0.1 定积分

0.1.1 建立积分递推

例题 0.1 计算 $\int_0^{\frac{\pi}{2}} \cos^n x\sin(nx)\mathrm{d}x, n\in\mathbb{N}$.

证明 利用分部积分和和差化积公式可得

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x\sin(nx)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x\cdot\cos x\sin(nx)\mathrm{d}x\\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x\cdot[\sin((n+1)x)+\sin((n-1)x)]\mathrm{d}x\\
&= \frac{I_{n-1}}{2}+\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x[\sin(nx)\cos x+\cos(nx)\sin x]\mathrm{d}x\\
&= \frac{I_{n-1}}{2}+\frac{I_n}{2}-\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x\cos(nx)\mathrm{d}\cos x\\
&= \frac{I_{n-1}+I_n}{2}-\frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos(nx)\mathrm{d}\cos^n x\\
&= \frac{I_{n-1}+I_n}{2}+\frac{1}{2n}-\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^n x\sin(nx)\mathrm{d}x\\
&= \frac{I_{n-1}+I_n}{2}+\frac{1}{2n}-\frac{I_n}{2}\\
&= \frac{I_{n-1}}{2}+\frac{1}{2n}.
\end{aligned}$$

故 $I_n=\frac{I_{n-1}}{2}+\frac{1}{2n}$, 则两边同乘 2^n(强行裂项)

$$2^nI_n=2^{n-1}I_{n-1}+\frac{2^{n-1}}{n}, n=1,2,\cdots$$

又注意到 $I_0=0$, 从而

$$2^nI_n=0+\sum_{k=1}^{n}\frac{2^{k-1}}{k}\Rightarrow I_n=\frac{1}{2^n}\sum_{k=1}^{n}\frac{2^{k-1}}{k}.$$

□

命题 0.1

证明:

(1) $\int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x=\begin{cases}0, & n\text{为偶数}\\ \pi, & n\text{为奇数}\end{cases}$.

(2) $\int_0^{\pi}\frac{\sin^2(nx)}{\sin^2 x}\mathrm{d}x=n\pi$.

(3) $\int_0^{\pi}\frac{\sin^2(nx)}{\sin x}\mathrm{d}x=\sum_{k=1}^{n}\frac{2}{2k-1}$.

笔记 提示:$\sin^2 x-\sin^2 y=\sin(x-y)\sin(x+y)$(证明见命题**??**).

证明

(1) 记 $I_n=\int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x$, 则

$$I_{n+2}-I_n=\int_0^{\pi}\frac{\sin((n+2)x)-\sin(nx)}{\sin x}\mathrm{d}x=\int_0^{\pi}\frac{2\cos((n+1)x)\sin x}{\sin x}\mathrm{d}x=2\int_0^{\pi}\cos((n+1)x)\mathrm{d}x=0.$$

于是

$$\int_0^\pi \frac{\sin(nx)}{\sin x}\,\mathrm{d}x = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$$

(2) 记 $I_n = \int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x}\,\mathrm{d}x \\ &= \int_0^\pi \frac{\sin((2n+1)x)}{\sin x}\,\mathrm{d}x \xlongequal{\text{命题 0.1(1)}} \pi. \end{aligned} \tag{1}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记 $I_n = \int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x$, 则

$$\begin{aligned} I_{n+1} - I_n &= \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin x}\,\mathrm{d}x \\ &= \int_0^\pi \sin((2n+1)x)\,\mathrm{d}x = \frac{1}{2n+1}\cos((2n+1)x)\Big|_\pi^0 = \frac{2}{2n+1}. \end{aligned} \tag{2}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1}\frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1}\frac{2}{2k+1} = \sum_{k=1}^{n}\frac{2}{2k-1}.$$

□

例题 0.2 设 $a > 1$, 计算积分 $\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x$.

注 很多情况下不需求出被积函数的原函数, 只需充分利用换元、分部积分以及被积函数的性质, 即可求出积分的值. 见下述解法二.

解 解法一:设 $a_0 = a > 1$. 构造数列如下:

$$a_{n+1} = 2a_n^2 - 1 \quad (n = 0, 1, \cdots),$$

则由例题**??**可知, 存在 $x_0 > 0$ 使得

$$a_0 = \mathrm{ch}(x_0), \quad a_n = \mathrm{ch}(2^n x_0),$$

其中 $\mathrm{ch}(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$. 可以解得

$$x_0 = \ln\left(a_0 + \sqrt{a_0^2 - 1}\right). \tag{3}$$

故

$$a_n = \frac{\mathrm{e}^{2^n x_0} + \mathrm{e}^{-2^n x_0}}{2}.$$

设

$$I_n = \int_0^\pi \ln(a_n - \cos x)\,\mathrm{d}x,$$

则

$$\begin{aligned} I_0 &= \int_0^\pi \ln(a_0 - \cos x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^\pi \ln(a_0 - \cos x)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \ln(a_0 + \cos x)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \ln(a_0^2 - \cos^2 x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\left(a_0^2 - \frac{1 + \cos 2x}{2}\right)\mathrm{d}x \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{a_1 - \cos 2x}{2}\right) \mathrm{d}x = \frac{1}{2}\int_0^{\pi} \ln\left(\frac{a_1 - \cos x}{2}\right) \mathrm{d}x = \frac{1}{2}I_1 - \frac{\pi}{2}\ln 2.$$

同理, 有

$$I_n = \frac{1}{2}I_{n+1} - \frac{\pi}{2}\ln 2. \tag{4}$$

由此递推公式, 可得

$$I_0 = \frac{1}{2^n}I_n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)\frac{\pi}{2}\ln 2. \tag{5}$$

因为

$$\begin{aligned} I_n &= \int_0^{\pi} \ln(a_n - \cos x)\,\mathrm{d}x = \int_0^{\pi} \ln\left(\frac{\mathrm{e}^{2^n x_0} + \mathrm{e}^{-2^n x_0}}{2} - \cos x\right) \mathrm{d}x \\ &= 2^n x_0 \pi + \int_0^{\pi} \ln\left(\frac{1 + \mathrm{e}^{-2^{n+1} x_0}}{2} - \mathrm{e}^{-2^n x_0}\cos x\right) \mathrm{d}x, \end{aligned}$$

所以

$$\frac{1}{2^n}I_n \to x_0\pi \quad (n \to +\infty).$$

故从式 (5) 可得

$$I_0 = x_0\pi - \pi\ln 2 = \pi\ln\left(\frac{a_0 + \sqrt{a_0^2 - 1}}{2}\right),$$

即所求的积分为

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right).$$

解法二:我们有

$$F(a) = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x = \int_0^{\pi} \ln(a - \cos x)\,\mathrm{d}x.$$

由定理??, 关于 a 求导得到

$$F'(a) = \int_0^{\pi} \frac{1}{a - \cos x}\,\mathrm{d}x \xlongequal{\text{万能公式}} \int_0^{+\infty} \frac{2}{a(1+t^2) - (1-t^2)}\,\mathrm{d}t = \frac{\pi}{\sqrt{a^2 - 1}}, \quad a > 1.$$

因此

$$F(a) = \int_1^a F'(a)\,\mathrm{d}a = \pi\ln\left(a + \sqrt{a^2 - 1}\right) + C, \quad a > 1.$$

结合

$$F(1) = 2\int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x = -\pi\ln 2.$$

可得

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right), \quad a > 1.$$

□

定理 0.1 (周期函数在任意一个周期上的定积分相同)

设 f 是 $\mathbb{R}$ 上的可积函数且周期为 $T > 0$, 则对 $\forall a \in \mathbb{R}$, 都有

$$\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x.$$

♡

证明 由条件可得

$$\int_a^{a+T} f(x)\mathrm{d}x = \int_a^{([\frac{a}{T}]+1)T} f(x)\mathrm{d}x + \int_{([\frac{a}{T}]+1)T}^{a+T} f(x)\mathrm{d}x$$

$$
\begin{aligned}
&= \int_a^{(\lfloor \frac{a}{T} \rfloor+1)T} f(x)\mathrm{d}x + \int_{\lfloor \frac{a}{T} \rfloor T}^{a} f(x+T)\mathrm{d}x \\
&= \int_a^{(\lfloor \frac{a}{T} \rfloor+1)T} f(x)\mathrm{d}x + \int_{\lfloor \frac{a}{T} \rfloor T}^{a} f(x)\mathrm{d}x \\
&= \int_{\lfloor \frac{a}{T} \rfloor T}^{(\lfloor \frac{a}{T} \rfloor+1)T} f(x)\mathrm{d}x \\
&= \int_0^T f\left(x + \lfloor \frac{a}{T} \rfloor T\right)\mathrm{d}x.
\end{aligned}
$$

□

定理 0.2 (点火公式)

1. 记 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\mathrm{d}x, \forall n \in \mathbb{N}$, 则

$$I_n = \frac{n-1}{n} I_{n-2}, \quad \forall n \geqslant 2.$$

从而

$$I_n = \begin{cases} \frac{(n-1)!!}{n!!} I_0 = \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & , n\text{为偶数} \\ \frac{(n-1)!!}{n!!} I_1 = \frac{(n-1)!!}{n!!} & , n\text{为奇数} \end{cases}. \tag{6}$$

2. 记 $J(m,n) = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\mathrm{d}x, \forall n,m \in \mathbb{N}$, 则

$$J(m,n) = \frac{m-1}{m+n} J(m-2,n), \quad \forall n,m \geqslant 2.$$

$$J(m,n) = \frac{n-1}{m+n} J(m,n-2), \quad \forall n,m \geqslant 2.$$

从而

$$J(m,n) = \begin{cases} \frac{(m-1)!!(n-1)!!}{(m+n)!!} & , m,n\text{不全为偶数} \\ \frac{(m-1)!!(n-1)!!}{(m+n)!!} \cdot \frac{\pi}{2} & , m,n\text{全为偶数} \end{cases}. \tag{7}$$

3.

