0.1 定积分

0.1.1 建立积分递推

例题 0.1 计算 $\displaystyle\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x, n \in \mathbb{N}$.

证明 利用分部积分和和差化积公式可得

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx)\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x[\sin(nx)\cos x + \cos(nx)\sin x]\mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(nx)\mathrm{d}\cos x \\
&= \frac{I_{n-1}+I_n}{2} - \frac{1}{2n}\int_0^{\frac{\pi}{2}} \cos(nx)\mathrm{d}\cos^n x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx)\mathrm{d}x \\
&= \frac{I_{n-1}+I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
&= \frac{I_{n-1}}{2} + \frac{1}{2n}.
\end{aligned}
$$

故 $I_n = \dfrac{I_{n-1}}{2} + \dfrac{1}{2n}$, 则两边同乘 2^n(强行裂项)

$$
2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \cdots
$$

又注意到 $I_0 = 0$, 从而

$$
2^n I_n = 0 + \sum_{k=1}^n \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n}\sum_{k=1}^n \frac{2^{k-1}}{k}.
$$

$\square$

命题 0.1

设 $n \in \mathbb{N}$, 证明:

(1) $\displaystyle\int_0^\pi \frac{\sin(nx)}{\sin x}\mathrm{d}x = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}$.

(2) $\displaystyle\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\mathrm{d}x = n\pi$.

(3) $\displaystyle\int_0^\pi \frac{\sin^2(nx)}{\sin x}\mathrm{d}x = \sum_{k=1}^n \frac{2}{2k-1}$.

♧ **笔记** 提示:$\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y)$(证明见命题 **??**).

证明

(1) 记 $I_n = \displaystyle\int_0^\pi \frac{\sin(nx)}{\sin x}\mathrm{d}x$, 则

$$
I_{n+2} - I_n = \int_0^\pi \frac{\sin((n+2)x) - \sin(nx)}{\sin x}\mathrm{d}x = \int_0^\pi \frac{2\cos((n+1)x)\sin x}{\sin x}\mathrm{d}x = 2\int_0^\pi \cos((n+1)x)\mathrm{d}x = 0.
$$

于是

$$\int_0^\pi \frac{\sin(nx)}{\sin x}\,\mathrm{d}x = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$$

(2) 记 $I_n = \displaystyle\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, 则

$$I_{n+1} - I_n = \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin^2 x}\,\mathrm{d}x$$
$$= \int_0^\pi \frac{\sin((2n+1)x)}{\sin x}\,\mathrm{d}x \xequal{\text{命题 0.1(1)}} \pi. \tag{1}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x = I_n = \pi + I_{n-1} = \cdots = (n-1)\pi + I_1 = n\pi.$$

(3) 记 $I_n = \displaystyle\int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x$, 则

$$I_{n+1} - I_n = \int_0^\pi \frac{\sin^2((n+1)x) - \sin^2(nx)}{\sin x}\,\mathrm{d}x = \int_0^\pi \frac{\sin x \cdot \sin((2n+1)x)}{\sin x}\,\mathrm{d}x$$
$$= \int_0^\pi \sin((2n+1)x)\,\mathrm{d}x = \frac{1}{2n+1}\cos((2n+1)x)\Big|_\pi^0 = \frac{2}{2n+1}. \tag{2}$$

于是

$$\int_0^\pi \frac{\sin^2(nx)}{\sin x}\,\mathrm{d}x = I_n = \frac{2}{2n-1} + I_{n-1} = \cdots = \sum_{k=1}^{n-1}\frac{2}{2k+1} + I_1 = \sum_{k=0}^{n-1}\frac{2}{2k+1} = \sum_{k=1}^n \frac{2}{2k-1}.$$

$\square$

例题 0.2 设 $a > 1$, 计算积分 $\displaystyle\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\,\mathrm{d}x$.

注 很多情况下不需求出被积函数的原函数, 只需充分利用换元、分部积分以及被积函数的性质, 即可求出积分的值. 见下述解法二.

解 解法一:设 $a_0 = a > 1$. 构造数列如下:

$$a_{n+1} = 2a_n^2 - 1 \quad (n = 0, 1, \cdots),$$

则由例题**??**可知, 存在 $x_0 > 0$ 使得

$$a_0 = \mathrm{ch}(x_0), \quad a_n = \mathrm{ch}(2^n x_0),$$

其中 $\mathrm{ch}(x) = \dfrac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$. 可以解得

$$x_0 = \ln\left(a_0 + \sqrt{a_0^2 - 1}\right). \tag{3}$$

故

$$a_n = \frac{\mathrm{e}^{2^n x_0} + \mathrm{e}^{-2^n x_0}}{2}.$$

设

$$I_n = \int_0^\pi \ln(a_n - \cos x)\,\mathrm{d}x,$$

则

$$I_0 = \int_0^\pi \ln(a_0 - \cos x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^\pi \ln(a_0 - \cos x)\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{2}} \ln(a_0 - \cos x)\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \ln(a_0 + \cos x)\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{2}} \ln(a_0^2 - \cos^2 x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\left(a_0^2 - \frac{1 + \cos 2x}{2}\right)\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{a_1 - \cos 2x}{2}\right) dx = \frac{1}{2}\int_0^{\pi} \ln\left(\frac{a_1 - \cos x}{2}\right) dx = \frac{1}{2}I_1 - \frac{\pi}{2}\ln 2.$$

同理, 有

$$I_n = \frac{1}{2}I_{n+1} - \frac{\pi}{2}\ln 2. \tag{4}$$

由此递推公式, 可得

$$I_0 = \frac{1}{2^n}I_n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right)\frac{\pi}{2}\ln 2. \tag{5}$$

因为

$$I_n = \int_0^{\pi} \ln(a_n - \cos x)\, dx = \int_0^{\pi} \ln\left(\frac{e^{2^n x_0} + e^{-2^n x_0}}{2} - \cos x\right) dx$$

$$= 2^n x_0 \pi + \int_0^{\pi} \ln\left(\frac{1 + e^{-2^{n+1}x_0}}{2} - e^{-2^n x_0}\cos x\right) dx,$$

所以

$$\frac{1}{2^n}I_n \to x_0\pi \quad (n \to +\infty).$$

故从式 (5) 可得

$$I_0 = x_0\pi - \pi\ln 2 = \pi\ln\left(\frac{a_0 + \sqrt{a_0^2 - 1}}{2}\right),$$

即所求的积分为

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\, dx = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right).$$

解法二: 我们有

$$F(a) = \int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\, dx = \int_0^{\pi} \ln(a - \cos x)\, dx.$$

由定理 **??**, 关于 a 求导得到

$$F'(a) = \int_0^{\pi} \frac{1}{a - \cos x}\, dx \xlongequal{\text{万能公式}} \int_0^{+\infty} \frac{2}{a(1 + t^2) - (1 - t^2)}\, dt = \frac{\pi}{\sqrt{a^2 - 1}}, \quad a > 1.$$

因此

$$F(a) = \int_1^a F'(a)\, da = \pi\ln\left(a + \sqrt{a^2 - 1}\right) + C, \quad a > 1.$$

结合

$$F(1) = 2\int_0^{\frac{\pi}{2}} \ln\sin x\, dx = -\pi\ln 2.$$

可得

$$\int_0^{\frac{\pi}{2}} \ln(a^2 - \cos^2 x)\, dx = \pi\ln\left(\frac{a + \sqrt{a^2 - 1}}{2}\right), \quad a > 1.$$

$\square$

定理 0.1 (周期函数在任意一个周期上的定积分相同)

设 f 是 $\mathbb{R}$ 上的可积函数且周期为 $T > 0$, 则对 $\forall a \in \mathbb{R}$, 都有

$$\int_a^{a+T} f(x)\, dx = \int_0^T f(x)\, dx.$$

♡

证明 由条件可得

$$\int_a^{a+T} f(x)\, dx = \int_a^{(\lfloor \frac{a}{T}\rfloor + 1)T} f(x)\, dx + \int_{(\lfloor \frac{a}{T}\rfloor + 1)T}^{a+T} f(x)\, dx$$

$$= \int_a^{(\lfloor \frac{a}{T} \rfloor + 1)T} f(x)\mathrm{d}x + \int_{\lfloor \frac{a}{T} \rfloor T}^a f(x + T)\mathrm{d}x$$

$$= \int_a^{(\lfloor \frac{a}{T} \rfloor + 1)T} f(x)\mathrm{d}x + \int_{\lfloor \frac{a}{T} \rfloor T}^a f(x)\mathrm{d}x$$

$$= \int_{\lfloor \frac{a}{T} \rfloor T}^{(\lfloor \frac{a}{T} \rfloor + 1)T} f(x)\mathrm{d}x$$

$$= \int_0^T f\left(x + \lfloor \frac{a}{T} \rfloor T\right) \mathrm{d}x.$$

$\square$

定理 0.2 (点火公式)

1. 记 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \mathrm{d}x, \forall n \in \mathbb{N}$, 则

$$I_n = \frac{n-1}{n} I_{n-2}, \quad \forall n \geqslant 2.$$

从而

$$I_n = \begin{cases} \dfrac{(n-1)!!}{n!!} I_0 = \dfrac{(n-1)!!}{n!!} \cdot \dfrac{\pi}{2} & , n\text{为偶数} \\ \dfrac{(n-1)!!}{n!!} I_1 = \dfrac{(n-1)!!}{n!!} & , n\text{为奇数} \end{cases}. \tag{6}$$

2. 记 $J(m,n) = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \mathrm{d}x, \forall n, m \in \mathbb{N}$, 则

$$J(m,n) = \frac{m-1}{m+n} J(m-2, n), \quad \forall n, m \geqslant 2.$$

$$J(m,n) = \frac{n-1}{m+n} J(m, n-2), \quad \forall n, m \geqslant 2.$$

从而

$$J(m,n) = \begin{cases} \dfrac{(m-1)!!(n-1)!!}{(m+n)!!} & , m, n\text{不全为偶数} \\ \dfrac{(m-1)!!(n-1)!!}{(m+n)!!} \cdot \dfrac{\pi}{2} & , m, n\text{全为偶数} \end{cases}. \tag{7}$$

3.

