0.1 **Cauchy-Riemann 方程**

> **定义 0.1**
>
> 设 $f(z) = u(x,y) + iv(x,y)$ 是定义在域 D 上的函数,$z_0 = x_0 + iy_0 \in D$. 我们说 f 在 z_0 处**实可微**,是指 u 和 v 作为 x, y 的二元函数在 (x_0, y_0) 处可微.
>
> ♣

> **命题 0.1**
>
> 设 $f : D \to \mathbb{C}$ 是定义在域 D 上的函数,$z_0 \in D$,那么 f 在 z_0 处实可微的充分必要条件是
>
> $$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \overline{z}}(z_0)\overline{\Delta z} + o(|\Delta z|). \tag{1}$$
>
> 成立, 其中
>
> $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
> $$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$
>
> ♠

证明 设 f 在 z_0 处实可微, 由二元实值函数可微的定义, 有

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|), \tag{2}$$

$$v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|), \tag{3}$$

这里,$|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. 于是

$$
\begin{aligned}
f(z_0 + \Delta z) - f(z_0) &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)) \\
&= \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|) + i\left(\frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|)\right) \\
&= \left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right)\Delta x + \left(\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)\right)\Delta y + o(|\Delta z|) \\
&= \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|).
\end{aligned}
$$

把 $\Delta x = \frac{1}{2}(\Delta z + \overline{\Delta z}), \Delta y = \frac{1}{2i}(\Delta z - \overline{\Delta z})$ 代入上式, 得

$$
\begin{aligned}
f(z_0 + \Delta z) - f(z_0) &= \frac{1}{2}\frac{\partial f}{\partial x}(x_0, y_0)(\Delta z + \overline{\Delta z}) - \frac{i}{2}\frac{\partial f}{\partial y}(x_0, y_0)(\Delta z - \overline{\Delta z}) + o(|\Delta z|) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f(x_0, y_0)\Delta z + \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f(x_0, y_0)\overline{\Delta z} + o(|\Delta z|).
\end{aligned}
$$

引进算子

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{4}$$

则上式可写为

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \overline{z}}(z_0)\overline{\Delta z} + o(|\Delta z|). \tag{5}$$

容易看出,(5)式和(2),(3)两式等价. □

注 为什么要像(4)式那样来定义算子 $\frac{\partial}{\partial z}$ 和 $\frac{\partial}{\partial \overline{z}}$ 呢？这是因为如果把复变函数 $f(z)$ 写成

$$f(x, y) = f\left(\frac{z + \overline{z}}{2}, -i\frac{z - \overline{z}}{2}\right),$$

把 $z, \bar{z}$ 看成独立变量, 分别对 z 和 $\bar{z}$ 求偏导数, 则得

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - \mathrm{i} \frac{\partial f}{\partial y} \right),$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + \mathrm{i} \frac{\partial f}{\partial y} \right).$$

这就是表达式(4)的来源. 这说明在进行微分运算时, 可以把 $z, \bar{z}$ 看成独立的变量.

现在很容易得到 f 在 z_0 处可微的条件了.

定理 0.1

设 f 是定义在域 D 上的函数, $z_0 \in D$, 那么 f 在 z_0 处可微的充要条件是 f 在 z_0 处实可微且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. 在可微的情况下, $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

证明 如果 f 在 z_0 处可微, 由(**??**)式得

$$f(z_0 + \Delta z) - f(z_0) = f'(z_0) \Delta z + o(|\Delta z|)$$

与(1)式比较就知道, f 在 z_0 处是实可微的, 而且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0, f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

反之, 若 f 在 z_0 处实可微, 且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$, 则由(1)式得

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0) \Delta z + o(|\Delta z|)$$

由此即知

$$\lim_{\Delta z} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial f}{\partial z}(z_0).$$

故 f 在 z_0 处可微, 而且 $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$. $\square$

定义 0.2 (Cauchy-Riemann 方程)

设 f 是定义在域 D 上的函数, $\frac{\partial f}{\partial \bar{z}} = 0$ 称为 **Cauchy – Riemann 方程**.