0.1.2 区间再现

定理 0.3 (区间再现恒等式)

当下述积分有意义时, 我们有

1.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\mathrm{d}x.$$

2.

$$\int_0^{\infty} f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^{\infty} f(x)\mathrm{d}x = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2}\right]\mathrm{d}x.$$

笔记 注意: 倒代换具有将 $[0,1]$ 转化为 $[1,+\infty)$ 的功能.

证明 证明是显然的.(第 1 问中最后一个等号是由轴对称得到的)

□

命题 0.2

证明

1. $\displaystyle\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
2. $\displaystyle\int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.$
3. $\displaystyle\int_0^{1} \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\ln 2.$

证明

1.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

2.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

3. 解法一:

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &\xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2}\mathrm{d}\tan\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\cdot\ln(1+\tan\theta)}{\sec^2\theta}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\frac{2}{1+\tan\theta}\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \ln 2\mathrm{d}\theta = \frac{\pi}{8}\ln 2.
\end{aligned}$$

解法二:考虑含参量积分

$$\varphi(\alpha) = \int_0^1 \frac{\ln(1+\alpha x)}{1+x^2}\mathrm{d}x, \quad \alpha\in[0,1].$$

显然 $\varphi(0)=0,\varphi(1)=I$, 且函数 $\dfrac{\ln(1+\alpha x)}{1+x^2}$ 在 $R=[0,1]\times[0,1]$ 上满足定理**??** 的条件, 于是

$$\varphi'(\alpha) = \int_0^1 \frac{x}{(1+x^2)(1+\alpha x)}\mathrm{d}x.$$

因为

$$\frac{x}{(1+x^2)(1+\alpha x)} = \frac{1}{1+\alpha^2}\left(\frac{\alpha+x}{1+x^2} - \frac{\alpha}{1+\alpha x}\right),$$

所以

$$\begin{aligned}\varphi'(\alpha) &= \frac{1}{1+\alpha^2}\left(\int_0^1 \frac{\alpha}{1+x^2}\,\mathrm{d}x + \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x - \int_0^1 \frac{\alpha}{1+\alpha x}\,\mathrm{d}x\right)\\ &= \frac{1}{1+\alpha^2}\left[\alpha \arctan x\Big|_0^1 + \frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1 - \ln(1+\alpha x)\Big|_0^1\right]\\ &= \frac{1}{1+\alpha^2}\left[\alpha\cdot\frac{\pi}{4} + \frac{1}{2}\ln 2 - \ln(1+\alpha)\right].\end{aligned}$$

因此

$$\begin{aligned}\int_0^1 \varphi'(\alpha)\,\mathrm{d}\alpha &= \int_0^1 \frac{1}{1+\alpha^2}\left[\frac{\pi}{4}\alpha + \frac{1}{2}\ln 2 - \ln(1+\alpha)\right]\mathrm{d}\alpha\\ &= \frac{\pi}{8}\ln\left(1+\alpha^2\right)\Big|_0^1 + \frac{1}{2}\ln 2 \arctan\alpha\Big|_0^1 - \varphi(1)\\ &= \frac{\pi}{8}\ln 2 + \frac{\pi}{8}\ln 2 - \varphi(1)\\ &= \frac{\pi}{4}\ln 2 - \varphi(1).\end{aligned}$$

另一方面,

$$\int_0^1 \varphi'(\alpha)\,d\alpha = \varphi(1) - \varphi(0) = \varphi(1),$$

所以 $I = \varphi(1) = \frac{\pi}{8}\ln 2.$

□

例题 0.3 计算

1. $\int_0^{\infty} \frac{\ln x}{x^2+a^2}\mathrm{d}x, a>0.$
2. $\int_0^{\infty} \frac{\ln x}{x^2+x+1}\mathrm{d}x.$
3. $\int_0^1 \frac{\ln x}{\sqrt{x-x^2}}\mathrm{d}x.$

解

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2}\mathrm{d}x \xlongequal{x=at} \frac{1}{a}\int_0^{+\infty} \frac{\ln(at)}{1+t^2}\mathrm{d}t = \frac{1}{a}\int_0^{+\infty} \frac{\ln a}{1+t^2}\mathrm{d}t + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t = \frac{\pi\ln a}{2a} + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t. \quad (8)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t \xlongequal{t=\frac{1}{x}} \int_0^{+\infty} \frac{\ln\frac{1}{x}}{1+\frac{1}{x^2}}\frac{1}{x^2}\mathrm{d}x = \int_0^{+\infty} \frac{-\ln x}{1+x^2}\mathrm{d}x \Longrightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t = 0.$$

于是再结合(8)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2}\mathrm{d}x = \frac{\pi\ln a}{2a}.$$

2.
$$\int_0^{\infty} \frac{\ln x}{x^2+x+1}\mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_0^{\infty} \frac{-\ln t}{1+\frac{1}{t}+\frac{1}{t^2}}\mathrm{d}\frac{1}{t} = \int_0^{+\infty} \frac{-\ln t}{1+t+t^2}\mathrm{d}t \Longrightarrow \int_0^{\infty} \frac{\ln x}{x^2+x+1}\mathrm{d}x = 0.$$

3.
$$\begin{aligned}\int_0^1 \frac{\ln x}{\sqrt{x-x^2}}\mathrm{d}x &\xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}} \frac{\ln\sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}}\mathrm{d}\sin^2 y\\ &= 4\int_0^{\frac{\pi}{2}} \ln\sin y\mathrm{d}y \xlongequal{\text{命题 0.2}} 4\cdot\left(-\frac{\pi}{2}\ln 2\right) = -2\pi\ln 2.\end{aligned}$$

□

例题 0.4

1. 对 $n\in\mathbb{N}$, 计算 $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}\mathrm{d}x.$

2. $\displaystyle\int_{-\pi}^{\pi}\frac{x\sin x\arctan e^x}{1+\cos^2 x}\mathrm{d}x$.
3. 对 $n\in\mathbb{N}$, 计算 $\displaystyle\int_0^{2\pi}\sin(\sin x+nx)\mathrm{d}x$.

解

1.

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{\sin(nx)}{(1+2^x)\sin x}\mathrm{d}x&=\int_{-\pi}^{0}\left[\frac{\sin(nx)}{(1+2^x)\sin x}+\frac{\sin(nx)}{(1+2^{-x})\sin x}\right]\mathrm{d}x=\int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\left(\frac{1}{1+2^x}+\frac{1}{1+2^{-x}}\right)\mathrm{d}x\\&=\int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\cdot\frac{2+2^x+2^{-x}}{2+2^x+2^{-x}}\mathrm{d}x=\int_{-\pi}^{0}\frac{\sin(nx)}{\sin x}\mathrm{d}x=\int_0^{\pi}\frac{\sin(nx)}{\sin x}\mathrm{d}x\xlongequal{\text{例题 0.1}}\begin{cases}0, n\text{为偶数}\\ \pi, n\text{为奇数}\end{cases}.\end{aligned}$$

2.

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{x\sin x\arctan e^x}{1+\cos^2 x}\mathrm{d}x&=\int_{-\pi}^{0}\left(\frac{x\sin x\arctan e^x}{1+\cos^2 x}+\frac{x\sin x\arctan e^{-x}}{1+\cos^2 x}\right)\mathrm{d}x=\int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}\left(\arctan e^x+\arctan e^{-x}\right)\mathrm{d}x\\&\xlongequal{\text{命题??(1)}}\int_{-\pi}^{0}\frac{x\sin x}{1+\cos^2 x}\cdot\frac{\pi}{2}\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x\\&=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\frac{x\sin x}{1+\cos^2 x}+\frac{(\pi-x)\sin x}{1+\cos^2 x}\right)\mathrm{d}x=\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x\\&=\frac{\pi^2}{2}\arctan\cos x\Big|_{\frac{\pi}{2}}^{0}=\frac{\pi^2}{2}\cdot\frac{\pi}{4}=\frac{\pi^3}{8}.\end{aligned}$$

3.

$$\begin{aligned}\int_0^{2\pi}\sin(\sin x+nx)\,\mathrm{d}x&=\int_0^{2\pi}\sin[\sin(2\pi-x)+n(2\pi-x)]\,\mathrm{d}x\\&=\int_0^{2\pi}\sin(-\sin x-nx)\,\mathrm{d}x=-\int_0^{2\pi}\sin(\sin x+nx)\,\mathrm{d}x\\&\Longrightarrow\int_0^{2\pi}\sin(\sin x+nx)\,\mathrm{d}x=0.\end{aligned}$$

□

例题 0.5 计算 $\displaystyle\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2)(1+x^{2019})}$.

