0.1 三角函数相关

0.1.1 三角函数

命题 0.1 (三角平方差公式)

$\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y) = \cos(y-x)\cos(y+x) = \cos^2 y - \cos^2 x.$

证明 首先, 我们有

$$\cos^2 x - \cos^2 y = 1 - \sin^2 x - \left(1 - \sin^2 y\right) = \sin^2 y - \sin^2 x.$$

接着, 我们有

$$\begin{aligned}\sin(x-y)\sin(x+y) &= (\sin x\cos y - \cos x\sin y)(\sin x\cos y + \cos x\sin y)\\ &= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y\\ &= \sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y\\ &= \sin^2 x - \sin^2 y;\end{aligned}$$

$$\begin{aligned}\cos(y-x)\cos(y+x) &= (\cos x\cos y + \sin x\sin y)(\cos x\cos y - \sin x\sin y)\\ &= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y\\ &= \cos^2 x\cos^2 y - (1-\cos^2 x)(1-\cos^2 y)\\ &= \cos^2 x - \cos^2 y.\end{aligned}$$

故结论得证. □

0.1.2 反三角函数

命题 0.2

(1) $\arctan x + \arctan\dfrac{1}{x} = \begin{cases}\dfrac{\pi}{2}, & x > 0\\ -\dfrac{\pi}{2}, & x < 0\end{cases}.$

(2) $\arctan x - \arctan y = \arctan\dfrac{x-y}{1+xy}, \forall x, y \in \mathbf{R}.$

证明

1. 令 $f(x) = \arctan x + \arctan\dfrac{1}{x}$, 则

$$f'(x) = \frac{1}{x^2+1} + \frac{1}{(\frac{1}{x})^2+1}(-\frac{1}{x^2}) = \frac{1}{x^2+1} - \frac{1}{x^2+1} = 0$$

故 $f(x)$ 为常函数, 于是就有 $f(x) = f(1) = \dfrac{\pi}{2}, \forall x > 0\,; f(x) = f(-1) = -\dfrac{\pi}{2}, \forall x < 0.$

2.

□

0.1.3 双曲三角函数

命题 0.3

(1) $\cosh x = \dfrac{e^x + e^{-x}}{2} \geqslant 1$,

(2) $\sinh x = \dfrac{e^x - e^{-x}}{2} \geqslant x$.

证明 可以分别利用均值不等式和求导进行证明. □

命题 0.4

1. $\cosh^2 x - \sinh^2 x = 1$.
2. $\cosh(2x) = 2\cosh^2 x - 1 = 1 - 2\sinh^2 x$.
3. $\sinh(2x) = 2\sinh x \cosh x$.

证明

□