# 0.1 三角函数相关

## 0.1.1 三角函数

**定理 0.1 (三角平方差公式)**

$\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y) = \cos(y-x)\cos(y+x) = \cos^2 y - \cos^2 x.$

**证明** 首先，我们有

$$\cos^2 x - \cos^2 y = 1 - \sin^2 x - \left(1 - \sin^2 y\right) = \sin^2 y - \sin^2 x.$$

接着，我们有

$$\begin{aligned}
\sin(x-y)\sin(x+y) &= (\sin x\cos y - \cos x\sin y)(\sin x\cos y + \cos x\sin y)\\
&= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y\\
&= \sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y\\
&= \sin^2 x - \sin^2 y;
\end{aligned}$$

$$\begin{aligned}
\cos(y-x)\cos(y+x) &= (\cos x\cos y + \sin x\sin y)(\cos x\cos y - \sin x\sin y)\\
&= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y\\
&= \cos^2 x\cos^2 y - (1-\cos^2 x)(1-\cos^2 y)\\
&= \cos^2 x - \cos^2 y.
\end{aligned}$$

故结论得证. □

**定理 0.2**

1.
$$\sin(n\theta) = \sum_{\substack{r=0\\2r+1\le n}} (-1)^r \binom{n}{2r+1} \cos^{n-2r-1}(\theta)\sin^{2r+1}(\theta).$$

2.
$$\cos(n\theta) = \sum_{\substack{r=0\\2r\le n}} (-1)^r \binom{n}{2r} \cos^{n-2r}(\theta)\sin^{2r}(\theta).$$

3.
$$\tan(n\theta) = \frac{\sum\limits_{\substack{r=0\\2r+1\le n}} (-1)^r \binom{n}{2r+1}\tan^{2r+1}(\theta)}{\sum\limits_{\substack{r=0\\2r\le n}} (-1)^r \binom{n}{2r}\tan^{2r}(\theta)}.$$

4.
$$\cos^n\theta = \begin{cases} \dfrac{1}{2^{n-1}}\sum\limits_{\substack{r=0\\2r<n}} \dbinom{n}{2r}\cos\left((n-2r)\theta\right) + \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}}, & n\text{为偶数}\\ \dfrac{1}{2^{n-1}}\sum\limits_{\substack{r=0\\2r<n}} \dbinom{n}{2r}\cos\left((n-2r)\theta\right), & n\text{为奇数} \end{cases}.$$

5.

$$\sin^n\theta=\begin{cases}\dfrac{(-1)^{\frac{n}{2}}}{2^{n-1}}\displaystyle\sum_{\substack{r=0\\2r<n}}(-1)^r\binom{n}{2r}\sin\left((n-2r)\theta\right), & n\text{为偶数}\\ \dfrac{(-1)^{\lfloor\frac{n}{2}\rfloor}}{2^{n-1}}\displaystyle\sum_{\substack{r=0\\2r<n}}(-1)^r\binom{n}{2r}\cos\left((n-2r)\theta\right)+\dfrac{1}{2^n}\binom{n}{\frac{n}{2}}, & n\text{为奇数}\end{cases}.$$

**笔记** 上述结论 4 表明: $\cos^n x$ 可以表示为 $1,\cos x,\cdots,\cos nx$ 的线性组合.

**证明** 具体证明见Expansions of sin(nx) and cos(nx).

□

### 0.1.2 反三角函数

**定理 0.3 (常用反三角函数性质)**

1.

$$\arcsin x+\arcsin y=\begin{cases}\arcsin\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) & ,xy<0\text{或}x^2+y^2\leqslant 1\\ \pi-\arcsin\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) & ,x>0,y>0,x^2+y^2>1\\ -\pi-\arcsin\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) & ,x<0,y<0,x^2+y^2>1\end{cases}.$$

2.