$\heartsuit$

0.1.2 区间再现

定理 0.3 (区间再现恒等式)

当下述积分有意义时, 我们有

1.
$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)]\mathrm{d}x.$$

2.
$$\int_0^\infty f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^\infty f(x)\mathrm{d}x = \int_0^1 \left[f(x) + \frac{f(\frac{1}{x})}{x^2} \right] \mathrm{d}x.$$

$\heartsuit$

笔记 注意: 倒代换具有将 $[0, 1]$ 转化为 $[1, +\infty)$ 的功能.

证明 证明是显然的.(第 1 问中最后一个等号是由轴对称得到的)

$\square$

命题 0.2

证明

1. $\displaystyle\int_0^{\frac{\pi}{2}} \ln \sin x \,\mathrm{d}x = -\frac{\pi}{2}\ln 2.$

2. $\displaystyle\int_0^{\frac{\pi}{2}} \ln \cos x \,\mathrm{d}x = -\frac{\pi}{2}\ln 2.$

3. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{8}\ln 2.$

证明

1.

$$
\begin{aligned}
I = \int_0^{\frac{\pi}{2}} \ln \sin x \,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \left[\ln \sin x + \ln \sin\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x \sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2}\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln \sin x \,\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
\Longrightarrow I &= \int_0^{\frac{\pi}{2}} \ln \cos x \,\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}
$$

2.

$$
\begin{aligned}
I = \int_0^{\frac{\pi}{2}} \ln \cos x \,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \ln \sin x \,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln \sin x + \ln \sin\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x \sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln \frac{1}{2}\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln \sin x \,\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
\Longrightarrow I &= \int_0^{\frac{\pi}{2}} \ln \cos x \,\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}
$$

3. 解法一:

$$
\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x \xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2}\,\mathrm{d}\tan\theta &= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta \cdot \ln(1+\tan\theta)}{\sec^2\theta}\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\frac{2}{1+\tan\theta}\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \ln 2\,\mathrm{d}\theta = \frac{\pi}{8}\ln 2.
\end{aligned}
$$

解法二:考虑含参量积分

$$
\varphi(\alpha) = \int_0^1 \frac{\ln(1+\alpha x)}{1+x^2}\,\mathrm{d}x, \quad \alpha \in [0,1].
$$

显然 $\varphi(0)=0, \varphi(1)=I$, 且函数 $\dfrac{\ln(1+\alpha x)}{1+x^2}$ 在 $R=[0,1]\times[0,1]$ 上满足定理**??** 的条件, 于是

$$
\varphi'(\alpha) = \int_0^1 \frac{x}{(1+x^2)(1+\alpha x)}\,\mathrm{d}x.
$$

因为

$$
\frac{x}{(1+x^2)(1+\alpha x)} = \frac{1}{1+\alpha^2}\left(\frac{\alpha+x}{1+x^2} - \frac{\alpha}{1+\alpha x}\right),
$$

所以

$$\varphi'(\alpha) = \frac{1}{1+\alpha^2}\left(\int_0^1 \frac{\alpha}{1+x^2}\,\mathrm{d}x + \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x - \int_0^1 \frac{\alpha}{1+\alpha x}\,\mathrm{d}x\right)$$

$$= \frac{1}{1+\alpha^2}\left[\alpha\arctan x\Big|_0^1 + \frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1 - \ln\left(1+\alpha x\right)\Big|_0^1\right]$$

$$= \frac{1}{1+\alpha^2}\left[\alpha\cdot\frac{\pi}{4} + \frac{1}{2}\ln 2 - \ln\left(1+\alpha\right)\right].$$

因此

$$\int_0^1 \varphi'(\alpha)\,\mathrm{d}\alpha = \int_0^1 \frac{1}{1+\alpha^2}\left[\frac{\pi}{4}\alpha + \frac{1}{2}\ln 2 - \ln\left(1+\alpha\right)\right]\,\mathrm{d}\alpha$$

$$= \frac{\pi}{8}\ln\left(1+\alpha^2\right)\Big|_0^1 + \frac{1}{2}\ln 2\arctan\alpha\Big|_0^1 - \varphi(1)$$

$$= \frac{\pi}{8}\ln 2 + \frac{\pi}{8}\ln 2 - \varphi(1)$$

$$= \frac{\pi}{4}\ln 2 - \varphi(1).$$

另一方面,

$$\int_0^1 \varphi'(\alpha)\,d\alpha = \varphi(1) - \varphi(0) = \varphi(1),$$

所以 $I = \varphi(1) = \dfrac{\pi}{8}\ln 2$.

$\square$

例题 0.3 计算

1. $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}\,\mathrm{d}x, a > 0.$
2. $\displaystyle\int_0^\infty \frac{\ln x}{x^2+x+1}\,\mathrm{d}x.$
3. $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x-x^2}}\,\mathrm{d}x.$

解

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2}\,\mathrm{d}x \xlongequal{x=at} \frac{1}{a}\int_0^{+\infty} \frac{\ln(at)}{1+t^2}\,\mathrm{d}t = \frac{1}{a}\int_0^{+\infty} \frac{\ln a}{1+t^2}\,\mathrm{d}t + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2}\,\mathrm{d}t = \frac{\pi\ln a}{2a} + \frac{1}{a}\int_0^{+\infty} \frac{\ln t}{1+t^2}\,\mathrm{d}t. \quad (8)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2}\,\mathrm{d}t \xlongequal{t=\frac{1}{x}} \int_0^{+\infty} \frac{\ln\frac{1}{x}}{1+\frac{1}{x^2}}\frac{1}{x^2}\,\mathrm{d}x = \int_0^{+\infty} \frac{-\ln x}{1+x^2}\,\mathrm{d}x \implies \int_0^{+\infty} \frac{\ln t}{1+t^2}\,\mathrm{d}t = 0.$$

于是再结合(8)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi\ln a}{2a}.$$

2.

$$\int_0^\infty \frac{\ln x}{x^2+x+1}\,\mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_0^\infty \frac{-\ln t}{1+\frac{1}{t}+\frac{1}{t^2}}\,\mathrm{d}\frac{1}{t} = \int_0^{+\infty} \frac{-\ln t}{1+t+t^2}\,\mathrm{d}t \implies \int_0^\infty \frac{\ln x}{x^2+x+1}\,\mathrm{d}x = 0.$$

3.

$$\int_0^1 \frac{\ln x}{\sqrt{x-x^2}}\,\mathrm{d}x \xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}} \frac{\ln\sin^2 y}{\sqrt{\sin^2 y(1-\sin^2 y)}}\,\mathrm{d}\sin^2 y$$

$$= 4\int_0^{\frac{\pi}{2}} \ln\sin y\,\mathrm{d}y \xlongequal{\text{命题 0.2}} 4\cdot\left(-\frac{\pi}{2}\ln 2\right) = -2\pi\ln 2.$$

$\square$

例题 0.4

1. 对 $n\in\mathbb{N}$, 计算 $\displaystyle\int_{-\pi}^\pi \frac{\sin(nx)}{(1+2^x)\sin x}\,\mathrm{d}x.$

2. $\displaystyle\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1 + \cos^2 x}\mathrm{d}x.$

3. 对 $n \in \mathbb{N}$, 计算 $\displaystyle\int_0^{2\pi} \sin(\sin x + nx)\mathrm{d}x.$

解

1.

$$\int_{-\pi}^{\pi} \frac{\sin (nx)}{(1+2^x)\sin x}\mathrm{d}x = \int_{-\pi}^{0} \left[\frac{\sin (nx)}{(1+2^x)\sin x} + \frac{\sin (nx)}{(1+2^{-x})\sin x} \right]\mathrm{d}x = \int_{-\pi}^{0} \frac{\sin (nx)}{\sin x} \left(\frac{1}{1+2^x} + \frac{1}{1+2^{-x}} \right)\mathrm{d}x$$

$$= \int_{-\pi}^{0} \frac{\sin (nx)}{\sin x} \cdot \frac{2 + 2^x + 2^{-x}}{2 + 2^x + 2^{-x}}\mathrm{d}x = \int_{-\pi}^{0} \frac{\sin (nx)}{\sin x}\mathrm{d}x = \int_{0}^{\pi} \frac{\sin (nx)}{\sin x}\mathrm{d}x \xlongequal{\text{例题 0.1}} \begin{cases} 0, n\text{为偶数} \\ \pi, n\text{为奇数} \end{cases}.$$

2.

$$\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1 + \cos^2 x}\mathrm{d}x = \int_{-\pi}^{0} \left(\frac{x \sin x \arctan e^x}{1 + \cos^2 x} + \frac{x \sin x \arctan e^{-x}}{1 + \cos^2 x} \right)\mathrm{d}x = \int_{-\pi}^{0} \frac{x \sin x}{1 + \cos^2 x} (\arctan e^x + \arctan e^{-x})\mathrm{d}x$$

$$\xlongequal{\text{命题??(1)}} \int_{-\pi}^{0} \frac{x \sin x}{1 + \cos^2 x} \cdot \frac{\pi}{2}\mathrm{d}x = \frac{\pi}{2} \int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x}\mathrm{d}x$$

$$= \frac{\pi}{2} \int_{0}^{\frac{\pi}{2}} \left(\frac{x \sin x}{1 + \cos^2 x} + \frac{(\pi - x) \sin x}{1 + \cos^2 x} \right)\mathrm{d}x = \frac{\pi^2}{2} \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\mathrm{d}x$$

$$= \frac{\pi^2}{2} \arctan \cos x \Big|_{\frac{\pi}{2}}^{0} = \frac{\pi^2}{2} \cdot \frac{\pi}{4} = \frac{\pi^3}{8}.$$

