0.1 Cauchy-Riemann 方程

定义 0.1

设 $f(z)=u(x,y)+\mathrm{i}v(x,y)$ 是定义在域 D 上的函数,$z_0=x_0+\mathrm{i}y_0\in D$. 我们说 f 在 z_0 处**实可微**, 是指 u 和 v 作为 x,y 的二元函数在 (x_0,y_0) 处可微.

命题 0.1

设 $f:D\to\mathbb{C}$ 是定义在域 D 上的函数,$z_0\in D$, 那么 f 在 z_0 处实可微的充分必要条件是

$$f(z_0+\Delta z)-f(z_0)=\frac{\partial f}{\partial z}(z_0)\Delta z+\frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z}+o(|\Delta z|). \tag{1}$$

成立, 其中

$$\begin{aligned}\frac{\partial}{\partial z}&=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right),\\ \frac{\partial}{\partial \bar{z}}&=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right).\end{aligned}$$

证明 设 f 在 z_0 处实可微, 由二元实值函数可微的定义, 有

$$u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)=\frac{\partial u}{\partial x}(x_0,y_0)\Delta x+\frac{\partial u}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|), \tag{2}$$

$$v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0)=\frac{\partial v}{\partial x}(x_0,y_0)\Delta x+\frac{\partial v}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|), \tag{3}$$

这里,$|\Delta z|=\sqrt{(\Delta x)^2+(\Delta y)^2}$. 于是

$$\begin{aligned}f(z_0+\Delta z)-f(z_0)&=u(x_0+\Delta x,y_0+\Delta y)-u(x_0,y_0)+\mathrm{i}(v(x_0+\Delta x,y_0+\Delta y)-v(x_0,y_0))\\ &=\frac{\partial u}{\partial x}(x_0,y_0)\Delta x+\frac{\partial u}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|)+\mathrm{i}\left(\frac{\partial v}{\partial x}(x_0,y_0)\Delta x+\frac{\partial v}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|)\right)\\ &=\left(\frac{\partial u}{\partial x}(x_0,y_0)+\mathrm{i}\frac{\partial v}{\partial x}(x_0,y_0)\right)\Delta x+\left(\frac{\partial u}{\partial y}(x_0,y_0)+\mathrm{i}\frac{\partial v}{\partial y}(x_0,y_0)\right)\Delta y+o(|\Delta z|)\\ &=\frac{\partial f}{\partial x}(x_0,y_0)\Delta x+\frac{\partial f}{\partial y}(x_0,y_0)\Delta y+o(|\Delta z|).\end{aligned}$$

把 $\Delta x=\frac{1}{2}(\Delta z+\overline{\Delta z})$,$\Delta y=\frac{1}{2\mathrm{i}}(\Delta z-\overline{\Delta z})$ 代入上式, 得

$$\begin{aligned}f(z_0+\Delta z)-f(z_0)&=\frac{1}{2}\frac{\partial f}{\partial x}(x_0,y_0)(\Delta z+\overline{\Delta z})-\frac{\mathrm{i}}{2}\frac{\partial f}{\partial y}(x_0,y_0)(\Delta z-\overline{\Delta z})+o(|\Delta z|)\\ &=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right)f(x_0,y_0)\Delta z+\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right)f(x_0,y_0)\overline{\Delta z}+o(|\Delta z|).\end{aligned}$$

引进算子

$$\begin{aligned}\frac{\partial}{\partial z}&=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right),\\ \frac{\partial}{\partial \bar{z}}&=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right),\end{aligned} \tag{4}$$

则上式可写为

$$f(z_0+\Delta z)-f(z_0)=\frac{\partial f}{\partial z}(z_0)\Delta z+\frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z}+o(|\Delta z|). \tag{5}$$

容易看出,(5)式和(2),(3)两式等价. □

注 为什么要像(4)式那样来定义算子 $\frac{\partial}{\partial z}$ 和 $\frac{\partial}{\partial \bar{z}}$ 呢？这是因为如果把复变函数 $f(z)$ 写成

$$f(x,y)=f\left(\frac{z+\bar{z}}{2},-\mathrm{i}\frac{z-\bar{z}}{2}\right),$$

把 $z, \bar{z}$ 看成独立变量, 分别对 z 和 $\bar{z}$ 求偏导数, 则得

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right).$$

这就是表达式(4)的来源. 这说明在进行微分运算时, 可以把 $z, \bar{z}$ 看成独立的变量.

现在很容易得到 f 在 z_0 处可微的条件了.

