t) \\ \vdots & \vdots & & \vdots & & \vdots \\ f_{n1}(t) & f_{n2}(t) & \cdots & \frac{d}{dt}f_{nj}(t) & \cdots & f_{nn}(t) \end{vmatrix}$$

证明 证法一 (数学归纳法):对阶数 $n$ 进行归纳. 当 $n=1$ 时结论显然成立. 假设 $n-1$ 阶时结论成立, 现证 $n$ 阶的情形.

将 $F(t)$ 按第一列展开得

$$F(t) = f_{11}(t)A_{11}(t) + f_{21}(t)A_{21}(t) + \cdots + f_{n1}(t)A_{n1}(t).$$

其中 $A_{i1}(t)$ 是元素 $f_{i1}(t)$ 的代数余子式.$(i = 1, 2, \cdots, n)$

从而由归纳假设可得

$$A_{i1}'(t) = \frac{d}{dt}A_{i1}(t) = \sum_{k=2}^{n} A_{i1}^{k}(t), i = 1, 2, \cdots, n.$$

$$\text{其中} A_{i1}^{k}(t) = \begin{vmatrix} f_{12}(t) & \cdots & \frac{d}{dt}f_{1k}(t) & \cdots & f_{1n}(t) \\ \vdots & & \vdots & & \vdots \\ f_{i-1,2}(t) & \cdots & \frac{d}{dt}f_{i-1,k}(t) & \cdots & f_{i-1,n}(t) \\ f_{i+1,2}(t) & \cdots & \frac{d}{dt}f_{i+1,k}(t) & \cdots & f_{i+1,n}(t) \\ \vdots & & \vdots & & \vdots \\ f_{n2}(t) & \cdots & \frac{d}{dt}f_{nk}(t) & \cdots & f_{nn}(t) \end{vmatrix}, k = 2, 3, \cdots, n.$$

于是, 我们就有

$$\begin{aligned} \frac{d}{dt}F(t) &= \frac{d}{dt}\left[f_{11}(t)A_{11}(t) + f_{21}(t)A_{21}(t) + \cdots + f_{n1}(t)A_{n1}(t)\right] \\ &= f_{11}'(t)A_{11}(t) + f_{21}'(t)A_{21}(t) + \cdots + f_{n1}'(t)A_{n1}(t) + f_{11}(t)A_{11}'(t) + f_{21}(t)A_{21}'(t) + \cdots + f_{n1}(t)A_{n1}'(t) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} f_{i1}'(t) A_{i1}(t) + f_{11}(t) \sum_{k=2}^{n} A_{11}^{k}(t) + f_{21}(t) \sum_{k=2}^{n} A_{21}^{k}(t) + \cdots + f_{n1}(t) \sum_{k=2}^{n} A_{n1}^{k}(t) \\
&= \sum_{i=1}^{n} f_{i1}'(t) A_{i1}(t) + \sum_{i=1}^{n} \left( f_{i1}(t) \sum_{k=2}^{n} A_{i1}^{k}(t) \right) \\
&= \sum_{i=1}^{n} f_{i1}'(t) A_{i1}(t) + \sum_{i=1}^{n} f_{i1}(t) \left( A_{i1}^{2} + A_{i1}^{3} + \cdots + A_{i1}^{n} \right) \\
&= \sum_{i=1}^{n} f_{i1}'(t) A_{i1}(t) + \sum_{i=1}^{n} f_{i1}(t) A_{i1}^{2} + \sum_{i=1}^{n} f_{i1}(t) A_{i1}^{3} + \cdots + \sum_{i=1}^{n} f_{i1}(t) A_{i1}^{n} \\
&= F_1(t) + F_2(t) + F_3(t) + \cdots + F_n(t) \\
&= \sum_{j=1}^{n} F_j(t).
\end{aligned}
$$

故由数学归纳法可知结论对任意正整数都成立.

证法二 (行列式的组合定义):由行列式的组合定义可得

$$
F(t) = \sum_{1 \le k_1, k_2, \cdots, k_n \le n} (-1)^{\tau(k_1 k_2 \cdots k_n)} f_{k_1 1}(t) f_{k_2 2}(t) \cdots f_{k_n n}(t).
$$

因此

$$
\begin{aligned}
\frac{d}{dt} F(t) = &\sum_{1 \le k_1, k_2, \cdots, k_n \le n} (-1)^{\tau(k_1 k_2 \cdots k_n)} f_{k_{11}}(t) f_{k_{22}}(t) \cdots f_{k_{nn}}(t) \\
&+ \sum_{1 \le k_1, k_2, \cdots, k_n \le n} (-1)^{\tau(k_1 k_2 \cdots k_n)} f_{k_{11}}(t) f\prime_{k_{22}}(t) \cdots f_{k_{nn}}(t) \\
&+ \cdots + \sum_{1 \le k_1, k_2, \cdots, k_n \le n} (-1)^{\tau(k_1 k_2 \cdots k_n)} f_{k_{11}}(t) f_{k_{22}}(t) \cdots f\prime_{k_{nn}}(t) \\
= &F_1(t) + F_2(t) + \cdots + F_n(t).
\end{aligned}
$$

□