3.

$$\int_{0}^{2\pi} \sin (\sin x + nx)\mathrm{d}x = \int_{0}^{2\pi} \sin [\sin (2\pi - x) + n (2\pi - x)]\mathrm{d}x$$

$$= \int_{0}^{2\pi} \sin (- \sin x - nx)\mathrm{d}x = - \int_{0}^{2\pi} \sin (\sin x + nx)\mathrm{d}x$$

$$\implies \int_{0}^{2\pi} \sin (\sin x + nx)\mathrm{d}x = 0.$$

$\square$

例题 0.5 计算 $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}x}{\left(1 + x^2\right)\left(1 + x^{2019}\right)}.$

解

$$\int_0^{+\infty} \frac{\mathrm{d}x}{\left(1 + x^2\right)\left(1 + x^{2019}\right)} = \int_0^{+\infty} \frac{\mathrm{d}x}{\left(1 + x^2\right)\left(1 + x^{2019}\right)}$$

$$\xlongequal{t = \frac{1}{x}} \int_0^{+\infty} \frac{t^{2019}\mathrm{d}t}{\left(1 + t^2\right)\left(1 + t^{2019}\right)} = \frac{1}{2} \int_0^{+\infty} \frac{\left(1 + x^{2019}\right)\mathrm{d}x}{\left(1 + x^2\right)\left(1 + x^{2019}\right)}$$

$$= \frac{1}{2} \int_0^{+\infty} \frac{\mathrm{d}x}{1 + x^2} = \frac{\pi}{4}.$$

$\square$

0.1.3 Frullani(傅汝兰尼) 积分

定理 0.4 (Frullani(傅汝兰尼) 积分)

设 $f \in C(0, +\infty)$.

1. 若存在极限

$$\lim_{x \to 0^+} f(x), \lim_{x \to +\infty} f(x), \tag{9}$$

则对 $a, b > 0$ 有

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x}\mathrm{d}x = \left[\lim_{x \to 0^+} f(x) - \lim_{x \to +\infty} f(x) \right] \ln \frac{b}{a}.$$

2. 若存在极限和积分

$$\lim_{x \to 0^+} f(x) = \alpha, \int_A^\infty \frac{f(x)}{x} \mathrm{d}x. \tag{10}$$

则对 $a, b > 0$, 有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \mathrm{d}x = \alpha \ln \frac{b}{a}.$$

3. 若存在极限和积分

$$\lim_{x \to +\infty} f(x) = \alpha, \int_0^1 \frac{f(x)}{x} \mathrm{d}x. \tag{11}$$

则对 $a, b > 0$, 有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \mathrm{d}x = \alpha \ln \frac{a}{b}.$$

4. 若 f 是周期 $T > 0$ 函数且 $\lim_{x \to 0^+} f(x)$ 存在, 则对 $a, b > 0$ 有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \mathrm{d}x = \left[\lim_{x \to 0^+} f(x) - \frac{1}{T} \int_0^T f(x)\mathrm{d}x \right] \ln \frac{b}{a}.$$

5. 若 f 满足 $\lim_{x \to 0^+} f(x)$, $\lim_{x \to +\infty} \frac{1}{x} \int_0^x f(y)\mathrm{d}y$ 存在, 则对 $a, b > 0$ 有

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \mathrm{d}x = \left[\lim_{x \to 0^+} f(x) - \lim_{x \to +\infty} \frac{1}{x} \int_0^x f(y)\mathrm{d}y \right] \ln \frac{b}{a}.$$

♡

✎ 笔记 傅汝兰尼积分有诸多变种, 无需记忆具体表达式, 知道有大概这么一个东西即可.

证明 不妨设 $b > a$.

1. 给定 $A > \delta > 0$, 考虑

$$\begin{aligned}
\int_\delta^A \frac{f(ax) - f(bx)}{x} \mathrm{d}x &= \int_\delta^A \frac{f(ax)}{x} \mathrm{d}x - \int_\delta^A \frac{f(bx)}{x} \mathrm{d}x \\
&= \int_{a\delta}^{aA} \frac{f(x)}{x} \mathrm{d}x - \int_{b\delta}^{bA} \frac{f(x)}{x} \mathrm{d}x \\
&= \int_{bA}^{aA} \frac{f(x)}{x} \mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x} \mathrm{d}x \\
&\xupdownarrow{积分中值定理} f(\theta_1) \int_{bA}^{aA} \frac{1}{x} \mathrm{d}x - f(\theta_2) \int_{b\delta}^{a\delta} \frac{1}{x} \mathrm{d}x,
\end{aligned}$$

这里 $\theta_1 \in (aA, bA), \theta_2 \in (a\delta, b\delta)$, 于是让 $A \to +\infty, \delta \to 0^+$, 由 (9), 我们知

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \mathrm{d}x = \left[\lim_{x \to 0^+} f(x) - \lim_{x \to +\infty} f(x) \right] \ln \frac{b}{a}.$$

2. 给定 $A > \delta > 0$, 考虑

$$\begin{aligned}
\int_\delta^A \frac{f(ax) - f(bx)}{x} \mathrm{d}x &= \int_\delta^A \frac{f(ax)}{x} \mathrm{d}x - \int_\delta^A \frac{f(bx)}{x} \mathrm{d}x \\
&= \int_{a\delta}^{aA} \frac{f(x)}{x} \mathrm{d}x - \int_{b\delta}^{bA} \frac{f(x)}{x} \mathrm{d}x \\
&= \int_{bA}^{aA} \frac{f(x)}{x} \mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x} \mathrm{d}x \\
&\xupdownarrow{积分中值定理} \int_{bA}^{aA} \frac{f(x)}{x} \mathrm{d}x - f(\theta) \int_{b\delta}^{a\delta} \frac{1}{x} \mathrm{d}x,
\end{aligned}$$

这里 $\theta \in (a\delta, b\delta)$, 于是让 $A \to +\infty, \delta \to 0^+$, 由 (10), 我们知

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \mathrm{d}x = \alpha \ln \frac{b}{a}.$$

3. 给定 $A > \delta > 0$, 考虑

$$\int_\delta^A \frac{f(ax) - f(bx)}{x}\mathrm{d}x = \int_\delta^A \frac{f(ax)}{x}\mathrm{d}x - \int_\delta^A \frac{f(bx)}{x}\mathrm{d}x$$

$$= \int_{a\delta}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{bA} \frac{f(x)}{x}\mathrm{d}x$$

$$= \int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x$$

$$\xLongequal{\text{积分中值定理}} f(\theta)\int_{bA}^{aA} \frac{1}{x}\mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x,$$

这里 $\theta \in (aA, bA)$, 于是让 $A \to +\infty, \delta \to 0^+$, 由(11), 我们知

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\mathrm{d}x = \alpha \ln \frac{a}{b}.$$

4. 给定 $A > \delta > 0$, 考虑

$$\int_\delta^A \frac{f(ax) - f(bx)}{x}\mathrm{d}x = \int_\delta^A \frac{f(ax)}{x}\mathrm{d}x - \int_\delta^A \frac{f(bx)}{x}\mathrm{d}x$$

$$= \int_{a\delta}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{bA} \frac{f(x)}{x}\mathrm{d}x$$

$$= \int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x - \int_{b\delta}^{a\delta} \frac{f(x)}{x}\mathrm{d}x$$

$$\xLongequal{\text{积分中值定理}} \int_{bA}^{aA} \frac{f(x)}{x}\mathrm{d}x - f(\theta)\int_{b\delta}^{a\delta} \frac{1}{x}\mathrm{d}x$$

$$= \int_b^a \frac{f(Ax)}{x}\mathrm{d}x - f(\theta)\int_{b\delta}^{a\delta} \frac{1}{x}\mathrm{d}x,$$

这里 $\theta \in (a\delta, b\delta)$. 现在

$$\lim_{\delta \to 0^+}\left(-f(\theta)\int_{b\delta}^{a\delta} \frac{1}{x}\mathrm{d}x\right) = \lim_{x \to 0^+} f(x)\ln\frac{b}{a}.$$

由 Riemann 引理, 我们有

$$\lim_{A \to +\infty}\int_b^a \frac{f(Ax)}{x}\mathrm{d}x = \int_b^a \frac{1}{x}\mathrm{d}x \cdot \frac{1}{T}\int_0^T f(x)\mathrm{d}x = -\frac{1}{T}\int_0^T f(x)\mathrm{d}x \cdot \ln\frac{b}{a},$$

这就证明了

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\mathrm{d}x = \left[\lim_{x \to 0^+} f(x) - \frac{1}{T}\int_0^T f(x)\mathrm{d}x\right]\ln\frac{b}{a}.$$

5. 上一问证明中把使用的 Riemann 引理用平均值极限版本的 Riemann 引理代替即可.

$\square$

0.1.4 化成多元累次积分 (换序)

> **命题 0.3**
>
> 证明:
> (1) $\displaystyle\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.
> (2) $\displaystyle\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$.
> (3) $\displaystyle\int_0^{+\infty} \frac{\sin^2 x}{x^2}\mathrm{d}x = \frac{\pi}{2}$.
> (4) $\displaystyle\int_0^\infty \sin x^2\mathrm{d}x, \int_0^\infty \cos x^2\mathrm{d}x = \sqrt{\frac{\pi}{8}}$.

🎗 **笔记** 本结果可以直接使用.

