

## 0.1 Vandermode 行列式

本节我们用  $V_n(x_1, x_2, \dots, x_n)$  表示  $n$  阶 Vandermonde 行列式.

### 定义 0.1

对  $1 \leq i \leq n$ ,  $V_n^{(i)}(x_1, x_2, \dots, x_n)$  表示删除  $V_n(x_1, x_2, \dots, x_n)$  的第  $i$  行  $(x_1^{i-1}, x_2^{i-1}, \dots, x_n^{i-1})$  之后新添第  $n$  行  $(x_1^n, x_2^n, \dots, x_n^n)$  所得  $n$  阶行列式.

### 定义 0.2

$\Delta_n(x_1, x_2, \dots, x_n)$  表示将  $V_n(x_1, x_2, \dots, x_n)$  的第  $n$  行换成  $(x_1^{n+1}, x_2^{n+1}, \dots, x_n^{n+1})$  所得  $n$  阶行列式.

**例题 0.1** 设初等对称多项式

$$\sigma_j = \sum_{1 \leq k_1 < k_2 < \dots < k_j \leq n} x_{k_1} x_{k_2} \dots x_{k_j}, j = 1, 2, \dots, n, \quad (1)$$

我们有

$$V_n^{(i)}(x_1, x_2, \dots, x_n) = \sigma_{n-i+1} V_n(x_1, x_2, \dots, x_n), i = 1, 2, \dots, n. \quad (2)$$

**证明** (加边法) 不妨设  $x_i, 1 \leq i \leq n$  互不相同. 设

$$D_n(x) \triangleq \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -x & x_1 & x_2 & \dots & x_n \\ (-x)^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-x)^n & x_1^n & x_2^n & \dots & x_n^n \end{vmatrix}.$$

由行列式性质我们知道  $D_n$  是  $n$  次多项式且有  $n$  个根  $-x_1, -x_2, \dots, -x_n$ . 于是我们有

$$D_n(x) = c(x + x_1)(x + x_2) \dots (x + x_n). \quad (3)$$

把  $D_n(x)$  按第一列展开得

$$D_n(x) = \sum_{i=1}^n V_n^{(i)}(x_1, x_2, \dots, x_n) x^{i-1} + V_n(x_1, x_2, \dots, x_n) x^n. \quad (4)$$

于是比较(3)式和(4)式最高次项系数, 我们有  $c = V_n(x_1, x_2, \dots, x_n)$ . 定义  $\sigma_0 = 1$ , 利用根和系数的关系 (Vieta 定理), 结合(3)式和(4)式得

$$D_n(x) = \sum_{i=1}^{n+1} \sigma_{n-i+1} V_n(x_1, x_2, \dots, x_n) x^{i-1} = \sum_{i=1}^n V_n^{(i)}(x_1, x_2, \dots, x_n) x^{i-1} + V_n(x_1, x_2, \dots, x_n) x^n,$$

比较上式等号两边  $x^i (1 \leq i \leq n)$  的系数就能得到(2). □

**例题 0.2** 证明:

$$\Delta_n(x_1, x_2, \dots, x_n) = \left( \sum_{k=1}^n x_k^2 + \sum_{1 \leq i < j \leq n} x_i x_j \right) V_n(x_1, x_2, \dots, x_n) \quad (5)$$

**证明** 不妨设  $x_i, 1 \leq i \leq n$  互不相同. 设  $n+1$  次多项式

$$P_{n+1}(x) \triangleq \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -x & x_1 & x_2 & \dots & x_n \\ (-x)^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-x)^{n-1} & x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ (-x)^{n+1} & x_1^{n+1} & x_2^{n+1} & \dots & x_n^{n+1} \end{vmatrix}$$

注意到有  $n$  个根  $-x_1, -x_2, \dots, -x_n$ 。我们用  $-x_{n+1}$  表示  $P_{n+1}$  第  $n+1$  个根。于是我们有

$$P_{n+1}(x) = c(x+x_1)(x+x_2)\cdots(x+x_n)(x+x_{n+1}). \quad (6)$$

将  $P_{n+1}(x)$  按第一列展开得

$$P_{n+1}(x) = -V_n(x_1, x_2, \dots, x_n)x^{n+1} + \Delta_n(x_1, x_2, \dots, x_n)x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 \quad (7)$$

