

## 0.1 Cauchy-Riemann 方程

### 定义 0.1

设  $f(z) = u(x, y) + iv(x, y)$  是定义在区域  $D$  上的函数,  $z_0 = x_0 + iy_0 \in D$ . 我们说  $f$  在  $z_0$  处实可微, 是指  $u$  和  $v$  作为  $x, y$  的二元函数在  $(x_0, y_0)$  处可微.

### 定理 0.1

如果  $f(z) = u(x, y) + iv(x, y)$  在  $z_0 = x_0 + iy_0$  处实可微, 则有极限

$$\lim_{\Delta x \rightarrow 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} = \frac{\partial u}{\partial x}(z_0) + i \frac{\partial v}{\partial x}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0),$$

$$\lim_{\Delta y \rightarrow 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{\Delta y} = \frac{\partial u}{\partial y}(z_0) + i \frac{\partial v}{\partial y}(z_0) = \frac{\partial u}{\partial y}(x_0, y_0) + i \frac{\partial v}{\partial y}(x_0, y_0),$$

分别记为  $\frac{\partial f}{\partial x}(z_0) = \frac{\partial f}{\partial x}(x_0, y_0)$ ,  $\frac{\partial f}{\partial y}(z_0) = \frac{\partial f}{\partial y}(x_0, y_0)$ , 也简记为

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}.$$

称为  $f$  的一阶偏导数. 与实变函数中的偏导数一样, 可同理定义  $f$  的  $k$  阶偏导数.

证明

□

### 定义 0.2

设  $z = x + iy$  是一个复数, 把  $z, \bar{z}$  看成独立变量, 定义算子

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (1)$$

**注** 为什么要像(1)式那样来定义算子  $\frac{\partial}{\partial z}$  和  $\frac{\partial}{\partial \bar{z}}$  呢? 这是因为如果把复变函数  $f(z)$  写成

$$f(x, y) = f\left(\frac{z + \bar{z}}{2}, -i\frac{z - \bar{z}}{2}\right),$$

把  $z, \bar{z}$  看成独立变量, 分别对  $z$  和  $\bar{z}$  求偏导数, 则得

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right),$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

这就是表达式(1)的来源. 这说明在进行微分运算时, 可以把  $z, \bar{z}$  看成独立的变量.

### 命题 0.1

设  $f: D \rightarrow \mathbb{C}$  是定义在区域  $D$  上的函数,  $z_0 \in D$ , 那么  $f$  在  $z_0$  处实可微的充分必要条件是

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z} + o(|\Delta z|). \quad (2)$$

成立.

**证明** 设  $f$  在  $z_0$  处实可微, 由二元实值函数可微的定义, 有

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|), \quad (3)$$

$$v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|), \quad (4)$$

这里,  $|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ . 于是

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)) \\ &= \frac{\partial u}{\partial x}(x_0, y_0)\Delta x + \frac{\partial u}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|) + i\left(\frac{\partial v}{\partial x}(x_0, y_0)\Delta x + \frac{\partial v}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|)\right) \\ &= \left(\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)\right)\Delta x + \left(\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)\right)\Delta y + o(|\Delta z|) \\ &= \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + o(|\Delta z|). \end{aligned}$$

把  $\Delta x = \frac{1}{2}(\Delta z + \overline{\Delta z})$ ,  $\Delta y = \frac{1}{2i}(\Delta z - \overline{\Delta z})$  代入上式, 得

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= \frac{1}{2}\frac{\partial f}{\partial x}(x_0, y_0)(\Delta z + \overline{\Delta z}) - \frac{i}{2}\frac{\partial f}{\partial y}(x_0, y_0)(\Delta z - \overline{\Delta z}) + o(|\Delta z|) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f(x_0, y_0)\Delta z + \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f(x_0, y_0)\overline{\Delta z} + o(|\Delta z|). \end{aligned}$$

从而上式可写为

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z} + o(|\Delta z|). \quad (5)$$

容易看出(5)式和(3)(4)两式等价.

□

### 定理 0.2

设  $f$  是定义在区域  $D$  上的函数,  $z_0 \in D$ , 那么  $f$  在  $z_0$  处可微的充要条件是  $f$  在  $z_0$  处实可微且  $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$ .  
并且  $f$  在可微的情况下,  $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ .

♥

**证明** 如果  $f$  在  $z_0$  处可微, 由(??)式得

$$f(z_0 + \Delta z) - f(z_0) = f'(z_0)\Delta z + o(|\Delta z|)$$

与(2)式比较就知道,  $f$  在  $z_0$  处是实可微的, 而且  $\frac{\partial f}{\partial \bar{z}}(z_0) = 0, f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ .

反之, 若  $f$  在  $z_0$  处实可微, 且  $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$ , 则由(2)式得

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + o(|\Delta z|)$$

由此即知

$$\lim_{\Delta z} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial f}{\partial z}(z_0).$$


故  $f$  在  $z_0$  处可微, 而且  $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ .

□

### 定义 0.3 (Cauchy-Riemann 方程)

设  $f$  是定义在区域  $D$  上的函数,  $\frac{\partial f}{\partial \bar{z}} = 0$  称为 **Cauchy - Riemann 方程**. 简称 **C. - R. 方程**

♣

 **笔记** 从这个方程可以得到  $f$  的实部和虚部应满足的条件.

### 定理 0.3 (Cauchy-Riemann 方程的等价定义)

设  $z = x + iy, f(z) = u(x, y) + iv(x, y)$ , 则

(i) Cauchy-Riemann 方程  $\frac{\partial f}{\partial \bar{z}} = 0$  等价于

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases} \quad (6)$$

(ii) Cauchy-Riemann 方程  $\frac{\partial f}{\partial \bar{z}} = 0$  等价于

$$\frac{\partial \bar{f}}{\partial z} = 0.$$

(iii) 令  $x = r \cos \theta, y = r \sin \theta$ , 进而  $z = r(\cos \theta + i \sin \theta), f(z) = u(r, \theta) + iv(r, \theta)$ , 则 Cauchy-Riemann 方程  $\frac{\partial f}{\partial \bar{z}} = 0$  等价于

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \end{cases}$$



**证明** (i) 由(1)式得

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right).$$

因此 Cauchy-Riemann 方程  $\frac{\partial f}{\partial \bar{z}} = 0$  就等价于

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$$

(ii) 又注意到

$$\frac{\partial \bar{f}}{\partial z} = \frac{\partial u}{\partial z} - i \frac{\partial v}{\partial z} = \frac{1}{2} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) - \frac{i}{2} \left( \frac{\partial v}{\partial x} - i \frac{\partial v}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) - \frac{i}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$

故 Cauchy-Riemann 方程  $\frac{\partial f}{\partial \bar{z}} = 0$  也等价于  $\frac{\partial \bar{f}}{\partial z} = 0$ .

(iii) 直接计算得

$$\frac{\partial (x, y)}{\partial (r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

再由定理 11.8 知

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial y}{\partial \theta} / \frac{\partial (x, y)}{\partial (r, \theta)} = \cos \theta, & \frac{\partial r}{\partial y} &= -\frac{\partial x}{\partial \theta} / \frac{\partial (x, y)}{\partial (r, \theta)} = \sin \theta, \\ \frac{\partial \theta}{\partial x} &= -\frac{\partial y}{\partial r} / \frac{\partial (x, y)}{\partial (r, \theta)} = -\frac{\sin \theta}{r}, & \frac{\partial \theta}{\partial y} &= \frac{\partial x}{\partial r} / \frac{\partial (x, y)}{\partial (r, \theta)} = \frac{\cos \theta}{r}. \end{aligned}$$

于是

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r} \cdot \cos \theta - \frac{\partial u}{\partial \theta} \cdot \frac{\sin \theta}{r}, \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r} \cdot \sin \theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta}{r}, \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r} \cdot \cos \theta - \frac{\partial v}{\partial \theta} \cdot \frac{\sin \theta}{r}, \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r} \cdot \sin \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos \theta}{r}. \end{aligned}$$