证明

(1) 注意到

$$\left(\int_0^{+\infty} e^{-x^2} \mathrm{d}x\right)^2 = \left(\int_0^{+\infty} e^{-x^2} \mathrm{d}x\right)\left(\int_0^{+\infty} e^{-y^2} \mathrm{d}y\right) \xop{把\int_0^{+\infty} e^{-y^2}\mathrm{d}y看作常数} \int_0^{+\infty} e^{-x^2}\left(\int_0^{+\infty} e^{-y^2} \mathrm{d}x\right) \mathrm{d}y$$

$$\xop{把e^{-x^2}看作常数} \int_0^{+\infty}\left(\int_0^{+\infty} e^{-(x^2+y^2)} \mathrm{d}x\right) \mathrm{d}y \xop{e^{-(x^2+y^2)}连续} \iint_{R^2} e^{-(x^2+y^2)} \mathrm{d}x\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{+\infty} r e^{-r^2} \mathrm{d}r = \frac{\pi}{2}\int_0^{+\infty} r e^{-r^2} \mathrm{d}r$$

$$= \frac{\pi}{4}\int_0^{+\infty} e^{-r^2} \mathrm{d}r^2 = \frac{\pi}{4}.$$

故 $\displaystyle\int_0^{\infty} e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$.

(2) 注意到

$$\int_0^{+\infty} \sin x\, e^{-yx} \mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx} \mathrm{d}x = \mathrm{Im}\int_0^{+\infty} e^{-(y-\mathrm{i})x} \mathrm{d}x = \mathrm{Im}\frac{1}{y-\mathrm{i}} = \mathrm{Im}\frac{y+\mathrm{i}}{y^2+1} = \frac{1}{y^2+1}.$$

因此就有

$$\int_0^{+\infty}\frac{\sin x}{x} \mathrm{d}x = \int_0^{+\infty} \sin x\left(\int_0^{+\infty} e^{-yx} \mathrm{d}y\right) \mathrm{d}x = \int_0^{+\infty} \mathrm{d}y\int_0^{+\infty} \sin x\, e^{-yx} \mathrm{d}x$$

$$= \int_0^{+\infty} \mathrm{d}y\left(\mathrm{Im}\int_0^{+\infty} e^{\mathrm{i}x-yx}\right) \mathrm{d}x = \int_0^{+\infty}\frac{1}{y^2+1} \mathrm{d}y = \frac{\pi}{2}.$$

当然本题也可以直接利用分部积分计算 $\displaystyle\int_0^{+\infty} \sin x\, e^{-yx} \mathrm{d}x = \frac{1}{y^2+1}$.

(3) 由分部积分可得

$$\int_0^{+\infty}\frac{\sin^2 x}{x^2} \mathrm{d}x = \frac{1}{2}\int_0^{+\infty}\frac{1-\cos 2x}{x^2} \mathrm{d}x = -\frac{1}{2}\int_0^{+\infty}(1-\cos 2x) \mathrm{d}\frac{1}{x}$$

$$= \frac{1}{2}\frac{1-\cos x}{x^2}\Big|_{+\infty}^0 + \frac{1}{2}\int_0^{+\infty}\frac{2\sin 2x}{x} \mathrm{d}x$$

$$= \int_0^{+\infty}\frac{\sin x}{x} \mathrm{d}x \xop{命题 0.3(2)} \frac{\pi}{2}.$$

(4) 注意到

$$\int_0^{+\infty} e^{-ax^2} \mathrm{d}x \xop{x=\frac{t}{\sqrt{a}}} \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-t^2} \mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a > 0.$$

并且 $-\mathrm{i} = e^{-\frac{\pi}{2}\mathrm{i}}$, 从而 $\sqrt{-\mathrm{i}} = e^{-\frac{\pi}{4}\mathrm{i}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}$. 于是

$$\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2) \mathrm{d}x = \int_0^{+\infty} e^{-\mathrm{i}x^2} \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\mathrm{i}}} = \frac{1}{2}\sqrt{-\mathrm{i}\pi}$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}.$$

故

$$\int_0^{+\infty} \cos x^2 \mathrm{d}x = \mathrm{Re}\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2) \mathrm{d}x = \mathrm{Re}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}},$$

$$\int_0^{+\infty} \sin x^2 \mathrm{d}x = \mathrm{Im}\int_0^{+\infty}(\cos x^2 - \mathrm{i}\sin x^2) \mathrm{d}x = \mathrm{Im}\left(\frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}\mathrm{i}\right) = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}.$$

$\square$

例题 0.6 计算 $\displaystyle\int_0^1 \sin\ln\frac{1}{x} \cdot \frac{x^b - x^a}{\ln x} \mathrm{d}x\ (b > a > 0)$.

证明

$$\int_0^1 \sin\ln\frac{1}{x} \cdot \frac{x^b - x^a}{\ln x}\mathrm{d}x = \int_0^1 \sin\ln\frac{1}{x}\left(\int_a^b x^y\mathrm{d}y\right)\mathrm{d}x = \int_a^b \mathrm{d}y\int_0^1 x^y\sin\ln\frac{1}{x}\mathrm{d}x$$

$$\xrightarrow{x=e^{-t}} \int_a^b \mathrm{d}y\int_{+\infty}^0 e^{-ty}\sin t\mathrm{d}e^{-t} = \int_a^b \mathrm{d}y\int_0^{+\infty} e^{-t(y+1)}\sin t\mathrm{d}t$$

$$\underset{\text{命题 0.3(2) 的证明过程}}{=\!=\!=\!=\!=} \int_a^b \frac{1}{1+(y+1)^2}\mathrm{d}y = \arctan(b+1) - \arctan(a+1).$$

$\square$

0.1.5 化成含参积分 (求导)

例题 0.7 设 $a, b \geqslant 0$ 且不全为 0, 计算 $\displaystyle\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$.

注 实际上, 根据 $a > b$ 时得到的结果, 可以看出 $F(a,b) = \pi\ln\dfrac{a+b}{2}$ 对 a, b 有轮换对称性, 故这个结果对其他情况显然也成立.

证明 设 $F(a,b) = \displaystyle\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x$, 当 $a > b$ 时, 则

$$\frac{\partial}{\partial b}F(a,b) = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial b}\ln\left(a^2\cos^2 x + b^2\sin^2 x\right)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{2b\sin^2 x}{a^2\cos^2 x + b^2\sin^2 x}\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \frac{2b\tan^2 x}{a^2 + b^2\tan^2 x}\mathrm{d}x = \int_0^{+\infty} \frac{2bt^2}{(a^2+b^2t^2)(1+t^2)}\mathrm{d}t$$

$$= \frac{1}{a^2-b^2}\int_0^{+\infty} \left(\frac{2a^2 b}{a^2+b^2t^2} - \frac{2b}{1+t^2}\right)\mathrm{d}t$$

$$= \frac{1}{a^2-b^2}\int_0^{+\infty} \frac{2a^2 b}{a^2+b^2t^2}\mathrm{d}t - \frac{2b}{a^2-b^2}\int_0^{+\infty} \frac{1}{1+t^2}\mathrm{d}t$$

$$= \frac{2b}{a^2-b^2}\int_0^{+\infty} \frac{1}{1+\left(\frac{b}{a}t\right)^2}\mathrm{d}t - \frac{b\pi}{a^2-b^2}$$

$$= \frac{2b}{a^2-b^2}\cdot\frac{a}{b}\cdot\frac{\pi}{2} - \frac{b\pi}{a^2-b^2} = \frac{\pi}{a+b}.$$

于是

$$F(a,b) = F(a,0) + \int_0^b \frac{\partial}{\partial b'}F(a,b')\mathrm{d}b' = F(a,0) + \int_0^b \frac{\pi}{a+b'}\mathrm{d}b'$$

$$= 2\int_0^{\frac{\pi}{2}} \ln(a\cos x)\mathrm{d}x + \pi\ln\frac{a+b}{a} \underset{\text{例题 0.2}}{=\!=\!=\!=} \pi\ln\frac{a+b}{2}.$$

当 $a < b$ 时, 类似可得 $F(a,b) = \pi\ln\dfrac{a+b}{2}$. 当 $a = b$ 时, $F(a,b) = \displaystyle\int_0^{\frac{\pi}{2}} \ln a^2\mathrm{d}x = \pi\ln a = \pi\ln\dfrac{a+b}{2}$.

综上, 对 $\forall a, b \geqslant 0$, 都有 $F(a,b) = \pi\ln\dfrac{a+b}{2}$.

$\square$

0.1.6 级数展开方法

积分和求和换序 $\displaystyle\sum_{n=1}^{\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x$, 等价于

$$\lim_{m\to\infty}\sum_{n=1}^{m}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \sum_{n=1}^{\infty} f_n(x)\mathrm{d}x.$$

又由于有限和随意交换, 因此上式等价于

$$\lim_{m \to \infty} \int_a^b \sum_{n=1}^m f_n(x) \mathrm{d}x = \int_a^b \sum_{n=1}^\infty f_n(x) \mathrm{d}x,$$

于是

$$\sum_{n=1}^\infty \int_a^b f_n(x) \mathrm{d}x = \int_a^b \sum_{n=1}^\infty f_n(x) \mathrm{d}x \Longleftrightarrow \lim_{m \to \infty} \int_a^b \sum_{n=m+1}^\infty f_n(x) \mathrm{d}x = 0.$$

例题 0.8 计算 $\int_0^\infty \dfrac{x}{1+e^x} \mathrm{d}x$.