其中  $a_{n-2}, \dots, a_0$  是某些与  $x_j$  有关的  $n$  阶行列式。比较(6)和(7)式的系数可知  $c = -V_n(x_1, x_2, \dots, x_n)$ 。于是结合(6)式, 并利用 Vieta 定理得

$$P_{n+1}(x) = -V_n(x_1, x_2, \dots, x_n)(x^{n+1} + \delta_1 x^n + \delta_2 x^{n-1} + \cdots + \delta_{n-1}) \quad (8)$$

这里  $\delta_j$  类似(1)式定义是  $x_1, x_2, \dots, x_n, x_{n+1}$  的初等对称多项式。比较(7)(8)式的  $x^{n-1}$  系数可得  $\Delta_n(x_1, x_2, \dots, x_n) = -\delta_2 V_n(x_1, x_2, \dots, x_n)$ 。因为  $P_{n+1}(x)$  没有  $x^n$  的项, 所以

$$\delta_1 = x_1 + x_2 + \cdots + x_{n+1} = 0 \Rightarrow x_{n+1} = -(x_1 + x_2 + \cdots + x_n).$$

从而

$$\begin{aligned} \delta_2 &= \sum_{1 \leq i < j \leq n+1} x_i x_j = \sum_{1 \leq i < j \leq n} x_i x_j + x_{n+1} \sum_{i=1}^n x_i \\ &= \sum_{1 \leq i < j \leq n} x_i x_j - (x_1 + x_2 + \cdots + x_n) \sum_{i=1}^n x_i \\ &= \sum_{1 \leq i < j \leq n} x_i x_j - \left( \sum_{i=1}^n x_i \right)^2 = - \sum_{i=1}^n x_i^2 - \sum_{1 \leq i < j \leq n} x_i x_j. \end{aligned}$$


现在就有(5)成立。 □

### 命题 0.1

设  $A = (a_{ij})_{n \times n}$ ,  $f_i(x) = a_{i1} + a_{i2}x + \cdots + a_{in}x^{n-1}$  ( $i = 1, 2, \dots, n$ ), 证明: 对任何复数  $x_1, x_2, \dots, x_n$ , 都有

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_1) & f_n(x_2) & \cdots & f_n(x_n) \end{vmatrix} = |A| \cdot V_n(x_1, x_2, \dots, x_n)$$

这里  $V_n(x_1, x_2, \dots, x_n)$  表示  $x_1, x_2, \dots, x_n$  的 Vandermonde 行列式。 ▲


 **笔记** 关键是利用命题??.

**证明** 直接由矩阵乘法观察知显然。 □

### 推论 0.1

设  $f_k(x) = x^k + a_{k1}x^{k-1} + a_{k2}x^{k-2} + \cdots + a_{kk}$ , 求下列行列式的值:

$$\begin{vmatrix} 1 & f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & f_1(x_n) & f_2(x_n) & \cdots & f_{n-1}(x_n) \end{vmatrix}.$$

 **笔记** 知道这类行列式化简的操作即可. 以后这种行列式化简操作不再作额外说明.

**注** 也可以由命题 0.1 直接得到.

**解 解法一:** 利用行列式的性质可得

$$\begin{vmatrix} 1 & f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & f_1(x_n) & f_2(x_n) & \cdots & f_{n-1}(x_n) \end{vmatrix} = \begin{vmatrix} 1 & x_1 + a_{11} & x_1^2 + a_{21}x_1 + a_{22} & \cdots & x_1^{n-1} + a_{n-1,1}x_1^{n-2} + \cdots + a_{n-1,n-2}x_1 + a_{n-1,n-1} \\ 1 & x_2 + a_{11} & x_2^2 + a_{21}x_2 + a_{22} & \cdots & x_2^{n-1} + a_{n-1,1}x_2^{n-2} + \cdots + a_{n-1,n-2}x_2 + a_{n-1,n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n + a_{11} & x_n^2 + a_{21}x_n + a_{22} & \cdots & x_n^{n-1} + a_{n-1,1}x_n^{n-2} + \cdots + a_{n-1,n-2}x_n + a_{n-1,n-1} \end{vmatrix}$$

$$\begin{vmatrix} -a_{ii}j_1 + j_{i+1}, i = 1, 2, \cdots, n-1 \\ -a_{i,i-1}j_2 + j_{i+1}, i = 2, 3, \cdots, n-1 \\ \cdots \\ -a_{i,i-(n-3)}j_{n-2} + j_{i+1}, i = n-2, n-1 \\ -a_{n-1,1}j_{n-1} + j_n \end{vmatrix} \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