从而

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \iff \frac{\partial u}{\partial r} \cdot \cos \theta - \frac{\partial u}{\partial \theta} \cdot \frac{\sin \theta}{r} = \frac{\partial v}{\partial r} \cdot \sin \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\cos \theta}{r}$$

$$\begin{aligned} &\iff \frac{\partial u}{\partial r} \cdot r \cos \theta - \frac{\partial u}{\partial \theta} \cdot \sin \theta = \frac{\partial v}{\partial r} \cdot r \sin \theta + \frac{\partial v}{\partial \theta} \cdot \cos \theta, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &\iff \frac{\partial u}{\partial r} \cdot \sin \theta + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta}{r} = -\frac{\partial v}{\partial r} \cdot \cos \theta + \frac{\partial v}{\partial \theta} \cdot \frac{\sin \theta}{r} \\ &\iff \frac{\partial u}{\partial r} \cdot r \sin \theta + \frac{\partial u}{\partial \theta} \cdot \cos \theta = -\frac{\partial v}{\partial r} \cdot r \cos \theta + \frac{\partial v}{\partial \theta} \cdot \sin \theta. \end{aligned}$$

由 (i) 可知 Cauchy-Riemann 方程等价于  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ , 故此时 Cauchy-Riemann 方程等价于

$$\begin{cases} \frac{\partial u}{\partial r} \cdot r \cos \theta - \frac{\partial u}{\partial \theta} \cdot \sin \theta = \frac{\partial v}{\partial r} \cdot r \sin \theta + \frac{\partial v}{\partial \theta} \cdot \cos \theta, \\ \frac{\partial u}{\partial r} \cdot r \sin \theta + \frac{\partial u}{\partial \theta} \cdot \cos \theta = -\frac{\partial v}{\partial r} \cdot r \cos \theta + \frac{\partial v}{\partial \theta} \cdot \sin \theta. \end{cases}$$

化简可得

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \\ \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \end{cases}$$

□

#### 定理 0.4

设  $f = u + iv$  是定义在区域  $D$  上的函数,  $z_0 = x_0 + iy_0 \in D$ , 那么  $f$  在  $z_0$  处可微的充要条件是  $u(x, y), v(x, y)$  在  $(x_0, y_0)$  处可微, 且在  $(x_0, y_0)$  处满足 Cauchy-Riemann 方程, 即

$$\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial \bar{f}}{\partial z} = 0 \iff \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}.$$

在可微的情况下, 有

$$f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

这里的偏导数都在  $(x_0, y_0)$  处取值.

♡

**证明** 由定理 0.2 和 Cauchy-Riemann 方程的等价定义容易得到充要条件的证明. 再由定理 0.2 中的  $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$  和 Cauchy-Riemann 方程的等价定义可得, 在可微的情况下, 有

$$\begin{aligned} f'(z_0) &= \frac{\partial f}{\partial z}(z_0) = \frac{1}{2} \left[ \frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0) \right] \\ &= \frac{1}{2} \left[ \frac{\partial u}{\partial x}(z_0) + i \frac{\partial v}{\partial x}(z_0) - i \left( \frac{\partial u}{\partial y}(z_0) + i \frac{\partial v}{\partial y}(z_0) \right) \right] \\ &= \frac{1}{2} \left[ \frac{\partial u}{\partial x}(z_0) + \frac{\partial v}{\partial y}(z_0) + i \left( \frac{\partial v}{\partial x}(z_0) - \frac{\partial u}{\partial y}(z_0) \right) \right] \\ &= \frac{1}{2} \left[ 2 \frac{\partial u}{\partial x}(z_0) + 2i \frac{\partial v}{\partial x}(z_0) \right] \\ &= \frac{\partial u}{\partial x}(z_0) + i \frac{\partial v}{\partial x}(z_0). \end{aligned}$$

再由 Cauchy-Riemann 方程的等价定义可得结论.

□

#### 定义 0.4

1. 设  $D$  是  $\mathbb{C}$  中的区域, 我们用  $C(D)$  记  $D$  上连续函数的全体, 用  $H(D)$  记  $D$  上全纯函数的全体.
2. 设  $f = u + iv$ , 我们用  $C^1(D)$  记有 1 阶连续偏导数的函数的全体. 用  $C^k(D)$  记在  $D$  上有  $k$  阶连续偏导数的函数的全体,  $C^\infty(D)$  记在  $D$  上有任意阶连续偏导数的函数的全体.