解 由于

$$\frac{1}{1+x} = \sum_{n=0}^\infty (-1)^n x^n, \quad 0 < x < 1.$$

并且 $0 < e^{-x} < 1$, 故

$$\int_0^{+\infty} \frac{x}{1+e^x} \mathrm{d}x = \int_0^{+\infty} \frac{xe^{-x}}{1+e^{-x}} \mathrm{d}x = \int_0^{+\infty} x \sum_{n=0}^\infty (-1)^n e^{-(n+1)x} \mathrm{d}x$$

$$\xlongequal{\text{换序}} \sum_{n=0}^\infty (-1)^n \int_0^{+\infty} xe^{-(n+1)x} \mathrm{d}x = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}$$

$$= \sum_{n=0}^\infty \frac{1}{(2n+1)^2} - \sum_{n=0}^\infty \frac{1}{(2n)^2}. \tag{12}$$

又因为 $\sum_{n=1}^\infty \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, 所以

$$\sum_{n=0}^\infty \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{24},$$

$$\sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \sum_{n=1}^\infty \frac{1}{n^2} - \sum_{n=0}^\infty \frac{1}{(2n)^2} = \frac{\pi^2}{8}.$$

故

$$\int_0^{+\infty} \frac{x}{1+e^x} \mathrm{d}x = \sum_{n=0}^\infty \frac{1}{(2n+1)^2} - \sum_{n=0}^\infty \frac{1}{(2n)^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.$$

下面证明(12)式换序成立, 等价于证明 $\lim\limits_{m \to +\infty} \int_0^{+\infty} \sum\limits_{n=m}^\infty x(-1)^n e^{-(n+1)x} \mathrm{d}x = 0$. 由交错级数不等式及 $xe^{-(n+1)x}$ 关于 n 非负递减, 对 $\forall m \in \mathbb{N}$, 都有

$$\int_0^{+\infty} \left| \sum_{n=m}^\infty x(-1)^n e^{-(n+1)x} \right| \mathrm{d}x \leqslant \int_0^{+\infty} xe^{-(m+1)x} \mathrm{d}x = -\frac{xe^{-(m+1)x}}{m+1} \Big|_0^{+\infty} + \frac{1}{m+1} \int_0^{+\infty} e^{-(m+1)x} \mathrm{d}x = \frac{1}{(m+1)^2}.$$

令 $m \to +\infty$, 得 $\lim\limits_{m \to +\infty} \int_0^{+\infty} \sum\limits_{n=m}^\infty x(-1)^n e^{-(n+1)x} \mathrm{d}x = 0$. 故(12)式换序成立.

$\square$

> **命题 0.4**
>
> 证明:
>
> (1) $\displaystyle\sum_{n=1}^\infty \frac{q^n \sin(nx)}{n} = \arctan \frac{q \sin x}{1 - q \cos x}, |q| \leqslant 1.$
>
> (2) $\displaystyle\sum_{n=1}^\infty \frac{q^n \cos(nx)}{n} = -\frac{1}{2} \ln(1 + q^2 - 2q \cos x), |q| \leqslant 1.$

(3) $\sum_{n=1}^{\infty} \dfrac{q^n \cos(nx)}{n!} = e^{q\cos x} \cos(q\sin x) - 1, |q| \leqslant 1, x \in \mathbb{R}$.

(4) $\sum_{n=1}^{\infty} \dfrac{q^n \sin(nx)}{n!} = e^{q\cos x} \sin(q\sin x), |q| \leqslant 1, x \in \mathbb{R}$.

笔记 在 $\mathbb{C}$ 上,

$$\text{Ln}z = \ln|z| + i(\arg z + 2k\pi), k \in \mathbb{Z}.$$

我们定义主值支

$$\ln z = \ln|z| + i\arg z.$$

本部分内容无需记忆,只需要大概有个可以算的感觉即可,实际做题中可以围绕这种级数给出构造.

证明 $\mathfrak{I}$ 表示取虚部,$\mathfrak{R}$ 表示取实部.

(1) 利用欧拉公式有

$$\sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n} = \mathfrak{I}\left(\sum_{n=1}^{\infty} \frac{q^n e^{inx}}{n}\right) = \mathfrak{I}\left(\sum_{n=1}^{\infty} \frac{(qe^{ix})^n}{n}\right) = \mathfrak{I}(-\ln(1 - qe^{ix}))$$

$$= -\mathfrak{I}\left(\ln|1 - qe^{ix}| + i\frac{-q\sin x}{1 - q\cos x}\right) = \arctan\frac{q\sin x}{1 - q\cos x}.$$

(2) 利用欧拉公式有

$$\sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n} = -\mathfrak{R}\left(\ln|1 - qe^{ix}| + i\frac{-q\sin x}{1 - q\cos x}\right) = -\frac{1}{2}\ln\left[(1 - q\cos x)^2 + q^2\sin^2 x\right]$$

$$= -\frac{1}{2}\ln(1 + q^2 - 2q\cos x).$$

(3) 利用欧拉公式有

$$\sum_{n=1}^{\infty} \frac{q^n \cos(nx)}{n!} = \mathfrak{R}\left(\sum_{n=1}^{\infty} \frac{(qe^{ix})^n}{n!}\right) = \mathfrak{R}\left(e^{qe^{ix}} - 1\right) = \mathfrak{R}\left(e^{q\cos x + iq\sin x} - 1\right)$$

$$= \mathfrak{R}\left(e^{q\cos x}\cos(q\sin x) - 1 + ie^{q\cos x}\sin(q\sin x)\right)$$

$$= e^{q\cos x}\cos(q\sin x) - 1.$$

(4) 利用 (3) 有

$$\sum_{n=1}^{\infty} \frac{q^n \sin(nx)}{n!} = \mathfrak{I}\left(e^{q\cos x}\cos(q\sin x) - 1 + ie^{q\cos x}\sin(q\sin x)\right)$$

$$= e^{q\cos x}\sin(q\sin x).$$

$\square$

例题 0.9 计算

1. $\displaystyle\int_0^{2\pi} e^{\cos x}\cos(\sin x)\mathrm{d}x$.

2. $\displaystyle\int_0^{\pi} \ln(1 - 2a\cos x + a^2)\mathrm{d}x, a \in (0, +\infty) \setminus \{1\}$.

注 由 1 的证明可得

$$e^{\cos x}\cos(\sin x) = \text{Re}\left(\sum_{n=0}^{\infty} \frac{(e^{ix})^n}{n!}\right) = \text{Re}\left(\sum_{n=0}^{\infty} \frac{e^{inx}}{n!}\right) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}.$$

实际上,上式就是命题 0.4(3) 的结论.

注 第 2 问也可以用含参积分求导的方法进行计算 (这个方法更容易想到).

证明

1.

$$\int_0^{2\pi} e^{\cos x} \cos(\sin x)\, dx = \operatorname{Re}\left(\int_0^{2\pi} e^{\cos x} e^{\mathrm{i}\sin x} dx\right) = \operatorname{Re}\left(\int_0^{2\pi} e^{\cos x + \mathrm{i}\sin x} dx\right)$$

$$= \operatorname{Re}\left(\int_0^{2\pi} e^{e^{\mathrm{i}x}} dx\right) = \operatorname{Re}\left[\int_0^{2\pi} \sum_{n=0}^{+\infty} \frac{(e^{\mathrm{i}x})^n}{n!} dx\right] = \operatorname{Re}\left[\sum_{n=0}^{+\infty} \int_0^{2\pi} \frac{(e^{\mathrm{i}x})^n}{n!} dx\right]$$

$$= \operatorname{Re}\left(\sum_{n=0}^{+\infty} \int_0^{2\pi} \frac{e^{\mathrm{i}nx}}{n!} dx\right) = \operatorname{Re}\left(\int_0^{2\pi} \frac{e^{\mathrm{i}\cdot 0\cdot x}}{n!} dx + \sum_{n=1}^{+\infty} \frac{e^{2\pi \mathrm{i}x} - 1}{\mathrm{i}n \cdot n!}\right)$$

$$= \operatorname{Re}\left(\int_0^{2\pi} 1 dx + 0\right) = 2\pi.$$

2. 注意到当 $a \in (0, 1)$ 时, 有

$$\sum_{n=1}^{\infty} \frac{a^n \cos(nx)}{n} = \operatorname{Re}\left[\sum_{n=1}^{\infty} \frac{(ae^{\mathrm{i}x})^n}{n}\right] = -\operatorname{Re}\left[\ln(1 - ae^{\mathrm{i}x})\right]$$

$$= -\operatorname{Re}\left[\ln|1 - ae^{\mathrm{i}x}| + \mathrm{i}\arg(1 - ae^{\mathrm{i}x})\right] = -\ln|1 - ae^{\mathrm{i}x}|$$

$$= -\ln|(1 - a\cos x) + a\mathrm{i}\sin x| = -\frac{1}{2}\ln(1 + a^2 - 2a\cos x).$$

于是当 $a \in (0, 1)$ 时, 就有

$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2) dx = -\frac{1}{2}\int_0^{\pi} \sum_{n=1}^{\infty} \frac{a^n \cos(nx)}{n} dx = 0.$$

若 $a > 1$, 则 $\dfrac{1}{a} \in (0, 1)$, 从而此时我们有

$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2) dx = \pi \ln a^2 + \int_0^{\pi} \ln\left(\frac{1}{a^2} - \frac{2}{a}\cos x + 1\right) dx = \pi \ln a^2 = 2\pi \ln a.$$

又由 $\ln(1 - 2a\cos x + a^2)$ 关于 a 的偏导存在可知 $\displaystyle\int_0^{\pi} \ln(1 - 2a\cos x + a^2) dx$ 关于 a 连续. 于是由

$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2) dx = 2\pi \ln a, \quad \forall a > 1.$$

可知当 $a = 1$ 时, 我们有

$$\int_0^{\pi} \ln(2 - 2\cos x) dx = \lim_{a\to 1^+} \int_0^{\pi} \ln(1 - 2a\cos x + a^2) dx = \lim_{a\to 1^+} (2\pi \ln a) = 0.$$

$\square$

定义 0.1 (多重对数函数-Li_2 函数)

定义

$$\mathrm{Li}_2(x) \triangleq \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad x \in [-1, 1].$$

♣

命题 0.5

(1) $\mathrm{Li}_2(x) + \mathrm{Li}_2(1 - x) = \dfrac{\pi^2}{6} - \ln x \cdot \ln(1 - x), x \in (0, 1)$.

