

## 0.1 非齐次 Cauchy 积分公式

### 定义 0.1

设  $z = x + iy$  是一个复数, 把  $z, \bar{z}$  看成独立变量, 定义微分  $dz, d\bar{z}$  的外积为

$$dz \wedge dz = 0, d\bar{z} \wedge d\bar{z} = 0, dz \wedge d\bar{z} = -d\bar{z} \wedge dz.$$

$dx, dy$  的外积定义与  $dz, d\bar{z}$  的外积定义一样, 即

$$dx \wedge dx = 0, dy \wedge dy = 0, dx \wedge dy = -dy \wedge dx.$$

定义面积元素  $dA = dx \wedge dy$ .

### 命题 0.1

设  $z = x + iy$  是一个复数, 则

$$dz \wedge d\bar{z} = -2idx \wedge dy = -2idA.$$

**证明** 由于  $dz = dx + idy, d\bar{z} = dx - idy$ , 所以

$$\begin{aligned} dz \wedge d\bar{z} &= (dx + idy) \wedge (dx - idy) \\ &= idy \wedge dx - idx \wedge dy \\ &= -2idx \wedge dy = -2idA. \end{aligned}$$

这里  $dA$  是面积元素.

□

### 定义 0.2

称  $z, \bar{z}$  的函数  $f(z, \bar{z})$  为零次微分形式,  $f_1(z, \bar{z})dz + f_2(z, \bar{z})d\bar{z}$  为一次微分形式,  $f(z, \bar{z})dz \wedge d\bar{z}$  为二次微分式.

♣

### 定义 0.3

定义算子  $\partial, \bar{\partial}$  如下:

$$\partial f = \frac{\partial f}{\partial z} dz, \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z},$$

这里

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \end{aligned}$$

定义算子  $d = \partial + \bar{\partial}$ , 即

$$df = \partial f + \bar{\partial} f = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \quad (1)$$

♣

### 定义 0.4

算子  $\partial, \bar{\partial}$  对一次微分形式的作用定义为

$$\begin{aligned} \partial(f_1(z, \bar{z})dz + f_2(z, \bar{z})d\bar{z}) &= \frac{\partial f_1}{\partial z} dz \wedge dz + \frac{\partial f_2}{\partial z} dz \wedge d\bar{z} = \frac{\partial f_2}{\partial z} dz \wedge d\bar{z}, \\ \bar{\partial}(f_1(z, \bar{z})dz + f_2(z, \bar{z})d\bar{z}) &= \frac{\partial f_1}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial f_2}{\partial \bar{z}} d\bar{z} \wedge d\bar{z} = -\frac{\partial f_1}{\partial \bar{z}} dz \wedge d\bar{z}. \end{aligned}$$

所以

$$d(f_1(z, \bar{z})dz + f_2(z, \bar{z})d\bar{z}) = \left( \frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial \bar{z}} \right) dz \wedge d\bar{z}. \quad (2)$$

算子  $\partial, \bar{\partial}$  作用在二次微分形式上的结果定义为零:

$$\begin{aligned}\partial(f(z, \bar{z})dz \wedge d\bar{z}) &= \frac{\partial f}{\partial z} dz \wedge dz \wedge d\bar{z} = 0, \\ \bar{\partial}(f(z, \bar{z})dz \wedge d\bar{z}) &= \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz \wedge d\bar{z} = 0,\end{aligned}$$

因而

$$d(f(z, \bar{z})dz \wedge d\bar{z}) = 0. \quad (3)$$

### 定义 0.5

$$d^2\omega = d(d\omega), \quad \partial^2\omega = \partial(\partial\omega), \quad \bar{\partial}^2\omega = \bar{\partial}(\bar{\partial}\omega), \quad \partial\bar{\partial}\omega = \partial(\bar{\partial}\omega), \quad \bar{\partial}\partial\omega = \bar{\partial}(\partial\omega).$$

### 命题 0.2

$$d^2 = 0, \quad \partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \bar{\partial}\partial + \partial\bar{\partial} = 0.$$

**证明** 当  $\omega$  是一个  $C^2$  函数时, 由 (1) 式和 (2) 式, 得

$$d^2\omega = d(d\omega) = d\left(\frac{\partial\omega}{\partial z}dz + \frac{\partial\omega}{\partial \bar{z}}d\bar{z}\right) = \left(\frac{\partial^2\omega}{\partial \bar{z}\partial z} - \frac{\partial^2\omega}{\partial z\partial \bar{z}}\right)dz \wedge d\bar{z} = 0.$$

当  $\omega$  是一个一次微分形式时, 由 (2) 式知  $d\omega$  是一个二次微分形式, 由 (3) 式即知  $d^2\omega = 0$ . 当  $\omega$  是一个二次微分形式时, 由 (3) 式知  $d^2\omega = 0$ . 总之, 不论  $\omega$  是零次、一次或二次微分形式, 都有  $d^2\omega = 0$ , 所以  $d^2 = 0$ .