**解法二:** 由命题 0.1 可得

$$\begin{vmatrix} 1 & f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & f_1(x_n) & f_2(x_n) & \cdots & f_{n-1}(x_n) \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & & \vdots \\ f_{n-1}(x_1) & f_{n-1}(x_2) & \cdots & f_{n-1}(x_n) \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 + a_{11} & x_2 + a_{11} & \cdots & x_n + a_{11} \\ x_1^2 + a_{21}x_1 + a_{22} & x_2^2 + a_{21}x_2 + a_{22} & \cdots & x_n^2 + a_{21}x_n + a_{22} \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} + \cdots + a_{n-1,n-1} & x_2^{n-1} + \cdots + a_{n-1,n-1} & \cdots & x_n^{n-1} + \cdots + a_{n-1,n-1} \end{vmatrix}$$

$$= V_n(x_1, x_2, \cdots, x_n) \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} = V_n(x_1, x_2, \cdots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

□

**例题 0.3** 计算

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 + 1 & x_2 + 1 & x_3 + 1 & \cdots & x_n + 1 \\ x_1^2 + x_1 & x_2^2 + x_2 & x_3^2 + x_3 & \cdots & x_n^2 + x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} + x_1^{n-2} & x_2^{n-1} + x_2^{n-2} & x_3^{n-1} + x_3^{n-2} & \cdots & x_n^{n-1} + x_n^{n-2} \end{vmatrix}$$

**证明** 由命题 0.1 我们知道

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 + 1 & x_2 + 1 & x_3 + 1 & \cdots & x_n + 1 \\ x_1^2 + x_1 & x_2^2 + x_2 & x_3^2 + x_3 & \cdots & x_n^2 + x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} + x_1^{n-2} & x_2^{n-1} + x_2^{n-2} & x_3^{n-1} + x_3^{n-2} & \cdots & x_n^{n-1} + x_n^{n-2} \end{vmatrix} = V_n(x_1, x_2, \cdots, x_n) \cdot \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}$$

$$= \prod_{1 \leq i < j \leq n} (x_j - x_i) \cdot \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$



## 命题 0.2

计算下列行列式的值:

$$|A| = \begin{vmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \cdots & a_1b_1^{n-2} & b_1^{n-1} \\ a_2^{n-1} & a_2^{n-2}b_2 & \cdots & a_2b_2^{n-2} & b_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2}b_n & \cdots & a_nb_n^{n-2} & b_n^{n-1} \end{vmatrix}.$$



解 若所有的  $a_i (i = 1, 2, \dots, n)$  都不为 0, 则有

$$\begin{aligned} |A| &= \begin{vmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \cdots & a_1b_1^{n-2} & b_1^{n-1} \\ a_2^{n-1} & a_2^{n-2}b_2 & \cdots & a_2b_2^{n-2} & b_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2}b_n & \cdots & a_nb_n^{n-2} & b_n^{n-1} \end{vmatrix} = \prod_{i=1}^n a_i^{n-1} \begin{vmatrix} 1 & \frac{b_1}{a_1} & \cdots & \frac{b_1^{n-2}}{a_1^{n-2}} & \frac{b_1^{n-1}}{a_1^{n-1}} \\ 1 & \frac{b_2}{a_2} & \cdots & \frac{b_2^{n-2}}{a_2^{n-2}} & \frac{b_2^{n-1}}{a_2^{n-1}} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \frac{b_n}{a_n} & \cdots & \frac{b_n^{n-2}}{a_n^{n-2}} & \frac{b_n^{n-1}}{a_n^{n-1}} \end{vmatrix} \\ &= \prod_{i=1}^n a_i^{n-1} \prod_{1 \leq i < j \leq n} \left( \frac{b_j}{a_j} - \frac{b_i}{a_i} \right) = \prod_{i=1}^n a_i^{n-1} \prod_{1 \leq i < j \leq n} \frac{a_ib_j - a_jb_i}{a_ja_i} = \prod_{1 \leq i < j \leq n} (a_ib_j - a_jb_i). \end{aligned}$$