(2) $\mathrm{Li}_2(1) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \mathrm{Li}_2(0) = 0, \quad \mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 \frac{1}{2}$.

♠

证明

(1) 记 $f(x) \triangleq \mathrm{Li}_2(x), F(x) \triangleq f(x) + f(1-x) + \ln x \ln(1-x)$. 则

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\frac{1}{x} \ln(1-x).$$

于是

$$F'(x) = -\frac{1}{x} \ln(1-x) + \frac{\ln x}{1-x} - \frac{\ln x}{1-x} + \frac{\ln(1-x)}{x} = 0.$$

故 $F(x) \equiv F(1) = f(0) + f(1) = 0 + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

(2) 显然 $\mathrm{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \mathrm{Li}_2(0) = 0.$ 由 (1) 可得

$$\mathrm{Li}_2\left(\frac{1}{2}\right) + \mathrm{Li}_2\left(\frac{1}{2}\right) = 2\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - \ln^2 \frac{1}{2} \implies \mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}.$$

□

例题 0.10 计算 $\int_0^{\frac{1}{2}} \frac{\ln x}{1-x} \, \mathrm{d}x$.

解

$$\int_0^{\frac{1}{2}} \frac{\ln x}{1-x} \, \mathrm{d}x = \int_{\frac{1}{2}}^1 \frac{\ln(1-x)}{x} \, \mathrm{d}x = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{2}}^1 x^{n-1} \, \mathrm{d}x$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{2^n n^2} = -\frac{\pi^2}{6} + \mathrm{Li}_2\left(\frac{1}{2}\right)$$

$$\stackrel{\text{命题 0.5}}{=\!=\!=\!=} -\frac{\pi^2}{12} - \frac{1}{2} \ln^2 \frac{1}{2}.$$

□

0.1.7 重积分计算

定理 0.5 (二重积分换序)

证明:

$$\int_a^b \mathrm{d}x \int_a^x f(x,y) \mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y) \mathrm{d}x, \tag{10}$$

其中 $f(x,y)$ 是在由直线 $y = a, x = b, y = x$ 所围成的三角形 (Δ) 上连续的任意函数.

证明

□

命题 0.6

设 $f(x)$ 在 $[a,b]$ 上连续, 试证: 对任意 $x \in (a,b)$, 有

$$\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_n} f(x_{n+1}) \mathrm{d}x_{n+1} = \frac{1}{n!} \int_a^x (x-y)^n f(y) \mathrm{d}y, \quad n = 1, 2, \cdots.$$

证明 当 $n = 1$ 时, 由二重积分换序可知

$$\int_a^x \mathrm{d}x_1 \int_a^{x_1} f(x_2) \mathrm{d}x_2 = \int_a^x \mathrm{d}x_2 \int_{x_2}^x f(x_2) \mathrm{d}x_1 = \int_a^x (x-x_2) f(x_2) \mathrm{d}x_2 = \int_a^x (x-y) f(y) \mathrm{d}y.$$

设原结论对 $n = k-1$ 的情形成立, 考虑 $n = k$ 的情形, 由归纳假设可知

$$\int_a^{x_1} \mathrm{d}x_1 \cdots \int_a^{x_k} f(x_{k+1}) \mathrm{d}x_{k+1} = \frac{1}{(k-1)!} \int_a^{x_1} (x_1 - y)^{k-1} f(y) \mathrm{d}y.$$

于是再利用二重积分换序得

$$\int_a^x \mathrm{d}x_1 \int_a^{x_1} \mathrm{d}x_2 \cdots \int_a^{x_k} f(x_{k+1})\mathrm{d}x_{k+1} = \frac{1}{(k-1)!}\int_a^x \mathrm{d}x_1 \int_a^{x_1}(x_1-y)^{k-1}f(y)\mathrm{d}y$$

$$= \frac{1}{(k-1)!}\int_a^x \mathrm{d}y \int_y^x (x_1-y)^{k-1}f(y)\mathrm{d}x_1$$

$$= \frac{1}{k!}\int_a^x (x-y)^k f(y)\mathrm{d}y.$$

故由数学归纳法知原结论成立.

$\square$

例题 0.11 求定义在星形区域 $D = \{(x,y) \mid x^{\frac{2}{3}} + y^{\frac{2}{3}} \leqslant 1\}$ 上满足 $f(1,0) = 1$ 的正值连续函数 f 使得 $\displaystyle\iint\limits_D \frac{f(x,y)}{f(y,x)}\mathrm{d}x\mathrm{d}y$ 达到最小, 并求出这个最小值.

解 对积分 $I = \displaystyle\iint\limits_D \frac{f(x,y)}{f(y,x)}\mathrm{d}x\mathrm{d}y$ 作变换 $x \to y,\ y \to x$, 由 D 的对称性, 知 $I = \displaystyle\iint\limits_D \frac{f(y,x)}{f(x,y)}\mathrm{d}x\mathrm{d}y$. 因而由均值不等式可得

$$I = \frac{1}{2}\iint\limits_D \left(\frac{f(x,y)}{f(y,x)} + \frac{f(y,x)}{f(x,y)} \right)\mathrm{d}x\mathrm{d}y \geqslant \iint\limits_D 1 \,\mathrm{d}x\mathrm{d}y = \sigma(D),$$

这里 $\sigma(D)$ 是 D 的面积.

$$I - \sigma(D) = \frac{1}{2}\iint\limits_D \left(\sqrt{\frac{f(x,y)}{f(y,x)}} - \sqrt{\frac{f(y,x)}{f(x,y)}} \right)^2 \mathrm{d}x\mathrm{d}y \geqslant 0.$$

$I = \sigma(D)$ 当且仅当 $f(x,y) = f(y,x)$. 故所求函数为所有满足 $f(x,y) = f(y,x)$ 及 $f(1,0) = 1$ 的连续正值函数.

D 的边界的参数方程为

$$x = \cos^3 \varphi, \quad y = \sin^3 \varphi \ (0 \leqslant \varphi \leqslant 2\pi),$$

故 I 的最小值为

$$\sigma(D) = \iint\limits_D 1\,\mathrm{d}x\mathrm{d}y = 4 \iint\limits_{\substack{0 \leqslant r \leqslant 1 \\ 0 \leqslant \varphi \leqslant \frac{\pi}{2}}} 3r\sin^2\varphi\cos^2\varphi\,\mathrm{d}r\mathrm{d}\varphi$$

$$= 6\int_0^{\frac{\pi}{2}} \sin^2\varphi\cos^2\varphi\,\mathrm{d}\varphi = \frac{3}{8}\pi.$$

所以所求最小值是 $\dfrac{3}{8}\pi$, 且当 $f(x,y) = f(y,x)$ 并满足 $f(1,0) = 1$ 时, 取到该最小值.

$\square$

例题 0.12 求证: $\displaystyle\iint\limits_{[0,1]^2} (xy)^{xy}\,\mathrm{d}x\mathrm{d}y = \int_0^1 t^t\,\mathrm{d}t.$

证明 首先化为累次积分

$$\iint\limits_{[0,1]^2} (xy)^{xy}\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^1 (xy)^{xy}\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^x \frac{t^t}{x}\,\mathrm{d}t = \int_0^1 \frac{f(x)}{x}\,\mathrm{d}x,$$

其中 $f(x) = \displaystyle\int_0^x t^t\,\mathrm{d}t$. 由分部积分,

$$\int_0^1 \frac{f(x)}{x}\,\mathrm{d}x = f(x)\ln x\Big|_0^1 - \int_0^1 x^x\ln x\,\mathrm{d}x = -\int_0^1 x^x\ln x\,\mathrm{d}x.$$

因为 $(x^x)' = x^x\ln x + x^x$, 所以

$$\int_0^1 x^x\ln x\,\mathrm{d}x = \int_0^1 \left((x^x)' - x^x \right)\mathrm{d}x = -\int_0^1 x^x\,\mathrm{d}x.$$

于是

$$\iint\limits_{[0,1]^2} (xy)^{xy} \, \mathrm{d}x\mathrm{d}y = \int_0^1 t^t \, \mathrm{d}t.$$

$\square$

例题 0.13 计算二重积分 $I = \iint\limits_{D} \mathrm{sgn}(x^2 - y^2 + 2) \, \mathrm{d}x\mathrm{d}y$，其中 $D = \{(x, y) \mid x^2 + y^2 \leqslant 4\}$。

解 设 D 在第一象限部分为 D_1，则由对称性

$$I = 4 \iint\limits_{D_1} \mathrm{sgn}(x^2 - y^2 + 2) \, \mathrm{d}x\mathrm{d}y.$$

设 D_2 是 D_1 中使得 $x^2 - y^2 + 2 < 0$ 的部分，D_3 是 D_1 中使得 $x^2 - y^2 + 2 \geqslant 0$ 的部分，则 $D_1 = D_2 \cup D_3$。因此

$$I = 4 \left[\iint\limits_{D_3} \mathrm{d}x\mathrm{d}y - \iint\limits_{D_2} \mathrm{d}x\mathrm{d}y \right] = 4[\sigma(D_3) - \sigma(D_2)]$$

$$= 4 \left[\frac{1}{4} \cdot \pi \cdot 2^2 - 2\sigma(D_2) \right] = 4\pi - 8\sigma(D_2),$$

其中 $\sigma(D_2), \sigma(D_3)$ 分别表示 D_2 和 D_3 的面积。在极坐标 $x = r\cos\varphi, y = r\sin\varphi$ 之下，D_2 为

$$\left\{ (r, \varphi) \mid \frac{\pi}{3} \leqslant \varphi \leqslant \frac{\pi}{2}, \sqrt{-\frac{2}{\cos 2\varphi}} \leqslant r \leqslant 2 \right\}.$$