♣

## 命题 0.2

- (1)  $H(D) \subseteq C(D)$ .  
 (2)  $C^1(D) \subseteq C(D)$ .  
 (3) 区域  $D$  上的全纯函数在  $D$  上有任意阶的连续偏导数, 并且有如下的包含关系:

$$H(D) \subseteq C^\infty(D) \subseteq C^k(D) \subseteq C^1(D) \subseteq C(D).$$

这里,  $k$  是大于 1 的自然数.

## 证明

- (1) 命题??告诉我们,  $H(D) \subseteq C(D)$ .  
 (2) 设  $f = u + iv$ , 记  $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ ,  $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$ . 我们用  $C^1(D)$  记  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  在  $D$  上连续的  $f$  的全体. 进而  $u, v$  关于  $x, y$  的偏导在  $D$  上都连续, 由多元微积分的知识知道,  $u, v$  在  $D$  上都可微. 于是对于任意  $f \in C^1(D)$ ,  $f$  在  $D$  上实可微, 从(5)式知道

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{\Delta z} + o(|\Delta z|).$$

令  $\Delta z \rightarrow 0$ , 则  $\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial f}{\partial z}(z_0)$ , 故  $f$  在  $D$  上连续, 因而  $C^1(D) \subseteq C(D)$ .

(3)

□

**例题 0.1** 研究函数  $f(z) = z^n$ ,  $n$  是自然数.

**解** 显然,  $\frac{\partial f}{\partial \bar{z}} = 0$ , 且  $f$  在整个平面上是实可微的. 因而,  $f$  是  $\mathbb{C}$  上的全纯函数, 而且

$$f'(z) = \frac{\partial f}{\partial z} = nz^{n-1}. \quad (7)$$

□

**例题 0.2** 研究函数  $f(z) = e^{-|z|^2}$ .

**解** 把  $f$  写为  $f(z) = e^{-z\bar{z}}$ , 于是  $\frac{\partial f}{\partial \bar{z}} = -e^{-z\bar{z}}z$ , 它只有在  $z = 0$  处才等于零. 因此,  $e^{-|z|^2}$  只有在  $z = 0$  处可微, 它在任何点处都不是全纯的. 但它对  $x, y$  有任意阶连续偏导数, 所以它是  $C^\infty(\mathbb{C})$  中的函数.

□

## 命题 0.3

设函数  $f(z) = u + iv$  在区域  $D$  内解析, 并满足下列条件之一:

- (1) 在  $D$  内  $f'(z) = 0$ .  
 (2)  $\overline{f(z)}$  在  $D$  内解析.  
 (3)  $|f(z)|$  在  $D$  内为常数.  
 (4)  $\operatorname{Re} f(z)$  或  $\operatorname{Im} f(z)$  在  $D$  内为常数.  
 (5)  $au + bv = c$ , 其中  $a, b$  与  $c$  是不全为零的实常数.

则  $f(z)$  是常数.

□

**证明** 设  $f(z) = u(x, y) + iv(x, y)$ .

- (1) 因为  $f'(z) = 0$ , 所以由定理 0.4 可知  $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0$ , 并且  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ . 于是  $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$ . 因此  $u, v$  都是常数, 故  $f$  是一常数.  
 (2) 若  $\overline{f(z)} = u(x, y) - iv(x, y)$  解析, 由 C. - R. 方程很容易得到  $u_x = u_y = v_x = v_y = 0$ , 从而  $u(x, y), v(x, y)$  在  $D$  内为常数, 故  $f(z)$  亦为常数.  
 (3) 若  $|f(z)| \equiv C = 0$ , 则显然  $f(z) \equiv 0$ . 若  $|f(z)| \equiv C \neq 0$ , 则  $f(z) \neq 0$ .

**证法一:** 由  $\overline{f(z)} \cdot f(z) \equiv C^2$ , 可推得  $\overline{f(z)} \equiv \frac{C^2}{f(z)}$  也是解析函数, 则利用命题 0.3(2) 可知  $f(z)$  在  $D$  内为常数.

**证法二:** 由  $|f(z)|^2 = C^2$  可得  $u^2 + v^2 = C^2 \neq 0$ , 分别对  $x, y$  微分, 再应用 C. - R. 方程, 讨论解二元一次齐次方

程组, 即得在  $D$  内  $u_x = v_y = u_y = v_x = 0$ .