解

$$\begin{aligned}\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2)(1+x^{2019})}&=\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^2)(1+x^{2019})}\\&\xlongequal{t=\frac{1}{x}}\int_0^{+\infty}\frac{t^{2019}\mathrm{d}t}{(1+t^2)(1+t^{2019})}=\frac{1}{2}\int_0^{+\infty}\frac{(1+x^{2019})\,\mathrm{d}x}{(1+x^2)(1+x^{2019})}\\&=\frac{1}{2}\int_0^{+\infty}\frac{\mathrm{d}x}{1+x^2}=\frac{\pi}{4}.\end{aligned}$$

□

0.1.3 Frullani(傅汝兰尼) 积分

定理 0.4 (Frullani(傅汝兰尼) 积分)

设 $f\in C(0,+\infty)$.

1. 若存在极限

$$\lim_{x\to 0^+}f(x),\ \lim_{x\to+\infty}f(x),\tag{9}$$

则对 $a,b>0$ 有

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to 0^+}f(x)-\lim_{x\to+\infty}f(x)\right]\ln\frac{b}{a}.$$

2. 若存在极限和积分

$$\lim_{x\to0^+}f(x)=\alpha,\ \int_A^{\infty}\frac{f(x)}{x}\mathrm{d}x. \tag{10}$$

则对 $a,b>0$, 有

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\alpha\ln\frac{b}{a}.$$

3. 若存在极限和积分

$$\lim_{x\to+\infty}f(x)=\alpha,\ \int_0^{1}\frac{f(x)}{x}\mathrm{d}x. \tag{11}$$

则对 $a,b>0$, 有

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\alpha\ln\frac{a}{b}.$$

4. 若 f 是周期 $T>0$ 函数且 $\lim\limits_{x\to0^+}f(x)$ 存在, 则对 $a,b>0$ 有

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to0^+}f(x)-\frac{1}{T}\int_0^T f(x)\mathrm{d}x\right]\ln\frac{b}{a}.$$

5. 若 f 满足 $\lim\limits_{x\to0^+}f(x),\ \lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(y)\mathrm{d}y$ 存在, 则对 $a,b>0$ 有

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to0^+}f(x)-\lim_{x\to+\infty}\frac{1}{x}\int_0^x f(y)\mathrm{d}y\right]\ln\frac{b}{a}.$$

♡

笔记 傅汝兰尼积分有诸多变种, 无需记忆具体表达式, 知道有大概这么一个东西即可.

证明 不妨设 $b>a$.

1. 给定 $A>\delta>0$, 考虑

$$\begin{aligned}\int_\delta^A\frac{f(ax)-f(bx)}{x}\mathrm{d}x&=\int_\delta^A\frac{f(ax)}{x}\mathrm{d}x-\int_\delta^A\frac{f(bx)}{x}\mathrm{d}x\\&=\int_{a\delta}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{bA}\frac{f(x)}{x}\mathrm{d}x\\&=\int_{bA}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{a\delta}\frac{f(x)}{x}\mathrm{d}x\\&\xlongequal{\text{积分中值定理}}f(\theta_1)\int_{bA}^{aA}\frac{1}{x}\mathrm{d}x-f(\theta_2)\int_{b\delta}^{a\delta}\frac{1}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta_1\in(aA,bA),\theta_2\in(a\delta,b\delta)$, 于是让 $A\to+\infty,\delta\to0^+$, 由(9), 我们知

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to0^+}f(x)-\lim_{x\to+\infty}f(x)\right]\ln\frac{b}{a}.$$

2. 给定 $A>\delta>0$, 考虑

$$\begin{aligned}\int_\delta^A\frac{f(ax)-f(bx)}{x}\mathrm{d}x&=\int_\delta^A\frac{f(ax)}{x}\mathrm{d}x-\int_\delta^A\frac{f(bx)}{x}\mathrm{d}x\\&=\int_{a\delta}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{bA}\frac{f(x)}{x}\mathrm{d}x\\&=\int_{bA}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{a\delta}\frac{f(x)}{x}\mathrm{d}x\\&\xlongequal{\text{积分中值定理}}\int_{bA}^{aA}\frac{f(x)}{x}\mathrm{d}x-f(\theta)\int_{b\delta}^{a\delta}\frac{1}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta\in(a\delta,b\delta)$, 于是让 $A\to+\infty,\delta\to0^+$, 由 (10), 我们知

$$\int_0^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\alpha\ln\frac{b}{a}.$$

3. 给定 $A>\delta>0$, 考虑

$$\begin{aligned}\int_{\delta}^{A}\frac{f(ax)-f(bx)}{x}\mathrm{d}x&=\int_{\delta}^{A}\frac{f(ax)}{x}\mathrm{d}x-\int_{\delta}^{A}\frac{f(bx)}{x}\mathrm{d}x\\&=\int_{a\delta}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{bA}\frac{f(x)}{x}\mathrm{d}x\\&=\int_{bA}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{a\delta}\frac{f(x)}{x}\mathrm{d}x\\&\xlongequal{\text{积分中值定理}}f(\theta)\int_{bA}^{aA}\frac{1}{x}\mathrm{d}x-\int_{b\delta}^{a\delta}\frac{f(x)}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta\in(aA,bA)$, 于是让 $A\to+\infty,\delta\to0^{+}$, 由(11), 我们知

$$\int_{0}^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\alpha\ln\frac{a}{b}.$$

4. 给定 $A>\delta>0$, 考虑

$$\begin{aligned}\int_{\delta}^{A}\frac{f(ax)-f(bx)}{x}\mathrm{d}x&=\int_{\delta}^{A}\frac{f(ax)}{x}\mathrm{d}x-\int_{\delta}^{A}\frac{f(bx)}{x}\mathrm{d}x\\&=\int_{a\delta}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{bA}\frac{f(x)}{x}\mathrm{d}x\\&=\int_{bA}^{aA}\frac{f(x)}{x}\mathrm{d}x-\int_{b\delta}^{a\delta}\frac{f(x)}{x}\mathrm{d}x\\&\xlongequal{\text{积分中值定理}}\int_{bA}^{aA}\frac{f(x)}{x}\mathrm{d}x-f(\theta)\int_{b\delta}^{a\delta}\frac{1}{x}\mathrm{d}x\\&=\int_{b}^{a}\frac{f(Ax)}{x}\mathrm{d}x-f(\theta)\int_{b\delta}^{a\delta}\frac{1}{x}\mathrm{d}x,\end{aligned}$$

这里 $\theta\in(a\delta,b\delta)$. 现在

$$\lim_{\delta\to0^{+}}\left(-f(\theta)\int_{b\delta}^{a\delta}\frac{1}{x}\mathrm{d}x\right)=\lim_{x\to0^{+}}f(x)\ln\frac{b}{a}.$$

由 Riemann 引理, 我们有

$$\lim_{A\to+\infty}\int_{b}^{a}\frac{f(Ax)}{x}\mathrm{d}x=\int_{b}^{a}\frac{1}{x}\mathrm{d}x\cdot\frac{1}{T}\int_{0}^{T}f(x)\mathrm{d}x=-\frac{1}{T}\int_{0}^{T}f(x)\mathrm{d}x\cdot\ln\frac{b}{a},$$

这就证明了

$$\int_{0}^{\infty}\frac{f(ax)-f(bx)}{x}\mathrm{d}x=\left[\lim_{x\to0^{+}}f(x)-\frac{1}{T}\int_{0}^{T}f(x)\mathrm{d}x\right]\ln\frac{b}{a}.$$

5. 上一问证明中把使用的 Riemann 引理用平均值极限版本的 Riemann 引理代替即可.

□

0.1.4 化成多元累次积分 (换序)

命题 0.3

证明:

(1) $\displaystyle\int_{0}^{\infty}e^{-x^{2}}\mathrm{d}x=\frac{\sqrt{\pi}}{2}.$

(2) $\displaystyle\int_{0}^{\infty}\frac{\sin x}{x}\mathrm{d}x=\frac{\pi}{2}.$

(3) $\displaystyle\int_{0}^{+\infty}\frac{\sin^{2}x}{x^{2}}\mathrm{d}x=\frac{\pi}{2}.$

(4) $\displaystyle\int_{0}^{\infty}\sin x^{2}\mathrm{d}x,\int_{0}^{\infty}\cos x^{2}\mathrm{d}x=\sqrt{\frac{\pi}{8}}.$

笔记 本结果可以直接使用.

