### 0.0.1 定积分

### 0.0.2 建立积分递推

**例题 0.1** 计算 $\int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx, n \in \mathbb{N}$.

**证明** 利用分部积分和和差化积公式可得

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot \cos x \sin(nx) \mathrm{d}x \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cdot [\sin((n+1)x) + \sin((n-1)x)] \mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x [\sin(nx) \cos x + \cos(nx) \sin x] \mathrm{d}x \\
&= \frac{I_{n-1}}{2} + \frac{I_n}{2} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(nx) \mathrm{d} \cos x \\
&= \frac{I_{n-1} + I_n}{2} - \frac{1}{2n} \int_0^{\frac{\pi}{2}} \cos(nx) \mathrm{d} \cos^n x \\
&= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) \mathrm{d}x \\
&= \frac{I_{n-1} + I_n}{2} + \frac{1}{2n} - \frac{I_n}{2} \\
&= \frac{I_{n-1}}{2} + \frac{1}{2n}.
\end{aligned}
$$

故 $I_n = \frac{I_{n-1}}{2} + \frac{1}{2n}$, 则两边同乘 $2^n$（强行裂项）

$$
2^n I_n = 2^{n-1} I_{n-1} + \frac{2^{n-1}}{n}, n = 1, 2, \cdots
$$

又注意到 $I_0 = 0$, 从而

$$
2^n I_n = 0 + \sum_{k=1}^{n} \frac{2^{k-1}}{k} \Rightarrow I_n = \frac{1}{2^n} \sum_{k=1}^{n} \frac{2^{k-1}}{k}.
$$

□

**例题 0.2**

1. $\int_0^{\pi} \frac{\sin(nx)}{\sin x} \mathrm{d}x = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.$
2.

**证明**

1. 记 $I_n = \int_0^{\pi} \frac{\sin(nx)}{\sin x} dx$, 则

$$
I_{n+2} - I_n = \int_0^{\pi} \frac{\sin((n+2)x) - \sin(nx)}{\sin x} dx = \int_0^{\pi} \frac{2\cos((n+1)x)\sin x}{\sin x} dx = 2 \int_0^{\pi} \cos((n+1)x) dx = 0.
$$

于是

$$
\int_0^{\pi} \frac{\sin(nx)}{\sin x} dx = I_n = I_{n-2} = \cdots = \begin{cases} I_0, & n\text{为偶数} \\ I_1, & n\text{为奇数} \end{cases} = \begin{cases} 0, & n\text{为偶数} \\ \pi, & n\text{为奇数} \end{cases}.
$$

2.

□

### 0.0.3 区间再现

**定理 0.1 (区间再现恒等式)**

当下述积分有意义时, 我们有

1.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\mathrm{d}x = \int_a^{\frac{a+b}{2}} [f(x)+f(a+b-x)]\mathrm{d}x.$$

2.

$$\int_0^{\infty} f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^{\infty} f(x)\mathrm{d}x = \int_0^1 \left[f(x)+\frac{f(\frac{1}{x})}{x^2}\right]\mathrm{d}x.$$

**笔记** 注意: 倒代换具有将 $[0,1]$ 转化为 $[1,+\infty)$ 的功能.

**证明** 证明是显然. □

**例题 0.3** 证明

1. $\int_0^{\frac{\pi}{2}} \ln\sin x dx = -\frac{\pi}{2}\ln 2.$
2. $\int_0^{\frac{\pi}{2}} \ln\cos x dx - \frac{\pi}{2}\ln 2.$
3. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2.$

**证明**

1.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

2.

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = \int_0^{\frac{\pi}{4}} \left[\ln\cos x + \ln\left(\frac{\pi}{2}-x\right)\right]\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos x\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\frac{1}{2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I \\
&\Longrightarrow I = \int_0^{\frac{\pi}{2}} \ln\cos x\mathrm{d}x = -\frac{\pi}{2}\ln 2.
\end{aligned}$$

3.

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x &\xlongequal{x=\tan\theta} \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan\theta^2}\mathrm{d}\tan\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\cdot\ln(1+\tan\theta)}{\sec^2\theta}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,\mathrm{d}\theta = \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\right]\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{8}} \left[\ln(1+\tan\theta) + \ln\frac{2}{1+\tan\theta}\right]\mathrm{d}\theta
\end{aligned}$$

$$= \int_0^{\frac{\pi}{8}} \ln 2 \mathrm{d}\theta = \frac{\pi}{8} \ln 2.$$

□

**例题 0.4** 计算

1. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2 + a^2} \mathrm{d}x, a > 0.$
2. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2 + x + 1} \mathrm{d}x.$
3. $\displaystyle\int_0^{1} \frac{\ln x}{\sqrt{x - x^2}} \mathrm{d}x.$

