0.1 一维 $\bar{\partial}$ 的解

定义 0.1

所谓一维 $\bar{\partial}$ 问题,是指在区域 D 上给定一个函数 f,要求函数 u,使得在 D 上有

$$\frac{\partial u(z)}{\partial \bar{z}} = f(z), \quad z \in D.$$

u 就称为 $\bar{\partial}$ 问题的解.

定义 0.2

设 φ 是 $\mathbb{C}$ 上的函数,使 φ 不取零值的点集的闭包称为 φ 的**支集**,记为 $\mathrm{supp}\varphi$,即

$$\mathrm{supp}\varphi = \overline{\{z \in \mathbb{C} : \varphi(z) \neq 0\}}.$$

引理 0.1

设 a 是 $\mathbb{C}$ 中任意一点,$0 < r < R$,则必存在 φ,满足下列条件:

(i) $\varphi \in C^\infty(\mathbb{C})$;

(ii) $\mathrm{supp}\varphi \subset \overline{B(a,R)}$;

(iii) 当 $z \in \overline{B(a,r)}$ 时,$\varphi(z) \equiv 1$;

(iv) 对于任意 $z \in \mathbb{C}$,$0 \leqslant \varphi(z) \leqslant 1$.

证明 令 $r < R_1 < R$ 和

$$h_1(z) = \begin{cases} e^{\frac{1}{|z-a|^2 - R_1^2}}, & z \in B(a,R_1); \\ 0, & z \notin B(a,R_1), \end{cases}$$

$$h_2(z) = \begin{cases} 0, & z \in \overline{B(a,r)}; \\ e^{\frac{1}{r^2 - |z-a|^2}}, & z \notin \overline{B(a,r)}, \end{cases}$$

那么 $h_1, h_2 \in C^\infty(\mathbb{C})$. 又令

$$\varphi(z) = \frac{h_1(z)}{h_1(z) + h_2(z)},$$

则 $\varphi \in C^\infty(\mathbb{C})$. 而且当 $z \in \overline{B(a,r)}$ 时,$\varphi(z) \equiv 1$; 当 $z \notin B(a,R_1)$ 时,$\varphi(z) \equiv 0$, 即 $\mathrm{supp}\varphi \subset B(a,R)$. 对于任意 $z \in \mathbb{C}$,$0 \leqslant \varphi(z) \leqslant 1$ 显然成立.φ 即为所求的函数.

□

定理 0.1

设 D 是 $\mathbb{C}$ 中的区域,$f \in C^1(D)$. 令

$$u(z) = \frac{1}{2\pi i} \int_D \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}, \quad z \in D, \tag{1}$$

则 $u \in C^1(D)$, 且对任意 $z \in D$, 有 $\dfrac{\partial u(z)}{\partial \bar{z}} = f(z)$.

笔记 在上面的证明中,容易看出,如果 $f \in C^\infty(D)$, 那么 $\bar{\partial}$ 问题的解 $u \in C^\infty(D)$.

证明 把 f 的定义扩充到整个复平面,对于 $z \notin D$, 定义 $f(z) = 0$. 这时,(1) 式可写为

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \xlongequal{\text{令 }\zeta = z + \eta} \frac{1}{2\pi i} \int_{\mathbb{C}} f(\zeta + \eta) \frac{1}{\eta} \mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}.$$

由 $f \in C^1(D)$, 可得 $u \in C^1(D)$.

现固定 $a \in D$, 我们证明
$$\frac{\partial u(a)}{\partial \bar{z}} = f(a).$$
为此, 取 $0 < \varepsilon < r$, 使得 $B(a,\varepsilon) \subset B(a,r) \subset D$. 根据引理 0.1, 存在 $\varphi \in C^\infty(\mathbb{C})$, 使得当 $z \in B(a,\varepsilon)$ 时,$\varphi(z) \equiv 1$; 而当 $z \notin B(a,r)$ 时,$\varphi(z) \equiv 0$. 记
$$u_1(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\varphi(\zeta)f(\zeta)}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta},$$
$$u_2(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(\zeta) - \varphi(\zeta)f(\zeta)}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta},$$
那么 $u = u_1 + u_2$. 由于当 $\zeta \in B(a,\varepsilon)$ 时,$f(\zeta) - \varphi(\zeta)f(\zeta) \equiv 0$, 所以
$$u_2(z) = \frac{1}{2\pi i}\int_{\mathbb{C}\setminus\overline{B(a,\varepsilon)}} \frac{(1-\varphi(\zeta))f(\zeta)}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$
因而, 当 $z \in B(a,\varepsilon)$ 时,u_2 是全纯函数, 所以由定理**??**可知 $\dfrac{\partial u_2}{\partial \bar{z}} = 0$. 于是, 在小圆盘 $B(a,\varepsilon)$ 上就有
$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}} = \frac{\partial u_1}{\partial \bar{z}} &\xlongequal{\text{令}\zeta = z+\eta} \frac{\partial}{\partial \bar{z}}\left\{\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{C}} \frac{\varphi(z+\eta)f(z+\eta)}{\eta}\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta}\right\} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{C}} \left\{\frac{\partial(\varphi f)}{\partial \zeta}\frac{\partial \zeta}{\partial \bar{z}} + \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right\}\frac{1}{\eta}\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{C}} \left\{\frac{\partial(\varphi f)}{\partial \zeta}\frac{\partial\,(z+\eta)}{\partial \bar{z}} + \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\frac{\partial\,(\bar{z}+\bar{\eta})}{\partial \bar{z}}\right\}\frac{1}{\eta}\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{C}} \left\{\frac{\partial(\varphi f)}{\partial \zeta}\cdot(0+0) + \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\cdot(1+0)\right\}\frac{1}{\eta}\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{C}} \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\frac{1}{\eta}\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{C}} \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\frac{1}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \\
&= \frac{1}{2\pi \mathrm{i}}\int_{B(a,r)} \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\frac{1}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.
\end{aligned} \tag{2}$$
最后一个等式成立是因为当 $\zeta \in \mathbb{C}\setminus\overline{B(a,r)}$ 时 $\varphi(\zeta) \equiv 0$. 又因为当 $\zeta \in \partial B(a,r)$ 时 $\varphi(\zeta) \equiv 0$, 所以根据非齐次 Cauchy 积分公式, 有
$$\varphi(z)f(z) = \frac{1}{2\pi i}\int_{B(a,r)} \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\frac{1}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$
因为当 $z \in B(a,\varepsilon)$ 时 $\varphi(z) = 1$, 所以
$$f(z) = \frac{1}{2\pi i}\int_{B(a,r)} \frac{\partial(\varphi f)}{\partial \bar{\zeta}}\frac{1}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}. \tag{3}$$
比较 (2) 式和 (3) 式, 即得
$$\frac{\partial u(z)}{\partial \bar{z}} = f(z).$$
特别地, 取 $z = a$, 即得
$$\frac{\partial u(a)}{\partial \bar{z}} = f(a).$$
由于 a 是 D 中的任意点, 所以 $\dfrac{\partial u(z)}{\partial \bar{z}} = f(z)$ 在 D 上成立.

□