因而

$$\sigma(D_2) = \iint\limits_{D_2} \mathrm{d}x\mathrm{d}y = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_{\sqrt{-\frac{2}{\cos 2\varphi}}}^{2} r \, \mathrm{d}r$$

$$= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(4 + \frac{2}{\cos 2\varphi} \right) \mathrm{d}\varphi = \frac{\pi}{3} + \frac{1}{2} \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{\cos\varphi} \mathrm{d}\varphi$$

$$= \frac{\pi}{3} - \frac{1}{2} \ln(2 + \sqrt{3}),$$

故

$$I = \frac{4\pi}{3} + 4\ln(2 + \sqrt{3}).$$

$\square$

例题 0.14 设 $D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$。求 $I = \iint\limits_{D} \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| \mathrm{d}x\mathrm{d}y$。

解 由极坐标变换 $x = r\cos\varphi, y = r\sin\varphi, 0 \leqslant r \leqslant 1, 0 \leqslant \varphi \leqslant 2\pi$，有

$$I = \iint\limits_{\substack{0\leqslant r\leqslant 1 \\ 0\leqslant\varphi\leqslant 2\pi}} \left| \frac{\cos\varphi + \sin\varphi}{\sqrt{2}} - r \right| r^2 \mathrm{d}r\mathrm{d}\varphi$$

$$= \iint\limits_{\substack{0\leqslant r\leqslant 1 \\ 0\leqslant\varphi\leqslant 2\pi}} \left| \sin\left(\varphi + \frac{\pi}{4}\right) - r \right| r^2 \mathrm{d}r\mathrm{d}\varphi = \iint\limits_{\substack{0\leqslant r\leqslant 1 \\ 0\leqslant\varphi\leqslant 2\pi}} |\sin\varphi - r| r^2 \mathrm{d}r\mathrm{d}\varphi$$

$$= \iint\limits_{\substack{0\leqslant r\leqslant 1 \\ 0\leqslant\varphi\leqslant\pi}} |\sin\varphi - r| r^2 \mathrm{d}r\mathrm{d}\varphi + \iint\limits_{\substack{0\leqslant r\leqslant 1 \\ \pi\leqslant\varphi\leqslant 2\pi}} |\sin\varphi - r| r^2 \mathrm{d}r\mathrm{d}\varphi$$

$$= \iint\limits_{\substack{0\leqslant r\leqslant 1 \\ 0\leqslant\varphi\leqslant\pi}} |\sin\varphi - r| r^2 \mathrm{d}r\mathrm{d}\varphi + \iint\limits_{\substack{0\leqslant r\leqslant 1 \\ 0\leqslant\varphi\leqslant\pi}} (\sin\varphi + r) r^2 \mathrm{d}r\mathrm{d}\varphi.$$

因此，有

$$I = \int_0^{\pi} \mathrm{d}\varphi \int_0^{\sin\varphi} (\sin\varphi - r) r^2 \mathrm{d}r + \int_0^{\pi} \mathrm{d}\varphi \int_{\sin\varphi}^1 (r - \sin\varphi) r^2 \mathrm{d}r$$

$$+ \int_0^\pi \mathrm{d}\varphi \int_0^{\sin\varphi} (\sin\varphi + r)r^2 \mathrm{d}r + \int_0^\pi \mathrm{d}\varphi \int_{\sin\varphi}^1 (\sin\varphi + r)r^2 \mathrm{d}r$$

$$= \int_0^\pi \mathrm{d}\varphi \int_0^{\sin\varphi} 2\sin\varphi \cdot r^2 \mathrm{d}r + \int_0^\pi \mathrm{d}\varphi \int_{\sin\varphi}^1 2r \cdot r^2 \mathrm{d}r$$

$$= \int_0^\pi \frac{2}{3}\sin^4\varphi \mathrm{d}\varphi + \int_0^\pi \frac{1}{2}(1 - \sin^4\varphi)\mathrm{d}\varphi$$

$$= \frac{1}{6}\int_0^\pi \sin^4\varphi \mathrm{d}\varphi + \frac{\pi}{2} = \frac{1}{6} \cdot \frac{3\pi}{8} + \frac{\pi}{2} = \frac{9}{16}\pi.$$

$\square$

例题 0.15 设 f 是定义在正方形 $S = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ 上的四阶连续可微函数, 在 S 的边界上为零, 并且

$$\left| \frac{\partial^4 f}{\partial x^2 \partial y^2} \right| \leqslant M.$$

求证:

$$\left| \iint_S f(x, y)\,\mathrm{d}x\mathrm{d}y \right| \leqslant \frac{1}{144}M.$$

证明 考虑函数 $g(x, y) = x(1 - x)y(1 - y)$. 易知

$$\frac{\partial^4 g}{\partial x^2 \partial y^2} = 4, \qquad \iint_S g(x, y)\,\mathrm{d}x\mathrm{d}y = \frac{1}{36}.$$

因为 f 在 S 的边界上为零, 所以 $\frac{\partial^2 f}{\partial y^2}$ 在 $x = 0$ 和 $x = 1$ 时为零. 于是

$$\iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g\,\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^1 \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g\,\mathrm{d}x$$

$$= \int_0^1 \mathrm{d}y \left(\frac{\partial^3 f}{\partial x \partial y^2} \cdot g \Big|_{x=0}^1 - \int_0^1 \frac{\partial^3 f}{\partial x \partial y^2} \cdot \frac{\partial g}{\partial x}\,\mathrm{d}x \right)$$

$$= -\int_0^1 \mathrm{d}y \int_0^1 \frac{\partial^3 f}{\partial x \partial y^2} \cdot \frac{\partial g}{\partial x}\,\mathrm{d}x$$

$$= -\int_0^1 \mathrm{d}y \left(\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial g}{\partial x} \Big|_{x=0}^1 - \int_0^1 \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x \right)$$

$$= \int_0^1 \mathrm{d}y \int_0^1 \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x$$

$$= \iint_S \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\mathrm{d}y.$$

同理, 由于 $\frac{\partial^2 g}{\partial x^2}$ 在 $y = 0$ 和 $y = 1$ 时为零, 作与上面类似的推导, 可得

$$\iint_S \frac{\partial^4 g}{\partial x^2 \partial y^2} \cdot f\,\mathrm{d}x\mathrm{d}y = \iint_S \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 g}{\partial x^2}\,\mathrm{d}x\mathrm{d}y.$$

因此

$$\iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g\,\mathrm{d}x\mathrm{d}y = \iint_S \frac{\partial^4 g}{\partial x^2 \partial y^2} \cdot f\,\mathrm{d}x\mathrm{d}y.$$

从而

$$\left| \iint_S f\,\mathrm{d}x\mathrm{d}y \right| = \frac{1}{4}\left| \iint_S 4f\mathrm{d}x\mathrm{d}y \right| = \frac{1}{4}\left| \iint_S \frac{\partial^4 g}{\partial x^2 \partial y^2} f\mathrm{d}x\mathrm{d}y \right|$$

$$= \frac{1}{4}\left| \iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2} \cdot g\,\mathrm{d}x\mathrm{d}y \right| \leqslant \frac{M}{4}\iint_S g\,\mathrm{d}x\mathrm{d}y = \frac{M}{144}.$$

$\square$

> **定理 0.6 (Poincaré(庞加莱) 不等式)**
>
> 设 φ, ψ 是 $[a, b]$ 上的连续函数, f 在区域 $D = \{(x, y) \mid a \leqslant x \leqslant b, \varphi(x) \leqslant y \leqslant \psi(x)\}$ 上连续可微, 且有 $f(x, \varphi(x)) = 0 \ (x \in [a, b])$. 则存在 $M > 0$, 使得
>
> $$\iint_D f^2(x, y)\,\mathrm{d}x\mathrm{d}y \leqslant M \iint_D (f'_y(x, y))^2\,\mathrm{d}x\mathrm{d}y.$$
>
> ♡

证明 由 Newton-Leibniz 公式和 Cauchy 不等式可得

$$f^2(x, y) = [f(x, y) - f(x, \varphi(x))]^2 = \left(\int_{\varphi(x)}^y \frac{\partial f}{\partial t}(x, t)\,\mathrm{d}t \right)^2$$

$$\leqslant (y - \varphi(x)) \int_{\varphi(x)}^y \left(\frac{\partial f}{\partial t}(x, t) \right)^2 \mathrm{d}t,$$

因此

$$\iint_D f^2(x, y)\,\mathrm{d}x\mathrm{d}y = \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} f^2(x, y)\,\mathrm{d}y$$

$$\leqslant \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} (y - \varphi(x))\,\mathrm{d}y \int_{\varphi(x)}^y \left(\frac{\partial f}{\partial t}(x, t) \right)^2 \mathrm{d}t$$

$$= \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \left(\frac{\partial f}{\partial t}(x, t) \right)^2 \mathrm{d}t \int_t^{\psi(x)} (y - \varphi(x))\,\mathrm{d}y$$

$$\leqslant \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \left(\frac{\partial f}{\partial t}(x, t) \right)^2 \mathrm{d}t \int_{\varphi(x)}^{\psi(x)} (y - \varphi(x))\,\mathrm{d}y$$

$$= \int_a^b \mathrm{d}x \int_{\varphi(x)}^{\psi(x)} \frac{1}{2}(\psi(x) - \varphi(x))^2 \left(\frac{\partial f}{\partial t}(x, t) \right)^2 \mathrm{d}t$$

$$\leqslant M \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \left(\frac{\partial f}{\partial t}(x, t) \right)^2 \mathrm{d}t \right) \mathrm{d}x$$

$$= M \iint_D \left(\frac{\partial f}{\partial y}(x, y) \right)^2 \mathrm{d}x\mathrm{d}y,$$

这里 M 是满足 $M > \max\limits_{a \leqslant x \leqslant b} \dfrac{1}{2}(\psi(x) - \varphi(x))^2$ 的常数. $\qquad\square$