根据上述证明, 同样可以证明

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \bar{\partial}\partial + \partial\bar{\partial} = 0.$$

□

### 定理 0.1 (Green 公式)

设  $\gamma_0, \gamma_1, \dots, \gamma_n$  是  $n+1$  条可求长简单闭曲线,  $\gamma_1, \dots, \gamma_n$  都在  $\gamma_0$  的内部,  $\gamma_1, \dots, \gamma_n$  中的每一条都在其他  $n-1$  条的外部,  $D$  是由这  $n+1$  条曲线围成的区域, 用  $\partial D$  记  $D$  的边界. 如果  $\omega = f_1(z, \bar{z})dz + f_2(z, \bar{z})d\bar{z}$  是区域  $D$  上的一个一次微分形式, 这里,  $f_1, f_2 \in C^1(\bar{D})$ , 那么

$$\int_{\partial D} \omega = \int_D d\omega. \quad (4)$$



**笔记** 公式 (4) 在高维空间中也成立, 通常称为 **Stokes 公式**, 这里只是它的一个特例.

**证明** 记  $f_1 = u_1 + iv_1, f_2 = u_2 + iv_2$ , 这里,  $u_1, v_1, u_2, v_2$  是  $z, \bar{z}$  的实值函数, 于是

$$\begin{aligned}\omega &= f_1 dz + f_2 d\bar{z} = (u_1 + iv_1)(dx + idy) + (u_2 + iv_2)(dx - idy) \\ &= \{(u_1 + u_2)dx + (-v_1 + v_2)dy\} + i\{(v_1 + v_2)dx + (u_1 - u_2)dy\}.\end{aligned} \quad (5)$$

由(2)式, 得

$$\begin{aligned}d\omega &= \left(\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial \bar{z}}\right)dz \wedge d\bar{z} = -\left\{\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u_2 + iv_2) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u_1 + iv_1)\right\}2idA \\ &= \left\{\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial x} - \frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\right) + i\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} - \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial y}\right)\right\}dA.\end{aligned} \quad (6)$$

因为  $f_1, f_2 \in C^1(\bar{D})$ , 所以  $u_1, u_2, v_1, v_2 \in C^1(\bar{D})$ . 于是根据实值函数的 Green 公式, 我们有

$$\int_{\partial D} (u_1 + u_2)dx + (-v_1 + v_2)dy = \int_D \left\{\frac{\partial}{\partial x}(-v_1 + v_2) - \frac{\partial}{\partial y}(u_1 + u_2)\right\}dA, \quad (7)$$

$$\int_{\partial D} (v_1 + v_2)dx + (u_1 - u_2)dy = \int_D \left\{\frac{\partial}{\partial x}(u_1 - u_2) - \frac{\partial}{\partial y}(v_1 + v_2)\right\}dA. \quad (8)$$

由等式(5)(6)(7)(8) 即得我们要证明的公式 (4).

□

**定理 0.2 (非齐次 Cauchy 积分公式 (Pompeiu 公式))**

设  $\gamma_0, \gamma_1, \dots, \gamma_n$  是  $n+1$  条可求长简单闭曲线,  $\gamma_1, \dots, \gamma_n$  都在  $\gamma_0$  的内部,  $\gamma_1, \dots, \gamma_n$  中的每一条都在其他  $n-1$  条的外部,  $D$  是由这  $n+1$  条曲线围成的区域, 用  $\partial D$  记  $D$  的边界. 如果  $f \in C^1(\bar{D})$ , 那么对任意  $z \in D$ , 有

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_D \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\zeta \wedge d\bar{\zeta}. \quad (9)$$

**笔记** 如果  $f \in H(D)$ , 那么由 Cauchy-Riemann 方程,  $\frac{\partial f}{\partial \bar{\zeta}} = 0$ , 这时公式 (9) 右端的第二项就消失了, 公式 (9) 就是 Cauchy 积分公式. 所以, 公式 (9) 是 Cauchy 积分公式在  $C^1$  函数类中的推广, 有时也称为**非齐次 Cauchy 积分公式**.

公式 (9) 首先是由 Pompeiu 在 1912 年证明的 (所以有时也称之为 Pompeiu 公式), 但长期以来似乎被人们遗忘了. 直到 1950 年, Grothendieck 和 Dolbeault 用它来解  $\bar{\partial}$  方程时, 人们才发现它的意义所在.

