0.1 Hermite(埃尔米特) 插值

0.1.1 重节点均差与 Taylor(泰勒) 插值

定理 0.1

设 $f \in C^n[a,b]$, $x_0, x_1, \cdots, x_n$ 为 $[a,b]$ 上的相异节点, 则 $f[x_0, x_1, \cdots, x_n]$ 是其变量的连续函数.

定义 0.1 (重节点均差)

如果 $[a,b]$ 上的节点 $x_0, x_1, \cdots, x_n$ 互异, 根据均差定义, 若 $f \in C^1[a,b]$, 则有

$$\lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

由此定义**重节点均差**

$$f[x_0, x_0] = \lim_{x \to x_0} f[x_0, x] = f'(x_0).$$

类似地可定义**重节点的二阶均差**, 当 $x_1 \neq x_0$ 时, 有

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}.$$

当 $x_1 \to x_0$ 时, 有

$$f[x_0, x_0, x_0] = \lim_{\substack{x_1 \to x_0 \\ x_2 \to x_0}} f[x_0, x_1, x_2] = \frac{1}{2} f''(x_0).$$

一般地, 可定义 **n 阶重节点的均差**, 由(??)式则得

$$f[x_0, x_0, \cdots, x_0] = \lim_{\substack{x_1 \to x_0 \\ \vdots \\ x_n \to x_0}} f[x_0, x_1, \cdots, x_n] = \frac{1}{n!} f^{(n)}(x_0). \tag{1}$$

定理 0.2

设 $f(x)$ 在 $[a,b]$ 上存在 n 阶连续导数, 且 (a,b) 上存在 $n+1$ 阶导数,x_0 为 $[a,b]$ 内一定点, 则对于任意的 $x \in [a,b]$, 在 x, x_0 之间存在一个数 ξ 使得

$$f(x) = P_n(x) + R_n(x),$$

其中

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \tag{2}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}, \quad \xi \in (a,b). \tag{3}$$

称 (2) 式为 **Taylor(泰勒) 插值多项式**, 它就是一个 **Hermite(埃尔米特) 插值多项式**.

注 实际上, 上述 Taylor 插值多项式和余项之和就是 f 在 x_0 点带 Lagrange 余项的 Taylor 展开式.

注 $P_n(x)$ 实际上是在点 x_0 附近逼近 $f(x)$ 的一个带导数的插值多项式, 它满足条件

$$P_n^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, 1, \cdots, n. \tag{4}$$

实际上 Taylor(泰勒) 插值是牛顿插值的极限形式, 是只在一点 x_0 处给出 $n+1$ 个插值条件 (4) 得到的 n 次埃尔米特插值多项式.

一般地只要给出 $m+1$ 个插值条件 (含函数值和导数值) 就可造出次数不超过 m 次的埃尔米特插值多项式, 由于导数条件各不相同, 这里就不给出一般的埃尔米特插值公式.

证明 任取 x_0 邻域中 $n+1$ 个互异点 $x_0, x_1, \cdots, x_n$ 作为插值点, 根据定理??可得到相应的牛顿均差插值多项式

$$P_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots$$

$$+ f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1}),$$

$$R_n(x) = f(x) - P_n(x) = f[x, x_0, \cdots, x_n]\omega_{n+1}(x),$$

在上述牛顿均差插值多项式中, 若令 $x_i \to x_0\ (i = 1, 2, \cdots, n)$, 则由 (1) 式可得 Taylor(泰勒) 多项式

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

其余项为

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}, \quad \xi \in (a, b).$$

□

0.1.2 两个典型的 Hermite 插值

定理 0.3

若已知四阶可导函数 f 在插值点 $x_i\ (i = 0, 1, 2)$ 上的值为 $f(x_i)\ (i = 0, 1, 2)$ 及一个导数值 $f'(x_1)$, 记 f 的三次插值多项式为 $P(x)$, 且满足条件

$$P(x_i) = f(x_i), \quad i = 0, 1, 2\text{及}P'(x_1) = f'(x_1).$$

则插值多项式可表示为

$$P(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + A(x - x_0)(x - x_1)(x - x_2),$$

其中

$$A = \frac{f'(x_1) - f[x_0, x_1] - (x_1 - x_0)f[x_0, x_1, x_2]}{(x_1 - x_0)(x_1 - x_2)}.$$

余项表达式为

$$R(x) = \frac{1}{4!}f^{(4)}(\xi)(x - x_0)(x - x_1)^2(x - x_2), \tag{5}$$

式中 ξ 位于 x_0, x_1, x_2 和 x 所界定的范围内.

♡

注 一般上述插值多项式的系数 A 都是用待定系数法求解, 并不直接套用上述 A 的公式. 即先待定 A, 得到插值多项式 $P(x)$, 再代入 $P'(x_1) = f'(x_1)$ 中解出 A.