若只有一个  $a_i$  为 0, 则将原行列式按第  $i$  行展开得到具有相同类型的  $n-1$  阶行列式

$$\begin{aligned} |A| &= \begin{vmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \cdots & a_1b_1^{n-2} & b_1^{n-1} \\ a_2^{n-1} & a_2^{n-2}b_2 & \cdots & a_2b_2^{n-2} & b_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_i^{n-1} & a_i^{n-2}b_i & \cdots & a_ib_i^{n-2} & b_i^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2}b_n & \cdots & a_nb_n^{n-2} & b_n^{n-1} \end{vmatrix} = \begin{vmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \cdots & a_1b_1^{n-2} & b_1^{n-1} \\ a_2^{n-1} & a_2^{n-2}b_2 & \cdots & a_2b_2^{n-2} & b_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_i^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2}b_n & \cdots & a_nb_n^{n-2} & b_n^{n-1} \end{vmatrix} \\ &\stackrel{\text{按第 } i \text{ 行展开}}{=} (-1)^{n+i} b_i^{n-1} \begin{vmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \cdots & a_1b_1^{n-2} \\ a_2^{n-1} & a_2^{n-2}b_2 & \cdots & a_2b_2^{n-2} \\ \vdots & \vdots & & \vdots \\ a_{i-1}^{n-1} & a_{i-1}^{n-2}b_{i-1} & \cdots & a_{i-1}b_{i-1}^{n-2} \\ a_{i+1}^{n-1} & a_{i+1}^{n-2}b_{i+1} & \cdots & a_{i+1}b_{i+1}^{n-2} \\ \vdots & \vdots & & \vdots \\ a_n^{n-1} & a_n^{n-2}b_n & \cdots & a_nb_n^{n-2} \end{vmatrix}. \end{aligned}$$

此时同理可得

$$\begin{aligned}
 |A| &= (-1)^{n+i} b_i^{n-1} \begin{vmatrix} a_1^{n-1} & a_1^{n-2}b_1 & \cdots & a_1b_1^{n-2} \\ a_2^{n-1} & a_2^{n-2}b_2 & \cdots & a_2b_2^{n-2} \\ \vdots & \vdots & & \vdots \\ a_{i-1}^{n-1} & a_{i-1}^{n-2}b_{i-1} & \cdots & a_{i-1}b_{i-1}^{n-2} \\ a_{i+1}^{n-1} & a_{i+1}^{n-2}b_{i+1} & \cdots & a_{i+1}b_{i+1}^{n-2} \\ \vdots & \vdots & & \vdots \\ a_n^{n-1} & a_n^{n-2}b_n & \cdots & a_nb_n^{n-2} \end{vmatrix} = (-1)^{n+i} b_i^{n-1} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} a_k^{n-1} \begin{vmatrix} 1 & \frac{b_1}{a_1} & \cdots & \frac{b_1^{n-2}}{a_1^{n-2}} \\ 1 & \frac{b_2}{a_2} & \cdots & \frac{b_2^{n-2}}{a_2^{n-2}} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{b_{i-1}}{a_{i-1}} & \cdots & \frac{b_{i-1}^{n-2}}{a_{i-1}^{n-2}} \\ 1 & \frac{b_{i+1}}{a_{i+1}} & \cdots & \frac{b_{i+1}^{n-2}}{a_{i+1}^{n-2}} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{b_n}{a_n} & \cdots & \frac{b_n^{n-2}}{a_n^{n-2}} \end{vmatrix} \\
 &= (-1)^{n+i} b_i^{n-1} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} a_k^{n-1} \prod_{\substack{1 \leq k < l \leq n \\ k, l \neq i}} \left( \frac{b_l}{a_l} - \frac{b_k}{a_k} \right) = (-1)^{n+i} b_i^{n-1} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} a_k^{n-1} \prod_{\substack{1 \leq k < l \leq n \\ k, l \neq i}} \frac{a_kb_l - a_lb_k}{a_ka_l} \\
 &= (-1)^{n+i} b_i^{n-1} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} a_k \cdot \prod_{\substack{1 \leq k < l \leq n \\ k, l \neq i}} (a_kb_l - a_lb_k) = (-1)^{n-i} b_i^{n-1} \prod_{\substack{1 \leq k \leq n \\ k \neq i}} a_k \cdot \prod_{\substack{1 \leq k < l \leq n \\ k, l \neq i}} (a_kb_l - a_lb_k) \\
 &= \prod_{1 \leq k < i} a_kb_i \prod_{i < l \leq n} (-a_lb_i) \cdot \prod_{\substack{1 \leq k < l \leq n \\ k, l \neq i}} (a_kb_l - a_lb_k) \\
 &= \prod_{1 \leq k < l \leq n} (a_kb_l - a_lb_k) \cdot (a_i = 0).
 \end{aligned}$$

