0.0.1 定积分

0.0.2 建立积分递推

例题 0.1 计算 $\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)dx, n \in \mathbb{N}$.

证明 利用分部积分和和差化积公式可得

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx)\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x[\sin(nx)\cos x + \cos(nx)\sin x]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(nx)\mathrm{d}\cos x \\
&= \frac{I_{n-1}+I_n}{2} - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos(nx)\mathrm{d}\cos^n x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
&= \frac{I_{n-1}}{2} + \frac{1}{2n}.
\end{aligned}$$

故 $I_n = \frac{I_{n-1}}{2} + \frac{1}{2n}$, 则两边同乘 2^n（强行裂项）

$$2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \cdots$$

又注意到 $I_0 = 0$, 从而

$$2^n I_n = 0 + \sum_{k=1}^{n} \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n}\sum_{k=1}^{n} \frac{2^{k-1}}{k}.$$

□

例题 0.2

1. $\int_0^{\pi} \frac{\sin(nx)}{\sin x}\mathrm{d}x = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$
2.

证明

1. 记 $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx$, 则

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin((n+2)x) - \sin(nx)}{\sin x} dx = \int_0^{\pi} \frac{2\cos((n+1)x)\sin x}{\sin x} dx = 2\int_0^{\pi} \cos((n+1)x)\, dx = 0.$$

于是

$$\int_0^{\pi} \frac{\sin(nx)}{\sin x} dx = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$$

2.

□

0.0.3 区间再现

定理 0.1 (区间再现恒等式)

当下述积分有意义时, 我们有

1.
$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\mathrm{d}x.$$

2.
$$\int_0^{\infty} f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^{\infty} f(x)\mathrm{d}x = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2}\right]\mathrm{d}x.$$

笔记 注意: 倒代换具有将 $[0,1]$ 转化为 $[1,+\infty)$ 的功能.

证明 证明是显然. □

例题 0.3 证明

1. $\displaystyle\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
2. $\displaystyle\int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x - \frac{\pi}{2}\ln 2.$
3. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln 2.$

证明

1.
$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

2.
$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

3.
$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &\xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2}\mathrm{d}\tan\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\cdot\ln(1+\tan\theta)}{\sec^2\theta}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\frac{2}{1+\tan\theta}\right]\mathrm{d}\theta
\end{aligned}$$

$$= \int_0^{\frac{\pi}{8}} \ln 2 \mathrm{d}\theta = \frac{\pi}{8} \ln 2.$$

□

例题 0.4 计算

1. $\int_0^{\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x, a>0.$
2. $\int_0^{\infty} \frac{\ln x}{x^2+x+1} \mathrm{d}x.$
3. $\int_0^{1} \frac{\ln x}{\sqrt{x-x^2}} \mathrm{d}x.$

解

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x \xlongequal{x=at} \frac{1}{a}\int_0^{+\infty} \frac{\ln(at)}{1+t^2} \mathrm{d}t = \frac{1}{a}\int_0^{+\infty} \frac{\ln a}{1+t^2} \mathrm{d}t + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t = \frac{\pi \ln a}{2a} + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t. \quad (1)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t \xlongequal{t=\frac{1}{x}} \int_0^{+\infty} \frac{\ln \frac{1}{x}}{1+\frac{1}{x^2}} \frac{1}{x^2} \mathrm{d}x = \int_0^{+\infty} \frac{-\ln x}{1+x^2} \mathrm{d}x \Longrightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t = 0.$$

于是再结合(1)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2} \mathrm{d}x = \frac{\pi \ln a}{2a}.$$

2.

$$\int_0^{\infty} \frac{\ln x}{x^2+x+1} \mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_0^{\infty} \frac{-\ln t}{1+\frac{1}{t}+\frac{1}{t^2}} \mathrm{d}\frac{1}{t} = \int_0^{+\infty} \frac{-\ln t}{1+t+t^2} \mathrm{d}t \Longrightarrow \int_0^{\infty} \frac{\ln x}{x^2+x+1} \mathrm{d}x = 0.$$