证明 由给定条件及牛顿均差插值多项式, 可确定次数不超过 3 的插值多项式. 由于此多项式通过点 $(x_0, f(x_0))$, $(x_1, f(x_1))$ 及 $(x_2, f(x_2))$, 故其形式为

$$P(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + A(x - x_0)(x - x_1)(x - x_2),$$

其中 A 为待定常数, 可由条件 $P'(x_1) = f'(x_1)$ 确定, 通过计算可得

$$A = \frac{f'(x_1) - f[x_0, x_1] - (x_1 - x_0)f[x_0, x_1, x_2]}{(x_1 - x_0)(x_1 - x_2)}.$$

为了求出余项 $R(x) = f(x) - P(x)$ 的表达式, 可设

$$R(x) = f(x) - P(x) = k(x)(x - x_0)(x - x_1)^2(x - x_2),$$

其中 $k(x)$ 为待定函数. 构造

$$\varphi(t) = f(t) - P(t) - k(x)(t - x_0)(t - x_1)^2(t - x_2),$$

显然 $\varphi(x_j) = 0\ (j = 0, 1, 2)$, 且 $\varphi'(x_1) = 0$, $\varphi(x) = 0$. 故 $\varphi(t)$ 在 (a, b) 内有 5 个零点 (二重根算两个). 假设 f 具有较好的可微性, 反复应用罗尔定理, 得 $\varphi^{(4)}(t)$ 在 (a, b) 内至少有一个零点 ξ, 故

$$\varphi^{(4)}(\xi) = f^{(4)}(\xi) - 4!k(x) = 0,$$

于是

$$k(x) = \frac{1}{4!}f^{(4)}(\xi),$$

余项表达式为

$$R(x)=\frac{1}{4!}f^{(4)}(\xi)(x-x_0)(x-x_1)^2(x-x_2),$$

式中 ξ 位于 x_0,x_1,x_2 和 x 所界定的范围内. □

例题 0.1 给定 $f(x)=x^{3/2}, x_0=\frac{1}{4}, x_1=1, x_2=\frac{9}{4}$, 试求 $f(x)$ 在 $\left[\frac{1}{4},\frac{9}{4}\right]$ 上的三次埃尔米特插值多项式 $P(x)$, 使它满足 $P(x_i)=f(x_i)\ (i=0,1,2), P'(x_1)=f'(x_1)$, 并写出余项表达式.

解 由所给节点可求出

$$f_0=f\left(\frac{1}{4}\right)=\frac{1}{8},\quad f_1=f(1)=1,\quad f_2=f\left(\frac{9}{4}\right)=\frac{27}{8},$$

$$f'(x)=\frac{3}{2}x^{1/2},\quad f'(1)=\frac{3}{2}.$$

利用牛顿均差插值, 先求均差表如表 1.

表 1: 均差表

x_i	f_i		
$\frac{1}{4}$	$\frac{1}{8}$		
1	1	$\frac{7}{6}$	$\frac{11}{30}$
$\frac{9}{4}$	$\frac{27}{8}$	$\frac{19}{10}$	

于是有 $f[x_0,x_1]=\frac{7}{6}, f[x_0,x_1,x_2]=\frac{11}{30}$. 故可令

$$\begin{aligned}P(x)=&\frac{1}{8}+\frac{7}{6}\left(x-\frac{1}{4}\right)+\frac{11}{30}\left(x-\frac{1}{4}\right)(x-1)\\&+A\left(x-\frac{1}{4}\right)(x-1)\left(x-\frac{9}{4}\right).\end{aligned}$$

再由条件 $P'(1)=f'(1)=\frac{3}{2}$ 可得

$$P'(1)=\frac{7}{6}+\frac{11}{30}\cdot\frac{3}{4}+A\cdot\frac{3}{4}\left(-\frac{5}{4}\right)=\frac{3}{2},$$

解出

$$A=-\frac{16}{15}\left(\frac{3}{2}-\frac{7}{6}-\frac{11}{40}\right)=-\frac{14}{225}.$$

于是所求的三次埃尔米特多项式为

$$\begin{aligned}P(x)&=\frac{1}{8}+\frac{7}{6}\left(x-\frac{1}{4}\right)+\frac{11}{30}\left(x-\frac{1}{4}\right)(x-1)-\frac{14}{225}\left(x-\frac{1}{4}\right)(x-1)\left(x-\frac{9}{4}\right)\\&=-\frac{14}{225}x^3+\frac{263}{450}x^2+\frac{233}{450}x-\frac{1}{25},\end{aligned}$$