$$\arcsin x-\arcsin y=\begin{cases}\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right) & ,xy\geqslant 0\text{或}x^2+y^2\leqslant 1\\ \pi-\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right) & ,x>0,y<0,x^2+y^2>1\\ -\pi-\arcsin\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right) & ,x<0,y>0,x^2+y^2>1\end{cases}.$$

3.

$$\arccos x+\arccos y=\begin{cases}\arccos\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right) & ,x+y\geqslant 0\\ 2\pi-\arccos\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right) & ,x+y<0\end{cases}.$$

4.

$$\arccos x-\arccos y=\begin{cases}-\arccos\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right) & ,x\geqslant y\\ \arccos\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right) & ,x<y\end{cases}.$$

5.

$$\arctan x+\arctan y=\begin{cases}\arctan\dfrac{x+y}{1-xy} & ,xy<1\\ \pi+\arctan\dfrac{x+y}{1-xy},x>0 & ,xy>1\\ -\pi+\arctan\dfrac{x+y}{1-xy},x<0 & ,xy>1\end{cases}.$$

6.

$$\arctan x-\arctan y=\begin{cases}\arctan\dfrac{x-y}{1+xy} & ,xy>-1\\ \pi+\arctan\dfrac{x-y}{1+xy} & ,x>0,xy<-1\\ -\pi+\arctan\dfrac{x-y}{1+xy} & ,x<0,xy<-1\end{cases}.$$

7.
$$2\arcsin x=\begin{cases}\arcsin\left(2x\sqrt{1-x^2}\right) & ,|x|\leqslant\dfrac{\sqrt{2}}{2}\\ \pi-\arcsin\left(2x\sqrt{1-x^2}\right) & ,\dfrac{\sqrt{2}}{2}<x\leqslant 1\\ -\pi-\arcsin\left(2x\sqrt{1-x^2}\right) & ,-1\leqslant x<-\dfrac{\sqrt{2}}{2}\end{cases}.$$

8.
$$2\arccos x=\begin{cases}\arccos\left(2x^2-1\right) & ,0\leqslant x\leqslant 1\\ 2\pi-\arccos\left(2x^2-1\right) & ,-1\leqslant x<0\end{cases}.$$

9.
$$2\arctan x=\begin{cases}\arctan\dfrac{2x}{1-x^2}, |x|\leqslant 1\\ \pi+\arctan\dfrac{2x}{1-x^2} & ,|x|>1\\ -\pi+\arctan\dfrac{2x}{1-x^2} & ,x<-1\end{cases}.$$

10.
$$\cos(n\arccos x)=\frac{\left(x+\sqrt{x^2-1}\right)^n+\left(x-\sqrt{x^2-1}\right)^n}{2}\ (n\geqslant 1).$$

证明

□

**命题 0.1**

$$\arctan x+\arctan\frac{1}{x}=\begin{cases}\dfrac{\pi}{2}, & x>0\\ -\dfrac{\pi}{2}, & x<0\end{cases}.$$

证明 令 $f(x)=\arctan x+\arctan\dfrac{1}{x}$, 则

$$f'(x)=\frac{1}{x^2+1}+\frac{1}{(\frac{1}{x})^2+1}(-\frac{1}{x^2})=\frac{1}{x^2+1}-\frac{1}{x^2+1}=0$$

故 $f(x)$ 为常函数, 于是就有 $f(x)=f(1)=\dfrac{\pi}{2},\forall x>0\,;f(x)=f(-1)=-\dfrac{\pi}{2},\forall x<0.$ □

### 0.1.3 双曲三角函数

**命题 0.2**

(1) $\cosh x=\dfrac{e^x+e^{-x}}{2}\geqslant 1,$

(2) $\sinh x=\dfrac{e^x-e^{-x}}{2}\geqslant x.$

证明 可以分别利用均值不等式和求导进行证明. □

**命题 0.3**

1. $\cosh^2 x-\sinh^2 x=1.$
2. $\cosh(2x)=2\cosh^2 x-1=1-2\sinh^2 x.$
3. $\sinh(2x)=2\sinh x\cosh x.$

证明

□