✍ **笔记** 从这个方程可以得到 f 的实部和虚部应满足的条件.

命题 0.2 (Cauchy-Riemann 方程的等价定义)

设 $z = x + \mathrm{i}y, f(z) = u(x, y) + \mathrm{i}v(x, y)$, 则 Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 等价于

(i)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases} \tag{6}$$

(ii)

$$\frac{\partial \overline{f}}{\partial z} = 0.$$

(iii) 令 $x = r\cos\theta, y = r\sin\theta$, 进而 $z = r(\cos\theta + \mathrm{i}\sin\theta), f(z) = u(r, \theta) + \mathrm{i}v(r, \theta)$, 则 Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 等价于

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r} \dfrac{\partial v}{\partial \theta}, \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r} \dfrac{\partial u}{\partial \theta}. \end{cases}$$

证明 (i) 由(4)式得

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial u}{\partial \overline{z}} + \mathrm{i}\frac{\partial v}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial u}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial v}{\partial x} + \mathrm{i}\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

因此,Cauchy-Riemann 方程 $\dfrac{\partial f}{\partial \overline{z}} = 0$ 就等价于

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases}$$

(ii) 又注意到

$$\frac{\partial \overline{f}}{\partial z} = \frac{\partial u}{\partial z} - \mathrm{i}\frac{\partial v}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}\right) - \frac{\mathrm{i}}{2}\left(\frac{\partial v}{\partial x} - \mathrm{i}\frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

故 Cauchy-Riemann 方程 $\dfrac{\partial f}{\partial \overline{z}} = 0$ 也等价于 $\dfrac{\partial \overline{f}}{\partial z} = 0$.

(iii) 令 $F = x - r\cos\theta, G = y - r\sin\theta$, 则

$$\begin{pmatrix} F_x & F_y & F_r & F_\theta \\ G_x & G_y & G_r & G_\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\cos\theta & r\sin\theta \\ 0 & 1 & -\sin\theta & -r\cos\theta \end{pmatrix}.$$

直接计算 Jacobi 行列式得

$$J = \frac{\partial(F,G)}{\partial(r,\theta)} = \begin{vmatrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix} = r,$$

于是

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,\theta)}\right] + \frac{\partial u}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,x)}\right] = \frac{\partial u}{\partial r}\cdot\cos\theta - \frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{r};$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,\theta)}\right] + \frac{\partial u}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,y)}\right] = \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r};$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,\theta)}\right] + \frac{\partial v}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,x)}\right] = \frac{\partial v}{\partial r}\cdot\cos\theta - \frac{\partial v}{\partial \theta}\cdot\frac{\sin\theta}{r};$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,\theta)}\right] + \frac{\partial v}{\partial \theta}\cdot\left[-\frac{1}{J}\frac{\partial(F,G)}{\partial(r,y)}\right] = \frac{\partial v}{\partial r}\cdot\sin\theta + \frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r}.$$

从而

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \iff \frac{\partial u}{\partial r}\cdot\cos\theta - \frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{r} = \frac{\partial v}{\partial r}\cdot\sin\theta + \frac{\partial v}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

$$\iff \frac{\partial u}{\partial r}\cdot r\cos\theta - \frac{\partial u}{\partial \theta}\cdot\sin\theta = \frac{\partial v}{\partial r}\cdot r\sin\theta + \frac{\partial v}{\partial \theta}\cdot\cos\theta,$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \iff \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cdot\cos\theta + \frac{\partial v}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$\iff \frac{\partial u}{\partial r}\cdot r\sin\theta + \frac{\partial u}{\partial \theta}\cdot\cos\theta = -\frac{\partial v}{\partial r}\cdot r\cos\theta + \frac{\partial v}{\partial \theta}\cdot\sin\theta.$$