**证明** 不妨设  $D$  是图 1 所示的二连通区域,  $D$  的边界  $\partial D$  由  $\gamma_0$  和  $\gamma_1$  组成. 任取  $z \in D$ , 因为  $f$  在  $z$  点连续, 故对任意  $\varepsilon > 0$ , 存在  $\delta > 0$ , 当  $|\zeta - z| < \delta$  时,  $|f(\zeta) - f(z)| < \varepsilon$ . 记  $\rho = \inf_{\zeta \in \partial D} |\zeta - z| > 0$ , 取  $\eta$ , 使得  $0 < \eta < \min(\rho, \delta)$ , 于是  $\overline{B(z, \eta)} \subset D$ . 记  $B_\eta = B(z, \eta)$ , 令  $G_\eta = D \setminus \overline{B_\eta}$ , 则  $G_\eta$  的边界  $\partial G_\eta$  由  $\gamma_0, \gamma_1$  和  $\partial B_\eta$  三条曲线组成. 考虑一次微分形式

$$\omega = \frac{f(\zeta)}{\zeta - z} d\zeta,$$

它在区域  $G_\eta$  上满足 Green 公式的条件, 因而有

$$\int_{\partial G_\eta} \omega = \int_{G_\eta} d\omega. \quad (10)$$

由于  $\frac{1}{\zeta - z}$  在  $G_\eta$  中全纯, 所以由定理 ?? 可知  $\frac{\partial}{\partial \bar{\zeta}} \left( \frac{1}{\zeta - z} \right) = 0$ . 因此

$$\bar{\partial} \omega = \frac{\partial}{\partial \bar{\zeta}} \left( \frac{f(\zeta)}{\zeta - z} \right) d\bar{\zeta} \wedge d\zeta \xrightarrow{\text{乘积求导法则}} \left\{ f(\zeta) \frac{\partial}{\partial \bar{\zeta}} \left( \frac{1}{\zeta - z} \right) + \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{1}{\zeta - z} \right\} d\bar{\zeta} \wedge d\zeta = \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta.$$

易知

$$\partial \omega = \frac{\partial}{\partial \zeta} \left( \frac{f(\zeta)}{\zeta - z} \right) d\zeta \wedge d\zeta = 0,$$

因而

$$d\omega = \partial \omega + \bar{\partial} \omega = \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta.$$

这样, (10) 式可以写成

$$\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial B_\eta} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{G_\eta} \frac{\partial f(\zeta)}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta. \quad (11)$$

注意

$$\int_{\partial B_\eta} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial B_\eta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + f(z) \int_{\partial B_\eta} \frac{d\zeta}{\zeta - z} \xrightarrow{\text{命题 ??}} \int_{\partial B_\eta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + 2\pi i f(z),$$

而

$$\left| \int_{\partial B_\eta} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq \int_{\partial B_\eta} \frac{|f(\zeta) - f(z)|}{|\zeta - z|} |d\zeta| \leq \frac{\varepsilon}{\eta} \cdot 2\pi\eta = 2\pi\varepsilon,$$

由此即得

$$\lim_{\eta \rightarrow 0} \int_{\partial B_\eta} \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z). \quad (12)$$

另一方面, 由  $f \in C^1(\bar{D})$  可知  $\frac{\partial f}{\partial \bar{\zeta}}$  在  $\bar{B}_\eta$  上连续, 故有常数  $M$ , 使得  $\left| \frac{\partial f}{\partial \bar{\zeta}} \right| \leq M$  在  $\bar{B}_\eta$  上成立. 于是

$$\left| \int_{B_\eta} \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta \right| \leq 2 \int_{B_\eta} \left| \frac{\partial f}{\partial \bar{\zeta}} \right| \frac{1}{|\zeta - z|} dA \leq 4M\pi\eta \rightarrow 0 (\eta \rightarrow 0).$$

因而

$$\lim_{\eta \rightarrow 0} \int_{G_\eta} \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta = \lim_{\eta \rightarrow 0} \left\{ \int_D \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta - \int_{B_\eta} \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta \right\} = \int_D \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z} d\bar{\zeta} \wedge d\zeta. \quad (13)$$

在等式 (11) 两端令  $\eta \rightarrow 0$ , 并利用 (12) 式和 (13) 式, 即得所要证明的公式 (9). □

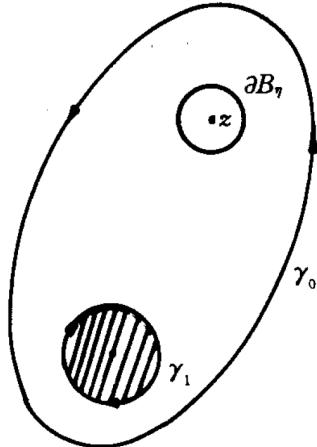


图 1