Real and Complex Analysis

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Chapter 1 Prologue:The Exponential Function

Definition 1.1 (Exponential Function)

Exponential function is defined , for every complex number z , by the formula

$$\exp\left(z\right)=\sum_{n=0}^{\infty}\frac{z^n}{n!}. \tag{1.1}$$

We define the number e to be $\exp(1)$*,and shall usually replace* $\exp(z)$ *by the customary shorter expression* e^z*.Note that* $e^0=\exp(0)=1$*,by*(1.1).

♣

Property

(a) *The series* (1.1) *converges absoultely for every z and converges uniformly on every bounded subset of the complex plane . Thus* exp *is a contionuous function .*

(b) *For all complex numbers a and b , we have*

$$\exp(a)\exp(b)=\exp(a+b). \tag{1.2}$$

Proof (a) is obvious . Next proof (b) .

The absoulte convergence of (1.1) and Cauchy's theorem shows that computation

$$\begin{aligned}
\exp\left(a\right)\exp\left(b\right) &= \sum_{k=0}^{\infty}\frac{a^k}{k!}\sum_{m=0}^{\infty}\frac{b^m}{m!} \xlongequal{\text{Add them in square order}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{a^kb^m}{k!m!} \\
&\xlongequal[\text{Add them up in diagonal order}]{Cauchy's\ theorem(\text{华师大数分下册}P20)} \sum_{n=0}^{\infty}\sum_{\substack{m+k=n\\ m,k\in\mathbb{N}}}\frac{a^kb^m}{k!m!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{a^kb^{n-k}}{k!\left(n-k\right)!} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!a^kb^{n-k}}{k!\left(n-k\right)!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\mathrm{C}_n^k a^kb^{n-k} \\
&= \sum_{n=0}^{\infty}\frac{\left(a+b\right)^n}{n!} = \exp\left(a+b\right).
\end{aligned}$$

Theorem 1.1 (The conclusions of exponential function)

(a) For every complex z we have $e^z\neq 0$.

(b) exp *is its own derivative :* $\exp'(z)=\exp(z)$.

(c) The resticton of exp *to the real axis is a monotonically increasing positive function , and*

$$e^x\to\infty\ as\ x\to\infty\ ,\ e^x\to 0\ as\ x\to-\infty.$$

(d)There exists a positive number π *such that* $e^{\frac{\pi}{2}}=i$ *and such that* $e^z=1$ *if and only if* $z/(2\pi i)$ *is an integer .*

(e) exp *is a periodic function , with period* $2\pi i$ *.*

(f) The mapping $t\to e^{iz}$ *maps tha the real axis onto unit circle .*

(g) If w *is a complex number and* $w\neq 0$ *, then* $w=e^z$ *for some* z *.*

♡

Proof