0.1 数值比较类

例题 0.1 证明如下积分不等式:

1. $\int_0^{\sqrt{2\pi}} \sin x^2 dx > 0.$
2. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+x^2} dx \geqslant \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+x^2} dx.$
3. $\int_0^1 \frac{\cos x}{\sqrt{1-x^2}} dx > \int_0^1 \frac{\sin x}{\sqrt{1-x^2}} dx.$

笔记 此类问题都是考虑分母更小的时候正的更多, 通过换元把负的区间转化到正的同一个区间.

证明

1.

$$\begin{aligned}\int_0^{\sqrt{2\pi}} \sin x^2 \mathrm{d}x &\xlongequal{x=\sqrt{y}} \int_0^{2\pi} \frac{\sin y}{2\sqrt{y}} \mathrm{d}y = \frac{1}{2}\int_0^{\pi} \frac{\sin y}{2\sqrt{y}} \mathrm{d}y + \frac{1}{2}\int_{\pi}^{2\pi} \frac{\sin y}{2\sqrt{y}} \mathrm{d}y \\ &= \frac{1}{2}\int_0^{\pi} \frac{\sin y}{2\sqrt{y}} \mathrm{d}y + \frac{1}{2}\int_0^{\pi} \frac{\sin(y+\pi)}{2\sqrt{y+\pi}} \mathrm{d}y \\ &= \frac{1}{2}\int_0^{\pi} \sin y \left(\frac{1}{2\sqrt{y}} - \frac{1}{2\sqrt{y+\pi}}\right) \mathrm{d}y > 0.\end{aligned}$$

2.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+x^2} \mathrm{d}x &= \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{1+x^2} \mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+x^2} \mathrm{d}x = \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\sin\left(\frac{\pi}{4}-x\right)}{1+x^2} \mathrm{d}x + \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{4}-x\right)}{1+x^2} \mathrm{d}x \\ &= \sqrt{2}\int_0^{\frac{\pi}{4}} \frac{\sin y}{1+\left(\frac{\pi}{4}-y\right)^2} \mathrm{d}y + \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(-y)}{1+\left(\frac{\pi}{4}+y\right)^2} \mathrm{d}y \\ &= \sqrt{2}\int_0^{\frac{\pi}{4}} \sin y \left[\frac{1}{1+\left(\frac{\pi}{4}-y\right)^2} - \frac{1}{1+\left(\frac{\pi}{4}+y\right)^2}\right] \mathrm{d}y > 0.\end{aligned}$$

3. 本题稍有不同, 注意到

$$\int_0^1 \frac{\cos x}{\sqrt{1-x^2}} dx \xlongequal{x=\sin y} \int_0^{\frac{\pi}{2}} \cos(\sin y) dy, \int_0^1 \frac{\sin x}{\sqrt{1-x^2}} dx \xlongequal{x=\cos y} \int_0^{\frac{\pi}{2}} \sin(\cos y) dy.$$

现在利用 $\sin x < x, \forall x \in (0, \frac{\pi}{2})$ 可得不等式链 $\cos\sin x > \cos x > \sin\cos x, \forall x \in (0, \frac{\pi}{2})$, 于是

$$\int_0^1 \frac{\cos x}{\sqrt{1-x^2}} dx > \int_0^1 \frac{\sin x}{\sqrt{1-x^2}} dx.$$

□

定理 0.1 (Jordan 不等式)

$sinx \geqslant \frac{2}{\pi}x, \forall x \in [0, \frac{\pi}{2}]$

证明 利用 $\sin x$ 的上凸性及割线放缩可得

$$\frac{\sin x - \sin 0}{x - 0} \geqslant \frac{\sin\frac{\pi}{2} - \sin x}{\frac{\pi}{2} - x}, \forall x \in \left[0, \frac{\pi}{2}\right].$$

□

例题 0.2 证明如下积分不等式

1. $\frac{\pi}{6} < \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}} dx < \frac{\pi}{4\sqrt{2}}.$
2. $\int_0^{\pi} e^{\sin^2 x} dx \geqslant \sqrt{e}\pi.$
3. $\frac{\pi}{2}e^{-R} < \int_0^{\frac{\pi}{2}} e^{-R\sin x} dx < \frac{\pi(1-e^{-R})}{2R}, R > 0.$

4. $\displaystyle\int_0^{n\pi}\frac{|\sin x|}{x}\mathrm{d}x>\frac{2}{\pi}\ln(n+1),n\geqslant 2.$

注 $(2n)!!=2^n\cdot n!$.

证明

1.

$$\frac{\pi}{6}=\int_0^1\frac{1}{\sqrt{4-x^2}}dx<\int_0^1\frac{1}{\sqrt{4-x^2-x^3}}dx<\int_0^1\frac{1}{\sqrt{4-x^2-x^2}}dx=\frac{\pi}{4\sqrt{2}}.$$

2.

$$\begin{aligned}\int_0^{\pi}e^{\sin^2 x}dx&=\int_0^{\pi}\sum_{n=0}^{\infty}\frac{\sin^{2n}x}{n!}\mathrm{d}x=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\pi}\sin^{2n}x\mathrm{d}x\\&=\pi\left[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!(2n)!!}\right]=\pi\left[1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n(n!)^2}\right]\\&\overset{(2n-1)!!\geqslant n!}{\geqslant}\pi\sum_{n=0}^{\infty}\frac{1}{2^n n!}=\sqrt{e}\pi.\end{aligned}$$

3.

$$\frac{\pi}{2}e^{-R}=\int_0^{\frac{\pi}{2}}e^{-R}dx<\int_0^{\frac{\pi}{2}}e^{-R\sin x}dx\overset{Jordan\text{不等式}}{<}\int_0^{\frac{\pi}{2}}e^{-\frac{2R}{\pi}x}dx=\frac{\pi(1-e^{-R})}{2R},R>0.$$

4.

$$\begin{aligned}\int_0^{n\pi}\frac{|\sin x|}{x}\mathrm{d}x&=\sum_{k=0}^{n-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin x|}{x}\mathrm{d}x\xlongequal{x=k\pi+y}\sum_{k=0}^{n-1}\int_0^{\pi}\frac{|\sin y|}{k\pi+y}\mathrm{d}y\\&>\sum_{k=0}^{n-1}\int_0^{\pi}\frac{|\sin y|}{(k+1)\pi}\mathrm{d}y=\frac{2}{\pi}\sum_{k=0}^{n-1}\frac{1}{k+1}\\&>\frac{2}{\pi}\sum_{k=0}^{n-1}\ln\left(1+\frac{1}{k+1}\right)=\frac{2}{\pi}\sum_{k=0}^{n-1}\left[\ln(k+2)-\ln(k+1)\right]\\&=\frac{2}{\pi}\ln(n+1).\end{aligned}$$

还可以使用积分放缩法处理 $\displaystyle\frac{2}{\pi}\sum_{k=0}^{n-1}\frac{1}{k+1}$, 如下所示:

$$\frac{2}{\pi}\sum_{k=0}^{n-1}\frac{1}{k+1}=\frac{2}{\pi}\sum_{k=0}^{n-1}\int_k^{k+1}\frac{1}{k+1}\mathrm{d}x\geqslant\frac{2}{\pi}\sum_{k=0}^{n-1}\int_k^{k+1}\frac{1}{x+1}\mathrm{d}x=\frac{2}{\pi}\int_0^n\frac{1}{x+1}\mathrm{d}x=\frac{2}{\pi}\ln(n+1).$$

□