证明

(1) 注意到

$$\begin{aligned}\left(\int_0^{+\infty} e^{-x^2}\mathrm{d}x\right)^2 &= \left(\int_0^{+\infty} e^{-x^2}\mathrm{d}x\right)\left(\int_0^{+\infty} e^{-y^2}\mathrm{d}y\right) \xlongequal{\text{把}\int_0^{+\infty} e^{-y^2}\mathrm{d}y\text{看作常数}} \int_0^{+\infty} e^{-x^2}\left(\int_0^{+\infty} e^{-y^2}\mathrm{d}x\right)\mathrm{d}y\\ &\xlongequal{\text{把}e^{-x^2}\text{看作常数}} \int_0^{+\infty}\left(\int_0^{+\infty} e^{-(x^2+y^2)}\mathrm{d}x\right)\mathrm{d}y \xlongequal{e^{-(x^2+y^2)}\text{连续}} \iint_{R^2} e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y\\ &= \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{+\infty} re^{-r^2}\mathrm{d}r = \frac{\pi}{2}\int_0^{+\infty} re^{-r^2}\mathrm{d}r\\ &= \frac{\pi}{4}\int_0^{+\infty} e^{-r^2}\mathrm{d}r^2 = \frac{\pi}{4}.\end{aligned}$$

故 $\int_0^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

(2) 注意到

$$\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{-(y-\mathrm{i})x}\,\mathrm{d}x = \mathrm{Im}\frac{1}{y-\mathrm{i}} = \mathrm{Im}\frac{y+\mathrm{i}}{y^2+1} = \frac{1}{y^2+1}.$$

因此就有

$$\begin{aligned}\int_0^{+\infty}\frac{\sin x}{x}\,\mathrm{d}x &= \int_0^{+\infty}\sin x\left(\int_0^{+\infty} e^{-yx}\,\mathrm{d}y\right)\mathrm{d}x = \int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x\\ &= \int_0^{+\infty}\mathrm{d}y\left(\mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\right)\mathrm{d}x = \int_0^{+\infty}\frac{1}{y^2+1}\,\mathrm{d}y = \frac{\pi}{2}.\end{aligned}$$

当然本题也可以直接利用分部积分计算 $\int_0^{+\infty}\sin xe^{-yx}\,\mathrm{d}x = \frac{1}{y^2+1}$.

(3) 由分部积分可得

$$\begin{aligned}\int_0^{+\infty}\frac{\sin^2 x}{x^2}\mathrm{d}x &= \frac{1}{2}\int_0^{+\infty}\frac{1-\cos 2x}{x^2}\mathrm{d}x = -\frac{1}{2}\int_0^{+\infty}(1-\cos 2x)\,\mathrm{d}\frac{1}{x}\\ &= \frac{1}{2}\frac{1-\cos x}{x^2}\Big|_{+\infty}^{0} + \frac{1}{2}\int_0^{+\infty}\frac{2\sin 2x}{x}\mathrm{d}x\\ &= \int_0^{+\infty}\frac{\sin x}{x}\mathrm{d}x \xlongequal{\text{命题 0.3(2)}} \frac{\pi}{2}.\end{aligned}$$

(4) 注意到

$$\int_0^{+\infty} e^{-ax^2}\,\mathrm{d}x \xlongequal{x=\frac{t}{\sqrt{a}}} \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{a}},\quad a>0.$$

并且 $-\mathrm{i} = e^{-\frac{\pi}{2}\mathrm{i}}$, 从而 $\sqrt{-\mathrm{i}} = e^{-\frac{\pi}{4}\mathrm{i}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}$. 于是

$$\begin{aligned}\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2)\,\mathrm{d}x &= \int_0^{+\infty} e^{-\mathrm{i}x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\mathrm{i}}} = \frac{1}{2}\sqrt{-\mathrm{i}\pi}\\ &= \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}.\end{aligned}$$

故

$$\int_0^{+\infty}\cos x^2\,\mathrm{d}x = \mathrm{Re}\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2)\,\mathrm{d}x = \mathrm{Re}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}},$$

$$\int_0^{+\infty}\sin x^2\,\mathrm{d}x = \mathrm{Im}\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2)\,\mathrm{d}x = \mathrm{Im}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}.$$

□

例题 0.6 计算 $\int_0^1 \sin\ln\frac{1}{x}\cdot\frac{x^b - x^a}{\ln x}\mathrm{d}x\ (b > a > 0)$.

证明

$$\int_0^1 \sin\ln\frac{1}{x}\cdot\frac{x^b-x^a}{\ln x}\mathrm{d}x=\int_0^1 \sin\ln\frac{1}{x}\left(\int_a^b x^y\mathrm{d}y\right)\mathrm{d}x=\int_a^b \mathrm{d}y\int_0^1 x^y\sin\ln\frac{1}{x}\mathrm{d}x$$

$$\xlongequal{x=e^{-t}}\int_a^b \mathrm{d}y\int_{+\infty}^0 e^{-ty}\sin t\mathrm{d}e^{-t}=\int_a^b \mathrm{d}y\int_0^{+\infty} e^{-t(y+1)}\sin t\mathrm{d}t$$

$$\xlongequal{\text{命题 0.3(2) 的证明过程}}\int_a^b \frac{1}{1+(y+1)^2}\mathrm{d}y=\arctan(b+1)-\arctan(a+1).$$

□

0.1.5 化成含参积分 (求导)

例题 0.7 设 $a,b\geqslant 0$ 且不全为 0, 计算 $\displaystyle\int_0^{\frac{\pi}{2}}\ln\left(a^2\cos^2x+b^2\sin^2x\right)\mathrm{d}x$.

注 实际上, 根据 $a>b$ 时得到的结果, 可以看出 $F(a,b)=\pi\ln\dfrac{a+b}{2}$ 对 a,b 有轮换对称性, 故这个结果对其他情况显然也成立.

证明 设 $F(a,b)=\displaystyle\int_0^{\frac{\pi}{2}}\ln\left(a^2\cos^2x+b^2\sin^2x\right)\mathrm{d}x$, 当 $a>b$ 时, 则

$$\begin{aligned}
\frac{\partial}{\partial b}F(a,b)&=\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial b}\ln\left(a^2\cos^2x+b^2\sin^2x\right)\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{2b\sin^2x}{a^2\cos^2x+b^2\sin^2x}\mathrm{d}x\\
&=\int_0^{\frac{\pi}{2}}\frac{2b\tan^2x}{a^2+b^2\tan^2x}\mathrm{d}x=\int_0^{+\infty}\frac{2bt^2}{(a^2+b^2t^2)(1+t^2)}\mathrm{d}t\\
&=\frac{1}{a^2-b^2}\int_0^{+\infty}\left(\frac{2a^2b}{a^2+b^2t^2}-\frac{2b}{1+t^2}\right)\mathrm{d}t\\
&=\frac{1}{a^2-b^2}\int_0^{+\infty}\frac{2a^2b}{a^2+b^2t^2}\mathrm{d}t-\frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+t^2}\mathrm{d}t\\
&=\frac{2b}{a^2-b^2}\int_0^{+\infty}\frac{1}{1+\left(\frac{b}{a}t\right)^2}\mathrm{d}t-\frac{b\pi}{a^2-b^2}\\
&=\frac{2b}{a^2-b^2}\cdot\frac{a}{b}\cdot\frac{\pi}{2}-\frac{b\pi}{a^2-b^2}=\frac{\pi}{a+b}.
\end{aligned}$$

于是

$$\begin{aligned}
F(a,b)&=F(a,0)+\int_0^b\frac{\partial}{\partial b'}F(a,b')\mathrm{d}b'=F(a,0)+\int_0^b\frac{\pi}{a+b'}\mathrm{d}b'\\
&=2\int_0^{\frac{\pi}{2}}\ln(a\cos x)\mathrm{d}x+\pi\ln\frac{a+b}{a}\xlongequal{\text{例题 0.2}}\pi\ln\frac{a+b}{2}.
\end{aligned}$$

当 $a<b$ 时, 类似可得 $F(a,b)=\pi\ln\dfrac{a+b}{2}$. 当 $a=b$ 时,$F(a,b)=\displaystyle\int_0^{\frac{\pi}{2}}\ln a^2\mathrm{d}x=\pi\ln a=\pi\ln\frac{a+b}{2}$.

综上, 对 $\forall a,b\geqslant 0$, 都有 $F(a,b)=\pi\ln\dfrac{a+b}{2}$.

□

0.1.6 级数展开方法

积分和求和换序 $\displaystyle\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x=\int_a^b\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x$, 等价于

$$\lim_{m\to\infty}\sum_{n=1}^{m}\int_a^b f_n(x)\mathrm{d}x=\int_a^b\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x.$$

又由于有限和随意交换, 因此上式等价于

$$\lim_{m\to\infty}\int_a^b\sum_{n=1}^{m}f_n(x)\mathrm{d}x=\int_a^b\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x,$$

于是

$$\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x=\int_a^b\sum_{n=1}^{\infty}f_n(x)\mathrm{d}x\Longleftrightarrow\lim_{m\to\infty}\int_a^b\sum_{n=m+1}^{\infty}f_n(x)\mathrm{d}x=0.$$

例题 0.8 计算 $\int_0^{\infty}\frac{x}{1+e^x}\mathrm{d}x$.