若至少有两个  $a_i = a_j = 0$ , 则第  $i$  行与第  $j$  行成比例, 因此行列式的值等于 0. 经过计算发现, 后面两种情形的答案都可以统一到第一种情形的答案.

$$\text{综上所述, } |A| = \prod_{1 \leq i < j \leq n} (a_ib_j - a_jb_i).$$

□

**结论 连乘号计算小结:**

$$(1) \prod_{1 \leq i < j \leq n} a_ia_j = \prod_{i=1}^n a_i^{n-1}.$$

$$\begin{aligned}
 \text{证明: } \prod_{1 \leq i < j \leq n} a_ia_j &= \underbrace{a_2a_1 \cdot a_3a_2a_3a_1 \cdot a_4a_3a_4a_2a_4a_1 \cdots a_ka_{k-1}a_ka_{k-2} \cdots a_ka_1}_{n-1 \text{ 组}} \cdots \underbrace{a_na_{n-1}a_na_{n-2} \cdots a_na_1}_{n-1 \text{ 对}} \\
 &\stackrel{\text{从左往右按组计数}}{=} a_1^{n-1} a_2^{1+n-2} a_3^{2+n-3} a_4^{3+n-4} \cdots a_k^{k-1+n-k} \cdots a_n^{n-1} = \prod_{i=1}^n a_i^{n-1}.
 \end{aligned}$$

$$(2) \prod_{\substack{1 \leq i < j \leq n \\ i, j \neq k}} a_ia_j = \prod_{\substack{1 \leq i \leq n \\ i \neq k}} a_i^{n-2}, \text{ 其中 } k \in [1, n] \cap \mathbb{N}_+.$$

$$\begin{aligned}
 \text{证明: } \prod_{\substack{1 \leq i < j \leq n \\ i, j \neq k}} a_ia_j &= \underbrace{a_2a_1 \cdot a_3a_2a_3a_1 \cdots a_{k-1}a_{k-2} \cdots a_{k-1}a_1}_{k-2 \text{ 对}} \cdot \underbrace{a_{k+1}a_{k-1} \cdots a_{k+1}a_1}_{k-1 \text{ 对}} \cdots \underbrace{a_na_{n-1} \cdots a_na_{k+1}a_na_{k-1} \cdots a_na_1}_{n-2 \text{ 对}} \\
 &\stackrel{\text{从左往右按组计数}}{=} a_1^{n-2} a_2^{1+n-3} a_3^{2+n-4} a_4^{3+n-4} \cdots a_{k-1}^{k-2+n-k} a_{k+1}^{k-1+n-k-1} \cdots a_n^{n-2} = \prod_{\substack{1 \leq i \leq n \\ i \neq k}} a_i^{n-2}.
 \end{aligned}$$

注意: 从第  $k-1$  组开始, 后面每组都比原来少一对 (后面每组均缺少原本含  $a_k$  的那一对).

**例题 0.4** 计算  $D_{n+1} = \begin{vmatrix} (a_0 + b_0)^n & (a_0 + b_1)^n & \cdots & (a_0 + b_n)^n \\ (a_1 + b_0)^n & (a_1 + b_1)^n & \cdots & (a_1 + b_n)^n \\ \vdots & \vdots & \ddots & \vdots \\ (a_n + b_0)^n & (a_n + b_1)^n & \cdots & (a_n + b_n)^n \end{vmatrix}$ .

**解** 由二项式定理可知

$$(a_i + b_j)^n = a_i^n + C_n^1 a_i^{n-1} b_j + \cdots + C_n^{n-1} a_i b_j^{n-1} + b_j^n, \text{ 其中 } i, j = 0, 1, \cdots, n.$$