例题 0.16 设 $a > 0$, $\Omega_n(a) : x_1 + x_2 + \cdots + x_n \leqslant a, x_i \geqslant 0 \ (i = 1, 2, \cdots, n)$. 求积分

$$I_n(a) = \int \cdots \int_{\Omega_n(a)} x_1 x_2 \cdots x_n \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

解 作变换 $x_i = at_i, i = 1, 2, \cdots, n$, 则

$$I_n(a) = a^{2n} \int \cdots \int_{\Omega_n(1)} t_1 t_2 \cdots t_n \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_n = a^{2n} I_n(1).$$

再用累次积分, 可得

$$I_n(1) = \int \cdots \int_{\Omega_n(1)} t_1 t_2 \cdots t_n \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_n$$

$$= \int_0^1 t_n \mathrm{d}t_n \int \cdots \int_{t_1 + t_2 + \cdots + t_{n-1} \leqslant 1 - t_n} t_1 \cdots t_{n-1} \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}$$

$$= \int_0^1 t_n I_{n-1}(1 - t_n) \mathrm{d}t_n = \int_0^1 t_n (1 - t_n)^{2(n-1)} I_{n-1}(1) \mathrm{d}t_n.$$

因此,

$$I_n(1) = \frac{1}{2n(2n-1)} I_{n-1}(1).$$

注意到 $I_1(1) = \int_0^1 t \mathrm{d}t = \dfrac{1}{2}$. 由上面的递推公式, 可得 $I_n(1) = \dfrac{1}{(2n)!}$. 故 $I_n(a) = \dfrac{a^{2n}}{(2n)!}$.

$\qquad\square$

0.1.8 其他

例题 0.17 证明积分 $\displaystyle\int_0^\infty e^{-ax^2-\frac{b}{x^2}}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}, a, b > 0$.

证明 当 $a = 1$ 时, 就有

$$\int_0^{+\infty} e^{-x^2-\frac{b}{x^2}}\mathrm{d}x = e^{-2\sqrt{b}}\int_0^{+\infty} e^{-\left(x-\frac{\sqrt{b}}{x}\right)^2}\mathrm{d}x \xlongequal{y=\frac{\sqrt{b}}{x}} e^{-2\sqrt{b}}\int_0^{+\infty} \frac{\sqrt{b}}{y^2}e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2}\mathrm{d}y$$

$$= \frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty}\left(1+\frac{\sqrt{b}}{y^2}\right)e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\mathrm{d}y = \frac{e^{-2\sqrt{b}}}{2}\int_0^{+\infty} e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}\mathrm{d}\left(y-\frac{\sqrt{b}}{y}\right)$$

$$= \frac{e^{-2\sqrt{b}}}{2}\int_{-\infty}^{+\infty} e^{-t^2}\mathrm{d}t = \frac{\sqrt{\pi}}{2}e^{-2\sqrt{b}}.$$

于是对 $\forall a > 0$, 就有

$$\int_0^{+\infty} e^{-ax^2-\frac{b}{x^2}}\mathrm{d}x = \frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-x^2-\frac{ab}{x^2}}\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}.$$

$\square$

例题 0.18 计算 $\displaystyle\int_0^\infty \frac{\cos(ax)}{1+x^2}\mathrm{d}x, a \in \mathbb{R}$.

注 本题可以用复变函数的方法 (留数定理) 来计算. 但是我们这里用基本的高等数学的方法来计算. $\displaystyle\int_0^\infty \frac{\sin(ax)}{1+x^2}\mathrm{d}x$ 这个积分没办法算出具体的初等数值.

证明

$$\int_0^{+\infty} \frac{\cos(ax)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(ax)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\cos(ax)\left(\int_0^{+\infty} e^{-(1+x^2)y}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} e^{-(1+x^2)y}\cos(ax)\mathrm{d}y\right)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^{+\infty} e^{-(1+x^2)y}\cos(ax)\mathrm{d}y\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-(1+x^2)y}\cos(ax)\mathrm{d}x\right)\mathrm{d}y = \frac{1}{2}\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-x^2 y}\cos(ax)\mathrm{d}x\right)\mathrm{d}y$$

$$= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-x^2 y + iax}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x^2-\frac{aix}{y}+\left(\frac{ai}{2y}\right)^2\right)-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x-\frac{ai}{2y}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2iy}\right)^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-y\left(x+\frac{a}{2iy}\right)^2-\frac{a^2}{4y}}\mathrm{d}\left(x+\frac{a}{2iy}\right)\right)\mathrm{d}y\right) = \frac{1}{2}\mathrm{Re}\left(\int_0^{+\infty} e^{-y}\left(\int_{-\infty}^{+\infty} e^{-yx^2-\frac{a^2}{4y}}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$= \frac{1}{2}\int_0^{+\infty} e^{-y-\frac{a^2}{4y}}\left(\int_{-\infty}^{+\infty} e^{-yx^2}\mathrm{d}x\right)\mathrm{d}y = \frac{\sqrt{\pi}}{2}\int_0^{+\infty} \frac{1}{\sqrt{y}}e^{-y-\frac{a^2}{4y}}\mathrm{d}y$$

$$\xlongequal{y=t^2} \sqrt{\pi}\int_0^{+\infty} e^{-t^2-\frac{a^2}{4t^2}}\mathrm{d}t \xlongequal{\text{例题 0.17}} \sqrt{\pi}\cdot\frac{\sqrt{\pi}}{2}e^{-|a|} = \frac{\pi}{2}e^{-|a|}.$$

$\square$

例题 0.19 计算 $\displaystyle\int_0^\infty \frac{1}{(1+x^8)^2}\mathrm{d}x$.

注 由命题**??**可知对 $\forall s > 0$, 都有

$$\frac{1}{t^s} = \frac{1}{\Gamma(s)}\int_0^{+\infty} y^{s-1}e^{-ty}\mathrm{d}y, \forall t \in \mathbb{R}.$$

本题的核心想法就是利用上式将 $\dfrac{z}{1+x^8}$ 转化成积分形式.

证明 注意到

$$\int_0^{+\infty} ye^{-(1+x^8)y}\mathrm{d}y \xlongequal{y=\frac{z}{1+x^8}} \frac{1}{(1+x^8)^2}\int_0^{+\infty} ze^{-z}\mathrm{d}z = \frac{1}{(1+x^8)^2},$$

因此

$$\int_0^{+\infty} \frac{1}{(1+x^8)^2} \mathrm{d}x = \int_0^{+\infty} \left(\int_0^{+\infty} y e^{-(1+x^8)y} \mathrm{d}y \right) \mathrm{d}x = \int_0^{+\infty} \left(\int_0^{+\infty} y e^{-(1+x^8)y} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^{+\infty} y e^{-y} \left(\int_0^{+\infty} e^{-x^8 y} \mathrm{d}x \right) \mathrm{d}y \xlongequal{x=y^{-\frac{1}{8}} z^{\frac{1}{8}}} \int_0^{+\infty} y e^{-y} \left(\int_0^{+\infty} y^{-\frac{1}{8}} e^{-z} \mathrm{d}z^{\frac{1}{8}} \right) \mathrm{d}y$$

$$= \frac{1}{8} \int_0^{+\infty} y^{\frac{7}{8}} e^{-y} \left(\int_0^{+\infty} z^{-\frac{7}{8}} e^{-z} \mathrm{d}z \right) \mathrm{d}y = \frac{1}{8} \int_0^{+\infty} y^{\frac{7}{8}} e^{-y} \Gamma\left(\frac{1}{8}\right) \mathrm{d}y$$

$$= \frac{1}{8} \Gamma\left(\frac{15}{8}\right) \Gamma\left(\frac{1}{8}\right) = \frac{1}{8} \cdot \frac{7}{8} \Gamma\left(\frac{7}{8}\right) \Gamma\left(\frac{1}{8}\right)$$

$$\xlongequal{\text{余元公式}} \frac{7\pi}{64 \sin\left(\frac{\pi}{8}\right)} = \frac{7\pi}{32\sqrt{2 - \sqrt{2}}}.$$

$\square$

例题 0.20 计算积分 $I = \int_{-1}^2 \frac{1+x^2}{1+x^4} \mathrm{d}x$.

注 在此例中 $I \neq F(2) - F(-1)$. 这是因为 F 并不是 f 在区间 $[-1, 2]$ 上的原函数.

解 在不包含 0 的区间上作变换 $t = x - \frac{1}{x}$ 得

$$\int \frac{1+x^2}{1+x^4} \mathrm{d}x = \int \frac{x - \frac{1}{x}}{2 + \left(x - \frac{1}{x}\right)^2} \mathrm{d}x = \int \frac{\mathrm{d}t}{2 + t^2}$$

$$= \frac{1}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \arctan \frac{x^2 - 1}{\sqrt{2}x} + C.$$

这说明在区间 $[-1, 0)$ 和 $(0, 2]$ 上, 函数 $f(x) = \frac{1+x^2}{1+x^4}$ 的一个原函数是

$$F(x) = \frac{1}{\sqrt{2}} \arctan \frac{x^2 - 1}{\sqrt{2}x}.$$

因此

$$\int_{-1}^0 f(x) \mathrm{d}x = F(0^-) - F(-1) = \frac{\pi}{2\sqrt{2}} - 0 = \frac{\pi}{2\sqrt{2}},$$

$$\int_0^2 f(x) \mathrm{d}x = F(2) - F(0^+) = \frac{1}{\sqrt{2}} \arctan \frac{3}{2\sqrt{2}} + \frac{\pi}{2\sqrt{2}}.$$

故

$$I = \int_{-1}^0 f(x) \mathrm{d}x + \int_0^2 f(x) \mathrm{d}x = \frac{\pi}{\sqrt{2}} + \frac{1}{\sqrt{2}} \arctan \frac{3}{2\sqrt{2}}.$$

$\square$