由 (i) 可知 Cauchy-Riemann 方程等价于 $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, 故此时 Cauchy-Riemann 方程等价于

$$\begin{cases} \dfrac{\partial u}{\partial r}\cdot r\cos\theta - \dfrac{\partial u}{\partial \theta}\cdot\sin\theta = \dfrac{\partial v}{\partial r}\cdot r\sin\theta + \dfrac{\partial v}{\partial \theta}\cdot\cos\theta, \\ \dfrac{\partial u}{\partial r}\cdot r\sin\theta + \dfrac{\partial u}{\partial \theta}\cdot\cos\theta = -\dfrac{\partial v}{\partial r}\cdot r\cos\theta + \dfrac{\partial v}{\partial \theta}\cdot\sin\theta. \end{cases}$$

化简可得

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}. \end{cases}$$

□

定理 0.2

设 $f = u + \mathrm{i}v$ 是定义在域 D 上的函数, $z_0 = x_0 + \mathrm{i}y_0 \in D$, 那么 f 在 z_0 处可微的充要条件是 $u(x,y), v(x,y)$ 在 (x_0, y_0) 处可微, 且在 (x_0, y_0) 处满足 Cauchy-Riemann 方程, 即

$$\frac{\partial f}{\partial \overline{z}} = 0, \qquad \frac{\partial \overline{f}}{\partial z} = 0, \qquad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\[2mm] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}.$$

在可微的情况下, 有

$$f'(z_0) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}.$$

这里的偏导数都在 (x_0, y_0) 处取值.

♡

证明 最后这个 $f'(z_0)$ 的表达式是从 定理 0.1 中的 $f'(z_0) = \dfrac{\partial f}{\partial z}(z_0)$ 和 Cauchy-Riemann 方程的等价定义得到的. □

定义 0.3

1. 设 D 是 $\mathbb{C}$ 中的域, 我们用 $C(D)$ 记 D 上连续函数的全体, 用 $H(D)$ 记 D 上全纯函数的全体.
2. 设 $f = u + \mathrm{i}v$, 记 $\dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + \mathrm{i}\dfrac{\partial v}{\partial x}, \dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + \mathrm{i}\dfrac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体.
3. 用 $C^k(D)$ 记在 D 上有 k 阶连续偏导数的函数的全体, $C^\infty(D)$ 记在 D 上有任意阶连续偏导数的函数的全体.

♣

命题 0.3

(1) $H(D) \subset C(D)$.

(2) $C^1(D) \subset C(D)$.

(3) 域 D 上的全纯函数在 D 上有任意阶的连续偏导数, 并且有如下的包含关系:

$$H(D) \subset C^\infty(D) \subset C^k(D) \subset C^1(D) \subset C(D).$$

这里, k 是大于 1 的自然数.

♠

证明

(1) 命题 **??** 告诉我们, $H(D) \subset C(D)$.

(2) 设 $f = u + \mathrm{i}v$, 记 $\dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + \mathrm{i}\dfrac{\partial v}{\partial x}, \dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + \mathrm{i}\dfrac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体. 进而 u, v 关于 x, y 的偏导在 D 上都连续, 由多元微积分的知识知道 u, v 在 D 上都可微. 于是对于任意 $f \in C^1(D), f$ 在 D 上实可微, 从 (5) 式知道

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \overline{z}}(z_0)\overline{\Delta z} + o(|\Delta z|).$$

令 $\Delta z \to 0$, 则 $\lim\limits_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$, 故 f 在 D 上连续, 因而 $C^1(D) \subset C(D)$.

(3)

□

例题 0.1 研究函数 $f(z) = z^n$, n 是自然数.

解 显然, $\dfrac{\partial f}{\partial \overline{z}} = 0$, 且 f 在整个平面上是实可微的. 因而, f 是 $\mathbb{C}$ 上的全纯函数, 而且

$$f'(z) = \frac{\partial f}{\partial z} = nz^{n-1}. \tag{7}$$

□

例题 0.2 研究函数 $f(z) = \mathrm{e}^{-|z|^2}$.