解 由于

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n,\quad 0<x<1.$$

并且 $0<e^{-x}<1$, 故

$$\begin{aligned}\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x&=\int_0^{+\infty}\frac{xe^{-x}}{1+e^{-x}}\mathrm{d}x=\int_0^{+\infty}x\sum_{n=0}^{\infty}(-1)^n e^{-(n+1)x}\mathrm{d}x\\&\xlongequal{\text{换序}}\sum_{n=0}^{\infty}(-1)^n\int_0^{+\infty}xe^{-(n+1)x}\mathrm{d}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}\\&=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(2n)^2}.\end{aligned}$$

又因为 $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, 所以

$$\sum_{n=0}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{24},$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=0}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{8}.$$

故

$$\int_0^{+\infty}\frac{x}{1+e^x}\mathrm{d}x=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{8}-\frac{\pi^2}{24}=\frac{\pi^2}{12}.$$

下面证明(??)式换序成立, 等价于证明 $\lim_{m\to+\infty}\int_0^{+\infty}\sum_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\mathrm{d}x=0$. 由交错级数不等式及 $xe^{-(n+1)x}$ 关于 n 非负递减, 对 $\forall m\in\mathbb{N}$, 都有

$$\int_0^{+\infty}\left|\sum_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\right|\mathrm{d}x\leqslant\int_0^{+\infty}xe^{-(m+1)x}\mathrm{d}x=-\frac{xe^{-(m+1)x}}{m+1}\bigg|_0^{+\infty}+\frac{1}{m+1}\int_0^{+\infty}e^{-(m+1)x}\mathrm{d}x=\frac{1}{(m+1)^2}.$$

令 $m\to+\infty$, 得 $\lim_{m\to+\infty}\int_0^{+\infty}\sum_{n=m}^{\infty}x(-1)^n e^{-(n+1)x}\mathrm{d}x=0$. 故(??)式换序成立.

□

命题 0.4

证明:

(1) $\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n}=\arctan\frac{q\sin x}{1-q\cos x},|q|\leqslant 1.$

(2) $\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n}=-\frac{1}{2}\ln(1+q^2-2q\cos x),|q|\leqslant 1.$

(3) $\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!}=e^{q\cos x}\cos(q\sin x)-1,|q|\leqslant 1,x\in\mathbb{R}.$

(4) $\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!}=e^{q\cos x}\sin(q\sin x),|q|\leqslant 1,x\in\mathbb{R}.$

笔记 在 $\mathbb{C}$ 上,

$$\mathrm{Ln}z=\ln|z|+i(\arg z+2k\pi),k\in\mathbb{Z}.$$

我们定义主值支

$$\ln z=\ln|z|+i\arg z.$$

本部分内容无需记忆,只需要大概有个可以算的感觉即可,实际做题中可以围绕这种级数给出构造.

证明 $\Im$ 表示取虚部,$\Re$ 表示取实部.

(1) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n}&=\Im\left(\sum_{n=1}^{\infty}\frac{q^ne^{inx}}{n}\right)=\Im\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n}\right)=\Im(-\ln(1-qe^{ix}))\\&=-\Im\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=\arctan\frac{q\sin x}{1-q\cos x}.\end{aligned}$$

(2) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n}&=-\Re\left(\ln|1-qe^{ix}|+i\frac{-q\sin x}{1-q\cos x}\right)=-\frac{1}{2}\ln\left[(1-q\cos x)^2+q^2\sin^2 x\right]\\&=-\frac{1}{2}\ln(1+q^2-2q\cos x).\end{aligned}$$

(3) 利用欧拉公式有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\cos(nx)}{n!}&=\Re\left(\sum_{n=1}^{\infty}\frac{(qe^{ix})^n}{n!}\right)=\Re\left(e^{qe^{ix}}-1\right)=\Re\left(e^{q\cos x+iq\sin x}-1\right)\\&=\Re\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)\\&=e^{q\cos x}\cos(q\sin x)-1.\end{aligned}$$

(4) 利用 (3) 有

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n\sin(nx)}{n!}&=\Im\left(e^{q\cos x}\cos(q\sin x)-1+ie^{q\cos x}\sin(q\sin x)\right)\\&=e^{q\cos x}\sin(q\sin x).\end{aligned}$$

□

例题 0.9 计算

1. $\int_0^{2\pi}e^{\cos x}\cos(\sin x)\mathrm{d}x.$
2. $\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x,a\in(0,+\infty)\setminus\{1\}.$

注 由 1 的证明可得

$$e^{\cos x}\cos(\sin x)=\mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{(e^{ix})^n}{n!}\right)=\mathrm{Re}\left(\sum_{n=0}^{\infty}\frac{e^{inx}}{n!}\right)=\sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}.$$

实际上,上式就是命题 0.4(3)的结论.

注 第 2 问也可以用含参积分求导的方法进行计算 (这个方法更容易想到).

证明

1.
$$\begin{aligned}\int_0^{2\pi} e^{\cos x}\cos(\sin x)\,\mathrm{d}x &= \mathrm{Re}\left(\int_0^{2\pi} e^{\cos x}e^{\mathrm{i}\sin x}\mathrm{d}x\right) = \mathrm{Re}\left(\int_0^{2\pi} e^{\cos x+\mathrm{i}\sin x}\mathrm{d}x\right)\\ &= \mathrm{Re}\left(\int_0^{2\pi} e^{e^{\mathrm{i}x}}\mathrm{d}x\right) = \mathrm{Re}\left[\int_0^{2\pi}\sum_{n=0}^{+\infty}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right] = \mathrm{Re}\left[\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{\left(e^{\mathrm{i}x}\right)^n}{n!}\mathrm{d}x\right]\\ &= \mathrm{Re}\left(\sum_{n=0}^{+\infty}\int_0^{2\pi}\frac{e^{\mathrm{i}nx}}{n!}\mathrm{d}x\right) = \mathrm{Re}\left(\int_0^{2\pi}\frac{e^{\mathrm{i}\cdot 0\cdot x}}{n!}\mathrm{d}x + \sum_{n=1}^{+\infty}\frac{e^{2\pi\mathrm{i}x}-1}{\mathrm{i}n\cdot n!}\right)\\ &= \mathrm{Re}\left(\int_0^{2\pi} 1\mathrm{d}x + 0\right) = 2\pi.\end{aligned}$$

2. 注意到当 $a\in(0,1)$ 时, 有
$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n} &= \mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{(ae^{\mathrm{i}x})^n}{n}\right] = -\mathrm{Re}\left[\ln(1-ae^{\mathrm{i}x})\right]\\ &= -\mathrm{Re}\left[\ln|1-ae^{\mathrm{i}x}| + \mathrm{i}\arg(1-ae^{\mathrm{i}x})\right] = -\ln|1-ae^{\mathrm{i}x}|\\ &= -\ln|(1-a\cos x)+a\mathrm{i}\sin x| = -\frac{1}{2}\ln(1+a^2-2a\cos x).\end{aligned}$$
于是当 $a\in(0,1)$ 时, 就有
$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = -\frac{1}{2}\int_0^{\pi}\sum_{n=1}^{\infty}\frac{a^n\cos(nx)}{n}\mathrm{d}x = 0.$$
若 $a>1$, 则 $\frac{1}{a}\in(0,1)$, 从而此时我们有
$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = \pi\ln a^2 + \int_0^{\pi}\ln\left(\frac{1}{a^2}-\frac{2}{a}\cos x+1\right)\mathrm{d}x = \pi\ln a^2 = 2\pi\ln a.$$
又由 $\ln(1-2a\cos x+a^2)$ 关于 a 的偏导存在可知 $\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x$ 关于 a 连续. 于是由
$$\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = 2\pi\ln a,\quad \forall a>1.$$
可知当 $a=1$ 时, 我们有
$$\int_0^{\pi}\ln(2-2\cos x)\mathrm{d}x = \lim_{a\to1^+}\int_0^{\pi}\ln(1-2a\cos x+a^2)\mathrm{d}x = \lim_{a\to1^+}(2\pi\ln a) = 0.$$
□

定义 0.1 (多重对数函数-Li_2 函数)

定义
$$\mathrm{Li}_2(x) \triangleq \sum_{n=1}^{\infty}\frac{x^n}{n^2},\quad x\in[-1,1].$$

命题 0.5

(1) $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x\cdot\ln(1-x), x\in(0,1)$.

(2) $\mathrm{Li}_2(1) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},\quad \mathrm{Li}_2(0) = 0,\quad \mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2\frac{1}{2}.$

证明

(1) 记 $f(x) \triangleq \mathrm{Li}_2(x), F(x) \triangleq f(x)+f(1-x)+\ln x\ln(1-x)$. 则

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}=-\frac{1}{x}\ln(1-x).$$

于是

$$F'(x)=-\frac{1}{x}\ln(1-x)+\frac{\ln x}{1-x}-\frac{\ln x}{1-x}+\frac{\ln(1-x)}{x}=0.$$

故 $F(x)\equiv F(1)=f(0)+f(1)=0+\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

(2) 显然 $\mathrm{Li}_2(1)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}, \mathrm{Li}_2(0)=0$. 由 (1) 可得

$$\mathrm{Li}_2\left(\frac{1}{2}\right)+\mathrm{Li}_2\left(\frac{1}{2}\right)=2\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{6}-\ln^2\frac{1}{2}\implies \mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.$$

□

例题 0.10 计算 $\int_0^{\frac{1}{2}}\frac{\ln x}{1-x}\mathrm{d}x$.