余项为

$$\begin{aligned}R(x)=f(x)-P(x)&=\frac{f^{(4)}(\xi)}{4!}\left(x-\frac{1}{4}\right)(x-1)^2\left(x-\frac{9}{4}\right)\\&=\frac{1}{4!}\cdot\frac{9}{16}\xi^{-5/2}\left(x-\frac{1}{4}\right)(x-1)^2\left(x-\frac{9}{4}\right),\quad \xi\in\left(\frac{1}{4},\frac{9}{4}\right).\end{aligned}$$

□

定理 0.4 (两点三次插值多项式)

若已知四阶可导函数 f 在插值点 x_k,x_{k+1} 上的值为 $y_k=f(x_k),\ y_{k+1}=f(x_{k+1})$ 及导数值为

$m_k = f'(x_k)$, $m_{k+1} = f'(x_{k+1})$, 记 f 的三次插值多项式为 $H_3(x)$, 且满足条件

$$\left.\begin{aligned} H_3(x_k) = y_k, \quad H_3(x_{k+1}) = y_{k+1}, \\ H_3'(x_k) = m_k, \quad H_3'(x_{k+1}) = m_{k+1}. \end{aligned}\right\} \tag{6}$$

则插值多项式可表示为

$$\begin{aligned} H_3(x) = & \left(1 + 2\frac{x - x_k}{x_{k+1} - x_k}\right)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 y_k + \left(1 + 2\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 y_{k+1} \\ & + (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 m_k + (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 m_{k+1}, \end{aligned} \tag{7}$$

其余项 $R_3(x) = f(x) - H_3(x)$ 可表示为

$$R_3(x) = \frac{1}{4!} f^{(4)}(\xi)(x - x_k)^2 (x - x_{k+1})^2, \quad \xi \in (x_k, x_{k+1}). \tag{8}$$

♡

证明 令

$$H_3(x) = \alpha_k(x) y_k + \alpha_{k+1}(x) y_{k+1} + \beta_k(x) m_k + \beta_{k+1}(x) m_{k+1}, \tag{9}$$

其中 $\alpha_k(x), \alpha_{k+1}(x), \beta_k(x), \beta_{k+1}(x)$ 是关于节点 x_k 及 x_{k+1} 的三次埃尔米特插值基函数, 它们应分别满足条件

$$\alpha_k(x_k) = 1, \quad \alpha_k(x_{k+1}) = 0, \quad \alpha_k'(x_k) = \alpha_k'(x_{k+1}) = 0;$$

$$\alpha_{k+1}(x_k) = 0, \quad \alpha_{k+1}(x_{k+1}) = 1, \quad \alpha_{k+1}'(x_k) = \alpha_{k+1}'(x_{k+1}) = 0;$$

$$\beta_k(x_k) = \beta_k(x_{k+1}) = 0, \quad \beta_k'(x_k) = 1, \quad \beta_k'(x_{k+1}) = 0;$$

$$\beta_{k+1}(x_k) = \beta_{k+1}(x_{k+1}) = 0, \quad \beta_{k+1}'(x_k) = 0, \quad \beta_{k+1}'(x_{k+1}) = 1.$$

根据给定条件可令

$$\alpha_k(x) = (ax + b)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2,$$

显然

$$\alpha_k(x_{k+1}) = \alpha_k'(x_{k+1}) = 0.$$

再利用

$$\alpha_k(x_k) = a x_k + b = 1,$$

及

$$\alpha_k'(x_k) = 2\frac{a x_k + b}{x_k - x_{k+1}} + a = 0,$$

解得

$$a = -\frac{2}{x_k - x_{k+1}}, \quad b = 1 + \frac{2 x_k}{x_k - x_{k+1}},$$

于是求得

$$\alpha_k(x) = \left(1 + 2\frac{x - x_k}{x_{k+1} - x_k}\right)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2. \tag{10}$$

同理可求得

$$\alpha_{k+1}(x) = \left(1 + 2\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2. \tag{11}$$

为求 $\beta_k(x)$, 由给定条件可令

$$\beta_k(x) = a(x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2,$$

直接由 $\beta_k'(x_k) = a = 1$ 得到

$$\beta_k(x) = (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2. \tag{12}$$

同理有

$$\beta_{k+1}(x) = (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2. \tag{13}$$

将 (10) 式 (13) 式的结果代入 (9) 式得

$$\begin{aligned} H_3(x) = &\left(1 + 2\frac{x - x_k}{x_{k+1} - x_k}\right)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 y_k + \left(1 + 2\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 y_{k+1} \\ &+ (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 m_k + (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 m_{k+1}, \end{aligned}$$

其余项 $R_3(x) = f(x) - H_3(x)$. 类似 (5) 式可得

$$R_3(x) = \frac{1}{4!} f^{(4)}(\xi)(x - x_k)^2(x - x_{k+1})^2, \quad \xi \in (x_k, x_{k+1}).$$

□