从而

$$\begin{aligned} D_{n+1} &= \begin{vmatrix} a_0^n + C_n^1 a_0^{n-1} b_0 + \cdots + C_n^{n-1} a_0 b_0^{n-1} + b_0^n & \cdots & a_0^n + C_n^1 a_0^{n-1} b_n + \cdots + C_n^{n-1} a_0 b_n^{n-1} + b_n^n \\ a_1^n + C_n^1 a_1^{n-1} b_0 + \cdots + C_n^{n-1} a_1 b_0^{n-1} + b_0^n & \cdots & a_1^n + C_n^1 a_1^{n-1} b_n + \cdots + C_n^{n-1} a_1 b_n^{n-1} + b_n^n \\ \vdots & \ddots & \vdots \\ a_{n-1}^n + C_n^1 a_{n-1}^{n-1} b_0 + \cdots + C_n^{n-1} a_{n-1} b_0^{n-1} + b_0^n & \cdots & a_{n-1}^n + C_n^1 a_{n-1}^{n-1} b_n + \cdots + C_n^{n-1} a_{n-1} b_n^{n-1} + b_n^n \\ a_n^n + C_n^1 a_n^{n-1} b_0 + \cdots + C_n^{n-1} a_n b_0^{n-1} + b_0^n & \cdots & a_n^n + C_n^1 a_n^{n-1} b_n + \cdots + C_n^{n-1} a_n b_n^{n-1} + b_n^n \end{vmatrix} \\ &= \begin{vmatrix} a_0^n & a_0^{n-1} & \cdots & a_0 & 1 \\ a_1^n & a_1^{n-1} & \cdots & a_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}^n & a_{n-1}^{n-1} & \cdots & a_{n-1} & 1 \\ a_n^n & a_n^{n-1} & \cdots & a_n & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ C_n^1 b_0 & C_n^1 b_1 & \cdots & C_n^1 b_{n-1} & C_n^1 b_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_n^{n-1} b_0^{n-1} & C_n^{n-1} b_1^{n-1} & \cdots & C_n^{n-1} b_{n-1}^{n-1} & C_n^{n-1} b_n^{n-1} \\ b_0^n & b_1^n & \cdots & b_{n-1}^n & b_n^n \end{vmatrix} \\ &\stackrel{\text{列倒排}}{=} (-1)^{\frac{n(n+1)}{2}} \begin{vmatrix} 1 & a_0 & \cdots & a_0^{n-1} & a_0^n \\ 1 & a_1 & \cdots & a_1^{n-1} & a_1^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} & a_n^n \end{vmatrix} \cdot \prod_{i=1}^{n-1} C_n^i \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ b_0 & b_1 & \cdots & b_{n-1} & b_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_0^{n-1} & b_1^{n-1} & \cdots & b_{n-1}^{n-1} & b_n^{n-1} \\ b_0^n & b_1^n & \cdots & b_{n-1}^n & b_n^n \end{vmatrix} \\ &= (-1)^{\frac{n(n+1)}{2}} \prod_{0 \leq j < i \leq n} (a_i - a_j) \prod_{i=1}^{n-1} C_n^i \prod_{0 \leq j < i \leq n} (b_i - b_j) = \prod_{i=1}^{n-1} C_n^i \prod_{0 \leq j < i \leq n} (a_j - a_i) (b_i - b_j). \end{aligned}$$

□

**例题 0.5** 求下列行列式的值:

$$|A| = \begin{vmatrix} 1 & \cos \theta_1 & \cos 2\theta_1 & \cdots & \cos (n-1)\theta_1 \\ 1 & \cos \theta_2 & \cos 2\theta_2 & \cdots & \cos (n-1)\theta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos \theta_n & \cos 2\theta_n & \cdots & \cos (n-1)\theta_n \end{vmatrix}.$$

**解** 由 De Moivre 公式及二项式定理, 可得

$$\begin{aligned} \cos k\theta + i \sin k\theta &= (\cos \theta + i \sin \theta)^k \\ &= \cos^k \theta + i C_k^1 \cos^{k-1} \theta \sin \theta - C_k^2 \cos^{k-2} \theta \sin^2 \theta + i C_k^3 \cos^{k-3} \theta \sin^3 \theta - \cdots \\ &= \cos^k \theta + i C_k^1 \cos^{k-1} \theta \sin \theta - C_k^2 \cos^{k-2} \theta (1 - \cos^2 \theta) + i C_k^3 \cos^{k-3} \theta \sin^3 \theta - \cdots \end{aligned}$$

比较实部可得

$$\begin{aligned} \cos k\theta &= \cos^k \theta (1 + C_k^2 + C_k^4 + \cdots) - C_k^2 \cos^{k-2} \theta + C_k^4 \cos^{k-4} \theta - \cdots \\ &= 2^{k-1} \cos^k \theta - C_k^2 \cos^{k-2} \theta + C_k^4 \cos^{k-4} \theta - \cdots \end{aligned}$$

利用这个事实, 依次将原行列式各列表示成  $\cos \theta_j (j = 2, 3, \cdots, n)$  的多项式.