解

$$\begin{aligned}\int_0^{\frac{1}{2}}\frac{\ln x}{1-x}\mathrm{d}x&=\int_{\frac{1}{2}}^{1}\frac{\ln(1-x)}{x}\mathrm{d}x=-\sum_{n=1}^{\infty}\frac{1}{n}\int_{\frac{1}{2}}^{1}x^{n-1}\mathrm{d}x\\&=-\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{2^n n^2}=-\frac{\pi^2}{6}+\mathrm{Li}_2\left(\frac{1}{2}\right)\\&\xlongequal{\text{命题 0.5}}-\frac{\pi^2}{12}-\frac{1}{2}\ln^2\frac{1}{2}.\end{aligned}$$

□

0.1.7 重积分计算

定理 0.5 (二重积分换序)

证明:

$$\int_a^b\mathrm{d}x\int_a^x f(x,y)\mathrm{d}y=\int_a^b\mathrm{d}y\int_y^b f(x,y)\mathrm{d}x, \tag{10}$$

其中 $f(x,y)$ 是在由直线 $y=a, x=b, y=x$ 所围成的三角形 (Δ) 上连续的任意函数.

证明

□

命题 0.6

设 $f(x)$ 在 $[a,b]$ 上连续, 试证: 对任意 $x\in(a,b)$, 有

$$\int_a^x\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_n}f(x_{n+1})\mathrm{d}x_{n+1}=\frac{1}{n!}\int_a^x(x-y)^n f(y)\mathrm{d}y,\quad n=1,2,\cdots.$$

证明 当 $n=1$ 时, 由二重积分换序可知

$$\int_a^x\mathrm{d}x_1\int_a^{x_1}f(x_2)\mathrm{d}x_2=\int_a^x\mathrm{d}x_2\int_{x_2}^x f(x_2)\mathrm{d}x_1=\int_a^x(x-x_2)f(x_2)\mathrm{d}x_2=\int_a^x(x-y)f(y)\mathrm{d}y.$$

设原结论对 $n=k-1$ 的情形成立, 考虑 $n=k$ 的情形, 由归纳假设可知

$$\int_a^{x_1}\mathrm{d}x_1\cdots\int_a^{x_k}f(x_{k+1})\mathrm{d}x_{k+1}=\frac{1}{(k-1)!}\int_a^{x_1}(x_1-y)^{k-1}f(y)\mathrm{d}y.$$

于是再利用二重积分换序得

$$\begin{aligned}\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_k} f(x_{k+1})\mathrm{d}x_{k+1} &= \frac{1}{(k-1)!}\int_a^x \mathrm{d}x_1 \int_a^{x_1} (x_1-y)^{k-1} f(y)\mathrm{d}y \\ &= \frac{1}{(k-1)!}\int_a^x \mathrm{d}y \int_y^{x} (x_1-y)^{k-1} f(y)\mathrm{d}x_1 \\ &= \frac{1}{k!}\int_a^x (x-y)^k f(y)\mathrm{d}y.\end{aligned}$$

故由数学归纳法知原结论成立.

□

例题 0.11 求定义在星形区域 $D=\{(x,y)\mid x^{\frac{2}{3}}+y^{\frac{2}{3}}\leqslant 1\}$ 上满足 $f(1,0)=1$ 的正值连续函数 f 使得 $\iint\limits_D \frac{f(x,y)}{f(y,x)}\mathrm{d}x\mathrm{d}y$ 达到最小, 并求出这个最小值.

解 对积分 $I=\iint\limits_D \frac{f(x,y)}{f(y,x)}\mathrm{d}x\mathrm{d}y$ 作变换 $x\to y,\ y\to x$, 由 D 的对称性, 知 $I=\iint\limits_D \frac{f(y,x)}{f(x,y)}\mathrm{d}x\mathrm{d}y$. 因而由均值不等式可得

$$I=\frac{1}{2}\iint\limits_D \left(\frac{f(x,y)}{f(y,x)}+\frac{f(y,x)}{f(x,y)}\right)\mathrm{d}x\mathrm{d}y \geqslant \iint\limits_D 1\,\mathrm{d}x\mathrm{d}y=\sigma(D),$$

这里 $\sigma(D)$ 是 D 的面积.

$$I-\sigma(D)=\frac{1}{2}\iint\limits_D \left(\sqrt{\frac{f(x,y)}{f(y,x)}}-\sqrt{\frac{f(y,x)}{f(x,y)}}\right)^2 \mathrm{d}x\mathrm{d}y \geqslant 0.$$

$I=\sigma(D)$ 当且仅当 $f(x,y)=f(y,x)$. 故所求函数为所有满足 $f(x,y)=f(y,x)$ 及 $f(1,0)=1$ 的连续正值函数.

D 的边界的参数方程为

$$x=\cos^3\varphi,\quad y=\sin^3\varphi\ (0\leqslant\varphi\leqslant 2\pi),$$

故 I 的最小值为

$$\begin{aligned}\sigma(D)=\iint\limits_D 1\,\mathrm{d}x\mathrm{d}y &= 4\iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\varphi\leqslant\frac{\pi}{2}}} 3r\sin^2\varphi\cos^2\varphi\,\mathrm{d}r\mathrm{d}\varphi \\ &= 6\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos^2\varphi\,\mathrm{d}\varphi=\frac{3}{8}\pi.\end{aligned}$$

所以所求最小值是 $\frac{3}{8}\pi$, 且当 $f(x,y)=f(y,x)$ 并满足 $f(1,0)=1$ 时, 取到该最小值.

□

例题 0.12 求证: $\iint\limits_{[0,1]^2}(xy)^{xy}\,\mathrm{d}x\mathrm{d}y=\int_0^1 t^t\,\mathrm{d}t.$

证明 首先化为累次积分

$$\iint\limits_{[0,1]^2}(xy)^{xy}\,\mathrm{d}x\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^1(xy)^{xy}\,\mathrm{d}y=\int_0^1\mathrm{d}x\int_0^x\frac{t^t}{x}\,\mathrm{d}t=\int_0^1\frac{f(x)}{x}\,\mathrm{d}x,$$

其中 $f(x)=\int_0^x t^t\,\mathrm{d}t$. 由分部积分,

$$\int_0^1\frac{f(x)}{x}\,\mathrm{d}x=f(x)\ln x\Big|_0^1-\int_0^1 x^x\ln x\,\mathrm{d}x=-\int_0^1 x^x\ln x\,\mathrm{d}x.$$

因为 $(x^x)'=x^x\ln x+x^x$, 所以

$$\int_0^1 x^x\ln x\,\mathrm{d}x=\int_0^1\left((x^x)'-x^x\right)\mathrm{d}x=-\int_0^1 x^x\,\mathrm{d}x.$$

于是

$$\iint\limits_{[0,1]^2} (xy)^{xy}\,\mathrm{d}x\mathrm{d}y = \int_0^1 t^t\,\mathrm{d}t.$$

□

例题 0.13 计算二重积分 $I = \iint\limits_D \operatorname{sgn}(x^2 - y^2 + 2)\,\mathrm{d}x\mathrm{d}y$, 其中 $D = \{(x,y) \mid x^2 + y^2 \leqslant 4\}$.

解 设 D 在第一象限部分为 D_1, 则由对称性

$$I = 4\iint\limits_{D_1} \operatorname{sgn}(x^2 - y^2 + 2)\,\mathrm{d}x\mathrm{d}y.$$

设 D_2 是 D_1 中使得 $x^2 - y^2 + 2 < 0$ 的部分, D_3 是 D_1 中使得 $x^2 - y^2 + 2 \geqslant 0$ 的部分, 则 $D_1 = D_2 \cup D_3$. 因此

$$\begin{aligned} I &= 4\left[\iint\limits_{D_3} \mathrm{d}x\mathrm{d}y - \iint\limits_{D_2} \mathrm{d}x\mathrm{d}y\right] = 4[\sigma(D_3) - \sigma(D_2)] \\ &= 4\left[\frac{1}{4}\cdot\pi\cdot 2^2 - 2\sigma(D_2)\right] = 4\pi - 8\sigma(D_2), \end{aligned}$$

其中 $\sigma(D_2), \sigma(D_3)$ 分别表示 D_2 和 D_3 的面积. 在极坐标 $x = r\cos\varphi, y = r\sin\varphi$ 之下, D_2 为

$$\left\{(r,\varphi) \mid \frac{\pi}{3} \leqslant \varphi \leqslant \frac{\pi}{2}, \sqrt{-\frac{2}{\cos 2\varphi}} \leqslant r \leqslant 2\right\}.$$

因而

$$\begin{aligned} \sigma(D_2) &= \iint\limits_{D_2} \mathrm{d}x\mathrm{d}y = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_{\sqrt{-\frac{2}{\cos 2\varphi}}}^{2} r\,\mathrm{d}r \\ &= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(4 + \frac{2}{\cos 2\varphi}\right)\mathrm{d}\varphi = \frac{\pi}{3} + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} \frac{1}{\cos\varphi}\mathrm{d}\varphi \\ &= \frac{\pi}{3} - \frac{1}{2}\ln(2+\sqrt{3}), \end{aligned}$$

故

$$I = \frac{4\pi}{3} + 4\ln(2+\sqrt{3}).$$

□

例题 0.14 设 $D = \{(x,y) \mid x^2 + y^2 \leqslant 1\}$. 求 $I = \iint\limits_D \left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right|\mathrm{d}x\mathrm{d}y$.