3.

$$\int_0^{1} \frac{\ln x}{\sqrt{x-x^2}} \mathrm{d}x \xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}} \frac{\ln \sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}} \mathrm{d}\sin^2 y$$

$$= 4\int_0^{\frac{\pi}{2}} \ln \sin y \mathrm{d}y \xlongequal{\text{例题 0.3}} 4 \cdot \left(-\frac{\pi}{2}\ln 2\right) = -2\pi \ln 2.$$

□

例题 0.5

1. 对 $n \in \mathbb{N}$, 计算 $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx.$
2. $\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} dx.$
3. 对 $n \in \mathbb{N}$, 计算 $\int_0^{2\pi} \sin(\sin x + nx) dx.$

解

1.

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} \mathrm{d}x = \int_{-\pi}^{0} \left[\frac{\sin(nx)}{(1+2^x)\sin x} + \frac{\sin(nx)}{(1+2^{-x})\sin x}\right] \mathrm{d}x = \int_{-\pi}^{0} \frac{\sin(nx)}{\sin x}\left(\frac{1}{1+2^x} + \frac{1}{1+2^{-x}}\right) \mathrm{d}x$$

$$= \int_{-\pi}^{0} \frac{\sin(nx)}{\sin x} \cdot \frac{2+2^x+2^{-x}}{2+2^x+2^{-x}} \mathrm{d}x = \int_{-\pi}^{0} \frac{\sin(nx)}{\sin x} \mathrm{d}x = \int_0^{\pi} \frac{\sin(nx)}{\sin x} \mathrm{d}x \xlongequal{\text{例题 0.2}} \begin{cases} 0, n\text{为偶数} \\ \pi, n\text{为奇数} \end{cases}.$$

2.

$$\int_{-\pi}^{\pi} \frac{x \sin x \operatorname{arc\,tan} e^x}{1+\cos^2 x} \mathrm{d}x = \int_{-\pi}^{0} \left(\frac{x \sin x \operatorname{arc\,tan} e^x}{1+\cos^2 x} + \frac{x \sin x \operatorname{arc\,tan} e^{-x}}{1+\cos^2 x}\right) \mathrm{d}x = \int_{-\pi}^{0} \frac{x \sin x}{1+\cos^2 x} (\operatorname{arc\,tan} e^x + \operatorname{arc\,tan} e^{-x}) \mathrm{d}x$$

$$\xlongequal{\text{命题??(1)}} \int_{-\pi}^{0} \frac{x \sin x}{1+\cos^2 x} \cdot \frac{\pi}{2} \mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \left(\frac{x \sin x}{1+\cos^2 x} + \frac{(\pi-x)\sin x}{1+\cos^2 x}\right) \mathrm{d}x = \frac{\pi^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} \mathrm{d}x$$

$$= \frac{\pi^2}{2}\arctan\cos x\Big|_{\frac{\pi}{2}}^{0} = \frac{\pi^2}{2}\cdot\frac{\pi}{4} = \frac{\pi^3}{8}.$$

3.

$$\begin{aligned}\int_0^{2\pi}\sin(\sin x + nx)\,\mathrm{d}x &= \int_0^{2\pi}\sin[\sin(2\pi - x) + n(2\pi - x)]\,\mathrm{d}x\\ &= \int_0^{2\pi}\sin(-\sin x - nx)\,\mathrm{d}x = -\int_0^{2\pi}\sin(\sin x + nx)\,\mathrm{d}x\\ &\Longrightarrow \int_0^{2\pi}\sin(\sin x + nx)\,\mathrm{d}x = 0.\end{aligned}$$

$\square$

0.0.4 化成含参积分/多元累次积分 (换序)

命题 0.1

证明:

1. $\displaystyle\int_0^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.
2. $\displaystyle\int_0^{\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.
3. $\displaystyle\int_0^{\infty}\sin x^2\mathrm{d}x, \int_0^{\infty}\cos x^2\mathrm{d}x = \sqrt{\frac{\pi}{8}}$.