(4) 若  $\operatorname{Re} f(z)$  在  $D$  内为常数, 可设  $u(x, y) \equiv C$ , 则  $u_x = u_y = 0$ , 由 C. - R. 方程, 得

$$v_x = -u_y \equiv 0, \quad u_x = v_y \equiv 0.$$

因此  $u(x, y) \equiv C_1, v(x, y) \equiv C_2, f(z) = C_1 + iC_2$  为常数. 若  $\operatorname{Im} f(z)$  在  $D$  内为常数, 同上可得证.

(5) 由条件可得  $au_x + bv_x = 0, au_y + bv_y = 0$ .

情况 1: 若  $a \neq 0, b = 0$  或  $a = 0, b \neq 0$ , 则再由命题 0.3(4) 可知  $u$  或  $v$  为常数.

情况 2: 若  $ab \neq 0$ , 则根据

$$au_x - bu_y = 0, \quad bu_x + au_y = 0,$$

注意到  $\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2 > 0$ , 系数矩阵满秩. 则上述方程组只有零解, 故  $u$  为常数. 同理  $v$  为常数. 故  $f(z)$  是常数.

□

### 定义 0.5 (调和函数)

设  $u$  是区域  $D$  上的实值函数, 如果  $u \in C^2(D)$ , 且对任意  $z \in D$ , 有

$$\Delta u(z) = \frac{\partial^2 u(z)}{\partial x^2} + \frac{\partial^2 u(z)}{\partial y^2} = 0, \quad (8)$$

就称  $u$  是  $D$  中的**调和函数**.  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  称为 **Laplace 算子**.

♣

### 命题 0.4

设  $u \in C^2(D)$ , 那么  $\Delta u = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}$ .

♠

**证明** 由(1)式, 有

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \end{aligned}$$

所以

$$\begin{aligned} \frac{\partial^2 u}{\partial z \partial \bar{z}} &= \frac{\partial}{\partial \bar{z}} \frac{\partial u}{\partial z} = \frac{1}{4} \left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) + i \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) \right] \\ &= \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{1}{4} \Delta u. \end{aligned}$$

□

### 定理 0.5

设  $f = u + iv \in H(D)$ , 那么  $u$  和  $v$  都是  $D$  上的调和函数.

♥

**证明** 因为  $f \in H(D)$ , 由定理 0.4, 有

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad \frac{\partial \bar{f}}{\partial z} = 0.$$

所以

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial^2 \bar{f}}{\partial z \partial \bar{z}} = 0.$$

于是, 由  $u = \frac{1}{2}(f + \bar{f})$  即得

$$\Delta u = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

同理可证  $\Delta v = 0$ .

□

### 定义 0.6 (共轭调和函数)

设  $u$  和  $v$  是  $D$  上的一对调和函数, 如果它们还满足 Cauchy-Riemann 方程

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \end{cases} \quad (9)$$

就称  $v$  为  $u$  的共轭调和函数.

♣

### 命题 0.5

全纯函数的实部和虚部就构成一对共轭调和函数.

♣

**证明** 由定理 0.5 和定理 0.4 立得.

□

### 定理 0.6

设  $u$  是单连通区域  $D$  上的调和函数, 则必存在  $u$  的共轭调和函数  $v$ , 使得  $u + iv$  是  $D$  上的全纯函数.

♥

**证明** 因为  $u$  满足 Laplace 方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

若令  $P = -\frac{\partial u}{\partial y}, Q = \frac{\partial u}{\partial x}$ , 则

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial y},$$

所以

$$Pdx + Qdy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

是一个全微分, 因而积分

$$\int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

与路径无关. 令

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy,$$

则

$$\begin{aligned} v(x, y) &= \int_{(x_0, y_0)}^{(x, y_0)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy + \int_{(x, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= \int_{x_0}^x -\frac{\partial u}{\partial y}dx + 0 + 0 + \int_{y_0}^y \frac{\partial u}{\partial x}dy \\ &= \int_{x_0}^x -\frac{\partial u}{\partial y}dx + \int_{y_0}^y \frac{\partial u}{\partial x}dy. \end{aligned}$$

那么

$$\begin{cases} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}. \end{cases}$$

所以, $v$  就是要求的  $u$  的共轭调和函数.

□