**解**

1. 注意到

$$\int_0^{+\infty} \frac{\ln x}{x^2 + a^2} \mathrm{d}x \xlongequal{x=at} \frac{1}{a} \int_0^{+\infty} \frac{\ln (at)}{1+t^2} \mathrm{d}t = \frac{1}{a} \int_0^{+\infty} \frac{\ln a}{1+t^2} \mathrm{d}t + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t = \frac{\pi \ln a}{2a} + \frac{1}{a} \int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t. \quad (1)$$

又注意到

$$\int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t \xlongequal{t=\frac{1}{x}} \int_0^{+\infty} \frac{\ln \frac{1}{x}}{1+\frac{1}{x^2}} \frac{1}{x^2} \mathrm{d}x = \int_0^{+\infty} \frac{-\ln x}{1+x^2} \mathrm{d}x \Longrightarrow \int_0^{+\infty} \frac{\ln t}{1+t^2} \mathrm{d}t = 0.$$

于是再结合(1)式可得

$$\int_0^{+\infty} \frac{\ln x}{x^2 + a^2} \mathrm{d}x = \frac{\pi \ln a}{2a}.$$

2. 

$$\int_0^{\infty} \frac{\ln x}{x^2 + x + 1} \mathrm{d}x \xlongequal{x=\frac{1}{t}} \int_0^{\infty} \frac{-\ln t}{1 + \frac{1}{t} + \frac{1}{t^2}} \mathrm{d}\frac{1}{t} = \int_0^{+\infty} \frac{-\ln t}{1 + t + t^2} \mathrm{d}t \Longrightarrow \int_0^{\infty} \frac{\ln x}{x^2 + x + 1} \mathrm{d}x = 0.$$

3. 

$$\int_0^{1} \frac{\ln x}{\sqrt{x - x^2}} \mathrm{d}x \xlongequal{x=\sin^2 y} \int_0^{\frac{\pi}{2}} \frac{\ln \sin^2 y}{\sqrt{\sin^2 y (1 - \sin^2 y)}} \mathrm{d} \sin^2 y$$

$$= 4 \int_0^{\frac{\pi}{2}} \ln \sin y \mathrm{d}y \xlongequal{\text{例题 0.3}} 4 \cdot \left(-\frac{\pi}{2} \ln 2\right) = -2\pi \ln 2.$$

□

**例题 0.5**

1. 对 $n \in \mathbb{N}$, 计算 $\displaystyle\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx.$
2. $\displaystyle\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1 + \cos^2 x} dx.$
3. 对 $n \in \mathbb{N}$, 计算 $\displaystyle\int_0^{2\pi} \sin(\sin x + nx) dx.$

**解**

1. 

$$\int_{-\pi}^{\pi} \frac{\sin (nx)}{(1+2^x)\sin x} \mathrm{d}x = \int_{-\pi}^{0} \left[\frac{\sin (nx)}{(1+2^x)\sin x} + \frac{\sin (nx)}{(1+2^{-x})\sin x}\right] \mathrm{d}x = \int_{-\pi}^{0} \frac{\sin (nx)}{\sin x} \left(\frac{1}{1+2^x} + \frac{1}{1+2^{-x}}\right) \mathrm{d}x$$

$$= \int_{-\pi}^{0} \frac{\sin (nx)}{\sin x} \cdot \frac{2+2^x+2^{-x}}{2+2^x+2^{-x}} \mathrm{d}x = \int_{-\pi}^{0} \frac{\sin (nx)}{\sin x} \mathrm{d}x = \int_{0}^{\pi} \frac{\sin (nx)}{\sin x} \mathrm{d}x \xlongequal{\text{例题 0.2}} \begin{cases} 0, n\text{为偶数} \\ \pi, n\text{为奇数} \end{cases}.$$

2. 

$$\int_{-\pi}^{\pi} \frac{x \sin x \arctan e^x}{1+\cos^2 x} dx \overset{\text{区间再现}}{=} \int_{-\pi}^{0} \frac{x \sin x [\arctan e^x + \arctan e^{-x}]}{1+\cos^2 x} dx = \frac{\pi}{2} \int_{-\pi}^{0} \frac{x \sin x}{1+\cos^2 x} dx$$

$$= \frac{\pi}{2} \int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx \overset{\text{区间再现}}{=} \frac{\pi}{2} \int_{0}^{\frac{\pi}{2}} \frac{\pi \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{2} \int_{0}^{1} \frac{1}{1+t^2} dx = \frac{\pi^3}{8}.$$

3. 

$$\int_0^{2\pi} \sin(\sin x + nx) dx \overset{\text{区间再现}}{=} \int_0^{2\pi} \sin(\sin(2\pi - x) + n(2\pi - x)) dx = -\int_0^{2\pi} \sin(\sin x + nx) dx = 0.$$