定理 0.1

设 f 是定义在域 D 上的函数,$z_0 \in D$, 那么 f 在 z_0 处可微的充要条件是 f 在 z_0 处实可微且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. 在可微的情况下,$f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

证明 如果 f 在 z_0 处可微, 由(??)式得

$$f(z_0 + \Delta z) - f(z_0) = f'(z_0)\Delta z + o(|\Delta z|)$$

与(1)式比较就知道,f 在 z_0 处是实可微的, 而且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0, f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

反之, 若 f 在 z_0 处实可微, 且 $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$, 则由(1)式得

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + o(|\Delta z|)$$

由此即知

$$\lim_{\Delta z}\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial f}{\partial z}(z_0).$$

故 f 在 z_0 处可微, 而且 $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$. □

定义 0.2 (Cauchy-Riemann 方程)

设 f 是定义在域 D 上的函数,$\frac{\partial f}{\partial \bar{z}} = 0$ 称为 **Cauchy − Riemann 方程**, 从这个方程可以得到 f 的实部和虚部应满足的条件. 设 $f = u + iv$, 则由(4)式得

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + \mathrm{i}\frac{\partial v}{\partial \bar{z}} \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial u}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \mathrm{i}\frac{\partial v}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\end{aligned}$$

因此,Cauchy-Riemann 方程 $\frac{\partial f}{\partial \bar{z}} = 0$ 就等价于

$$\begin{cases}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.\end{cases} \tag{6}$$

定理 0.2

设 $f = u + iv$ 是定义在域 D 上的函数,$z_0 = x_0 + iy_0 \in D$, 那么 f 在 z_0 处可微的充要条件是 $u(x, y)$,$v(x, y)$ 在 (x_0, y_0) 处可微, 且在 (x_0, y_0) 处满足

$$\begin{cases}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\end{cases}$$

在可微的情况下, 有

$$f'(z_0)=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-\mathrm{i}\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}-\mathrm{i}\frac{\partial u}{\partial y}.$$

这里的偏导数都在 (x_0, y_0) 处取值.

证明 最后这个 $f'(z_0)$ 的表达式是从 定理 0.1中的 $f'(z_0)=\dfrac{\partial f}{\partial z}(z_0)$ 和 Cauchy-Riemann 方程(6)得到的. □

定义 0.3

1. 设 D 是 $\mathbb{C}$ 中的域, 我们用 $C(D)$ 记 D 上连续函数的全体, 用 $H(D)$ 记 D 上全纯函数的全体.
2. 设 $f=u+\mathrm{i}v$, 记 $\dfrac{\partial f}{\partial x}=\dfrac{\partial u}{\partial x}+\mathrm{i}\dfrac{\partial v}{\partial x}, \dfrac{\partial f}{\partial y}=\dfrac{\partial u}{\partial y}+\mathrm{i}\dfrac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体.
3. 用 $C^k(D)$ 记在 D 上有 k 阶连续偏导数的函数的全体,$C^\infty(D)$ 记在 D 上有任意阶连续偏导数的函数的全体.

命题 0.2

(1) $H(D)\subset C(D)$.

(2) $C^1(D)\subset C(D)$.

(3) 域 D 上的全纯函数在 D 上有任意阶的连续偏导数, 并且有如下的包含关系:

$$H(D)\subset C^\infty(D)\subset C^k(D)\subset C^1(D)\subset C(D)$$

这里,k 是大于 1 的自然数.

证明

(1) 命题??告诉我们,$H(D)\subset C(D)$.

(2) 设 $f=u+\mathrm{i}\mathrm{i}v$, 记 $\dfrac{\partial f}{\partial x}=\dfrac{\partial u}{\partial x}+\mathrm{i}\dfrac{\partial v}{\partial x}, \dfrac{\partial f}{\partial y}=\dfrac{\partial u}{\partial y}+\mathrm{i}\dfrac{\partial v}{\partial y}$. 我们用 $C^1(D)$ 记 $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ 在 D 上连续的 f 的全体. 进而 u, v 关于 x, y 的偏导在 D 上都连续, 由多元微积分的知识知道,u, v 在 D 上都可微. 于是对于任意 $f\in C^1(D)$,f 在 D 上实可微, 从(5)式知道

$$f(z_0+\Delta z)-f(z_0)=\frac{\partial f}{\partial z}(z_0)\Delta z+\frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z}+o(|\Delta z|).$$

令 $\Delta z\to 0$, 则 $\lim\limits_{\Delta z\to 0} f(z_0+\Delta z)=f(z_0)$, 故 f 在 D 上连续, 因而 $C^1(D)\subset C(D)$.

(3)

□

例题 0.1

证明

□