笔记 本结果可以直接使用.

证明

1. 注意到

$$\left(\int_0^{+\infty} e^{-x^2}\mathrm{d}x\right)^2 = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \int_0^{\frac{\pi}{2}}\int_0^{+\infty} re^{-r^2}\mathrm{d}r\mathrm{d}\theta = \frac{\pi}{4}\int_0^{+\infty} e^{-r^2}\mathrm{d}r^2 = \frac{\pi}{4}.$$

故 $\displaystyle\int_0^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

2. 注意到

$$\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x - yx}\,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{-(y-\mathrm{i})x}\,\mathrm{d}x = \mathrm{Im}\frac{1}{y-\mathrm{i}} = \mathrm{Im}\frac{y+\mathrm{i}}{y^2+1} = \frac{1}{y^2+1}.$$

因此就有

$$\begin{aligned}\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x &= \int_0^{+\infty}\sin x\left(\int_0^{+\infty} e^{-yx}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x\\ &= \int_0^{+\infty}\mathrm{d}y\left(\mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\right)\mathrm{d}x = \int_0^{+\infty}\frac{1}{y^2+1}\,\mathrm{d}y = \frac{\pi}{2}.\end{aligned}$$

当然本题也可以直接利用分部积分计算 $\displaystyle\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x = \frac{1}{y^2+1}$.

3. 注意到

$$\int_0^{+\infty} e^{-ax^2}\,\mathrm{d}x \xlongequal{x=\frac{t}{\sqrt{a}}} \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{a}},\quad a > 0.$$

并且 $-\mathrm{i} = e^{-\frac{\pi}{2}\mathrm{i}}$, 从而 $\sqrt{-\mathrm{i}} = e^{-\frac{\pi}{4}\mathrm{i}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}$. 于是

$$\begin{aligned}\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2)\,\mathrm{d}x &= \int_0^{+\infty} e^{-\mathrm{i}x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\mathrm{i}}} = \frac{1}{2}\sqrt{-\mathrm{i}\pi}\\ &= \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}.\end{aligned}$$

故

$$\int_0^{+\infty}\cos x^2\,\mathrm{d}x=\mathrm{Re}\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x=\mathrm{Re}\left(\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}=\sqrt{\frac{\pi}{8}},$$
$$\int_0^{+\infty}\sin x^2\,\mathrm{d}x=\mathrm{Im}\int_0^{+\infty}(\cos x^2-\mathrm{i}\sin x^2)\,\mathrm{d}x=\mathrm{Im}\left(\frac{\sqrt{2\pi}}{4}-\frac{\sqrt{2\pi}}{4}\mathrm{i}\right)=\frac{\sqrt{2\pi}}{4}=\sqrt{\frac{\pi}{8}}.$$

□

例题 0.6 计算 $\int_0^1\sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}dx\ (b>a>0)$.

证明

$$\int_0^1\sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x=\int_0^1\sin\ln\frac{1}{x}\left(\int_a^b x^y\mathrm{d}y\right)\mathrm{d}x=\int_a^b\mathrm{d}y\int_0^1 x^y\sin\ln\frac{1}{x}\mathrm{d}x$$
$$\xlongequal{x=e^{-t}}\int_a^b\mathrm{d}y\int_{+\infty}^0 e^{-ty}\sin t\mathrm{d}e^{-t}=\int_a^b\mathrm{d}y\int_0^{+\infty}e^{-t(y+1)}\sin t\mathrm{d}t$$
$$\xlongequal{\text{命题 0.1(2) 的证明过程}}\int_a^b\frac{1}{1+(y+1)^2}\mathrm{d}y=\arctan(b+1)-\arctan(a+1).$$

□