0.1 Fourier 积分不等式

定理 0.1 (Fourier 积分不等式)

若 $f(x) \in C^1[a,b]$, 则

(1)

$$\int_a^b |f(x)|^2 \mathrm{d}x - \frac{1}{b-a}\left(\int_a^b f(x)\mathrm{d}x\right)^2 \leq \frac{(b-a)^2}{\pi^2}\int_a^b |f'(x)|^2 \mathrm{d}x,$$

等号成立条件为

$$f(x) = c_1 + c_2 \cos\left(\frac{\pi(x-a)}{b-a}\right), c_1, c_2 \in \mathbb{R}.$$

(2) 若 $f(a) = f(b)$, 则

$$\int_a^b |f(x)|^2 \mathrm{d}x - \frac{1}{b-a}\left(\int_a^b f(x)\mathrm{d}x\right)^2 \leq \frac{(b-a)^2}{4\pi^2}\int_a^b |f'(x)|^2 \mathrm{d}x,$$

等号成立条件是

$$f(x) = c_1 + c_2 \cos\left(\frac{2\pi x}{b-a}\right) + c_3 \sin\left(\frac{2\pi x}{b-a}\right), c_1, c_2, c_3 \in \mathbb{R}.$$

(3) 若 $f(a) = f(b) = 0$, 则

$$\int_a^b |f(x)|^2 \mathrm{d}x \leq \frac{(b-a)^2}{\pi^2}\int_a^b |f'(x)|^2 \mathrm{d}x,$$

等号成立条件是

$$f(x) = c \sin\left(\frac{\pi(x-a)}{b-a}\right), c \in \mathbb{R}.$$

♡

证明

(1) 把 $f(x)$ 延拓到 $[2a-b, b]$, 使得 $f(x) = f(2a-x), x \in [a,b)$, 则 $f(b) = f(2a-b), f \in C[2a-b,b]$ 且分段可微, 因此设 $f(x)$ 有傅立叶级数

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n(x-a)}{b-a}\right),$$

进而由 Fourier 级数的逐项微分定理可得

$$f'(x) \sim -\frac{\pi}{b-a}\sum_{n=1}^{\infty}[n a_n \sin\left(\frac{\pi n(x-a)}{b-a}\right)].$$

这里

$$a_n = \frac{1}{b-a}\int_{2a-b}^b f(x)\cos\left(\frac{\pi n(x-a)}{b-a}\right)\mathrm{d}x, n \in \mathbb{N}_0.$$

我们由 Parseval 等式可得

$$\int_{2a-b}^b |f(x)|^2 \mathrm{d}x = (b-a)\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right],$$

$$\int_{2a-b}^b |f'(x)|^2 \mathrm{d}x = \frac{\pi^2}{b-a}\sum_{n=1}^{\infty} n^2 a_n^2.$$

从而有

$$\int_{2a-b}^b |f(x)|^2 \mathrm{d}x - \frac{1}{2(b-a)}\left(\int_{2a-b}^b f(x)\mathrm{d}x\right)^2 \leq \frac{(b-a)^2}{\pi^2}\int_{2b-a}^b |f'(x)|^2 \mathrm{d}x.$$

利用对称, 就有

$$\int_a^b |f(x)|^2 \mathrm{d}x - \frac{1}{(b-a)}\left(\int_a^b f(x)\mathrm{d}x\right)^2 \leq \frac{(b-a)^2}{\pi^2}\int_a^b |f'(x)|^2 \mathrm{d}x,$$

等号成立条件为

$$f(x) = c_1 + c_2 \cos\left(\frac{\pi(x-a)}{b-a}\right), c_1, c_2 \in \mathbb{R}.$$

(2) 设

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi nx}{b-a}\right) + b_n \sin\left(\frac{2\pi nx}{b-a}\right)\right),$$
$$f'(x) \sim \frac{2\pi}{b-a}\sum_{n=1}^{\infty}\left(-na_n \sin\left(\frac{2\pi nx}{b-a}\right) + nb_n \cos\left(\frac{2\pi nx}{b-a}\right)\right),$$

这里

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2\pi nx}{b-a}\right)\mathrm{d}x,$$
$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2\pi nx}{b-a}\right)\mathrm{d}x.$$

由 Parseval 等式, 我们有

$$\int_a^b |f(x)|^2\mathrm{d}x = \frac{b-a}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right],$$
$$\int_a^b |f'(x)|^2\mathrm{d}x = \frac{2\pi^2}{b-a}\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2).$$

因此

$$\int_a^b |f(x)|^2\mathrm{d}x - \frac{1}{b-a}\left(\int_a^b f(x)\mathrm{d}x\right)^2 \le \frac{(b-a)^2}{4\pi^2}\int_a^b |f'(x)|^2\mathrm{d}x,$$

等号成立条件是

$$f(x) = c_1 + c_2\cos\left(\frac{2\pi x}{b-a}\right) + c_3\sin\left(\frac{2\pi x}{b-a}\right).$$

(3) 令

$$f(x) = -f(2a-x), x \in [2a-b, a),$$

则 $f(x) \in C^1[2a-b, b]$. 设 $f(x)$ 有傅立叶级数

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n(x-a)}{b-a}\right),$$
$$f'(x) \sim \frac{\pi}{b-a}\sum_{n=1}^{\infty} nb_n \cos\left(\frac{\pi n(x-a)}{b-a}\right),$$

这里

$$b_n = \frac{1}{b-a}\int_{2a-b}^b f(x)\sin\left(\frac{\pi n(x-a)}{b-a}\right)\mathrm{d}x, n \in \mathbb{N}_0.$$

我们由 Parseval 等式可得

$$\int_{2a-b}^b |f(x)|^2\mathrm{d}x = (b-a)\sum_{n=1}^{\infty} b_n^2,$$
$$\int_{2a-b}^b |f'(x)|^2\mathrm{d}x = \frac{\pi^2}{b-a}\sum_{n=1}^{\infty} n^2 b_n^2.$$

从而有

$$\int_{2a-b}^b |f(x)|^2\mathrm{d}x \le \frac{(b-a)^2}{\pi^2}\int_{2b-a}^b |f'(x)|^2\mathrm{d}x.$$

利用对称, 我们有

$$\int_a^b |f(x)|^2\mathrm{d}x \le \frac{(b-a)^2}{\pi^2}\int_a^b |f'(x)|^2\mathrm{d}x,$$

等号成立条件是

$$f(x) = c\sin\left(\frac{\pi(x-a)}{b-a}\right).$$

□