解 由极坐标变换 $x = r\cos\varphi, y = r\sin\varphi, 0 \leqslant r \leqslant 1, 0 \leqslant \varphi \leqslant 2\pi$, 有

$$\begin{aligned} I &= \iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\varphi\leqslant 2\pi}} \left|\frac{\cos\varphi + \sin\varphi}{\sqrt{2}} - r\right| r^2\mathrm{d}r\mathrm{d}\varphi \\ &= \iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\varphi\leqslant 2\pi}} \left|\sin\left(\varphi + \frac{\pi}{4}\right) - r\right| r^2\mathrm{d}r\mathrm{d}\varphi = \iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\varphi\leqslant 2\pi}} |\sin\varphi - r|\, r^2\mathrm{d}r\mathrm{d}\varphi \\ &= \iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\varphi\leqslant \pi}} |\sin\varphi - r|\, r^2\mathrm{d}r\mathrm{d}\varphi + \iint\limits_{\substack{0\leqslant r\leqslant 1\\ \pi\leqslant\varphi\leqslant 2\pi}} |\sin\varphi - r|\, r^2\mathrm{d}r\mathrm{d}\varphi \\ &= \iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\varphi\leqslant \pi}} |\sin\varphi - r|\, r^2\mathrm{d}r\mathrm{d}\varphi + \iint\limits_{\substack{0\leqslant r\leqslant 1\\ 0\leqslant\varphi\leqslant \pi}} (\sin\varphi + r) r^2\mathrm{d}r\mathrm{d}\varphi. \end{aligned}$$

因此, 有

$$I = \int_0^{\pi} \mathrm{d}\varphi \int_0^{\sin\varphi} (\sin\varphi - r) r^2\mathrm{d}r + \int_0^{\pi} \mathrm{d}\varphi \int_{\sin\varphi}^{1} (r - \sin\varphi) r^2\mathrm{d}r$$

$$\begin{aligned}
&+\int_0^\pi \mathrm{d}\varphi\int_0^{\sin\varphi}(\sin\varphi+r)r^2\mathrm{d}r+\int_0^\pi \mathrm{d}\varphi\int_{\sin\varphi}^1(\sin\varphi+r)r^2\mathrm{d}r\\
&=\int_0^\pi \mathrm{d}\varphi\int_0^{\sin\varphi}2\sin\varphi\cdot r^2\mathrm{d}r+\int_0^\pi \mathrm{d}\varphi\int_{\sin\varphi}^1 2r\cdot r^2\mathrm{d}r\\
&=\int_0^\pi \frac{2}{3}\sin^4\varphi\mathrm{d}\varphi+\int_0^\pi\frac{1}{2}(1-\sin^4\varphi)\mathrm{d}\varphi\\
&=\frac{1}{6}\int_0^\pi\sin^4\varphi\mathrm{d}\varphi+\frac{\pi}{2}=\frac{1}{6}\cdot\frac{3\pi}{8}+\frac{\pi}{2}=\frac{9}{16}\pi.
\end{aligned}$$

□

例题 0.15 设 f 是定义在正方形 $S=\{(x,y)\mid 0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1\}$ 上的四阶连续可微函数, 在 S 的边界上为零, 并且

$$\left|\frac{\partial^4 f}{\partial x^2\partial y^2}\right|\leqslant M.$$

求证:

$$\left|\iint_S f(x,y)\,\mathrm{d}x\mathrm{d}y\right|\leqslant\frac{1}{144}M.$$

证明 考虑函数 $g(x,y)=x(1-x)y(1-y)$. 易知

$$\frac{\partial^4 g}{\partial x^2\partial y^2}=4,\quad \iint_S g(x,y)\,\mathrm{d}x\mathrm{d}y=\frac{1}{36}.$$

因为 f 在 S 的边界上为零, 所以 $\dfrac{\partial^2 f}{\partial y^2}$ 在 $x=0$ 和 $x=1$ 时为零. 于是

$$\begin{aligned}
&\iint_S\frac{\partial^4 f}{\partial x^2\partial y^2}\cdot g\,\mathrm{d}x\mathrm{d}y=\int_0^1\mathrm{d}y\int_0^1\frac{\partial^4 f}{\partial x^2\partial y^2}\cdot g\,\mathrm{d}x\\
&=\int_0^1\mathrm{d}y\left(\frac{\partial^3 f}{\partial x\partial y^2}\cdot g\bigg|_{x=0}^1-\int_0^1\frac{\partial^3 f}{\partial x\partial y^2}\cdot\frac{\partial g}{\partial x}\,\mathrm{d}x\right)\\
&=-\int_0^1\mathrm{d}y\int_0^1\frac{\partial^3 f}{\partial x\partial y^2}\cdot\frac{\partial g}{\partial x}\,\mathrm{d}x\\
&=-\int_0^1\mathrm{d}y\left(\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial g}{\partial x}\bigg|_{x=0}^1-\int_0^1\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\right)\\
&=\int_0^1\mathrm{d}y\int_0^1\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\\
&=\iint_S\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

同理, 由于 $\dfrac{\partial^2 g}{\partial x^2}$ 在 $y=0$ 和 $y=1$ 时为零, 作与上面类似的推导, 可得

$$\iint_S\frac{\partial^4 g}{\partial x^2\partial y^2}\cdot f\,\mathrm{d}x\mathrm{d}y=\iint_S\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\mathrm{d}y.$$

因此

$$\iint_S\frac{\partial^4 f}{\partial x^2\partial y^2}\cdot g\,\mathrm{d}x\mathrm{d}y=\iint_S\frac{\partial^4 g}{\partial x^2\partial y^2}\cdot f\,\mathrm{d}x\mathrm{d}y.$$

从而

$$\begin{aligned}
&\left|\iint_S f\,\mathrm{d}x\mathrm{d}y\right|=\frac{1}{4}\left|\iint_S 4f\mathrm{d}x\mathrm{d}y\right|=\frac{1}{4}\left|\iint_S\frac{\partial^4 g}{\partial x^2\partial y^2}f\mathrm{d}x\mathrm{d}y\right|\\
&=\frac{1}{4}\left|\iint_S\frac{\partial^4 f}{\partial x^2\partial y^2}\cdot g\,\mathrm{d}x\mathrm{d}y\right|\leqslant\frac{M}{4}\iint_S g\,\mathrm{d}x\mathrm{d}y=\frac{M}{144}.
\end{aligned}$$

□

定理 0.6 (Poincaré(庞加莱) 不等式)

设 φ,ψ 是 $[a,b]$ 上的连续函数, f 在区域 $D=\{(x,y)\mid a\leqslant x\leqslant b,\varphi(x)\leqslant y\leqslant\psi(x)\}$ 上连续可微, 且有 $f(x,\varphi(x))=0\ (x\in[a,b])$. 则存在 $M>0$, 使得

$$\iint_D f^2(x,y)\,\mathrm{d}x\mathrm{d}y\leqslant M\iint_D (f_y'(x,y))^2\,\mathrm{d}x\mathrm{d}y.$$

证明 由 Newton-Leibniz 公式和 Cauchy 不等式可得

$$\begin{aligned}f^2(x,y)=[f(x,y)-f(x,\varphi(x))]^2&=\left(\int_{\varphi(x)}^{y}\frac{\partial f}{\partial t}(x,t)\,\mathrm{d}t\right)^2\\&\leqslant(y-\varphi(x))\int_{\varphi(x)}^{y}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t,\end{aligned}$$

因此

$$\begin{aligned}&\iint_D f^2(x,y)\,\mathrm{d}x\mathrm{d}y=\int_a^b\mathrm{d}x\int_{\varphi(x)}^{\psi(x)}f^2(x,y)\,\mathrm{d}y\\&\leqslant\int_a^b\mathrm{d}x\int_{\varphi(x)}^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\int_{\varphi(x)}^{y}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\\&=\int_a^b\mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\int_t^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\\&\leqslant\int_a^b\mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\int_{\varphi(x)}^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\\&=\int_a^b\mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\frac{1}{2}(\psi(x)-\varphi(x))^2\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\\&\leqslant M\int_a^b\left(\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\mathrm{d}t\right)\mathrm{d}x\\&=M\iint_D\left(\frac{\partial f}{\partial y}(x,y)\right)^2\mathrm{d}x\mathrm{d}y,\end{aligned}$$

这里 M 是满足 $M>\max\limits_{a\leqslant x\leqslant b}\frac{1}{2}(\psi(x)-\varphi(x))^2$ 的常数.