再利用行列式的性质,可依次将第 3, 4,  $\dots$ ,  $n$  列消去最高次项外的其他项,从而得到

$$|A| = \begin{vmatrix} 1 & \cos \theta_1 & 2 \cos^2 \theta_1 & \cdots & 2^{n-2} \cos^{n-1} \theta_1 \\ 1 & \cos \theta_2 & 2 \cos^2 \theta_2 & \cdots & 2^{n-2} \cos^{n-1} \theta_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos \theta_n & 2 \cos^2 \theta_n & \cdots & 2^{n-2} \cos^{n-1} \theta_n \end{vmatrix} = 2^{\frac{1}{2}(n-1)(n-2)} \begin{vmatrix} 1 & \cos \theta_1 & \cos^2 \theta_1 & \cdots & \cos^{n-1} \theta_1 \\ 1 & \cos \theta_2 & \cos^2 \theta_2 & \cdots & \cos^{n-1} \theta_2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos \theta_n & \cos^2 \theta_n & \cdots & \cos^{n-1} \theta_n \end{vmatrix}$$

$$= 2^{\frac{1}{2}(n-1)(n-2)} \prod_{1 \leq i < j \leq n} (\cos \theta_j - \cos \theta_i).$$

□

**结论** 组合式计算常用公式:

$$(1) C_n^m = C_{n-1}^m + C_{n-1}^{m-1}$$

$$(2) C_n^0 + C_n^2 + \cdots = C_n^1 + C_n^3 + \cdots = 2^{n-1}$$

证明:(1)

$$C_n^m = \frac{n!}{m!(n-m)!} = \frac{(n-1)!(n-m+m)}{m!(n-m)!} = \frac{(n-1)!(n-m)}{m!(n-m)!} + \frac{(n-1)!m}{m!(n-m)!}$$

$$= \frac{(n-1)!}{m!(n-m-1)!} + \frac{(n-1)!}{(m-1)!(n-m)!} = C_{n-1}^m + C_{n-1}^{m-1}$$

(2)(i) 当  $n$  为奇数时, 由  $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$ , 可得

$$C_n^0 + C_n^2 + C_n^4 \cdots + C_n^{n-1} = C_{n-1}^0 + C_{n-1}^1 + C_{n-1}^2 + C_{n-1}^3 + C_{n-1}^4 \cdots + C_{n-1}^{n-2} + C_{n-1}^{n-1}$$

$$C_n^1 + C_n^3 + C_n^5 \cdots + C_n^n = C_{n-1}^0 + C_{n-1}^1 + C_{n-1}^2 + C_{n-1}^3 + C_{n-1}^4 + C_{n-1}^5 + \cdots + C_{n-1}^{n-1} + C_{n-1}^n$$

由于  $C_{n-1}^n = 0$ , 再对比上面两式每一项可知, 上面两式相等.

而上面两式相加, 得  $C_n^0 + C_n^1 + C_n^2 \cdots + C_n^{n-1} + C_n^n = (1+1)^n = 2^n$ .

故  $C_n^0 + C_n^2 + C_n^4 \cdots + C_n^{n-1} = C_n^1 + C_n^3 + C_n^5 \cdots + C_n^n = 2^{n-1}$ .

(ii) 当  $n$  为偶数时, 由  $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$ , 可得

$$C_n^0 + C_n^2 + C_n^4 \cdots + C_n^n = C_{n-1}^0 + C_{n-1}^1 + C_{n-1}^2 + C_{n-1}^3 + C_{n-1}^4 \cdots + C_{n-1}^{n-1} + C_{n-1}^n$$

$$C_n^1 + C_n^3 + C_n^5 \cdots + C_n^{n-1} = C_{n-1}^0 + C_{n-1}^1 + C_{n-1}^2 + C_{n-1}^3 + C_{n-1}^4 + C_{n-1}^5 + \cdots + C_{n-1}^{n-2} + C_{n-1}^{n-1}$$

由于  $C_{n-1}^n = 0$ , 再对比上面两式每一项可知, 上面两式相等.


而上面两式相加, 得  $C_n^0 + C_n^1 + C_n^2 \cdots + C_n^{n-1} + C_n^n = (1+1)^n = 2^n$ .

故  $C_n^0 + C_n^2 + C_n^4 \cdots + C_n^{n-1} = C_n^1 + C_n^3 + C_n^5 \cdots + C_n^n = 2^{n-1}$ .

