

0.1 常用积分公式

0.1.1 不定积分

1. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C \ (a > 0).$
2. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \ (a > 0).$ 3. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C \ (a > 0).$
3. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C \ (a > 0).$
4. $\int \ln x dx = x \ln x - x + C.$
5. (1) $\int \sec x dx = \ln |\sec x + \tan x| + C;$
(2) $\int \csc x dx = \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C = \ln |\csc x - \cot x| + C = \ln \left| \tan \frac{x}{2} \right| + C.$
6. $\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} \left[x \sqrt{x^2 \pm a^2} \pm a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \right] + C \ (a > 0);$
 $\int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left[x \sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} \right] + C \ (a > 0).$
7. $\int e^{ax} \cos bxdx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C \ (ab \neq 0);$
 $\int e^{ax} \sin bxdx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C \ (ab \neq 0).$
8. $\int x \cos nxdx = \frac{1}{n^2} \cos nx + \frac{x}{n} \sin nx + C \ (n \neq 0);$
 $\int x \sin nxdx = \frac{1}{n^2} \sin nx - \frac{x}{n} \cos nx + C \ (n \neq 0).$
9. 记 $I(m, n) = \int \cos^m x \sin^n x dx, \forall n, m \in \mathbb{N}$, 则

$$I(m, n) = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} I(m-2, n) \quad (m \geq 2, n \geq 0);$$

$$= -\frac{\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} I(m, n-2) \quad (m \geq 0, n \geq 2).$$

证明

- 1.
- 2.
- 3.
- 4.

5. (1)

(2) 第一种:

$$\begin{aligned} \int \csc x dx &= \int \frac{\sin x}{1 - \cos^2 x} dx = \int \frac{1}{\cos^2 x - 1} d(\cos x) = \frac{1}{2} \int \frac{1}{\cos x - 1} - \frac{1}{\cos x + 1} d(\cos x) \\ &= \frac{1}{2} \ln |\cos x - 1| - \frac{1}{2} \ln |\cos x + 1| + C = \frac{1}{2} \ln \left| \frac{\cos x - 1}{\cos x + 1} \right| + C. \end{aligned}$$

第二种:

$$\int \csc x dx = \frac{1}{2} \ln |\cos x - 1| - \frac{1}{2} \ln |\cos x + 1| + C = \ln \left| \tan \frac{x}{2} \right| + C.$$

第三种:

$$\begin{aligned}\int \csc x dx &= \int \frac{\csc x (\csc x - \cot x)}{\csc x - \cot x} dx = \ln |\csc x - \cot x| + C \\ &= \frac{\csc^2 x - \cot^2 x = 1}{\csc x + \cot x} \ln \left| \frac{1}{\csc x + \cot x} \right| + C = -\ln |\csc x + \cot x| + C.\end{aligned}$$

6.

7.

8.

9.

□

0.1.2 定积分

1. 记 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx, \forall n \in \mathbb{N}$, 则

$$I_n = \frac{n-1}{n} I_{n-2}, \quad \forall n \geq 2.$$

从而

$$I_n = \begin{cases} \frac{(n-1)!!}{n!!} I_0 = \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2}, & n \text{ 为偶数} \\ \frac{(n-1)!!}{n!!} I_1 = \frac{(n-1)!!}{n!!}, & n \text{ 为奇数} \end{cases}. \quad (1)$$

2. 记 $J(m, n) = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx, \forall n, m \in \mathbb{N}$, 则

$$J(m, n) = \frac{m-1}{m+n} J(m-2, n), \quad \forall n, m \geq 2.$$

$$J(m, n) = \frac{n-1}{m+n} J(m, n-2), \quad \forall n, m \geq 2.$$

从而

$$J(m, n) = \begin{cases} \frac{(m-1)!!(n-1)!!}{(m+n)!!}, & m, n \text{ 不全为偶数} \\ \frac{(m-1)!!(n-1)!!}{(m+n)!!} \cdot \frac{\pi}{2}, & m, n \text{ 全为偶数} \end{cases}. \quad (2)$$

3.

注 公式(1)(2)通常称为“点火公式”。