0.1 常用积分公式

0.1.1 不定积分

1. $\displaystyle\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C\,(a>0).$
2. $\displaystyle\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C\,(a>0).$ 3. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C\,(a>0).$
3. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C\,(a>0).$
4. $\displaystyle\int \ln x\mathrm{d}x = x\ln x - x + C.$
5. $\displaystyle\int \sec x\mathrm{d}x = \ln|\sec x+\tan x| + C;$
   $\displaystyle\int \csc x\mathrm{d}x = -\ln|\csc x+\cot x| + C.$
6. $\displaystyle\int \sqrt{x^2\pm a^2}\mathrm{d}x = \frac{1}{2}\left[x\sqrt{x^2\pm a^2} \pm a^2\ln\left|x+\sqrt{x^2\pm a^2}\right|\right] + C\,(a>0);$
   $\displaystyle\int \sqrt{a^2-x^2}\mathrm{d}x = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right] + C\,(a>0).$
7. $\displaystyle\int e^{ax}\cos bx\mathrm{d}x = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C\,(ab\neq 0);$
   $\displaystyle\int e^{ax}\sin bx\mathrm{d}x = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C\,(ab\neq 0).$
8. $\displaystyle\int x\cos nx\mathrm{d}x = \frac{1}{n^2}\cos nx + \frac{x}{n}\sin nx + C\,(n\neq 0);$
   $\displaystyle\int x\sin nx\mathrm{d}x = \frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx + C\,(n\neq 0).$
9. 记 $\displaystyle I(m,n) = \int \cos^m x\sin^n x\mathrm{d}x, \forall n,m\in\mathbb{N}$, 则

$$\begin{aligned} I(m,n) &= \frac{\cos^{m-1}x\sin^{n+1}x}{m+n} + \frac{m-1}{m+n}I(m-2,n) \quad (m\geq 2, n\geq 0);\\ &= -\frac{\cos^{m+1}x\sin^{n-1}x}{m+n} + \frac{n-1}{m+n}I(m,n-2) \quad (m\geq 0, n\geq 2). \end{aligned}$$

0.1.2 定积分

1. 记 $\displaystyle I_n = \int_0^{\frac{\pi}{2}} \sin^n x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\mathrm{d}x, \forall n\in\mathbb{N}$, 则

$$I_n = \frac{n-1}{n}I_{n-2}, \quad \forall n\geqslant 2.$$

从而

$$I_n = \begin{cases} \dfrac{(n-1)!!}{n!!}I_0 = \dfrac{(n-1)!!}{n!!}\cdot\dfrac{\pi}{2} & ,n\text{为偶数}\\ \dfrac{(n-1)!!}{n!!}I_1 = \dfrac{(n-1)!!}{n!!} & ,n\text{为奇数} \end{cases}. \tag{1}$$

2. 记 $\displaystyle J(m,n) = \int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\mathrm{d}x, \forall n,m\in\mathbb{N}$, 则

$$J(m,n) = \frac{m-1}{m+n}J(m-2,n), \quad \forall n,m\geqslant 2.$$

$$J(m,n)=\frac{n-1}{m+n}J(m,n-2),\quad \forall n,m\geqslant 2.$$

从而

$$J(m,n)=\begin{cases}\dfrac{(m-1)!!(n-1)!!}{(m+n)!!} & ,m,n\text{不全为偶数}\\ \dfrac{(m-1)!!(n-1)!!}{(m+n)!!}\cdot\dfrac{\pi}{2} & ,m,n\text{全为偶数}\end{cases}. \tag{2}$$

3.

注 公式(1)(2)通常称为“点火公式”.