综上所述,  $C_n^0 + C_n^2 + \cdots = C_n^1 + C_n^3 + \cdots = 2^{n-1}$ .

**例题 0.6** 求下列行列式式的值:

$$|A| = \begin{vmatrix} \sin \theta_1 & \sin 2\theta_1 & \cdots & \sin n\theta_1 \\ \sin \theta_2 & \sin 2\theta_2 & \cdots & \sin n\theta_2 \\ \vdots & \vdots & & \vdots \\ \sin \theta_n & \sin 2\theta_n & \cdots & \sin n\theta_n \end{vmatrix}.$$

 **笔记** 可以利用上一题类似的方法求解. 但我们给出另外一种解法, 目的是直接利用上一题的结论.

**解** 根据和差化积公式, 可得

$$\sin k\theta - \sin(k-2)\theta = 2 \sin \theta \cos(k-1)\theta, k = 2, 3, \dots, n.$$

再结合上一题结论, 可得

$$|A| = \begin{vmatrix} \sin \theta_1 & \sin 2\theta_1 & \cdots & \sin n\theta_1 \\ \sin \theta_2 & \sin 2\theta_2 & \cdots & \sin n\theta_2 \\ \vdots & \vdots & & \vdots \\ \sin \theta_n & \sin 2\theta_n & \cdots & \sin n\theta_n \end{vmatrix} = \begin{vmatrix} \sin \theta_1 & 2 \sin \theta_1 \cos \theta_1 & \cdots & 2 \sin \theta_1 \cos(n-1)\theta_1 \\ \sin \theta_2 & 2 \sin \theta_2 \cos \theta_2 & \cdots & 2 \sin \theta_2 \cos(n-1)\theta_2 \\ \vdots & \vdots & & \vdots \\ \sin \theta_n & 2 \sin \theta_n \cos \theta_n & \cdots & 2 \sin \theta_n \cos(n-1)\theta_n \end{vmatrix}$$

$$\begin{aligned}
&= 2^{n-1} \prod_{i=1}^n \sin \theta_i \begin{vmatrix} \cos \theta_1 & \cos 2\theta_1 & \cdots & \cos(n-1)\theta_1 \\ \cos \theta_2 & \cos 2\theta_2 & \cdots & \cos(n-1)\theta_2 \\ \vdots & \vdots & & \vdots \\ \cos \theta_n & \cos 2\theta_n & \cdots & \cos(n-1)\theta_n \end{vmatrix} = 2^{\frac{1}{2}(n-2)(n-1)+n-1} \prod_{i=1}^n \sin \theta_i \prod_{1 \leq i < j \leq n} (\cos \theta_j - \cos \theta_i) \\
&= 2^{\frac{1}{2}n(n-1)} \prod_{i=1}^n \sin \theta_i \prod_{1 \leq i < j \leq n} (\cos \theta_j - \cos \theta_i).
\end{aligned}$$

□

**命题 0.3 (多项式根的有限性)**

设多项式

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

若  $f(x)$  有  $n+1$  个不同的根  $b_1, b_2, \dots, b_{n+1}$ , 即  $f(b_1) = f(b_2) = \cdots = f(b_{n+1}) = 0$ , 求证:  $f(x)$  是零多项式, 即  $a_n = a_{n-1} = \cdots = a_1 = a_0 = 0$ .

◆

**证明** 由  $f(b_1) = f(b_2) = \cdots = f(b_{n+1}) = 0$ , 可知  $x_0 = a_0, x_1 = a_1, \dots, x_{n-1} = a_{n-1}, x_n = a_n$  是下列线性方程组的解:

$$\begin{cases} x_0 + b_1 x_1 + \cdots + b_1^{n-1} x_{n-1} + b_1^n x_n = 0, \\ x_0 + b_2 x_1 + \cdots + b_2^{n-1} x_{n-1} + b_2^n x_n = 0, \\ \dots\dots\dots \\ x_0 + b_{n+1} x_1 + \cdots + b_{n+1}^{n-1} x_{n-1} + b_{n+1}^n x_n = 0. \end{cases}$$

上述线性方程组的系数行列式是一个 Vandermode 行列式, 由于  $b_1, b_2, \dots, b_{n+1}$  互不相同, 所以系数行列式不等于零. 由 Crammer 法则可知上述方程组只有零解. 即有  $a_n = a_{n-1} = \cdots = a_1 = a_0 = 0$ . □