□

例题 0.16 设 $a>0,\Omega_n(a):x_1+x_2+\cdots+x_n\leqslant a,x_i\geqslant0\ (i=1,2,\cdots,n)$. 求积分

$$I_n(a)=\int\cdots\int_{\Omega_n(a)}x_1x_2\cdots x_n\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

解 作变换 $x_i=at_i,i=1,2,\cdots,n$, 则

$$I_n(a)=a^{2n}\int\cdots\int_{\Omega_n(1)}t_1t_2\cdots t_n\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n=a^{2n}I_n(1).$$

再用累次积分, 可得

$$\begin{aligned}I_n(1)&=\int\cdots\int_{\Omega_n(1)}t_1t_2\cdots t_n\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n\\&=\int_0^1t_n\mathrm{d}t_n\int\cdots\int_{t_1+t_2+\cdots+t_{n-1}\leqslant1-t_n}t_1\cdots t_{n-1}\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\\&=\int_0^1t_nI_{n-1}(1-t_n)\mathrm{d}t_n=\int_0^1t_n(1-t_n)^{2(n-1)}I_{n-1}(1)\mathrm{d}t_n.\end{aligned}$$

因此,

$$I_n(1)=\frac{1}{2n(2n-1)}I_{n-1}(1).$$

注意到 $I_1(1)=\int_0^1t\mathrm{d}t=\frac{1}{2}$. 由上面的递推公式, 可得 $I_n(1)=\frac{1}{(2n)!}$. 故 $I_n(a)=\frac{a^{2n}}{(2n)!}$.

□

0.1.8 其他

例题 0.17 证明积分 $\int_0^{\infty} e^{-ax^2-\frac{b}{x^2}}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}, a, b > 0.$

证明 当 $a = 1$ 时, 就有

$$\begin{aligned}\int_0^{+\infty} e^{-x^2-\frac{b}{x^2}}\mathrm{d}x &= e^{-2\sqrt{b}}\int_0^{+\infty} e^{-\left(x-\frac{\sqrt{b}}{x}\right)^2}\mathrm{d}x \xlongequal{y=\frac{\sqrt{b}}{x}} e^{-2\sqrt{b}}\int_0^{+\infty}\frac{\sqrt{b}}{y^2}e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2}\mathrm{d}y\\ &= \frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty}\left(1+\frac{\sqrt{b}}{y^2}\right)e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\mathrm{d}y = \frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty} e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\mathrm{d}\left(y-\frac{\sqrt{b}}{y}\right)\\ &= \frac{e^{-2\sqrt{b}}}{2}\int_{-\infty}^{+\infty} e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}e^{-2\sqrt{b}}.\end{aligned}$$

于是对 $\forall a > 0$, 就有

$$\int_0^{+\infty} e^{-ax^2-\frac{b}{x^2}}\mathrm{d}x = \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-x^2-\frac{ab}{x^2}}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}.$$

□

例题 0.18 计算 $\int_0^{\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x, a \in \mathbb{R}.$

注 本题可以用复变函数的方法 (留数定理) 来计算. 但是我们这里用基本的高等数学的方法来计算. $\int_0^{\infty}\frac{\sin(ax)}{1+x^2}\mathrm{d}x$ 这个积分没办法算出具体的初等数值.

证明

$$\begin{aligned}\int_0^{+\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos(ax)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\cos(ax)\left(\int_0^{+\infty} e^{-(1+x^2)y}\mathrm{d}y\right)\mathrm{d}x\\ &= \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} e^{-(1+x^2)y}\cos(ax)\mathrm{d}y\right)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^{+\infty} e^{-(1+x^2)y}\cos(ax)\mathrm{d}y\mathrm{d}x\\ &= \frac{1}{2}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-(1+x^2)y}\cos(ax)\mathrm{d}x\right)\mathrm{d}y = \frac{1}{2}\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-x^2y}\cos(ax)\mathrm{d}x\right)\mathrm{d}y\\ &= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-x^2y+\mathrm{i}ax}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x^2-\frac{aix}{y}+\left(\frac{ai}{2y}\right)^2\right)-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\ &= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x-\frac{ai}{2y}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2iy}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\ &= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2iy}\right)^2-\frac{a^2}{4y}}\mathrm{d}\left(x+\frac{a}{2iy}\right)\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-yx^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)\\ &= \frac{1}{2}\int_0^{+\infty} e^{-y-\frac{a^2}{4y}}\left(\int_{-\infty}^{+\infty} e^{-yx^2}\mathrm{d}x\right)\mathrm{d}y = \frac{\sqrt{\pi}}{2}\int_0^{+\infty}\frac{1}{\sqrt{y}}e^{-y-\frac{a^2}{4y}}\mathrm{d}y\\ &\xlongequal{y=t^2} \sqrt{\pi}\int_0^{+\infty} e^{-t^2-\frac{a^2}{4t^2}}\mathrm{d}t \xlongequal{\text{例题 0.17}} \sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2}e^{-|a|} = \frac{\pi}{2}e^{-|a|}.\end{aligned}$$

□

例题 0.19 计算 $\int_0^{\infty}\frac{1}{(1+x^8)^2}\mathrm{d}x.$

注 由命题??可知对 $\forall s > 0$, 都有

$$\frac{1}{t^s} = \frac{1}{\Gamma(s)}\int_0^{+\infty} y^{s-1}e^{-ty}\mathrm{d}y, \forall t \in \mathbb{R}.$$

本题的核心想法就是利用上式将 $\frac{z}{1+x^8}$ 转化成积分形式.

证明 注意到

$$\int_0^{+\infty} ye^{-(1+x^8)y}\mathrm{d}y \xlongequal{y=\frac{z}{1+x^8}} \frac{1}{(1+x^8)^2}\int_0^{+\infty} ze^{-z}\mathrm{d}z = \frac{1}{(1+x^8)^2},$$

因此

$$
\begin{aligned}
\int_0^{+\infty}\frac{1}{(1+x^8)^2}\mathrm{d}x &= \int_0^{+\infty}\left(\int_0^{+\infty} ye^{-(1+x^8)y}\mathrm{d}y\right)\mathrm{d}x = \int_0^{+\infty}\left(\int_0^{+\infty} ye^{-(1+x^8)y}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^{+\infty} ye^{-y}\left(\int_0^{+\infty} e^{-x^8 y}\mathrm{d}x\right)\mathrm{d}y \xlongequal{x=y^{-\frac{1}{8}}z^{\frac{1}{8}}} \int_0^{+\infty} ye^{-y}\left(\int_0^{+\infty} y^{-\frac{1}{8}}e^{-z}\mathrm{d}z^{\frac{1}{8}}\right)\mathrm{d}y \\
&= \frac{1}{8}\int_0^{+\infty} y^{\frac{7}{8}}e^{-y}\left(\int_0^{+\infty} z^{-\frac{7}{8}}e^{-z}\mathrm{d}z\right)\mathrm{d}y = \frac{1}{8}\int_0^{+\infty} y^{\frac{7}{8}}e^{-y}\Gamma\left(\frac{1}{8}\right)\mathrm{d}y \\
&= \frac{1}{8}\Gamma\left(\frac{15}{8}\right)\Gamma\left(\frac{1}{8}\right) = \frac{1}{8}\cdot\frac{7}{8}\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{1}{8}\right) \\
&\xlongequal{??} \frac{7\pi}{64\sin\left(\frac{\pi}{8}\right)} = \frac{7\pi}{32\sqrt{2-\sqrt{2}}}.
\end{aligned}
$$

□

例题 0.20 计算积分 $I=\int_{-1}^{2}\frac{1+x^2}{1+x^4}\mathrm{d}x$.

注 在此例中 $I\neq F(2)-F(-1)$. 这是因为 F 并不是 f 在区间 $[-1,2]$ 上的原函数.

解 在不包含 0 的区间上作变换 $t=x-\frac{1}{x}$ 得

$$
\begin{aligned}
\int\frac{1+x^2}{1+x^4}\mathrm{d}x &= \int\frac{x-\frac{1}{x}}{2+\left(x-\frac{1}{x}\right)^2}\mathrm{d}x = \int\frac{\mathrm{d}t}{2+t^2} \\
&= \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C = \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}+C.
\end{aligned}
$$

这说明在区间 $[-1,0)$ 和 $(0,2]$ 上, 函数 $f(x)=\frac{1+x^2}{1+x^4}$ 的一个原函数是

$$
F(x)=\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x}.
$$

因此

$$
\begin{aligned}
\int_{-1}^{0}f(x)\,\mathrm{d}x &= F(0^-)-F(-1) = \frac{\pi}{2\sqrt{2}}-0=\frac{\pi}{2\sqrt{2}}, \\
\int_{0}^{2}f(x)\,\mathrm{d}x &= F(2)-F(0^+) = \frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}}+\frac{\pi}{2\sqrt{2}}.
\end{aligned}
$$

故

$$
I=\int_{-1}^{0}f(x)\,\mathrm{d}x+\int_{0}^{2}f(x)\,\mathrm{d}x=\frac{\pi}{\sqrt{2}}+\frac{1}{\sqrt{2}}\arctan\frac{3}{2\sqrt{2}}.
$$

□