解 把 f 写为 $f(z) = \mathrm{e}^{-z\overline{z}}$, 于是 $\dfrac{\partial f}{\partial \overline{z}} = -\mathrm{e}^{-z\overline{z}}z$, 它只有在 $z = 0$ 处才等于零. 因此, $\mathrm{e}^{-|z|^2}$ 只有在 $z = 0$ 处可微, 它在任

何点处都不是全纯的. 但它对 x, y 有任意阶连续偏导数, 所以它是 $C^\infty(\mathbb{C})$ 中的函数. □

定义 0.4 (调和函数)

设 u 是域 D 上的实值函数, 如果 $u \in C^2(D)$, 且对任意 $z \in D$, 有

$$\Delta u(z) = \frac{\partial^2 u(z)}{\partial x^2} + \frac{\partial^2 u(z)}{\partial y^2} = 0, \tag{8}$$

就称 u 是 D 中的**调和函数**. $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ 称为 **Laplace 算子**.

命题 0.4

设 $u \in C^2(D)$, 那么 $\Delta u = 4 \dfrac{\partial^2 u}{\partial z \partial \overline{z}}$.

证明 由(4)式, 有

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - \mathrm{i} \frac{\partial}{\partial y} \right),$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + \mathrm{i} \frac{\partial}{\partial y} \right),$$

所以

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = \frac{\partial}{\partial \overline{z}} \frac{\partial u}{\partial z} = \frac{1}{4} \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} - \mathrm{i} \frac{\partial u}{\partial y} \right) + \mathrm{i} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} - \mathrm{i} \frac{\partial u}{\partial y} \right) \right]$$

$$= \frac{1}{4} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{1}{4} \Delta u.$$

□

定理 0.3

设 $f = u + \mathrm{i}v \in H(D)$, 那么 u 和 v 都是 D 上的调和函数.

证明 因为 $f \in H(D)$, 由定理 0.2, 有

$$\frac{\partial f}{\partial \overline{z}} = 0, \quad \frac{\partial \overline{f}}{\partial z} = 0.$$

所以

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} = \frac{\partial^2 \overline{f}}{\partial z \partial \overline{z}} = 0.$$

于是, 由 $u = \dfrac{1}{2}(f + \overline{f})$ 即得

$$\Delta u = 4 \frac{\partial^2 u}{\partial z \partial \overline{z}} = 0.$$

同理可证 $\Delta v = 0$. □

定义 0.5 (共轭调和函数)

设 u 和 v 是 D 上的一对调和函数, 如果它们还满足 Cauchy-Riemann 方程

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \end{cases} \tag{9}$$

就称 v 为 u 的**共轭调和函数**.

> **命题 0.5**
>
> 全纯函数的实部和虚部就构成一对共轭调和函数. ♠

证明 由定理 0.3 和定理 0.2 立得. □

> **定理 0.4**
>
> 设 u 是单连通域 D 上的调和函数, 则必存在 u 的共轭调和函数 v, 使得 $u + \mathrm{i}v$ 是 D 上的全纯函数. ♡

证明 因为 u 满足 Laplace 方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

若令 $P = -\dfrac{\partial u}{\partial y}, Q = \dfrac{\partial u}{\partial x}$, 则

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial y},$$

所以

$$P\mathrm{d}x + Q\mathrm{d}y = -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$$

是一个全微分, 因而积分

$$\int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y$$

与路径无关. 令

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y,$$

则

$$
\begin{aligned}
v(x,y) &= \int_{(x_0,y_0)}^{(x,y_0)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y + \int_{(x,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y \\
&= \int_{x_0}^{x} -\frac{\partial u}{\partial y}\mathrm{d}x + 0 + 0 + \int_{y_0}^{y} \frac{\partial u}{\partial x}\mathrm{d}y \\
&= \int_{x_0}^{x} -\frac{\partial u}{\partial y}\mathrm{d}x + \int_{y_0}^{y} \frac{\partial u}{\partial x}\mathrm{d}y.
\end{aligned}
$$

那么

$$
\begin{cases}
\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}, \\
\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}.
\end{cases}
$$

所以, v 就是要求的 u 的共轭调和函数. □