0.1 特殊级数

命题 0.1 ($\frac{\sin x}{x}$ 因式分解)

对任意 x 都有 $\dfrac{\sin x}{x}=\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$, 这里 x 可以为复数.

证明

□

命题 0.2

1. $\cot x=\dfrac{\cos x}{\sin x}=\dfrac{1}{x}+\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right)$.
2. $\tan x=\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)\frac{\pi}{2}-x}-\frac{1}{(2n-1)\frac{\pi}{2}+x}\right)$.
3. $\dfrac{1}{\sin^2 x}=\dfrac{1}{x^2}+\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{(x+n\pi)^2}+\frac{1}{(x-n\pi)^2}\right)=\sum_{n\in\mathbb{Z}}\frac{1}{(x+n\pi)^2}$.
4. $\dfrac{1}{\sin x}=\dfrac{1}{x}+\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right)$.

证明

1. 对命题 0.1中级数两边同时取对数得

$$\ln\frac{\sin x}{x}=\sum_{n=1}^{\infty}\ln\left(1-\frac{x^2}{n^2\pi^2}\right).$$

两边同时求导得

$$\frac{\cos x}{\sin x}-\frac{1}{x}=\sum_{n=1}^{\infty}\frac{-\frac{2x}{n^2\pi^2}}{1-\frac{x^2}{n^2\pi^2}}=\sum_{n=1}^{\infty}\frac{2x}{x^2-n^2\pi^2}=\sum_{n=1}^{\infty}\left(\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right).$$

故

$$\cot x=\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{x+n\pi}+\frac{1}{x-n\pi}\right).$$

2. 由第 1 问及 $\cot\left(\frac{\pi}{2}-x\right)=\tan x$ 可得

$$\begin{aligned}\tan x=\cot\left(\frac{\pi}{2}-x\right)&=\frac{1}{\frac{\pi}{2}-x}+\sum_{n=1}^{\infty}\left(\frac{1}{\frac{\pi}{2}-x+n\pi}+\frac{1}{\frac{\pi}{2}-x-n\pi}\right)\\&=\frac{1}{\frac{\pi}{2}-x}+\sum_{n=1}^{\infty}\left(\frac{1}{(2n+1)\frac{\pi}{2}-x}-\frac{1}{(2n-1)\frac{\pi}{2}+x}\right)\\&=\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)\frac{\pi}{2}-x}-\frac{1}{(2n-1)\frac{\pi}{2}+x}\right).\end{aligned}$$

3.
4.

□